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Some Cyclic Group Actions on a Homotopy Sphere and the Parallelizability of its Orbit Spaces

By

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§ 1. Introduction

In this paper, we introduce a way to define a free cyclic group action on a homotopy sphere and examine the stable parallelizability of its orbit spaces. J. Ewing et al [3] answered the stable parallelizability problem for the classical lens space, that is, the orbit space of the standard sphere under a linear cyclic group action.

Let $w_1, w_2, \dots, w_{n+1}$ be positive rational numbers. A polynomial $f(z_1, z_2, \dots, z_{n+1})$ is called a *weighted homogeneous polynomial* of type $(w_1, w_2, \dots, w_{n+1})$ if it can be expressed as a linear combination of monomials

$$z_1^{i_1} z_2^{i_2} \cdots z_{n+1}^{i_{n+1}}$$

for which $\sum_{j=1}^{n+1} i_j/w_j = 1$. This is equivalent to the requirement that

$$f(e^{c/w_1} z_1, e^{c/w_2} z_2, \dots, e^{c/w_{n+1}} z_{n+1}) = e^c f(z_1, z_2, \dots, z_{n+1})$$

for every complex number c.

Throughout this paper, we assume that all weighted homogeneous polynomials have an isolated critical point at the origin. For example, the Brieskorn polynomial

$$f(z_1, z_2, \dots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \dots + z_{n+1}^{a_{n+1}},$$

all $a_i \geq 2$, is a weighted homogeneous polynomial of weights $w = (a_1,$

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$a_2, \dots, a_{n+1})$.

Set $\sum_w = f^{-1}(0) \cap S^{2n+1}$, and consider the Milnor fibering $g\colon S^{2n+1} - \sum_w \longrightarrow S^1$ defined by

$$g(z_1, \dots, z_{n+1}) = f(z_1, \dots, z_{n+1}) / |f(z_1, \dots, z_{n+1})|,$$

then each fiber $F_t = g^{-1}(e^{it})$ is a smooth parallelizable $2n$-dimensional manifold with the homotopy type of a bouquet of n-spheres. We can obtain $S^{2n+1} - \sum_w$ from $F \times [0, 1]$ by identifying $F \times 0$ and $F \times 1$ by a homeomorphism $h\colon F \longrightarrow F$ called the characteristic map. Denote the characteristic polynomial of

$$h_*\colon H_n(F; \boldsymbol{C}) \longrightarrow H_n(F; \boldsymbol{C})$$

by

$$\Delta(t) = \text{determinant } (tI_* - h_*),$$

where I is the identity map of F. This characteristic map h_* and its characteristic polynomial $\Delta(t)$ are fundamental topological invariants. Brieskorn [2] computed $\Delta(t)$ for varieties defined by Brieskorn polynomials, and Milnor and Orlik [9] did it for weighted homogeneous polynomials.

The following theorem answers whether or not the $2n-1$ dimensional manifold $\sum_w = f^{-1}(0) \cap S^{2n+1}$ is a topological sphere.

Theorem ([8], Section. 8). *For $n \geq 3$, the followings are equivalent:*

i) *$\sum_w$ is a topological sphere.*

ii) $H_{n-1}(\sum_w) = 0$.

iii) *The intersection pairing $H_n(F; \boldsymbol{Z}) \otimes H_n(F; \boldsymbol{Z}) \longrightarrow \boldsymbol{Z}$ has determinant ± 1.*

iv) $\Delta(1) = \pm 1$.

Furthermore, if $\sum_w$ is a topological sphere, the diffeomorphism class of $\sum_w$ is completely determined by the signature of the intersection pairing

$$H_n(F; \boldsymbol{Z}) \otimes H_n(F; \boldsymbol{Z}) \longrightarrow \boldsymbol{Z}$$

if n is even. If n is odd, $\sum_w$ is

the standard sphere if $\Delta(-1) = \pm 1 \pmod 8$,
the Kervaire sphere if $\Delta(-1) = \pm 3 \pmod 8$.

Let $\sum_w = f^{-1}(0) \cap S^{2n+1}$ be a topological sphere, where f is a weighted homogeneous polynomial of weight $w=(w_1, w_2, \ldots, w_{n+1})$, say $w_i=u_i/v_i$ in irreducible form for $i=1,2,\ldots,n+1$, and let p be an odd prime number relatively prime to each u_i. To define a free cyclic group $\boldsymbol{Z}_p$-action on $\sum_w$, choose natural numbers b_i such that $b_i=h/w_i=hv_iu_i^{-1}$ (mod p) for some $h\neq 0$ (mod p) and $(b_i, p)=1$ for all i, where (b_i, p) denotes the greatest common divisor of b_i and p. Now, we define a map T on $\sum_w$ by

$$T(z_1, z_2, \ldots, z_{n+1})=(\zeta^{b_1}z_1, \zeta^{b_2}z_2, \ldots, \zeta^{b_{n+1}}z_{n+1}),$$

where $\zeta=e^{2\pi i/p}$. Then

$$f(T(z_1, z_2, \ldots, z_{n+1}))=\zeta^h f(z_1, z_2, \ldots, z_{n+1}).$$

This is a well-defined free action on $\sum_w$ generating the cyclic group $\boldsymbol{Z}_p$. Denote its orbit space by $L(p;w;b)$. Note that we may assume that $h=1$ (mod p), i. e., $w_ib_i=1$ (mod p) for all i by taking a suitable generator T of $\boldsymbol{Z}_p$.

§ 2. An Algebraic Characterization of Stable Parallelizability

Define a $\boldsymbol{Z}_p$-action on $\sum_w \times \boldsymbol{C}$ by $T'(z, \eta)=(T(z), \zeta\eta)$, where ζ and $T(z)$ are the same as above, so that the natural projection from $\sum_w \times \boldsymbol{C}$ to $\sum_w$ is equivariant, that is, it commutes with the $\boldsymbol{Z}_p$-actions. By taking quotients, one can get the canonical complex line bundle γ over $L(p;w;b)$. Similarly, one can get $\gamma^b=\gamma\otimes\gamma\otimes\ldots\otimes\gamma$, ($b$ times) with a $\boldsymbol{Z}_p$-action on $\sum_w \times \boldsymbol{C}$ given by $T'(z, \eta)=(T(z), \zeta^b\eta)$. It can be proved easily that

$$\gamma^{b_1}\oplus\gamma^{b_2}\oplus\ldots\oplus\gamma^{b_{n+1}}=\textstyle\sum_w \times \boldsymbol{C}^{n+1}/T\times T,$$

where

$$(T\times T)(z, (\eta_1, \eta_2, \ldots, \eta_{n+1}))=(T(z), (\zeta^{b_1}\eta_1, \zeta^{b_2}\eta_2, \ldots, \zeta^{b_{n+1}}\eta_{n+1})).$$

To reduce the question of stable parallelizability of the orbit space $L(p;w;b)$ to a purely algebraic one, we first describe the tangent bundle of $L(p;w;b)$.

Theorem 2.1. *Over* $L(p;w;b)$, $\tau\oplus\varepsilon\oplus\mathrm{re}(\gamma)$ *is isomorphic to*

$$\mathrm{re}(\gamma^{b_1}\oplus\gamma^{b_2}\oplus\ldots\oplus\gamma^{b_{n+1}}),$$

where τ *denotes the tangent bundle,* ε *the trivial* 1*-dimensional real bundle* *over* $L(p;w;b)$, *and* re *the realification of a bundle.*

Proof. Let $\tau(.)$ denote the tangent bundle and $\nu(.)$ the normal bundle of the space (.) in $\boldsymbol{C}^{n+1}$, then the trivial bundle $\sum_w \times \boldsymbol{C}^{n+1}$ is isomorphic to

$$\tau(\textstyle\sum_w)\oplus\nu(\sum_w)=\tau(\sum_w)\oplus\nu(S^{2n+1})\oplus\nu(f^{-1}(0))$$

over $\sum_w$. But $\nu(S^{2n+1})$ is trivial and grad f is a cross section of $\nu(f^{-1}(0))$, so that $\nu(f^{-1}(0))=\boldsymbol{C}\cdot$grad f. Define

$$\Phi\colon \tau(\textstyle\sum_w)\oplus\boldsymbol{R}\oplus\boldsymbol{C}\longrightarrow\sum_w\times\boldsymbol{C}^{n+1}$$

by

$$\Phi(v_z, r, \eta)=(z, v+rz+\eta \text{ grad } f(z)),$$

where v_z denotes a tangent vector at z and $\boldsymbol{R}$, $\boldsymbol{C}$ represent the trivial bundles $\boldsymbol{R}\times\sum_w$, $\boldsymbol{C}\times\sum_w$ respectively. By using ζ grad $f(Tz)=T$(grad $f(z)$), we can see that Φ is an equivariant isomorphism from $\tau(\sum_w)\oplus\boldsymbol{R}\oplus\boldsymbol{C}$ with $\boldsymbol{Z}_p$-action given by $dT\oplus I\oplus(\cdot\zeta)$ to $\sum_w\times\boldsymbol{C}^{n+1}$ with $\boldsymbol{Z}_p$-action given by $T\times T$. Therefore, by taking quotients, it is proved.

Remark. In Theorem 2.1, if $L(p;w;b)$ is defined as an orbit space of a Brieskorn sphere, then we have

$$\tau\oplus\varepsilon\oplus\mathrm{re}(\gamma)\simeq\mathrm{re}(\gamma^{b_1}\oplus\gamma^{b_2}\oplus\dots\oplus\gamma^{b_{n+1}}).$$

This is the correction of Orlik's theorem 3 ([12], p. 252).

Recall that the standard lens space $L^{2n-1}(p)$ is defined as the orbit space of S^{2n-1} by the linear action. Since the principal $\boldsymbol{Z}_p$-bundles

$$S^{2n-1}\longrightarrow L^{2n-1}(p) \text{ and } \textstyle\sum_w\longrightarrow L(p;w;b)$$

are $2n-1$ universal, there are maps

$$f\colon L^{2n-1}(p)\longrightarrow L(p;w;b) \text{ and } g\colon L(p;w;b)\longrightarrow L^{2n-1}(p)$$

such that the induced bundles $f^*\gamma=\gamma$ and $g^*\gamma=\gamma$, where γ is the canonical bundle over the suitable orbit space. Hence, Theorem 2.1 implies the following:

Lemma 2.2. *The space* $L(p;w;b)$ *is stably parallelizable if and only if* re(γ) *is stably isomorphic to*

$$\mathrm{re}(\gamma^{b_1})\oplus\mathrm{re}(\gamma^{b_2})\oplus\dots\oplus\mathrm{re}(\gamma^{b_{n+1}})$$

over the standard lens space $L^{2n-1}(p)$, *where* γ *represents the canonical bundle over* $L^{2n-1}(p)$.

Recall that the mod p cohomology ring of the standard lens space $L^{2n-1}(p)$ is the tensor product

$$H^*(L^{2n-1}(p)\,;\boldsymbol{Z}_p)\simeq \Lambda(u)\otimes \boldsymbol{Z}_p[v]/(v^n)$$

of the exterior algebra $\Lambda(u)$ and the truncated polynomial ring generated by v, where $\deg u=1$, $\deg v=2$, and $\beta^*_p(u)=v$ for the Bockstein isomorphism

$$\beta^*_p\colon H^1(L^{2n-1}(p)\,;\boldsymbol{Z}_p)\longrightarrow H^2(L^{2n-1}(p)\,;\boldsymbol{Z}_p).$$

Lemma 2.3. *If the space* $L(p\,;w\,;b)$ *is stably parallelizable, then*

$$1+v^2=(1+b_1^2v^2)(1+b_2^2v^2)\cdots(1+b_{n+1}^2v^2)$$

in $\boldsymbol{Z}_p[v]/(v^n)$.

Proof. By Lemma 2.2 and the hypothesis, the mod p reduction of the total Pontrjagin class of $\mathrm{re}(\gamma)$ is equal to that of

$$\mathrm{re}(\gamma^{b_1})\oplus\mathrm{re}(\gamma^{b_2})\oplus\dots\oplus\mathrm{re}(\gamma^{b_{n+1}}),$$

where γ is the canonical line bundle over $L^{2n-1}(p)$. Let ξ be the canonical line bundle over $CP(n-1)$, then the induced bundle $\pi^*(\xi)$ over $L^{2n-1}(p)$ is clearly the line bundle γ, where $\pi\colon L^{2n-1}(p)\longrightarrow CP(n-1)$ is the natural projection. Note that $H^*(CP(n-1)\,;\boldsymbol{Z}_p)\simeq\boldsymbol{Z}_p[w]/(w^n)$. The Gysin sequence of the principal bundle $S^1\longrightarrow L^{2n-1}(p)\longrightarrow CP(n-1)$ with $\boldsymbol{Z}_p$ coefficients is

$$\longrightarrow H^1(CP(n-1))\xrightarrow{\pi^*}H^1(L^{2n-1}(p))\longrightarrow H^0(CP(n-1))$$
$$\longrightarrow H^2(CP(n-1))\xrightarrow{\pi^*}H^2(L^{2n-1}(p))\longrightarrow H^1(CP(n-1)),$$

in which $H^2(CP(n-1))\xrightarrow{\pi^*}H^2(L^{2n-1}(p))$ must be an isomorphism. By the naturality of Chern classes,

$$c_1(\gamma)=c_1(\pi^*(\xi))=\pi^*(c_1(\xi))=\pi^*(w)=v.$$

The first Pontrjagin class $P_1(\mathrm{re}(\gamma))$ comes from the identity

$$1-P_1(\mathrm{re}(\gamma))=(1-c_1(\gamma))(1+c_1(\gamma))=1-v^2.$$

Hence, the total Pontrjagin class of $\mathrm{re}(\gamma)$ in mod p is $P(\mathrm{re}(\gamma))=1+P_1(\mathrm{re}(\gamma))=1+v^2$. Since $c_1(\mu\otimes\nu)=c_1(\mu)+c_1(\nu)$ for any line bundles μ, ν,

$$P_1(\mathrm{re}(\gamma^{b_j}))=(c_1(\gamma^{b_j}))^2=(b_jc_1(\gamma))^2=b_j^2v^2.$$

Therefore, $P(\mathrm{re}(\gamma^{b_j}))=1+b_j^2v^2$, and

$$\begin{aligned}1+v^2&=P(\mathrm{re}(\gamma))\\&=P(\mathrm{re}(\gamma^{t_1})\oplus\mathrm{re}(\gamma^{b_2})\oplus\dots\oplus\mathrm{re}(\gamma^{b_{n+1}}))\\&=P(\mathrm{re}(\gamma^{b_1}))\cdot P(\mathrm{re}(\gamma^{b_2}))\cdots P(\mathrm{re}(\gamma^{b_{n+1}}))\\&=(1+b_1^2v^2)(1+b_2^2v^2)\cdots(1+b_{n+1}^2v^2)\end{aligned}$$

in $\mathbb{Z}_p[v]/(v^n)$, by the product formular of the Pontrjagin class.

From theorem 2.1, one can also get the total Pontrjagin and Stiefel-Whitney classes of the space $L(p;w;b)$.

Corollary 2.4.

$$P(L(p;w;b))=(1+v^2)^{-1}\prod_{i=1}^{n+1}(1+b_i^2v^2),$$

$$w(L(p;w;b))=(1+u)^{-1}\prod_{i=1}^{n+1}(1+b_iu),$$

where v is a prefered generator for $H^2(L(p;w;b);\mathbb{Z})$, so that the total Chern class of γ is $1+v$, and u is its mod 2 *reduction.*

In [5], $\widetilde{KO}(L^{2n-1}(p))$ is computed. Setting $\bar{\sigma}=\mathrm{re}(\gamma)-2$, the p-torsion part of $\widetilde{KO}(L^{2n-1}(p))$ is a direct summand of cyclic groups generated by $\bar{\sigma}^i$, $1\leq i\leq(p-1)/2$, where if $n-1=s(p-1)+r$, $0\leq r<p-1$, the order of $\bar{\sigma}^i$ is p^{s+1} for $i\leq[r/2]$ and p^s for $i>[r/2]$.

Lemma 2.5. *If $L(p;w;b)$ is stably parallelizable, then $n-1$ is less than p.*

Proof. Let $L(p;w;b)$ be stably parallelizable, then $\mathrm{re}(\gamma)$ is stably isomorphic to

$$\mathrm{re}(\gamma^{b_1})\oplus\mathrm{re}(\gamma^{b_2})\oplus\dots\oplus\mathrm{re}(\gamma^{b_{n+1}})$$

over the standard lens space $L^{2n-1}(p)$, which gives

$$\mathrm{re}(\gamma)-2=(\mathrm{re}(\gamma^{b_1})-2)+\dots+(\mathrm{re}(\gamma^{b_{n+1}})-2)$$

in $\widetilde{KO}(L^{2n-1}(p))$. Since $\widetilde{KO}(L^{2n-1}(p))$ is abelian, we can assume that $b_1\leq b_2\leq\dots\leq b_{n+1}$. By taking the diffeomorphic copy of $L(p;w;b)$ under the complex conjugation of the i-th coordinate, if it is needed, we may assume that $b_1 \leq b_2 \leq \ldots \leq b_{n+1} \leq (p-1)/2$. Let $n-1=s(p-1)+r$, $0 \leq r < p-1$, and set $\bar{\sigma}=\mathrm{re}(\gamma)-2$, then $\bar{\sigma}^i$, $1 \leq i \leq (p-1)/2$, are generators of the cyclic subgroups of the p-torsion part of $\widetilde{KO}(L^{2n-1}(p))$, and their orders are p^s or p^{s+1}. On the other hand,

$$(\mathrm{re}(\gamma^{b_1})-2)+(\mathrm{re}(\gamma^{b_2})-2)+\ldots+(\mathrm{re}(\gamma^{b_{n+1}})-2)$$

can be written as a polynomial of $\bar{\sigma}$. So, we can set

$$\bar{\sigma}=\alpha_{b_{n+1}}+\alpha_{b_{n+1}-1}\bar{\sigma}+\ldots+\alpha_0\bar{\sigma}^{b_{n+1}}$$

with some coefficients α_i's, so that $\alpha_{b_{n+1}-1}=1 \pmod{p^s}$, and all other coefficients are divided by p^s. And α_0 is also the number of b_j's such that $b_j=b_{n+1}$ in $b_1 \leq b_2 \leq \ldots \leq b_{n+1}$, because

$$\mathrm{re}(\gamma^{b_{n+1}})-2=\bar{\sigma}^{b_{n+1}}+\text{terms of lower degree of } \bar{\sigma}.$$

Similarly, for any b with $1 < b \leq (p-1)/2$, the number of copies of $\mathrm{re}(\gamma^b)-2$ in

$$(\mathrm{re}(\gamma^{b_1})-2)+(\mathrm{re}(\gamma^{b_2})-2)+\ldots+(\mathrm{re}(\gamma^{b_{n+1}})-2)$$

must be divided by p^s. Now, let β be the number of copies of $\mathrm{re}(\gamma)-2$ in

$$(\mathrm{re}(\gamma^{b_1})-2)+(\mathrm{re}(\gamma^{b_2})-2)+\ldots+(\mathrm{re}(\gamma^{b_{n+1}})-2),$$

then $\beta+\beta'=\alpha_{b_{n+1}-1}=1 \pmod{p^s}$, where β' is the coefficient of $\bar{\sigma}$ in the polynomial of $\bar{\sigma}$ for

$$(\mathrm{re}(\gamma^{b_1})-2)+\ldots+(\mathrm{re}(\gamma^{b_{n+1}})-2)-\beta(\mathrm{re}(\gamma)-2).$$

On the other hand, β' is divided by p^s, so $\beta=1 \pmod{p^s}$. Since the total number of b_i's is $n+1$, $\beta+hp^s=n+1$ for some h, so $n=s(p-1)+r+1 \pmod{p^s}$. The only possibility is $s=0$, or $s=1$ and $r=0$. In both cases, $n-1$ is less than p.

The next lemma will be useful to prove the main theorem.

Lemma 2.6 ([3]). *Let ξ, η be oriented vector bundles over a finite CW complex X, and suppose that*

i) $\dim(X) < 2p+2$, *p an odd prime, and*

ii) $H^{4*}(X;\mathbf{Z})$ *has no q-torsion for any $q < p$.*

If their Pontrjagin classes $P(\xi)$, $P(\eta)$ are equal, then $(\xi-\eta)-(\dim\xi-\dim\eta) \in \widetilde{KO}(X)$ is a 2-torsion element.

Theorem 2.7. *The space $L(p;w;b)$ is stably parallelizable if and only if*

i) $n-1$ *is less than* p, *and*

ii) $(1+b_1^2v^2)(1+b_2^2v^2)\cdots(1+b_{n+1}^2v^2)=1+v^2$ *in* $\boldsymbol{Z}_p[v]/(v^n)$, *or equivalently*

$$b_1^{2j}+b_2^{2j}+\ldots+b_{n+1}^{2j}=1 \pmod p \quad \textit{for } j=1,2,\ldots,[(n-1)/2].$$

Proof. The "only if" part comes from Lemmas 2.3–2.5. Let us assume i) and ii). Then, the mod p Pontrjagin class of $\mathrm{re}(\gamma)$ is equal to that of $\mathrm{re}(\gamma^{b_1})\oplus\mathrm{re}(\gamma^{b_2})\oplus\ldots\oplus\mathrm{re}(\gamma^{b_{n+1}})$. By Lemma 2.6,

$$\mathrm{re}(-\gamma\oplus\gamma^{b_1}\oplus\gamma^{b_2}\oplus\ldots\oplus\gamma^{b_{n+1}})-2n$$

is a 2-torsion element in $\widetilde{KO}(L^{2n-1}(p))$. But it is clearly in the image of

$$\mathrm{re}\colon \tilde{K}(L^{2n-1}(p))\longrightarrow\widetilde{KO}(L^{2n-1}(p)),$$

which does not contain any 2-torsion element. So it must be a zero element. Therefore, $\mathrm{re}(\gamma)$ is stably isomorphic to $\mathrm{re}(\gamma^{b_1})\oplus\mathrm{re}(\gamma^{b_2})\oplus\cdots\oplus\mathrm{re}(\gamma^{b_{n+1}})$ over $L^{2n-1}(p)$, and $L(p;w;b)$ is stably parallelizable.

§ 3. Some Examples

Milnor and Orlik [9] gave the computation of $\Delta(1)$ as follows: Let $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$ denote the group with the multiplication. To each monic polynomial

$$(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_k),\ \alpha_i\in\boldsymbol{C}^*,$$

assign the divisor

$$\begin{aligned}\text{divisor }&((t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_k))\\ &=\langle\alpha_1\rangle+\langle\alpha_2\rangle+\cdots+\langle\alpha_k\rangle\end{aligned}$$

as an element of the rational group ring $\boldsymbol{QC}^*$. Denote

$$\begin{aligned}\Lambda_k&=\text{divisor}(t^k-1)\\ &=\langle 1\rangle+\langle\xi\rangle+\ldots+\langle\xi^{k-1}\rangle,\end{aligned}$$

where $\xi=e^{2\pi i/k}$. Note that $\Lambda_a\Lambda_b=(a,b)\Lambda_{[a,b]}$, where $[a,b]$ denotes their least common multiple and (a,b) the greatest common divisor. Then, for a weighted homogeneous polynomial $f(z_1,z_2,\ldots,z_{n+1})$ of type $w=(w_1,w_2,\ldots,w_{n+1})$, the characteristic polynomial $\Delta(t)$ =determinant

$(tI_* - h_*)$ of the linear transformation $h_*: H_n(F;\mathbf{C}) \longrightarrow H_n(F;\mathbf{C})$ is determined by

$$\text{divisor } \Delta = (v_1^{-1}\Lambda_{u_1} - 1)(v_2^{-1}\Lambda_{u_2} - 1) \dots (v_{n+1}^{-1}\Lambda_{u_{n+1}} - 1),$$

where $w_i = u_i/v_i$, $i = 1, 2, \dots, n+1$, is the expression in irreducible form.

To make the computation of $\Delta(1)$ easy, we cite two Milnor-Orlik's theorems.

Theorem 3.1 ([9]). *By using $\Lambda_a\Lambda_b = (a, b)\Lambda_{[a,b]}$, divisor Δ can be expressed as a linear combination of the divisors Λ_r. Let*

$$\text{divisor } \Delta = a_1\Lambda_1 + a_2\Lambda_2 + \dots + a_s\Lambda_s,$$

and define two numbers $k(\Delta)$ and $\rho(\Delta)$ by the formular

$$k(\Delta) = a_1 + a_2 + \dots + a_s, \text{ and } \rho(\Delta) = 2^{a_2}3^{a_3} \cdots s^{a_s}.$$

Then, $k(\Delta)$ and $\rho(\Delta)$ are non-negative integers, and

$$\Delta(1) = \rho(\Delta) \qquad \text{if } k(\Delta) = 0,$$
$$\Delta(1) = 0 \qquad \text{if } k(\Delta) \neq 0.$$

Theorem 3.2 ([9]). *Let*

$$f(z_1, \dots, z_{n+1}) = f_1(z_1, \dots, z_k) + f_2(z_{k+1}, \dots, z_{n+1})$$

where f_1 and f_2 are weighted homogeneous polynomials, and let Δ_1 and Δ_2 be the characteristic polynomials associated to f_1 and f_2. For the weight $w = (w_1, \dots, w_k, \dots, w_{n+1})$, express $w_i = u_i/v_i$, $i = 1, 2, \dots, n+1$, in an irreducible form. Suppose that each of the numbers $u_1, \dots, u_k$ is relatively prime to each of $u_{k+1}, \dots, u_{n+1}$. Then the numbers $k(\Delta)$, $\rho(\Delta)$ corresponded to the polynomial $f = f_1 + f_2$ are determined by the integers $k_j = k(\Delta_j)$ and $\rho_j = \rho(\Delta_j)$ corresponded to f_j, $j = 1, 2$ according to the formulars

$$k(\Delta) = k_1k_2 \text{ and } \rho(\Delta) = \rho_1^{k_2}\rho_2^{k_1}.$$

The next theorems show how one can construct topological spheres using the weighted homogeneous polynomial.

Theorem 3.3. *Let $g(z_1, z_2, \dots, z_m)$ be a weighted homogeneous polynomial with weight $w = (w_1, w_2, \dots, w_m)$, $w_i = u_i/v_i$ as before, $i = 1, 2, \dots, m$. Choose any two positive integers w_{m+1} and w_{m+2} such that $(w_{m+j}, u_i) = 1$ for all $i = 1, 2, \dots, m; j = 1, 2$. Then a polynomial f defined by*

$$f(z_1, \ldots, z_m, z_{m+1}, z_{m+2}) = g(z_1, \ldots, z_m) + z_{m+1}^{w_{m+1}} + z_{m+2}^{w_{m+2}}$$

is also a weighted homogeneous polynomial of weight (w_i), *and* $\sum_w = f^{-1}(0) \cap S^{2m+3}$ *is a topological sphere.*

Proof. Let k, $k(g)$, k_1, k_2 and $\rho, \rho(g), \rho_1, \rho_2$ be numbers defined in Theorem 3.1 associated to $f, g, z_{m+1}^{w_{m+1}}, z_{m+2}^{w_{m+2}}$ respectively. Clearly, divisor $\Delta_i = \Lambda_{w_i} - 1$, for $i=1,2$, so that $k_i = 1-1=0$. Hence $k = k_1 k_2 k(g) = 0$, and then $\Delta(1) = \rho = (\rho_g^{k_1} \rho_1^{k(g)})^{k_2} \rho_1^{k(g)k_1} = 1$. Therefore, $\sum_w$ is a topological sphere.

Theorem 3.4 ([11]). *Let* $g(z)$ *be a weighted homogeneous polynomial in* $\mathbf{C}^n$ *with an isolated critial point at the origin, and let* $f(z, w)$ *be a weighted homogeneous polynomial in* $\mathbf{C}^n \times \mathbf{C}^2$ *defined by* $f(z, w) = g(z) + w_1 w_2$. *Then* $g^{-1}(0) \cap S^{2n-1}$ *is a topological sphere if and only if* $f^{-1}(0) \cap S^{2n+3}$ *is a topological sphere. (Here,* $n > 3$).

We conclude with an example. Let

$$f(z_1, z_2, \ldots, z_7) = f_1(z_1, \ldots, z_5) + f_2(z_6, z_7),$$

where

$$f_1(z_1, z_2, \ldots, z_5) = z_1^3 + z_2^{6k-1} + z_3^2 + z_4^2 + z_5^2,$$
$$f_2(z_6, z_7) = z_6 z_7.$$

Then, f is a weighted homogeneous polynomial with weight $(w_i) = (3, 6k-1, 2, 2, 2, 1/2, 1/2)$. By Theorem 3.4, $\sum_w = f^{-1}(0) \cap S^{13}$ is an 11-dimensional topological sphere. First, we are interested in the diffeomorphic type of this sphere $\sum_w$. Let F, F_1, and F_2 be the fibre in the Milnor's fibering corresponding to the polynomials f, f_1, and f_2 respectively. Then F, F_1, and F_2 are diffeomorphic to $f^{-1}(1)$, $f_1^{-1}(1)$, and $f_2^{-1}(1)$ respectively (cf. [8], Lemma 9.4.), and $f^{-1}(1)$ is homotopy equivalent to the join $f_1^{-1}(1) * f_2^{-1}(1)$ (cf. [11]). Note that $f_2^{-1}(1)$ has the same homotopy type as S^1. Hence,

$$\begin{aligned} H_6(F; \mathbf{Z}) &= H_6(F_1 * F_2; \mathbf{Z}) \\ &= \sum_{i+j=5} \tilde{H}_i(F_1; \mathbf{Z}) \otimes \tilde{H}_j(F_2; \mathbf{Z}) \oplus \sum_{p+q=4} \tilde{H}_p(F_1; \mathbf{Z}) * \tilde{H}_p(F_2; \mathbf{Z}) \\ &= H_4(F_1; \mathbf{Z}) \otimes H_1(F_2; \mathbf{Z}) = H_4(F_1; \mathbf{Z}). \end{aligned}$$

(See [7] for the 2nd isomorphism). Hence, the signature of the intersection pairing of F is equal to that of F_1. Also it is well-known that $f_1^{-1}(0) \cap S^9 = k \cdot g_2$ and the signature of F_1 is equal to $8k$, where g_2 is a generator of the cyclic group of all 28 7-dimensional homotopy spheres. Therefore, we get $\sum_w = f^{-1}(0) \cap S^{13} = k \cdot g_3$ for a generator of the cyclic group of all 992 11-dimensional homotopy spheres.

To get a cyclic group action on these spheres which induces stably parallelizable orbit spaces, it is required to choose a prime p and numbers b_1's such that

$$w_1 b_1 = w_2 b_2 = \dots = w_7 b_7 \pmod p,$$
$$b_1^2 + b_2^2 + \dots + b_7^2 = 1 \pmod p,$$
$$b_1^4 + b_2^4 + \dots + b_7^4 = 1 \pmod p.$$

Hence,

$$(1/3)^2 + b_2^2 + 3(1/2)^2 = -7 \pmod p,$$
$$(1/3)^4 + b_2^4 + 3(1/2)^4 = -31 \pmod p$$

must be satisfied. Accordingly, $120524 = 0 \pmod p$, so $p = 29$ or $p = 1039$.

For example, if $p = 29$, then we can take

$$(b_1, b_2, \dots, b_7) = (10, 1, 15, 15, 15, 2, 2),$$

and then, for $k = 10 + 29q, q = 1, 2, \dots, 992$, $\sum_w$ represent all 992 11-dimensional homotopy spheres. Furthermore, on these 992 homotopy spheres, the cyclic group action defined by the given b_i's is well defined, and all their orbit spaces are stably parallelizable.

Bibliography

[1] Atiyah, M. F., *K-theory*, Benjamin Inc., New York, 1967.

[2] Brieskorn, E., Beiespiele zur Differential-Topologie von Singulatitäten, *Inventiones Math.*, **2** (1966), 1-14.

[3] Ewing, J., S. Moolgavkar, L. Smith and R. S. Stong, Stable parallelizability of lens spaces, *J. of Pure and Applied Algebra*, **10** (1977), 177-191.

[4] Hirzebruch, F., Singularities and exotic spheres, *Séminaire Bourbaki*, 19e année, (1966/67), no **314**.

[5] Kambe, T., The structure of K_A-rings of the lens space and their applications, *J. Math. Soc. Japan*, **18** (1966), 135-146.

[6] Kervaire M. and J. Milnor, Groups of homotopy spheres I, *Annals of Math.*, **77** (1963), 504-537.

[7] Milnor J., Constructions of universal bundle II, *Annals of Math.*, **63** (1956), 430-436.

[8] ——————, *Singular points of complex hypersurfaces*, Annals of Math., Studies 61, Princeton University Press, New Jersey, 1968.

[9] —————— and P. Orlik, Isolated singularities defined by weighted homogeneous polynomials, *Topology*, **9** (1969), 385-393.

[10] ———— and J. Stasheff, *Characteristic Classes*, Annals of Math., Studies 76, Princeton Univ. Press, New Jersey, 1974.

[11] Oka M., On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials, *Topology*, **12** (1973), 19–32.

[12] Orlik P., Smooth homotopy lens spaces, *Michigan Math. J.*, **16** (1969), 245–255.