J. Eur. Math. Soc. 24, 3801–3827 (2022) DOI 10.4171/JEMS/1237



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# Integral $p$-adic Hodge filtrations in low dimension and ramification

Received February 19, 2019

**Abstract.** Given an integral $p$-adic variety, we observe that if the integral Hodge–de Rham spectral sequence behaves nicely, then the special fiber knows the Hodge numbers of the generic fiber. Applying recent advancements of integral $p$-adic Hodge theory, we show that such a nice behavior is guaranteed if the $p$-adic variety can be lifted to an analogue of second Witt vectors and satisfies some bound on dimension and ramification index. This is a (ramified) mixed characteristic analogue of results due to Deligne–Illusie, Fontaine–Messing, and Kato. Lastly, we discuss an example illustrating the necessity of the aforementioned lifting condition, which is of independent interest.

**Keywords.** De Rham cohomology, prismatic cohomology, $p$-adic Hodge theory

## 1. Introduction

Given a smooth proper scheme $\mathcal{X}$ over some $p$-adic ring of integers $\mathcal{O}_K$, can we tell the Hodge diamond of its generic fiber $X$ by simply staring at the geometry of the special fiber $\mathcal{X}_0$? In general, there is no hope of this being true. But surely if one puts some constraints, this will be true.

There are two typical pathological phenomenons concerning Hodge and de Rham cohomology groups in an integral $p$-adic situation: one being torsions in the cohomology groups, the other being the non-degeneracy of the integral Hodge–de Rham spectral sequence. In this paper we convey the idea that, for the question asked at the beginning, the trouble comes from the second phenomenon. To be more precise, we define virtual Hodge numbers for any smooth proper variety in characteristic $p$ (see Definition 3.1). In Proposition 3.4, we observe that if the integral Hodge–de Rham spectral sequence of a lift $\mathcal{X}$ degenerates saturatedly (see Definition 2.3), then the virtual Hodge numbers of $\mathcal{X}_0$ agree with the Hodge numbers of the generic fiber $X$.

Then we show the following:

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*Mathematics Subject Classification (2020)*: Primary 14F30; Secondary 14F40, 14G22

**Theorem 1.1** (Main Theorem). *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be a smooth proper formal scheme, and let $\mathrm{W}(\kappa)$ be the Witt ring of the residue field of $\mathcal{O}_K$. Let $\mathfrak{S} := \mathrm{W}(\kappa)[\![u]\!]$ be the Breuil–Kisin prism associated with $\mathcal{O}_K$ and let $E$ be an associated Eisenstein polynomial* [6, Example 1.3 (3)]. *Assume that*

(1) *there is a lift of $\mathcal{X}$ over $\mathfrak{S}/(E^2)$; and*

(2) *the relative dimension of $\mathcal{X}$ and the ramification index $e$ of $\mathcal{O}_K$ satisfy the inequality $\dim(\mathcal{X}_0) \cdot e < p - 1$.*

*Then the Hodge–de Rham spectral sequence for $\mathcal{X}$ is split degenerate. In particular, we have equality of* (*virtual*) *Hodge numbers:*

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) = h^{i,j}(X).$$

One may think of this result as a mixed characteristic analogue of a theorem by Deligne–Illusie [8]. There is a similar statement with weaker conclusion when $\dim X$ exceeds the bound in (2) (see Porism 3.11). When $\mathcal{O}_K$ is the Witt ring of a perfect field, condition (1) is automatic and our result can also be deduced from Fontaine–Messing's result [10] or Kato's result [12]. We summarize the implications between various relevant conditions in Section 3.4. The proof of the main theorem uses recent theory of prismatic cohomology due to Bhatt–Scholze [6] along with a result of Min [15]. For more details, see Section 3.

Lastly, one may wonder if either condition (1) or (2) is really necessary. Previously in [14] we have constructed a pair of relative 3-folds over $\mathbb{Z}_p[\zeta_p]$ with isomorphic special fiber such that their generic fibers have different Hodge numbers, showing that the conclusion of Theorem 1.1 is not true in general. While it is unclear if condition (2) is really necessary, in Section 4, which is independent of other sections, we extensively discuss an example illustrating the necessity of condition (1).

**Theorem 1.2** (see Theorem 4.14). *There exists a smooth projective relative* 4*-fold $\mathcal{X}$ over a ramified degree* 2 *extension $\mathcal{O}_K$ of $\mathbb{Z}_p$ such that both its Hodge–de Rham and Hodge–Tate spectral sequences are non-degenerate. Moreover, the Hodge/conjugate filtrations are non-split as $\mathcal{O}_K$-modules.*

The idea of construction, which may be traced back to W. Lang's work [13] and Raynaud's [16], is as follows. The exotic group scheme $\alpha_p$ admits liftings over ramified $p$-adic rings of integers. We choose a lift $G$ over a degree 2 ramified $p$-adic ring of integers, then we study the Hodge–de Rham and Hodge–Tate spectral sequences of $BG$, the classifying stack of $G$. With the aid of various computations in [2], we find out that both are non-degenerate starting at degree 3 with non-split Hodge/conjugate filtrations starting at degree 2. In the end, we take an approximation of $BG$ to get the desired example.

## 2. Preliminaries on spectral sequences over DVRs

This section is a general discussion of spectral sequences associated with a filtered bounded perfect complex over a DVR.

**Notation 2.1.** Throughout this section, let $R$ be a DVR with a uniformizer $\pi$. Denote $K := R[1/\pi]$ and $\kappa := R/\pi$. Let $(C, \mathrm{Fil}^\bullet)$ be a filtered object in $D^b_{\mathrm{Coh}}(R)$. We assume that the filtration on $C$ is exhaustive and complete.

Given a finitely generated $R$-module $M$, we denote by $M_{\mathrm{tor}}$ the torsion submodule in $M$, and we denote its torsion-free quotient by $M_{\mathrm{tf}} := M/M_{\mathrm{tor}}$.

**Remark 2.2.** We do not assume this filtration to be either increasing or decreasing, as it is modeling both Hodge–de Rham and Hodge–Tate spectral sequences.

From $(C, \mathrm{Fil}^\bullet)$ we naturally get a spectral sequence converging from $H^i(\mathrm{Gr}^j)$ to $H^i(C)$. From now on, we will call it "the spectral sequence" if no confusion seems to arise. In the following definition, we refine the classical notion of the spectral sequence being degenerate.

**Definition 2.3.** (1) We say the spectral sequence *degenerates* or is *degenerate* if for all pairs of integers $(i, j)$, the natural map $\mathrm{H}^i(\mathrm{Fil}^j) \to \mathrm{H}^i(C)$ is an injection.

(2) We say the spectral sequence *degenerates saturatedly* or is *saturated degenerate* if it degenerates and the induced injection $\mathrm{H}^i(\mathrm{Fil}^j)_{\mathrm{tf}} \to \mathrm{H}^i(C)_{\mathrm{tf}}$ is saturated.

(3) We say the spectral sequence *degenerates splittingly* or is *split degenerate* if it degenerates and the induced injection $\mathrm{H}^i(\mathrm{Fil}^j) \to \mathrm{H}^i(C)$ splits.

Recall that an injection/inclusion of torsion-free $R$-modules $N \subset M$ is said to be *saturated* if $\pi N = N \cap \pi M$ or, what is the same, $M/N$ is $\pi$-torsion-free.

**Remark 2.4.** It is obvious that the spectral sequence being split degenerate implies it being saturated degenerate, and both imply it is degenerate.

In the case of Hodge–de Rham or Hodge–Tate spectral sequences (over mixed characteristic DVRs), we know that they degenerate after inverting $p$ (see [17, Corollary 1.8] and [5, Theorem 1.7]). In this scenario, we have a condition on the $E_\infty$-page of the spectral sequence characterizing the spectral sequence being saturated or split degenerate:

**Proposition 2.5.** *Suppose that the spectral sequence degenerates after inverting $\pi$. Then*

(1) *the spectral sequence is saturated degenerate if and only if*

$$\mathrm{length}(H^i(C)_{\mathrm{tor}}) = \sum_j \mathrm{length}(H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}) \quad \text{for all } i;$$

(2) *the spectral sequence is split degenerate if and only if there is an abstract isomorphism of $R$-modules*

$$H^i(C)_{\mathrm{tor}} \simeq \bigoplus_j H^i(\mathrm{Gr}^j C)_{\mathrm{tor}} \quad \text{for all } i.$$

Note that since we assumed that $\mathrm{Fil}^\bullet$ is exhaustive and saturated, the summation process is finite.

*Proof of Proposition* 2.5 (1). First notice that

$$\operatorname{length}(H^i(C)_{\mathrm{tor}}) \le \sum_j \operatorname{length}(\mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}) \le \sum_j \operatorname{length}(H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}),$$

where the second inequality comes from the fact that the spectral sequence degenerates after inverting $\pi$ (so $\mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}$ must be a subquotient of $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$). Therefore the equality condition implies equality between $\mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}$ and $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$. In other words, every element in $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$ is a permanent cycle. Since the spectral sequence degenerates after inverting $\pi$, we know that all the differentials in the spectral sequence are torsion. Combining these two, we see that all the differentials are forced to be zero, which exactly means that the spectral sequence must degenerate.

Now we have reduced the statement of (1) to: assume the spectral sequence degenerates; then it is saturated degenerate if and only if the equality of lengths of $\pi$-torsions holds, and this statement is handled in the following lemma.

**Lemma 2.6.** *Let $M$ be a finitely generated $R$-module with an exhaustive and saturated filtration $\mathrm{F}^\bullet$. Then $\mathrm{F}^i_{\mathrm{tf}} \subset M_{\mathrm{tf}}$ is saturated for all $i$ if and only if*

$$\operatorname{length}(M_{\mathrm{tor}}) = \sum_i \operatorname{length}(\mathrm{Gr}^i)_{\mathrm{tor}}).$$

*Proof of Lemma* 2.6. First observe that for any $i$ we have an inequality

$$\operatorname{length}(M_{\mathrm{tor}}) \le \operatorname{length}(\mathrm{F}^i_{\mathrm{tor}}) + \operatorname{length}((M/\mathrm{F}^i)_{\mathrm{tor}}),$$

with equality holding if and only if the map

$$M_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}}$$

is surjective. Hence we see that the equality in the statement is equivalent to this map being surjective for all $i$.

Applying the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{F}^i_{\mathrm{tor}} & \longrightarrow & \mathrm{F}^i & \longrightarrow & \mathrm{F}^i_{\mathrm{tf}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{\mathrm{tor}} & \longrightarrow & M & \longrightarrow & M_{\mathrm{tf}} & \longrightarrow & 0 \end{array}$$

yields an exact sequence

$$0 \to M_{\mathrm{tor}}/\mathrm{F}^i_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}} \to (M_{\mathrm{tf}}/\mathrm{F}^i_{\mathrm{tf}})_{\mathrm{tor}} \to 0.$$

Here we used the fact that $\mathrm{F}^i \cap M_{\mathrm{tor}} = \mathrm{F}^i_{\mathrm{tor}}$. This short exact sequence says exactly that the surjectivity of $M_{\mathrm{tor}} \to (M/\mathrm{F}^i)_{\mathrm{tor}}$ is equivalent to $M_{\mathrm{tf}}/\mathrm{F}^i_{\mathrm{tf}}$ being torsion-free, which concludes the proof of this lemma. ■

This completes the proof of Proposition 2.5 (1). ■

Before proving the second part of Proposition 2.5, let us briefly discuss the condition of an extension of finitely generated torsion $R$-modules being split.

**Definition 2.7.** Let $M$ be a finitely generated torsion $R$-module. Write

$$M = \bigoplus_{i=1}^{l} R/\pi^{n_i}$$

where $n_1 \leq \cdots \leq n_l$. Then the *characteristic polygon* of $M$, denoted by $\mathcal{P}_M$, is the graph of the piecewise linear function defined on $[0, l]$ passing through $(0, 0)$, with the $i$-th segment of horizontal span 1 and slope $n_i$.

**Remark 2.8.** It is easy to see that the width of $\mathcal{P}_M$ is given by $\dim_\kappa(M/\pi M)$, and the end points are $(0, 0)$ and $(\dim_\kappa(M/\pi M), \mathrm{length}(M))$.

Given two finitely generated torsion $R$-modules, we would like to compare the characteristic polygons of an extension class and that of their direct sums.

**Example 2.9** (see also [7, p. 502]). Consider an extension

$$0 \to N = R/\pi^l \to M \to R/\pi^m \to 0.$$

Then we must have either $M \simeq R/\pi^{l+m}$ or $M \simeq R/\pi^n \oplus R/\pi^{m+l-n}$, where $\min\{n, m+l-n\} \leq \min\{l, m\}$ with equality if and only if the short exact sequence splits.

We observe that the former case corresponds to $N/\pi \to M/\pi$ being not injective. In the latter case we see that $\mathcal{P}_M$ is always lower than or equal to $\mathcal{P}_{N \oplus M/N}$, and equality holds exactly when the extension class splits. Here by $\mathcal{P}_{N \oplus M/N}$ we mean the characteristic polygon associated with the module $N \oplus M/N$.

Inspired by this example, we give the following criterion characterizing split short exact sequences of finitely generated torsion $R$-modules.

**Proposition 2.10.** *Let $M$ be a finitely generated torsion $R$-module, and $N \subset M$ a submodule. Suppose $N/\pi \to M/\pi$ is an injection. Then $\mathcal{P}_M$ is lower than or equal to $\mathcal{P}_{N \oplus M/N}$, with equality holding if and only if $N \subset M$ splits.*

*Proof.* First assume we can prove the statement when $N$ is cyclic (generated by one element). Write $N = N_1 \oplus N_2$. Inducting on the dimension of $N/\pi$, we see that

$$\mathcal{P}_M \leq \mathcal{P}_{N_1 \oplus M/N_1} \leq \mathcal{P}_{N_1 \oplus N_2 \oplus M/N} = \mathcal{P}_{N \oplus M/N},$$

with equality holding if and only if both $N_1 \subset M$ and $N_2 \subset M/N_1$ split, or equivalently $N = N_1 \oplus N_2 \subset M$ splits. Therefore we reduce to the case where $N = R/\pi^n$ is cyclic.

Dually, we may induct on the dimension of $(M/N)/\pi$. By the same argument as above, we may also assume that $M/N$ is cyclic. Now we have reduced the statement to the case where both $N$ and $M/N$ are cyclic, which is discussed in Example 2.9. ∎

We may extend this discussion to a multi-filtered situation, which is useful when considering spectral sequences. Let us record one consequence of Proposition 2.10.

**Corollary 2.11.** *Let* $(M, \mathrm{F}^\bullet)$ *be a finitely generated torsion* $R$*-module with an exhaustive and complete filtration. Suppose that we have an abstract isomorphism*

$$M \simeq \bigoplus_i \mathrm{Gr}^i.$$

*Then all* $\mathrm{F}^i \subset M$ *are direct summands.*

*Proof.* Without loss of generality, assume that the filtration is increasing. Now $M \simeq \bigoplus_i \mathrm{Gr}^i$ is a Noetherian $R$-module, and since the filtration is exhaustive and complete, we may assume that $\mathrm{F}^i = 0$ for $i < 0$ and that there is an integer $N$ such that $\mathrm{F}^j = M$ whenever $j > N$.

Next let us show that the natural map $\mathrm{F}^i/\pi \to \mathrm{F}^{i+1}/\pi$ is injective for all $i$. Due to right exactness of reduction modulo $\pi$, we have a chain of inequalities

$$\begin{aligned}\dim_\kappa(M/\pi) &\leq \dim_\kappa(\mathrm{F}^{N-1}/\pi) + \dim_\kappa(\mathrm{Gr}^N/\pi) \leq \cdots \\ &\leq \dim_\kappa(\mathrm{F}^i/\pi) + \sum_{j=i+1}^{N} \dim_\kappa(\mathrm{Gr}^j/\pi) \leq \cdots \leq \sum_{j=0}^{N} \dim_\kappa(\mathrm{Gr}^j/\pi) = \dim_\kappa(M/\pi),\end{aligned}$$

where the last equality follows from the condition that $M \simeq \bigoplus_i \mathrm{Gr}^i$. Therefore all the inequalities in the above chain must in fact be equalities, which is equivalent to saying all the maps $\mathrm{F}^i/\pi \to \mathrm{F}^{i+1}/\pi$ are injective.

Now Proposition 2.10 implies that

$$\mathcal{P}_M \leq \mathcal{P}_{\mathrm{Gr}^0 \oplus M/\mathrm{F}^0} \leq \mathcal{P}_{\mathrm{Gr}^0 \oplus \mathrm{Gr}^1 \oplus M/\mathrm{F}^1} \leq \cdots \leq \mathcal{P}_{\bigoplus_i \mathrm{Gr}^i},$$

and our condition forces all the inequalities above to be equalities. Hence applying Proposition 2.10 again yields what we want. ∎

Now we turn to the proof of (the "if" part of) Proposition 2.5 (2).

*Proof of Proposition* 2.5 (2). By validity of (1), we see that in this situation, the spectral sequence is already saturated degenerate. Therefore it suffices to show that the induced filtration $H^i(\mathrm{Fil}^j C)_{\mathrm{tor}} = \mathrm{Fil}^j H^i(C)_{\mathrm{tor}}$ on $H^i(C)_{\mathrm{tor}}$ is split for all $i$.

Notice that in our proof of (1), we established that the graded pieces of this filtration are exactly given by $H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}$. Now our condition implies that we have an abstract isomorphism

$$H^i(C)_{\mathrm{tor}} \simeq \bigoplus_j \mathrm{Gr}^j H^i(C)_{\mathrm{tor}}.$$

Applying Corollary 2.11, we see that $H^i(\mathrm{Fil}^j C)_{\mathrm{tor}} = \mathrm{Fil}^j H^i(C)_{\mathrm{tor}} \subset H^i(C)_{\mathrm{tor}}$ is split for all $i$. ∎

Inspired by Proposition 2.5, we make the following further definition concerning the torsion part of various pages of the spectral sequence.

**Definition 2.12.** (1) We say the spectral sequence *has saturated degenerate torsion in degree $i$* if

$$\operatorname{length}(H^i(C)_{\mathrm{tor}}) = \sum_j \operatorname{length}(H^i(\mathrm{Gr}^j C)_{\mathrm{tor}}).$$

(2) We say the spectral sequence *has split degenerate torsion in degree $i$* if

$$H^i(C)_{\mathrm{tor}} \simeq \bigoplus_j H^i(\mathrm{Gr}^j C)_{\mathrm{tor}} \quad \text{as } R\text{-modules.}$$

The following proposition is similar to Proposition 2.5.

**Proposition 2.13.** *Suppose that the spectral sequence $(C, \mathrm{Fil}^\bullet)$ degenerates after inverting $\pi$. Let $i$ be an integer.*

(1) *If the spectral sequence has saturated degenerate torsion in degree $i$, then*

$$H^i(\mathrm{Gr}^j)_{\mathrm{tor}} \cong \mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}.$$

*Consequently, the maps $H^i(\mathrm{Fil}^j) \to H^i(C)$ are injective and the induced injection $H^i(\mathrm{Fil}^j)_{\mathrm{tf}} \to H^i(C)_{\mathrm{tf}}$ is saturated for all $j$.*

(2) *If the spectral sequence has split degenerate torsion in degree $i$, then the induced maps $H^i(\mathrm{Fil}^j) \to H^i(C)$ are split for all $j$.*

*Proof.* For (1): In the proof of Proposition 2.5 (1) we have established that the equality of lengths of torsions implies the claimed identifications. Since the spectral sequence is assumed to be degenerate after inverting $\pi$, we immediately see that the maps $H^i(\mathrm{Fil}^j) \to H^i(C)$ are injective.

Next we look at the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i(\mathrm{Fil}^j)_{\mathrm{tor}} & \longrightarrow & H^i(C)_{\mathrm{tor}} & \longrightarrow & H^i(C/\mathrm{Fil}^j)_{\mathrm{tor}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^i(\mathrm{Fil}^j) & \longrightarrow & H^i(C) & \longrightarrow & H^i(C/\mathrm{Fil}^j) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^i(\mathrm{Fil}^j)_{\mathrm{tf}} & \longrightarrow & H^i(C)_{\mathrm{tf}} & \longrightarrow & H^i(C/\mathrm{Fil}^j)_{\mathrm{tf}} & &
\end{array}$$

Notice that the first two rows are exact, and the snake lemma gives us an exact sequence

$$0 \to H^i(\mathrm{Fil}^j)_{\mathrm{tf}} \to H^i(C)_{\mathrm{tf}} \to H^i(C/\mathrm{Fil}^j)_{\mathrm{tf}},$$

which implies that $H^i(C)_{\mathrm{tf}}/H^i(\mathrm{Fil}^j)_{\mathrm{tf}}$, as a submodule of $H^i(C/\mathrm{Fil}^j)_{\mathrm{tf}}$, is torsion-free.

For (2): Using what we have proved in (1), all we need to show is that the inclusion $H^i(\mathrm{Fil}^j)_{\mathrm{tor}} \to H^i(C)_{\mathrm{tor}}$ splits, which follows from the identifications $H^i(\mathrm{Gr}^j)_{\mathrm{tor}} \cong \mathrm{Gr}^j(H^i(C))_{\mathrm{tor}}$ and Corollary 2.11. ∎

One can also define what it means for the spectral sequence to have saturated/split degenerate torsion in a range of degrees. Proposition 2.13 can be immediately generalized to this generality.

The following proposition will be used in later sections.

**Proposition 2.14.** *Let $M$ be a finitely generated $R$-module, with $N \subset M$ a submodule. Suppose that $N_{\mathrm{tf}} \subset M_{\mathrm{tf}}$ is saturated. Then*

$$\dim_K N[1/\pi] = \dim_\kappa \mathrm{Im}(N/\pi \to M_{\mathrm{tf}}/\pi).$$

*Proof.* Neither side changes when we pass from $M$ to $M_{\mathrm{tf}}$. Hence we may assume $M$ is torsion-free, in which case $N$ is a direct summand by being a saturated submodule, and the statement becomes trivial. ∎

The next two lemmas have been pointed out by the anonymous referee; we thank them for this suggestion.

**Lemma 2.15.** *Let $U \to V$ be a map of $R$-complexes. Let $n$ be an integer and assume that $H^{n+1}(U) \to H^{n+1}(V)$ is injective. Then we have an identification of subspaces in $H^n(V)/\pi$:*

$$\begin{array}{c} \mathrm{Im}(H^n(U)/\pi \to H^n(V)/\pi) \\ \big\downarrow \cong \\ \mathrm{Im}(H^n(U/\pi) \to H^n(V/\pi)) \cap H^n(V)/\pi \end{array}$$

*Proof.* Let us denote the cone of $U \to V$ by $C$. The assumption yields an exact sequence

$$H^n(U) \to H^n(V) \to H^n(C) \to 0.$$

Modulo $\pi$, and using right exactness, we get

$$H^n(U)/\pi \to H^n(V)/\pi \to H^n(C)/\pi \to 0.$$

Now we have the following diagram, with horizontal lines exact and vertical arrows injective:

$$\begin{array}{ccccc} H^n(U)/\pi & \longrightarrow & H^n(V)/\pi & \longrightarrow & H^n(C)/\pi \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H^n(U/\pi) & \longrightarrow & H^n(V/\pi) & \longrightarrow & H^n(C/\pi) \end{array}$$

A simple diagram chasing now gives the claimed identification. ∎

Combining Proposition 2.14 and Lemma 2.15 gives

**Lemma 2.16.** *Let $U \to V$ be a map of $R$-complexes. Let $n$ be an integer. Assume that*

(1) *the image of $H^n(U) \to H^n(V)_{\mathrm{tf}}$ is saturated inside $H^n(V)_{\mathrm{tf}}$; and*

(2) *the map $H^{n+1}(U) \to H^{n+1}(V)$ is injective.*

*Then*

$$\begin{aligned}&\dim_K \operatorname{Im}(H^n(U)[1/\pi] \to H^n(V)[1/\pi])\\ &\qquad = \dim_\kappa \operatorname{Im}\Big(\big(\operatorname{Im}(H^n(U/\pi) \to H^n(V/\pi)) \cap H^n(V)/\pi\big) \to H^n(V)_{\mathrm{tf}}/\pi\Big).\end{aligned}$$

*Proof.* We use Proposition 2.14, where $N \subset M$ are given by the image of $H^n(U)$ inside $H^n(V)$. We then use Lemma 2.15 to rewrite the right hand side of the equality. ■

For our later purposes, we also need to discuss the relation of cohomologies of a perfect $R$-complex and cohomologies of its dual.

**Lemma 2.17.** *Let $U$ be a perfect $R$-complex. Consider $V = \mathrm{RHom}_R(U, R[0])$. Then for any integer $i$ we have canonical identifications:*

$$H^{-i+1}(V)_{\mathrm{tor}} \cong \mathrm{Hom}_R(H^i(U)_{\mathrm{tor}}, K/R) \quad \text{and} \quad H^{-i}(V)_{\mathrm{tf}} \cong \mathrm{Hom}_R(H^i(U)_{\mathrm{tf}}, R).$$

*Proof.* We have an $E_2$ spectral sequence:

$$\mathrm{Ext}^i_R(H^j(U), R) \Longrightarrow H^{i-j}(V).$$

Since $R$ is a DVR, the terms on the second page vanish unless $i \in \{0, 1\}$. Hence the spectral sequence degenerates for degree reasons, and we get a natural short exact sequence

$$0 \to \mathrm{Ext}^1_R(H^i(U), R) \to H^{-i+1}(V) \to \mathrm{Hom}_R(H^{i-1}(U), R) \to 0.$$

Now we finish by recalling that given a finitely generated $R$-module $M$, one has canonical identifications

$$\mathrm{Hom}_R(M, R) = \mathrm{Hom}_R(M_{\mathrm{tf}}, R) \quad \text{and} \quad \mathrm{Ext}^1_R(M, R) = \mathrm{Hom}_R(M_{\mathrm{tor}}, K/R). \qquad ■$$

## 3. Consequences of recent developments in integral $p$-adic Hodge theory

In this section we concentrate on the $p$-adic situation.

### 3.1. *Notations and setup*

Throughout this section, let $K$ be a complete $p$-adic field with a chosen uniformizer $\pi$, ring of integers $\mathcal{O}_K$ and perfect residue field $\kappa := \mathcal{O}_K/(\pi)$. Recall that $\mathcal{O}_K$ contains the ring of Witt vectors of $\kappa$, and the degree of the fraction field extension $K_0 := \mathrm{W}(\kappa)[1/p] \subset K$ is called the *ramification index* of $K$ and denoted by $e$. This paper concerns low ramification situation, in particular we assume that $e \leq p - 1$, therefore the ideal $(\pi) \subset \mathcal{O}_K$ has a unique divided power structure.

Consider $\mathfrak{S} := \mathrm{W}(\kappa)[\![u]\!]$ with a surjection $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$, where $u$ is sent to the chosen uniformizer $\pi$. The kernel of this surjection is generated by an Eisenstein polynomial $I = (E(u))$. Define the Frobenius $\phi\colon \mathfrak{S} \to \mathfrak{S}$ that extends the Frobenius on $\mathrm{W}(\kappa)$ and sends $u$ to $u^p$; since $\mathfrak{S}$ is $p$-torsion-free, this puts a unique $\delta$-structure on $\mathfrak{S}$. The pair $(\mathfrak{S}, I)$ is a Breuil–Kisin type prism (see [6, Example 1.3 (3)]).

Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathcal{O}_K$. We denote the special fiber of $\mathcal{X}$ by $\mathcal{X}_0 := \mathcal{X} \times_{\mathcal{O}_K} \kappa$ and the (rigid) generic fiber of $\mathcal{X}$ by $X := \mathcal{X} \times_{\mathcal{O}_K} K$. We call $\mathcal{X}$ a *lifting of* $\mathcal{X}_0$ *over* $\mathcal{O}_K$. In the case where $e = 1$, i.e. $\mathcal{O}_K = \mathrm{W}(\kappa)$, we call $\mathcal{X}$ an *unramified lift of* $\mathcal{X}_0$.

### 3.2. *Virtual Hodge numbers*

Recall a natural identification $\mathrm{R\Gamma}_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa)) \otimes^{\mathbb{L}}_{\mathrm{W}(\kappa)} \kappa \simeq \mathrm{R\Gamma}_{dR}(\mathcal{X}_0/\kappa)$, which implies that we have a natural injection

$$H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \hookrightarrow H^i_{\mathrm{dR}}(\mathcal{X}_0/\kappa)$$

for all $i$. Therefore we may regard $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p$ as a natural subquotient of $H^i_{\mathrm{dR}}(\mathcal{X}_0/\kappa)$.

**Definition 3.1** (Virtual Hodge numbers). The Hodge filtration on $H^i_{\mathrm{dR}}(\mathcal{X}_0/\kappa)$ induces a natural filtration on the subquotient $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p$. The *virtual Hodge numbers* of $\mathcal{X}_0$ are given by

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) := \dim_\kappa \mathrm{Fil}^i(H^{i+j}_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p) - \dim_\kappa \mathrm{Fil}^{i+1}(H^{i+j}_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p).$$

Unwinding the definition, we have the following description of the $i$-th induced filtration on $H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p$:

$$\mathrm{Im}\big(\mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}_0/\kappa}) \to H^n(\Omega^\bullet_{\mathcal{X}_0/\kappa})) \cap H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \to H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p\big). \tag{$\boxdot$}$$

Note that this definition *only depends* on the smooth proper variety $\mathcal{X}_0$ in characteristic $p$.

**Remark 3.2.** It is worth pointing out that in this definition, we may replace the Witt vectors by any ring of integers $\mathcal{O}_K$ as long as $e \leq p-1$. By the de Rham–crystalline comparison and the base change formula of crystalline cohomology [4, Corollary 7.3 and Theorem 7.8], we have natural identifications:

- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa)) \otimes_{\mathrm{W}(\kappa)} \mathcal{O}_K \cong H^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$,
- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}} \otimes_{\mathrm{W}(\kappa)} \mathcal{O}_K \cong H^i_{\mathrm{dR}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}$,
- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p \cong H^i_{\mathrm{dR}}(\mathcal{X}_0/\mathcal{O}_K)/\pi$, and
- $H^i_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p \cong H^i_{\mathrm{dR}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi$.

These filtrations are only objects in characteristic $p$. The goal of this subsection is to show that if the integral $p$-adic Hodge filtration behaves nicely, then these filtrations can tell us something about the rational Hodge filtrations.

**Proposition 3.3.** *Let* $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ *be as in Section 3.1. Assume that the integral Hodge–de Rham spectral sequence of* $\mathcal{X}$ *degenerates. Then we have identifications of subspaces* *in* $H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})/\pi \cong H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p$*:*

$$\begin{array}{c}\mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})/\pi) \\ \downarrow \cong \\ \mathrm{Im}(H^n(\Omega^{\geq i}_{\mathcal{X}_0/\kappa}) \to H^n(\Omega^\bullet_{\mathcal{X}_0/\kappa})) \cap H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))/p\end{array}$$

*Proof.* This is a direct application of Lemma 2.15. We write $U = \mathrm{R}\Gamma(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})$ and $V = \mathrm{R}\Gamma(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})$. Then the cone $C$ equals $\mathrm{R}\Gamma(\Omega^{\leq i-1}_{\mathcal{X}/\mathcal{O}_K})$. The assumption on the Hodge–de Rham spectral sequence implies the condition required in Lemma 2.15. Together with the identification spelled out in Remark 3.2, we get the claimed identification. ∎

Recall the definition of a spectral sequence being saturated degenerate in Definition 2.3. The following is an immediate consequence of Lemma 2.16 applying to all the Hodge filtrations of the de Rham complex, but let us repeat the proof one more time.

**Proposition 3.4.** *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be as in Section 3.1. Assume that the integral Hodge–de Rham spectral sequence of $\mathcal{X}$ is saturated degenerate. Then we have equality of* (*virtual*) *Hodge numbers:*

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) = h^{i,j}(X).$$

*Proof.* According to the definitions, we need to show that the dimension of the $i$-th filtration on $H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathrm{W}(\kappa))_{\mathrm{tf}}/p \cong H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi$ agrees with the dimension of the $i$-th filtration on $H^n_{\mathrm{dR}}(X)$.

By Proposition 3.3, we may rewrite formula (⊡) for the $i$-th filtration on $H^n_{\mathrm{crys}}(\mathcal{X}_0/\mathcal{O}_K)_{\mathrm{tf}}/\pi \cong H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})_{\mathrm{tf}}/\pi$ as

$$\mathrm{Im}\big(H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})/\pi \to H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})_{\mathrm{tf}}/\pi\big).$$

Our assumption implies that the submodule $H^n(\Omega^{\geq i}_{\mathcal{X}/\mathcal{O}_K}) \subset H^n(\Omega^\bullet_{\mathcal{X}/\mathcal{O}_K})$ meets the condition of Proposition 2.14. This gives the claimed equality of the dimensions of filtrations. ∎

### *3.3. Main Theorem*

In this subsection, we explain the proof of the Main Theorem 1.1, which we repeat below:

**Theorem 3.5.** *Let $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_K)$ be a smooth proper formal scheme. Assume that*

(1) *there is a lift of $\mathcal{X}$ over $\mathfrak{S}/(E^2)$; and*

(2) *the relative dimension of $\mathcal{X}$ and the ramification index satisfy* $\dim(\mathcal{X}_0) \cdot e < p - 1$.

*Then the Hodge–de Rham spectral sequence for $\mathcal{X}$ is split degenerate. In particular, we have equality of* (*virtual*) *Hodge numbers:*

$$\mathfrak{h}^{i,j}(\mathcal{X}_0) = h^{i,j}(X).$$

**Remark 3.6.** (1) Modulo $(u)$, the surjection $\mathfrak{S}/(E^2) \twoheadrightarrow \mathcal{O}_K$ becomes $\mathrm{W}_2(\kappa) \twoheadrightarrow \kappa$ (note that $E$ is an Eisenstein polynomial in $u$). So we may view this theorem as a mixed-characteristic analogue of a theorem by Deligne–Illusie [8, Corollaire 2.4].

(2) Our result implies that a smooth proper variety $\mathcal{X}_0$ in positive characteristic knows Hodge numbers of the generic fiber of a (formal) lifting provided (i) the lifting can be further lifted to $\mathfrak{S}/(E^2)$, and (ii) $\dim(\mathcal{X}_0) \cdot e < p-1$.

(3) Condition (1) of Theorem 3.5 is not so easy to verify. There are two cases we can think of in which this condition is automatically guaranteed, the first being that $\mathcal{X}$ is an unramified lift (for then the surjection $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ admits a section), in which case a stronger statement follows from the work of Fontaine–Messing [10] or Kato [12] (see Remark 3.12). The second case is when $\mathcal{X}_0$ has unobstructed deformation theory, e.g. when $H^2(\mathcal{X}_0, \mathrm{T}) = 0$, by deformation-theoretic considerations.

(4) Similar to the situation of Deligne–Illusie's statement, there are examples showing the necessity of condition (1). In fact one example comes from (ramified) liftings of (counter-)examples to Deligne–Illusie's statement in characteristic $p$ due to Antieau–Bhatt–Mathew [2]; see Section 4 and more precisely Theorem 4.14.

The main ingredients that go into the proof are some recent developments in integral $p$-adic Hodge theory. So we first introduce these ingredients.

Recently Bhatt–Scholze [6] developed prismatic cohomology theory to unite many (if not all) known $p$-adic cohomology theories one may attach to a $p$-adic smooth formal scheme over a certain class of $p$-adic base rings. While their theory is in a much broader context, we shall specialize their results to the situation of interest for this paper: they introduced prismatic site $(\mathcal{X}/\mathfrak{S})_{\Delta}$ on which there are two structure sheaves $\mathcal{O}_{\Delta}$ and $\overline{\mathcal{O}}_{\Delta} := \mathcal{O}_{\Delta}/E$. The cohomology of $\mathcal{O}_{\Delta}$ on $(\mathcal{X}/\mathfrak{S})_{\Delta}$ is denoted by $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$. There is a natural map of ringed topoi (see [6, Construction 4.4])

$$\nu\colon \mathrm{Shv}((\mathcal{X}/\mathfrak{S})_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \to \mathrm{Shv}(\mathcal{X}_{\text{ét}}, \mathcal{O}_{\mathcal{X}}).$$

The properties we need of these objects are summarized in the following:

**Theorem 3.7** (Bhatt–Scholze).

(1) ([6, Corollary 15.4]) *There is a canonical isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \cong \mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \phi_* \mathcal{O}_K,$$

*where $\phi_*\mathcal{O}_K$ is $\mathcal{O}_K$ viewed as an $\mathfrak{S}$-module (or even algebra) via the composite $\mathfrak{S} \xrightarrow{\phi} \mathfrak{S} \to \mathcal{O}_K = \mathfrak{S}/(E)$.*

(2) ([6, Theorem 4.10]) *There is a canonical isomorphism*

$$\Omega^i_{\mathcal{X}/\mathcal{O}_K}\{-i\} \cong R^i \nu_* \overline{\mathcal{O}}_{\Delta}.$$

*Here $(-)\{j\} := (-) \otimes_{\mathcal{O}_K} ((E)/(E^2))^{\otimes j}$. This induces an increasing filtration (called the conjugate filtration) on $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K$ giving rise to an $E_2$ spectral sequence*

(*called the Hodge–Tate spectral sequence*)

$$E_2^{i,j} = H^i(\mathcal{X}, \Omega^j_{\mathcal{X}/\mathcal{O}_K})\{-j\} \Longrightarrow H^{i+j}_{\mathrm{HT}}(\mathcal{X}/\mathcal{O}_K) := H^{i+j}(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K).$$

(⚁)

(3) ([6, Remark 4.13 and Proposition 4.14], [1, Proposition 3.2.1]) *The map*

$$\mathcal{O}_{\mathcal{X}} \to \tau^{\leq 1} R\nu_* \overline{\mathcal{O}}_{\Delta}$$

*splits if and only if* $\mathcal{X}$ *lifts to* $\mathfrak{S}/(E^2)$.

Lastly, we need a result of Min [15] concerning the $\mathfrak{S}$-module structure of $H^i_{\Delta}(\mathcal{X}/\mathfrak{S})$ in the case of small $i$ and low ramification index:

**Theorem 3.8** (Min [15, Corollary 5.4 and Theorem 5.11]). *When* $i \cdot e < p - 1$, *we have an abstract isomorphism of* $\mathfrak{S}$*-modules*

$$H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \cong \mathfrak{S}^{n_i} \oplus \bigoplus_{j \in J} \mathfrak{S}/p^{n_j},$$

*where* $J$ *is a finite set. Moreover,* $H^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$ *is* $u$*-torsion-free.*

Using the result of Min and our analysis of spectral sequences in the previous section, we may relate the behavior of the Hodge–de Rham and the Hodge–Tate spectral sequences. Let $T$ be the largest integer with $T \cdot e < p - 1$, which is the threshold given by Min's theorem above.

**Corollary 3.9.** *Let* $i \leq T$ *be an integer. We have two equivalences:*

(1) *The Hodge–de Rham spectral sequence having saturated degenerate torsion in degree* $i$ *is equivalent to the Hodge–Tate spectral sequence having saturated degenerate torsion in degree* $i$.

(2) *The Hodge–de Rham spectral sequence having split degenerate torsion in degree* $i$ *is equivalent to the Hodge–Tate spectral sequence having split degenerate torsion in degree* $i$.

*Proof.* Since $\mathcal{O}_K$ and $\phi_* \mathcal{O}_K$ are of flat dimension 1 as $\mathfrak{S}$-modules, we have short exact sequences

$$0 \to H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathcal{O}_K \to H^i(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K) \to \mathrm{Tor}_1^{\mathfrak{S}}(H^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S}), \mathcal{O}_K) \to 0,$$

$$0 \to H^i_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} \phi_* \mathcal{O}_K \to H^i(\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \phi_* \mathcal{O}_K) \to \mathrm{Tor}_1^{\mathfrak{S}}(H^{i+1}_{\Delta}(\mathcal{X}/\mathfrak{S}), \phi_* \mathcal{O}_K) \to 0.$$

Notice that $\mathfrak{S}$ and $\mathfrak{S}/p^{\ell}$ are, as $\mathfrak{S}$-modules, Tor-independent with $\mathcal{O}_K$ and $\phi_* \mathcal{O}_K$. Moreover, the $\mathrm{Tor}_1$ term is given by $E(u)$-torsions of $H^{T+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$ and $\phi^* H^{T+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$. Recall the relation [6, Theorem 17.2] between Breuil–Kisin prismatic cohomology and $A_{\mathrm{inf}}$-cohomology by Bhatt–Morrow–Scholze [5]. We also need to invole the fact that the $A_{\mathrm{inf}}$-cohomology groups after inverting $p$ are always finite free over $A_{\mathrm{inf}}[1/p]$

[5, Definition 1.5 and Theorem 1.8]. The result of [15, Lemma 5.3] and the $u$-torsion-freeness of $H^{T+1}_{\Delta}(\mathcal{X}/\mathfrak{S})$ imply the $\mathrm{Tor}_1$ terms above when $i = T$ also vanish.

Therefore in the situation considered, by Theorems 3.7 (1, 2) and 3.8, we have an abstract isomorphism of $\mathcal{O}_K$-modules

$$H^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \simeq H^i_{\mathrm{HT}}(\mathcal{X}/\mathcal{O}_K) \quad \text{for all } i \leq T.$$

Moreover, we know that both the Hodge–de Rham and Hodge–Tate spectral sequences degenerate after inverting $\pi$ (see [17, Corollary 1.8] and [5, Theorem 1.7]). Hence we reduce these two statements respectively to Definition 2.12 (1, 2), by observing that the starting pages of these two spectral sequences are formed by abstractly isomorphic $\mathcal{O}_K$, modules (after switching bi-degrees). ∎

Finally, we are ready to prove Theorem 1.1.

*Proof of Theorem* 1.1. By anti-symmetrizing (cf. [8, step (a) of Theorem 2.1]) the section

$$\Omega^1_{\mathcal{O}_\mathcal{X}}\{-1\}[-1] \to \tau^{\leq 1} R\nu_* \overline{\mathcal{O}}_{\Delta}$$

given by the lift of $\mathcal{X}$ to $\mathfrak{S}/(E^2)$ (see Theorem 3.7 (3)), we see that the conjugate filtration splits (note that our constraint on $\dim X$ in particular implies the relative dimension is smaller than $p$):

$$\bigoplus_{i=0}^{\dim X} \Omega^i_{\mathcal{O}_\mathcal{X}}\{-i\}[-i] \simeq R\nu_* \overline{\mathcal{O}}_{\Delta}.$$

Therefore we see that the Hodge–Tate spectral sequence (⚁) has split degenerate torsion in all degrees, and thus in particular in the range $(-\infty, T]$. By Corollary 3.9 the Hodge–de Rham spectral sequence must also have split degenerate torsion in the same range of degrees.

Now we invoke the duality statements for de Rham and Hodge cohomologies. By Poincaré duality for de Rham cohomology [3, Chapter VII, Théorème 2.1.3] we have an identification

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \cong \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K), \mathcal{O}_K[-2d]),$$

where $d = \dim \mathcal{X}_0$ and we have implicitly identified de Rham and crystalline cohomologies. Similarly, by Serre duality for Hodge cohomology [18, Tag 0BRT, Tag 0AU3, and Tag 0E9Z] we have an identification

$$\mathrm{R}\Gamma(\Omega^{d-i}_{\mathcal{X}/\mathcal{O}_K}) \cong \mathrm{RHom}_R(\mathrm{R}\Gamma(\Omega^i_{\mathcal{X}/\mathcal{O}_K}), \mathcal{O}_K[-d]).$$

Using Lemma 2.17 and what we obtained in the previous paragraph, we can now conclude that the Hodge–de Rham spectral sequence has split degenerate torsion in degrees in $[2d + 1 - T, +\infty)$.

The union $(-\infty, T] \cup [2d + 1 - T, +\infty)$ covers all integers exactly when $d \leq T$. Therefore condition (2) now implies that the Hodge–de Rham spectral sequence is split degenerate by Proposition 2.5 (2). The last statement concerning numerical equalities follows from Proposition 3.4. ∎

**Remark 3.10.** An ongoing project of Bhatt–Lurie establishes a duality statement for prismatic cohomology. Assuming their result, together with Min's result, one gets a more uniform proof of the Main Theorem. Indeed, these results together imply that under the assumption (2) all the prismatic cohomology of $(\mathcal{X}/\mathfrak{S})_{\Delta}$ is of the shape $\mathfrak{S}^n \oplus \bigoplus_{j\in J} \mathfrak{S}/p^{n_j}$. Notice, however, that their duality statement does not improve the bound on dimension, because their duality statement is over the 2-*dimensional* ring $\mathfrak{S}$.

In the situation where $\dim X$ exceeds the bound, our argument produces the following. Recall that $T$ denotes the largest integer satisfying $T \cdot e < p - 1$.

**Porism 3.11.** *Let $\mathcal{X}$ be a smooth proper formal scheme over $\mathcal{O}_K$ which lifts to $\mathfrak{S}/(E^2)$. Then the differentials in the Hodge–de Rham spectral sequence with target of total degree $\leq T$ are zero, and the induced Hodge filtrations on de Rham cohomology of degree $\leq T$ are split. Hence $\mathfrak{h}^{i,j}(\mathcal{X}_0) = H^{i,j}(X)$ for $i + j \leq T - 1$ (or equivalently $(i + j + 1) \cdot e < p - 1$).*

*Proof.* By the first paragraph of the proof of Theorem 1.1, the Hodge–de Rham spectral sequence has split degenerate torsion up to degree $T$. The statement about Hodge filtrations being split follows from Proposition 2.13. The statement about equality of numbers follows from Lemma 2.16; notice that in applying Lemma 2.16, we need the map of next cohomological degree to be injective, which lowers the range of application by 1. ∎

In this proof, we are not using the duality statement. In particular, following this proof, if $2\dim(\mathcal{X}_0) \cdot e < p - 1$, then we do not need to invoke the duality statement in the proof of Theorem 1.1.

When $e = 1$, namely $\mathcal{X}_0$ has an unramified lifting, the result of Fontaine–Messing and Kato gives something slightly more.

**Remark 3.12.** It is a result of Fontaine–Messing [10, Corollary 2.7 (iii)] and Kato [12, Chapter II, Proposition 2.5 (1)] that given an unramified lift $\mathcal{X}$, the Hodge–de Rham spectral sequence degenerates up to degree $p - 1$, that is all the differentials with *target* of total degree $\leq p - 1$ are zero. Moreover, they showed (see [10, Corollary 2.7 (ii) and Remark 2.8 (ii)] and [12, Chapter II, Proposition 2.5 (2)]) that the integral Hodge filtrations on $H^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$, where $0 \leq i \leq p - 1$, are equipped with divided Frobenius structure and altogether these form so-called Fontaine–Laffaille modules [9]. In particular, by a result of Wintenberger [20], the Hodge filtrations are split submodules in the range $0 \leq i \leq p - 1$. Hence these results imply the following:

**Corollary 3.13** (Corollary of [10, Corollary 2.7] or [12, Chapter II, Proposition 2.5]). *Let $\mathcal{X}$ be an unramified lift of a smooth proper variety $\mathcal{X}_0$. Then the degree $\leq p - 2$ Hodge numbers of $X$ are determined by $\mathcal{X}_0$.*

Comparing those results concerning unramified liftings with our result, we see that our approach (specialized to unramified liftings) so far can only prove analogous facts with $\mathcal{X}_0$ of one dimension less. This is due to the fact that Min's result was established by proving the $i$-th prismatic cohomology of $\mathcal{X}/\mathfrak{S}$ shares the same structure as the $i$-th étale cohomology of the (geometric) generic fiber $X$, which (in general) only holds when $i \cdot e < p - 1$. Perhaps this obstacle may be overcome in the unramified case, which would recover Fontaine–Messing's or Kato's result in this particular direction.

### *3.4. Summary*

Given $\mathcal{X}/\mathcal{O}_K$ as in Section 3.1, let $T$ be the largest integer satisfying $T \cdot e < p - 1$. We list several conditions on $\mathcal{X}/\mathcal{O}_K$:

**C.1.** The formal scheme $\mathcal{X}$ lifts to $\mathfrak{S}/(E^2)$.

**C.2.** The Hodge–Tate spectral sequence has split degenerate torsion up to degree $T$.

**C.3.** The Hodge–Tate spectral sequence has saturated degenerate torsion up to degree $T$.

**C.4.** The Hodge–de Rham spectral sequence has split degenerate torsion up to degree $T$.

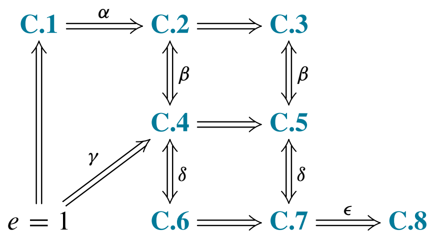
**C.5.** The Hodge–de Rham spectral sequence has saturated degenerate torsion up to degree $T$.

**C.6.** The Hodge–de Rham spectral sequence degenerates splittingly.

**C.7.** The Hodge–de Rham spectral sequence degenerates saturatedly.

**C.8.** The virtual Hodge numbers of $\mathcal{X}_0$ equal the Hodge numbers of $X$.

The relations between these conditions are summarized in the following diagram:




Below we remind the readers under what condition (and why) we have some of these implications:

- $\alpha$ holds when $\dim X \leq p - 1$ and follows from [6, Remark 4.13 and Proposition 4.14].
- $\beta$ follows from Min's work [15] together with the analysis of relevant spectral sequences (see Corollary 3.9).
- $\gamma$ holds provided $\dim X \leq p - 2$ and follows from the work of either Fontaine–Messing [10, Corollary 2.7] or Kato [12, Chapter II, Proposition 2.5].
- $\delta$ holds when $\dim \mathcal{X}_0 \leq T$ by duality of de Rham and Hodge cohomologies.
- $\epsilon$ always holds and is the content of Proposition 3.4.

## 4. Lifting the example of Antieau–Bhatt–Mathew

One might wonder if it is really necessary to have both of conditions (1) and (2) in Theorem 1.1, or even any condition at all, in order for the Hodge–de Rham spectral sequence to behave nicely. We mention that in [14] we found pairs of relatively 3-dimensional smooth projective schemes over $\mathbb{Z}_p[\zeta_p]$ such that their special fibers are isomorphic but the degree 2 Hodge numbers of their generic fibers are different. Therefore these give rise to examples where the Hodge–de Rham spectral sequence is not saturated degenerate by Proposition 3.4. The cohomological degree times the ramification index of these examples are twice of $p-1$, so these examples do not satisfy condition (2) in Theorem 1.1. Moreover, the author suspects that these examples do not satisfy condition (1) either.

While it is unclear whether condition (2) in Theorem 1.1 is really necessary, in this last section we would like to illustrate, by an example, the necessity of condition (1). More precisely, we shall construct smooth proper schemes over degree 2 ramified extensions of $\mathbb{Z}_p$ such that the Hodge–de Rham spectral sequencies are not degenerate (starting at degree 3), and the Hodge filtrations are non-saturated (starting at degree 2). The idea is to approximate the classifying stack of a lift of $\alpha_p$ (which only exists over a ramified ring of integers), and the key computations and techniques are already in [2].

We remark that in a concise paper by W. Lang [13], examples in positive characteristic which admit liftings to ramified DVRs but violate Hodge–de Rham degeneration have been found. Lang used exactly the idea of approximating $B\alpha_p$, we learned from loc. cit. that this idea goes back to Raynaud [16]. In the sense that our Theorem 1.1 may be thought of as a generalization of Deligne–Illusie's result, our example here can also be thought of as a generalization of Lang's.

### *4.1. Recollection of* [19]

In this subsection, we give a preliminary discussion of group schemes of order $p$ over $p$-adic base rings. Fix a scheme $S$ over $\mathbb{Z}_p$. Recall that in [19], the authors made a detailed study of finite flat group schemes of order $p$ over $S$; let us summarize their results.

Firstly, all such group schemes are commutative [19, Theorem 1]. Secondly, for each $p$ there is a unit $\omega \in \mathbb{Z}_p^*$ (denoted as $\omega_{p-1}$ in *loc. cit.*); see [19, Remark on p. 11] for a recursive formula defining it, and a bijection between

(1) isomorphism classes of finite flat order $p$ group schemes over $S$, and
(2) isomorphism classes of triples $(\mathcal{L}, a, b)$ where $\mathcal{L}$ is a line bundle on $S$, elements $a$ and $b$ are sections of line bundles: $a \in \Gamma(S, \mathcal{L}^{\otimes(p-1)})$ and $b \in \Gamma(S, \mathcal{L}^{\otimes(1-p)})$, and they satisfy the relation $a \otimes b = p\omega$.

Here we have identified $\mathcal{L}^{\otimes(p-1)} \otimes \mathcal{L}^{\otimes(1-p)} \cong \mathcal{O}_S$ [19, Theorem 2]. The group associated with $(\mathcal{L}, a, b)$ is denoted by $\mathrm{G}_{a,b}^{\mathcal{L}}$, with underlying scheme structure given by $\underline{\mathrm{Spec}}(\mathcal{O}_S \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-p+1})$ where the ring structure comes from $\mathcal{L}^{-p} \xrightarrow{a} \mathcal{L}^{-1}$ [19, p. 12]. So $\mathrm{G}_{a,b}^{\mathcal{L}}$ is an étale group scheme if and only if $a \in \Gamma(S, \mathcal{L}^{\otimes(p-1)})$ is an invertible section [19, p. 16, Remark 6]. Moreover, the Cartier dual of $\mathrm{G}_{a,b}^{\mathcal{L}}$ is[1] given by $\mathrm{G}_{b,a}^{\mathcal{L}^{-1}}$ [19, p. 15, Remark 2].

[1]Note that they are commutative group schemes by the first sentence of this paragraph.

**Example 4.1.** When $S = \mathrm{Spec}(\mathbb{F}_p)$, there is only one line bundle on $S$, namely $\mathcal{O}_S$. Furthermore, $p = 0$ on $S$. Hence we see that group schemes of order $p$ over $S$ are classified by pairs $(a, b) \in \mathbb{F}_p^2$ with the constraint that $ab = 0$. Note that these pairs have no non-trivial automorphism as any invertible element $u \in \mathbb{F}_p^*$ satisfies $u^{p-1} = 1$. There are three possibilities:

(1) $a \neq 0$, which forces $b = 0$, corresponding to a form of the étale group scheme $\mathbb{Z}/p$. When $a = 1$, it is $\mathbb{Z}/p$.

(2) Dually, we can have $b \neq 0$ and $a = 0$, corresponding to a form of $\mu_p$. It is $\mu_p$ when $b = 1$.

(3) Lastly, if $a = b = 0$, we get $\alpha_p$.

### 4.2. *A stacky example*

Now we specialize to the case where $S = \mathrm{Spec}(\mathcal{O}_K)$ is given by the valuation ring of a $p$-adic field. There is no non-trivial line bundle on a local scheme such as $S$. In order to lift $\alpha_p$ from the residue field of $\mathcal{O}_K$, it suffices to find an element $\pi \in \mathfrak{m}$ such that $p/\pi \in \mathfrak{m}$. Here $\mathfrak{m}$ denotes the maximal ideal in $\mathcal{O}_K$. We see that $\mathcal{O}_K$ cannot be absolutely unramified, and as long as it is ramified, we may find such an element $\pi$. From now on, let us fix such a choice of $\mathcal{O}_K$ and $\pi$.

**Notation 4.2.** Let $K$ be a degree 2 ramified extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, a uniformizer $\pi$ in the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$. Then $\pi' := p\omega/\pi$ is a uniformizer as well. Denote $S := \mathrm{Spec}(\mathcal{O}_K)$ and let $G := G_{\pi,\pi'}^{\mathcal{O}_S}$ be the lift of $\alpha_p$ over $S$ corresponding to $(\pi, \pi')$.

In the following we shall study the Hodge–Tate and Hodge–de Rham spectral sequence of $BG$. Note that $BG$ is a smooth proper stack over $\mathrm{Spec}(\mathcal{O}_K)$ with special fiber $BG \times_{\mathcal{O}_K} \mathbb{F}_p \cong B\alpha_p$. The following computation of Antieau–Bhatt–Mathew is very useful.

**Proposition 4.3** (see [2, Proposition 4.10]). *If $p > 2$, the Hodge cohomology group of $B\alpha_p$ is given by*

$$H^*(B\alpha_p, \wedge^* L_{B\alpha_p/\mathbb{F}_p}) \cong E(\alpha) \otimes P(\beta) \otimes E(s) \otimes P(u)$$

*where $E(-)$ (resp. $P(-)$) denotes the exterior (resp. polynomial) algebra on the designated generator. Here $\alpha \in H^1(B\alpha_p, \mathcal{O})$, $\beta \in H^2(B\alpha_p, \mathcal{O})$, $s \in H^0(B\alpha_p, L_{B\alpha_p/\mathbb{F}_p})$ and $u \in H^1(B\alpha_p, L_{B\alpha_p/\mathbb{F}_p})$. For $p = 2$ we replace $E(\alpha) \otimes P(\beta)$ with $P(\alpha)$.*

**Lemma 4.4.** *The cotangent complex of $BG$ is $L_{BG/\mathcal{O}_K} \simeq \mathcal{O}/(\pi)[-1]$.*

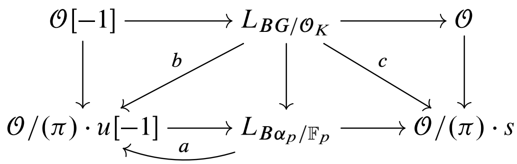
*Proof.* Observe that the equation of the underlying scheme of $G$ is given by $x^p - \pi x$, hence we know that $L_{G/\mathcal{O}_K} \simeq \mathcal{O}_G/(\pi)$. Therefore the underlying coLie complex of $G$ is also $\mathcal{O}_G/(\pi)$. As $G$ is commutative, our statement follows from [11, Proposition 4.4].[2] ■

[2] Note that the cotangent complex of $BG$ is the coLie complex shifted by $-1$.

**Remark 4.5.** In the proof of [2, Proposition 4.10], the authors showed that the Postnikov tower $\mathcal{O}_{B\alpha_p} \to L_{B\alpha_p/\mathbb{F}_p} \xrightarrow{a} \mathcal{O}_{B\alpha_p}[-1]$ of the cotangent complex of $B\alpha_p$ splits: $L_{B\alpha_p/\mathbb{F}_p} \simeq \mathcal{O}_{B\alpha_p} \oplus \mathcal{O}_{B\alpha_p}[-1]$. In our situation, we get a triangle in $D(BG)$:

$$\mathcal{O}[-1] \to L_{BG/\mathcal{O}_K} \to \mathcal{O},$$

where the connecting morphism is multiplication by $\pi$. Specializing to the special fiber $B\alpha_p$, we get a diagram



(⚂)

where $b$ is the identification $L_{BG/\mathcal{O}_K} \simeq \mathcal{O}/(\pi)[-1]$. This gives a particular choice of the splitting of $L_{B\alpha_p/\mathbb{F}_p} \simeq \mathcal{O}_{B\alpha_p} s \oplus \mathcal{O}_{B\alpha_p} u[-1]$, where the classes $s$ and $u$ are as in the statement of the aforementioned Proposition 4.3. Here let us name the map $\mathrm{sp}\colon L_{BG/\mathcal{O}_K} \xrightarrow{b\oplus c} \mathcal{O}_{B\alpha_p} u[-1] \oplus \mathcal{O}_{B\alpha_p} s \simeq L_{B\alpha_p/\mathbb{F}_p}$, for we will use it later.

Next let us compute the Hodge cohomology groups of $BG$ and identify the algebra structure.

**Proposition 4.6.** *For any pair $(i, j)$ of integers, we have*

$$H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) = \begin{cases} \mathcal{O}_K, & i = j = 0, \\ \mathbb{F}_p, & j = 0,\ i = 2m > 0 \text{ or } 0 < j \le i, \\ 0, & \text{otherwise.} \end{cases}$$

*Therefore specialization maps give rise to injections*

$$\mathrm{sp}\colon H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \hookrightarrow H^i(B\alpha_p, \wedge^j L_{B\alpha_p/\mathbb{F}_p})$$

*whenever $i + j > 0$. Moreover, these injections are compatible with multiplication and differentials, and give an identification*

$$H^*(BG, \wedge^* L_{BG/\mathcal{O}_K}) = \begin{cases} (\mathcal{O}_K[\beta, u] \otimes E(\tau))/(\pi\tau, \pi\beta, \pi u), & p > 2, \\ (\mathcal{O}_K[\beta, u, \tau])/(\pi\tau, \pi\beta, \pi u, \tau^2 - \beta u^2), & p = 2, \end{cases}$$

*where $\beta \in H^2(BG, \mathcal{O})$ and $u \in H^1(BG, L_{BG/\mathcal{O}_K})$ both specialize to the designated elements in the Hodge ring of $B\alpha_p$, and $\tau \in H^2(BG, L_{BG/\mathcal{O}_K})$ specializes to $\alpha u + \beta s$ (up to scaling $s$ by a unit).*

*Proof.* First we begin with the computation of cohomology of $\mathcal{O}$. Similar to the first paragraph of [2, proof of Proposition 4.10], we have $H^*(BG, \mathcal{O}) = \mathrm{Ext}^*_{\mathcal{O}_K[y]/(y^p - \pi' y)}(\mathcal{O}_K, \mathcal{O}_K)$ by Cartier duality. Here we used the fact that the Cartier dual of $G_{\pi,\pi'}$ is $G_{\pi',\pi}$ whose underlying scheme structure is $\mathrm{Spec}(\mathcal{O}_K[y]/(y^p - \pi' y))$ with its identity section given by $y = 0$. Using the standard resolution

$$\Big(\cdots \to \mathcal{O}_K[y]/(y^p - \pi' y) \xrightarrow{y^{p-1}-\pi'} \mathcal{O}_K[y]/(y^p - \pi' y) \xrightarrow{y} \mathcal{O}_K[y]/(y^p - \pi' y)\Big) \simeq \mathcal{O}_K$$

one verifies the computation when $j = 0$.

When $j > 0$, just observe that

$$\wedge^* L_{BG/\mathcal{O}_K} = \wedge^* (\mathcal{O}/(\pi)[-1]) = \mathrm{Sym}^*(\mathcal{O}/(\pi))[-*].$$

Therefore we get

$$H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) = H^i(BG, \mathrm{Sym}^j \mathcal{O}/(\pi)[-j]) = H^{i-j}(B\alpha_p, \mathcal{O}),$$

which verifies the computation when $j > 0$ via Proposition 4.3.

The second statement follows from the fact that $H^i(BG, \wedge^j L_{BG/\mathcal{O}_K})$ are all $\pi$-torsion when $i + j > 0$ by the first sentence. In particular, by dimension considerations we see that the induced map $H^2(BG, \mathcal{O}) \to H^2(B\alpha_p, \mathcal{O})$ must be an isomorphism. Hence we may pick a generator $\beta \in H^2(BG, \mathcal{O})$ which lifts the designated generator in $H^2(B\alpha_p, \mathcal{O})$.

Next we deal with the statement concerning images of other specialization maps. Since the map $b$ in (⚄) is an identification, we see that the $b$ component of

$$H^*(BG, L_{BG/\mathcal{O}_K}) \xrightarrow{\mathrm{sp}=b\oplus c} H^*(B\alpha_p, L_{B\alpha_p/\mathbb{F}_p}) = H^{*-1}(B\alpha_p, \mathcal{O}) \cdot u \oplus H^*(B\alpha_p, \mathcal{O}) \cdot s$$

is always an isomorphism. In particular we can choose generators of $H^1(BG, L_{BG/\mathcal{O}_K})$ and $H^2(BG, L_{BG/\mathcal{O}_K})$ corresponding to $u$ and $\alpha \cdot u$ under $b$. Let us denote each generator by $u$ and $\tau$ respectively.

The map $c$ factors as the composition $L_{BG/\mathcal{O}_K} \to \mathcal{O} \to \mathcal{O}/(\pi) \cdot s$. The first map fits in the triangle

$$\mathcal{O}[-1] \to L_{BG/\mathcal{O}_K} \cong \mathcal{O}/\pi[-1] \to \mathcal{O}$$

with connecting morphism being multiplication by $\pi$. Since $H^1(BG, \mathcal{O})$ is shown to be zero, we see that the map $c$ on $H^1(BG, L_{BG/\mathcal{O}_K})$ factors through zero, hence $c(u) = 0$. Still by the computation of $H^*(BG, \mathcal{O})$, the long exact sequence of cohomology associated with the above triangle gives an isomorphism $H^2(BG, L_{BG/\mathcal{O}_K}) \to H^2(BG, \mathcal{O})$. The map $\mathcal{O} \to \mathcal{O}/(\pi) \cdot s$ is just reduction modulo $\pi$, hence induces an injection $H^2(BG, \mathcal{O})/\pi \to H^2(B\alpha_p, \mathcal{O}/(\pi) \cdot s)$. Now the cohomology group $H^2(BG, \mathcal{O})$ is shown to be $\mathcal{O}_K/\pi$ and the dimension of $H^2(B\alpha_p, \mathcal{O}/(\pi) \cdot s)$ is 1, so we see that the map $H^2(BG, \mathcal{O})/\pi \to H^2(B\alpha_p, \mathcal{O}/(\pi) \cdot s)$ is an isomorphism. Therefore $c(\tau) = \beta \cdot s$ (up to a unit). Putting these together, we have

$$\mathrm{sp}(u) = b(u) + c(u) = u \quad \text{and} \quad \mathrm{sp}(\tau) = b(\tau) + c(\tau) = \alpha \cdot u + \beta \cdot s.$$

For the last sentence, let us just prove the case when $p > 2$; the case of $p = 2$ can be proved in the same way. First we observe that we have

$$(\mathcal{O}_K[\beta, u] \otimes E(\tau)) \xrightarrow{f} H^*(BG, \wedge^* L_{BG/\mathcal{O}_K}) \xrightarrow{\mathrm{sp}} H^*(B\alpha_p, \wedge^* L_{B\alpha_p/\mathbb{F}_p}),$$

with $\beta$, $u$ and $\tau$ as in the statement. The map $f$ must kill the relations $\pi\tau, \pi\beta, \pi u$, as the positive degree Hodge groups of $BG$ are $\pi$-torsion. After quotienting out the relations, we get an injection

$$(\mathcal{O}_K[\beta, u] \otimes E(\tau))/(\pi\tau, \pi\beta, \pi u) \xrightarrow{\mathrm{sp}\circ f} H^*(B\alpha_p, \wedge^* L_{B\alpha_p/\mathbb{F}_p})$$

on the positive degree part because of Proposition 4.3. Hence the map $f$ induces an injection

$$(\mathcal{O}_K[\beta, u] \otimes E(\tau))/(\pi\tau, \pi\beta, \pi u) \xrightarrow{f} H^*(BG, \wedge^* L_{BG/\mathcal{O}_K}).$$

By explicitly comparing dimensions of each bi-degree parts, one concludes that $f$ must also be surjective, hence an isomorphism. ∎

Finally, we can understand the Hodge–de Rham spectral sequence of $BG$ with the aid of [2, Proposition 4.12].

**Proposition 4.7.** *In the Hodge–de Rham spectral sequence of $BG$, we have (up to unit) $d_1(\tau) = u^2$ and $d_1(\beta) = d_1(u) = 0$ for all $p$. The de Rham cohomology of $BG$ is given by*

$$H^*_{\mathrm{dR}}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K[\beta']/(p\beta'),$$

*where $\beta'$ has degree 2.*

*Proof.* The first sentence follows from [2, proof of Proposition 4.12] and the fact that specialization gives an injection

$$\mathrm{sp}\colon H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \hookrightarrow H^i(B\alpha_p, \wedge^j L_{B\alpha_p/\mathbb{F}_p})$$

which is compatible with multiplication and differentials. Indeed, in that proof, it is shown that in the special fiber we have $d_1(\alpha) = u$ (up to a unit) and all other generators are killed by differentials. Hence the specialization of our $d_1(\tau)$ must be $d_1(\alpha u + \beta s) = d_1(\alpha)u = u^2$ (up to a unit). Since specialization is injective in positive cohomological degrees (by Proposition 4.6), we conclude that $d_1(\tau) = u^2$.

Using the fact that $d_1$ is a differential, we see that on the $E_2$-page the non-zero entries are

$$E_2^{i,j} = \begin{cases} \mathcal{O}_K, & i = j = 0, \\ \mathbb{F}_p \cdot \beta^n, & i = 0,\ j = 2n > 0, \\ \mathbb{F}_p \cdot \beta^n u, & i = 1,\ j = 2n+1 > 0. \end{cases}$$

In particular, there is no room for non-zero differentials, hence the spectral sequence degenerates on the $E_2$-page. In particular, we see that the length of de Rham cohomology is as described in the statement of this proposition. To pin down the $\mathcal{O}_K$-module structure of $H^*_{\mathrm{dR}}(BG/\mathcal{O}_K)$, we use the fact that $H^i_{\mathrm{dR}}(BG/\mathcal{O}_K)/\pi$ injects into $H^i_{\mathrm{dR}}(B\alpha_p/\mathbb{F}_p)$, which is always one-dimensional for $i \geq 0$ due to [2, Proposition 4.10].

Lastly, pick a preimage of $\beta$ under $H^2_{\mathrm{dR}}(BG/\mathcal{O}_K) \twoheadrightarrow H^2(BG, \mathcal{O})$; denote it by $\beta' \in H^2_{\mathrm{dR}}(BG/\mathcal{O}_K)$. Since $H^*_{\mathrm{dR}}(BG/\mathcal{O}_K) \to H^*_{\mathrm{dR}}(B\alpha_p/\mathbb{F}_p)$ is a map preserving multiplication, we see that $\beta'^n$ is a generator of $H^{2n}_{\mathrm{dR}}(BG/\mathcal{O}_K)$. This finishes the proof of the ring structure on $H^*_{\mathrm{dR}}(BG/\mathcal{O}_K)$. ■

Similarly, we can understand the Hodge–Tate spectral sequence of $BG$ with the aid of [2, Remark 4.13].

**Proposition 4.8.** *In the Hodge–Tate spectral sequence of $BG$, we have (up to a unit) $d_2(\tau) = \beta^2$ and $d_2(\beta) = d_2(u) = 0$ for all $p$. The Hodge–Tate cohomology of $BG$ is given by*

$$H^*_{\mathrm{HT}}(BG/\mathcal{O}_K) \simeq \mathcal{O}_K[u']/(pu'),$$

*where $u'$ has degree 2.*

*Proof.* Recall that in characteristic $p$, the conjugate spectral sequence comes from the canonical filtration on the de Rham complex of affine opens. Similarly when working with a prism $(A, I)$, the Hodge–Tate spectral sequence comes from the canonical filtration on the sheaf $\overline{\Delta}_{-/(A/I)}$. When the prism is $(\mathbb{Z}_p, (p))$, one can ignore the Frobenius twist, and hence identify the sheaf $\overline{\Delta}_{-/\mathbb{F}_p}$ with the sheaf (relative to $\mathbb{F}_p$) of de Rham complex (by either of [6, Theorem 1.8 (1) or (3)]). Therefore in this case, the Hodge–Tate spectral sequence is identified with the conjugate spectral sequence.

Reduction modulo $u$ gives rise to a map $(\mathfrak{S}, (E)) \to (\mathbb{Z}_p, (p))$ of prisms. Since $u$ is mapped to $\pi$ under $\mathfrak{S} \to \mathcal{O}_K$, we see that the Hodge–Tate spectral sequence of $BG$ specializes, under reduction modulo $\pi$, to the conjugate spectral sequence of $B\alpha_p$.

Now the differentials in the conjugate spectral sequence of $B\alpha_p$ are understood in [2, Remark 4.13]. Using that remark, the proof of this proposition is almost the same as the proof of Proposition 4.7, except we now have $d_2(\alpha) = d_2(\beta) = d_2(u) = 0$ and $d_2(s) = \beta$ in the special fiber. The multiplicative structure is justified by the fact that de Rham and Hodge–Tate cohomologies are the same over $\mathbb{F}_p$. Hence the even degree part of the Hodge–Tate cohomology of $B\alpha_p$ is also a polynomial algebra with a degree 2 generator. ■

In particular, both the Hodge–de Rham and Hodge–Tate spectral sequences are non-degenerate with non-zero differentials starting at degree 3, and the Hodge (resp. conjugate) filtrations on de Rham (resp. Hodge–Tate) cohomology is not split starting at degree 2. When the prime is $p \geq 11$, these give rise to stacky examples satisfying condition (2) of the main theorem, which violates the conclusion. The obstruction to lifting $G$ to $\mathrm{Spec}(\mathfrak{S}/(E^2))$ specializes (modulo $u$) to the obstruction to lifting $\alpha_p$ to $\mathrm{Spec}(W_2)$, which is non-zero.

Let us take a closer look at degree 2. By Proposition 4.7 we have a short exact sequence

$$0 \to H^1(BG, L_{BG/\mathcal{O}_K}) = \mathbb{F}_p \cdot u \to H^2_{\mathrm{dR}}(BG/\mathcal{O}_K) = (\mathcal{O}_K/p) \cdot \beta' \\ \to H^2(BG, \mathcal{O}_{BG}) = \mathbb{F}_p \cdot \beta \to 0.$$

Therefore $\beta'$ lifts $\beta$, and $\pi \cdot \beta' = u$, up to units. The latter also "explains" why $u^2 = 0$ in de Rham cohomology (as $(\pi^2) = (p)$ by our assumption of $e = 2$). Similarly by

Proposition 4.8 we have a short exact sequence

$$0 \to H^2(BG, \mathcal{O}_{BG}) = \mathbb{F}_p \cdot \beta \to H^2_{\mathrm{HT}}(BG/\mathcal{O}_K) = (\mathcal{O}_K/p) \cdot \beta' \\ \to H^1(BG, L_{BG/\mathcal{O}_K}) = \mathbb{F}_p \cdot u \to 0.$$

Now up to units, $u'$ lifts $u$ and $\beta = \pi \cdot u'$. Again, $\beta^2 = 0$ can be seen from the fact that $p$ divides $\pi^2$ (actually they only differ by a unit in $\mathcal{O}_K$).

We can also determine the prismatic cohomology of $BG$ using Proposition 4.8. Before that, we need a few words about prismatic cohomology of a smooth proper stack.

**Remark 4.9.** Throughout this remark, let us focus on the Breuil–Kisin prism $(\mathfrak{S}, (E))$ associated with $\pi \in \mathcal{O}_K$.

(1) Bhatt–Scholze showed that the prismatic cohomology satisfies (quasi-)syntomic descent [6, Theorem 1.15 (2)]. Hence the presheaf, valued in the derived $\infty$-category $D(\mathfrak{S})$, on $\mathrm{Syn}^{\mathrm{op}}_{\mathcal{O}_K}$ (the syntomic site of $\mathcal{O}_K$) sending $R$ to $\mathrm{R\Gamma}_{\Delta}(\mathrm{Spf}(\widehat{R})/\mathfrak{S})$ is a sheaf. Here $\widehat{R}$ denotes the $p$-adic completion of $R$. Since our group scheme $G$ is syntomic over $\mathcal{O}_K$, we know that $BG$ over $\mathcal{O}_K$ is a syntomic stack (see [2, Notation 2.1]). In [2, Construction 2.7], one finds a definition of $\mathrm{R\Gamma}_{\Delta}(BG/\mathfrak{S})$. Concretely, given any syntomic cover $U \to BG$ with $U$ a syntomic $\mathcal{O}_K$-scheme, we have

$$\mathrm{R\Gamma}_{\Delta}(BG/\mathfrak{S}) \simeq \lim_{[m]\in\Delta} \mathrm{R\Gamma}_{\Delta}(\widehat{U}^m/\mathfrak{S}).$$

Here $\widehat{U}^m$ denotes the $p$-adic formal completion of the $(m+1)$-fold fiber product of $U \to BG$. The syntomic sheaf property exactly guarantees that this formula does not depend on the choice of the syntomic cover $U \to BG$.

(2) All the results stated previously concerning prismatic cohomology of smooth proper formal schemes (e.g. a natural Frobenius structure, Theorem 3.7 and Theorem 3.8) still hold verbatim for $\mathrm{R\Gamma}_{\Delta}(BG/\mathfrak{S})$. This is because most of the statements in [6] are shown by proving their analogues for affine formal schemes. Let us show that all the prismatic cohomology groups of $BG$ are finitely generated.

Since the sheaf $\Delta_{-/\mathfrak{S}}$ is derived $(p, E)$-complete, the resulting cohomology groups $H^*_{\Delta}(BG/\mathfrak{S})$ are also derived $(p, E)$-complete, as derived completeness is preserved under taking limit. Since the Hodge–Tate cohomology groups of $BG/\mathfrak{S}$ are finitely generated over $\mathcal{O}_K$ (by Proposition 4.8), the derived Nakayama Lemma implies that all the cohomology groups $H^*_{\Delta}(BG/\mathfrak{S})$ are also finitely generated over $\mathfrak{S}$.

(3) We claim that $H^n_{\Delta}(BG/\mathfrak{S})$ is a Breuil–Kisin module [5, Definition 4.1] for all $n$. This follows from the fact that $BG$ is a quasi-compact separated smooth stack over $\mathcal{O}_K$ and the same argument laid out in [6]. Below let us spell out the argument for the sake of being rigorous.

For any affine smooth formal scheme $\mathcal{X}$ over $\mathcal{O}_K$, the Frobenius on its prismatic cohomology has an isogeny property [6, Theorem 1.15 (4)]: the Frobenius induces a canonical isomorphism

$$\phi^*_{\mathfrak{S}} \mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S}) \xrightarrow[\phi_{\mathcal{X}}]{\cong} L\eta_E \mathrm{R\Gamma}_{\Delta}(\mathcal{X}/\mathfrak{S}).$$

We direct readers to [5, Section 6] for a discussion of the $L\eta$ functor. In particular, [5, Lemma 6.9] implies that, for any $n$, we have a functorial map

$$\psi^n_{\mathcal{X}}\colon \tau^{\leq n}\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \to \tau^{\leq n}\phi^*_{\mathfrak{S}}\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$$

such that its composition with the Frobenius in either direction is given by multiplying by $E^n$.

Since the group scheme $G$ is finite flat over $\mathcal{O}_K$, we know that $BG$ is quasi-compact, separated, and smooth over $\mathcal{O}_K$. For a justification of the smoothness, see [18, Tag 0DLS]. Therefore we may find a smooth cover $X \to BG$ with $X$ being an affine scheme and smooth over $\mathcal{O}_K$. Together with the separatedness of $BG \to \mathcal{O}_K$, all of the $\widehat{X}^m$'s are smooth affine formal schemes over $\mathcal{O}_K$.

Using the smooth cover $X \to BG$ in the last paragraph, we have

$$\mathrm{R}\Gamma_{\Delta}(BG/\mathfrak{S}) \simeq \lim_{[m]\in\Delta} \mathrm{R}\Gamma_{\Delta}(\widehat{X}^m/\mathfrak{S}).$$

The Frobenius is the totalization of the Frobenius on each of the $\mathrm{R}\Gamma_{\Delta}(\widehat{X}^m/\mathfrak{S})$. Fix a positive integer $n$, using the relation between canonical truncation and limit, we get

$$\tau^{\leq n}\mathrm{R}\Gamma_{\Delta}(BG/\mathfrak{S}) \simeq \tau^{\leq n} \lim_{[m]\in\Delta} \tau^{\leq n}\mathrm{R}\Gamma_{\Delta}(\widehat{X}^m/\mathfrak{S}).$$

Because $\phi_{\mathfrak{S}}$ is finite flat, we also have

$$\tau^{\leq n}\phi^*_{\mathfrak{S}}\mathrm{R}\Gamma_{\Delta}(BG/\mathfrak{S}) \simeq \tau^{\leq n} \lim_{[m]\in\Delta} \tau^{\leq n}\phi^*_{\mathfrak{S}}\mathrm{R}\Gamma_{\Delta}(\widehat{X}^m/\mathfrak{S}).$$

Lastly we totalize the maps $\psi^n_{\widehat{X}^m}$ to get a map

$$\psi^n_{BG}\colon \tau^{\leq n}\mathrm{R}\Gamma_{\Delta}(BG/\mathfrak{S}) \to \tau^{\leq n}\phi^*_{\mathfrak{S}}\mathrm{R}\Gamma_{\Delta}(BG/\mathfrak{S}).$$

This map composed with the Frobenius in either direction is given by multiplying by $E^n$, as it is so for all of the maps $\psi^n_{\widehat{X}^m}$. In particular, we see that $H^n_{\Delta}(BG/\mathfrak{S})$ is a Breuil–Kisin module (of height $n$).

Finally, we are ready to compute the Breuil–Kisin prismatic cohomology of $BG$.

**Proposition 4.10.** *The prismatic cohomology of $BG$ is given by*

$$H^*_{\Delta}(BG/\mathfrak{S}) \simeq \mathfrak{S}[\tilde{u}]/(p\tilde{u}),$$

*where $\tilde{u}$ has degree 2.*

Before giving the proof, we need an auxiliary lemma.

**Lemma 4.11.** *Let $M$ be a cyclic torsion Breuil–Kisin module over $\mathfrak{S}$ with no $E$-torsion. Then there is an integer $n$ such that $M \simeq \mathfrak{S}/(p^n)$.*

*Proof.* Say $M = \mathfrak{S}/I$. First we know that $M[1/p] = 0$ [5, Proposition 4.3]. Since $M$ has no $E$-torsion, we see that $M$ contains no non-zero finite $\mathfrak{S}$-submodule. Let $n$ be the smallest integer such that $p^n \in I$. It suffices to show that for any non-unit $f \in \mathfrak{S} - (p)$, the smallest integer $m$ such that $p^m f \in I$ is $n$. Suppose otherwise; then $m < n$. Then the image of

$$\mathfrak{S}/(f, p) \xrightarrow{\cdot p^{n-1}} M$$

is a non-zero (as $p^{n-1} \notin I$) finite (as the image of $f$ in $\mathfrak{S}/(p) = k[\![u]\!]$ is non-zero and non-unit) submodule, which we have argued is impossible. Therefore we must have $n = m$. ∎

*Proof of Proposition* 4.10. The Hodge–Tate specialization gives us short exact sequences

$$0 \to H^*_{\Delta}(BG/\mathfrak{S})/(E) \to H^*_{\mathrm{HT}}(BG/\mathcal{O}_K) \to H^{*+1}_{\Delta}(BG/\mathfrak{S})[E] \to 0,$$

where $M[E]$ denotes the $E$-torsion of an $\mathfrak{S}$-module $M$.

We make the following claim:

(1) the odd degree prismatic cohomology groups of $BG$ are zero; and

(2) the positive even degree prismatic cohomology groups of $BG$ are cyclic and $E$-torsion-free.

Indeed, by our Proposition 4.8, we see that $H^{\mathrm{odd}}_{\Delta}(BG/\mathfrak{S})/(E) = 0$. Since $H^i_{\Delta}(BG/\mathfrak{S})$ is $E$-complete, this gives (1) above. Using the above short exact sequence and vanishing of odd degree Hodge–Tate cohomology established in Proposition 4.8, we find that the positive even degree prismatic cohomology groups of $BG$ are $E$-torsion-free. Then we use Proposition 4.8 again, to see that for any $i > 0$ we get an isomorphism $H^{2i}_{\Delta}(BG/\mathfrak{S})/(E) \cong \mathfrak{S}/p \cdot \tilde{u}^i$. Therefore for each $i$ we can find a map $\mathfrak{S} \to H^{2i}_{\Delta}(BG/\mathfrak{S})$, which is surjective modulo $E$. Since $\mathfrak{S}$ itself is derived $(p, E)$-complete, the cokernel of this map is both $E$-adically complete and vanishes modulo $E$. These two together imply that the cokernel vanishes, in other words, the chosen map $\mathfrak{S} \to H^{2i}_{\Delta}(BG/\mathfrak{S})$ is surjective, which shows (2) above.

For any $m$ we know that $H^{2m}_{\Delta}(BG/\mathfrak{S})$ is cyclic and $E$-torsion-free, hence it is either free or isomorphic to $\mathfrak{S}/(p^n)$ for some $n$ (by the above lemma). To see that, we must be in the latter case with $n$ being 1, we use the fact that it is so under the Hodge–Tate specialization. Powers of any generator in $H^2_{\Delta}(BG/\mathfrak{S})$ are generators of the corresponding prismatic cohomology group, as it is so after the Hodge–Tate specialization (using again $E$-adic completeness of these prismatic cohomology groups). ∎

**Remark 4.12.** We do not know how the Frobenius acts on the prismatic cohomology groups. Since the geometric generic fiber of $BG$ is $B\mathbb{Z}/p$, we at least know that the Frobenius is not identically zero, by étale specialization of the prismatic cohomology [6, Theorem 1.8 (4)]. On the other hand, since the Frobenius is zero for $B\alpha_p$, we know that it cannot be surjective on $H^2_{\Delta}(BG/\mathfrak{S})$. Since $((\mathbb{F}_p[\![u]\!])^*)^{p-1} = (1 + (u), \times)$, we see that after choosing an appropriate generator, the Frobenius on $H^2_{\Delta}(BG/\mathfrak{S}) \simeq \mathfrak{S}/(p) \cong \mathbb{F}_p[\![u]\!]$ sends 1 to $\gamma \cdot u^d$, where $\gamma \in \mathbb{F}_p^*$ and $d$ is a positive integer. It would be interesting to understand the relation between our choice[3] of $\pi$ and the values $\gamma$ and $d$.

[3] Recall that we need to make such a choice in order to lift $\alpha_p$.

### 4.3. Approximating $BG$

In this last subsection, let us show that the pathologies of $BG$ are inherited by approximations of $BG$, so that in the end we can get some scheme examples. For this purpose, it suffices to follow [2, Section 6].

**Proposition 4.13** (see also [2, Theorem 1.2]). *For any integer $d \geq 0$, there exists a smooth projective $\mathcal{O}_K$-scheme $\mathcal{X}$ of dimension $d$ together with a map $\mathcal{X} \to BG$ such that the pullback $H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \to H^i(\mathcal{X}, \wedge^j L_{\mathcal{X}/\mathcal{O}_K})$ is injective for $i + j \leq d$.*

*Proof.* We simply follow the first paragraph of [2, Section 6, proof of Theorem 1.2]. By a standard argument (see e.g. [5, 2.7–2.9]), we can find an integral representation $V$ of $G$ and a $d$-dimensional complete intersection $\mathcal{Y} \subset \mathbb{P}(V)$ such that $\mathcal{Y}$ is stable under the $G$-action, the action is free, and $\mathcal{X} := \mathcal{Y}/G \simeq [\mathcal{Y}/G]$ is smooth and projective over $\mathcal{O}_K$ together with a map $\mathcal{X} \to BG$. We see that the special fiber of this map induces injections on the corresponding Hodge cohomology groups. Now we observe that the composite map

$$H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \to H^i(B\alpha_p, \wedge^j L_{B\alpha_p/\mathbb{F}_p}) \to H^i(\mathcal{X}_0, \Omega^j_{\mathcal{X}_0/\mathbb{F}_p})$$

is injective when $i + j \leq d$ (as a composite of two injective maps) and factors through $H^i(BG, \wedge^j L_{BG/\mathcal{O}_K}) \to H^i(\mathcal{X}, \Omega^j_{\mathcal{X}/\mathcal{O}_K})$. Hence the latter map must also be injective when $i + j \leq d$. ∎

By choosing $d = 4$ and using Propositions4.7 and 4.8, we arrive at the following theorem.

**Theorem 4.14.** *There exists a smooth projective relative 4-fold $\mathcal{X}$ over a ramified degree two extension $\mathcal{O}_K$ of $\mathbb{Z}_p$ such that both Hodge–de Rham and Hodge–Tate spectral sequences are non-degenerate. Moreover, the Hodge/conjugate filtrations are non-split as $\mathcal{O}_K$-modules.*

*Acknowledgements.* The author cannot overstate how much this paper owes to Bhargav Bhatt for so many helpful discussions. He is also grateful to Yu Min for useful conversations. The author would also like to thank Johan de Jong, Remy van Dobben de Bruyn, Arthur-César Le Bras, Tong Liu, Qixiao Ma, Emanuel Reinecke, and Jason Starr, for their interest in and correspondence about this project. The author thanks anonymous referees, for reminding the author of using duality statements, pointing out many references as well as several precious comments.

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