J. Eur. Math. Soc. 24, 3679–3702 (2022) DOI 10.4171/JEMS/1263

Chang-Yeon Chough

Proper base change for étale sheaves of spaces

Received September 30, 2018

Abstract. The goal of this paper is to prove the proper base change theorem for truncated coherent étale sheaves of spaces, generalizing the usual proper base change in étale cohomology. As applications, we show that the profinite étale topological type functor commutes with finite products and the symmetric powers of proper algebraic spaces over a separably closed field, respectively. In particular, the commutativity of the étale fundamental groups with finite products is extended to all higher homotopy groups.

Keywords. Proper base change, étale homotopy, infinity-categories, étale topologies, shape theory

1. Introduction

1.1. One of the fundamental results of étale cohomology theory is the proper base change theorem [25, Exposé XII.5.1]: Consider a cartesian square

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>{g}>> Y \end{CD} \tag{1.1.1}$$

in the category of schemes. Let F be a torsion étale sheaf of abelian groups on X. If f is proper, then for each integer $n \geq 0$, the canonical base change morphism

$$g^* R^n f_* F \to R^n f'_*(g'^* F)$$

is an isomorphism in $\mathrm{D}^+(Y'_{\text{ét}})$.

In this paper, we study a nonabelian version of the proper base change theorem: In the simple case of set-valued étale sheaves, the push-pull transformation

$$g^* f_* \to f'_* g'^*$$

Chang-Yeon Chough: Center for Quantum Structures in Modules and Spaces, Seoul National University, Seoul 08826, Republic of Korea; chough@snu.ac.kr

Mathematics Subject Classification (2020): Primary 14A30; Secondary 14F35, 18N60, 14F20, 55P55

is an isomorphism of functors from the category of set-valued étale sheaves on X to the category of set-valued étale sheaves on Y'. Extending to the ∞-categories of étale sheaves which take values in the ∞-category $\mathcal{S}$ of spaces, there is a commutative diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\acute{e}t}(X') & \xrightarrow{g'_*} & \mathcal{S}\mathrm{hv}_{\acute{e}t}(X) \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{S}\mathrm{hv}_{\acute{e}t}(Y') & \xrightarrow{g_*} & \mathcal{S}\mathrm{hv}_{\acute{e}t}(Y) \end{array} \tag{1.1.2}$$

The main result of this paper is the following:

Theorem 1.2. *Suppose we are given a cartesian square (1.1.1) of quasi-compact and quasi-separated schemes. Let F be a truncated coherent object of $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$. If f is proper, then the push-pull morphism*

$$\theta : g^* f_* F \to f'_* g'^* F$$

is an equivalence in $\mathcal{S}\mathrm{hv}_{\acute{e}t}(Y')$.

Remark 1.3. Suppose we are given a pullback square of topological spaces

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If f is a proper and separated, then the canonical base change map is an isomorphism in the derived category $\mathrm{D}^+(Y')$ of abelian category of sheaves of abelian groups on Y' (see [24, Exposé Vbis.4.1.1]). Likewise, in the simple case, the push-pull transformation is an isomorphism of functors from the category of sheaves of sets on X to the category of sheaves of sets on Y' (see, for example, [23, Tag 0D90]). The proper base change statements for topological spaces were unified and generalized into a single statement concerning sheaves of spaces by Jacob Lurie [14, 7.3.1.18]: Suppose the above diagram is a fibered square of locally compact Hausdorff spaces. If f is proper, then the push-pull transformation is an isomorphism of functors between the ∞-categories of $\mathcal{S}$-valued sheaves.

1.4. However, a nonabelian version in the algebro-geometric setting is subject to some restrictions; if it were true for every $\mathcal{S}$-valued sheaves, we would have the proper base change for any (not necessarily torsion) étale sheaves of abelian groups by the same argument as in [14, 7.3.1.19], which is not the case (see [25, Exposé XII.2]). Consequently, we need to impose some finiteness condition on étale sheaves of spaces. Indeed, we will show that the nonabelian proper base change theorem holds for those étale sheaves of spaces which are truncated coherent; see [16, A.7.4.4]. The proof will be parallel to the proof of the classical one: use passage to limit argument to reduce to the case where Y is a strictly henselian local ring and g is the inclusion of the closed point (cf. [25, Exposé XII.6.1]).

Remark 1.5. The proof cannot be reduced to the case of 0-truncated sheaves, the usual proper base change for set-valued sheaves. This is because 0-truncation functors do not necessarily commute with pushforward functors. We may therefore need a new approach as in the proof of Lurie's generalization for topological spaces. In the special case where F is locally constant constructible (see [16, E.2.5.1]), the key observation from [16, E.2.3.3] is that for geometric morphisms of ∞-topoi, the associated morphisms of locally constant constructible objects are completely determined by the maps of profinite shapes of the ∞-topoi.

Remark 1.6. Fix an integer $n \geq -2$. By virtue of [16, A.2.3.2], the proofs of Lemma 4.10 and Proposition 4.14 show that Theorem 1.2 holds for all filtered colimits of n-truncated and coherent objects of $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$. In the special case where $n = 0$, [25, Exposé IX.2.7.2] guarantees that we can recover [25, Exposé XII.5.1] (see also Remark 4.7).

Remark 1.7. According to the work of Clark Barwick, Saul Glasman, and Peter Haine [3, 7.3.3], the push-pull morphism is an equivalence for all truncated objects of $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ in the setting where $\mathcal{S}\mathrm{hv}_{\text{ét}}(X) \times \mathcal{S}\mathrm{hv}_{\text{ét}}(Y)$ is used in place of $\mathcal{S}\mathrm{hv}_{\text{ét}}(X')$ in the diagram (1.1.2); see also [3, 7.3.4] for a consequence of combining Theorem 1.2 and [3, 7.3.3].

1.8. Our applications of the nonabelian proper base change theorem (Theorem 1.2) come from étale homotopy theory. Let $f : X \to \operatorname{Spec} k$ and $g : Y \to \operatorname{Spec} k$ be morphisms of schemes, where k is a separably closed field. Assume f is proper and g is locally of finite type. It is well known from [20, Exposé X.1.7] that the natural map

$$\pi_1(X \times_k Y) \to \pi_1(X) \times \pi_1(Y)$$

is an isomorphism of profinite groups (here we omit the base points). This raises the natural question of whether it can be extended to higher homotopy groups. Or, more precisely, if the natural map

$$\widehat{\mathrm{h}}(X \times_k Y) \to \widehat{\mathrm{h}}(X) \times \widehat{\mathrm{h}}(Y) \tag{1.8.1}$$

is an equivalence of profinite spaces, where $\widehat{\mathrm{h}}$ denotes the profinite étale topological type functor (see [6, 3.2.6, 4.1.4]). By cohomological criteria [1, 4.3], it suffices to show that for each integer $n \geq 0$ and for each local coefficient system of finite abelian groups on $\widehat{\mathrm{h}}(X) \times \widehat{\mathrm{h}}(Y)$, the associated map of n-th cohomology groups is an isomorphism. Here we encounter a subtlety: in order to apply the Künneth formula in étale cohomology [25, Exposé XVII.5.4.3], we want to restrict our attention to the case where we are given local systems of finite abelian groups on $\widehat{\mathrm{h}}(X)$ and $\widehat{\mathrm{h}}(Y)$, rather than on their product $\widehat{\mathrm{h}}(X) \times \widehat{\mathrm{h}}(Y)$. However, Theorem 1.2 can provide a new perspective to understand the subtlety and the map (1.8.1). To this end, we shift from étale topological types to (∞-)shapes (see the proof of [6, 3.2.13] for the equivalence of these two approaches and [5, 1.3] for the equivalence of profinite completions in the ∞-category theory and in the model category theory). For each scheme T and the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(T)$ of $\mathcal{S}$-valued étale sheaves on T, the associated shape $\mathrm{Sh}(T)$ is defined to be the composite $\pi_* \circ \pi^* : \mathcal{S} \to \mathcal{S}$, where $\pi_* : \mathcal{S}\mathrm{hv}_{\text{ét}}(T) \to \mathcal{S}$ denotes the essentially unique geometric morphism of ∞-topoi. The advantage of this ∞-categorical perspective is an intermediate object $\mathrm{Sh}(X) \circ \mathrm{Sh}(Y)$—the composition of pro-spaces—which helps us to understand the homotopy type of $X \times_k Y$: there is a commutative triangle of pro-spaces

$$\begin{array}{ccc} & \mathrm{Sh}(X \times_k Y) & \\ \nearrow & & \searrow \\ \mathrm{Sh}(X) \circ \mathrm{Sh}(Y) & \longrightarrow & \mathrm{Sh}(X) \times \mathrm{Sh}(Y) \end{array} \tag{1.8.2}$$

Here the bottom map is under our control, and the left diagonal map is very closely related to the nonabelian proper base change theorem (Theorem 1.2) by the definition of shapes: in the setting of (5.3.1), the left diagonal is nothing but the canonical map $p_* p^* q_* q^* \to p_* q'_* p'^* q^*$. Using this idea, we will show in Theorem 5.3 that (1.8.1) is an equivalence.

Remark 1.9. The intermediate object $\mathrm{Sh}(X) \circ \mathrm{Sh}(Y)$ and the maps in (1.8.2) are naturally defined in the ∞-categorical setup, but hardly seen in the classical étale homotopy theory or in its model-categorical refinement.

1.10. A second application concerns symmetric powers of algebraic spaces. Using the qfh topology of schemes, Marc Hoyois proved that for a quasi-projective scheme X over a separably closed field k and a prime ℓ different from the characteristic of k, the natural map

$$\mathrm{Sh}(\mathrm{Sym}^n X) \to \mathrm{Sym}^n(\mathrm{Sh}(X)) \tag{1.10.1}$$

of [11, 5.7] is a $\mathbb{Z}/\ell$-homological equivalence; see [11, 5.6]. One of the main steps in the proof is that the map (1.8.1) before taking the profinite completions induces a $\mathbb{Z}/\ell$-homological equivalence for finite type k-schemes, which can be reduced to the Künneth formula in étale cohomology (see [11, 5.1]). As mentioned above, however, this is not the case for its profinite version in Theorem 5.3. Using Theorem 5.3 as an essential piece, we prove in Theorem 6.12 that (1.10.1) is a profinite equivalence for proper k-algebraic spaces X.

Remark 1.11. Under the extra assumption that X is geometrically normal, the commutativity of the profinite étale topological types and the symmetric powers (Theorem 6.12) recovers [26, Theorem 1] of Arnav Tripathy. In contrast with Tripathy's proof involving a concrete discussion of the étale fundamental groups of symmetric powers, our approach is very formal.

1.12. Conventions. We assume that the reader is familiar with the basic theory of ∞-categories as developed in [14]. We follow the set-theoretic convention of [14]. Let $*$ denote the final object of an ∞-category $\mathcal{C}$, if it exists.

2. Preliminaries on shapes and profinite completion

2.1. Let us give a quick review of pro-categories in ∞-category theory (see [16, A.8.1] for more details). Let $\mathcal{C}$ be an accessible ∞-category which admits finite limits. Let

$\mathrm{Pro}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}}$ denote the full subcategory spanned by those functors $\mathcal{C} \to \mathcal{S}$ which are accessible and preserve finite limits. We refer to it as the *∞-category of pro-objects of $\mathcal{C}$*. It has the following universal property: Let $\mathcal{D}$ be an ∞-category which admits small cofiltered limits, and let $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \subseteq \mathrm{Fun}(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ denote the full subcategory spanned by those functors which preserve small cofiltered limits. By virtue of [16, A.8.1.6], the Yoneda embedding induces an equivalence of ∞-categories

$$\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

2.2. According to [16, E.0.7.8], an object $X \in \mathcal{S}$ is defined to be *π-finite* if it satisfies the following conditions:

(i) The space X is n-truncated for some integer $n \geq -2$.

(ii) The set $\pi_0 X$ is finite.

(iii) For each $x \in X$ and each integer $m \geq 1$, the group $\pi_m(X, x)$ is finite.

Let $\mathcal{S}_\pi \subseteq \mathcal{S}$ denote the full subcategory spanned by the π-finite spaces. We refer to the associated pro-category $\mathrm{Pro}(\mathcal{S}_\pi)$ as the *∞-category of profinite spaces*.

2.3. The universal property of pro-categories applied to the fully faithful embedding $i : \mathcal{S}_\pi \hookrightarrow \mathcal{S}$ extends it to a fully faithful embedding $\mathrm{Pro}(i) : \mathrm{Pro}(\mathcal{S}_\pi) \to \mathrm{Pro}(\mathcal{S})$ of pro-categories. By virtue of [16, E.2.1.3], composition with i induces a forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi)$ which is left adjoint to $\mathrm{Pro}(i)$. We refer to it as the *profinite completion functor*.

2.4. Let $\mathcal{X}$ be an ∞-topos. For an essentially unique geometric morphism $\pi_* : \mathcal{X} \to \mathcal{S}$, the composition $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$ is an object of $\mathrm{Pro}(\mathcal{S})$, which is referred to as the *shape of* $\mathcal{X}$ and denoted by $\mathrm{Sh}(\mathcal{X})$. Note that [16, E.2.2.1] supplies a left adjoint functor $\mathrm{Sh} : \infty\mathcal{T}\mathrm{op} \to \mathrm{Pro}(\mathcal{S})$, where $\infty\mathcal{T}\mathrm{op}$ denotes the ∞-category of ∞-topoi of [14, 6.3.1.5]. Composing with the forgetful functor $\mathrm{Pro}(\mathcal{S}) \to \mathrm{Pro}(\mathcal{S}_\pi)$, one obtains a functor $\mathrm{Sh}_\pi : \infty\mathcal{T}\mathrm{op} \to \mathrm{Pro}(\mathcal{S}_\pi)$. We refer to the image $\mathrm{Sh}_\pi(\mathcal{X})$ of an ∞-topos $\mathcal{X}$ under this functor as the *profinite shape of* $\mathcal{X}$.

Definition 2.5. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We say that f_* is a *profinite shape equivalence* if it induces an equivalence $\mathrm{Sh}_\pi(\mathcal{X}) \to \mathrm{Sh}_\pi(\mathcal{Y})$ of profinite spaces.

2.6. According to [16, E.2.5.1], an object F of an ∞-topos $\mathcal{X}$ is *locally constant constructible* if there exists a finite collection $\{X_i \in \mathcal{X}\}_{1 \leq i \leq n}$ of objects such that the map $\coprod X_i \to *$ is an effective epimorphism, a collection $\{Y_i\}_{1 \leq i \leq n}$ of π-finite spaces, and equivalences $\pi_i^* Y_i \simeq F \times X_i$ in the ∞-topos $\mathcal{X}_{/X_i}$ for $1 \leq i \leq n$, where $\pi_i^* : \mathcal{S} \to \mathcal{X}_{/X_i}$ is the essentially unique geometric morphism. Let $\mathcal{X}^{\mathrm{lcc}} \subseteq \mathcal{X}$ denote the full subcategory spanned by the locally constant constructible objects. If $f^* : \mathcal{Y} \to \mathcal{X}$ is a geometric morphism of ∞-topoi, then it carries locally constant constructible objects to locally constant constructible objects. In what follows, it is important to note that locally constant constructible objects completely determine profinite shape equivalences; according to [16, E.2.3.3], the pushforward f_* is a profinite shape equivalence if and only if the restriction functor $f^* : \mathcal{Y}^{\mathrm{lcc}} \to \mathcal{X}^{\mathrm{lcc}}$ is an equivalence of ∞-categories.

2.7. According to the proof of [6, 3.2.13], for locally noetherian schemes, the ∞-categorical counterparts of the étale topological types of [6, 3.2.6] (or Artin–Mazur's étale homotopy types; see [2, 6.0.5] and [6, 3.2.11]) are the shapes of the hypercompletions [14, p. 669] of the ∞-topoi of $\mathcal{S}$-valued étale sheaves on the schemes; see also [5, 1.3] for the compatibility of profinite completions in model category theory and in ∞-category theory. In this paper, we will mainly work with the shapes of ∞-topoi rather than of the hypercompletions of ∞-topoi. When applying our results to étale topological types, there is no harm in doing so because profinite étale topological types are of interest to us:

Lemma 2.8. *Let $\mathcal{X}$ be an ∞-topos. Then the canonical map*

$$\mathrm{Sh}_\pi(\mathcal{X}^\wedge) \to \mathrm{Sh}_\pi(\mathcal{X})$$

is an equivalence of profinite spaces, where $\mathcal{X}^\wedge$ denotes the hypercompletion of $\mathcal{X}$.

Proof. By evaluating at π-finite spaces, the statement follows immediately from [14, 6.5.2.9], which asserts that $\tau_{\leq n}\,\mathcal{X} \subseteq \mathcal{X}^\wedge$ for every integer $n \geq -2$. ■

3. Limits of ∞-topoi and global sections

Throughout this section, we fix a diagram of ∞-topoi satisfying the following condition:

($*$) Let $\mathcal{I}$ be a filtered ∞-category and let $p : \mathcal{I}^{\mathrm{op}} \to \infty\mathcal{T}\mathrm{op}$ be a diagram of ∞-topoi $\{\mathcal{X}_i\}$. Assume that each $\mathcal{X}_i$ is coherent [16, A.2.1.6] and that for each of the transition morphisms $p_{ij\,*} : \mathcal{X}_j \to \mathcal{X}_i$, the restriction of $p_{ij\,*}$ to $\tau_{\leq n-1}\,\mathcal{X}_j$—the full subcategory of $\mathcal{X}_j$ spanned by the $(n-1)$-truncated objects—commutes with filtered colimits for all integers $n \geq 0$.

Remark 3.1. By virtue of [14, 5.3.1.18], we may assume that $\mathcal{I}$ is a filtered partially ordered set I throughout the rest of this section.

3.2. The following analogue of [24, Exposé VI.5.1] supplies a simple criterion which can be used to verify condition ($*$) in many cases of interest:

Theorem 3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be small ∞-categories which admit finite limits and which are equipped with finitary Grothendieck topologies. Let $f : \mathcal{C} \to \mathcal{D}$ be a continuous functor which commutes with finite limits. Let $p_* : \mathcal{S}\mathrm{hv}(\mathcal{D}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the induced geometric morphism of ∞-topoi. Then for each integer $n \geq 0$, the restriction of p_* to $\tau_{\leq n-1}\,\mathcal{S}\mathrm{hv}(\mathcal{D})$ commutes with filtered colimits.*

Proof. Let $\mathcal{J}$ be a filtered ∞-category, and let $\{F_\alpha\}_{\alpha\in\mathcal{J}}$ be a compatible family of $(n-1)$-truncated sheaves on $\mathcal{D}$. We claim that the canonical map

$$\operatorname*{colim}_{\alpha\in\mathcal{J}} p_* F_\alpha \to p_* \operatorname*{colim}_{\alpha\in\mathcal{J}} F_\alpha$$

is an equivalence of sheaves on $\mathcal{D}$. Using [14, 1.2.4.1], it suffices to show that for each $G \in \mathcal{S}\mathrm{hv}(\mathcal{C})$, the induced map

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(G, \operatorname{colim} p_* F_\alpha) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(G, p_* \operatorname{colim} F_\alpha)$$

is a weak homotopy equivalence. Invoking the finitary assumption, [16, A.3.1.3] guarantees that the composition of the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ with the sheafification $L : \mathcal{P}(\mathcal{C}) \to \mathcal{S}\mathrm{hv}(\mathcal{C})$ carries each object $C \in \mathcal{C}$ to a coherent object $L(j(C))$ of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. Applying [16, A.2.3.1] to the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})_{/L(j(C))}$, one deduces that the restriction of $\mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(L(j(C)), -)$ to $\tau_{\leq n-1}\, \mathcal{S}\mathrm{hv}(\mathcal{C})$ commutes with filtered colimits. Since $\mathcal{P}(\mathcal{C})$ is generated under small colimits by the Yoneda embedding, one can then reduce to establishing that the canonical map

$$\operatorname{colim} \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(L(j(C)), p_* F_\alpha) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}(\mathcal{C})}(L(j(C)), p_* \operatorname{colim} F_\alpha)$$

is an equivalence for each $C \in \mathcal{C}$. By the adjunction (p^*, p_*), the desired equivalence is a consequence of [16, A.2.3.1] applied to $\mathcal{S}\mathrm{hv}(\mathcal{D})_{/Lj(f(C))}$. ∎

Remark 3.4. Theorem 3.3 also implies that the diagrams of ∞-topoi satisfying condition $(*)$ exist in great abundance. According to [16, A.7.5.2], the ∞-category of coherent ∞-topoi is defined to be the subcategory of $\infty\mathcal{T}\mathrm{op}$ whose objects are coherent ∞-topoi and whose morphisms are functors $f^* : \mathcal{Y} \to \mathcal{X}$ which carry coherent objects to coherent objects [16, A.2.1.6] (see also [14, 6.3.1.5]). Note that [16, A.7.5.3] supplies an equivalence of ∞-categories from the ∞-category of bounded ∞-pretopoi to the opposite of the full subcategory of the ∞-category of coherent ∞-topoi spanned by the bounded coherent ∞-topoi, given on objects by $\mathcal{C} \mapsto \mathcal{S}\mathrm{hv}(\mathcal{C})$ (here $\mathcal{C}$ is equipped with the effective epimorphism topology [16, A.6.2.4]); see [16, A.7.4.1] and [16, A.7.1.2] for the definitions of bounded ∞-pretopoi and bounded ∞-topoi, respectively. If $\mathcal{C} \to \mathcal{D}$ is a morphism of bounded ∞-pretopoi (equipped with the effective epimorphism topology), then it satisfies the assumptions of Theorem 3.3 (in a larger universe), thereby the equivalence of ∞-categories guarantees that every cofiltered diagram of bounded coherent ∞-topoi satisfies condition $(*)$.

3.5. Let $\mathcal{X}$ denote the cofiltered limit of the diagram p in condition $(*)$. Let $I^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the diagram of $\mathcal{X}_i$ obtained via the embedding $\infty\mathcal{T}\mathrm{op} \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$, where $\widehat{\mathcal{C}\mathrm{at}}_\infty$ denotes the ∞-category of (not necessarily small) ∞-categories of [14, 3.0.0.5]. Then this diagram classifies a cartesian fibration $q : Z \to I$. Using [14, 3.3.3.2] and [14, 6.3.3.1], the underlying ∞-category of the ∞-topos $\mathcal{X}$ can be identified with the ∞-category of cartesian sections of q. Let $\pi_* : \mathcal{X} \to \mathcal{S}$ and $\pi_{i*} : \mathcal{X}_i \to \mathcal{S}$ denote the unique (up to homotopy) geometric morphisms, respectively. Let $p_{i*} : \mathcal{X} \to \mathcal{X}_i$ denote the geometric morphism associated to the limit ∞-topos $\mathcal{X}$. The virtue of condition $(*)$ is that one can describe p_i^* as in the case of ordinary categories (see, for example, [17, Lemma 2]):

Lemma 3.6. *Fix $i \in I$ and an integer $n \geq 0$. Let F be an $(n-1)$-truncated object of $\mathcal{X}_i$. Then the section of q determined by $p_i^* F$ is given explicitly by the formula $j \mapsto \operatorname{colim}_{k \geq i,j}\, p_{jk*} p_{ik}^* F$.*

Proof. Condition $(*)$ guarantees that the section determined by the formula is cartesian, so it can be viewed as an object of $\mathcal{X}$. Note that the pushforward p_{i*} sends an object $\{G_j\}$ of $\mathcal{X}$ to its i-th component G_i. Therefore, it suffices to observe that there is a chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathcal{X}}\left(\left\{\operatorname*{colim}_{k\geq i,j} p_{jk*}p_{ik}^{*}F\right\},\{G_j\}\right) &\simeq \lim_{j\geq i}\lim_{k\geq j}\mathrm{Map}_{\mathcal{X}_j}(p_{jk*}p_{ik}^{*}F, p_{jk*}G_k)\\
&\simeq \lim_{k\geq i}\mathrm{Map}_{\mathcal{X}_k}(p_{ik}^{*}F, G_k)\\
&\simeq \mathrm{Map}_{\mathcal{X}_i}(F, G_i),
\end{aligned}$$

where we use the fact that if $\mathcal{I}$ is a filtered ∞-category (see [14, 5.3.1.7]), then for each object $i \in \mathcal{I}$, the undercategory $\mathcal{I}_{i/}$ is also filtered (see [14, 5.3.1.21]) and the forgetful functor $\mathcal{I}_{i/} \to \mathcal{I}$ is cofinal in the sense of [14, 4.1.1.1]; see also [14, 4.1.1.8] and [14, 5.4.5.9]. Note also that the second equivalence uses the fact that the diagonal embedding $\mathcal{I} \to \mathrm{Fun}(\Delta^1, \mathcal{I})$ is cofinal; see [14, 5.3.1.22]. ∎

Corollary 3.7. *In the situation of Lemma 3.6, the canonical map*

$$\operatorname*{colim}_{j\geq i}\mathrm{Map}_{\mathcal{X}_j}(*, p_{ij}^{*}F) \to \mathrm{Map}_{\mathcal{X}}(*, p_i^{*}F)$$

is a homotopy equivalence.

Proof. By virtue of Lemma 3.6 and the adjunction (p_i^*, p_{i*}), $\mathrm{Map}_{\mathcal{X}}(*, p_i^*F)$ can be identified with $\mathrm{Map}_{\mathcal{X}_i}(*, \mathrm{colim}_{k\geq i}\, p_{ik*}p_{ik}^*F)$. Note that each p_{ik}^*F is $(n-1)$-truncated. Invoking our assumption on p_{ik*} that it commutes with filtered colimits of $(n-1)$-truncated objects (and using the adjunction (p_{ik}^*, p_{ik*})), the map in the statement can be identified with the map

$$\operatorname*{colim}_{j\geq i}\mathrm{Map}_{\mathcal{X}_j}(*, p_{ij}^{*}F) \to \mathrm{Map}_{\mathcal{X}_k}\left(*, \operatorname*{colim}_{k\geq i} p_{ik}^{*}F\right).$$

Since $\mathcal{X}_i$ is assumed to be coherent, the desired result follows from [16, A.2.3.1]. ∎

3.8. Let X be a scheme. The usual small étale topology on X induces a Grothendieck topology on the nerve of the category $\text{Ét}(X)$ of étale X-schemes. Let $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ and $\mathcal{P}_{\text{ét}}(X)$ denote the ∞-categories of sheaves and presheaves of spaces on $\mathrm{N}(\text{Ét}(X))$, respectively. Note that the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ is 1-localic; see [14, 6.4.5.8]. Now assume that X is quasi-compact and quasi-separated. Let $\text{Ét}(X)^{\mathrm{fp}} \subseteq \text{Ét}(X)$ denote the full subcategory spanned by those étale X-schemes which are of finite presentation. With respect to the induced Grothendieck topology on $\mathrm{N}(\text{Ét}(X)^{\mathrm{fp}})$, the associated geometric morphism of ∞-topoi (which is given by composition with the inclusion) is an equivalence since the ∞-topoi are 1-localic and the equivalence holds for the 0-truncations. Specifically, one may assume henceforth that $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ is induced by a finitary Grothendieck topology [16, A.3.1.1]. Consequently, $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ is both coherent and locally coherent [16, A.2.1.6] by virtue of [16, A.3.1.3].

3.9. Let I be an ordinary filtered category. Let $\{X_i\}_{i \in I^{\mathrm{op}}}$ be a compatible family of quasi-compact and quasi-separated schemes with affine transition maps. It follows immediately from Theorem 3.3 that the diagram of ∞-topoi $\{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_i)\}$ satisfies condition $(*)$; see also 3.8. Set $X = \lim X_i$. There is an equivalence $\mathcal{S}\mathrm{hv}_{\text{ét}}(X) \simeq \lim_{i \in I^{\mathrm{op}}} \mathcal{S}\mathrm{hv}_{\text{ét}}(X_i)$ in $\infty\mathcal{T}\mathrm{op}$ as the ∞-topoi are all 1-localic and the equivalence holds for the usual 1-topoi by virtue of [24, Exposé VII.5.6 and Exposé VI.8.2.3].

Remark 3.10. If we work in the setting of spectral algebraic geometry, the fact that the diagram of coherent ∞-topoi $\{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_i)\}$ appearing in 3.9 satisfies condition $(*)$ can also be proven as follows: If $f : X \to Y$ is a morphism of quasi-compact quasi-separated schemes, then [16, 2.3.4.2] guarantees that f is ∞-quasi-compact when regarded as a morphism of schematic spectral Deligne–Mumford stacks [16, 1.6.7] (that is, the pull-back functor $f^* : \mathcal{S}\mathrm{hv}_{\text{ét}}(Y) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(X)$ carries coherent objects to coherent objects; see [16, 2.3.2.2]). By virtue of [16, 2.3.2.3], the pushforward functor $f_* : \tau_{\leq n-1}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(X) \to \tau_{\leq n-1}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(Y)$ commutes with filtered colimits for each integer $n \geq 0$.

4. Proper base change for étale sheaves of spaces

4.1. Suppose we are given a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{q'_*} & \mathcal{C} \\ {\scriptstyle p'_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathcal{D}' & \xrightarrow{q_*} & \mathcal{D} \end{array}$$

Assume that q_* and q'_* admit left adjoints, which we denote by q^* and q'^*, respectively. Consider the composition

$$q^* p_* \to q^* p_* q'_* q'^* \xrightarrow{\sim} q^* q_* p'_* q'^* \to p'_* q'^*,$$

where the first and third arrows are induced by a unit for the adjunction (q'^*, q'_*) and a counit for the adjunction (q^*, q_*), respectively. This composite is referred to as the *push-pull transformation* (sometimes called the *Beck–Chevalley transformation*).

4.2. The purpose of this section is to prove the nonabelian proper base change theorem (Theorem 1.2) whose proof is deferred until the end of the section. As in the case of the usual proper base change theorem, we will show that the morphism θ in Theorem 1.2 is an equivalence by looking at stalks.

4.3. We begin with the special case of Theorem 1.2 where the morphism g in (1.1.1) is a point of Y. Suppose we are given a morphism $\operatorname{Spec} k \to S$ of schemes, where k is a separably closed field. Let s denote the set-theoretic image of the morphism and let $\bar{s}$ denote the morphism. Consider a pullback diagram of quasi-compact quasi-separated schemes

$$\begin{array}{ccc} X_{\overline{s}} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \xrightarrow{\overline{s}} & S \end{array}$$

where f is proper. We have a commutative diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}(X_{\overline{s}}) & \xrightarrow{p'_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(X) \\ f'_* \downarrow & & \downarrow f_* \\ \mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k) & \xrightarrow{p_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(S) \end{array}$$

4.4. For each object $F \in \mathcal{S}\mathrm{hv}_{\text{ét}}(S)$, let $F_{\overline{s}}$ denote the pullback p^*F. To give an explicit description of the pullbacks, consider the continuous functor

$$\delta : \text{Ét}(S) \to \text{Ét}(\operatorname{Spec} k) : S' \mapsto \operatorname{Spec} k \times_S S'.$$

Then the restriction functor

$$\hat{p}_* : \mathcal{P}_{\text{ét}}(\operatorname{Spec} k) \xrightarrow{\circ \mathrm{N}\,\delta^{\mathrm{op}}} \mathcal{P}_{\text{ét}}(S)$$

admits a left adjoint $\hat{p}^*$ given by the left Kan extension along $\mathrm{N}\,\delta^{\mathrm{op}}$ [14, 4.3.3.7], defined on objects by the formula

$$(\hat{p}^* F)(Y) = \operatorname*{colim}_{(S',\rho) \in I_Y^{\mathrm{op}}} F(S'). \tag{4.4.1}$$

Here I_Y is the ordinary category whose objects are pairs (S', ρ), where $S' \in \text{Ét}(S)$ and $\rho : Y \to \delta(S')$ is a morphism in $\text{Ét}(\operatorname{Spec} k)$, and whose morphisms $(S'', \rho') \to (S', \rho)$ are morphisms $a : S'' \to S'$ in $\text{Ét}(S)$ such that $\delta(a) \circ \rho' = \rho$.

Note that if $Y = \operatorname{Spec} k$, then one can identify I_Y with the category $I_{\overline{s}}$ of étale neighborhoods of the geometric point $\overline{s} : \operatorname{Spec} k \to S$.

Lemma 4.5. *Let G be an object of $\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$. Then the canonical map*

$$\alpha_G : \mathrm{Map}_{\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)}(*, G) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)}(*, LG)$$

is a weak homotopy equivalence, where $L : \mathcal{P}_{\text{ét}}(\operatorname{Spec} k) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$ denotes the sheafification functor.

Proof. Let Γ and γ denote the global section functors on $\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$ and $\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$, respectively. We prove by induction that the map $\alpha_G : \gamma G \to \Gamma L G$, which is induced by the unit for the adjunction between $\mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$ and $\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$, is n-connective for every $n \geq 0$. The case $n = 0$ follows from the property of sheafification that every section of LG locally comes from that of G (up to equivalence) and that every étale covering of the final object of $\text{Ét}(\operatorname{Spec} k)$ admits a refinement by the identity on the final object.

Suppose that $n > 0$. It suffices to prove that for every pair of maps $\eta, \eta' : * \to \gamma G$, the induced map

$$\beta_G : * \times_{\gamma G} * \to * \times_{\Gamma L G} *$$

of fiber products is $(n-1)$-connective (note that giving a map $* \to \gamma G$ is equivalent to giving a map $* \to G$ because γ is given by evaluation at the final object of Ét(Spec k)). Let $H = * \times_G * \in \mathcal{P}_{\text{ét}}(\operatorname{Spec} k)$. Using the fact that γ, Γ, and L all commute with finite limits, one can identify β_G with α_H; the desired result now follows from the inductive hypothesis. ■

4.6. Let $\mathcal{X}$ be an ∞-topos. We say that an object $F \in \mathcal{X}$ is *truncated* if it is n-truncated for some integer $n \geq -2$. According to [16, A.2.1.6], an object $F \in \mathcal{X}$ is defined to be *coherent* if the ∞-topos $\mathcal{X}_{/F}$ of [14, 6.3.5.1] is n-coherent for every integer n (see [16, A.2.0.12]).

Remark 4.7. Let X be a quasi-compact and quasi-separated scheme. Then the ∞-topos $\mathcal{S}\text{hv}_{\text{ét}}(X)$ is coherent (see 3.8). By virtue of [15, 2.3.25], the class of truncated coherent objects of $\mathcal{S}\text{hv}_{\text{ét}}(X)$ can be viewed as an $\mathcal{S}$-valued counterpart to the class of constructible objects of the usual étale topos $\tau_{\leq 0}\,\mathcal{S}\text{hv}_{\text{ét}}(X)$ (see [25, Exposé IX.2.3 and Exposé IX.2.4]). Here we use the fact that $\tau_{\leq 0}\,\mathcal{S}\text{hv}_{\text{ét}}(X)$ can be identified with the ordinary category of étale sheaves of sets on X; see [14, 7.2.2.17].

4.8. Let $\mathcal{X}$ be an ∞-topos. Let $F \in \mathcal{X}$ be an object. Let $s : F^{S^n} \to F$ denote the map induced by the evaluation at a fixed based point of the n-sphere S^n (see [14, p. 658] for the definition of F^{S^n}). According to [14, 6.5.1.1], $\pi_n F$ is defined to be the 0-truncation of s, regarded as an object of $\mathcal{X}_{/F}$. If $p : * \to F$ is a morphism in $\mathcal{X}$, we let ΩF denote the fiber product $* \times_F *$ determined by p. For each integer $n \geq 1$, we define $\Omega^n F$ by applying this procedure repeatedly. Using [14, 6.5.1.3, 6.5.1.5], we can identify the image of $\pi_n F \in \mathcal{X}_{/F}$ under the pullback functor $p^* : \mathcal{X}_{/F} \to \mathcal{X}$ (which carries an object G in $\mathcal{X}_{/F}$ to the fiber product $* \times_F G \in \mathcal{X}$) with the 0-truncation $\tau_{\leq 0}\Omega^n F$ of [14, 5.5.6.18]. Combining this observation with [16, A.2.1.8, A.2.4.4], we deduce that if $*$ and F are coherent objects of $\mathcal{X}$, then $p^*\pi_n F \in \mathcal{X}$ is coherent.

4.9. We will need the following lemmas to prove the special case of the proper base change theorem:

Lemma 4.10. *Let $k \subset K$ be an extension of separably closed fields and let X be a proper scheme over k. Let $X_K = X \times_{\operatorname{Spec} k} \operatorname{Spec} K$. For each truncated coherent object F of $\mathcal{S}\text{hv}_{\text{ét}}(X)$, the canonical map*

$$\alpha_F : \operatorname{Map}_{\mathcal{S}\text{hv}_{\text{ét}}(X)}(*, F) \to \operatorname{Map}_{\mathcal{S}\text{hv}_{\text{ét}}(X_K)}(*, F_K)$$

is a weak homotopy equivalence, where F_K denotes the pullback of F to $\mathcal{S}\text{hv}_{\text{ét}}(X_K)$.

Proof. Let $n \geq -2$ be an integer such that F is n-truncated. Our proof will proceed by induction on n. The case $n < 0$ is immediate. If $n = 0$, the desired result is a consequence of the first part of [25, Exposé XII.5.4]. Let us therefore assume that $n > 0$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)}(*, F) & \longrightarrow & \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)}(*, F_K) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)}(*, \tau_{\leq n-1} F) & \longrightarrow & \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)}(*, \tau_{\leq n-1} F_K) \end{array}$$

Since the pullback of $\tau_{\leq n-1}F$ to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)$ can be identified with $\tau_{\leq n-1}F_K$ (see [14, 5.5.6.28]), the lower horizontal map can be identified with $\alpha_{\tau_{\leq n-1}F}$, and is therefore a weak homotopy equivalence by the inductive hypothesis (here we use the fact that $\tau_{\leq n-1}F \in \mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ is coherent; see [16, A.2.4.4]). Consequently, in order to prove that the upper horizontal map α_F is a homotopy equivalence, it suffices to show that it induces a homotopy equivalence after passing to the homotopy fibers of the vertical maps over any point $\eta : * \to \tau_{\leq n-1}F$ of $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)}(*, \tau_{\leq n-1}F) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)}(*, \tau_{\leq n-1}F_K)$. This is equivalent to the assertion that the canonical map α_G is a weak homotopy equivalence, where G denotes the fiber of the truncation map $F \to \tau_{\leq n-1}F$ over the point η (so that G is both n-truncated and n-connective). Note that G is coherent by virtue of [16, A.2.4.4, A.2.1.8].

We first consider the case where the domain of the map α_G is empty. In this case, we wish to show that the codomain of α_G is also empty. Assume otherwise. Fix an algebraic closure $\overline{K}$ of K. Let k^{alg} denote the algebraic closure of k in $\overline{K}$, so that k^{alg} itself is an algebraically closed field. Since K and k are separably closed, the field extensions $K \subseteq \overline{K}$ and $k \subseteq k^{\mathrm{alg}}$ are purely inseparable. Let $X_{\overline{K}} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{K}$ and $X_{k^{\mathrm{alg}}} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k^{\mathrm{alg}}$. Using [24, Exposé VIII.1.1] (see also [14, 6.4.5.8]), we can identify $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{\overline{K}})$ and $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{k^{\mathrm{alg}}})$ with $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)$ and $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$, respectively. Write $\overline{K}$ as a filtered colimit of finite type k^{alg}-algebras $\{A_i\}_{i \in I}$; see, for example, [23, Tag 00QN]. Let X_i denote the fiber product $X_{k^{\mathrm{alg}}} \times_{\mathrm{Spec}\, k^{\mathrm{alg}}} \mathrm{Spec}\, A_i$, so that $X_{\overline{K}} \simeq \lim_{i \in I^{\mathrm{op}}} X_i$. According to 3.9, there is an equivalence $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{\overline{K}}) \simeq \lim_{i \in I^{\mathrm{op}}} \mathcal{S}\mathrm{hv}_{\acute{e}t}(X_i)$ in the ∞-category $\infty\mathcal{T}\mathrm{op}$ of ∞-topoi. Combining this with our assumption that the space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)}(*, G_K) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{\overline{K}})}(*, G_{\overline{K}})$ is nonempty (here G_K and $G_{\overline{K}}$ denote the pullbacks of G to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)$ and $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{\overline{K}})$, respectively), we deduce that there exists some index j such that $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_j)}(*, G_j)$ is nonempty, where G_j denotes the pullback of G to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_j)$. We can therefore choose a global section $p_j : * \to G_j$. Using the fact that k^{alg} is algebraically closed and Hilbert's Nullstellensatz (see, for example, [23, Tag 00FV]), we see that there exists a homomorphism $A_j \to k^{\mathrm{alg}}$ of k^{alg}-algebras and therefore the global section p_j determines a global section $* \to G_{k^{\mathrm{alg}}}$, where $G_{k^{\mathrm{alg}}}$ denotes the pullback of G to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{k^{\mathrm{alg}}})$. Using the identification $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X) \simeq \mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{k^{\mathrm{alg}}})$, we obtain a global section $* \to G$, contradicting our assumption that $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)}(*, G)$ is empty.

We now treat the case where G admits a global section $p : * \to G$. In this case, we can identify G with the Eilenberg–MacLane object $K(A, n)$ of [14, 7.2.2.1], where A denotes the pullback $p^*\pi_n G$ of $\pi_n G$ to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$; see [14, 7.2.2.12]. Let A_K denote the pullback of A to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)$, so that the pullback of $K(A,n)$ to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)$ can be identified with the Eilenberg–MacLane object $K(A_K,n)$ of $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)$; see [14, 6.5.1.4]. We note that A is a group object of the ordinary topos $\tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ for $n \geq 1$, which is commutative for $n \geq 2$ (see [14, p. 659]), and that A (when regarded as an object of the ordinary topos $\tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$) is constructible in the classical sense of [25, Exposé IX.2.3] by virtue of Remark 4.7 and 4.8; in particular, its stalks are finite. It then follows from [25, Exposé XII.5.4] and [14, 7.2.2.17] that $\alpha_{K(A,n)}$ is a weak homotopy equivalence, as desired. ■

Remark 4.11. Since $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ is a coherent ∞-topos (see 3.8), it follows from [16, E.2.5.7 and E.2.5.5] that every locally constant constructible object of $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ in the sense of [16, E.2.5.1] is truncated coherent. In the special case where F is locally constant constructible, using profinite shapes, we can give an alternative proof of Lemma 4.10 as follows: according to [25, Exposé XII.5.4] (see also [10, 2.15]), the projection $X_K \to X$ induces an equivalence $\mathrm{Sh}_\pi(X_K) \to \mathrm{Sh}_\pi(X)$ of profinite spaces. Then [16, E.2.3.3] guarantees that the functor $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)^{\mathrm{lcc}} \to \mathcal{S}\mathrm{hv}_{\acute{e}t}(X_K)^{\mathrm{lcc}}$ is an equivalence of ∞-categories, thereby completing the proof.

Lemma 4.12. *Let R be a henselian local ring and let X be a proper scheme over R with closed fiber X_k. For each truncated coherent object $F \in \mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$, the canonical map*

$$\alpha_F : \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)}(*, F) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_k)}(*, F_k)$$

is a weak homotopy equivalence, where F_k denotes the pullback of F to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_k)$.

Proof. As in the proof of Lemma 4.10 (using [25, Exposé XII.5.5] in place of [25, Exposé XII.5.4]), we can reduce to the case where F is n-truncated and n-connective for some $n > 0$. Suppose first that F admits a global section. Arguing as in the proof of Lemma 4.10 (again using [25, Exposé XII.5.5] in place of [25, Exposé XII.5.4]), we deduce that α_F is a weak homotopy equivalence as desired.

Now suppose that the domain of the map α_F is empty; we wish to show that its codomain is also empty. We first treat the case $n \geq 2$. According to [14, p. 659], we can regard $\pi_n F$ as an abelian group object of the ordinary topos $\tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)_{/F}$; see [14, 6.5.1.1]. Since F is 2-connective, [14, 7.2.1.13] guarantees that the pullback functor $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X) \to \mathcal{S}\mathrm{hv}_{\acute{e}t}(X)_{/F}$ (which is a right adjoint to the projection) restricts to an equivalence $\tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X) \to \tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)_{/F}$ of ∞-categories, so that we can identify $\pi_n F$ with an abelian group object $A \in \tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$. In other words, F can be viewed as an n-gerbe banded by A in the sense of [14, p. 731]. Since F is coherent, $\pi_n F$ is a coherent object of $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)_{/F}$; that is, $A \times F \in \mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ is coherent. Note that the projection $A \times F \to A$ is 1-connective by virtue of [14, 6.5.1.16]. It then follows from [16, A.2.4.1] that A is 1-coherent (see [16, A.2.0.12]). Since A is 0-truncated, we deduce from [16, A.2.3.5] that $A \in \mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ is coherent, hence constructible as an object of the ordinary topos $\tau_{\leq 0}\,\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ (see Remark 4.7). According to [14, 7.2.2.27] (see also [14, 7.2.2.17]), there is a canonical bijection between the set of equivalence classes of n-gerbes on $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X)$ banded by A (see [14, p. 731]) and $\mathrm{H}^{n+1}_{\acute{e}t}(X, A)$; moreover, [14, 7.2.2.28] guarantees that an n-gerbe banded by A admits a global section if and only if the associated cohomology class vanishes (and similarly for the pullback of A to $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_k)$). The desired result now follows from the third part of [25, Exposé XII.5.5].

We now treat the case $n = 1$. Using [14, 6.5.2.14] (see also [14, 6.5.2.9]), we can regard F as an ordinary gerbe over the small étale site Ét(X) in the sense of [9, III.2.1.1] (see 3.8); let L denote the lien associated to F, regarded as an ordinary gerbe (see [9, IV.2.2.2]). Since F is 1-truncated, we observe that $\pi_1 F \in \tau_{\leq 0}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(X)_{/F}$ can be identified with the pullback of the diagonal map $F \to F \times F$ along itself. Combining this observation with our assumption that F is coherent and [16, A.3.1.3], we deduce that for each pair $(f : U \to X, x \in F(U))$, where f is an étale morphism of schemes for which U is quasi-compact and quasi-separated, the ordinary sheaf $\underline{\mathrm{Aut}}_x$ of groups on U (see, for example, [18, 3.4.7]) is constructible; see Remark 4.7. Let $\mathrm{H}^2_{\text{ét}}(X, L)$ denote the set of L-equivalence classes of L-gerbes as defined in [9, IV.3.1.1] (see also [9, IV.2.2.2.1 and p. 216]), and define $\mathrm{H}^2_{\text{ét}}(X_k, L_k)$ similarly (here L_k denotes the inverse image of L in the sense of [9, V.1.2.2]). Note that we have a map $\mathrm{H}^2_{\text{ét}}(X, L) \to \mathrm{H}^2_{\text{ét}}(X_k, L_k)$. Since $\underline{\mathrm{Aut}}_x$ is constructible for each pair (f, x) as above, the inverse image under this map of the subset of $\mathrm{H}^2_{\text{ét}}(X_k, L_k)$ consisting of those L_k-equivalence classes of L_k-gerbes which are neutral in the sense of [9, IV.3.1.1] (that is, those gerbes which admit a global section) is the subset of $\mathrm{H}^2_{\text{ét}}(X, L)$ consisting of those L-equivalence classes of L-gerbes which are neutral by virtue of the proof of [13, 7.1], which allows us to remove the noetherian assumption in [9, VII.2.2.4] so that [9, VII.2.2.6] holds without the noetherian assumption. It then follows that F admits a global section if and only if F_k admits a global section, as desired. ■

4.13. Note that the essentially unique geometric morphism

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k) \to \mathcal{S} : F \mapsto \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)}(*, F)$$

is an equivalence of ∞-categories as $\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k)$ is a 1-localic ∞-topos and the equivalence holds for the usual 1-topoi. Under this equivalence, the pushforward functor f'_* in 4.3 can be identified with the global section functor on the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X_{\overline{s}})$.

We now formulate and give a proof of the special case of the proper base change theorem under this identification:

Proposition 4.14. *In the situation of 4.3, for each truncated coherent object F in $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$, the push-pull morphism*

$$\theta : (f_* F)_{\overline{s}} \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(X_{\overline{s}})}(*, p'^* F)$$

is an equivalence in the ∞-category $\mathcal{S}$.

Proof. Recall that the strict henselization of S at $\overline{s}$ is defined as follows:

$$\mathcal{O}_{S,\overline{s}} = \operatorname*{colim}_{(U,\overline{u}) \in I^{\mathrm{op}}_{\overline{s}}} \Gamma(U, \mathcal{O}_U) \tag{4.14.1}$$

where $I_{\overline{s}}$ is the category of étale neighborhoods of $\overline{s}$ (note that it is cofiltered). Fix a separable closure $k(s)^{\mathrm{sep}}$ of $k(s)$ in k (see 4.3). Consider a commutative diagram of schemes

$$\begin{array}{ccccccc} X_{\overline{s}} & \longrightarrow & X' & \longrightarrow & X_{(\overline{s})} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow f \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec}(k(s)^{\mathrm{sep}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{S,\overline{s}}) & \longrightarrow & S \end{array}$$

where all squares are cartesian. We first note that Lemmas 4.12 and 4.10 supply canonical equivalences

$$\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})}(*, F_{(\overline{s})}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X')}(*, F') \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{\overline{s}})}(*, p'^{*}F) \qquad (4.14.2)$$

where $F_{(\overline{s})}$ and F' denote the pullbacks of F to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})$ and $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X')$, respectively. Here we use the fact that $F_{(\overline{s})}$ and F' are truncated coherent objects; see Remark 3.10.

We now claim that there is a canonical equivalence

$$(f_*F)_{\overline{s}} \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})}(*, F_{(\overline{s})}). \qquad (4.14.3)$$

Under the identification $\mathcal{S}\mathrm{hv}_{\acute{e}t}(\operatorname{Spec} k) \simeq \mathcal{S}$, the stalk $(f_*F)_{\overline{s}}$ is identified with its global section which is equivalent to the global section of the presheaf $\hat{p}^*(f_*F)$ by virtue of Lemma 4.5. Let $I_{\overline{s}}^{\mathrm{aff}} \subseteq I_{\overline{s}}$ denote the full subcategory spanned by the affine étale neighborhoods of $\overline{s}$. Then the inclusion $I_{\overline{s}}^{\mathrm{aff}} \to I_{\overline{s}}$ is a cofinal functor between cofiltered categories (cf. 3.9). Combining this observation, [14, 4.1.1.8], and formula (4.4.1), we obtain an equivalence

$$(f_*F)_{\overline{s}} \simeq \operatorname*{colim}_{(U,\overline{u}) \in I_{\overline{s}}^{\mathrm{aff,op}}} F(X \times_S U).$$

On the other hand, applying the cofinality argument to (4.14.1), we obtain an isomorphism of schemes

$$X_{(\overline{s})} \simeq \lim_{(U,\overline{u}) \in I_{\overline{s}}^{\mathrm{aff}}} X \times_S U.$$

Since the projection $X \times_S U \to X$ is étale, one can identify $F(X \times_S U)$ with the space $\mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X \times_S U)}(*, F_{X \times_S U})$, where $F_{X \times_S U}$ is the restriction of F to $\mathcal{S}\mathrm{hv}_{\acute{e}t}(X \times_S U)$. Consequently, the desired equivalence (4.14.3) will follow if the canonical map

$$\operatorname*{colim}_{(U,\overline{u}) \in I_{\overline{s}}^{\mathrm{aff,op}}} \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X \times_S U)}(*, F_{X \times_S U}) \to \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\acute{e}t}(X_{(\overline{s})})}(*, F_{(\overline{s})})$$

is an equivalence; choosing any index $(U, \overline{u})$ and applying the cofinality of the forgetful functor $(I_{\overline{s}}^{\mathrm{aff,op}})_{(U,\overline{u})/} \to I_{\overline{s}}^{\mathrm{aff,op}}$ (see the proof of Lemma 3.6), this is guaranteed by Corollary 3.7. By construction, the composition of (4.14.3) with (4.14.2) can be identified with the push-pull morphism θ, thereby completing the proof. ∎

4.15. Let $\mathcal{X}$ be an ∞-topos which is locally n-coherent [16, A.2.0.12] for all $n \geq 0$ (for example, a coherent ∞-topos), let $\mathcal{X}^{\wedge}$ be the hypercompletion of $\mathcal{X}$, and let $L : \mathcal{X} \to \mathcal{X}^{\wedge}$ be a left adjoint to the inclusion. Combining [14, 5.5.6.16, 6.5.2.9] with [16, A.2.2.2], we see that the functor L induces an equivalence from the full subcategory of $\mathcal{X}$ spanned by the truncated coherent objects to the full subcategory of $\mathcal{X}^\wedge$ spanned by the truncated coherent objects.

4.16. We are now ready to prove the proper base change theorem (Theorem 1.2):

Proof of Theorem 1.2. Using 4.15, we may work with the hypercompletions of ∞-topoi and the associated geometric morphisms in (1.1.2). Since $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')$ is locally coherent by virtue of 3.8 and [16, A.3.1.3], so is its hypercompletion $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')^\wedge$ (see [16, A.2.2.2]). Consequently, the ∞-categorical Deligne Completeness Theorem [16, A.4.0.5] guarantees that $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')^\wedge$ has enough points. Therefore, one can check that θ is an equivalence after pulling back along a geometric morphism $p_* : \mathcal{S} \to \mathcal{S}\mathrm{hv}_{\text{ét}}(Y')$ (given by a point of $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')^\wedge$ followed by the fully faithful geometric morphism $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')^\wedge \to \mathcal{S}\mathrm{hv}_{\text{ét}}(Y')$). Since $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')$ is a 1-localic ∞-topos and points of the étale 1-topos of Y' are determined by geometric points $\bar{s} : \operatorname{Spec} k \to Y'$, where k is a separably closed field [24, Exposé VIII.7.9], one can assume that the point p of $\mathcal{S}\mathrm{hv}_{\text{ét}}(Y')$ is induced by a geometric point $\bar{s}$ of Y'. Consider the commutative diagram of ∞-topoi

$$\begin{array}{ccccc}
\mathcal{S}\mathrm{hv}_{\text{ét}}(X'_{\bar{s}}) & \xrightarrow{p'_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(X') & \xrightarrow{g'_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(X) \\
\downarrow{\scriptstyle f''_*} & & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\
\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k) & \xrightarrow{p_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(Y') & \xrightarrow{g_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(Y)
\end{array}$$

where the left square is associated to the cartesian square defining $X'_{\bar{s}} = \operatorname{Spec} k \times_{Y'} X'$. In this case, $p^*\theta$ fits into a commutative triangle

$$\begin{array}{ccccc}
 & & p^* f'_* g'^* F & & \\
 & \nearrow^{p^*\theta} & & \searrow^{\theta'} & \\
p^* g^* f_* F & & \xrightarrow{\theta''} & & f''_* p'^* g'^* F
\end{array}$$

where θ' and θ'' are equivalences by virtue of the special case of the proper base change theorem (Proposition 4.14). Consequently, $p^*\theta$ is an equivalence as desired. ∎

5. Application: profinite shapes of products

5.1. For each scheme X, let $\mathrm{Sh}(X)$ and $\mathrm{Sh}_\pi(X)$ denote the shape and profinite shape of the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$, respectively.

5.2. Let X be a locally noetherian scheme, and let $\bar{x} : \operatorname{Spec} K \to X$ be a geometric point of X. For each integer $n \geq 0$, the *n-th homotopy (pro-)group* $\pi_n(X, \bar{x})$ of the pointed scheme $(X, \bar{x})$ is defined to be the n-th homotopy pro-group of the étale topological type of X with the associated base point (here the étale topological type of X can be replaced by the shape of the hypercompletion of the ∞-topos $\mathcal{S}\mathrm{hv}_{\text{ét}}(X)$; see the proof of [6, 3.2.13]).

It follows from [6, 3.2.11] that the fundamental groups of locally noetherian schemes are isomorphic to the enlarged étale fundamental groups in the sense of [22, Exposé X.6]. In particular, their profinite completions are isomorphic to the étale fundamental groups in the sense of [20, Exposé V.7]. By the universal property of profinite completions, there is a natural map

$$\widehat{\pi}_n(X, x) \to \pi_n(\widehat{\mathrm{h}}(X), x) \tag{5.2.1}$$

of profinite groups, where $\widehat{\pi}_n(X, x)$ denotes the profinite completion of $\pi_n(X, \overline{x})$ and $\widehat{\mathrm{h}}(X)$ denotes the profinite étale topological type of X. When $n = 1$, (5.2.1) is an isomorphism by virtue of [1, 3.7]. From this point of view, the following theorem generalizes the commutativity of étale fundamental groups (or equivalently, fundamental groups of profinite étale topological types) with finite products of proper schemes over a separably closed field [20, Exposé X.1.7] to all higher homotopy groups of profinite étale topological types of such schemes:

Theorem 5.3. *Let k be a separably closed field, let $f : X \to \operatorname{Spec} k$ be a quasi-compact and quasi-separated morphism of schemes, and let $g : Y \to \operatorname{Spec} k$ be a proper morphism of schemes. Then the canonical map*

$$\mathrm{Sh}_\pi(X \times_k Y) \to \mathrm{Sh}_\pi(X) \times \mathrm{Sh}_\pi(Y)$$

is an equivalence of profinite spaces.

Proof. There is a commutative diagram of ∞-topoi

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}(X \times_k Y) & \xrightarrow{p'_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(Y) \\ \Big\downarrow{\scriptstyle q'_*} & & \Big\downarrow{\scriptstyle q_*} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}(X) & \xrightarrow{p_*} & \mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname{Spec} k) \end{array} \tag{5.3.1}$$

We also have a commutative triangle of pro-spaces of (1.8.2):

$$\begin{array}{ccccc} & & \mathrm{Sh}(X \times_k Y) & & \\ & \nearrow & & \searrow & \\ \mathrm{Sh}(X) \circ \mathrm{Sh}(Y) & & \longrightarrow & & \mathrm{Sh}(X) \times \mathrm{Sh}(Y) \end{array}$$

Using the fact that profinite completions commute with finite limits, the desired equivalence will follow if one can show that both the left diagonal and the bottom maps in (1.8.2) are equivalences after applying the profinite completion functor. For the left diagonal map, we need to show that for each π-finite space A, the canonical map $p_* p^* q_* q^* A \to p_* q'_* p'^* q^* A$ is an equivalence of spaces; since g is proper and the sheaf $q^* A$ is locally constant constructible, this is a consequence of the proper base change for étale sheaves of spaces (Theorem 1.2).

For the bottom map, choose cofiltered systems $\{A_i \in \mathcal{S}\}_{i \in \mathcal{I}}$ and $\{B_j \in \mathcal{S}\}_{j \in \mathcal{J}}$ in such a way that $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are equivalent to the cofiltered limits of the functors corepresented by A_i and B_j, respectively. Let A be a π-finite space. The bottom map evaluated at the final object of $\mathcal{S}$ is an equivalence, so we may assume that A is n-truncated for some $n \geq -1$. There is a chain of equivalences

$$\begin{aligned}(\mathrm{Sh}(X) \circ \mathrm{Sh}(Y))(A) &\simeq \mathrm{Sh}(X)\Big(\operatorname*{colim}_{j \in \mathcal{J}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}}(B_j, A)\Big) \\ &\simeq \operatorname*{colim}_{j \in \mathcal{J}^{\mathrm{op}}} \mathrm{Sh}(X)(\mathrm{Map}_{\mathcal{S}}(B_j, A)) \\ &\simeq \operatorname*{colim}_{j \in \mathcal{J}^{\mathrm{op}}} \operatorname*{colim}_{i \in \mathcal{I}^{\mathrm{op}}} \mathrm{Map}_{\mathcal{S}}(A_i, \mathrm{Map}_{\mathcal{S}}(B_j, A)) \\ &\simeq (\mathrm{Sh}(X) \times \mathrm{Sh}(Y))(A),\end{aligned}$$

where the second equivalence follows from [16, A.2.3.1], which guarantees that the pro-space $\mathrm{Sh}(X)$ commutes with filtered colimits of n-truncated objects of $\mathcal{S}$. ■

5.4. It is usually not true that (5.2.1) is an isomorphism for higher homotopy groups. Nevertheless, in some cases Theorem 5.3 can be stated in terms of the homotopy groups of (étale topological types of) schemes rather than the homotopy groups of the profinite étale topological types of the schemes:

Corollary 5.5. *In the situation of Theorem 5.3, assume further that X and Y are geometrically normal and that f is locally of finite type. Let $\overline{(x, y)} : \operatorname{Spec} K \to X \times_k Y$ be a geometric point of $X \times_k Y$. Let $\overline{x}$ and $\overline{y}$ denote the induced geometric points on X and Y, respectively. Then for each integer $n \geq 0$, the canonical map*

$$\pi_n(X \times_k Y, \overline{(x, y)}) \to \pi_n(X, \overline{x}) \times \pi_n(Y, \overline{y})$$

is an isomorphism of profinite groups.

Proof. By virtue of Theorem 5.3, it will suffice to show that the homotopy groups of $\mathrm{Sh}_\pi(X)$ in the sense of [16, E.5.2.1] are isomorphic to the homotopy groups of X (and similarly for Y and $X \times_k Y$). As we will see in Lemma 6.5 (see also [6, 3.2.11]), the homotopy groups of $\mathrm{Sh}_\pi(X)$ are isomorphic to the homotopy groups of the profinite étale topological type $\widehat{\mathrm{h}}(X)$. Combining [6, 3.2.7, 3.2.11] with [19, 2.33], we see that they are isomorphic to the homotopy groups of the Artin–Mazur profinite completion of Artin–Mazur's étale homotopy type of X (see [1, p. 114] and [19, p. 604]). Since X and Y are geometrically normal, $X \times_k Y$ is normal. Invoking the fact [1, 11.1] that Artin–Mazur's étale homotopy types of X, Y, and $X \times_k Y$ are profinite (that is, their homotopy groups are all profinite), the desired result now follows by using [6, 3.2.11] again. ■

6. Application: profinite shapes of symmetric powers

6.1. Let X be a connected CW-complex. For each integer $n \geq 0$, the *n-th symmetric power* $\mathrm{Sym}^n X$ is defined as the quotient space X^n/S_n of the natural action of the symmetric group S_n on the n-fold product of X with itself. The classical Dold–Thom theorem [7, 6.10] shows that for each integer $i > 0$, the natural map

$$H_i(X;\mathbb{Z}) \to \pi_i\Big(\operatorname*{colim}_{n\geq 0} \operatorname{Sym}^n X\Big)$$

is an isomorphism.

In the setting of algebraic geometry, Arnav Tripathy proved that for a proper, normal, and connected algebraic space X over a separably closed field, there is a ♮-isomorphism [1, 4.2] between $\operatorname{Sym}^n \mathrm{h}(X)$ and $\mathrm{h}(\operatorname{Sym}^n X)$ when regarded as pro-objects in the homotopy category of those connected pointed CW-complexes whose homotopy groups are all finite, where h denotes the étale topological type functor of [6, 3.2.6]; see [26, Theorem 1]. In particular, the Dold–Thom theorem in the algebro-geometric setting holds.

6.2. The purpose of this section is to generalize [26, Theorem 1] without the assumption that X is normal. Unlike Tripathy who made a detailed study of the étale fundamental group of $\operatorname{Sym}^n X$ and used Deligne's computation of the cohomology of $\operatorname{Sym}^n X$, we will use the qfh topology where the behavior of symmetric powers of algebraic spaces becomes categorical. We remark that the idea of using the qfh topology in the study of étale homotopy types appears in the work of Hoyois [11].

As we will use various topologies associated to algebraic spaces, let us make it clear what the associated shapes mean:

Definition 6.3. Let $\mathcal{X}$ be an ∞-topos. The *shape of an object* $F \in \mathcal{X}$ is the shape of the ∞-topos $\mathcal{X}_{/F}$. The *profinite shape of* $F \in \mathcal{X}$ is the profinite completion of the shape of F. Let $\mathrm{Sh}(F)$ and $\mathrm{Sh}_\pi(F)$ denote the shape and the profinite shape of F, respectively.

6.4. Let T be an ordinary topos, and let F be an object of T. The author defined the topological type $\mathrm{h}(F)$ and the profinite topological type $\widehat{\mathrm{h}}(F)$ of F in [6, 2.3.2] and [6, 4.1.4], respectively; they are compatible with the definitions above in the following sense:

Lemma 6.5. *Let $\mathcal{X}$ be a* 1*-localic ∞-topos, and let F be an object of $\mathcal{X}$. Then the shape* $\mathrm{Sh}((\mathcal{X}_{/F})^\wedge)$ *of the hypercomplete ∞-topos $(\mathcal{X}_{/F})^\wedge$ is equivalent to the topological type* $\mathrm{h}(F)$ *of F as an object of the* 1*-topos* $\mathrm{Disc}(\mathcal{X})$ *under the equivalence of the model-categorical and the ∞-categorical pro-spaces* [2, 6.0.1]*. Moreover, the profinite shape* $\mathrm{Sh}_\pi(F)$ *of $F \in \mathcal{X}$ is equivalent to the profinite topological type* $\widehat{\mathrm{h}}(F)$ *of* $F \in \mathrm{Disc}(\mathcal{X})$ *under the equivalence of the model-categorical and the ∞-categorical profinite spaces* [2, 7.4.9].

Proof. The desired result follows by combining [2, 6.0.4] with [6, 2.3.17] (which shows that $\mathrm{h}(F)$ is equivalent to the topological type of the 1-topos $\mathrm{Disc}(\mathcal{X})_{/F}$). For profinite completions, use [5, 1.3] and Lemma 2.8. ■

6.6. Let S be a scheme. There exists a Grothendieck topology on the category of S-schemes (Sch/S) which can be characterized as follows: for each S-scheme T, a collection of S-morphisms $\{f_i : T_i \to T\}$ is a covering if and only if it is an *h covering* in the sense of [23, Tag 0ETS] and each f_i is locally quasi-finite. We refer to it as the *big qfh* *topology* on S. In the special case where T is noetherian, the collection is finite, and each f_i is quasi-compact, [23, Tag 0ETT] guarantees that one can recover the usual definition of qfh coverings of [27, 3.1.2]: a finite collection $\{f_i : T_i \to T\}$ of morphisms of schemes is a *qfh covering* if each f_i is quasi-finite and finite type and the induced map $\coprod T_i \to T$ is universally submersive (see, for example, [21, 3.9]). Let $\mathcal{T}_{\text{ét}}$ and $\mathcal{T}_{\text{qfh}}$ denote the ∞-categories of $\mathcal{S}$-valued sheaves with respect to the big étale and the big qfh topologies on (the nerve of) (Sch/S), respectively. The identity functor $(\text{Sch}/S)_{\text{ét}} \to (\text{Sch}/S)_{\text{qfh}}$ is continuous and commutes with finite limits (see [23, Tag 0ETK]), and therefore induces a geometric morphism $i_* : \mathcal{T}_{\text{qfh}} \to \mathcal{T}_{\text{ét}}$ of ∞-topoi.

Let X be an algebraic space over a scheme S. Regarding X as a 0-truncated object of $\mathcal{T}_{\text{ét}}$, one can define its shape $\text{Sh}(X)$ and profinite shape $\text{Sh}_\pi(X)$ (see Definition 6.3). In the case where X is a scheme, these definitions do not conflict with 5.1 (cf. [6, 3.2.7, 3.2.11]).

6.7. Fix an integer $n \geq 0$. Let S_n denote the symmetric group on n letters. Let $\text{Sub}(S_n)$ denote the partially ordered set of subgroups of S_n. For each subgroup $H \leq S_n$, let $o(H)$ denote the set of orbits of the induced action of H on the n letters.

Let $\mathcal{C}$ be an ∞-category which admits colimits and finite limits. According to [11, 2.2], the n-th symmetric power functor $\text{Sym}^n : \mathcal{C} \to \mathcal{C}$ is defined by the formula

$$\text{Sym}^n X = \operatorname*{colim}_{H \in \text{Sub}(S_n)} X^{o(H)},$$

where $X^{o(H)}$ denotes the $|o(H)|$-fold product of X with itself; when $\mathcal{C} = \mathcal{S}$, it follows from [8, 4.A.3] that for a CW-complex X, the above construction can be identified with the quotient space of the n-fold product of X with itself by the natural action of S_n, and when $\mathcal{C}$ is the nerve of an ordinary category, the symmetric powers behave like categorical quotients in the sense that $\text{Sym}^n X$ is equivalent to the coequalizer of the diagram

$$S_n \times X^n \rightrightarrows X^n,$$

where the two maps are the natural action of S_n on the n-fold product X^n of X with itself and the projection onto the second factor.

6.8. Let X be an algebraic space which is locally of finite type and separated over a scheme S. Associated to the natural action of S_n on the n-fold product X^n of X over S is the groupoid $S_n \times X^n \rightrightarrows X^n$ in algebraic spaces over S. It follows from [21, 5.3] that a GC quotient $q : X^n \to \text{Sym}^n X$ of the groupoid exists (see [21, 3.17] for the definition of a GC quotient). Let us now assume that the base scheme S is locally noetherian. The advantage of the qfh topology over the étale topology is that the GC quotient and the diagonal morphism,

$$q : X^n \to \text{Sym}^n X, \quad j : S_n \times X^n \to X^n \times_{\text{Sym}^n X} X^n,$$

are qfh coverings of algebraic spaces. In other words, the pullbacks $i^* q$ and $i^* j$ are epimorphisms in the category of qfh sheaves of sets on S.

To avoid confusion, let us denote the n-th symmetric power (in the sense of 6.7) of the qfh sheaf i^*X in the ∞-topos $\mathcal{T}_{\mathrm{qfh}}$ by $\mathrm{Sym}^n_{\mathrm{qfh}}(i^*X)$ for the remainder of this section.

Proposition 6.9. *Let S be a locally noetherian scheme. Let X be an algebraic space which is locally of finite type and separated over S. Then the canonical map*

$$\mathrm{Sym}^n_{\mathrm{qfh}}(i^*X) \to i^*(\mathrm{Sym}^n X)$$

is an equivalence in $\mathcal{T}_{\mathrm{qfh}}$.

Proof. We proceed as in the proof of [11, 4.6]. It follows from [11, 2.4] that $\tau_{\leq 0}\mathcal{T}_{\mathrm{qfh}} \subseteq \mathcal{T}_{\mathrm{qfh}}$ is stable under symmetric powers. Combining this with the fact that the pullback i^* preserves 0-truncated objects, our task amounts to proving the equivalence in the usual 1-topos of qfh sheaves of sets on S. Since i^* preserves finite limits, it suffices to show that the diagram

$$S_n \times X^n \rightrightarrows X^n \longrightarrow \mathrm{Sym}^n X$$

of big étale sheaves of sets on S defining $\mathrm{Sym}^n X$ pulls back to a coequalizer of qfh sheaves of sets on S. Invoking the fact that i^*q is an (effective) epimorphism, i^* applied to the diagram

$$X^n \times_{\mathrm{Sym}^n X} X^n \rightrightarrows X^n \longrightarrow \mathrm{Sym}^n X$$

which is induced by the fiber product $X^n \times_{\mathrm{Sym}^n X} X^n$ is a coequalizer of qfh sheaves of sets on S. We complete the proof by observing that one still has a coequalizer diagram after replacing $i^*(X^n \times_{\mathrm{Sym}^n X} X^n)$ of the coequalizer by $i^*(S_n \times X^n)$, using the epimorphism i^*j. ∎

Remark 6.10. In the situation of Proposition 6.9, if we work with the étale topology, the n-th symmetric power of X in the sense of 6.7 is a poorly behaved construction and is different from the GC quotient $\mathrm{Sym}^n X$ in general. Nevertheless, Proposition 6.9 shows that these two notions coincide after passing to the qfh topology: that is, the pullback of the GC quotient $\mathrm{Sym}^n X$ to $\mathcal{T}_{\mathrm{qfh}}$ is equivalent to the categorical quotient of the n-fold product of the qfh sheaf i^*X by the natural action of S_n (see 6.7).

Lemma 6.11 (cf. [11, 4.2]). *Let X be a quasi-compact, quasi-separated, and locally noetherian algebraic space over a scheme S. Then the natural map*

$$\mathrm{Sh}_\pi(i^*X) \to \mathrm{Sh}_\pi(X)$$

is an equivalence of profinite spaces.

Proof. Using Lemma 6.5 and the profinite hypercover descent [6, 4.2.9], we may assume that X is a scheme. Consider a morphism of 1-topoi induced by the identity functor $(\mathrm{Sch}/X)_{\text{ét}} \to (\mathrm{Sch}/X)_{\mathrm{qfh}}$ which is continuous and commutes with finite limits. By virtue of [12, Theorem 1] and its proof, the morphism of topoi induces isomorphisms on non-abelian first cohomology groups for locally constant sheaves of finite groups, and on global sections for locally constant sheaves of finite sets. On the other hand, [27, 3.4.4] guarantees that it induces isomorphisms on abelian cohomology groups for locally constant sheaves of finite abelian groups, which completes the proof. ■

Theorem 6.12. *Let X be a proper algebraic space over a separably closed field k. Then the natural map*

$$\mathrm{Sh}_\pi(\mathrm{Sym}^n X) \to \mathrm{Sym}^n(\mathrm{Sh}_\pi(X)),$$

which is induced by the map appearing in [11, 5.7]*, is an equivalence of profinite spaces.*

Proof. By virtue of [11, 5.7], we have a commutative square of profinite spaces

$$\begin{array}{ccc} \mathrm{Sh}_\pi(\mathrm{Sym}^n_{\mathrm{qfh}}(i^*X)) & \longrightarrow & \mathrm{Sym}^n(\mathrm{Sh}_\pi(i^*X)) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_\pi(\mathrm{Sym}^n X) & \longrightarrow & \mathrm{Sym}^n(\mathrm{Sh}_\pi(X)) \end{array}$$

It follows from Proposition 6.9 and Lemma 6.11 that the vertical arrows are equivalences. To complete the proof, it suffices to show that the top horizontal arrow is an equivalence. The commutativity of profinite shapes and products (Theorem 5.3, which also holds for algebraic spaces by virtue of Lemma 6.5 and [6, 4.2.9]) combined with Lemma 6.11 guarantees that for each $H \leq S_n$, the canonical map

$$\mathrm{Sh}_\pi((i^*X)^{o(H)}) \to (\mathrm{Sh}_\pi(i^*X))^{o(H)}$$

is an equivalence of profinite spaces. Passing to the colimit over $H \in \mathrm{Sub}(S_n)$, the bottom map in the following commutative diagram of profinite spaces is an equivalence:

$$\begin{array}{ccccc} & & \mathrm{Sh}_\pi(\mathrm{Sym}^n_{\mathrm{qfh}}(i^*X)) & & \\ & \nearrow & & \searrow & \\ \operatorname{colim} \mathrm{Sh}_\pi(\mathcal{T}_{\mathrm{qfh}/i^*X^{o(H)}}) & & \longrightarrow & & \mathrm{Sym}^n(\mathrm{Sh}_\pi(i^*X)) \end{array}$$

Invoking the fact that the profinite shape functor preserves colimits (because it is a left adjoint) and that the functor $\mathcal{T}_{\mathrm{qfh}}^{\mathrm{op}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which carries each object $F \in \mathcal{T}_{\mathrm{qfh}}$ to the ∞-category $\mathcal{T}_{\mathrm{qfh}/F}$ preserves small limits [14, 6.1.3.9], the left diagonal arrow is an equivalence, and therefore so is the right diagonal as desired. ■

Remark 6.13. (i) Assume further that X is connected and geometrically normal. By virtue of [1, 11.1], $\mathrm{h}(X)$ and $\mathrm{h}(\mathrm{Sym}^n X)$ can be regarded as pro-objects in the homotopy category of those connected CW-complexes whose homotopy groups are all finite. As Hoyois pointed out to the author, it is not clear whether they are profinite in the sense of [19, §2.7]. Nevertheless, both a $\natural$-isomorphism of profinite spaces in the sense of Artin–Mazur and a weak equivalence of profinite spaces in the sense of Gereon Quick [19, 2.6] are precisely those maps which induce isomorphisms on all homotopy groups. Therefore, using [19, 2.33] and Lemma 6.5, one can recover [26, Theorem 1], which supplies a $\natural$-isomorphism between $\mathrm{Sym}^n \mathrm{h}(X)$ and $\mathrm{h}(\mathrm{Sym}^n X)$.

(ii) In comparison with [4, 10.0.4]—which gives an alternative proof of [26, Theorem 1]—in the thesis of the author, the theorem above holds without the assumption that X is geometrically normal, and does not depend on the computation of the étale fundamental group of $\mathrm{Sym}^n X$ by Indranil Biswas and Amit Hogadi (see [4, 9.3.5]). It also fills in the gap in the proof of [4, 9.2.11] via Theorem 5.3.

Acknowledgments. The author is thankful to Clark Barwick, Peter J. Haine, Marc Hoyois, and Martin Olsson for their helpful comments and conversations. The author is also sincerely grateful to an anonymous referee for an enormous number of very detailed, helpful comments and suggestions.

Funding. This work was supported by the Institute for Basic Science (IBS-R003-D1) and the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2020R1A5A1016126).

References

[1] Artin, M., Mazur, B.: Etale Homotopy. Lecture Notes in Math. 100, Springer, Berlin (1969) Zbl 0182.26001 MR 0245577

[2] Barnea, I., Harpaz, Y., Horel, G.: Pro-categories in homotopy theory. Algebr. Geom. Topol. **17**, 567–643 (2017) Zbl 1360.18025 MR 3604386

[3] Barwick, C., Glasman, S., Haine, P.: Exodromy. arXiv:1807.03281v7 (2020)

[4] Cho, C.-Y.: Topological types of algebraic stacks. Ph.D. thesis, Univ. of California, Berkeley (2016) MR 3553672

[5] Chough, C.-Y.: An equivalence of profinite completions. Preprint, https://sites.google.com/view/chough (2018)

[6] Chough, C.-Y.: Topological types of algebraic stacks. Int. Math. Res. Notices **2021**, 7799–7849 Zbl 07398536 MR 4259160

[7] Dold, A., Thom, R.: Quasifaserungen und unendliche symmetrische Produkte. Ann. of Math. (2) **67**, 239–281 (1958) Zbl 0091.37102 MR 97062

[8] Dror Farjoun, E.: Cellular Spaces, Null Spaces and Homotopy Localization. Lecture Notes in Math. 1622, Springer, Berlin (1996) Zbl 0842.55001 MR 1392221

[9] Giraud, J.: Cohomologie non abélienne. Grundlehren Math. Wiss. 179, Springer, Berlin (1971) Zbl 0226.14011 MR 0344253

[10] Hoyois, M.: Higher Galois theory. J. Pure Appl. Algebra **222**, 1859–1877 (2018) Zbl 1403.18003 MR 3763287

[11] Hoyois, M.: The étale symmetric Künneth theorem. arXiv:1810.00351 (2018)

[12] Jardine, J. F.: Finite group torsors for the qfh topology. Math. Z. **244**, 859–871 (2003) Zbl 1050.14012 MR 2000462

[13] Laszlo, Y., Moreau, A.: Exposé XX. Rigidité. In: Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, Astérisque **363-364**, 491–528 (2014) Zbl 1320.14028 MR 3329791

[14] Lurie, J.: Higher Topos Theory. Ann. of Math. Stud. 170, Princeton Univ. Press, Princeton, NJ (2009) Zbl 1175.18001 MR 2522659

[15] Lurie, J.: Rational and p-adic homotopy theory. Preprint, https://www.math.ias.edu/~lurie/papers/DAG-XIII.pdf (2011)

[16] Lurie, J.: Spectral algebraic geometry. Preprint, https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf (2018)

[17] Moerdijk, I., Vermeulen, J. J. C.: Proof of a conjecture of A. Pitts. J. Pure Appl. Algebra **143**, 329–338 (1999) Zbl 0944.18004 MR 1731050

[18] Olsson, M.: Algebraic Spaces and Stacks. Amer. Math. Soc. Colloq. Publ. 62, Amer. Math. Soc., Providence, RI (2016) Zbl 1346.14001 MR 3495343

[19] Quick, G.: Profinite homotopy theory. Doc. Math. **13**, 585–612 (2008) Zbl 1173.55008 MR 2466189

[20] Revêtements étales et groupe fondamental. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1),, Lecture Notes in Math. 224, Springer, Berlin (1971) Zbl 0234.14002 MR 0354651

[21] Rydh, D.: Existence and properties of geometric quotients. J. Algebraic Geom. **22**, 629–669 (2013) Zbl 1278.14003 MR 3084720

[22] Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Lecture Notes in Math. 152, Springer, Berlin (1970) Zbl 0209.24201 MR 0274459

[23] The Stacks Project Authors. Stacks Project. https://stacks.math.columbia.edu (2018)

[24] Théorie des topos et cohomologie étale des schémas. Tome 2. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Math. 270, Springer, Berlin (1972) Zbl 0237.00012 MR 0354653

[25] Théorie des topos et cohomologie étale des schémas. Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Math. 305, Springer, Berlin (1973) Zbl 0245.00002 MR 0354654

[26] Tripathy, A.: The symmetric power and etale realization functors commute. Ph.D. thesis, Stanford Univ. (2016) MR 4172224

[27] Voevodsky, V.: Homology of schemes. Selecta Math. (N.S.) **2**, 111–153 (1996) Zbl 0871.14016 MR 1403354