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Nearby cycles and semipositivity in positive characteristic

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Abstract. We study restriction of logarithmic Higgs bundles to the boundary divisor and we construct the corresponding nearby-cycles functor in positive characteristic. As applications we prove some strong semipositivity theorems for analogs of complex polarized variations of Hodge structures and their generalizations. This implies, e.g., semipositivity for the relative canonical divisor of a semistable reduction in positive characteristic and it gives some new strong results generalizing semipositivity even for complex varieties.

Keywords. Higgs bundles, nearby cycles, positive characteristic, semipositivity

Introduction

Let X be a smooth projective variety defined over an algebraically closed field k of characteristic p and let D be a simple normal crossing divisor on X . In this introduction we assume that (X, D) lifts to the ring $W_2(k)$ of Witt vectors of length at most 2.

A logarithmic Higgs sheaf on (X, D) is a pair (E, θ) consisting of a coherent \mathcal{O}_X -module and an \mathcal{O}_X -linear map $\theta : E \rightarrow E \otimes \Omega_X(\log D)$ such that $\theta \wedge \theta = 0$. Equivalently, replacing θ by $\hat{\theta} : T_X(\log D) \otimes E \rightarrow E$ one can consider a logarithmic Higgs sheaf on (X, D) as a $\text{Sym}^\bullet T_X(\log D)$ -module, which is coherent when considered as an \mathcal{O}_X -module.

Let $\text{MinHIG}^0(X, D)$ be the category of locally free logarithmic Higgs sheaves of rank $r \leq p$ on (X, D) , which have vanishing Chern classes in $H_{\text{et}}^{2*}(X, \mathbb{Q}_l)$ for some $l \neq p$ and are semistable. In this case semistable means slope H -semistable with respect to some ample divisor H , but one can show that the category $\text{MinHIG}^0(X, D)$ does not depend on the choice of H . One can also replace slope semistability by Gieseker semistability and the category remains the same.

Let Y be an irreducible component of D and let $\iota : Y \rightarrow X$ be the corresponding embedding. One of the main aims of this paper is to prove the following theorem:

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Theorem 0.1. *Let (E, θ) be an object of $\text{MinHIG}^0(X, D)$. The restriction of $(E, \hat{\theta})$ to Y defines a semistable $\text{Sym}^\bullet i^* T_X(\log D)$ -module. Moreover, this restriction can be deformed to an element of $\text{MinHIG}^0(Y, D^Y)$, where D^Y is the restriction of the divisor $D - Y$ to Y .*

The precise statement of this theorem is contained in Theorem 3.11 and Corollary 3.13. In fact, we prove a more general version that works also for Higgs sheaves (or modules with an integrable connection) with non-vanishing Chern classes.

Together with the restriction theorem for curves not contained in the boundary divisor D (see Theorem 4.7) this gives an inductive procedure for studying restriction of elements of $\text{MinHIG}^0(X, D)$ to curves. In particular, it implies the following theorem (see Definition 5.1 for the definition of a strongly liftable morphism).

Theorem 0.2. *Let (E, θ) be an object of $\text{MinHIG}^0(X, D)$. Let C be a smooth projective curve and let $v : C \rightarrow (X, D)$ be a separable morphism that is strongly liftable to $W_2(k)$. Then the induced $\text{Sym}^\bullet v^* T_X(\log D)$ -module $v^* E$ is semistable. In particular, if G is a subsheaf of the kernel of $v^* \theta : v^* E \rightarrow v^* E \otimes v^* \Omega_X(\log D)$ then $\deg G \leq 0$.*

This theorem has an obvious analogue in characteristic zero (see Theorem 5.4). But even the last part of this theorem was not known in characteristic zero. Already this part implies essentially all known semipositivity results (see below) for Higgs bundles or complex polarized variations of Hodge structures due to Fujita [9], Kawamata [15], Zuo [36], Fujino–Fujisawa [8, Theorem 5.21], Brunbarbe [3, Theorems 1.8 and 4.5], [4, Theorem 1.2] and many others. Note that almost all the proofs of such results are analytic and use Hodge theory. A notable exception is Arapura’s proof of [1, Theorem 2] that uses reduction to positive characteristic. However, his proof uses vanishing theorems and it does not give any semipositivity results in positive characteristic.

We say that a sheaf E on (X, D) is W_2 -nef if for any smooth projective curve C and any morphism $v : C \rightarrow (X, D)$ that is strongly liftable to $W_2(k)$ (see Definition 5.1), all quotients of $v^* E$ have a non-negative degree.

The following corollary is a direct analogue of [4, Theorem 1.2] in positive characteristic. In fact, it implies its generalization from the polystable to the semistable case.

Corollary 0.3. *Let (E, θ) be an object of $\text{MinHIG}^0(X, D)$. If E' is a locally split subsheaf of E contained in the kernel of θ then its dual $(E')^*$ is W_2 -nef on (X, D) .*

Over complex numbers a typical example of application of such a result is to semipositivity of direct images of relative canonical sheaves. This happens also in positive characteristic and we prove the following result (see Corollary 5.10 for a more precise version).

Corollary 0.4. *Let X and Y be smooth projective varieties and let B be a normal crossing divisor on Y . Let $f : X \rightarrow Y$ be a surjective morphism of relative dimension d , which has semistable reduction along B . Set $D = f^{-1}(B)$. Assume that there exists a lifting*

$\tilde{f} : (\tilde{X}, \tilde{D}) \rightarrow (\tilde{Y}, \tilde{B})$ of f to $W_2(k)$ with \tilde{f} a semistable reduction along \tilde{B} . Assume that $p > d + \dim Y$. Then $R^j f_*(\omega_{X/Y}(D))$ is a W_2 -nef locally free sheaf on (Y, B) for all integers $j \geq 0$.

This is a positive characteristic analogue of various semipositivity results due to Griffiths [10], Fujita [9], Kawamata [15], Fujino–Fujisawa [8] and others.

In positive characteristic p there are well-known examples due to L. Moret-Bailly (see [34, Exposé 8]), who exhibited for any integer $n \geq 1$ and any p a family of smooth abelian surfaces $f : X \rightarrow \mathbb{P}^1$ such that $f_*\omega_{X/\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(pn)$. In particular, one needs to add some additional assumptions to be able to get semipositivity results. The only known results on semipositivity in positive characteristic concern either $f_*(\omega_{X/Y}^m(mD))$ for $m \gg 0$ (see [27] in case $\omega_{X/Y}(D)$ is f -nef, or [7] in the case of relative dimension 1 or 2), or they deal with $f_*\omega_{X/Y}$ adding very strong assumptions on the fibers (see [33] for the case $\dim X = 2$ and $\dim Y = 1$, and [28, Theorem 6.4] for a rather complicated statement).

One of the important results that we prove is the following theorem that is a special case of Theorem 2.1.

Theorem 0.5. *Let E be a rank r reflexive sheaf with $c_1(E) = 0$ (in $H_{\text{et}}^2(X, \mathbb{Q}_l)$ for some $l \neq p$) and $c_2(E)H^{n-2} = 0$. Assume that E has a filtration M_\bullet such that all factors of the filtration are torsion free of rank $\leq p$ with $\mu_H(\text{Gr}_j^M E) = \mu_H(E)$. Assume also that each factor has the structure of a slope H -semistable sheaf with an integrable logarithmic connection on (X, D) . Then E is locally free and it has vanishing Chern classes in $H_{\text{et}}^{2*}(X, \mathbb{Q}_l)$ for any $l \neq p$. Moreover, every quotient $\text{Gr}_j^M E$ is locally free and has vanishing Chern classes in $H_{\text{et}}^{2*}(X, \mathbb{Q}_l)$.*

This result can be thought of as an analogue of a graded version of Schmid’s nilpotent orbit theorem (see Remark 4.5). In the case $D = 0$ Theorem 0.5 gives [24, Theorem 11] and fills in a gap in its proof. The stronger version, Theorem 2.2, generalizes Theorem 0.5 to the case of Higgs sheaves with possibly non-trivial Chern classes and in the case $D = 0$ it is indispensable for the proofs of [32, Theorem 3.6, Corollary-Definition 3.8 and Theorem 3.10]. In this last case Theorem 2.2 allows computing higher Chern classes of twisted preperiodic Higgs bundles.

The structure of the paper is as follows. In Section 1 we recall some results and prove a few auxiliary results. In Section 2 we prove Theorem 0.5 in case $D = 0$. In Section 3 we construct a nearby-cycles functor and we check that it preserves some semistability conditions. We also finish the proof of Theorem 0.5 in case $D \neq 0$. In Section 4 we give a few applications of Theorem 0.5. Section 5 is devoted to applications of these results to semistability and semipositivity of restriction of Higgs bundles to curves. We also give some geometric applications to semipositivity of direct images of relative canonical sheaves. The appendix contains a proof of the functoriality of the inverse Cartier transform in the logarithmic case.

Notation

Let X be a smooth variety defined over an algebraically closed field k and let D be a normal crossing divisor on X . We often view D as a closed subscheme of X given locally by one equation but by abuse of notation we also identify D with the corresponding Weil divisor and write $D = 0$ instead of $D = \emptyset$. All normal crossing divisors in the paper are reduced simple normal crossing divisors. Sometimes we add “simple” to stress the place where we need to use this assumption.

Let us recall that a *logarithmic Higgs sheaf* is a pair (E, θ) consisting of a coherent \mathcal{O}_X -module and an \mathcal{O}_X -linear map $\theta : E \rightarrow E \otimes \Omega_X(\log D)$ such that $\theta \wedge \theta = 0$. A *system of logarithmic Hodge sheaves* is a Higgs sheaf (E, θ) with a decomposition $E = \bigoplus E^{p,q}$ such that θ maps $E^{p,q}$ into $E^{p-1,q+1} \otimes \Omega_X(\log D)$.

In this paper, whenever X is projective and we say that a logarithmic Higgs sheaf (E, θ) is slope H -semistable for some ample H , we always implicitly assume that E is torsion free. Let us recall that a system of logarithmic Hodge sheaves is slope H -semistable as a system of logarithmic Hodge sheaves if and only if it is slope H -semistable as a logarithmic Higgs sheaf (see [23, Corollary 3.5]).

Now let S be any scheme. We say that (X, D) is a *smooth log pair over S* if X is a smooth S -scheme and D is a relatively simple normal crossing divisor over S . We say that $f : (Y, B) \rightarrow (X, D)$ is a morphism of smooth log pairs if $f : Y \rightarrow X$ is a morphism and the support of B contains the support of $f^{-1}(D)$.

If E is a coherent sheaf of rank r on a smooth projective variety X then we denote by $\text{ch}(E)$ the Chern character of E . This is defined as an element of the rational Chow ring $\text{CH}^*(X) \otimes \mathbb{Q}$ but in this paper we abuse notation and denote by $\text{ch}(E)$ the image of this class by the cycle map and we treat it as an element of the étale cohomology ring $H_{\text{ét}}^*(X, \mathbb{Q}_l)$, where l is different from the characteristic of the base field (or an element of $H^*(X, \mathbb{Q})$ in the case of complex manifolds). We denote by $\Delta(E)$ the discriminant of E defined as $2rc_2(E) - (r-1)c_1^2(E)$. In the case of surfaces we use the degree map \int_X to identify the cohomology group $H_{\text{ét}}^4(X, \mathbb{Q}_l)$ (or $H^4(X, \mathbb{Q})$) with \mathbb{Q}_l (respectively, \mathbb{Q}) and we think of $\Delta(E)$ as an integer. Similarly, in higher dimensions the top degree intersections like $\Delta(E)H^{\dim X-2}$ denote the degree of the cycle $\Delta(E)H^{\dim X-2}$.

1. Preliminaries

1.1. Logarithmic Higgs sheaves

In this subsection we recall a few results on semistable logarithmic Higgs sheaves. Throughout this subsection we fix the following notation.

Let X be a smooth variety of dimension n defined over an algebraically closed field k of characteristic p . Let D be a normal crossing divisor on X .

Let us recall the following theorem due to Ogus and Vologodsky [26] in the usual case and Schepler [29] in the logarithmic case (see also [25, Theorem 2.5] and [18, Appendix]):

Theorem 1.1. *Assume that (X, D) is liftable to $W_2(k)$ and fix such a lifting (\tilde{X}, \tilde{D}) . There exists a Cartier transform $C_{(\tilde{X}, \tilde{D})}$, which defines an equivalence between the category of torsion free \mathcal{O}_X -modules with an integrable logarithmic connection whose logarithmic p -curvature is nilpotent of level $\leq p - 1$ and the residues are nilpotent of order $\leq p$, and the category of torsion free logarithmic Higgs \mathcal{O}_X -modules with a nilpotent Higgs field of level $\leq p - 1$.*

From now on in this subsection we assume that X is projective and we fix an ample divisor H on X . Let us recall the following boundedness result for logarithmic Higgs sheaves.

Theorem 1.2. *Fix some number Δ and a class $c \in H_{\text{et}}^2(X, \mathbb{Q}_l)$ for some $l \neq p$. The family of slope H -semistable logarithmic Higgs sheaves (E, θ) such that E is reflexive with fixed rank r , $c_1(E) = c$ and $\Delta(E)H^{n-2} \leq \Delta$ is bounded.*

Proof. By [24, Lemma 5] one can find a constant C such that for any rank r slope H -semistable logarithmic Higgs sheaves (E, θ) we have $\mu_{\max, H}(E) \leq \mu(E) + (r - 1)C$. Hence the result follows from [21, Theorem 3.4]. ■

Note that in the above theorem it is not sufficient to fix r , $c_1(E)H^{n-1}$ and $\Delta(E)H^{n-2}$. We will also need the following theorem, which is a special case of [23, Theorem 5.5].

Theorem 1.3. *Let (E, θ) be a slope H -semistable logarithmic Higgs sheaf. Then there exists a decreasing filtration $E = N^0 \supset N^1 \supset \dots \supset N^m = 0$ such that $\theta(N^i) \subset N^{i-1} \otimes \Omega_X(\log D)$ and the associated graded is a slope H -semistable system of logarithmic Hodge sheaves.*

The result of [23, Theorem 5.5] (see also [19, Theorem A.4] in the case of flat torsion free sheaves) also implies the following result:

Theorem 1.4. *If (E, ∇) is a slope H -semistable sheaf with an integrable logarithmic connection then there exists a canonical Griffiths transverse filtration $E = S^0 \supset S^1 \supset \dots \supset S^m = 0$ such that the associated graded system of logarithmic Hodge sheaves is slope H -semistable.*

The canonical filtration S^\bullet from Theorem 1.4 is called *Simpson's filtration* of (E, ∇) . This notion is used in the following generalization of [23, Theorem 5.12] and [20, Theorem 2.2] (see [25, Theorem 3.1]).

Theorem 1.5. *Assume the pair (X, D) admits a lifting (\tilde{X}, \tilde{D}) to $W_2(k)$. If (E, θ) is a slope H -semistable system of logarithmic Hodge sheaves of rank $r \leq p$ then there exists*

a canonically defined Higgs–de Rham sequence

$$\begin{array}{ccccc}
 & & (V_0, \nabla_0) & & (V_1, \nabla_1) \\
 & \nearrow^{C_{(\bar{X}, \bar{D})}^{-1}} & & \searrow^{\text{Gr}_{S_0}} & \nearrow^{C_{(\bar{X}, \bar{D})}^{-1}} & & \searrow^{\text{Gr}_{S_1}} \\
 (E_0, \theta_0) = (E, \theta) & & & & (E_1, \theta_1) & & \dots
 \end{array}$$

in which each (V_i, ∇_i) is slope H -semistable and (E_{i+1}, θ_{i+1}) is the slope H -semistable system of logarithmic Hodge sheaves associated to (V_i, ∇_i) via its Simpson’s filtration S_i^\bullet .

The following theorem is a generalization of [24, Theorem 10] to the logarithmic case. We skip its proof as it is the same as in the non-logarithmic case.

Theorem 1.6. *Assume the pair (X, D) is liftable to $W_2(k)$. Let d_0 be a non-negative integer such that $T_X(-\log D) \otimes \mathcal{O}_X(d_0 H)$ is globally generated. Let (E, θ) be a slope H -stable logarithmic Higgs sheaf of rank $r \leq p$. Take an integer*

$$d > \frac{r-1}{r} \Delta(E) H^{n-2} + \frac{1}{r(r-1)H^n}.$$

Moreover, if $r > 2$, assume also that $d > 2(r-1)^2 d_0$. Let $Y \in |dH|$ be a smooth divisor such that E_Y has no torsion and $D \cap Y$ is a normal crossing divisor on Y . Then the logarithmic Higgs sheaf (E_Y, θ_Y) induced from (E, θ) via restricting to Y and composition $E_Y \rightarrow E_Y \otimes \Omega_X(\log D)|_Y \rightarrow E_Y \otimes \Omega_Y(\log D \cap Y)$ is slope H_Y -stable.

Corollary 1.7. *Assume the pair (X, D) is liftable to $W_2(k)$ and let d_0 be as in the previous theorem. Let (E, θ) be a slope H -semistable logarithmic Higgs sheaf of rank $r \leq p$ and let d be an integer satisfying the same inequalities as in the previous theorem. Then for a general divisor $Y \in |dH|$ the restriction (E_Y, θ_Y) is slope H_Y -semistable.*

Proof. Let M_\bullet be a Jordan–Hölder filtration of (E, θ) . By definition this means that all the quotients $\text{Gr}_i^M E$ are slope H -stable logarithmic Higgs sheaves with slopes $\mu_H(\text{Gr}_i^M E)$ equal to $\mu_H(E)$. Existence of such a filtration for logarithmic slope H -semistable logarithmic Higgs sheaves is standard and follows by the same arguments as for the usual slope H -stable sheaves (see, e.g., [12, §1.5 and §1.6]). To simplify notation let us set $E_i = \text{Gr}_i^M E$ (we consider it as a logarithmic Higgs sheaf and not only a sheaf) and $r_i = \text{rk} E_i$. Since E_i and E_j have the same slope, we have $(\frac{c_1 E_i}{r_i} - \frac{c_1 E_j}{r_j}) H = 0$ and the Hodge index theorem implies that

$$\begin{aligned}
 \frac{\Delta(E) H^{n-2}}{r} &= \sum \frac{\Delta(E_i) H^{n-2}}{r_i} - \frac{1}{r} \sum_{i < j} r_i r_j \left(\frac{c_1 E_i}{r_i} - \frac{c_1 E_j}{r_j} \right)^2 H^{n-2} \\
 &\geq \sum \frac{\Delta(E_i) H^{n-2}}{r_i}.
 \end{aligned}$$

Therefore our assumptions on d imply that we can apply Theorem 1.6 to each quotient E_i . So if we choose a smooth divisor $Y \in |mH|$ such that $D \cap Y$ is a normal crossing divisor

on Y and $(E_i)_Y$ has no torsion for every i then the restricted logarithmic Higgs sheaf $(E_i)_Y$ is slope H -stable and hence (E_Y, θ_Y) is slope H_Y -semistable. Note that a general $Y \in |mH|$ satisfies the above assumptions by Bertini's theorem and Lemma 1.13 below. ■

Remark 1.8. In Theorem 1.6 and Corollary 1.7 we can replace a logarithmic Higgs sheaf with a sheaf with an integrable logarithmic connection. The proofs of the results remain the same.

Let us also recall Bogomolov's inequality for logarithmic Higgs sheaves (see [25, Theorem 3.3] for a more general version).

Theorem 1.9. *Assume that (X, D) admits a lifting to $W_2(k)$. Then for any slope H -semistable logarithmic Higgs sheaf (E, θ) of rank $r \leq p$ we have*

$$\Delta(E)H^{n-2} \geq 0.$$

Remark 1.10. The above theorem also holds for sheaves with an integrable logarithmic connection. Indeed, if (E, ∇) is a rank $r \leq p$ slope H -semistable sheaf with an integrable logarithmic connection and S^\bullet is its Simpson's filtration then by the above theorem,

$$\Delta(E)H^{n-2} = \Delta(\text{Gr}_S E)H^{n-2} \geq 0.$$

1.2. Higher discriminants

Let us fix a smooth projective variety X defined over an arbitrary algebraically closed field k .

Let E be a rank $r > 0$ coherent sheaf on X . Let us fix a prime l different from the characteristic of the base field k and let us write

$$\log(\text{ch}(E)) = \log r + \sum_{i \geq 1} (-1)^{i+1} \frac{1}{i!r^i} \Delta_i(E)$$

for some classes $\Delta_i(E) \in H_{\text{et}}^{2i}(X, \mathbb{Q}_l)$, which we call *higher discriminants* of E (we can also use $\Delta_i(E) \in H^*(X, \mathbb{Q})$ in the case of complex manifolds). These discriminants are polynomials in Chern classes of E with integral coefficients. They are variants of Drezet's logarithmic invariants (with somewhat different normalization to get integral coefficients and $\Delta_2(E) = \Delta(E)$). Note that for any line bundle L we have $\Delta_i(E \otimes L) = \Delta_i(E)$ for $i \geq 2$. This follows immediately from the fact that

$$\log(\text{ch}(E \otimes L)) = \log(\text{ch}(E) \cdot \text{ch}(L)) = \log(\text{ch}(E)) + c_1(L).$$

In the following we often use this property of discriminants without further notice.

Lemma 1.11. *The following conditions in $H_{\text{et}}^*(X, \mathbb{Q}_l)$ (or in $H^*(X, \mathbb{Q})$ in the case of complex manifolds) are equivalent:*

$$(1) \quad r^i c_i(E) = \binom{r}{i} c_1(E)^i \text{ for all } i \geq 1,$$

- (2) $\Delta_i(E) = 0$ for all $i \geq 2$,
- (3) $\log \text{ch}(E) = \log r + c_1(E)/r$.

Proof. Equivalence of (2) and (3) is clear as $\Delta_1(E) = c_1(E)$. For simplicity of notation let us assume that E is locally free; the proof in the general case is the same except that we need to replace E by its class in $K^0(X)$ and do all the computations in Grothendieck’s K -group.

By the Bloch–Gieseker covering trick [2, Lemma 2.1] there exists a smooth projective variety \tilde{X} and a finite flat surjective covering $f : \tilde{X} \rightarrow X$ together with a line bundle L such that $f^*(\det E)^{-1} = L^{\otimes r}$. Set $\tilde{E} := f^*E \otimes L$. Note that $c_1(\tilde{E}) = 0$ and

$$\Delta_i(\tilde{E}) = \Delta_i(f^*E) \quad \text{for all } i \geq 2.$$

Since f induces an injection $H_{\text{et}}^*(X, \mathbb{Q}_l) \rightarrow H_{\text{et}}^*(\tilde{X}, \mathbb{Q}_l)$, condition (2) is equivalent to the vanishing of $\Delta_i(\tilde{E})$ for all $i \geq 1$, i.e., to the equality $\log \text{ch}(\tilde{E}) = \log r$. Clearly, this is equivalent to $c_i(\tilde{E}) = 0$ for all $i \geq 1$.

For all $i \geq 0$ we have

$$c_i(\tilde{E}) = c_i(f^*E \otimes L) = \sum_{j=0}^i \binom{r-j}{i-j} c_1(L)^{i-j} c_j(f^*E).$$

Using $c_1(L) = -\frac{1}{r}c_1(f^*E)$ and the fact that the map $H_{\text{et}}^*(X, \mathbb{Q}_l) \rightarrow H_{\text{et}}^*(\tilde{X}, \mathbb{Q}_l)$ is injective, we see that the second condition is equivalent to the equalities

$$\sum_{j=0}^i \binom{r-j}{i-j} (-c_1(E))^{i-j} r^j c_j(E) = 0 \tag{1.1}$$

for all $i = 1, \dots, r$. This follows from the fact that the equalities (1.1) for $i \leq m$ are equivalent to the equalities

$$r^i c_i(E) = \binom{r}{i} c_1(E)^i \tag{1.2}$$

for $i = 1, \dots, m$. We prove this by induction on m . For $m = 1$ it is clear, so let us assume it holds for $1, \dots, m - 1$. We can assume that (1.2) holds for $i < m$. Then we have

$$\begin{aligned} & \sum_{i=0}^m \binom{r-i}{m-i} (-c_1(E))^{m-i} r^i c_i(E) \\ &= r^m c_m(E) - \binom{r}{m} c_1(E)^m + \sum_{i=0}^{m-1} (-1)^{m-i} \binom{r-i}{m-i} \binom{r}{i} c_1(E)^m \\ &= r^m c_m(E) - \binom{r}{m} c_1(E)^m + \binom{r}{m} c_1(E)^m \cdot \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} \\ &= r^m c_m(E) - \binom{r}{m} c_1(E)^m. \end{aligned}$$

This proves that under our assumptions, (1.1) for $i = m$ is equivalent to (1.2) for $i = m$. ■

1.3. Criterion for local freeness and restriction to divisors

Let X be an integral noetherian scheme and let E be a coherent sheaf of \mathcal{O}_X -modules. Let $S(E)$ be the set of points $x \in X$ such that E_x is not a free $\mathcal{O}_{X,x}$ -module. We call $S(E)$ the *singular set* of E .

Let us define the function $\varphi : X \rightarrow \mathbb{Z}$ by $\varphi(x) = \dim_{k(x)}(E \otimes k(x))$. Let η be the generic point of X . For a point $x \in X$, by [11, Chapter II, Lemma 8.9], E_x is a free $\mathcal{O}_{X,x}$ -module if and only if $\varphi(x) = \varphi(\eta)$. On the other hand, by Nakayama's lemma the function φ is upper semicontinuous (see [11, Chapter III, Example 12.7.2]), so $S(E) = \{x \in X : \varphi(x) > \varphi(\eta)\}$ is closed.

In the following we say that E is *locally free outside a finite number of points* if $S(E)$ is a finite set of points.

Now let X be a smooth projective variety of dimension n defined over an arbitrary algebraically closed field k . We will use several times the following criterion for local freeness of graded sheaves associated to filtrations.

Lemma 1.12. *Assume that $n \geq 3$ and let V be a reflexive sheaf on X with a filtration $N^m = 0 \subset N^{m-1} \subset \dots \subset N^0 = V$ such that each N^i is saturated in V . Let $W = \bigoplus N^i/N^{i-1}$ be the associated graded and assume that*

- (1) *the reflexivization W^{**} of W is locally free, and*
- (2) *W is locally free outside a finite number of points.*

Then both V and W are locally free.

Proof. It is sufficient to prove the lemma for $m = 2$; the general case then follows easily by induction on the length m of the filtration.

Assuming $m = 2$ our assumptions imply that N^1 is locally free and we have a short exact sequence

$$0 \rightarrow N^0/N^1 \rightarrow (N^0/N^1)^{**} \rightarrow T \rightarrow 0$$

for some sheaf T supported on a finite number of points. By assumption we also know that $(N^0/N^1)^{**}$ is locally free. Let us note that by Serre's duality, $\text{Ext}^2(T, N^1)$ is dual to $\text{Ext}^{n-2}(N^1, T \otimes \omega_X) = H^{n-2}(T \otimes \omega_X \otimes (N^1)^*) = 0$ as $n \geq 3$. Hence by the long Ext exact sequence, the canonical map $\text{Ext}^1((N^0/N^1)^{**}, N^1) \rightarrow \text{Ext}^1(N^0/N^1, N^1)$ is surjective. Therefore there exists a coherent sheaf \tilde{V} such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N^1 & \longrightarrow & V & \longrightarrow & N^0/N^1 & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N^1 & \longrightarrow & \tilde{V} & \longrightarrow & (N^0/N^1)^{**} & \longrightarrow & 0 \end{array}$$

But since V is reflexive and \tilde{V} is locally free, the map $V \rightarrow \tilde{V}$ is an isomorphism (as it is an isomorphism outside the support of T). Hence $T = 0$ and $W^{**} = W$. This immediately implies the required assertion. \blacksquare

We will also need the following lemmas allowing us to keep track of singularities of sheaves when restricting to divisors.

Lemma 1.13. *Let Λ be a base point free linear system on X and let E be a coherent \mathcal{O}_X -module.*

- (1) *If E is reflexive and $Y \in \Lambda$ is integral then E_Y is a torsion free \mathcal{O}_Y -module.*
- (2) *If E is torsion free (resp. reflexive) and $Y \in \Lambda$ is general then E_Y is also torsion free (resp. reflexive) as an \mathcal{O}_Y -module.*

The above lemma follows from [12, Lemma 1.1.12 and Corollary 1.1.14].

Lemma 1.14. *Let E be a rank r torsion free sheaf on X and let Y be an integral divisor on X such that E_Y is locally free. Then $S(E) \cap Y = \emptyset$, i.e., E is locally free at all points of Y . Moreover, if Y is ample then E is locally free outside a finite number of points.*

Proof. Since every torsion free sheaf on a smooth curve is locally free, we can assume that the dimension n of X is greater than 1. Since $S(E)$ has codimension ≥ 2 in X , there exists a codimension 1 point $y \in Y - S(E)$. Let η be the generic point of X and η' the generic point of Y . Since E_Y is locally free, $E_{Y,y}$ is a free $\mathcal{O}_{Y,y}$ -module and hence

$$\dim_{k(y)} E_{Y,y} \otimes_{\mathcal{O}_{Y,y}} k(y) = \dim_{k(\eta')} E_{Y,y} \otimes_{\mathcal{O}_{Y,y}} k(\eta') = \text{rk} E_Y.$$

By the choice of y the $\mathcal{O}_{X,y}$ -module E_y is free and hence

$$\dim_{k(y)} E_y \otimes_{\mathcal{O}_{X,y}} k(y) = \dim_{k(\eta)} E_y \otimes_{\mathcal{O}_{X,y}} k(\eta) = r.$$

Since $E_y \otimes_{\mathcal{O}_{X,y}} k(y) \simeq E_{Y,y} \otimes_{\mathcal{O}_{Y,y}} k(y)$ we see that E_Y has rank r . By assumption for any point $z \in Y$ the $\mathcal{O}_{Y,z}$ -module $E_{Y,z}$ is free, so we get

$$\dim_{k(z)} E_z \otimes_{\mathcal{O}_{X,z}} k(z) = \dim_{k(z)} E_{Y,z} \otimes_{\mathcal{O}_{Y,z}} k(z) = \dim_{k(\eta')} E_{Y,z} \otimes_{\mathcal{O}_{Y,z}} k(\eta') = r.$$

Then [11, Chapter II, Lemma 8.9] implies that E_z is a free $\mathcal{O}_{X,z}$ -module, which proves the first assertion.

Now let us assume that Y is ample. The singular set $S(E)$ is a closed subset of X and $S(E) \cap Y = \emptyset$, so it does not have any irreducible components of dimension ≥ 1 . So $S(E)$ is zero-dimensional. ■

2. Local freeness

In this section we fix the following notation. Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k of characteristic p and let D be a simple normal crossing divisor on X . We assume that $D \subset X$ admits a lifting to $W_2(k)$. We also fix an ample divisor H on X .

The main aim of this section is to prove a large part of the following generalization of Theorem 0.5 (the proof will be finished in Subsection 3.5):

Theorem 2.1. *Let E be a rank r reflexive sheaf with $\Delta(E)H^{n-2} = 0$. Assume that E has a filtration M_\bullet such that all factors of the filtration are torsion free of rank $\leq p$ with $\mu_H(\text{Gr}_j^M E) = \mu_H(E)$. Also assume that each factor has the structure of a slope H -semistable logarithmic Higgs sheaf (or of a slope H -semistable sheaf with an integrable logarithmic connection) on (X, D) . Then E is locally free and*

$$c_m(E) = \binom{r}{m} \left(\frac{c_1(E)}{r} \right)^m$$

in $H_{\text{et}}^{2m}(X, \mathbb{Q}_l)$ for all $m \geq 1$ and any $l \neq p$. Moreover, every quotient $\text{Gr}_j^M E$ is locally free and for all $m \geq 1$ we have

$$c_m(\text{Gr}_j^M E) = \binom{r_j}{m} \left(\frac{c_1(E)}{r} \right)^m$$

in $H_{\text{et}}^{2m}(X, \mathbb{Q}_l)$, where $r_j = \text{rk Gr}_j^M E$.

This theorem is a strong version of the following theorem to which we will reduce its proof.

Theorem 2.2. *Let (E, θ) (resp. (E, ∇)) be a rank $r \leq p$ slope H -semistable logarithmic Higgs sheaf (resp. a rank $r \leq p$ slope H -semistable sheaf with a logarithmic connection). Then the following conditions are equivalent:*

- (1) $\Delta(E)H^{n-2} = 0$ and E is reflexive,
- (2) $\Delta(E)H^{n-2} = 0$ and E is locally free,
- (3) $c_m(E) = \binom{r}{m} \left(\frac{c_1(E)}{r} \right)^m$ in $H_{\text{et}}^{2m}(X, \mathbb{Q}_l)$ for all $m \geq 1$ and any $l \neq p$.

Remarks 2.3. (1) To simplify notation, in Theorems 2.1 and 2.2 we deal with only one polarization although one can also replace H by a collection of ample divisors as in, e.g., [24, Theorem 10].

(2) A special case of the implication (3) \Rightarrow (2) in Theorem 2.2 was proven in [20, Proposition 3.12] using Faltings’s result on Fontaine modules.

(3) Theorem 2.2 implies that all the sheaves E_i and V_i appearing in the canonical Higgs–de Rham sequence of a system of logarithmic Hodge sheaves, which satisfies the equivalent conditions of Theorem 2.2 and has a nilpotent Higgs field, are locally free.

Theorem 2.2 generalizes [24, Theorem 11] to the case of logarithmic Higgs sheaves with possibly non-trivial Chern classes. It also generalizes [32, Theorems 3.6 and 3.10], which deal with systems of Hodge sheaves of rank $r < p$ on X defined over $k = \bar{\mathbb{F}}_p$. In this last case Theorem 2.2 allows one to compute higher Chern classes of twisted preperiodic Higgs bundles. Let us also remark that a special case of the above result was implicitly used in [1, proof of Theorem 3]. More precisely, in [1, proofs of Lemmas 4.4 and 4.5] (needed for [1, Theorem 3]) the author implicitly uses that $B(E, \theta)$ is locally free if (E, θ) is locally free. He applies [1, Lemma 4.3] to $B(E, \theta)$ and that lemma does not apply if $B(E, \theta)$ is not locally free. It is easy to find examples for which (E, θ) is semistable, E is locally free but $B(E, \theta)$ is not even reflexive. In particular, in both

[1, Lemma 4.4] and [1, Lemma 4.5] one needs to assume that (E, θ) is semistable with vanishing Chern classes and then use our Theorem 2.2. Note also that at the time of writing [1], Theorem 2.2 was not stated in the logarithmic case that was used there. In the logarithmic case, even if $k = \bar{\mathbb{F}}_p$ and one has the vanishing of all Chern classes, the method of proof of local freeness from [20, Proposition 3.12] does not apply.

The strategy of our proof of Theorem 2.2 in the case $D = 0$ is modelled on the proof of [24, Theorem 11]. Unfortunately, the latter proof contains a serious gap: it is not clear that the family $\{(E_i, \theta_i)\}$ of Higgs sheaves considered in the proof is bounded a priori the sheaves E_i need not be reflexive. However, if one assumes that in [24, Theorem 11] all Chern classes vanish, then the arguments there show that E is locally free. This is already sufficient for almost all the applications mentioned in [24] (except for Corollary 6 there, which also needs an additional assumption on vanishing Chern classes; one also needs to slightly adjust the proof of [24, Corollary 5]).

In general, one can easily find examples of Higgs–de Rham sequences starting with a locally free sheaf for which other sheaves in the sequence are not reflexive. This causes several complications that we need to overcome. The same error appeared independently in the first version of [32, Theorem 3.10], where the authors claimed existence of a certain map on the open subset of the moduli space of semistable sheaves, parameterizing reflexive sheaves. However, in the case of [32, Theorem 3.10], it is not so easy to adjust the arguments adding additional assumptions (this would require at least our Lemma 1.11 and repeating the proof of [24, Theorem 11]). So in this case Theorem 2.2 offers the only available proof.

A new idea appearing in the general proof of Theorem 2.2, compared to the case $D = 0$, is that we need to use a nearby cycles functor to prove local freeness of the restriction of E to the irreducible components of D .

2.1. Reduction from filtrations to sheaves

In this subsection we show how to reduce the proof of Theorem 2.1 to Theorem 2.2. Before we do that let us prove a few independent lemmas:

Lemma 2.4. *Let E be a rank r reflexive sheaf with $\Delta(E)H^{n-2} = 0$. Assume that E has a filtration M_\bullet such that all factors of the filtration are torsion free of rank $\leq p$ with $\mu_H(\text{Gr}_j^M E) = \mu_H(E)$. Assume also that each factor has the structure of a slope H -semistable logarithmic Higgs sheaf (or of a slope H -semistable sheaf with an integrable logarithmic connection) on (X, D) . Then for all j we have $\Delta(\text{Gr}_j^M E)H^{n-2} = 0$ and $c_1(\text{Gr}_j^M E)/\text{rk}(\text{Gr}_j^M E) = c_1(E)/r$ in $H_{\text{ét}}^2(X, \mathbb{Q}_l)$ for any $l \neq p$.*

Proof. To simplify notation let us set

$$E_i = \text{Gr}_i^M E \quad \text{and} \quad r_i = \text{rk} E_i.$$

Since $\mu_H(E_i) = \mu_H(E_j)$, we have $(c_1 E_i / r_i - c_1 E_j / r_j)H = 0$. So the Hodge index

theorem and Theorem 1.9 imply that

$$\begin{aligned} 0 &= \frac{\Delta(E)H^{n-2}}{r} = \sum \frac{\Delta(E_i)H^{n-2}}{r_i} - \frac{1}{r} \sum_{i < j} r_i r_j \left(\frac{c_1(E_i)}{r_i} - \frac{c_1(E_j)}{r_j} \right)^2 H^{n-2} \\ &\geq \sum \frac{\Delta(E_i)H^{n-2}}{r_i} \geq 0. \end{aligned}$$

Hence $\Delta(E_i)H^{n-2} = 0$ and $(c_1(E_i)/r_i - c_1(E_j)/r_j)^2 H^{n-2} = 0$ for all i and j . By assumption we also have $(c_1(E_i)/r_i - c_1(E_j)/r_j)H^{n-1} = \mu_H(E_i) - \mu_H(E_j) = 0$.

Now let us recall that by [16, Theorem 9.6.3] if B is a divisor on X such that $BH^{n-1} = B^2H^{n-2} = 0$, then the class of B in the group of divisors on X modulo algebraic equivalence is torsion. Using the cycle map we obtain the equality $B = 0$ in $H_{\text{ét}}^2(X, \mathbb{Q}_l)$ for any $l \neq p$.

Applying this fact to $B = (c_1(E_i)/r_i - c_1(E_j)/r_j)$ we get the required equalities. ■

Lemma 2.5. *Let (E, θ) (resp. (E, ∇)) be a rank $r \leq p$ slope H -semistable logarithmic Higgs sheaf (resp. sheaf with an integrable logarithmic connection) with $\Delta(E)H^{n-2} = 0$. Then θ (resp. ∇) extends uniquely to a logarithmic Higgs field $\tilde{\theta}$ (resp. an integrable logarithmic connection $\tilde{\nabla}$) on the reflexivization E^{**} so that $(E^{**}, \tilde{\theta})$ (resp. $(E^{**}, \tilde{\nabla})$) is slope H -semistable. Moreover, $\Delta(E^{**})H^{n-2} = 0$ and the canonical map $E \rightarrow E^{**}$ is an isomorphism outside a closed subset of codimension ≥ 3 .*

Proof. The equality $\theta \wedge \theta = 0$ implies $\theta^{**} \wedge \theta^{**} = 0$, so $\tilde{\theta} := \theta^{**}$ is a logarithmic Higgs field. In the second case we can extend ∇ , e.g., in the following way. Let us set $U := X - S(E)$ and let $j : U \hookrightarrow X$ be the corresponding embedding. Then $E^{**} = j_*(j^*E)$ and we can define $\tilde{\nabla}$ by $\tilde{\nabla} = j_*\nabla j^*$. It is easy to see that this is a logarithmic connection. It is integrable, because $\tilde{\nabla} \wedge \tilde{\nabla} : E^{**} \rightarrow E^{**} \otimes \Omega_X^2(\log D)$ is an \mathcal{O}_X -linear map extending $\nabla_U \wedge \nabla_U = 0$.

Note that for any torsion free sheaf G the line bundles $\det(G^{**})$ and $\det(G)$ are isomorphic on $X - S(G)$ and $S(G)$ has codimension ≥ 2 . So $\det(G^{**}) \simeq \det(G)$ and $c_1(G^{**}) = c_1(G)$. Now for any subsheaf $G \subset E^{**}$ we have $(E \cap G)^{**} = G^{**}$ as both sheaves are reflexive and equal outside codimension ≥ 2 . So the sheaf E contains the subsheaf $E \cap G$ of the same slope as G . This shows that passing to the reflexivization preserves slope H -semistability (and also slope H -stability).

To prove the last part note that the canonical map $E \rightarrow E^{**}$ is injective as by assumption E is torsion free. Let T be the cokernel of this map. Without any loss of generality we can assume that H is very ample. After restricting to a general complete intersection surface $Y \in |H| \cap \cdots \cap |H|$, we get a short exact sequence

$$0 \rightarrow E_Y \rightarrow (E^{**})_Y \rightarrow T_Y \rightarrow 0.$$

Since T is supported in codimension ≥ 2 , T_Y is supported on a finite number of points. We have

$$0 = \Delta(E)H^{n-2} = \Delta(E_Y) = \Delta((E^{**})_Y) + h^0(Y, T_Y) = \Delta(E^{**})H^{n-2} + h^0(Y, T_Y).$$

Since (E^{**}, θ^{**}) is slope H -semistable, by Bogomolov’s inequality for logarithmic Higgs bundles (see Theorem 1.9 and Remark 1.10) we have $\Delta(E^{**})H^{n-2} \geq 0$. Hence we get $\Delta(E^{**})H^{n-2} = 0$ and $h^0(Y, T_Y) = 0$. Since T_Y is supported on a finite number of points, we get $T_Y = 0$. It follows that T is supported in codimension ≥ 3 . ■

Lemma 2.6. *Replacing H by some multiple we can assume that any $Y \in |H|$ is liftable to $\tilde{Y} \subset \tilde{X}$. Moreover, for any closed point $x \in U := X - \text{Supp } D$ a general divisor $Y \in |H|$ passing through x is smooth and the divisor $D + Y$ is a normal crossing divisor. Then $D_Y = D \cap Y$ is a normal crossing divisor on Y and the pair (Y, D_Y) is liftable to $W_2(k)$.*

Proof. The first part follows from [24, proof of Theorem 11]. Replacing H by a multiple we can also assume that H is very ample. As in [6, proof of Theorem 3.1] we can also assume that the subsystem $\Lambda \subset |H|$ consisting of all divisors containing x has x as its scheme-theoretic base locus. If $\pi : X' \rightarrow X$ is the blow up of x then, replacing H if necessary by its multiple, we can also assume that $\pi^*H - E$ is very ample (see [11, Chapter II, Proposition 7.10]). Let $\{D_i\}_{i \in I}$ be the irreducible components of D viewed as reduced closed subschemes of X . Let us set $D'_i = \pi^{-1}(D_i)$ and $D' = \pi^{-1}(D)$. By Bertini’s theorem for any $J \subset I$, a general $Y' \in |\pi^*H - E|$ intersects all irreducible components of $\bigcap_{j \in J} D'_j$ along smooth divisors. Then $D' + Y'$ is a normal crossing divisor on X' . By [6, Theorem 2.1] the image of a general divisor $Y' \in |\pi^*H - E|$ is smooth and it is a general divisor in Λ . Hence for general $Y \in \Lambda$, $D + Y$ is a normal crossing divisor on X . Moreover, $\tilde{D} + \tilde{Y} \subset \tilde{X}$ is its lifting to $W_2(k)$. This implies that also $(\tilde{Y}, \tilde{Y} \cap \tilde{D})$ lifts (Y, D_Y) to $W_2(k)$. ■

Lemma 2.7. *Theorem 2.2 in dimension $\leq n$ implies Theorem 2.1 in dimension $\leq n$.*

Proof. The proof is by induction on the dimension n of X . If $n = 1$ then the assertion follows from the fact that torsion free sheaves on a smooth curve are locally free. Assume that the implication holds for varieties of dimension less than n and let X be of dimension n .

First we consider the case in which each factor of the filtration from Theorem 2.1 has the structure of a slope H -semistable logarithmic Higgs sheaf. Let us write E_j for $\text{Gr}_j^M E$ and r_j for its rank. Replacing H by its multiple, by Lemma 2.6 we can assume that for all $d \geq 1$ and a general divisor $Y \in |dH|$ the pair $(Y, D_Y = D \cap Y)$ is log smooth and liftable to $W_2(k)$. By Corollary 1.7 (or simply by the Mehta–Ramanathan restriction theorem for logarithmic Higgs sheaves) applied to each quotient of the filtration M_\bullet , for large d and for a general section $Y \in |dH|$, the restriction of each quotient E_j to Y is a slope H_Y -semistable logarithmic Higgs sheaf and the restriction E_Y is reflexive (here we use Lemma 1.13). Hence by the induction assumption each $(E_j)_Y$ is locally free. So by Lemma 1.14 each E_j is locally free outside a finite number of points of X .

Since by Lemma 2.4 we have $\Delta(E_j)H^{n-2} = 0$, Theorem 2.2 applied to X implies that all E_j^{**} are locally free. Hence the assumptions of Lemma 1.12 are satisfied and we conclude that E and all quotients E_j are locally free. By Theorem 2.2 this implies that

$$c_m(E_j) = \binom{r_j}{m} \left(\frac{c_1(E_j)}{r_j} \right)^m,$$

which together with the equality $c_1(E_j) = \frac{r_j}{r} c_1(E)$ finishes the proof of the last part of Theorem 2.1. Then a simple computation of Chern classes shows that

$$c_m(E) = \binom{r}{m} \left(\frac{c_1(E)}{r} \right)^m.$$

Now let us consider the case where each factor of the filtration M_\bullet from Theorem 2.1 has the structure of a slope H -semistable sheaf with an integrable logarithmic connection. The same arguments as above allow us to prove that for general Y as above, the restriction $(\mathrm{Gr}^M E)_Y$ is locally free (here we use Remark 1.8 instead of Corollary 1.7). So $\mathrm{Gr}^M E$ is locally free outside a finite number of points and by Lemma 1.12 it is sufficient to prove that $(\mathrm{Gr}^M E)^{**}$ is locally free. Then $\mathrm{Gr}^M E$ is locally free and we can finish as in the case of logarithmic Higgs sheaves.

Let us set $E_j = \mathrm{Gr}_j^M E$ and $r_j = \mathrm{rk} E_j$. For general Y the restriction $(M_i)_Y$ is a subsheaf of E_Y , so $(M_\bullet)_Y$ is a filtration of E_Y . If $n > 2$ then as above we have $\Delta((E_j)_Y)H^{n-3} = 0$ and hence by the induction assumption applied to E_Y we have

$$c_m((E_j)_Y) = \binom{r_j}{m} \left(\frac{c_1(E_Y)}{r} \right)^m \quad \text{for all } m \geq 1.$$

For $n = 2$ such equalities are clear as we need to check them only for $m = 1$.

Let us recall that by Lemma 2.4 we have $\Delta(E_j)H^{n-2} = 0$. So by Lemma 2.5, $\bar{E}_j := (E_j)^{**}$ is a slope H -semistable sheaf with an integrable logarithmic connection and we have $\Delta(\bar{E}_j)H^{n-2} = 0$. Theorem 1.4 allows us to construct a filtration S_j^\bullet of \bar{E}_j such that the associated graded $\mathrm{Gr}_{S_j} \bar{E}_j$ is a slope H -semistable Higgs sheaf with $\Delta(\mathrm{Gr}_{S_j} \bar{E}_j)H^{n-2} = 0$. Again using Lemma 2.5, we see that $(\mathrm{Gr}_{S_j} \bar{E}_j)^{**}$ satisfies condition (1) of Theorem 2.1 and hence it is locally free.

Note that for general Y as above we have $(S_j^i)_Y \subset (\bar{E}_j)_Y$ and $(\mathrm{Gr}_{S_j} \bar{E}_j)_Y = \mathrm{Gr}_{(S_j)_Y} (\bar{E}_j)_Y$. Since E_j is locally free along Y , we have $(\bar{E}_j)_Y = (E_j)_Y$. Therefore the Chern classes of $(\mathrm{Gr}_{S_j} \bar{E}_j)_Y$ satisfy condition (3) of Theorem 2.2 and thus $(\mathrm{Gr}_{S_j} \bar{E}_j)_Y$ is locally free. So by Lemma 1.14 the sheaf $\mathrm{Gr}_{S_j} \bar{E}_j$ is locally free outside a finite number of points. Now we can use Lemma 1.12 to conclude that \bar{E}_j is locally free. This proves that $(\mathrm{Gr}^M E)^{**} = \bigoplus_j \bar{E}_j$ is locally free as required. ■

2.2. Local freeness for sheaves with trivial boundary

In this subsection we prove Theorem 2.2 for $D = 0$. The proof for $D \neq 0$ will be given in Subsection 3.5 after some study of monodromy filtrations along the irreducible components of D . The main difference between the cases $D = 0$ and $D \neq 0$ lies in the proof of Claim 2.2.2, which is much more complicated if $D \neq 0$.

Proof of Theorem 2.2 for $D = 0$. We prove the assertion by induction on the dimension n of X . Let us assume that $n = 2$. Then the equivalence of (1) and (2) is obvious since every reflexive sheaf on a smooth surface is locally free. The fact that (2) implies (3) is also obvious as the equality in (3) for $m = 1$ is trivial and for $m = 2$ it is equivalent to $\Delta(E) = 0$. The fact that (3) implies (1) follows from Lemma 2.5.

Now let us assume that $n \geq 3$ and that the equivalence of conditions (1)–(3) holds for varieties of dimension less than n . Replacing H by its multiple we can assume the following properties:

- (*) For every closed point $x \in X$, a general divisor $Y \in |H|$ passing through x is smooth and liftable to $W_2(k)$ (see Lemma 2.6).
- (**) For every slope H -stable Higgs sheaf (F, θ_F) (resp. a sheaf (F, ∇_F) with an integrable connection) of rank $\leq p$ on X and for every smooth divisor $Y \in |H|$ such that F_Y is torsion free, the restriction $(F_Y, (\theta_F)_Y)$ (resp. $(F_Y, (\nabla_F)_Y)$) is slope H_Y -stable (see Theorem 1.6 and Remark 1.8).
- (***) For every slope H -semistable Higgs sheaf (F, θ_F) (resp. a sheaf (F, ∇_F) with an integrable connection) of rank $\leq p$ on X and for a general divisor $Y \in |H|$, the restriction $(F_Y, (\theta_F)_Y)$ (resp. $(F_Y, (\nabla_F)_Y)$) is slope H_Y -semistable (see Corollary 1.7 and Remark 1.8).

First let us prove that (1) implies (2) and (3). Let (E, θ) (resp. (E, ∇)) be a reflexive rank $r \leq p$ slope H -semistable Higgs sheaf (resp. sheaf with an integrable connection) with $\Delta(E)H^{n-2} = 0$.

CLAIM 2.2.1. *We have $\Delta_i(E) = 0$ for $2 \leq i < n$.*

Proof. By (***), for a general divisor $Y \in |H|$, the restriction (E_Y, θ_Y) (resp. (E_Y, ∇_Y)) is slope H_Y -semistable. By Lemma 1.13 we also know that E_Y is reflexive. Since $\Delta(E_Y)H_Y^{n-3} = d \cdot \Delta(E)H^{n-2} = 0$, by the induction assumption E_Y is locally free and $c_m(E_Y) = \binom{r}{m} \left(\frac{c_1(E_Y)}{r}\right)^m$ in $H_{\text{et}}^{2m}(Y, \mathbb{Q}_l)$ for all $m \geq 1$ and any $l \neq p$. By Lemma 1.11 this implies $\Delta_i(E_Y) = 0$ for $2 \leq i < n$. By the Lefschetz hyperplane theorem, the inclusion $Y \hookrightarrow X$ induces injections $H_{\text{et}}^{2i}(X, \mathbb{Q}_l) \rightarrow H_{\text{et}}^{2i}(Y, \mathbb{Q}_l)$ for $i < n$, which proves the claim. ■

CLAIM 2.2.2. *If (E, θ) (resp. (E, ∇)) is slope H -stable then E is locally free.*

Proof. Since E is reflexive, for any smooth hypersurface $Y \in |H|$ the restriction E_Y is torsion free (see Lemma 1.13). So by (*) for every closed point $x \in X$ we can find $Y \in |H|$ passing through x such that Y is smooth and liftable to $W_2(k)$. By (**), for any such Y the restriction (E_Y, θ_Y) (resp. (E_Y, ∇_Y)) is slope H_Y -stable. Since $\Delta_i(E) = 0$ for $2 \leq i < n$ we get $\Delta_i(E_Y) = 0$ for $2 \leq i \leq \dim Y = n - 1$. So E_Y satisfies (3) and our induction assumption implies that E_Y is locally free. Then Lemma 1.14 implies that E is locally free at all points of Y . This shows that E is locally free on X . ■

Now we can prove that E is always locally free. Let M_\bullet be a Jordan–Hölder filtration of (E, θ) (or (E, ∇)) and let us set $E_i = \text{Gr}_i^M E$ and $r_i = \text{rk} E_i$. Then by Lemma 2.4 we know that $\Delta(E_i)H^{n-2} = 0$ and for all i we have $c_1(E_i) = \frac{r_i}{r} c_1(E)$ in $H_{\text{ét}}^2(X, \mathbb{Q}_l)$ for $l \neq p$. By (*) and (**) for a general smooth hypersurface $Y \in |H|$ the restriction $(E_i)_Y$ is slope H_Y -stable. Since E is reflexive, by Lemma 1.13 the restriction E_Y is also reflexive for general Y . Therefore by the induction assumption E_Y is locally free. Moreover, our induction assumption and Lemma 2.7 imply that all the factors $(E_i)_Y$ are also locally free. So by Lemma 1.14 all E_i are locally free outside a finite number of points.

However, we also know that E_i^{**} is slope H -stable and $\Delta(E_i^{**})H^{n-2} = 0$. So by Claim 2.2.2 all sheaves E_i^{**} are locally free. Hence we can apply Lemma 1.12 to conclude that E is locally free. This finishes the proof that (1) implies (2).

To finish the proof that (1) implies (3) note that by Theorem 1.3 there exists a decreasing filtration $E = N^0 \supset N^1 \supset \dots \supset N^m = 0$ such that $\theta(N^i) \subset N^{i-1} \otimes \Omega_X$ (in the case of integrable connections we use Simpson’s filtration) and the associated graded system (E_0, θ_0) of logarithmic Hodge sheaves is slope H -semistable. Let us recall that by Claim 2.2.1 we already know that $\Delta_i(E) = 0$ for $2 \leq i < n$. Hence $\Delta_i(E_0) = 0$ for $2 \leq i < n$. So by (*), (**) and Lemma 1.11, for a general divisor $Y \in |H|$ the restriction $(E_0)_Y$ satisfies (3) on Y . So by the induction assumption $(E_0)_Y$ is locally free, which by Lemma 1.14 implies that E_0 is locally free outside a finite number of points. Since we already know that (1) implies (2), we see that E_0^{**} is locally free. Then Lemma 1.12 implies that E_0 is locally free.

Now let us consider the canonical Higgs–de Rham sequence starting with (E_0, θ_0) (see Theorem 1.5)

$$\begin{array}{ccccc}
 & & (V_0, \nabla_0) & & (V_1, \nabla_1) & & \\
 & \nearrow^{C^{-1}} & & \searrow^{\text{Gr}_{S_0}} & & \nearrow^{C^{-1}} & \searrow^{\text{Gr}_{S_1}} \\
 (E_0, \theta_0) & & & & (E_1, \theta_1) & & \dots
 \end{array}$$

where for simplicity we write C^{-1} to denote the inverse Cartier transform. By definition each (V_m, ∇_m) is slope H -semistable and each (E_{m+1}, θ_{m+1}) is the slope H -semistable logarithmic system of Hodge sheaves associated to (V_i, ∇_i) via Simpson’s filtration.

CLAIM 2.2.3. *All E_m are locally free.*

Proof. Each $(E_m^{**}, \tilde{\theta}_m)$ is slope H -semistable, and since (1) implies (2) it is also locally free. Note also that $\Delta_i(E_m) = p^{im} \Delta(E) = 0$ for $i < n$, so the same argument as in the case of E_0 shows that each sheaf E_m is locally free outside a finite number of points. Now we prove by induction on m that E_m and V_m are locally free. For $m = 0$ we already know that $E_0 = E$ is locally free and hence so is $V_0 = C^{-1}(E_0)$. So let us assume that V_{m-1} is locally free. Then Lemma 1.12 implies that E_m is locally free and hence so is $V_m = C^{-1}(E_m)$, which finishes the induction. ■

Now let us write $p^m = r s_m + q_m$ for some non-negative integers s_m and $0 \leq q_m < r$. Let us set $(G_m, \theta_{G_m}) := (E_m, \tilde{\theta}_m) \otimes \det E^{-s_m}$. By [24, Lemma 2] we have

$\Delta_i(G_m) = \Delta_i(E_m) = p^{im} \Delta_i(E)$ for $1 \leq i \leq n$, so $\Delta(G_m) = 0$. Note also that $c_1(G_m) = q_m c_1(E)$ can take only finitely many values, so Theorem 1.2 implies that the family $\{(G_m, \theta_{G_m})\}_{m \geq 0}$ of locally free slope H -semistable Higgs sheaves is bounded. In particular, the set $\{\Delta_n(G_m)\}_{m \geq 0} = \{p^{nm} \Delta_n(E)\}_{m \geq 0}$ is finite. Hence $\Delta_n(E) = 0$, which finishes the proof of vanishing of $\Delta_i(E)$ for all $2 \leq i \leq n$. Now Lemma 1.11 implies that for all $m \geq 1$,

$$c_m(E) = \frac{\binom{r}{m}}{r^m} c_1(E)^m$$

in $H_{\text{et}}^{2m}(X, \mathbb{Q}_l)$. This finishes the proof that (1) implies (3).

Clearly (2) implies (1), so it is sufficient to prove that (3) implies (1). Let us consider a rank $r \leq p$ slope H -semistable Higgs sheaf (E, θ) such that $c_m(E) = \frac{\binom{r}{m}}{r^m} c_1(E)^m$ for all $m \geq 2$. By Lemma 2.5 we have $\Delta(E^{**})H^{n-2} = 0$. Since (1) implies (3) we know that $(E^{**}, \tilde{\theta})$ satisfies $c_m(E^{**}) = \frac{\binom{r}{m}}{r^m} c_1(E^{**})^m$ for all $m \geq 1$. As in the proof of Lemma 2.5 we see that $c_1(E^{**}) = c_1(E)$. So our assumptions imply that $c_m(E^{**}) = c_m(E)$ for all $m \geq 1$. Since E and E^{**} have the same rank, the Riemann–Roch theorem implies that the Hilbert polynomials of E and E^{**} are equal. Let $T = E^{**}/E$. Then the short exact sequence

$$0 \rightarrow E(m) \rightarrow E^{**}(m) \rightarrow T(m) \rightarrow 0$$

shows that the Hilbert polynomial of T is trivial. So $T = 0$ and E is reflexive. For a sheaf with an integrable connection the proof of (3) \Rightarrow (1) is exactly the same. ■

3. Nearby cycles

The main aim of this section is to understand the restriction of an integrable logarithmic connection (or a logarithmic Higgs sheaf) to the boundary divisor. We need this in order to prove Theorem 2.2 in the case of non-trivial boundary. The proof is completed in Subsection 3.5.

For Hodge structures on complex varieties the restriction to the boundary divisor is understood by means of a nearby-cycle functor for the category of real graded-polarized families of mixed Hodge structures (see [3, Section 4]). Here we use a different approach that allows us to keep more information about the restrictions. As in [3] this construction is related to the standard constructions of a nearby-cycles functor going back to Grothendieck, Deligne and Saito.

In this section we will use some basic facts and definitions related to Lie algebroids, for which we refer to [23].

3.1. Nearby-cycles functor

Let X be a smooth projective variety of dimension n defined over an algebraically closed field k . Let D be a simple normal crossing divisor on X and let Y be an irreducible component of D .

Let $\iota : Y \hookrightarrow X$ be the canonical embedding. We define a Lie algebroid \mathbb{L}_Y on Y as the triple $(L, [\cdot, \cdot], \alpha)$, where $L = \iota^*T_X(\log D)$ is a locally free \mathcal{O}_Y -module with the Lie algebra structure induced from the standard Lie algebra structure on T_X , and the anchor map $\alpha : L \rightarrow \text{Der}_k(\mathcal{O}_Y) = T_Y$ is the canonical map induced by ι . The anchor map induces a k -derivation $d_{\Omega_{\mathbb{L}_Y}} : \mathcal{O}_Y \rightarrow \Omega_{\mathbb{L}_Y} = L^*$.

Giving an \mathbb{L}_Y -module structure $\mathbb{L}_Y \rightarrow \text{End}_k E$ on a coherent \mathcal{O}_Y -module E is equivalent to giving an integrable $d_{\Omega_{\mathbb{L}_Y}}$ -connection $\nabla_{\mathbb{L}_Y} : E \rightarrow E \otimes_{\mathcal{O}_Y} \Omega_{\mathbb{L}_Y}$ (see [23, Lemma 3.2]).

The usefulness of the above construction comes from the fact that the restriction to Y defines an obvious functor

$$\Psi_Y : \text{MIC}(X, D) \rightarrow \mathbb{L}_Y\text{-Mod}$$

from the category $\text{MIC}(X, D)$ of coherent \mathcal{O}_X -modules with an integrable logarithmic connection on (X, D) to the category $\mathbb{L}_Y\text{-Mod}$ of coherent \mathcal{O}_Y -modules with an \mathbb{L}_Y -module structure. If (V, ∇) is a coherent \mathcal{O}_X -module with an integrable connection on (X, D) then $\Psi_Y(V, \nabla)$ is defined as the restriction ι^*V of V to Y , and the \mathbb{L}_Y -module structure is given by the integrable $d_{\Omega_{\mathbb{L}_Y}}$ -connection $\iota^*\nabla : \iota^*V \rightarrow \iota^*V \otimes \iota^*\Omega_X(\log D)$. By an abuse of notation we will often write $(V_Y, \nabla|_Y)$ to denote $\Psi_Y(V, \nabla)$.

Let \mathbb{L}_Y^0 be the trivial Lie algebroid underlying \mathbb{L}_Y (i.e., we consider the same $L = \iota^*T_X(\log D)$ but with zero Lie bracket and zero anchor map). As above, we get the functor

$$\Phi_Y : \text{HIG}(X, D) \rightarrow \mathbb{L}_Y^0\text{-Mod}$$

from the category $\text{HIG}(X, D)$ of coherent \mathcal{O}_X -modules with a logarithmic Higgs field on (X, D) to the category $\mathbb{L}_Y^0\text{-Mod}$ of coherent \mathcal{O}_Y -modules with an \mathbb{L}_Y^0 -module structure. Note that $\mathbb{L}_Y^0\text{-Mod}$ is the same as the category of coherent \mathcal{O}_Y -modules with a $\text{Sym}^\bullet(\iota^*T_X(\log D))$ -module structure. As above, we will often write $(E_Y, \theta|_Y)$ for $\Phi_Y(E, \theta)$.

3.2. General monodromy filtrations

Let Y be a smooth projective variety of dimension n defined over an algebraically closed field k . Let \mathbb{L} be a smooth Lie algebroid on Y/k and let E be an \mathbb{L} -module.

Let $N : E \rightarrow E$ be a nilpotent endomorphism of \mathbb{L} -modules. By [5, Proposition 1.6.1], N induces on E a unique finite increasing filtration M_\bullet by \mathbb{L} -submodules such that

- (1) $N(M_i) \subset M_{i-2}$ for all i ,
- (2) N^i induces an isomorphism $\text{Gr}_i^M E \xrightarrow{\simeq} \text{Gr}_{-i}^M E$ for all $i \geq 0$.

We call M_\bullet the *monodromy filtration* for the \mathbb{L} -module E .

Let us define the j -th primitive part $P_j(E)$ of E as the kernel of $N^{j+1} : \text{Gr}_j^M E \rightarrow \text{Gr}_{-j-2}^M E$ for $j \geq 0$ and $P_j(E) = 0$ for $j < 0$. Then by [5, (1.6.4)] we have the decomposition into primitive parts

$$\text{Gr}_j^M E = \bigoplus_{i \geq \max(0, -j)} N^i P_{j+2i}(E) \simeq \bigoplus_{i \geq \max(0, -j)} P_{j+2i}(E). \tag{3.1}$$

Lemma 3.1. *If E is torsion free (as an \mathcal{O}_Y -module) then all quotients $\text{Gr}_j^M E$ are also torsion free.*

Proof. The proof is by induction on the rank of E . If E has rank 1 then N is nilpotent if and only if $N = 0$, so the filtration is trivial.

Now let us assume that the assertion holds for all sheaves of rank less than the rank of E . If $N = 0$ the assertion is trivial, so we can assume that $N \neq 0$. Since for $j \geq 0$ the map N^j induces an isomorphism $\text{Gr}_j^M E \xrightarrow{\cong} \text{Gr}_{-j}^M E$, the image $N^j P_j(E)$ is the kernel of $N : \text{Gr}_{-j}^M E \rightarrow \text{Gr}_{-j-2}^M E$. By [5, Corollaire (1.6.6)] the associated graded of the filtration induced by M_\bullet on $\ker N$ satisfies

$$\text{Gr}_{-j}^M(\ker N) \xrightarrow{\cong} N^j P_j(E) \simeq P_j(E).$$

But $\ker N \subset E$ is torsion free and since N is nilpotent, the rank of $\ker N$ is less than the rank of E . So by the induction assumption all quotients $\text{Gr}_{-j}^M(\ker N)$ are torsion free. Hence all $P_j(E)$ are torsion free and by the decomposition (3.1) all $\text{Gr}_j^M E$ are also torsion free. ■

Now let us fix an ample divisor H on Y . If an \mathbb{L} -module E is slope H -semistable then we always assume that it is torsion free as an \mathcal{O}_Y -module. The following lemma proves that the monodromy filtration (or the filtration by primitive cohomology) of a slope H -semistable \mathbb{L} -module can always be refined to a Jordan–Hölder filtration.

Lemma 3.2. *Let E be a slope H -semistable \mathbb{L} -module. Then every quotient $\text{Gr}_j^M E$ of the monodromy filtration M_\bullet of E is slope H -semistable with $\mu_H(\text{Gr}_j^M E) = \mu_H(E)$. Moreover, all $P_j(E)$ are slope H -semistable with $\mu_H(P_j(E)) = \mu_H(E)$.*

Proof. The proof is by induction on the rank of E . For rank 1 the assertion is clear so assume that it holds for all sheaves of rank less than the rank of E .

Let d be the largest integer such that $M_{-d} \neq 0$. Since E is slope H -semistable we have $\mu_H(M_{-d}) \leq \mu_H(E)$. But N^d induces an isomorphism

$$\text{Gr}_d^M E = E/M_{d-1} \xrightarrow{\cong} \text{Gr}_{-d}^M E = M_{-d}$$

and by slope H -semistability of E we get $\mu_H(M_{-d}) = \mu_H(E/M_{d-1}) \geq \mu_H(E)$. Hence $\mu_H(M_{-d}) = \mu_H(E)$ and M_{-d} is slope H -semistable. So $E/M_{d-1} \simeq M_{-d}$ is also slope H -semistable with $\mu_H(E/M_{d-1}) = \mu_H(E)$. This shows that M_{d-1} is slope H -semistable with $\mu_H(M_{d-1}) = \mu_H(E)$. Note also that M_{d-1}/M_{-d} is torsion free by Lemma 3.1. Since $\mu_H(M_{-d}) = \mu_H(M_{d-1}) = \mu_H(E)$, this implies that M_{d-1}/M_{-d} is slope H -semistable with $\mu_H(M_{d-1}/M_{-d}) = \mu_H(E)$. But N induces on M_{d-1}/M_{-d} a nilpotent endomorphism whose quotients coincide with the remaining quotients of the monodromy filtration M_\bullet of E . Hence by the induction assumption all $\text{Gr}_j^M E$ are slope H -semistable with $\mu_H(\text{Gr}_j^M E) = \mu_H(E)$.

The second assertion follows immediately from the first one and the decomposition (3.1) of $\text{Gr}_j^M E$ into primitive parts. ■

3.3. Residue maps

Let X be a smooth projective variety of dimension n defined over an algebraically closed field k . Let D be a simple normal crossing divisor on X and let Y be an irreducible component of D . We can write $D = D' + Y$ for some divisor D' which does not contain Y . In the following we denote D' by $D - Y$ and set $D^Y = (D - Y)|_Y$.

Note that \mathbb{L}_Y (see Subsection 3.1) is equipped with the canonical map $\text{Res} : \Omega_{\mathbb{L}_Y} = \iota^* \Omega_X(\log D) \rightarrow \mathcal{O}_Y$ given by the Poincaré residue. Using it for any \mathbb{L}_Y -module E we can define the *residue endomorphism* Res_E as a composition

$$E \xrightarrow{\nabla_{\mathbb{L}_Y}} E \otimes_{\mathcal{O}_Y} \Omega_{\mathbb{L}_Y} \xrightarrow{\text{Id}_E \otimes \text{Res}} E \otimes_{\mathcal{O}_Y} \mathcal{O}_Y = E.$$

Since $\text{Res} \circ d_{\Omega_{\mathbb{L}_Y}} = 0$, this endomorphism is \mathcal{O}_Y -linear. It is easy to check that Res_E is an endomorphism of \mathbb{L}_Y -modules. In the same way we can define the residue endomorphism of an \mathbb{L}_Y^0 -module.

Let $\mathbb{L}_Y\text{-Mod}_0$ (resp. $\mathbb{L}_Y^0\text{-Mod}_0$) be the full subcategory of $\mathbb{L}_Y\text{-Mod}$ (resp. $\mathbb{L}_Y^0\text{-Mod}$) containing as objects all \mathbb{L}_Y -modules E (resp. \mathbb{L}_Y^0 -modules) with $\text{Res}_E = 0$.

Similarly, let $\mathbb{L}_Y\text{-Mod}_{\text{nil}}$ (resp. $\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}$) be the full subcategory of $\mathbb{L}_Y\text{-Mod}$ (resp. $\mathbb{L}_Y^0\text{-Mod}$) containing as objects all \mathbb{L}_Y -modules E (resp. \mathbb{L}_Y^0 -modules E) that have nilpotent residue Res_E .

Lemma 3.3. *The category $\mathbb{L}_Y\text{-Mod}_0$ is equivalent to the category $\text{MIC}(Y, D^Y)$. Similarly, the category $\mathbb{L}_Y^0\text{-Mod}_0$ is equivalent to the category $\text{HIG}(Y, D^Y)$. Moreover, we have natural functors*

$$\Upsilon : \mathbb{L}_Y\text{-Mod}_{\text{nil}} \rightarrow \text{MIC}(Y, D^Y)$$

given by sending E to $\text{Gr}^W E$, where W_\bullet is the monodromy filtration of Res_E , and

$$\Upsilon^0 : \mathbb{L}_Y^0\text{-Mod}_{\text{nil}} \rightarrow \text{HIG}(Y, D^Y)$$

given by sending E to $\text{Gr}^M E$, where M_\bullet is the monodromy filtration of Res_E .

Proof. The short exact sequence

$$0 \rightarrow \Omega_Y(\log D^Y) \rightarrow \iota^* \Omega_X(\log D) \xrightarrow{\text{Res}} \mathcal{O}_Y \rightarrow 0$$

shows that an \mathbb{L}_Y -module E with $\text{Res}_E = 0$ gives rise to a canonically defined integrable logarithmic connection $E \rightarrow E \otimes_{\mathcal{O}_Y} \Omega_Y(\log D^Y)$. Conversely, if (V, ∇) is an element of $\text{MIC}(Y, D^Y)$ then ∇ defines an integrable $d_{\Omega_{\mathbb{L}_Y}}$ -connection, so we get an \mathbb{L}_Y -module V with $\text{Res}_V = 0$. If E is an \mathbb{L}_Y^0 -module with $\text{Res}_E = 0$ then the same argument shows that E is a logarithmic Higgs sheaf on (Y, D^Y) . This shows the first part of the lemma.

Now let us assume that E is an \mathbb{L}_Y -module with nilpotent $N = \text{Res}_E$. Let W_\bullet be the corresponding monodromy filtration (in the category of \mathbb{L}_Y -modules). Note that the composition $W_i \xrightarrow{\text{Res}_{W_i}} W_i \rightarrow W_i/W_{i-1}$ is zero as $N(W_i) \subset W_{i-1}$. Hence $\text{Res}_{\text{Gr}_i^W E} = 0$ and each quotient $\text{Gr}_i^W E$ is endowed with an integrable logarithmic connection ∇_i^W on

(Y, D^Y) . Similarly, for an \mathbb{L}_Y^0 -module E with nilpotent $N = \text{Res}_E$ all quotients $\text{Gr}_i^M E$ of the monodromy filtration M_\bullet have canonically defined structure of a logarithmic Higgs sheaf $(\text{Gr}_i^M E, \theta_i^M)$ on (Y, D^Y) . ■

Let D_i be an irreducible component of D different from Y . Let $\text{Res}_{D_i} : \Omega_X(\log D) \rightarrow \mathcal{O}_{D_i}$ be the Poincaré residue along D_i . Pulling it back to Y we get an \mathcal{O}_Y -linear map $\text{Res}_Y^{D_i} : \Omega_{\mathbb{L}_Y} = \iota^* \Omega_X(\log D) \rightarrow \mathcal{O}_{D_i^Y}$, where $D_i^Y = D_i \cap Y$. Now for any \mathbb{L}_Y -module E we consider the composition

$$E \xrightarrow{\nabla_{\mathbb{L}_Y}} E \otimes_{\mathcal{O}_Y} \Omega_{\mathbb{L}_Y} \xrightarrow{\text{Id}_E \otimes \text{Res}_Y^{D_i}} E \otimes_{\mathcal{O}_Y} \mathcal{O}_{D_i^Y} = E_{D_i^Y}.$$

One can easily check that this map is \mathcal{O}_Y -linear and it factors through the restriction map $E \rightarrow E_{D_i^Y}$. Therefore it defines a map $\text{Res}_E^{D_i} : E_{D_i^Y} \rightarrow E_{D_i^Y}$ that we call the *residue map of E along D_i* . In the same way we can define the residue maps along D_i for any \mathbb{L}_Y^0 -module E .

Remark 3.4. Let (V, ∇) be an object of $\text{MIC}(X, D)$ and let $E = \Psi_Y(V, \nabla)$ be the corresponding \mathbb{L}_Y -module. Then the residue map $\text{Res}_E : E \rightarrow E$ coincides with the residue map $\text{Res}_Y \nabla : V_Y \rightarrow V_Y$. Similarly, for any irreducible component D_i of $D - Y$ the residue map $\text{Res}_E^{D_i} : E_{D_i^Y} \rightarrow E_{D_i^Y}$ coincides with the restriction of the residue map $\text{Res}_{D_i} \nabla : V_{D_i} \rightarrow V_{D_i}$ to D_i^Y .

3.4. Compatibility of the Cartier transform with monodromy filtrations

Let X be a smooth projective variety of dimension n defined over an algebraically closed field k and let D be a simple normal crossing divisor on X . Let Y be an irreducible component of D .

Let $Z = \mathbb{V}(\mathcal{O}_Y(-Y))$ be the total space of the normal bundle of $\iota : Y \hookrightarrow X$ and let $\pi : Z \rightarrow Y$ be the canonical projection. Let $s : Y \rightarrow Z$ be the zero section and let Y_0 be its image.

Lemma 3.5. *Set $D^Z = Y_0 + \pi^{-1}(D^Y)$. The short exact sequence*

$$0 \rightarrow \pi^* \Omega_Y(\log D^Y) \rightarrow \Omega_Z(\log D^Z) \rightarrow \Omega_{Z/Y}(\log Y_0) = \mathcal{O}_Z \rightarrow 0$$

is the pullback of

$$0 \rightarrow \Omega_Y(\log D^Y) \rightarrow \iota^* \Omega_X(\log D) \xrightarrow{\text{Res}_Y} \mathcal{O}_Y \rightarrow 0.$$

Proof. Let us recall that the extension class of

$$0 \rightarrow \Omega_Y \rightarrow \iota^* \Omega_X(\log Y) \xrightarrow{\text{Res}_Y} \mathcal{O}_Y \rightarrow 0$$

in $\text{Ext}^1(\mathcal{O}_Y, \Omega_Y) = H^1(\Omega_Y)$ is equal to the Atiyah class of $\mathcal{O}_Y(-Y)$, which is also the image of the class of $\mathcal{O}_Y(-Y)$ in $H^1(\mathcal{O}_Y^*)$ under the map $H^1(\mathcal{O}_Y^*) \rightarrow H^1(\Omega_Y)$. Hence

by [35, Proposition 3.3] the pullback of the above sequence to Z induces

$$0 \rightarrow \pi^* \Omega_Y \rightarrow \Omega_Z(\log Y_0) \rightarrow \mathcal{O}_Z \rightarrow 0.$$

Let $\{D_i\}$ be the divisors corresponding to the irreducible components of $D - Y$. Now the required assertion follows from the following standard exact sequences:

$$\begin{aligned} 0 \rightarrow \Omega_Y \rightarrow \Omega_Y(\log D^Y) \rightarrow \bigoplus \mathcal{O}_{D_i \cap Y} \rightarrow 0, \\ 0 \rightarrow i^* \Omega_X(\log Y) \rightarrow i^* \Omega_X(\log D) \rightarrow \bigoplus \mathcal{O}_{D_i \cap Y} \rightarrow 0, \\ 0 \rightarrow \Omega_Z(\log Y_0) \rightarrow \Omega_Z(\log D^Z) \rightarrow \bigoplus \mathcal{O}_{\pi^{-1}(D_i \cap Y)} \rightarrow 0. \quad \blacksquare \end{aligned}$$

An alternative proof of the lemma can be obtained, e.g., by directly making a local calculation and checking equality of the corresponding gluing conditions (cf. [35, proof of Proposition 3.3]).

Let (V, ∇) be a coherent \mathcal{O}_X -module with an integrable logarithmic connection $\nabla : V \rightarrow V \otimes \Omega_X(\log D)$. After restricting to Y we see that V_Y acquires an integrable $\Omega_{\mathbb{L}_Y}$ -connection. After a further pullback to Z we get an induced integrable logarithmic connection

$$\nabla' : \pi^* V_Y \rightarrow \pi^* V_Y \otimes \Omega_Z(\log D^Z).$$

The same construction allows us to associate to a logarithmic Higgs sheaf (E, θ) on (X, D) a logarithmic Higgs sheaf $(\pi^* E_Y, \theta')$ on (Z, D^Z) . Note that if θ is nilpotent then so is θ' .

Remark 3.6. One could naively hope that one can work with logarithmic connections on projective varieties by pulling back the \mathbb{L}_Y -module $(V_Y, \nabla|_Y)$ via $\varphi : T = \mathbb{P}(\mathcal{O}_Y(-Y) \oplus \mathcal{O}_Y) \rightarrow Y$. Indeed, one has a short exact sequence

$$\begin{aligned} 0 \rightarrow \varphi^* \Omega_Y(\log D^Y) \rightarrow \Omega_T(\log Y_0 + Y_\infty + \pi^{-1}(D^Y)) \\ \rightarrow \Omega_{T/Y}(\log Y_0 + Y_\infty) = \mathcal{O}_T \rightarrow 0, \end{aligned}$$

where $Y_\infty = T - Z$ is the image of the infinity section. But if $p \neq 2$ then

$$0 \rightarrow \varphi^* \Omega_Y(\log D^Y) \rightarrow \varphi^*(i^* \Omega_X(\log D)) \rightarrow \mathcal{O}_T \rightarrow 0$$

defines a different extension class. This can be seen by computing the extension class of both sequences after restricting to Y_∞ . This forces us to deal with non-projective varieties, where the difficulty is that one cannot directly apply Theorem 2.2.

Let us assume that the base field k has characteristic p and (X, D) is liftable to $W_2(k)$. Let us fix a lifting (\tilde{X}, \tilde{D}) . This lifting induces a lifting (\tilde{Y}, \tilde{D}^Y) of (Y, D^Y) to $W_2(k)$ and also a compatible lifting (\tilde{Z}, \tilde{D}^Z) of (Z, D^Z) to $W_2(k)$.

The following lemma is functoriality of Cartier transforms in a situation that is not covered by Theorem 6.4 (we do not even have a map $(Z, D^Z) \rightarrow (X, D)$).

Lemma 3.7. *Let (E, θ) be a reflexive logarithmic Higgs sheaf on (X, D) with a nilpotent Higgs field of level $\leq p - 1$. If $(V, \nabla) = C_{(\tilde{X}, \tilde{D})}^{-1}(E, \theta)$ then we have a canonical isomorphism $(\pi^*V_Y, \nabla') \simeq C_{(\tilde{Z}, \tilde{D}^Z)}^{-1}(\pi^*E_Y, \theta')$, and the diagram*

$$\begin{array}{ccc}
 (\pi^*V_Y, \nabla') & \xrightarrow{\simeq} & C_{(\tilde{Z}, \tilde{D}^Z)}^{-1}(\pi^*E_Y, \theta') \\
 \pi^* \text{Res}_Y \nabla \downarrow & & \downarrow C_{(\tilde{Z}, \tilde{D}^Z)}^{-1}(\pi^* \text{Res}_Y \theta) \\
 (\pi^*V_Y, \nabla') & \xrightarrow{\simeq} & C_{(\tilde{Z}, \tilde{D}^Z)}^{-1}(\pi^*E_Y, \theta')
 \end{array}$$

is commutative.

Proof. Since E is reflexive and Y is smooth, E_Y is torsion free. Since π is flat, π^*E_Y is also torsion free, so we can apply $C_{(\tilde{Z}, \tilde{D}^Z)}^{-1}$ to (π^*E_Y, θ') . We will use the notation introduced in the proof of Theorem 6.4.

There exists an affine covering $\{\tilde{U}_\alpha\}_{\alpha \in I}$ of \tilde{X} such that for each \tilde{U}_α we have a system of logarithmic coordinates, i.e., x_1, \dots, x_n such that $\tilde{D} \cap \tilde{U}_\alpha$ is given by $\prod_{i=1}^{n_0} x_i = 0$, with $x_1 = 0$ giving $\tilde{Y} \cap \tilde{U}_\alpha$. We can assume that $\mathcal{O}_{\tilde{U}_\alpha}(-\tilde{Y})$ is trivial and choose for each α its generator t . Let us also choose standard log Frobenius liftings $\tilde{F}_{U_\alpha} : \tilde{U}_\alpha \rightarrow \tilde{U}_\alpha$ so that $\tilde{F}_{U_\alpha}^*(x_i) = x_i^p$. Then the projection $\tilde{V}_\alpha := \pi^{-1}(\tilde{U}_\alpha \cap \tilde{Y}) \rightarrow \tilde{U}_\alpha \cap \tilde{Y}$ corresponds to the projection $(\tilde{U}_\alpha \cap \tilde{Y}) \times_{W_2(k)} \text{Spec } W_2(k)[t] \rightarrow \tilde{U}_\alpha \cap \tilde{Y}$. We choose a logarithmic Frobenius lifting of $(\tilde{V}_\alpha, \tilde{D}^Z \cap \tilde{V}_\alpha)$ to be $\tilde{F}_{V_\alpha} = \tilde{F}_{\tilde{U}_\alpha \cap \tilde{Y}} \times \tilde{F}_{\text{Spec } k[t]}$, where $\tilde{F}_{\text{Spec } k[t]}$ is given by $t \mapsto t^p$. Note that $\tilde{D}^Z \cap \tilde{V}_\alpha$ is given by $\prod_{i=1}^{n_0} x_i t = 0$. We can locally write

$$\theta|_{U_\alpha} = \sum_{i=1}^{n_0} \theta_i \otimes \frac{dx_i}{x_i} + \sum_{i=n_0+1}^n \theta_i \otimes dx_i,$$

where $\theta_i : E_{U_\alpha} \rightarrow E_{U_\alpha}$ are some commuting endomorphisms. This allows us to identify $C_{(\tilde{Z}, \tilde{D}^Z)}^{-1}(\pi^*E_Y, \theta')$. Over each \tilde{V}_α we have $F^*(\pi^*(E_{U_\alpha \cap Y}))$ with the connection given by

$$\nabla_\alpha := \nabla_{\text{can}} + (\text{Id} \otimes \zeta_\alpha) \circ (F^*\pi^*(\theta|_Y)),$$

where $\zeta_\alpha = \frac{d\tilde{F}_{V_\alpha}}{p}$. The isomorphism from Lemma 3.5 is locally given by $\pi^*(\frac{dx_1}{x_1}|_Y) = \frac{dt}{t}$, $\pi^*(\frac{dx_i}{x_i}|_Y) = \frac{dx_i}{x_i}$ for $2 \leq i \leq n_0$ and $\pi^*(dx_i|_Y) = dx_i$ for $n_0 \leq i \leq n$. So we get

$$\begin{aligned}
 \nabla_\alpha &:= \nabla_{\text{can}} + F^*\pi^*(\theta_1|_Y) \otimes \frac{dt}{t} + \sum_{i=2}^{n_0} F^*\pi^*(\theta_i|_Y) \otimes \frac{dx_i}{x_i} \\
 &+ \sum_{i=n_0+1}^n F^*\pi^*(\theta_i|_Y) \otimes x_i^{p-1} dx_i.
 \end{aligned}$$

On the other hand, locally on U_α , (V, ∇) can be identified with $F^*E_{U_\alpha}$ with the connection given by

$$\nabla|_{U_\alpha} := \nabla_{\text{can}} + (\text{Id} \otimes \zeta'_\alpha) \circ (F^*\theta),$$

where $\zeta'_\alpha = \frac{d\tilde{F}_{U_\alpha}}{p}$. Writing down this formula in local coordinates we get

$$\nabla|_{U_\alpha} = \nabla_{\text{can}} + \sum_{i=1}^{n_0} F^*\theta_i \otimes \frac{dx_i}{x_i} + \sum_{i=n_0+1}^n F^*\theta_i \otimes x_i^{p-1} dx_i.$$

Using the equality $\nabla_{\text{can}} = \pi^*(\nabla_{\text{can}}|_Y)$ and the above formulas we get $\nabla_\alpha = \pi^*(\nabla|_{U_\alpha \cap Y})$. Checking equality of gluing conditions is similar and left to the reader.

Finally, note that since the isomorphism $(\pi^*V_Y, \nabla') \simeq C_{(\tilde{Z}, \tilde{D}_Z)}^{-1}(\pi^*E_Y, \theta')$ is functorial with respect to open embeddings $V_\alpha \subset Z$, it is sufficient to check the commutativity of the diagram only locally. In the local situation this follows easily from the local equalities

$$\text{Res}_Y \nabla|_{U_\alpha} = F^*(\text{Res}_Y \theta|_{U_\alpha}). \quad \blacksquare$$

Let (E, θ) be a logarithmic Higgs sheaf on (X, D) with a nilpotent Higgs field. Let M_\bullet be the monodromy filtration for $\text{Res}_Y \theta$. Then each quotient $\text{Gr}_i^M E_Y$ is endowed with a nilpotent logarithmic Higgs field θ_i^M on (Y, D^Y) .

Let (V, ∇) be an object of $\text{MIC}(X, D)$. Assume that the residue $\text{Res}_Y \nabla$ is nilpotent and let W_\bullet be the monodromy filtration for $\text{Res}_Y \nabla$. Then each quotient $\text{Gr}_i^W V_Y$ is endowed with a nilpotent integrable logarithmic connection ∇_i^W on (Y, D^Y) .

Proposition 3.8. *Let (E, θ) be a reflexive logarithmic Higgs sheaf on (X, D) with a nilpotent Higgs field of level $\leq p - 1$ and let $(V, \nabla) = C_{(X, \tilde{D})}^{-1}(E, \theta)$. Let M_\bullet be the monodromy filtration for $\text{Res}_Y \theta$ and let W_\bullet be the monodromy filtration for $\text{Res}_Y \nabla$. Then $(\text{Gr}_i^M E_Y, \theta_i^M)$ is a torsion free logarithmic Higgs sheaf on (Y, D^Y) with a nilpotent Higgs field of level $\leq p - 1$ and we have*

$$(\text{Gr}_i^W V_Y, \nabla_i^W) = C_{(\tilde{Y}, \tilde{D}^Y)}^{-1}(\text{Gr}_i^M E_Y, \theta_i^M).$$

Proof. Note that π^*M_\bullet is a filtration of (π^*E_Y, θ') by logarithmic Higgs submodules on (Z, D^Z) . Moreover, quotients of this filtration are logarithmic Higgs modules on $(Z, \pi^{-1}(D^Y))$. Similarly, π^*W_\bullet is a filtration of (π^*V_Y, ∇') by integrable logarithmic connections on (Z, D^Z) and the quotients are objects of $\text{MIC}(Z, \pi^{-1}(D^Y))$.

Lemma 3.7 and uniqueness of the monodromy filtrations imply that

$$(\pi^*W_i, \nabla'_i) = C_{(\tilde{Z}, \tilde{D}_Z)}^{-1}(\pi^*M_i, \theta'_i),$$

where ∇'_i and θ'_i denote the restrictions of ∇' and θ' to the corresponding subsheaves. But this implies that

$$\pi^*(\text{Gr}_i^W V_Y, \nabla_i^W) = C_{(\tilde{Z}, \tilde{D}_Z)}^{-1} \pi^*(\text{Gr}_i^M E_Y, \theta_i^M) = C_{(\tilde{Z}, \pi^{-1}(D^Y))}^{-1} \pi^*(\text{Gr}_i^M E_Y, \theta_i^M).$$

Pulling back this equality by the zero section $s : (Y, D^Y) \rightarrow (Z, \pi^{-1}(Y))$ and using functoriality of the Cartier transform, we get the required assertion. \blacksquare

3.5. Local freeness for sheaves: the general case

In this subsection we give a full proof of Theorem 2.2. The strategy is similar to that in Section 2.2.

Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k of characteristic p and let D be a simple normal crossing divisor on X . We assume that $D \subset X$ admits a lifting to $W_2(k)$. Replacing H by its multiple we can also assume the following properties:

- (*) For every closed point $x \in U := X - \text{Supp } D$ and a general divisor $Y \in |H|$ passing through x , the pair $(Y, D_Y = D \cap Y)$ is log smooth and liftable to $W_2(k)$ (see Lemma 2.6).
- (**) For every slope H -stable logarithmic Higgs sheaf (F, θ_F) (resp. a sheaf (F, ∇_F) with an integrable logarithmic connection) of rank $\leq p$ on (X, D) and for every smooth divisor $Y \in |H|$ such that F_Y is torsion free and $D_Y = D \cap Y$ is a normal crossing divisor on Y , the restriction $(F_Y, (\theta_F)_Y)$ (resp. $(F_Y, (\nabla_F)_Y)$) is slope H_Y -stable (see Theorem 1.6 and Remark 1.8).
- (***) For every slope H -semistable logarithmic Higgs sheaf (F, θ_F) (resp. a sheaf (F, ∇_F) with an integrable logarithmic connection) of rank $\leq p$ on X and for a general divisor $Y \in |H|$, the restriction $(F_Y, (\theta_F)_Y)$ (resp. $(F_Y, (\nabla_F)_Y)$) is slope H_Y -semistable (see Corollary 1.7 and Remark 1.8).

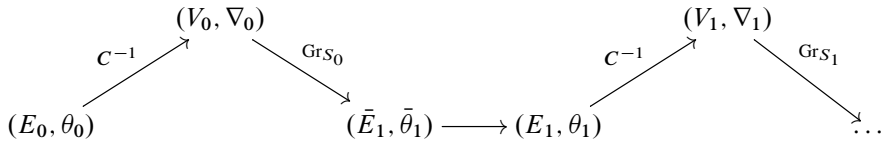
The proof of the following lemma is very similar to that of Claim 2.2.2.

Lemma 3.9. *Let (E, θ) be a reflexive rank $r \leq p$ slope H -stable logarithmic Higgs sheaf with $\Delta(E)H^{n-2} = 0$. If Theorem 2.2 holds for varieties of dimension less than n then E is locally free on U and locally free outside a finite number of points of X .*

Proof. If $n = 2$ then every reflexive sheaf on X is locally free, so we can assume that $n \geq 3$. The same arguments as for Claim 2.2.1 show that $\Delta_i(E)H^{n-2} = 0$ for $2 \leq i < n$ (here we use (***)). Since E is reflexive, the restriction E_Y is torsion free for any smooth hypersurface $Y \in |H|$ (see Lemma 1.13). So by (*) and (**), for every closed point $x \in U$ we can find $Y \in |H|$ passing through x such that the pair $(Y, D_Y = D \cap Y)$ is log smooth and liftable to $W_2(k)$ and the restriction (E_Y, θ_Y) is slope H_Y -stable. Since $\Delta_i(E) = 0$ for $2 \leq i \leq n$ we get $\Delta_i(E_Y) = 0$ for $2 \leq i \leq \dim Y = n - 1$. So E_Y satisfies condition (3) of Theorem 2.2 and our assumption implies that E_Y is locally free. Then Lemma 1.14 implies that E is locally free at all points of Y (and locally free outside a finite number of points of X). This shows that E is locally free on U . ■

Lemma 3.10. *Let (E_0, θ_0) be a reflexive rank $r \leq p$ slope H -stable system of logarithmic Hodge sheaves with $\Delta(E_0)H^{n-2} = 0$. If Theorem 2.2 holds for varieties of dimension less than n then E_0 is locally free. Moreover, for every irreducible component Y of D the \mathbb{L}_Y^0 -module $((E_0)_Y, \theta_0|_Y)$ is slope H_Y -semistable.*

Proof. Let us construct a certain sequence analogous to the canonical Higgs–de Rham sequence of (E_0, θ_0) in the following way. We set $(V_0, \nabla_0) := C_{(\bar{X}, \bar{D})}^{-1}(E_0, \theta_0)$. Let S_0^\bullet be Simpson’s (decreasing) filtration on (V_0, ∇_0) and let $(\bar{E}_1 = \text{Gr}_{S_0} V_0, \bar{\theta}_1)$ be the associated system of Hodge sheaves. Then we set $(E_1, \theta_1) := ((\bar{E}_1)^{**}, \bar{\theta}_1^{**})$ and then we repeat the procedure. In this way we get the following sequence:



in which each logarithmic Higgs sheaf (E_j, θ_j) is reflexive rank $r \leq p$ slope H -semistable with $\Delta(E_j)H^{n-2} = 0$. This follows by induction as $\Delta(\bar{E}_j)H^{n-2} = \Delta(V_j)H^{n-2} = p^2 \Delta(E_{j-1})H^{n-2} = 0$ and then Lemma 2.5 gives $\Delta(E_j)H^{n-2} = 0$. An easy induction also shows that $c_1(E_j) = p^j c_1(E_0)$ for all $j \geq 0$.

By Lemma 3.9 every E_m is locally free outside a finite number of points of X . Since by Lemma 2.5, \bar{E}_m is isomorphic to E_m outside a closed subset of codimension ≥ 3 , \bar{E}_m is locally free outside a closed subset of codimension ≥ 3 .

Let Y be an irreducible component of D . Let E'_0 be an \mathbb{L}_Y^0 -submodule of the \mathbb{L}_Y^0 -module $((E_0)_Y, \theta_0|_Y)$. Then by Lemma 3.7,

$$V'_0 = s^* C_{(\bar{Z}, \bar{D}^Z)}^{-1}(E'_0) \subset s^* C_{(\bar{Z}, \bar{D}^Z)}^{-1}(\pi^*(E_0)_Y, \theta'_0) = s^*(\pi^*(V_0)_Y, \nabla'_0) = ((V_0)_Y, \nabla_0|_Y),$$

i.e., V'_0 is an \mathbb{L}_Y -submodule of $((V_0)_Y, \nabla_0|_Y)$.

We have a filtration \bar{S}_Y^\bullet of $(V_0)_Y$ defined by $\bar{S}_Y^j := \text{im}((S_0^j)_Y \rightarrow (V_0)_Y)$. Since \bar{E}_1 is locally free outside a closed subset of codimension ≥ 3 , $(\bar{E}_1)_Y$ is locally free on Y outside a closed subset of codimension ≥ 2 . Therefore we have equalities of sheaves on Y ,

$$(\text{Gr}_{\bar{S}_Y}((V_0)_Y))^{**} = ((\text{Gr}_{S_0} V_0)_Y)^{**} = ((\bar{E}_1)_Y)^{**} = ((E_1)_Y)^{**},$$

as all sheaves are reflexive and isomorphic on the set where $(\bar{E}_1)_Y$ is locally free. Now $V'_0 \subset (V_0)_Y$ has a filtration induced from \bar{S}_Y^\bullet and the reflexivization of the associated graded is a subsheaf of $((E_1)_Y)^{**}$ that after intersecting with $(E_1)_Y$ gives an \mathbb{L}_Y^0 -submodule $E'_1 \subset ((E_1)_Y, \theta_1|_Y)$ such that $\mu_{H_Y}(E'_1) = p \mu_{H_Y}(E'_0)$.

Repeating the above procedure allows us to construct a sequence $\{E'_m\}_{m \geq 0}$ of \mathbb{L}_Y^0 -modules such that $E'_m \subset ((E_m)_Y, \theta_m|_Y)$ and $\mu_{H_Y}(E'_m) = p^m \mu_{H_Y}(E'_0)$.

Now let us write $p^m = r s_m + q_m$ for some non-negative integers s_m and $0 \leq q_m < r$. Let us set $(G_m, \theta_{G_m}) := (E_m, \theta_m) \otimes \det E_0^{-s_m}$. Then $\Delta_i(G_m) = \Delta_i(E_m) = 0$ for $2 \leq i < \min(n, 3)$ and $c_1(G_m) = q_m c_1(E_0)$ can take only finitely many values. So Theorem 1.2 implies that the family $\{(G_m, \theta_{G_m})\}_{m \geq 0}$ of reflexive slope H -semistable logarithmic Higgs sheaves is bounded. It follows that the family $\{(G_m)_Y\}_{m \geq 0}$ of sheaves is also bounded. Therefore the numbers

$$\begin{aligned}
 \mu_{H_Y}(E'_m \otimes \det(E_0)_Y^{-s_m}) &= p^m \mu_{H_Y}(E'_0) - r s_m \mu_{H_Y}((E_0)_Y) \\
 &= p^m (\mu_{H_Y}(E'_0) - \mu_{H_Y}((E_0)_Y)) + q_m \mu_{H_Y}((E_0)_Y)
 \end{aligned}$$

are uniformly bounded from above. Hence we get $\mu_{H_Y}(E'_0) \leq \mu_{H_Y}((E_0)_Y)$, i.e., the \mathbb{L}_Y^0 -module $((E_0)_Y, \theta_0|_Y)$ is slope H_Y -semistable. But we know that $\Delta_i((E_0)_Y) = 0$ for $i \geq 2$, so Lemma 1.11 and our assumption show that $(E_0)_Y$ is locally free. By Lemma 1.14 this implies that E_0 is locally free along Y . This proves that E_0 is locally free along $\text{Supp } D$. Since by Lemma 3.9 it is also locally free along $U = X - \text{Supp } D$, it is locally free everywhere. ■

Proof of Theorem 2.2 for $D \neq 0$. The proof in dimension 2 is the same as in Section 2.2 and we can proceed by induction on the dimension n of X . The same arguments as in the proof of (1) \Rightarrow (2) in Section 2.2 show that Lemma 3.10 holds if (E_0, θ_0) is only slope H -semistable.

Now let (E, θ) (resp. (E, ∇)) be a reflexive rank $r \leq p$ slope H -semistable logarithmic Higgs sheaf (resp. sheaf with an integrable logarithmic connection) with $\Delta(E)H^{n-2} = 0$. By Theorem 1.3 there exists a decreasing Griffiths transverse filtration N^\bullet of E such that the associated graded $(\bar{E}_0, \bar{\theta}_0) := \text{Gr}_N(E, \theta)$ is a slope H -semistable system of logarithmic Hodge sheaves (in particular, $\bar{\theta}_0$ is nilpotent). In the case of logarithmic connections we use Simpson’s filtration S^\bullet instead of N^\bullet . Let us set $(E_0, \theta_0) = ((\bar{E}_0)^{**}, \bar{\theta}_0^{**})$. Then Lemma 2.5 implies that $\Delta(E_0)H^{n-2} = 0$ and (E_0, θ_0) is slope H -semistable. If Theorem 2.2 holds for varieties of dimension less than n then the above arguments show that E_0 is locally free.

By Claim 2.2.1 we have $\Delta_i(E)H^{n-2} = 0$ for $2 \leq i < n$, so also $\Delta_i(\bar{E}_0)H^{n-2} = 0$ for $2 \leq i < n$. Then for a general divisor $Y \in |H|$, the restriction $(\bar{E}_0)_Y$ satisfies condition (3) of Theorem 2.2, so $(\bar{E}_0)_Y$ is locally free. Hence by Lemma 1.14, \bar{E}_0 is locally free outside a finite number of points and Lemma 1.12 implies that E is locally free. This proves that (1) implies (2). The rest of the proof of Theorem 2.2 is the same as in Section 2.2. ■

3.6. Nearby cycles in positive characteristic

Let X be a smooth projective variety of dimension n defined over an algebraically closed field k of characteristic p and let D be a simple normal crossing divisor on X . In this subsection we also assume that (X, D) is liftable to $W_2(k)$ and we fix a lifting (\tilde{X}, \tilde{D}) .

Let H be an ample divisor on X and let us fix a class $\mu \in H_{\text{et}}^2(X, \mathbb{Q}_l)$ for some $l \neq p$. We define the category $\text{MinHIG}^\mu(X, D)$ of minimally semistable Higgs sheaves of slope μ as the full subcategory of the category $\text{HIG}(X, D)$ of logarithmic Higgs sheaves on (X, D) , whose objects are pairs (E, θ) , where

- E is a locally free \mathcal{O}_X -module of rank $r \leq p$,
- (E, θ) is slope H -semistable,
- $c_1(E) = r\mu$ (i.e., the slope of E is equal to μ),
- $\Delta(E)H^{n-2} = 0$ (i.e., E has a minimal possible discriminant).

By Theorem 2.2 for any object (E, θ) of $\text{MinHIG}^\mu(X, D)$ we have $c_m(E) = \binom{r}{m} \mu^m$ for all $m \geq 1$. Taking in Theorem 4.7 the identity as f , we see that the above category does not depend on the choice of the polarization H .

Unfortunately, $\text{MinHIG}^\mu(X, D)$ is not abelian as it does not contain direct sums of objects. However, by Theorem 2.2 it satisfies all the other axioms of an abelian category. In particular, it contains kernels, images and cokernels (cf. [25, Corollary 5]) and any morphism in this category admits a canonical decomposition.

Let Y be an irreducible component of D and let us fix a class $\eta \in H_{\text{et}}^2(Y, \mathbb{Q}_l)$ for some $l \neq p$. Let us define the category $\text{Min-}\mathbb{L}_Y^0\text{-Mod}^\eta$ as the full subcategory of the category $\mathbb{L}_Y^0\text{-Mod}$ (defined in Section 3.1), whose objects E satisfy the following conditions:

- as an \mathcal{O}_Y -module, E is locally free of rank $r \leq p$,
- E is slope H_Y -semistable (as an \mathbb{L}_Y^0 -module),
- $c_1(E) = r\eta$ and $\Delta(E)H_Y^{n-3} = 0$.

Replacing \mathbb{L}_Y^0 by \mathbb{L}_Y in the above definition one can also define the category $\text{Min-}\mathbb{L}_Y\text{-Mod}^\eta$.

Theorem 3.11. *Let Y be an irreducible component of D . Then $\Phi_Y : \text{HIG}(X, D) \rightarrow \mathbb{L}_Y^0\text{-Mod}$ induces the functor*

$$\Phi_Y^\mu : \text{MinHIG}^\mu(X, D) \rightarrow \text{Min-}\mathbb{L}_Y^0\text{-Mod}^{\mu_Y},$$

where μ_Y is the image of μ under the restriction map $H_{\text{et}}^2(X, \mathbb{Q}_l) \rightarrow H_{\text{et}}^2(Y, \mathbb{Q}_l)$.

Proof. Let (E, θ) be an object of $\text{MinHIG}^\mu(X, D)$. By Theorem 1.3 there exists a decreasing Griffiths transverse filtration N^\bullet of E such that the associated graded $(E_0, \theta_0) := \text{Gr}_N(E, \theta)$ is a slope H -semistable system of logarithmic Hodge sheaves (in particular, θ_0 is nilpotent). Moreover, by Theorem 2.2, E_0 is locally free. By Lemma 3.10 we know that the \mathbb{L}_Y^0 -module $((E_0)_Y, \theta_0|_Y)$ is semistable. Then by openness of semistability, $(E_Y, \theta|_Y)$ is also a semistable \mathbb{L}_Y^0 -module. ■

Let $\text{Min-}\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}^\eta$ be the full subcategory of $\text{Min-}\mathbb{L}_Y^0\text{-Mod}^\eta$ whose objects are \mathbb{L}_Y^0 -modules E with Res_E nilpotent. Replacing \mathbb{L}_Y^0 by \mathbb{L}_Y we get the definition of $\text{Min-}\mathbb{L}_Y\text{-Mod}_{\text{nil}}^\eta$.

Theorem 3.12. *Fix a class $\eta \in H_{\text{et}}^2(Y, \mathbb{Q}_l)$ for some $l \neq p$. The functor $\Upsilon^0 : \mathbb{L}_Y^0\text{-Mod}_{\text{nil}} \rightarrow \text{HIG}(Y, D^Y)$ from Lemma 3.3 induces the functor*

$$\Upsilon_\eta^0 : \text{Min-}\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}^\eta \rightarrow \text{MinHIG}^\eta(Y, D^Y).$$

In particular, for any object E of $\text{Min-}\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}^\eta$ we have for all $m \geq 1$,

$$c_m(E) = \binom{r}{m} \eta^m$$

in $H_{\text{et}}^{2m}(Y, \mathbb{Q}_l)$.

Proof. Let E be an object of $\text{Min-}\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}^\eta$. We need to prove that every quotient $(\text{Gr}_j^M E, \theta_j)$ of the monodromy filtration M_\bullet of E is locally free, slope H_Y -semistable

with $c_m(\text{Gr}_j^M E) = \binom{r_j}{m} \eta^m$ for all $m \geq 1$, where $r_j = \text{rk Gr}_j^M E$. This also implies that $c_m(E) = \binom{r}{m} \eta^m$ for all $m \geq 1$.

By Lemma 3.2 we know that every quotient $\text{Gr}_j^M E$ of the monodromy filtration M_\bullet of E is slope H_Y -semistable (as an \mathbb{L}_Y^0 -module) with $\mu_{H_Y}(\text{Gr}_j^M E) = \mu_{H_Y}(E)$. We also know that $\text{Gr}_j^M E$ is endowed with a natural logarithmic Higgs field θ_j^M on (Y, D^Y) , coming from the \mathbb{L}_Y^0 -action and triviality of the residue of $\text{Gr}_j^M E$. Since any logarithmic Higgs subsheaf of $(\text{Gr}_j^M E, \theta_j)$ has a canonical structure of an \mathbb{L}_Y^0 -submodule, the pair $(\text{Gr}_j^M E, \theta_j)$ is slope H_Y -semistable. Therefore by Theorem 2.1 all quotients $\text{Gr}_j^M E$ are locally free with $c_m(\text{Gr}_j^M E) = \binom{r_j}{m} \eta^m$. ■

Corollary 3.13. *Any element in the essential image of the functor*

$$\Phi_Y^0 : \text{MinHIG}^\mu(X, D) \rightarrow \text{Min-}\mathbb{L}_Y^0\text{-Mod}^{\mu_Y}$$

has a filtration whose quotients are elements of $\text{MinHIG}^{\mu_Y}(Y, D^Y)$.

Proof. Assume that an object M of $\text{Min-}\mathbb{L}_Y^0\text{-Mod}^{\mu_Y}$ is isomorphic to $\Phi_Y^0(E, \theta)$ for some (E, θ) in $\text{MinHIG}^\mu(X, D)$. In the last part of the proof of Theorem 3.11 we showed that there exists a decreasing Griffiths transverse filtration N^\bullet of (E, θ) such that $\Phi_Y^0(\text{Gr}_N(E, \theta))$ is an object of $\text{Min-}\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}^\eta$. In particular, by Theorem 3.12 every quotient in the monodromy filtration of $\Phi_Y^0(\text{Gr}_N(E, \theta))$ is an element of $\text{MinHIG}^{\mu_Y}(Y, D^Y)$. The proof finishes by remarking that N^\bullet induces an analogous filtration on M . ■

4. Applications of local freeness results

In this section we give some applications of Theorem 2.2.

4.1. Local freeness in positive characteristic

Let X be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field k of characteristic p and let D be a simple normal crossing divisor on X . We assume that $D \subset X$ admits a lifting to $W_2(k)$. We also fix an ample divisor H on X .

Lemma 2.7 and Theorem 2.2 immediately imply the following corollary:

Corollary 4.1. *Let (E, θ) be a rank $r \leq p$ slope H -semistable logarithmic Higgs sheaf. Assume that E is reflexive and $\Delta(E)H^{n-2} = 0$. If (G, θ_G) is a rank s factor in a slope H -Jordan–Hölder filtration of (E, θ) then it is locally free and for all $m \geq 1$ we have*

$$c_m(G) = \binom{s}{m} \left(\frac{c_1(E)}{r} \right)^m$$

in $H_{\text{ét}}^{2m}(X, \mathbb{Q}_l)$ for $l \neq p$.

The following corollary is a direct generalization of [24, Theorem 11] to the logarithmic case.

Corollary 4.2. *Let (E, θ) be a rank $r \leq p$ slope H -semistable logarithmic Higgs sheaf with $\text{ch}_1(E)H^{n-1} = 0$ and $\text{ch}_2(E)H^{n-2} = 0$. Assume that either E is reflexive or the normalized Hilbert polynomial of E is the same as that of \mathcal{O}_X . Then (E, θ) has a filtration whose quotients are locally free slope H -stable logarithmic Higgs sheaves with vanishing Chern classes.*

Proof. By Theorem 1.9 we have $\Delta(E)H^{n-2} \geq 0$. So by the Hodge index theorem we get

$$0 = 2r\text{ch}_2(E)H^{n-2} = c_1(E)^2H^{n-2} - \Delta(E)H^{n-2} \leq c_1(E)^2H^{n-2} \leq \frac{(c_1(E)H^{n-1})^2}{H^n} = 0.$$

Hence we have $\Delta(E)H^{n-2} = 0$ and $c_1(E)^2H^{n-2} = 0$. Since $c_1(E)H^{n-1} = 0$ this implies that $c_1(E) = 0$ (see the proof of Lemma 2.4). If E is reflexive then the corollary follows directly from Corollary 4.1. In the other case we argue as in the proof that (3) implies (1) in Theorem 2.2. Namely, E^{**} satisfies condition (1) of Theorem 2.2 and hence $c_m(E^{**}) = 0$ for all $m \geq 1$. Then the Hilbert polynomials of E and E^{**} are equal. So the Hilbert polynomial of $T = E^{**}/E$ is trivial. This implies that $T = 0$ and E is reflexive, which reduces us to the previous case. ■

4.2. Local freeness in characteristic zero

If X is a smooth projective variety then we denote by $N^m(X)$ the space of numerical classes of cycles of codimension m with rational coefficients. Grothendieck's standard conjectures suggest that vanishing of some class of cycles $H^{2m}(X, \mathbb{Q})$ is equivalent to vanishing of its image in $N^m(X)$, but this is not known to hold in general. However, checking vanishing of classes in $N^m(X)$ is easier as one only needs to deal with some intersection numbers. In general, one can also prove vanishing of rational Chern classes in the theorems below but this requires more work and the use of the hard Lefschetz theorem.

By a standard spreading-out argument, Theorem 2.1, Theorem 2.2 and Corollary 4.2 imply the following generalization of [31, Theorem 2] to the logarithmic case.

Theorem 4.3. *Let X be a smooth projective variety defined over a field of characteristic zero and let D be a normal crossing divisor on X . Let H be an ample divisor on X and let (E, θ) be a slope H -semistable logarithmic Higgs sheaf with $\text{ch}_1(E)H^{n-1} = 0$ and $\text{ch}_2(E)H^{n-2} = 0$. Then the following conditions are equivalent:*

- (1) E is reflexive,
- (2) E is locally free,
- (3) the normalized Hilbert polynomial of E is the same as that of \mathcal{O}_X ,
- (4) E has vanishing numerical Chern classes, i.e., $c_m(E) = 0$ in $N^m(X)$ for all $m \geq 1$,
- (5) (E, θ) has a filtration whose quotients are locally free slope H -stable logarithmic Higgs sheaves with vanishing rational Chern classes.

Remark 4.4. The same theorems show that in the above theorem we can replace a logarithmic Higgs sheaf by a sheaf with an integrable logarithmic connection.

Remark 4.5. Let (V, ∇) be a polarized variation of Hodge structures on $X - D$ with unipotent monodromy along the irreducible components of D . Let $(\tilde{V}, \tilde{\nabla})$ be Deligne’s canonical extension of (V, ∇) with nilpotent residues along the irreducible components of D . Then Schmid’s nilpotent orbit theorem implies that the Hodge filtration on V extends to a filtration of \tilde{V} with locally free subquotients.

Note that it is easy to see that \tilde{V} has vanishing Chern classes in the de Rham cohomology of X (as all residues are nilpotent) and hence it also has vanishing Chern classes in $H^{2*}(X, \mathbb{Q})$. Similarly, all the subobjects of $(\tilde{V}, \tilde{\nabla})$ have vanishing rational Chern classes. In particular, $(\tilde{V}, \tilde{\nabla})$ is slope semistable. Therefore Theorem 1.4 gives Simpson’s filtration such that the associated graded (E, θ) is slope semistable. Since E has vanishing rational Chern classes, Theorem 4.3 implies that E is locally free. Note that [23, Corollary 5.6] implies that the associated graded of Simpson’s filtration of $(\tilde{V}, \tilde{\nabla})$ coincides with the associated graded of the filtration obtained by Schmid’s theorem. Moreover, if the associated graded (E, θ) is slope stable then the corresponding filtrations coincide.

Again, using spreading out, Theorem 2.2 implies the following theorem. However, we also give a different proof by deducing it from Theorem 4.3 that was already known in the non-logarithmic case ($D = 0$). Note also that Corollary 4.2 can be proven in a somewhat simpler way than Theorem 2.2. The difference is that if we follow the proofs of (1) \Rightarrow (2) and (1) \Rightarrow (3) in Theorem 2.2 under the assumptions of Corollary 4.2 then we do not need to consider the family $\{G_m\}_{m \geq 0}$ and we can work directly with $\{E_m\}_{m \geq 0}$. However, it should be stressed that similar arguments to those below (showing that Theorem 4.6 follows from Theorem 4.3) do not allow one to deduce Theorem 2.2 from Corollary 4.2 in positive characteristic. This is caused by the use of coverings that usually do not preserve liftability to $W_2(k)$. Another problem is that such covers are sometimes necessarily inseparable, in which case pullback does not preserve semistability.

Theorem 4.6. *Let X be a smooth projective variety of dimension $n \geq 2$ defined over a field of characteristic zero and let D be a normal crossing divisor on X . Let H be an ample divisor on X and let (E, θ) be a slope H -semistable logarithmic Higgs sheaf with $\Delta(E)H^{n-2} = 0$. If E is reflexive then it is locally free and*

$$c_m(E) = \frac{\binom{r}{m}}{r^m} c_1(E)^m \quad \text{in } N^m(X) \text{ for all } m \geq 1.$$

Moreover, each rank s factor (G, θ_G) of a slope H -Jordan–Hölder filtration of (E, θ) is locally free with

$$c_m(G) = \frac{\binom{s}{m}}{r^m} c_1(E)^m \quad \text{in } N^m(X) \text{ for all } m \geq 1.$$

Proof. By a variant of the Bloch–Gieseker covering trick (see [17, Proposition 2.67]) there exists a smooth projective variety \tilde{X} and a finite flat surjective covering $f : \tilde{X} \rightarrow X$

together with a line bundle L such that $f^*(\det E)^{-1} = L^{\otimes r}$ and the pullback $\tilde{D} = (f^*D)_{\text{red}}$ is a simple normal crossing divisor. Let us define a logarithmic Higgs sheaf $(\tilde{E}, \tilde{\theta} : \tilde{E} \rightarrow \tilde{E} \otimes \Omega_{\tilde{X}}(\log \tilde{D}))$ by $(\tilde{E}, \tilde{\theta}) := f^*(E, \theta) \otimes L$. Note that \tilde{E} is reflexive, $c_1(\tilde{E}) = 0$ and

$$\Delta(\tilde{E})(f^*H)^{n-2} = \Delta(f^*E)(f^*H)^{n-2} = \deg f \cdot \Delta(E)H^{n-2} = 0.$$

Hence \tilde{E} is a slope f^*H -semistable logarithmic Higgs sheaf with $\text{ch}_1(\tilde{E})(f^*H)^{n-1} = 0$ and $\text{ch}_2(\tilde{E})(f^*H)^{n-2} = 0$. By Theorem 4.3, \tilde{E} is locally free and it has vanishing Chern classes. Therefore by the flat descent E is also locally free and we have

$$0 = \Delta_m(\tilde{E}) = f^*(\Delta_m(E)) \quad \text{for all } m \geq 2.$$

Using the fact that f induces an injection $N^m(X) \rightarrow N^m(\tilde{X})$, we get vanishing of $\Delta_m(E)$ for all $m \geq 2$. Hence by Lemma 1.11 we get $r^m c_m(E) = \binom{r}{m} c_1(E)^m$.

Now let (G, θ_G) be a rank s factor of a slope H -Jordan–Hölder filtration of (E, θ) . Then $f^*(G, \theta_G) \otimes L$ is a successive extension of some factors of a slope f^*H -Jordan–Hölder filtration of $(\tilde{E}, \tilde{\theta})$. In particular, it has a filtration whose quotients are locally free slope f^*H -stable logarithmic Higgs sheaves with vanishing numerical Chern classes. It follows that G is locally free, $c_1(f^*G) = -sc_1(L) = \frac{s}{r}c_1(f^*E)$ and

$$c_m(G) = \frac{\binom{s}{m}}{s^m} c_1(G)^m = \frac{\binom{s}{m}}{r^m} c_1(E)^m \quad \text{for all } m \geq 1. \quad \blacksquare$$

4.3. Restriction theorem

The following theorem generalizes [24, Theorem 12] to the logarithmic case and to arbitrary (Y, B) .

Theorem 4.7. *Let X be a smooth projective variety of dimension n defined over an algebraically closed field k of characteristic p and let D be a normal crossing divisor on X . Let H be an ample divisor on X and let E be a locally free \mathcal{O}_X -module of rank $r \leq p$ with $\Delta(E)H^{n-2} = 0$. Assume that a logarithmic Higgs sheaf (E, θ) is slope H -semistable. Let $f : (Y, B) \rightarrow (X, D)$ be a proper morphism of smooth log pairs that has a good lifting to $W_2(k)$ (see Definition 6.1). Then the induced logarithmic Higgs sheaf*

$$f^*(E, \theta) = (f^*E, f^*E \xrightarrow{f^*\theta} f^*E \otimes f^*\Omega_X(\log D) \xrightarrow{\text{Id}_{f^*E} \otimes df} f^*E \otimes \Omega_Y(\log B))$$

is slope A -semistable for any ample divisor A on Y .

Proof. By Theorem 1.3 we can deform (E, θ) to a slope H -semistable system of Hodge sheaves (E_0, θ_0) . Moreover, by Theorem 2.2, E_0 is locally free. If $f^*(E_0, \theta_0)$ is semistable then by openness of semistability, $f^*(E, \theta)$ is also semistable. So without loss of generality one can assume that (E, θ) is a system of Hodge sheaves. The rest of the proof is the same as that of [24, Theorem 12], using Theorem 1.5 instead of [24, Theorem 5]. Here we also need to apply functoriality of the inverse Cartier transform in the logarithmic case (see Theorem 6.4). \blacksquare

Applying the above theorem to iterates of the Frobenius morphism we get the following corollary:

Corollary 4.8. *In the notation of the above theorem assume that $(Y, B) = (X, D)$ and f is the Frobenius morphism. Then E is strongly A -semistable for any ample divisor A on X .*

In the formulation of [24, Theorem 12] the author forgot to explicitly state the assumption on existence of a compatible lifting of C and X (even though it was used in the proof). The next example shows that this assumption is really necessary.

Example 4.9. Here we give an example of a smooth projective surface that is liftable to the Witt ring $W(k)$ and a slope semistable Higgs sheaf which is not semistable after restricting to the normalization of some projective curve on this surface.

Let us consider a smooth complex projective surface X which is a quotient of the product $\mathbb{H} \times \mathbb{H}$ of the upper half-planes by an irreducible, torsion free, cocompact lattice G in $\mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$. Then $\Omega_X = L \oplus M$, where $L^2 = M^2 = 0$, and L and M are strictly nef (see [22, Lemma 4.5]).

Let us consider a Higgs bundle (E, θ) , where $E = L \oplus \mathcal{O}_X$ and θ is given by the canonical inclusion $L \rightarrow \Omega_X$. This Higgs bundle corresponds to the representation $\rho : \pi_1(X) \rightarrow \mathrm{PGL}(2, \mathbb{C})$ obtained by composing the inclusion $G \subset \mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$ with projection onto the first factor and embedding into the complexification. Then the Higgs bundle $(E', \theta') := \mathrm{Sym}^2(E, \theta) \otimes (\det E, \det \theta)^{-1}$ corresponds to the composition of ρ with the adjoint representation $\mathrm{PGL}(2, \mathbb{C}) \rightarrow \mathrm{SL}(3, \mathbb{C})$. In particular, since this representation is irreducible, the Higgs bundle (E', θ') is slope stable (with respect to any polarization) and it has vanishing rational Chern classes. This can also be checked directly from the definition of stability. More precisely, (E', θ') is a system of Hodge sheaves $E^{2,0} \oplus E^{1,1} \oplus E^{0,2} = L \oplus \mathcal{O}_X \oplus L^{-1}$ with θ given by the canonical inclusions $E^{2,0} = L \rightarrow E^{1,1} \otimes \Omega_X = L \oplus M$ and $E^{1,1} = \mathcal{O}_X \rightarrow E^{0,2} \otimes \Omega_X = \mathcal{O}_X \oplus (M \otimes L^{-1})$ onto the first factor. This system has only two non-trivial saturated subsystems of Hodge sheaves given by $E^{0,2} = L^{-1}$ and $E^{1,1} \oplus E^{0,2} = \mathcal{O}_X \oplus L^{-1}$. In particular, (E', θ') is slope H -stable if and only if $LH > 0$.

By openness of stability, the reduction of (E', θ') modulo almost all primes is stable. Again this can be easily seen directly, because ampleness is an open condition and $LH > 0$ implies an analogous inequality for the reductions. Note also that for almost all reductions, X_s lifts to $W(k(s))$.

Now for a large number of primes (of positive density) the reduction of L is not nef (see [22, Example 5.6]). For such s there exists an irreducible curve C_s with $L_s \cdot C_s < 0$. Let $\nu_s : \bar{C}_s \rightarrow C_s$ be the normalization. Then $\nu_s^*(E'_s, \theta'_s)$ is not semistable because it has degree zero and it contains a Higgs subbundle $(\nu_s^* L_s^{-1}, 0)$ of positive degree. This shows that $\nu_s : \bar{C}_s \rightarrow X_s$ cannot be compatibly lifted to $W_2(k(s))$, even though both \bar{C}_s and X_s can be lifted to $W(k(s))$.

By the usual spreading-out technique, Theorem 4.7 implies the following corollary.

Corollary 4.10. *Let X be a smooth projective variety of dimension n defined over an algebraically closed field k of characteristic zero. Let D be a normal crossing divisor on X and let H be an ample divisor on X . Let E be a locally free \mathcal{O}_X -module with $\Delta(E)H^{n-2} = 0$. If a logarithmic Higgs sheaf (E, θ) is slope H -semistable then for every smooth projective curve C not contained in D and a morphism $f : C \rightarrow X$ the Higgs bundle $f^*(E, \theta)$ is semistable.*

Remark 4.11. In the case of complex projective manifolds and $D = 0$ the above corollary follows from Simpson's correspondence. A rough sketch of proof is as follows. A slope semistable Higgs bundle with vanishing rational Chern classes corresponds to a local system on X . So for any morphism $f : C \rightarrow X$ we get an induced local system on C . This again corresponds to a slope semistable Higgs bundle on C . By functoriality of Simpson's correspondence this is the pullback of the original Higgs bundle. The general case with $\Delta(E)H^{n-2} = 0$ can be reduced to the above by taking $\text{End } E$ and using [31, Theorem 2] (or Theorem 4.3). More precisely, if $\Delta(E)H^{n-2} = 0$ then $c_1(\text{End } E) = 0$ and $\Delta(\text{End } E)H^{n-2} = 0$, so $(\text{End } E, \theta_{\text{End } E})$ is semistable with vanishing rational Chern classes. Then $f^*(\text{End } E, \theta_{\text{End } E})$ is semistable, which implies semistability of $f^*(E, \theta)$.

5. Semistability and semipositivity

In this section we prove Theorem 0.2 and show some of its applications mentioned in the introduction.

5.1. General results on semistability

Unless otherwise stated, in this subsection (X, D) stands for a smooth log pair defined over an algebraically closed field k of positive characteristic. We assume that the pair (X, D) is liftable to $W_2(k)$ and we fix its lifting (\tilde{X}, \tilde{D}) .

Let C be a smooth projective curve and let $\nu : C \rightarrow X$ be a separable morphism. Let D' be the sum of those irreducible components of D that do not contain $\nu(C)$ and let $D'_C = (\nu^{-1}(D'))_{\text{red}}$.

Definition 5.1. We say that $\nu : C \rightarrow (X, D)$ is *strongly liftable to $W_2(k)$* if there exists a good lifting $\tilde{\nu} : (\tilde{C}, \tilde{D}'_C) \rightarrow (\tilde{X}, \tilde{D})$ (see Definition 6.1) of $\nu : (C, D'_C) \rightarrow (X, D')$ such that for every irreducible component Y of D containing C , $\tilde{\nu}$ factors through $\tilde{C} \rightarrow \tilde{Y}$.

In the above definition we write $\nu : C \rightarrow (X, D)$ to keep in mind that being strongly liftable to $W_2(k)$ depends not only on $\nu : C \rightarrow X$ but also on the choice of the normal crossing divisor D (in fact, it also depends on the choice of lifting (\tilde{X}, \tilde{D}) of (X, D)).

Theorem 5.2. *Let (E, θ) be an object of $\text{MinHIG}^\mu(X, D)$. Let C be a smooth projective curve and let $\nu : C \rightarrow (X, D)$ be a morphism that is strongly liftable to $W_2(k)$. Then the induced $\text{Sym}^\bullet \nu^* T_X(\log D)$ -module $\nu^* E$ is semistable. In particular, if G is a subsheaf of the kernel of $\nu^* \theta : \nu^* E \rightarrow \nu^* E \otimes \nu^* \Omega_X(\log D)$ then $\mu(G) \leq \mu(\nu^* E)$.*

Proof. The proof is by induction on the dimension of X . In dimension $n = 1$ the assertion follows from Theorem 4.7, so let us assume that $n \geq 2$. As in the proof of Theorem 3.11, there exists a decreasing Griffiths transverse filtration N^\bullet of E such that the associated graded $(E_0, \theta_0) := \text{Gr}_N(E, \theta)$ is an object of $\text{MinHIG}^\mu(X, D)$ with nilpotent θ_0 . Since $\text{Gr}_{\nu^*N}(\nu^*E, \nu^*\theta) = (\nu^*E_0, \nu^*\theta_0)$, by openness of semistability, if the $\text{Sym}^\bullet \nu^*T_X(\log D)$ -module ν^*E_0 is semistable then the $\text{Sym}^\bullet \nu^*T_X(\log D)$ -module ν^*E is semistable. So in the following we can assume that θ is nilpotent.

If $\nu(C)$ is not contained in D then $\nu^*(E, \theta)$ is semistable by Theorem 4.7 (for this we do not need nilpotence of θ). Since any $\text{Sym}^\bullet \nu^*T_X(\log D)$ -submodule of ν^*E defines a Higgs subsheaf of $\nu^*(E, \theta)$, this implies that ν^*E is semistable as a $\text{Sym}^\bullet \nu^*T_X(\log D)$ -module.

If $\nu(C)$ is contained in D then we choose an irreducible component Y of D containing $\nu(C)$ and as before we set $D^Y = (D - Y)|_Y$. By definition of strong liftability, the morphism $C \rightarrow (Y, D^Y)$ is also strongly liftable to $W_2(k)$. By Theorem 3.11, $E' := \Phi_Y^0(E, \theta)$ is an element of $\text{Min-}\mathbb{L}_Y^0\text{-Mod}_{\text{nil}}^{\mu_Y}$. By Theorem 3.12 we know that E' has a filtration M_\bullet whose associated graded $E'' = \text{Gr}^M E'$ is an element of $\text{MinHIG}^{\mu_Y}(Y, D^Y)$. Hence by the induction assumption the induced $\text{Sym}^\bullet \nu^*T_Y(\log D^Y)$ -module ν^*E'' is semistable. Equivalently, ν^*E'' is semistable as a $\nu^*\mathbb{L}_Y^0$ -module.

But ν^*M_\bullet is a filtration of ν^*E' by $\nu^*\mathbb{L}_Y^0$ -submodules and the associated graded is equal to ν^*E'' (here we use the fact that E'' is locally free). So by openness of semistability, ν^*E' is semistable as a $\nu^*\mathbb{L}_Y^0$ -module. This is equivalent to saying that ν^*E is semistable as a $\text{Sym}^\bullet \nu^*T_X(\log D)$ -module, which finishes the induction step.

The last part of the theorem follows from the fact that $\ker \nu^*\theta$ with trivial action is a $\text{Sym}^\bullet \nu^*T_X(\log D)$ -submodule of ν^*E . ■

Corollary 5.3. *Let (E, θ) be an object of $\text{MinHIG}^0(X, D)$. If E' is a locally split subsheaf of E contained in the kernel of θ then its dual $(E')^*$ is W_2 -nef.*

Proof. If E' is a locally split subsheaf of E then for any smooth projective curve C and any morphism $\nu : C \rightarrow X$, ν^*E' is a subsheaf of ν^*E . Moreover, the image of $\nu^*(\ker \theta)$ in ν^*E is contained in $\ker \nu^*\theta$, so $\nu^*E' \subset \ker \nu^*\theta$. So if ν is separable and liftable to $W_2(k)$, then by the above theorem any subsheaf of ν^*E' has a non-positive degree. Passing to the dual of ν^*E' , we get the required assertion. ■

Standard spreading-out arguments show that Theorem 5.2 implies the following result:

Theorem 5.4. *Let (E, θ) be a locally free logarithmic Higgs sheaf on a smooth log pair (X, D) defined over an algebraically closed field of characteristic zero. Assume that it has vanishing Chern classes in $H^{2*}(X, \mathbb{Q})$ and it is slope semistable with respect to some ample polarization. Let $\nu : C \rightarrow X$ be any morphism from some smooth projective curve. Then the induced $\text{Sym}^\bullet \nu^*T_X(\log D)$ -module ν^*E is semistable. In particular, if G is a subsheaf of the kernel of $\nu^*\theta : \nu^*E \rightarrow \nu^*E \otimes \nu^*\Omega_X(\log D)$ then $\deg G \leq 0$.*

Remark 5.5. (1) For the first part of Theorem 5.4 one can replace the assumption that E has vanishing Chern classes with the assumption that $r^i c_m(E) = \binom{r}{m} (c_1(E))^m$ for all $m \geq 2$ in $H^{2*}(X, \mathbb{Q})$.

(2) In Theorem 5.4 the assertion holds if we replace the curve C by any smooth polarized variety. This immediately follows from the fact that semistability on a general complete intersection curve implies semistability on the original variety.

(3) A posteriori one can see that it is possible to obtain the proof of the above theorem without passing to positive characteristic. In case (E, θ) comes from a real graded-polarized family of mixed Hodge structures it is possible to use Mochizuki’s version of Simpson’s correspondence to adapt Brunebarbe’s proof [3, Theorem 4.5] to obtain the above theorem. This strategy can also be generalized to deal with arbitrary systems of logarithmic Hodge bundles. The general case needs a logarithmic version of [31, Theorem 2] (cf. Theorem 2.2), which again can be obtained using Mochizuki’s results. Passing to non-zero μ as in Theorem 5.2 can be done using Theorem 4.6.

Theorem 5.4 implies the following result generalizing [4, Theorem 1.2] from the polystable to the semistable case:

Corollary 5.6. *Let (E, θ) be a locally free logarithmic Higgs sheaf on a smooth log pair (X, D) defined over an algebraically closed field of characteristic zero. Assume that it has vanishing Chern classes in $H^{2*}(X, \mathbb{Q})$ and it is slope semistable with respect to some ample polarization. If E' is a locally split subsheaf of E contained in the kernel of θ then its dual $(E')^*$ is nef.*

5.2. Geometric applications

In this subsection we give several geometric applications of Corollary 5.3 in more or less increasing degree of generality showing how to adjust some arguments. We fix the following notation. Let X and Y be smooth projective varieties defined over an algebraically closed field k of characteristic p and let $f : X \rightarrow Y$ be a surjective k -morphism of relative dimension d . Moreover, i and j are arbitrary non-negative integers.

Corollary 5.7. *Assume that f is smooth $d < p$ and there exists a lifting $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ of f to $W_2(k)$. Then $(R^i f_*^{\text{dR}} \mathcal{O}_X, \nabla_{\text{GM}})$, where ∇_{GM} is the Gauss–Manin connection, is a locally free semistable sheaf with an integrable connection and vanishing Chern classes. In particular, $R^j f_* \omega_{X|Y}$ is a W_2 -nef locally free sheaf on Y .*

Proof. By [26, Theorem 4.17] we have a canonical isomorphism

$$C_{\tilde{Y}}^{-1}(\text{Gr}_F R^i f_*^{\text{dR}} \mathcal{O}_X, \kappa) \simeq (R^i f_*^{\text{dR}} \mathcal{O}_X, \nabla_{\text{GM}}),$$

where F^\bullet is the Hodge filtration and κ is the associated graded (i.e., the cup product with the Kodaira–Spencer mapping). If Y is projective then the above isomorphism implies that both $(R^i f_*^{\text{dR}} \mathcal{O}_X, \nabla_{\text{GM}})$ and $(\text{Gr}_F R^i f_*^{\text{dR}} \mathcal{O}_X, \kappa)$ are semistable as we have a periodic Higgs–de Rham sequence of $(\text{Gr}_F R^i f_*^{\text{dR}} \mathcal{O}_X, \kappa)$ (here we use [24, Proposition 1]). So

Corollary 0.3 implies that the first non-zero piece of the Hodge filtration of $R^i f_*^{\text{dR}} \mathcal{O}_X$, i.e., $R^{i-d} f_* \omega_{X/Y}$, is a W_2 -nef locally free sheaf on Y . ■

Remark 5.8. In the complex case the above corollary is precisely the result of Griffiths (see [10, Corollary 7.8]), who showed that if $f : X \rightarrow Y$ is a smooth morphism of smooth projective varieties, then the direct image $f_* \omega_{X/Y}$ of the relative canonical bundle is locally free and nef.

Corollary 5.9. *Let D be a divisor on X which is a union of divisors, each of which is smooth over Y , and which have normal crossings relative to Y . Assume that f is smooth, $d < p$ and there exists a lifting $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ of f to $W_2(k)$ and a compatible lifting \tilde{D} of D . Then $(R^i f_* \Omega_{X/Y}^\bullet(\log D), \nabla_{\text{GM}})$ is semistable with vanishing Chern classes. In particular, $R^j f_* \omega_{X/Y}(D)$ is a W_2 -nef locally free sheaf on Y .*

Proof. The proof is the same as that of Corollary 5.7 except that we need to reformulate Katz’s [14, Theorem 3.2] using the inverse Cartier transform (cf. [26, Example 3.17 and Remark 3.19]). In this way we get a canonical isomorphism

$$C_{\tilde{Y}}^{-1}(\text{Gr}_F R^i f_* \Omega_{X/Y}^\bullet(\log D), \kappa) \simeq (R^i f_* \Omega_{X/Y}^\bullet(\log D), \nabla_{\text{GM}}). \quad \blacksquare$$

One can also get similar theorems in the case of “unipotent local monodromies”, e.g., for semistable reductions. Before stating the corresponding result let us recall the definition of a semistable reduction (see [13, Definition 1.1]). Let S be a scheme and let X and Y be smooth S -schemes, $f : X \rightarrow Y$ an S -morphism, $B \subset Y$ a normal crossing divisor relative to S , and $D := X \times_Y B$. We say that $f : X \rightarrow Y$ is *semistable* (or f has a *semistable reduction along B*) if locally in the étale topology on X , f is a product of S -morphisms of the following type:

- (1) the projection $\pi_1 : \mathbb{A}_S^n \rightarrow \mathbb{A}_S^1, B = 0$,
- (2) $h : \mathbb{A}_S^n = \text{Spec } \mathcal{O}_S[x_1, \dots, x_n] \rightarrow \mathbb{A}_S^1 = \text{Spec } \mathcal{O}_S[y], h^* y = x_1 \dots x_n, B = V(y)$.

Corollary 5.10. *Let B be a normal crossing divisor on Y and assume that f has a semistable reduction along B . Set $D = f^{-1}(B)$. Assume that there exists a lifting $\tilde{f} : (\tilde{X}, \tilde{D}) \rightarrow (\tilde{Y}, \tilde{B})$ of f to $W_2(k)$ with \tilde{f} a semistable reduction along \tilde{B} . Assume that $p > d + \dim Y$. Then*

$$(R^i f_* \Omega_{X/Y}^\bullet(\log D/B), \nabla_{\text{GM}})$$

is a semistable locally free \mathcal{O}_Y -module with an integrable logarithmic connection on (Y, B) . In particular, $R^j f_ \omega_{X/Y}(D)$ is a W_2 -nef locally free sheaf on (Y, B) .*

Proof. Again the proof is the same as that of Corollary 5.7, except that now one needs to use [13, Theorem 4.7] and check that the corresponding result describes an isomorphism

$$C_{(\tilde{Y}, \tilde{B})}^{-1}(\text{Gr}_F R^i f_* \Omega_{X/Y}^\bullet(\log D/B), \kappa) \simeq (R^i f_* \Omega_{X/Y}^\bullet(\log D/B), \nabla_{\text{GM}}).$$

The assumptions of this theorem are satisfied due to [13, Corollary 2.4] and our assumption $p > d + \dim Y$. We leave checking the cumbersome details to the interested reader. ■

In characteristic zero, the above result is almost the same as [15, Theorem 5].

Remark 5.11. One can also combine Corollaries 5.9 and 5.10 using [13, §4.22]). It is also possible to further generalize these results and deal with pushforwards of Fontaine modules as in [26, Theorem 4.17] and the corresponding log versions.

6. Appendix: functoriality of the inverse Cartier transform

In this appendix we prove the functoriality of the inverse Cartier transform. In the non-logarithmic case functoriality follows from [26, Theorem 3.22]. Unfortunately, although it seems very likely that an analogue of this result holds in the logarithmic case, this part of their paper was never generalized.

In the following, instead of dealing with a general theory that would demand a lot of space and additional notation, we deal only with the simple cases used in the paper. Instead of using the general framework of [29] that follows [26], we use an explicit description of the Ogus–Vologodsky correspondence provided in [19] and [18, Appendix].

Let k be an algebraically closed field of positive characteristic and let $f : (Y, B) \rightarrow (X, D)$ be a k -morphisms of smooth log pairs over k .

Definition 6.1. We say that f has a *good lifting to $W_2(k)$* if f lifts to a morphism of smooth log pairs $\tilde{f} : (\tilde{Y}, \tilde{B}) \rightarrow (\tilde{X}, \tilde{D})$ over $W_2(k)$ such that locally in the étale topology on \tilde{X} , \tilde{f} admits compatible liftings of the Frobenius morphisms, i.e., we can cover \tilde{X} with images of étale $W_2(k)$ -morphisms $\tilde{U} \rightarrow \tilde{X}$ and \tilde{Y} with images of étale $W_2(k)$ -morphisms $\tilde{V} \rightarrow \tilde{Y}$ so that

- (1) there exists $\tilde{F}_U : \tilde{U} \rightarrow \tilde{U}$ lifting the Frobenius morphism F_U , where $U = \tilde{U} \otimes_{W_2(k)} k$, so that $\tilde{F}_U^{-1}(\tilde{D}) = p\tilde{D}$,
- (2) there exists $\tilde{F}_V : \tilde{V} \rightarrow \tilde{V}$ lifting the Frobenius morphism F_V , where $V = \tilde{V} \otimes_{W_2(k)} k$, so that $\tilde{F}_V^{-1}(\tilde{B}) = p\tilde{B}$,
- (3) there exists $\tilde{f}_V : \tilde{V} \rightarrow \tilde{U}$ lifting \tilde{f} such that the diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{f}_V} & \tilde{U} \\ \tilde{F}_V \downarrow & & \downarrow \tilde{F}_U \\ \tilde{V} & \xrightarrow{\tilde{f}_V} & \tilde{U} \end{array}$$

is commutative.

In this case we say that \tilde{f} is a *good lifting of f to $W_2(k)$* .

Clearly, if \tilde{f} is an open embedding then it is a good lifting. Similarly, a composition of good liftings is a good lifting. It is also easy to see that the standard Frobenius morphism given by raising elements to the p -th power gives the following proposition:

Proposition 6.2. *Assume f lifts to a morphism of smooth log pairs $\tilde{f} : (\tilde{Y}, \tilde{B}) \rightarrow (\tilde{X}, \tilde{D})$ over $S = \text{Spec } W_2(k)$ such that locally in the étale topology on \tilde{X} , \tilde{f} is a composition of products of S -morphisms of the following type:*

- (1) *the projection $\pi_1 : \mathbb{A}_S^n \rightarrow \mathbb{A}_S^1, \tilde{B} = 0, \tilde{D} = 0,$*
- (2) *the embedding $i_1 : \mathbb{A}_S^1 \rightarrow \mathbb{A}_S^n, \tilde{B} = 0, \tilde{D} = 0,$*
- (3) *$h : \mathbb{A}_S^m = \text{Spec } \mathcal{O}_S[y_1, \dots, y_m] \rightarrow \mathbb{A}_S^n = \text{Spec } \mathcal{O}_S[x_1, \dots, x_n], \tilde{B} = V(\prod_{i=1}^m y_i),$
 $\tilde{D} = V(\prod_{j=1}^n x_j)$ and for $j = 1, \dots, n$ we have*

$$h^*(x_j) = \prod_{i=1}^m y_i^{a_{ij}},$$

where a_{ij} are some non-negative integers.

Then \tilde{f} is a good lifting of f to $W_2(k)$.

Remark 6.3. It is easy to see that any log-smooth lifting \tilde{f} of f to $W_2(k)$ is a good lifting. One can also see that almost every reduction of a morphism of smooth log pairs from characteristic zero to positive characteristic gives rise to a good lifting. For example, in the case $\tilde{B} = 0$ and $\tilde{D} = 0$ one can decompose any morphism of smooth schemes over an algebraically closed field into a composition of a closed embedding and a smooth morphism. A smooth closed subvariety of a smooth variety is locally in the étale topology a product of maps of type 2, and a smooth morphism is locally in the étale topology a product of maps of type 1.

Let $\text{HIG}_{\leq p-1}^{\text{lf}}(X, D)$ be the full subcategory of $\text{HIG}(X, D)$ consisting of locally free logarithmic Higgs sheaves with nilpotent Higgs field of level $\leq p - 1$. Let $\text{MIC}_{\leq p-1}(X, D)$ be the full subcategory of $\text{MIC}(X, D)$ consisting of \mathcal{O}_X -modules with an integrable logarithmic connection whose logarithmic p -curvature is nilpotent of level $\leq p - 1$ and the residues are nilpotent of order $\leq p$.

Theorem 6.4.¹ *Let $f : (Y, B) \rightarrow (X, D)$ be a morphism of smooth log pairs that has a good lifting $\tilde{f} : (\tilde{Y}, \tilde{B}) \rightarrow (\tilde{X}, \tilde{D})$ to $W_2(k)$. Then we have an isomorphism of functors*

$$f^* \circ C_{(\tilde{X}, \tilde{D})}^{-1} \simeq C_{(\tilde{Y}, \tilde{B})}^{-1} \circ f^* : \text{HIG}_{\leq p-1}^{\text{lf}}(X, D) \rightarrow \text{MIC}_{\leq p-1}(Y, B).$$

Proof. Step 1. Let us first assume that there exist global compatible logarithmic liftings of the Frobenius morphism on X and Y , i.e.,

- (1) there exists $\tilde{F}_X : \tilde{X} \rightarrow \tilde{X}$ lifting the Frobenius morphism F_X so that

$$\tilde{F}_X^* \mathcal{O}_{\tilde{X}}(-\tilde{D}) = \mathcal{O}_{\tilde{X}}(-p\tilde{D}),$$

¹After sending the preprint, the author was informed by K. Zuo that together with R. Sun and J. Yang they checked compatibility of the inverse Cartier transform for double covers of \mathbb{P}^1 . Recently, a more general result was proven by M. Sheng [30].

(2) there exists $\tilde{F}_Y : \tilde{Y} \rightarrow \tilde{Y}$ lifting the Frobenius morphism F_Y so that

$$\tilde{F}_Y^* \mathcal{O}_{\tilde{Y}}(-\tilde{B}) = \mathcal{O}_{\tilde{Y}}(-p\tilde{B}),$$

(3) the diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \tilde{F}_Y \downarrow & & \downarrow \tilde{F}_X \\ \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{X} \end{array}$$

is commutative.

The first condition implies that there exists a uniquely defined ζ_X such that the diagram

$$\begin{array}{ccc} \tilde{F}_X^* \Omega_{\tilde{X}}^1(\log \tilde{D}) & \xrightarrow{d\tilde{F}_X} & \Omega_{\tilde{X}}^1(\log \tilde{D}) \\ \downarrow & & \uparrow p \\ F_X^* \Omega_X^1(\log D) & \xrightarrow{\zeta_X} & \Omega_X^1(\log D) \end{array}$$

is commutative. The second condition gives ζ_Y with a similar diagram for (\tilde{Y}, \tilde{B}) . The third condition shows that we have a commutative diagram

$$\begin{array}{ccccc} \tilde{F}_Y^* \tilde{f}^* \Omega_{\tilde{X}}^1(\log \tilde{D}) & \xlongequal{\quad} & \tilde{f}^* \tilde{F}_X^* \Omega_{\tilde{X}}^1(\log \tilde{D}) & \xrightarrow{\tilde{f}^*(d\tilde{F}_X)} & \tilde{f}^* \Omega_{\tilde{X}}^1(\log \tilde{D}) \\ & \searrow \tilde{F}_Y^*(d\tilde{f}) & & & \downarrow d\tilde{f} \\ & & \tilde{F}_Y^* \Omega_{\tilde{Y}}^1(\log \tilde{B}) & \xrightarrow{d\tilde{F}_Y} & \Omega_{\tilde{Y}}^1(\log \tilde{B}) \end{array}$$

Together with the previous two diagrams this shows that the diagram

$$\begin{array}{ccccc} F_Y^* f^* \Omega_X^1(\log D) & \xlongequal{\quad} & f^* F_X^* \Omega_X^1(\log D) & \xrightarrow{f^*(\zeta_X)} & f^* \Omega_X^1(\log D) \\ & \searrow F_Y^*(df) & & & \downarrow df \\ & & F_Y^* \Omega_Y^1(\log B) & \xrightarrow{\zeta_Y} & \Omega_Y^1(\log B) \end{array}$$

is also commutative. Now let (E, θ) be an object of $\text{HIG}_{\leq p-1}^{\text{lf}}(X, D)$ and let us write $f^*(E, \theta) = (f^*E, \theta_Y)$. Then we set $C_{(\tilde{X}, \tilde{D})}^{-1}(E, \theta) = (F_X^*E, \nabla)$, where

$$\nabla := \nabla_{\text{can}} + (\text{Id}_{F_X^*E} \otimes \zeta_X) \circ (F_X^*\theta)$$

and ∇_{can} is the canonical connection on F_X^*E appearing in Cartier's descent theorem (i.e., ∇_{can} is differentiation along the fibers of the Frobenius morphism). Similarly, we can define $C_{(\tilde{Y}, \tilde{B})}^{-1}$. Since $f^*(F_X^*E, \nabla_{\text{can}}) = (F_Y^*(f^*E), \nabla_{\text{can}})$, the above diagram shows that

$$\begin{aligned} f^* C_{(\tilde{X}, \tilde{D})}^{-1}(E, \theta) &= f^*(F_X^*E, \nabla) = (F_Y^* f^*E, \nabla_{\text{can}} + (\text{Id}_{F_Y^* f^*E} \otimes \zeta_Y) \circ (F_Y^*\theta_Y)) \\ &= C_{(\tilde{Y}, \tilde{B})}^{-1} f^*(E, \theta). \end{aligned}$$

Step 2. Now let us assume that we have two pairs $(\tilde{F}_X^1, \tilde{F}_Y^1)$ and $(\tilde{F}_X^2, \tilde{F}_Y^2)$ of compatible global logarithmic liftings of the Frobenius morphism on X and Y . There exists an \mathcal{O}_X -linear map h_{12}^X such that the following diagram is commutative:

$$\begin{array}{ccc}
 \mathcal{O}_{\tilde{X}} & \xrightarrow{(\tilde{F}_X^2)^* - (\tilde{F}_X^1)^*} & p\tilde{F}_* \mathcal{O}_{\tilde{X}} \\
 \downarrow & & \uparrow \\
 \mathcal{O}_X & & p \simeq \\
 \downarrow d & & \uparrow \\
 \Omega_X^1(\log D) & \xrightarrow{h_{12}^X} & F_* \mathcal{O}_X
 \end{array}$$

By abuse of notation we let $h_{12}^X : F_* \Omega_X(\log D) \rightarrow \mathcal{O}_X$ be adjoint to h_{12}^X . Similarly, one can define $h_{12}^Y : F_* \Omega_Y(\log B) \rightarrow \mathcal{O}_Y$. It is straightforward to check that we have a commutative diagram

$$\begin{array}{ccccc}
 F_Y^* f^* \Omega_X^1(\log D) & \xlongequal{\quad} & f^* F_X^* \Omega_X^1(\log D) & \xrightarrow{f^*(h_{12}^X)} & f^* \mathcal{O}_X \\
 \searrow F_Y^*(df) & & & & \parallel \\
 & & F_Y^* \Omega_Y^1(\log B) & \xrightarrow{h_{12}^Y} & \mathcal{O}_Y
 \end{array}$$

Now let us define a map

$$\tau_{12}^X : F^* E \xrightarrow{F^* \theta} F^* E \otimes F^*(\Omega_X(\log D)) \xrightarrow{\text{Id} \otimes h_{12}^X} F^* E.$$

Similarly we define $\tau_{12}^Y : F^*(f^* E) \rightarrow F^*(f^* E)$. The above diagram shows that $\tau_{12}^Y = f^* \tau_{12}^X$.

Step 3. Now we consider the general situation. Let (E, θ) be an object of $\text{HIG}_{\leq p-1}^{\text{lf}}(X, D)$. By assumption there exist étale coverings $\{\tilde{U}_\alpha\}_{\alpha \in I}$ of \tilde{X} and $\{\tilde{V}_\alpha\}_{\alpha \in I}$ of \tilde{Y} such that we have compatible logarithmic liftings $(\tilde{F}_{X,\alpha}, \tilde{F}_{Y,\alpha})$ of the Frobenius morphisms $F_{X,\alpha} : U_\alpha \rightarrow U_\alpha$ and $F_{Y,\alpha} : V_\alpha \rightarrow V_\alpha$.

Let us recall the construction of $(M, \nabla) = C_{(\tilde{X}, \tilde{D})}^{-1}(E, \theta) \in \text{MIC}(X, D)$ after [19] and [18, Appendix]. Over each U_α we define $(M_\alpha, \nabla_\alpha)$ by using Step 1 and setting

$$(M_\alpha, \nabla_\alpha) := C_{(\tilde{U}_\alpha, \tilde{D} \cap \tilde{U}_\alpha)}^{-1}(E, \theta).$$

Over $U_{\alpha\beta} = U_\alpha \times_X U_\beta$ we can use two liftings $\tilde{F}_{X,\alpha}|_{U_{\alpha\beta}}$ and $\tilde{F}_{X,\beta}|_{U_{\alpha\beta}}$ of the Frobenius morphism $F : U_{\alpha\beta} \rightarrow U_{\alpha\beta}$ to define $\tau_{\alpha\beta}^X : F^*(E_{U_{\alpha\beta}}) \rightarrow F^*(E_{U_{\alpha\beta}})$ as in Step 2. Then we glue $(M_\alpha, \nabla_\alpha)$ and (M_β, ∇_β) over $U_{\alpha\beta}$ to a global object $(M, \nabla) \in \text{MIC}(X, D)$ using

$$g_{\alpha\beta}^X := \exp(\tau_{\alpha\beta}^X) = \sum_{i=0}^{p-1} \frac{(\tau_{\alpha\beta}^X)^i}{i!}.$$

Here we use the fact that the categories of quasi-coherent sheaves in the Zariski and étale toposes of X are equivalent (and we can replace a connection by an appropriate

\mathcal{O}_X -linear map using Grothendieck's description of connections). We can also define $\zeta_{Y,\alpha}$, $\tau_{\alpha\beta}^Y$ and $g_{\alpha\beta}^Y$. We already know that

$$f^*(M_\alpha, \nabla_\alpha) = f^*C_{(\tilde{U}_\alpha, \tilde{D} \cap \tilde{U}_\alpha)}^{-1}(E_{U_\alpha}, \theta|_{U_\alpha}) = C_{(\tilde{V}_\alpha, \tilde{B} \cap \tilde{V}_\alpha)}^{-1}(f^*(E, \theta)|_{V_\alpha})$$

and $\tau_{\alpha\beta}^Y = f^*\tau_{\alpha\beta}^X$. In particular, $g_{\alpha\beta}^Y = f^*g_{\alpha\beta}^X$, which shows that gluing maps agree and

$$f^*C_{(\tilde{X}, \tilde{D})}^{-1}(E, \theta) = C_{(\tilde{Y}, \tilde{B})}^{-1}f^*(E, \theta). \quad \blacksquare$$

Remark 6.5. The above isomorphism of functors holds more generally without restricting to locally free logarithmic Higgs sheaves. We added this assumption only to ensure that $\mathcal{T}or_1(f^*E, F_*\mathcal{O}_{B_i}) = 0$ for all irreducible components B_i of B . This allows us to conclude that the image is in $\text{MIC}_{\leq p-1}(Y, B)$.

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References

- [1] Arapura, D.: Kodaira–Saito vanishing via Higgs bundles in positive characteristic. *J. Reine Angew. Math.* **755**, 293–312 (2019) Zbl [1468.14044](#) MR [4015235](#)
- [2] Bloch, S., Gieseker, D.: The positivity of the Chern classes of an ample vector bundle. *Invent. Math.* **12**, 112–117 (1971) Zbl [0212.53502](#) MR [297773](#)
- [3] Brunebarbe, Y.: Symmetric differentials and variations of Hodge structures. *J. Reine Angew. Math.* **743**, 133–161 (2018) Zbl [1403.14026](#) MR [3859271](#)
- [4] Brunebarbe, Y.: Semi-positivity from Higgs bundles. arXiv:[1707.08495v1](#) (2017)
- [5] Deligne, P.: La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.* **52** 137–252 (1980) Zbl [0456.14014](#) MR [601520](#)
- [6] Diaz, S., Harbater, D.: Strong Bertini theorems. *Trans. Amer. Math. Soc.* **324**, 73–86 (1991) Zbl [0744.14004](#) MR [986689](#)
- [7] Ejiri, S.: Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers. *J. Algebraic Geom.* **26**, 691–734 (2017) Zbl [1400.14026](#) MR [3683424](#)
- [8] Fujino, O., Fujisawa, T.: Variations of mixed Hodge structure and semipositivity theorems. *Publ. RIMS Kyoto Univ.* **50**, 589–661 (2014) Zbl [1305.14004](#) MR [3273305](#)
- [9] Fujita, T.: On Kähler fiber spaces over curves. *J. Math. Soc. Japan* **30**, 779–794 (1978) Zbl [0393.14006](#) MR [513085](#)
- [10] Griffiths, P. A.: Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. *Inst. Hautes Études Sci. Publ. Math.* **38**, 125–180 (1970) Zbl [0367.14001](#) MR [282990](#)
- [11] Hartshorne, R.: *Algebraic Geometry*. Grad. Texts in Math. 52, Springer, New York (1977) Zbl [0367.14001](#) MR [0463157](#)
- [12] Huybrechts, D., Lehn, M.: *The Geometry of Moduli Spaces of Sheaves*. 2nd ed., Cambridge Math. Library, Cambridge Univ. Press, Cambridge (2010) Zbl [1206.14027](#) MR [2665168](#)
- [13] Illusie, L.: Réduction semi-stable et décomposition de complexes de de Rham à coefficients. *Duke Math. J.* **60**, 139–185 (1990) Zbl [0708.14014](#) MR [1047120](#)

- [14] Katz, N. M.: Algebraic solutions of differential equations (p -curvature and the Hodge filtration). *Invent. Math.* **18**, 1–118 (1972) Zbl [0278.14004](#) MR [337959](#)
- [15] Kawamata, Y.: Characterization of abelian varieties. *Compos. Math.* **43**, 253–276 (1981) Zbl [0471.14022](#) MR [622451](#)
- [16] Kleiman, S. L.: The Picard scheme. In: *Fundamental Algebraic Geometry*, Math. Surveys Monogr. 123, Amer. Math. Soc., Providence, RI, 235–321 (2005) Zbl [1085.14001](#) MR [2223410](#)
- [17] Kollár, J., Mori, S.: *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Math. 134, Cambridge Univ. Press, Cambridge (1998) Zbl [0926.14003](#) MR [1658959](#)
- [18] Lan, G., Sheng, M., Yang, Y., Zuo, K.: Uniformization of p -adic curves via Higgs–de Rham flows. *J. Reine Angew. Math.* **747**, 63–108 (2019) Zbl [1439.14115](#) MR [3905130](#)
- [19] Lan, G., Sheng, M., Zuo, K.: Nonabelian Hodge theory in positive characteristic via exponential twisting. *Math. Res. Lett.* **22**, 859–879 (2015) Zbl [1326.14016](#) MR [3350108](#)
- [20] Lan, G., Sheng, M., Zuo, K.: Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups. *J. Eur. Math. Soc.* **21**, 3053–3112 (2019) Zbl [1444.14048](#) MR [3994100](#)
- [21] Langer, A.: Moduli spaces of sheaves and principal G -bundles. In: *Algebraic Geometry (Seattle, 2005)*, Part 1, Proc. Sympos. Pure Math. 80, Amer. Math. Soc., Providence, RI, 273–308 (2009) Zbl [1179.14010](#) MR [2483939](#)
- [22] Langer, A.: On positivity and semistability of vector bundles in finite and mixed characteristics. *J. Ramanujan Math. Soc.* **28A**, 287–309 (2013) Zbl [1295.14032](#) MR [3115197](#)
- [23] Langer, A.: Semistable modules over Lie algebroids in positive characteristic. *Doc. Math.* **19**, 509–540 (2014) Zbl [1330.14017](#) MR [3218782](#)
- [24] Langer, A.: Bogomolov’s inequality for Higgs sheaves in positive characteristic. *Invent. Math.* **199**, 889–920 (2015) Zbl [1348.14048](#) MR [3314517](#)
- [25] Langer, A.: The Bogomolov–Miyaoaka–Yau inequality for logarithmic surfaces in positive characteristic. *Duke Math. J.* **165**, 2737–2769 (2016) Zbl [1386.14160](#) MR [3551772](#)
- [26] Ogus, A., Vologodsky, V.: Nonabelian Hodge theory in characteristic p . *Publ. Math. Inst. Hautes Études Sci.* **106**, 1–138 (2007) Zbl [1140.14007](#) MR [2373230](#)
- [27] Patakfalvi, Z.: Semi-positivity in positive characteristics. *Ann. Sci. École Norm. Sup. (4)* **47**, 991–1025 (2014) Zbl [1326.14015](#) MR [3294622](#)
- [28] Patakfalvi, Z.: On subadditivity of Kodaira dimension in positive characteristic over a general type base. *J. Algebraic Geom.* **27**, 21–53 (2018) Zbl [1397.14030](#) MR [3722689](#)
- [29] Schepler, D. K.: Logarithmic nonabelian Hodge theory in characteristic p . *arXiv:0802.1977* (2008)
- [30] Sheng, M.: Twisted functoriality in nonabelian Hodge theory in positive characteristic. *arXiv:2105.01385* (2021)
- [31] Simpson, C. T.: Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.* **75**, 5–95 (1992) Zbl [0814.32003](#) MR [1179076](#)
- [32] Sun, R., Yang, J., Zuo, K.: Projective crystalline representations of étale fundamental groups and twisted periodic Higgs–de Rham flow. *arXiv:1709.01485* (2017)
- [33] Szpiro, L.: Sur le théorème de rigidité de Parsin et Arakelov. In: *Journées de Géométrie Algébrique de Rennes (Rennes, 1978)*, Vol. II, Astérisque **64**, 169–202 (1979) Zbl [0425.14005](#) MR [563470](#)
- [34] Szpiro, L. (ed.): *Séminaire sur les pincesaux de courbes de genre au moins deux*. Astérisque **86** (1981) Zbl [0463.00009](#) MR [0642675](#)
- [35] Wahl, J. M.: Equisingular deformations of normal surface singularities. I. *Ann. of Math. (2)* **104**, 325–356 (1976) Zbl [0358.14007](#) MR [422270](#)
- [36] Zuo, K.: On the negativity of kernels of Kodaira–Spencer maps on Hodge bundles and applications. *Asian J. Math.* **4**, 279–301 (2000) Zbl [0983.32020](#) MR [1803724](#)