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On the monodromy conjecture for non-degenerate hypersurfaces

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Abstract. The monodromy conjecture is an umbrella term for several conjectured relationships between poles of zeta functions, monodromy eigenvalues and roots of Bernstein–Sato polynomials in arithmetic geometry and singularity theory. Even the weakest of these relations – the Denef–Loeser conjecture on topological zeta functions – is open for surface singularities.

We prove it for a wide class of multidimensional singularities that are non-degenerate with respect to their Newton polyhedra, including all such singularities of functions of four variables.

A crucial difference from the known case of three variables is the existence of degenerate singularities arbitrarily close to a non-degenerate one. Thus, even aiming at the study of non-degenerate singularities, we have to go beyond this setting.

We develop new tools to deal with such multidimensional phenomena, and conjecture how the proof for non-degenerate singularities of arbitrarily many variables might look like.

Keywords. Monodromy conjecture, Newton polytope

Contents

1. Introduction . . . 3874
2. The monodromy conjecture for the topological zeta function . . . 3878
   2.1. The conjecture . . . 3878
   2.2. The topological zeta function and Newton polyhedra . . . 3879
   2.3. The monodromy zeta function and Newton polyhedra . . . 3881
3. Candidate poles of the topological zeta function and B-facets . . . 3882
   3.1. B_1-faces . . . 3883
   3.2. Critical edges . . . 3884
   3.3. Some non-contributing configurations of B_1-facets . . . 3885
   3.4. B_2-facets . . . 3888

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4. Fake poles of the topological zeta function in arbitrary dimension . . . 3890
4.1. Contributions . . . 3891
4.2. The main theorem: the plan of the proof . . . 3892
4.3. Intersections of B-facets . . . 3893
4.4. Bases and apices . . . 3895
4.5. Sprouts and cancellation of contributions . . . 3896
4.6. Critical cones . . . 3898
4.7. A tubular neighborhood of the critical subfan . . . 3900
5. Fake poles of the topological zeta function in dimension 4 . . . 3904
5.1. The main theorem . . . 3904
5.2. Very critical cones . . . 3907
5.3. A tubular neighborhood of the critical subfan revisited . . . 3909
5.4. Generalizing the notion of B-facets . . . 3912
6. Eigenvalues of monodromy and corners . . . 3914
6.1. Motivation and results . . . 3914
6.2. Hypermodular functions . . . 3917
6.3. Reduction to the case $k = n - 1$ and $r = n - 1$ . . . 3918
6.4. The proof for $r = n - 1$ . . . 3919
7. On non-convenient Newton polyhedra . . . 3925
7.1. Nearby singularities at coordinate lines . . . 3925
7.2. Nearby singularities outside coordinate lines . . . 3927
8. The monodromy conjecture for $n = 4$ . . . 3929
8.1. 2- and 3-dimensional cases . . . 3930
8.2. Dimension 4: Looking for the eigenvalue t_0 outside the origin . . . 3931
8.3. Dimension 4: Triangulating . . . 3934
8.4. Monodromy conjecture for non-degenerate singularities of four variables . . . 3937
8.5. Exotic families . . . 3940
9. Appendix: some elements of lattice geometry . . . 3945
References . . . 3947

1. Introduction

Over the fields $\mathbb{R}$ and $\mathbb{C}$ it is well-known that the poles of the local zeta function associated to a polynomial f are contained in the set of roots of the Bernstein–Sato polynomial and their integer shifts. By a celebrated theorem of Kashiwara and Malgrange, this implies that for any such pole $s_0 \in \mathbb{Q}$ the complex number $\exp(2\pi i s_0) \in \mathbb{C}$ is an eigenvalue of the monodromies of the complex hypersurface defined by f. Igusa predicted a similar beautiful relationship between the poles of p-adic integrals and the complex monodromies. This is now called the monodromy conjecture (see the papers of Denef [7], Nicaise [24] and Denef and Loeser [11] for excellent reviews on this subject). Later in [9], Denef and Loeser introduced the local topological zeta function $Z_{\mathrm{top},f}(s)$ associated to f and proposed a weaker version of the monodromy conjecture. However, even this weaker version, proposed thirty years ago, is proved so far without any restrictions in dimension 2 only [20].

For other important contributions to this Denef–Loeser conjecture, see for example [21] by Loeser, where he studies the monodromy conjecture for non-degenerate singularities satisfying some non-resonance conditions. The result holds for isolated singularities.

In [18], the second author and Van Proeyen restrict to non-degenerate surface singularities and then prove the monodromy conjecture without further conditions. In [2, 3], Artal Bartolo–Cassou-Noguès–Luengo–Melle Hernández prove the monodromy conjecture for some particular classes of singularities for which they establish an explicit formula for the topological zeta function. It was also proved for hyperplane arrangements by Budur, Mustata and Teitler [6]. In [30, 32], Veys obtained various results in dimension 3. Moreover, the case of homogeneous and isolated quasihomogeneous singularities was proved by Blanco–Budur–van der Veer [4] and Rodrigues–Veys [27]. Other important contributions to the Denef–Loeser conjecture and related results include [5, 15, 19, 23, 31].

Generalizing the Igusa zeta function to an ideal and using the notion of Verdier monodromy, one can similarly formulate the monodromy conjecture for ideals. At the level of ideals, the conjecture has only been proven in full generality for ideals in two variables [28]. Very recently Mustaţă [22] showed that the monodromy conjecture for polynomials implies the monodromy conjecture for ideals.

The aim of the present paper is to explore to what extent the results of [18] hold true in higher dimensions, and what we are missing to step one dimension higher for non-degenerate singularities.

A crucial difference that we observe in dimension 4 is the existence of degenerate singularities arbitrarily close to a non-isolated non-degenerate singularity. So, even aiming at the study of non-degenerate singularities, we have to go beyond the setting of Newton polyhedra and toric resolutions at some point.

This is in contrast to all preceding results on non-isolated singularities: in the three-dimensional setting of [18], all singularities close to a non-degenerate one are non-degenerate, and, in the setting of [3], all singularities close to a quasi-ordinary one are quasi-ordinary.

The paper consists of three parts. In Sections 3–5, we study configurations of facets of the Newton polyhedron that do not ensure the existence of the corresponding pole of the topological zeta function. In Sections 6–7, we study configurations of faces that, on the contrary, always non-trivially contribute to the multiplicity of the expected monodromy eigenvalue. Finally, in the last section, we use these results to prove the monodromy conjecture for singularities of non-degenerate functions of four variables.

Theorem 1.1. *The Denef–Loeser monodromy conjecture* (*and moreover its polyhedral version from* [13]) *holds true for all non-degenerate hypersurface singularities of four variables.*

Our proof of this theorem admits the following conjectural generalization to arbitrary dimension. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function non-degenerate with respect to its Newton polyhedron $\Gamma_+(f)$.

Definition 1.2. (1) A bounded facet F of the Newton polyhedron $\Gamma_+(f)$ *produces the number* s_0 if the affine span of F is given by an equation $a_1 v_1 + \cdots + a_n v_n = q$ with coprime coefficients a_i such that $s_0 = -(a_1 + \cdots + a_n)/q$.

(2) A bounded face F of $\Gamma_+(f)$ is said to produce the number s_0 if every bounded facet of $\Gamma_+(f)$ containing F produces s_0.

We say that a polytope P is *inscribed* into a polytope Q if $\dim P = \dim Q$, $P \subset Q$ and $\operatorname{vert} P \subset \operatorname{vert} Q$ (where vert denotes the set of vertices).

Conjecture 1.3. *Let $\mathcal{F}$ be the set of all faces of the Newton polyhedron $\Gamma_+(f) \subset \mathbb{R}^n$, producing the number s_0, and let $\mathcal{V}$ be the set of all of their vertices having at least one coordinate equal to* 1.

(1) *Assume there exists a function $b : \mathcal{V} \to \{1, \ldots, n\}$, assigning to every vertex $v \in \mathcal{V}$ the index of one of its unit coordinates, such that every simplex Δ inscribed into a face from $\mathcal{F}$ has some vertex $v \in \mathcal{V}$ for which the other vertices,* $\operatorname{vert} \Delta \setminus \{v\}$, *belong to the $b(v)$-th coordinate hyperplane. Then s_0 is not a pole of the topological zeta function of f.*

(2) *If such a function b does not exist, then* $\exp(2\pi i s_0)$ *is an eigenvalue of the monodromy of f at some point near the origin* (*and moreover a nearby tropical monodromy eigenvalue of the polyhedron $\Gamma_+(f)$ in the sense of* [13]).

Both of these statements belong to polyhedral geometry, and together they imply the monodromy conjecture for non-degenerate singularities in arbitrary dimension. When proving Theorem 1.1, we actually prove both parts of this conjecture for $n = 4$ in Sections 5 and 8 respectively.

A key step in proving the first part would be to combinatorially classify faces that can appear in the aforementioned family $\mathcal{F}$. We call them *B-faces*.

Definition 1.4. (1) A lattice simplex in $\mathbb{R}^n$ with the standard coordinate system $v_1, \ldots, v_n$ is called a B_1*-simplex* with respect to the i-th coordinate if one of its vertices lies in the plane $v_i = 1$ and the others in the plane $v_i = 0$.

(2) A lattice polytope in $\mathbb{R}^n$ is called a *B-polytope* if every lattice simplex that it contains is a B_1-simplex.

For $n = 4$, B-faces are classified in Lemma 5.18: besides B_1-pyramids that are well known from the 3-dimensional case, we detect another combinatorial type, which we call B_2-faces. As soon as the classification is done in any dimension, we believe that the general technique from Section 4 can be extended to arbitrary dimension, though this is still non-trivial, because the fact that the family $\mathcal{F}$ from the conjecture entirely consists of B-faces does not ensure the existence of the function b: see e.g. the phenomenon of B-borders (Definition 5.1) in dimension 4. As to the second part of the conjecture, one step in its proof for $n = 4$ is already done for arbitrary dimension (see Section 6).

The structure of the paper is as follows. In Section 2 we recall the exact statement of the monodromy conjecture and the notion of non-degenerate singularity.

In Section 3, as a generalization of the notion of B_1-facets in [18], we introduce B_1-facets of the Newton polyhedron $\Gamma_+(f)$ (Definition 3.10) and discover so called

B_2-facets (Definition 3.9) that behave similarly, although do not exist in the lower-dimensional setting.

In Section 4, we show that many configurations of B_1- and B_2-facets alone never ensure the existence of the corresponding pole of the topological zeta function (Theorem 4.3). In the course of the proof we introduce an important notion of a critical face of the Newton polyhedron (Definition 4.27). Its role in the proof indicates that it might be possible to find a similarly important notion of a critical stratum of the exceptional divisor in the context of arbitrary singularities and their non-toric resolutions.

In Section 5, we apply the tools from the preceding two sections to completely classify configurations of facets of 4-dimensional Newton polyhedra that never ensure the existence of the corresponding pole of the topological zeta function (Theorem 5.2). Besides the previously found configurations, we discover so called B^2-borders (Definition 5.1).

In Section 6, following the strategy of [18], we prove that the candidate poles of $Z_{\text{top},f}(s)$ contributed by certain non-B_1-facets of $\Gamma_+(f)$ always yield monodromy eigenvalues. Most notably, we obtain Theorem 6.4.

Its proof relies upon the new notion of a hypermodular function (Definition 6.8), which is inspired by supermodular functions in convex geometry and analysis, and may be of independent interest.

As a corollary, we can confirm the monodromy conjecture of Denef–Loeser for many non-degenerate hypersurfaces in higher dimensions (see e.g. Theorem 6.6).

In Section 7 we prove that singularities adjacent to a Newton non-degenerate singularity along a coordinate line are themselves Newton non-degenerate (see Proposition 7.2).

However, we notice that starting from dimension 4, not all singularities adjacent to Newton non-degenerate singularities are non-degenerate themselves (see Example 7.5). In particular, even in dimension 4 it is not possible to prove the Denef–Loeser conjecture for Newton non-degenerate singularities within the framework of non-degenerate singularities.

To this end, the first author introduced in [13] the notion of tropical nearby monodromy eigenvalues and the corresponding monodromy conjecture, which implies the Denef–Loeser conjecture and (in contrast to the latter) turns into a purely combinatorial statement on the Newton polyhedron for non-degenerate singularities. This tool helps to study monodromy eigenvalues of singularities that are adjacent to a singularity with a given resolution.

In particular, it allows us to prove in Section 8 the monodromy conjecture for non-degenerate functions of four variables: if the sought monodromy eigenvalue is not a tropical monodromy eigenvalue outside the origin, this imposes lots of restrictions on the combinatorial structure of the Newton polyhedron, and a detailed study of this structure shows that the sought monodromy eigenvalue is a root of the monodromy zeta function at the origin.

Remark 1.5. Note that the order of this reasoning is opposite to the one usually seen in the literature: first try to find the sought monodromy eigenvalue at the origin, then in the case of trouble switch to nearby singularities. We proceed in the following order:

(1) try to find the sought monodromy eigenvalue outside the origin;

(2) notice that we can fail only if the Newton polyhedron has certain combinatorial properties allowing one to triangulate it naturally (in a sense);

(3) if this occurs, then the resulting natural triangulation allows one to find the sought monodromy eigenvalue at the origin.

One could speculate that it is reasonable to expect the same in the general (non-toric) setting: the absence of the sought monodromy eigenvalue outside the origin ensures the existence of a certain geometric/deformation-theoretic structure on the resolution space of the singularity, whose combinatorial counterpart is the aforementioned triangulation, and which likewise allows finding the sought monodromy eigenvalue at the origin.

2. The monodromy conjecture for the topological zeta function

In this section, we recall the monodromy conjecture for the topological zeta function and related results.

2.1. The conjecture

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a non-trivial analytic function. We assume that f is defined on an open neighborhood X of the origin $0 \in \mathbb{C}^n$. Let $\pi : Y \to X$ be an embedded resolution of the complex hypersurface $f^{-1}(0) \subset X$ and E_j $(j \in J)$ the irreducible components of the normal crossing divisor $\pi^{-1}(f^{-1}(0)) \subset Y$. For $j \in J$ we denote by N_j (resp. $\nu_j - 1$) the multiplicity of the divisor associated to $f \circ \pi$ (resp. $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$) along $E_j \subset Y$. For a non-empty subset $I \subset J$ we set

$$E_I = \bigcap_{i \in I} E_i, \quad E_I^\circ = E_I \setminus \bigcup_{j \notin I} E_j.$$

In [9] Denef and Loeser defined the local topological zeta function $Z_{\mathrm{top},f}(s) \in \mathbb{C}(s)$ associated to f (at the origin) by

$$Z_{\mathrm{top},f}(s) = \sum_{I \neq \emptyset} \chi(E_I^\circ \cap \pi^{-1}(0)) \prod_{i \in I} \frac{1}{N_i s + \nu_i},$$

where $\chi(\cdot)$ denotes the topological Euler characteristic. More precisely, they introduced $Z_{\mathrm{top},f}(s)$ by p-adic integrals and showed by algebraic methods that it does not depend on the choice of the embedded resolution $\pi : Y \to X$. Later in [10] and [11], they redefined $Z_{\mathrm{top},f}(s)$ by using the motivic zeta function of f and re-proved this independence from π more elegantly. For a point $x \in f^{-1}(0) \cap X$ let $F_x \subset X \setminus f^{-1}(0)$ be the Milnor fiber of f at x and $\Phi_{j,x} : H^j(F_x; \mathbb{C}) \xrightarrow{\sim} H^j(F_x; \mathbb{C})$ $(j \in \mathbb{Z}_+ := \{m \in \mathbb{Z} \mid m \geq 0\})$ the Milnor monodromies associated to it. Then the monodromy conjecture of Denef–Loeser for the local topological zeta function $Z_{\mathrm{top},f}(s)$ is stated as follows.

Monodromy Conjecture (Denef–Loeser [9, Conjecture 3.3.2]): Assume that $s_0 \in \mathbb{C}$ is a pole of $Z_{\mathrm{top},f}(s)$. Then $\exp(2\pi i s_0) \in \mathbb{C}$ is an eigenvalue of the monodromy $\Phi_{j,x} : H^j(F_x;\mathbb{C}) \xrightarrow{\sim} H^j(F_x;\mathbb{C})$ for some point $x \in f^{-1}(0) \cap X$ in a neighborhood of $0 \in \mathbb{C}^n$ and some $j \geq 0$.

In [9] the authors also formulated an even stronger conjecture concerning the Bernstein–Sato polynomial $b_f(s)$ of f: they conjectured that the poles of $Z_{\mathrm{top},f}(s)$ are roots of $b_f(s)$.

From now on, we assume that f is a non-trivial polynomial on $\mathbb{C}^n$ such that $f(0) = 0$ and recall the results of Denef–Loeser [9, Section 5] and Varchenko [29]. For $f(x) = \sum_{v\in\mathbb{Z}^n_+} c_v x^v \in \mathbb{C}[x_1,\dots,x_n]$, its *support* $\operatorname{supp} f \subset \mathbb{Z}^n_+$ is the subset

$$\operatorname{supp} f = \{v \in \mathbb{Z}^n_+ \mid c_v \neq 0\} \subset \mathbb{Z}^n_+.$$

We denote by $\Gamma_+(f) \subset \mathbb{R}^n_+$ the convex hull of $\bigcup_{v\in\operatorname{supp} f}(v + \mathbb{R}^n_+)$ in $\mathbb{R}^n_+$. It is called the *Newton polyhedron* of f at the origin $0 \in \mathbb{C}^n$. The polynomial f such that $f(0) = 0$ is called *convenient* if $\Gamma_+(f)$ intersects the positive part of any coordinate axis of $\mathbb{R}^n$.

Definition 2.1 (Kouchnirenko [17]). The polynomial f is *non-degenerate* (*at the origin* $0 \in \mathbb{C}^n$) if for any compact face $\tau \prec \Gamma_+(f)$ the complex hypersurface

$$\{x \in (\mathbb{C}^*)^n \mid f_\tau(x) = 0\}$$

in $(\mathbb{C}^*)^n$ is smooth, where we set

$$f_\tau(x) = \sum_{v\in\tau\cap\mathbb{Z}^n_+} c_v x^v \in \mathbb{C}[x_1,\dots,x_n].$$

It is well-known that generic polynomials having a fixed Newton polyhedron are non-degenerate (see for example [26, Chapter V, Section 2]).

2.2. *The topological zeta function and Newton polyhedra*

In what follows, we assume that the reader is familiar with basic facts and notions of integer lattice geometry; see Appendix for some digest. For $u = (u_1,\dots,u_n) \in \mathbb{R}^n_+$ we set

$$N(u) = \min_{v\in\Gamma_+(f)} \langle u, v\rangle, \qquad \nu(u) = |u| = \sum_{i=1}^n u_i$$

and

$$F(u) = \{v \in \Gamma_+(f) \mid \langle u, v\rangle = N(u)\} \prec \Gamma_+(f).$$

We call $F(u)$ the *supporting face* of the vector $u \in \mathbb{R}^n_+$ on $\Gamma_+(f)$. To a face $\tau \prec \Gamma_+(f)$ one can associate a dual cone

$$\tau^\circ = \overline{\{u \in \mathbb{R}^n_+ \mid F(u) = \tau\}} \subset \mathbb{R}^n_+.$$

Note that τ° is an $(n - \dim \tau)$-dimensional rational polyhedral convex cone in $\mathbb{R}^n_+$. The subdivision of $\mathbb{R}^n_+$ into the cones τ° $(\tau \prec \Gamma_+(f))$ satisfies the axiom of fans (for the definition, see [14, 25]) and is called the *dual fan* of $\Gamma_+(f)$. Let $\Delta = \mathbb{R}_+ a(1) + \cdots + \mathbb{R}_+ a(l)$ $(a(i) \in \mathbb{Z}^n_+)$ be a rational simplicial cone in $\mathbb{R}^n_+$, where the lattice vectors $a(i)$ are linearly independent over $\mathbb{R}$ and primitive. Let $\mathrm{aff}(\Delta) \simeq \mathbb{R}^l$ be the affine span of Δ in $\mathbb{R}^n$ and $s(\Delta) \subset \Delta$ the l-dimensional lattice simplex whose vertices are $a(1), \dots, a(l)$ and the origin $0 \in \Delta \subset \mathbb{R}^n_+$. We denote by $\mathrm{mult}(\Delta) \in \mathbb{Z}_{>0}$ the *l-dimensional normalized volume* $\mathrm{Vol}_{\mathbb{Z}}(s(\Delta))$ of $s(\Delta)$, i.e. $l!$ times the usual volume of $s(\Delta)$ with respect to the affine lattice $\mathrm{aff}(\Delta) \cap \mathbb{Z}^n \simeq \mathbb{Z}^l$ in $\mathrm{aff}(\Delta)$. By using the integer $\mathrm{mult}(\Delta)$ we set

$$J_\Delta(s) = \frac{\mathrm{mult}(\Delta)}{\prod_{i=1}^{l} \{N(a(i))s + \nu(a(i))\}} \in \mathbb{C}(s).$$

For a face $\tau \prec \Gamma_+(f)$ we choose a decomposition $\tau^\circ = \bigcup_{i=1}^r \Delta_i$ of the dual cone τ° into rational simplicial cones Δ_i of dimension $l = \dim \tau^\circ$ such that $\dim(\Delta_i \cap \Delta_j) < l$ $(i \neq j)$ and set

$$J_\tau(s) = \sum_{i=1}^{r} J_{\Delta_i}(s) \in \mathbb{C}(s).$$

By the following result of Denef–Loeser [9], the rational function $J_\tau(s)$ does not depend on the choice of the decomposition of τ°. Let us set

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n_+.$$

Lemma 2.2 (see the proof of [9, Lemme 5.1.1]). *We have an equality*

$$J_\tau(s) = \int_{\tau^\circ} \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du,$$

where P is a point in τ and du is the l-dimensional volume form on the affine span $\mathrm{aff}(\tau^\circ) \simeq \mathbb{R}^l$ *for which the volume of the parallelepiped spanned by a basis of the affine lattice* $\mathrm{aff}(\tau^\circ) \cap \mathbb{Z}^n \simeq \mathbb{Z}^l$ *is equal to* 1.

It is also well-known that one can decompose τ° into rational simplicial cones without adding new edges. Then we have the following formula for $Z_{\mathrm{top},f}(s)$.

Theorem 2.3 (Denef–Loeser [9, Théorème 5.3 (ii)]). *Assume that $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ is non-degenerate. Then*

$$Z_{\mathrm{top},f}(s) = \sum_{\gamma} J_\gamma(s) + \frac{s}{s+1} \sum_{\tau} (-1)^{\dim \tau} \mathrm{Vol}_{\mathbb{Z}}(\tau) \cdot J_\tau(s),$$

where in the sum $\sum_\gamma$ (resp. $\sum_\tau$) the face $\gamma \prec \Gamma_+(f)$ (resp. $\tau \prec \Gamma_+(f)$) ranges through the vertices of $\Gamma_+(f)$ (resp. the compact ones such that $\dim \tau \geq 1$) *and* $\mathrm{Vol}_{\mathbb{Z}}(\tau) \in \mathbb{Z}_{>0}$ *is* *the* $(\dim \tau)$*-dimensional normalized volume of* τ *with respect to the affine lattice* $\mathrm{aff}(\tau) \cap \mathbb{Z}^n \simeq \mathbb{Z}^{\dim \tau}$ *in* $\mathrm{aff}(\tau) \simeq \mathbb{R}^{\dim \tau}$.

Recall that a face τ of $\Gamma_+(f)$ is called a *facet* if $\dim \tau = n - 1$. For a facet $\tau \prec \Gamma_+(f)$ let $a(\tau) = (a(\tau)_1, \dots, a(\tau)_n) \in \tau^\circ \cap \mathbb{Z}^n_+$ be its primitive conormal vector and set

$$N(\tau) = \min_{v \in \Gamma_+(f)} \langle a(\tau), v \rangle, \quad \nu(\tau) = |a(\tau)| = \sum_{i=1}^{n} a(\tau)_i = \langle a(\tau), \mathbf{1} \rangle.$$

We call $N(\tau)$ the *lattice distance* from τ to $0 \in \mathbb{R}^n$. It follows from Theorem 2.3 that any pole $s_0 \neq -1$ of $Z_{\mathrm{top},f}(s)$ is contained in the finite set

$$\left\{ -\frac{\nu(\tau)}{N(\tau)} \;\middle|\; \tau \prec \Gamma_+(f) \text{ is a facet not lying in a coordinate hyperplane} \right\} \subset \mathbb{Q}.$$

Its elements are called *candidate poles* of $Z_{\mathrm{top},f}(s)$.

If τ is a simplicial facet, the normalized volume $\mathrm{Vol}_{\mathbb{Z}}(\tau)$ is equal to the multiplicity of the cone spanned by the vertices, divided by $N(\tau)$.

2.3. *The monodromy zeta function and Newton polyhedra*

Finally, we recall the result of Varchenko [29]. For a polynomial $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ such that $f(0) = 0$, its *monodromy zeta function* $\zeta_{f,0}(t) \in \mathbb{C}(t)$ at the origin $0 \in \mathbb{C}^n$ is defined by

$$\zeta_{f,0}(t) = \prod_{j \in \mathbb{Z}_+} \{\det(\mathrm{id} - t\Phi_{j,0})\}^{(-1)^j} \in \mathbb{C}(t).$$

Similarly one can also define $\zeta_{f,x}(t) \in \mathbb{C}(t)$ for any point $x \in f^{-1}(0)$. Then by considering the decomposition of the nearby cycle perverse sheaf $\psi_f(\mathbb{C}_{\mathbb{C}^n})[n-1]$ with respect to the monodromy eigenvalues of f and the concentrations of its components at generic points $x \in f^{-1}(0)$ (see e.g. [12, 16]), in order to prove the monodromy conjecture, it suffices to show that for any pole $s_0 \in \mathbb{C}$ of $Z_{\mathrm{top},f}(s)$ the complex number $\exp(2\pi i s_0)$ is a root or a pole of $\zeta_{f,x}(t)$ for some point $x \in f^{-1}(0)$ in a neighborhood of $0 \in \mathbb{C}^n$ (see Denef [8, Lemma 4.6]).

For a subset $S \subset \{1, \dots, n\}$ we define a coordinate subspace $\mathbb{R}^S \simeq \mathbb{R}^{|S|}$ of $\mathbb{R}^n$ by

$$\mathbb{R}^S = \{v = (v_1, \dots, v_n) \in \mathbb{R}^n \mid v_i = 0 \text{ for any } i \notin S\}$$

and set

$$\mathbb{R}^S_+ = \mathbb{R}^S \cap \mathbb{R}^n_+ \simeq \mathbb{R}^{|S|}_+.$$

For a compact face $\tau \prec \Gamma_+(f)$ we take the minimal coordinate subspace $\mathbb{R}^S$ of $\mathbb{R}^n$ containing τ and set $s_\tau = |S|$. If τ satisfies the condition $\dim \tau = s_\tau - 1$ we set

$$\zeta_\tau(t) = (1 - t^{N(\tau)})^{\mathrm{Vol}_{\mathbb{Z}}(\tau)} \in \mathbb{C}[t],$$

where $N(\tau) \in \mathbb{Z}_{>0}$ is the lattice distance (for the definition, see Appendix) from the affine hyperplane $\mathrm{aff}(\tau) \simeq \mathbb{R}^{\dim \tau}$ in $\mathbb{R}^S$ to $0 \in \mathbb{R}^S$. Let $a(\tau) \in \tau^\circ \cap \mathbb{Z}^S_+$ be the primitive conormal vector of $\tau \subset \mathbb{R}^S$ whose value on τ is equal to $N(\tau) > 0$.

Lemma 2.4. *Let $\tau \prec \Gamma_+(f)$, S and $a(\tau)$ be as above and let $\alpha \in \mathbb{Q}$. For $\beta \in \mathbb{Q}$ we define a hyperplane $L(\beta)$ in $\mathbb{R}^S$ by*

$$L(\beta) = \{v \in \mathbb{R}^S \mid \langle a(\tau), v \rangle = \beta \cdot N(\tau)\}.$$

Then the complex number $\lambda = \exp(2\pi i \alpha)$ is a root of the polynomial $\zeta_\tau(t)$ if and only if the hyperplane $L(\alpha) \subset \mathbb{R}^S$ is rational, i.e. $L(\alpha) \cap \mathbb{Z}^S \neq \emptyset$.

Proof. Note that $0 \in L(0)$, $\tau \subset L(1) = \mathrm{aff}(\tau)$ and the hyperplanes $L(\beta)$ ($\beta \in \mathbb{Q}$) are parallel to each other. The lattice distance $N(\tau) > 0$ is equal to the number of (mutually parallel) "rational" hyperplanes $L(\beta)$ ($\beta \in \mathbb{Q}, 0 < \beta < 1$) between $L(0)$ and $L(1)$ plus 1. Then the assertion immediately follows from this geometric interpretation of $N(\tau)$. ■

Theorem 2.5 (Varchenko [29]). *Assume that $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ is non-degenerate. Then*

$$\zeta_{f,0}(t) = \prod_\tau \{\zeta_\tau(t)\}^{(-1)^{\dim \tau}},$$

where in the product the face τ ranges through the compact faces of $\Gamma_+(f)$ satisfying $\dim \tau = s_\tau - 1$.

Definition 2.6. We say that a face τ of $\Gamma_+(f)$ is a *V-face* (or a Varchenko face) if it is compact and $\dim \tau = s_\tau - 1$.

Definition 2.7. (1) We say that a candidate pole $s_0 \in \mathbb{C}$ of $Z_{\mathrm{top},f}(s)$ is *contributed by a facet* $\tau \prec \Gamma_+(f)$ or that τ *contributes* s_0 if $s_0 = -\nu(\tau)/N(\tau)$.

(2) Let σ be a V-face in $\Gamma_+(f)$. We say that σ *contributes to* (*the multiplicity of*) $t_0 \in \mathbb{C}$ if t_0 is a root of the polynomial $\zeta_\sigma(t)$.

3. Candidate poles of the topological zeta function and B-facets

In this section, we develop new tools (Lemmas 3.3 and 3.6) to detect configurations of facets contributing fake poles of $Z_{\mathrm{top},f}(s)$ – so that once a candidate pole is contributed only by facets from this configuration, then it is definitely fake.

As a first example, we define B_1-pyramid facets (Definition 3.1) of the Newton polyhedron $\Gamma_+(f)$. Our definition is a straightforward generalization of that of Lemahieu–Van Proeyen [18]. However, starting from dimension $n = 4$, there exist many other combinatorial types of facets and configurations that may contribute fake poles. In particular, we introduce so called B_2-facets (Definition 3.9) and detect some non-contributing configurations of B_1- and B_2-facets (see Propositions 3.7, 3.8 and 3.11). The proofs of these facts are intended to motivate general constructions in the next section, where we prove a more general Theorem 4.3.

From now on, we introduce the following convention on figures in this paper: whenever we depict some configuration of cones in $\mathbb{R}^n$, we draw its projectivization, resulting in an $(n-1)$-dimensional figure.

3.1. B_1-faces

For a subset $S \subset \{1,\ldots,n\}$ let $\pi_S : \mathbb{R}^n_+ \to \mathbb{R}^{S^c}_+ \simeq \mathbb{R}^{n-|S|}_+$ be the natural projection. We say that a polyhedron τ in $\mathbb{R}^n_+$ is *non-compact for* $S \subset \{1,\ldots,n\}$ if the Minkowski sum $\tau + \mathbb{R}^S_+$ is contained in τ.

Definition 3.1 (cf. Lemahieu–Van Proeyen [18]). Let τ be a polyhedron in $\mathbb{R}^n_+$.

(1) We say that τ is a *B_1-pyramid of compact type for the variable v_i* if τ is a compact pyramid over the base $\gamma = \tau \cap \{v_i = 0\}$ and its unique vertex $P \prec \tau$ such that $P \notin \gamma$ has height 1 over the hyperplane $\{v_i = 0\} \subset \mathbb{R}^n_+$.

(2) We say that τ is a *B_1-pyramid of non-compact type* if there exists a non-empty subset $S \subset \{1,\ldots,n\}$ such that τ is non-compact for S and $\pi_S(\tau) \subset \mathbb{R}^{S^c}_+ \simeq \mathbb{R}^{n-|S|}_+$ is a B_1-pyramid of compact type for some variable v_i $(i \notin S)$.

(3) We say that τ is a *B_1-pyramid* if it is a B_1-pyramid of compact or non-compact type.

(4) We say that a face τ of the Newton polyhedron $\Gamma_+(f)$ is a *B_1-face* if it is a B_1-pyramid. In particular, B_1-faces of dimension $n-1$ and 1 will be called *B_1-facets* and *B_1-segments* respectively.

We shall see later in this section that B_1-facets alone tend not to contribute poles to the topological zeta function.

Remark 3.2. The fact that B_1-facets might not give rise to eigenvalues of monodromy was already discovered by Loeser (see [21, Remark 6.3]). The condition he requires on the facets rules out among others all B_1-facets. Let us recall this condition.

For two distinct facets τ and τ' of $\Gamma_+(f)$ let $\beta(\tau,\tau') \in \mathbb{Z}$ be the greatest common divisor of the 2×2 minors of the matrix $(a(\tau), a(\tau')) \in M(n,2;\mathbb{Z})$. Recall that $a(\tau) \in \tau^\circ \cap \mathbb{Z}^n$ is the primitive conormal vector of τ, and $\beta(\tau,\tau') \in \mathbb{Z}$ is equal to the lattice area of the triangle spanned by $a(\tau)$ and $a(\tau')$.

If $N(\tau) \neq 0$ (e.g. if τ is compact) we set

$$\lambda(\tau,\tau') = \nu(\tau') - \frac{\nu(\tau)}{N(\tau)} N(\tau') \in \mathbb{Q}, \quad \mu(\tau,\tau') = \frac{\lambda(\tau,\tau')}{\beta(\tau,\tau')} \in \mathbb{Q}.$$

In [21] the author considered only compact facets τ of $\Gamma_+(f)$ which satisfy the following technical condition:

"For any facet $\tau' \prec \Gamma_+(f)$ such that $\tau' \neq \tau$ and $\tau' \cap \tau \neq \emptyset$ we have $\mu(\tau,\tau') \notin \mathbb{Z}$."

He showed that if f is non-degenerate, the candidate pole of $Z_{\mathrm{top},f}(s)$ associated to such a compact facet τ is a root of the local Bernstein–Sato polynomial of f. Now let $\tau \prec \Gamma_+(f)$ be a facet containing a B_1-pyramid of compact type for the variable v_i and set $\gamma = \tau \cap \{v_i = 0\}$. Let $\tau_0 \prec \Gamma_+(f)$ be the unique (non-compact) facet such that $\gamma \prec \tau_0$, $\tau_0 \neq \tau$ and $\tau_0 \subset \{v_i = 0\}$. Then we can easily show that $\beta(\tau,\tau_0) = 1$. Indeed, by a rotation of $\mathbb{R}^n$ which preserves the hyperplane $\{v_i = 0\} \simeq \mathbb{R}^{n-1}$, we can reduce the problem to the case $n = 2$. Moreover, since $\nu(\tau_0) = 1$, $N(\tau_0) = 0$ and $\lambda(\tau,\tau_0) = 1$ we obtain $\mu(\tau,\tau_0) = 1$; that is, such a facet τ does not satisfy the above-mentioned condition of [21].

The atypical behavior of candidate poles of $Z_{\mathrm{top},f}(s)$ associated to B_1-facets essentially arises from the following simple computation (cf. Lemma 2.2). For a subcone C of the dual cone τ° to a k-dimensional face τ of the Newton polyhedron $\Gamma_+(f)$, define the contribution of C to the topological ζ-function $Z_{\mathrm{top,f}}$ as

$$\int_C \exp(-N(u)s - \langle u, \mathbf{1}\rangle)\, du^{(n)}$$

for $k = 0$ (see Lemma 2.2 for the details) and otherwise

$$(-1)^k \operatorname{Vol}_{\mathbb{Z}}(\tau)\frac{s}{s+1}\int_C \exp(-N(u)s - \langle u, \mathbf{1}\rangle)\, du^{(n-k)},$$

where $N(\cdot)$ is the support function of the Newton polyhedron, and $du^{(m)}$ is the m-dimensional lattice volume form. This definition is chosen so that the topological ζ-function of f equals the sum of the contributions of the dual cones to all bounded faces of $\Gamma_+(f)$.

Lemma 3.3. *Assume that a B_1-face τ of $\Gamma_+(f)$ is the convex hull of its base γ in the coordinate hyperplane $\{v_n = 0\}$ and its apex $P = (*, \dots, *, 1)$. Furthermore, assume that $C \subset \gamma^\circ$ is the convex hull of a rational polyhedral subcone $C' \subset \tau^\circ$ and the n-th coordinate axis $O_n = \mathbb{R}_+(0, \dots, 0, 1) \subset \mathbb{R}^n_+$. Then the sum of the contributions from the cones $C \subset \gamma^\circ$ and $C' \subset \tau^\circ$ to $Z_{\mathrm{top},f}(s)$ is equal to*

$$\int_C \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_n$$

if τ is a B_1-segment, and is 0 *otherwise.*

Proof. In the second case, the contributions from C and C' are equal up to sign and cancel each other. Indeed, if we decompose C' into simplicial cones $\Delta'_i \subset C'$ and take the convex hulls $\Delta_i \subset C$ of them and the coordinate axis $O_n = \mathbb{R}_+ \cdot (0, \dots, 0, 1)$, then by using the condition $P = (*, \dots, *, 1)$ we can easily show that $\mathrm{mult}(\Delta_i) = \mathrm{mult}(\Delta'_i)$. In the first case, we may assume that C' is simplicial and $\mathrm{mult}(C) = \mathrm{mult}(C')$. Let $a(i) \in C' \cap \mathbb{Z}^n_+$ $(1 \le i \le n-1)$ be the primitive vectors on the edges of the $(n-1)$-dimensional cone C'. Then the sum of the contributions is equal to

$$\frac{\mathrm{mult}(C)}{\prod_{i=1}^{n-1}\{N(a(i))s + \nu(a(i))\}} - \frac{s}{s+1}\frac{\mathrm{mult}(C)}{\prod_{i=1}^{n-1}\{N(a(i))s + \nu(a(i))\}}$$
$$= \frac{\mathrm{mult}(C)}{\{\langle e_n, P\rangle s + \langle e_n, \mathbf{1}\rangle\} \cdot \prod_{i=1}^{n-1}\{\langle a(i), P\rangle s + \langle a(i), \mathbf{1}\rangle\}},$$

where $e_n = (0, \dots, 0, 1)$. The right hand side is equal to the sought integral by the proof of Lemma 2.2. ■

3.2. *Critical edges*

We now introduce our main tool to prove that a given number is not a pole of the topological zeta function.

Lemma 3.4. *Assume that $s_0 \neq -1$. Then for the points P in the summit of a (possibly non-compact) B_1-facet of $\Gamma_+(f)$ the equation $\langle u, P\rangle s_0 + \langle u, \mathbf{1}\rangle = 0$ is non-trivial; so it defines a hyperplane L_P in $\mathbb{R}^n$.*

Proof. If it is trivial, then $P = -\mathbf{1}/s_0$. Since $s_0 \neq -1$, none of the coordinates of P is equal to 1, so it is not in the summit. ■

Definition 3.5. We say that a closed set $C \subset \mathbb{R}^n$ is an *n-dimensional polyhedral cone* if it is a union of finitely many n-dimensional closed convex polyhedral cones. A ray R on the boundary ∂C of an n-dimensional polyhedral cone C in $\mathbb{R}^n$ is called an *edge* of C if, in an arbitrarily small neighborhood of a point of rel.int $R = R \setminus \{0\}$, the cone C is not affinely isomorphic to a product $\mathbb{R}^2 \times C'$ for some subset $C' \subset \mathbb{R}^{n-2}$. Moreover, for $s_0 \in \mathbb{R}$ and a point $P \in \mathbb{R}^n$ we say that a ray R in $\mathbb{R}^n$ is *critical* with respect to the pair (s_0, P) if for its generator $u \in R$ we have $\langle u, P\rangle s_0 + \langle u, \mathbf{1}\rangle = 0$, i.e. $u \perp (s_0 P + \mathbf{1})$.

Lemma 3.6. *Let $C \subset \mathbb{R}^n$ be an n-dimensional polyhedral cone in $\mathbb{R}^n$. Assume that for $s_0 \in \mathbb{R}$ and a point $P \in \mathbb{R}^n$ no edge of C is critical with respect to (s_0, P). Then the integral*

$$\int_C \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_n \tag{3.1}$$

is a rational function of s holomorphic at $s = s_0 \in \mathbb{C}$.

Proof. Subdivide C into simplicial cones. This subdivision is combinatorially stable under small perturbation of each ray R within its ambient face of C. Since no edge in this ambient face is critical with respect to (s_0, P), the same is true for almost all rays in the face. We can thus perturb the rays in the subdivision in their ambient faces so that all of them become non-critical with respect to (s_0, P). Then by integrating $\exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)$ over each of the simplicial cones in the resulting subdivision, we obtain a rational function holomorphic at $s = s_0 \in \mathbb{C}$ (see Lemma 2.2). ■

3.3. Some non-contributing configurations of B_1-facets

We first show that a candidate pole contributed by a unique facet is always fake, once this facet is B_1. Then we discuss what happens in other cases (when the same candidate pole is contributed by several B_1-facets or by a non-B_1-facet).

The following result is not used in what follows and, on the contrary, is a special case of the subsequent Theorem 4.3. Nevertheless, we prefer to give it an independent proof, keeping things as simple and explicit as possible. This proof is a good illustration of a more general construction (of so called sprouts) used later on to prove Theorem 5.2 leading to the monodromy conjecture in dimension 4.

Proposition 3.7. *Assume that f is non-degenerate and let $\tau \prec \Gamma_+(f)$ be a B_1-facet. Assume also that the candidate pole*

$$s_0 = -\frac{\nu(\tau)}{N(\tau)} \neq -1$$

of $Z_{\mathrm{top},f}(s)$ is contributed only by τ. Then s_0 is fake, i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.

Proof. Since the proof for B_1-pyramids of non-compact type is similar, we prove the assertion only for B_1-pyramids of compact type. Without loss of generality we may assume that τ is a compact pyramid over the base $\gamma = \tau \cap \{v_n = 0\}$ and its unique vertex $P \prec \tau$ such that $P \notin \gamma$ has height 1 over the hyperplane $\{v_n = 0\} \subset \mathbb{R}^n$. Let $A_1, \dots, A_m$ ($m \geq n-1$) be the vertices of the $(n-2)$-dimensional polytope γ. For $1 \leq i \leq m$ we denote the dual cone A_i° of $A_i \prec \Gamma_+(f)$ by C_{A_i}. Similarly we set $C_P = P^\circ$. Let $a(\tau) \in \mathbb{Z}_+^n$ be the primitive vector on the ray τ°. Then we have

$$a(\tau) \in \operatorname{Int}(C_P \cup C_{A_1} \cup \cdots \cup C_{A_m}).$$

Note that $C_P \cup C_{A_1} \cup \cdots \cup C_{A_m}$ is an n-dimensional polyhedral cone in the sense of Definition 3.5. In order to construct a nice n-dimensional polyhedral subcone $\Box$ of $C_P \cup C_{A_1} \cup \cdots \cup C_{A_m}$ such that $a(\tau) \in \operatorname{Int}\Box$, we shall introduce a new dummy vector $b \in \operatorname{Int} C_P \cap \mathbb{Z}_+^n$ satisfying the condition

$$-\frac{\nu(b)}{N(b)} \neq s_0 \tag{3.2}$$

in the following way. First, by our assumption $s_0 \neq -1$ and Lemma 3.4, for the apex $P = (*, *, \dots, *, 1) \in \mathbb{Z}_+^n$ of the B_1-pyramid τ the equation $\langle u, P\rangle s_0 + \langle u, \mathbf{1}\rangle = 0$ is non-trivial. It thus defines a hyperplane L_P in $\mathbb{R}^n$. Then by taking a primitive vector $b \in \operatorname{Int} C_P \cap \mathbb{Z}_+^n$ such that $b \notin L_P$ we get the desired condition $N(b)s_0 + \nu(b) \neq 0$. Let $\tau_0 \prec \Gamma_+(f)$ be the unique facet such that $\gamma \prec \tau_0$, $\tau_0 \neq \tau$ and $\tau_0 \subset \{v_n = 0\}$. Then the primitive vector $a(\tau_0) \in \mathbb{Z}_+^n$ on the ray τ_0° is given by $a(\tau_0) = (0, 0, \dots, 0, 1)$. For $1 \leq i \leq m$ let $\sigma_i \prec \tau$ be the edge of τ connecting the two points P and A_i and $F_i \prec C_P$ the corresponding facet of the cone C_P containing $\tau^\circ = \mathbb{R}_+ a(\tau) \prec C_P$. All the facets of C_P containing τ° are obtained in this way. Since the point $A_i \prec \sigma_i$ is a vertex of τ_0, its dual cone $C_{A_i} = A_i^\circ$ contains not only F_i but also the ray $\tau_0^\circ = \mathbb{R}_+ a(\tau_0)$. For $1 \leq i \leq m$ set

$$F_i^\sharp = \mathbb{R}_+ a(\tau_0) + F_i, \quad F_i^\flat = \mathbb{R}_+ b + F_i.$$

In Figure 1 we present the transversal hyperplane sections of the cones F_i, $F_i^\sharp$ and $F_i^\flat$.

Then by our construction, $\Box = \bigcup_{i=1}^m (F_i^\sharp \cup F_i^\flat)$ is an n-dimensional polyhedral cone in $\mathbb{R}^n$ and satisfies the desired condition $a(\tau) \in \operatorname{Int}\Box$.

By Lemma 3.3 and the argument in [18, Case 1 of the proof of Proposition 14], the contribution to $Z_{\mathrm{top},f}$ from the dual cones $C_P, C_{A_1}, \dots, C_{A_m}, F_1, \dots, F_m$ of $P, A_1, \dots, A_m, PA_1, \dots, PA_m$ respectively is equal to $Z_{\mathrm{top},f}$ modulo functions holomorphic at s_0. We thus obtain

$$Z_{\mathrm{top},f}(s) \equiv \int_\Box \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_n$$

modulo functions holomorphic at s_0. By our assumption and condition (3.2) no edge of the polyhedral cone $\Box$ is critical with respect to (s_0, P) in the sense of Definition 3.5. Then by Lemma 3.6 the rational function

$$\int_\Box \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_n$$

of s is holomorphic at s_0. This implies that also $Z_{\mathrm{top},f}(s)$ is holomorphic there. ∎

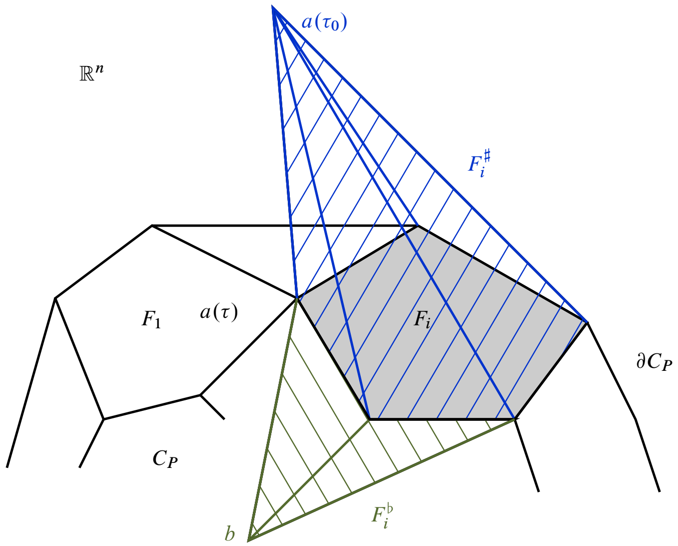

Fig. 1. The proof of Proposition 3.7.

We now discuss what happens when the same candidate pole is contributed by several B_1-facets.

If two B_1-facets for different variables are adjacent (i.e. have a common $(n-2)$-dimensional face) and contribute the same candidate pole, this may happen to be an actual pole of the topological ζ-function. This is always so for $n = 3$ (see [18]), but may fail starting from $n = 4$ (see B^2-borders in Theorem 5.2).

On the contrary, two adjacent B_1-faces for the same variable cannot alone yield an actual pole (similarly to [18, Proposition 14] for $n = 3$). The following result is a higher-dimensional analogue of the one in the proof of [18, Proposition 14].

Proposition 3.8. *Assume that f is non-degenerate and let $\tau_1, \dots, \tau_k \prec \Gamma_+(f)$ be B_1-facets such that*

$$s_0 := -\frac{\nu(\tau_1)}{N(\tau_1)} = \cdots = -\frac{\nu(\tau_k)}{N(\tau_k)} \neq -1$$

and their common candidate pole $s_0 \in \mathbb{Q}$ of $Z_{\mathrm{top},f}(s)$ is contributed only by them. Assume also that if τ_i and τ_j $(i \neq j)$ have a common facet then they are B_1-pyramids for the same variable. Then s_0 is fake, i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.

Proof. If τ_i and τ_j $(i \neq j)$ do not have a common facet, then by the proof of Proposition 3.7 after a suitable subdivision of the dual fan of $\Gamma_+(f)$ into rational simplicial cones we can calculate their contributions to $Z_{\mathrm{top},f}(s)$ separately. So we may assume that the B_1-facets $\tau_1, \dots, \tau_k$ have the common apex $P \in \tau_1 \cap \cdots \cap \tau_k$. For simplicity, we shall only treat the case where $k = 2$, τ_1 (resp. τ_2) is a compact B_1-pyramid over the base $\gamma_1 = \tau_1 \cap \{v_n = 0\}$ (resp. $\gamma_2 = \tau_2 \cap \{v_n = 0\}$) and $\tau_1 \cap \tau_2$ is the (unique) common facet of τ_1 and τ_2; the proofs for the other cases are similar.

Let $\tau_0 \prec \Gamma_+(f)$ be the unique facet of $\Gamma_+(f)$ such that $\gamma_1, \gamma_2 \prec \tau_0$, $\tau_0 \neq \tau_i$ $(i = 1, 2)$ and $\tau_0 \subset \{v_n = 0\}$. We denote by P the common apex of τ_1 and τ_2. Since $\tau_1 \cap \tau_2$ is a common facet of τ_1 and τ_2, there exists a 2-dimensional face of the dual cone C_P of $P \prec \Gamma_+(f)$ containing both $\tau_1^\circ = \mathbb{R}_+ a(\tau_1)$ and $\tau_2^\circ = \mathbb{R}_+ a(\tau_2)$. As in the proof of Proposition 3.7, let $F_1, \dots, F_m$ be the facets of C_P containing the ray τ_1° or τ_2° and subdivide $F_1 \cup \cdots \cup F_m \subset \partial C_P$ into rational simplicial cones without adding new edges. Let $\Delta_1, \dots, \Delta_r$ be the $(n-1)$-dimensional simplicial cones thus obtained in $F_1 \cup \cdots \cup F_m \subset \partial C_P$ and containing τ_1° or τ_2°. As in the proof of Proposition 3.7, we take a new primitive vector $b \in \operatorname{Int} C_P \cap \mathbb{Z}^n_+$ such that

$$-\frac{\nu(b)}{N(b)} \neq s_0. \tag{3.3}$$

For $1 \leq i \leq r$ set

$$\Delta_i^\sharp = \mathbb{R}_+ a(\tau_0) + \Delta_i, \quad \Delta_i^\flat = \mathbb{R}_+ b + \Delta_i.$$

Then

$$\square := \bigcup_{i=1}^r (\Delta_i^\sharp \cup \Delta_i^\flat)$$

is an n-dimensional polyhedral cone in $\mathbb{R}^n$ such that

$$a(\tau_1), a(\tau_2) \in \operatorname{Int} \square.$$

By Lemma 3.3 (or the argument in [18, Case 1 of the proof of Proposition 14]) we obtain an equality

$$Z_{\mathrm{top},f}(s) \equiv \int_\square \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_n$$

modulo functions holomorphic at s_0. By our assumption and the condition (3.3) no edge of the polyhedral cone $\square$ is critical with respect to (s_0, P) in the sense of Definition 3.5. Then by Lemma 3.6 the rational function

$$\int_\square \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_n$$

of s is holomorphic at s_0. ■

3.4. B_2-facets

We now discuss what happens when a candidate pole is contributed by a non-B_1-facet. Such an s_0 is always allowed to be a genuine pole for $n = 3$, because $\exp(2\pi i s_0)$ is always a nearby monodromy eigenvalue (see [18]). However, this is not the case starting from $n = 4$ for some non-B_1-facets. In this subsection, we introduce one such example.

Definition 3.9. For all $n \geq 4$ we define *B_2-facets* $\tau \prec \Gamma_+(f)$ to be non-B_1 compact facets whose projection to a certain $(n-2)$-dimensional coordinate plane coincides with the standard $(n-2)$-dimensional simplex.

In particular, for $n = 4$, a facet τ of $\Gamma_+(f)$ is a B_2-facet if and only if, up to reordering the coordinates, it has the vertices A, B, P, Q, X, Y of the form

$$\begin{cases} A = (1, 0, *, *), \\ B = (1, 0, *, *), \\ P = (0, 1, *, *), \\ Q = (0, 1, *, *), \\ X = (0, 0, *, *), \\ Y = (0, 0, *, *), \end{cases}$$

as in Figure 2 below (it can be degenerate so that $X = Y$).

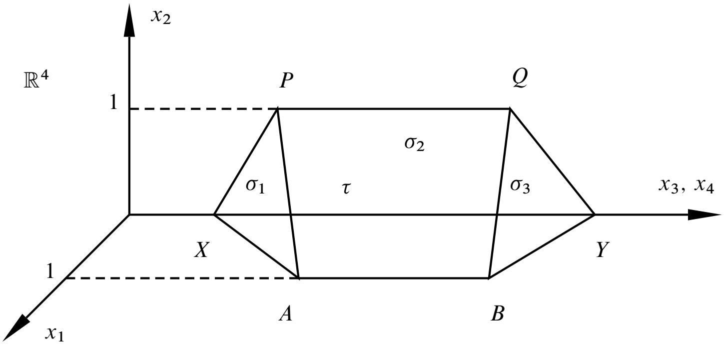

Fig. 2. A B_2-facet in dimension 4.

Note that the facet τ splits into two B_1-pyramids for different variables whose intersection does not contain any 1-dimensional V-face. For example we have the decomposition $\tau = AXPQ \cup QXABY$.

Definition 3.10. A facet τ of $\Gamma_+(f)$ is called a *B-facet* if it is a B_1-facet or a B_2-facet.

Proposition 3.11. *In the case $n \geq 4$ assume that f is non-degenerate and let $\tau \prec \Gamma_+(f)$ be a B_2-facet. Assume also that the candidate pole*

$$s_0 = -\frac{\nu(\tau)}{N(\tau)} \neq -1$$

of $Z_{\mathrm{top},f}(s)$ is contributed only by τ. Then s_0 is fake.

Proof. We prove the assertion only for $n = 4$. The proof for the general case $n \geq 4$ is similar. In the notation of Figure 2, we define facets $\sigma_1, \sigma_2, \sigma_3$ of τ by $\sigma_1 = XAP$, $\sigma_2 = PABQ$, $\sigma_3 = YQB$ respectively. As in [18, proof of Proposition 14] let $\tau_i \prec \Gamma_+(f)$ $(1 \leq i \leq 3)$ be the unique facet such that $\sigma_i \prec \tau_i$ and $\tau_i \neq \tau$. Moreover, for $i = 1, 2$ let $\rho_i \prec \Gamma_+(f)$ be the unique facet such that $\rho_i \subset \{v_i = 0\} \simeq \mathbb{R}^3$, $\tau \cap \{v_i = 0\} \prec \rho_i$ and $\rho_i \neq \tau$. Since the dimension of the B_2-facet τ is 3, the three segments PQ, AB and XY are parallel. This implies that their dual cones are on the same hyperplane $H \simeq \mathbb{R}^3$ in $\mathbb{R}^4$.

Having the four facets τ, τ_1, τ_2, ρ_1 containing the vertex P we define a 4-dimensional simplicial cone $\Delta_P \subset \mathbb{R}^4$ by

$$\Delta_P = \mathbb{R}_+ a(\tau) + \mathbb{R}_+ a(\tau_1) + \mathbb{R}_+ a(\tau_2) + \mathbb{R}_+ a(\rho_1).$$

Similarly we define 4-dimensional simplicial cones $\Delta_A, \Delta_X, \Delta_Q, \Delta_B, \Delta_Y \subset \mathbb{R}^4$ for the vertices A, X, Q, B, Y and set

$$\square_P = \Delta_P \cup \Delta_A \cup \Delta_X, \quad \square_Q = \Delta_Q \cup \Delta_B \cup \Delta_Y.$$

Then by the above-mentioned property of H, $\square_P \cap H$ (resp. $\square_Q \cap H$) is a facet of $\square_P$ (resp. $\square_Q$) and $\square_P \cap H = \square_Q \cap H = \square_P \cap \square_Q$. The dual ray τ° of τ is contained in $\square_P \cap H = \square_Q \cap H = \square_P \cap \square_Q$, but it is an edge of neither $\square_P$ nor $\square_Q$. Moreover, $\square := \square_P \cup \square_Q$ is a 4-dimensional polyhedral cone such that $a(\tau) \in \operatorname{Int} \square$. By Lemmas 2.2 and 3.3 and the argument in [18, Case 1 of the proof of Proposition 14], the contribution to $Z_{\mathrm{top},f}$ from the dual cones of P, A, X, PA, PX and Q, B, Y, QB, QY is equal to $Z_{\mathrm{top},f}$ modulo functions holomorphic at s_0. For example, the sum of the contributions to $Z_{\mathrm{top},f}$ from the dual cones of $\sigma_1 = XAP$ (resp. $\sigma_3 = YQB$) and XA (resp. YB) is zero. The same is true for the dual cones of the three faces $\sigma_2 = PABQ$, PQ and AB. Here we use the fact that the normalized area of the quadrilateral face $\sigma_2 = PABQ$ of τ is equal to the sum of the lengths of the segments PQ and AB (see Lemma 4.22 for higher-dimensional cases).

Moreover, by Lemma 2.2 it suffices to consider the contribution of the subcones $\Delta_P, \Delta_A, \Delta_X, \Delta_P \cap \Delta_A, \Delta_P \cap \Delta_X$ and $\Delta_Q, \Delta_B, \Delta_Y, \Delta_Q \cap \Delta_B, \Delta_Q \cap \Delta_Y$. Then by applying Lemma 3.3 to the pair of cones $\Delta_A \subset A^\circ$ and $\Delta_P \cap \Delta_A \subset (PA)^\circ$ (resp. $\Delta_X \subset X^\circ$ and $\Delta_P \cap \Delta_X \subset (PX)^\circ$) etc., we obtain

$$\begin{aligned} &Z_{\mathrm{top},f}(s) \\ &\equiv \int_{\square_P} \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_4 + \int_{\square_Q} \exp(-\langle u, Q\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_4 \end{aligned}$$

modulo functions holomorphic at s_0. By Lemma 3.6 the rational function

$$\int_{\square_P} \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du_1 \cdots du_4$$

of s is holomorphic at s_0. The same is true for the integral over $\square_Q$. Hence $Z_{\mathrm{top},f}(s)$ is holomorphic at s_0. ∎

4. Fake poles of the topological zeta function in arbitrary dimension

In view of the observations from the preceding section, the following result does not look unexpected. From now on, by *faces* we mean faces of the Newton polyhedron $\Gamma_+(f)$, unless explicitly stated otherwise.

Definition 4.1. (1) A one-element set $\{i\} \subset \{1, \dots, n\}$ is called a *base direction for a* B_1*-facet* τ if the i-th coordinate equals 1 for one vertex of τ, and equals 0 for the other vertices.

(2) An $(n-2)$-element set $I \subset \{1, \dots, n\}$ is called a *base direction for a* B_2*-facet* if its projection to the I-th coordinate plane is the standard $(n-2)$-dimensional simplex.

Note that a B_1-facet may have more than one base direction.

Definition 4.2. A collection of B-facets is said to be *consistent* if their base directions can be chosen so that

a pair of B-facets in the collection have a common facet

$\Longrightarrow$ the intersection of their base directions is non-empty.

Theorem 4.3. *Assume that f is non-degenerate and does not have a Morse singularity at $0 \in \mathbb{C}^n$. Let $\tau_1, \dots, \tau_k \prec \Gamma_+(f)$ be B-facets such that*

$$s_0 = -\frac{\nu(\tau_1)}{N(\tau_1)} = \cdots = -\frac{\nu(\tau_k)}{N(\tau_k)} \neq -1$$

and their common candidate pole $s_0 \in \mathbb{Q}$ of $Z_{\mathrm{top},f}(s)$ is contributed only by them. If we can choose their base directions to be consistent, then s_0 is fake, i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.

We allow ourselves to exclude Morse singularities from consideration, because the monodromy conjecture for Morse singularities is clear.

This section is devoted to the proof of Theorem 4.3.

In the course of the proof we introduce some new tools that will be used in the next section, which is devoted to a sharper version of Theorem 4.3, completely classifying configurations of B-faces contributing fake poles for $n = 4$. This classification is a key point in the proof of the monodromy conjecture for $n = 4$.

4.1. Contributions

Definition 4.4. Let $S \subset \mathbb{R}^n_+$ be a *polyhedral set*, that is, a disjoint union of the relative interiors of some (finitely many) closed convex polyhedral cones in $\mathbb{R}^n_+$. Then we define its *contribution* $Z(S)(s) \in \mathbb{C}(s)$ (*to the topological zeta function* $Z_{\mathrm{top,f}}$) by

$$\int_S \exp(-N(u)s - \langle u, \mathbf{1}\rangle)\, du_{\mathbb{R}^n_+} + \frac{s}{s+1} \sum_\tau (-1)^{\dim \tau} \operatorname{Vol}_{\mathbb{Z}}(\tau) \int_{S \cap \tau^\circ} \exp(-N(u)s - \langle u, \mathbf{1}\rangle)\, du_{\tau^\circ},$$

where τ ranges through all positive-dimensional compact faces of $\Gamma_+(f)$, $N(\cdot)$ is the support function of $\Gamma_+(f)$, and du_C is the lattice volume form on a rational polyhedral cone C (so that no components of dimension smaller than $\dim C$ in $S \cap C$ affect the integral with respect to du_C).

Remark 4.5. (1) As a function of S, the contribution to the topological zeta function is an additive measure.

(2) By Lemma 2.2, the contribution $Z(\mathbb{R}^n_+)$ of the open positive quadrant $\mathbb{R}^n_+$ equals the topological zeta function of a generic f with the given Newton polyhedron $\Gamma_+(f)$.

(3) We do not assume the argument S to be closed or open, because $Z(S)$ changes significantly when passing to the closure or the interior of S, and indeed we shall need sets S that are neither closed nor open.

4.2. The main theorem: the plan of the proof

(I) Very loosely, the proof of Theorem 4.3 will consist in constructing a particular subdivision of $\mathbb{R}^n_+$ into pieces σ_i such that $Z(\sigma_i)$ has no pole at s_0. The boldest hope would be to choose σ_i so that the key Lemma 3.6 is directly applicable to every σ_i, i.e.:

– Every σ_i is contained in the dual cone to some vertex P of the Newton polyhedron;

– No edge of σ_i is critical with respect to (s_0, P) in the sense of Lemma 3.6.

(II) Unfortunately, in general Step (I) is not realistic as written, because some of the edges of the dual cone P° will be critical, and they also have to be edges for some σ_i. These critical edges are exactly the ones dual to the contributing facets τ_i of the Newton polyhedron.

Fortunately, all such facets τ_i are B-facets under the assumptions of Theorem 4.3. This will help us to surround every such critical edge τ_i° by a conic neighborhood σ_i' (called a *sprout*) such that Lemma 3.6 is still applicable to it:

– The contribution $Z(\sigma_i')$ can be written as the integral from Lemma 3.6 for some appropriate vertex P_i (despite the fact that σ_i' is not contained in any individual cone of the form P° anymore!).

– No edge of σ_i' is critical with respect to (s_0, P_i).

(III) Unfortunately, upon choosing neighborhoods in Step (II), we face the next (and the last) obstacle: the complement to $\bigcup_i \sigma_i'$ in any cone P° may have new critical edges in the boundary of P° (different from the edges of P° itself). This happens because some cones (called *critical cones*) in the dual fan Σ_0 entirely (!) consist of critical rays, so every 0-dimensional intersection of a face of σ_i' with such a critical cone C will create a new critical edge of the complement of σ_i' in any cone containing C.

Example 4.6. Recall that here and in what follows, we draw the projectivization of the fan Σ_0 rather than the fan itself. On the left of Figure 3 below, the critical 1-dimensional cone τ_1°, whose dual facet τ_1 is a B_1-pyramid with apex P, is surrounded by a conical neighborhood σ_1' (shown in bold). Since the dashed segment is a critical 2-dimensional cone, its intersection point with the boundary of σ_1' is a critical ray that is not an edge of σ_1', but is a critical ray of the complement to σ_1' in the 3-dimensional cone P°.

Fortunately, choosing the neighborhoods σ_i' wisely, we can ensure that no new critical edge of the complement of an individual σ_i' is an edge of the complement to the

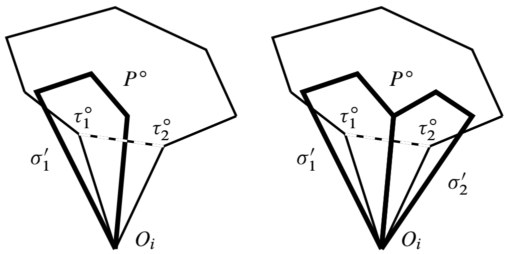

Fig. 3. Good neighborhoods of critical cones.

whole $\bigcup_i \sigma_i'$ (see the right of Figure 3). For instance, this is done in detail in the proof of Proposition 3.8 for the case of only two facets contributing the pole s_0.

In the general setting, this will be done by using the geometry of so called *delimiter planes* and will allow us to literally apply Step (I) of our plan to the complement of $\bigcup_i \sigma_i'$. Warning: the resulting cones σ_i' will not form a fan in the sense that $\sigma_i' \cap \sigma_j'$ may not be a face of σ_i' and σ_j'.

We introduce all the aforementioned objects in the subsequent subsections.

4.3. Intersections of B-facets

Lemma 4.7. *Assume that f is non-degenerate and does not have a Morse singularity at $0 \in \mathbb{C}^n$. If two B-facets of the Newton polyhedron $\Gamma_+(f)$ have a common $(n-2)$-dimensional face, then it is a B_1-pyramid.*

The proof of this lemma requires the following observation.

Lemma 4.8. *If a homogeneous polynomial g of degree 2 on $\mathbb{C}^4$ is non-degenerate with respect to its Newton polyhedron and* $\operatorname{supp} g$ *contains the points* $(1,1,0,0)$ *and* $(0,0,1,1)$*, then g is non-degenerate as a quadratic form.*

Upon publishing the first preprint version of this paper, this lemma was beautifully generalized to arbitrary dimension in [33].

Proof of Lemma 4.8. The non-degeneracy of g as a quadratic form is equivalent to the smoothness of the hypersurface $V = \{g = 0\}$ of $\mathbb{P}^3$ defined by g. Indeed, the kernel of the symmetric matrix associated to the quadratic form corresponds to the singular locus of $V \subset \mathbb{P}^3$. Recall that $\mathbb{P}^3$ is naturally a toric variety on which the complex torus $T = (\mathbb{C}^*)^3$ acts with 15 orbits. First of all, by the non-degeneracy of g with respect to its Newton polyhedron, the hypersurface $V = \{g = 0\}$ of $\mathbb{P}^3$ is smooth on $T \subset \mathbb{P}^3$. So we have only to analyze it at other points x in $\mathbb{P}^3 \setminus T$. We shall do it step by step, considering x in each of the other 14 T-orbits in $\mathbb{P}^3$. We denote the Newton polytope of g by $Q \subset \{v_1 + v_2 + v_3 + v_4 = 2\}$.

(1) Assume that $x = (1:0:0:0)$. If the point $(2,0,0,0)$ is in $\operatorname{supp} g$, then $g(x) \neq 0$. Otherwise, we have $dg(x) \neq 0$, because $(1,1,0,0) \in \operatorname{supp} g$ and hence $(\partial g/\partial x_2)(x) \neq 0$. This implies that the hypersurface $V = \{g = 0\}$ of $\mathbb{P}^3$ is smooth at $x \in V$.

(2) Assume that x is in one of the other three 0-dimensional T-orbits in $\mathbb{P}^3$. Then the reasoning is as in (1).

(3) Assume that $x = (s : t : 0 : 0)$ $(s, t \neq 0)$. Then, since the face $F = Q \cap [(2,0,0,0),(0,2,0,0)]$ of Q is non-empty (containing at least $(1,1,0,0)$) and the restriction g_F of g to the face $F \prec Q$ defines a smooth hypersurface in the 1-dimensional T-orbit in $\mathbb{P}^3$ associated to $[(2,0,0,0),(0,2,0,0)]$ (by the non-degeneracy of g), the hypersurface $V = \{g = 0\}$ of $\mathbb{P}^3$ is smooth at x.

(4) Assume that $x = (0 : 0 : s : t)$ $(s, t \neq 0)$ or x is in one of the four 2-dimensional T-orbits in $\mathbb{P}^3$. Then the reasoning is as in (3).

(5) Assume that $x = (s : 0 : t : 0)$ $(s, t \neq 0)$. If $\operatorname{supp} g \cap [(2,0,0,0),(0,0,2,0)]$ is not empty, then the reasoning is as in (3). Otherwise, we have $g(x) = 0$. Assume that we also have $dg(x) = 0$. Then in particular $(\partial g/\partial x_2)(x) = (\partial g/\partial x_4)(x) = 0$. From these identities, we see that the restrictions g_A, g_B of g to the segments $A = [(1,1,0,0),(0,1,1,0)]$ and $B = [(1,0,0,1),(0,0,1,1)]$ vanish at x, thus they are multiples of the same linear function. So the restriction g_P of g to the parallelogram $P = \operatorname{conv}(A \cup B)$ is a product of two linear functions. This implies that g_P defines a singular hypersurface in $(\mathbb{C}^*)^4$. This would contradict the non-degeneracy of g with respect to its Newton polyhedron.

(6) Assume that x is in one of the other three 1-dimensional T-orbits in $\mathbb{P}^3$. Then the reasoning is as in (5). ■

Lemma 4.9. *Assume that $n = 4$, and two B_2-facets of $\Gamma_+(f)$ have a common quadrilateral face. Then f has a Morse singularity at $0 \in \mathbb{C}^4$.*

Proof. Projectivizing the ambient space of $\Gamma_+(f)$, we see the positive octant as a tetrahedron, and the two B_2-facets in it as two polytopes from Figure 4 with a common quadrilateral face. This is only possible if the common quadrilateral is a parallelogram, whose edges are parallel to two opposite edges of the tetrahedron, and whose vertices are contained in the four other edges, as shown in the picture.

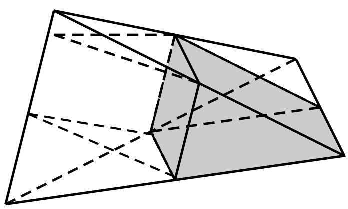

Fig. 4. B_2-facets with a common quadrilateral face.

Thus, reordering coordinates if necessary, we can assume that the vertices are of the form $(*,*,0,0)$, $(0,*,*,0)$, $(0,0,*,*)$ $(*,0,0,*)$, and all the stars are equal to 1 by the definition of B_2-facets. By Lemma 4.8, the quadratic part of f is non-degenerate, and hence f has a Morse singularity at $0 \in \mathbb{C}^4$. ■

Proof of Lemma 4.7. Assume that an $(n-2)$-dimensional non-B_1-pyramid F is contained in two B-facets F_1 and F_2. First, note that F is not a V-face, otherwise one of F_1 and F_2 were contained in the boundary of $\mathbb{R}^n_+$, which cannot happen for a B-facet.

Second, note that neither F_1 nor F_2 can be a B_1-facet, because every $(n-2)$-dimensional face of a B_1-facet is either a V-face or a B_1-pyramid.

Finally, the only $(n-2)$-dimensional non-B_1 non-V-face of a B_2-facet F_i is combinatorially isomorphic to the product of a segment and an $(n-3)$-simplex (let us call it the *front face* of the B_2-facet). So the only exception from the statement of the lemma could come from two B_2-facets with different base directions and the common front face. However, for $n > 4$ this is impossible, because two products of a segment and an $(n-3)$-simplex cannot be non-trivially combinatorially isomorphic, and for $n = 4$ this is excluded by Lemma 4.9. ∎

4.4. *Bases and apices*

Definition 4.10. The *star* of a cone C *in the dual fan of* $\Gamma_+(f)$ is the set of all cones containing C.

For each B_1-face τ contributing the candidate pole s_0 we can choose its apex and preferred base to be its vertex P and a number $i \in \{1, \dots, n\}$ respectively, such that the i-th coordinate of P equals 1 and the i-th coordinates of the other vertices of τ are 0. Note that we may have several options for this choice.

We now fix once and for all the choice of apices P_τ and preferred bases b_τ

- for all B_1-facets τ contributing the candidate pole s_0, and
- for all B_1-facets τ of B_2-facets contributing the candidate pole s_0 (by a B_1-facet of a facet σ we mean a codimension 2 face of the Newton polyhedron that belongs to σ and is a B_1-pyramid).

Moreover, in the setting of Theorem 4.3, we can choose the apices and the preferred bases consistently, so that the preferred base of every aforementioned face τ belongs to the base direction of every B-facet containing τ. This in particular ensures that

- any two B_1-facets intersecting in a codimension 2 B_1-face have the same apex and preferred base, and
- if a codimension 2 B_1-face τ' is at the same time a facet of a B_1-facet τ and of a B_2-facet (so that we have chosen the preferred base and apex for it), then τ' and τ have the same apex and preferred base.

Definition 4.11. Let L be the 2-dimensional coordinate plane along which the projection of a B_2-facet τ equals the standard simplex, and let $l \subset \mathbb{R}^n$ be the 1-dimensional vector space parallel to the intersection of L with the affine span of τ.

The dual hyperplane to l will be denoted by $D_\tau = D_{\tau^\circ}$ and called the *delimiter* of τ.

In particular, if $n = 4$, then, in the notation of Figure 2, the delimiter is the 3-dimensional plane normal to the three parallel segments. Furthermore, the hyperplane H in the proof of Proposition 3.11 is nothing but the delimiter of τ.

Remark 4.12. Most B_2-facets have a unique V-edge normal to the delimiter. However, we allow this edge to degenerate into a vertex (such B_2-facets are said to be *degenerate*).

However, various attributes of this V-edge make natural sense even for degenerate B_2-facets. For instance, "the length of the V-edge of τ" and "the dual cone of the V-edge of τ" refer to 0 and $D_\tau \cap V^\circ$ respectively for a degenerate B_2-facet τ with a vertex V instead of the V-edge. In what follows, this small abuse of terminology never causes confusion.

Remark 4.13. Every B_2-facet (including the degenerate ones) has three distinguished facets (i.e. $(n-2)$-dimensional faces): exactly one non-simplicial non-V-facet and two B_1-facets.

For instance, in the 4-dimensional setting of Figure 2, they are denoted by σ_2, σ_1 and σ_3 respectively.

Definition 4.14. (1) We shall say that a vertex P of a B_2-facet τ and a number $i \in \{1, \ldots, n\}$ are an *apex* and a *preferred base* of τ *on the side of a face* $F \prec \tau$, if P and i are the apex and the preferred base of a B_1-facet σ of τ such that $\sigma \cap F \neq \emptyset$. If τ has a unique preferred base on the side of F (that is, if σ is uniquely defined by the condition $\sigma \cap F \neq \emptyset$), this preferred base will be denoted by $b_\tau^F = b_{\tau^\circ}^F$.

(2) Similarly, if a cone C in the star of τ° does not intersect the delimiter D_τ, then the apex and the preferred base of the B_2-facet τ on the side of C are defined as the apex and the preferred base of the B_1-facet of τ whose dual cone is not separated from C by the delimiter.

(3) For conformity, we shall say that a vertex P of a B_1-facet τ and a number i are its apex and preferred base on the side of a face $F \prec \tau$ (or a cone C in the star of τ°) if P and i are the apex and the preferred base of τ (independently of F and C).

Example 4.15. For instance, let $n = 4$, consider the B_2-facet τ in Figure 2, and assume that (in the notation of this figure) the preferred base and apex for the B_1-triangle σ_1 are 2 and P, while those for σ_3 are 1 and B. Then

- P is the apex of τ on the side of σ_1 and all of its faces,
- B is the apex of τ on the side of σ_3 and all of its faces,
- both B and P are apices of τ on the side of its other 7 faces.

Further, the delimiter D_τ is the hyperplane normal to the segment XY. It divides the dual space into two half-spaces, containing the dual cones to the triangles σ_1 and σ_3; let us call them *left* and *right* respectively. If a cone C is in the left (respectively right) half-space, then P (respectively B) is the apex of τ on the side of C.

Lemma 4.16. *If a face F is contained in a B-facet τ and not contained in a coordinate hyperplane, then every apex of τ on the side of F is contained in F.*

4.5. Sprouts and cancellation of contributions

Let C be a (not necessarily convex) polyhedral cone in the star of τ°, where τ is a B-facet. In the case of a B_2-facet, we additionally assume that C does not intersect the delimiter D_τ. Let P and i be the apex and the preferred base of τ on the side of C. Note that they uniquely determine each other, and that we have chosen them once and for all in the preceding subsection. Recall that the standard basis in $\mathbb{R}^n$ is denoted by $e_1, \ldots, e_n$.

Definition 4.17. The union of $C \cap \text{rel.int}\, P^\circ$ and all 2-dimensional cones, generated by e_i and a point of $C \cap \partial P^\circ$, is denoted by $S_{C,\tau}$ and is called the *sprout* of C. The vertex P is denoted by $R_{C,\tau}$ and is called the *root* of C.

For example, if τ is a B_1-facet, then the cone $\square$ in the proof of Proposition 3.7 is a sprout of some cone.

Remark 4.18. By the definition, if C is a closed n-dimensional polyhedral cone (in the sense of Definition 3.5), containing τ° in its interior, then so is the sprout. On the other hand, convexity of C does not imply convexity of the sprout: see Figure 5.

Example 4.19. Figure 5 gives the simplest example of a cone C and its sprout in the star of the ray τ°.

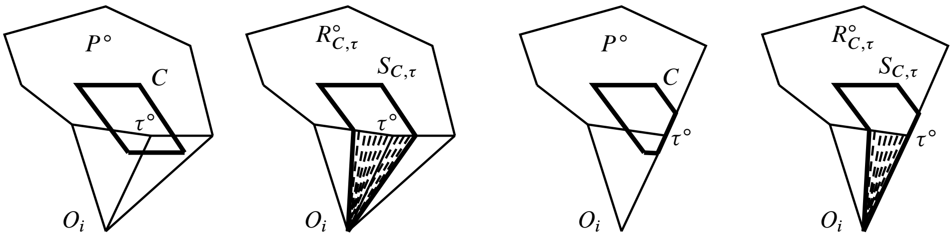

Fig. 5. The sprout of the cone C (the case of compact τ on the left and non-compact τ on the right).

Lemma 4.20. *The contribution of a sprout is simple:*

$$Z(S_{C,\tau}) = \int_{S_{C,\tau}} \exp(-\langle u, R_{C,\tau}\rangle s - \langle u, \mathbf{1}\rangle)\, du.$$

Proof. The contribution of $C \cap \text{rel.int}\, P^\circ$ equals the integral of $\exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)$ by definition. Triangulating the set $C \cap \partial P^\circ$ and, for every simplicial cone T of this triangulation, applying Lemma 3.3 to the contribution of the cone generated by T and e_i, we conclude that the rest of $S_{C,\tau}$ also contributes the integral of $\exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)$. ∎

Let τ be a B_2-facet, and assume without loss of generality that the projection of τ along the (e_{n-1}, e_n)-coordinate plane is the $(n-2)$-dimensional standard simplex S with vertices $0, e_1, \ldots, e_{n-2}$. The preimage of the facet $(e_1 \ldots e_{n-2})$ of this simplex under this projection intersects τ in its unique non-simplicial non-V-facet τ_0. Choose a ray r in the (relatively open) dual cone to τ_0.

Definition 4.21. The $(n-1)$-dimensional (relatively open) cone $S_{r,\tau}$ generated by $r, e_1, \ldots, e_{n-2}$ is called *the delimiter sprout* of r.

Lemma 4.22. *The contribution of the delimiter sprout $Z(S_{r,\tau})$ is equal to* 0 (*independently of the choice of the ray r*).

Proof. For a subset $I \subset \{1,\dots,n-2\}$, introduce the following notation:

- C_I is the cone generated by τ° and e_i $(i \in I)$,
- $\widetilde{C}_I$ is the cone spanned by C_I and r,
- τ_I is the preimage of the simplex with vertices e_i, $i \in I$, under the projection $\tau \to S$.
- $\tilde{\tau}_I$ is the preimage of the simplex with vertices 0 and e_i, $i \in I$, under the projection $\tau \to S$.

Then the contribution of the delimiter sprout splits into those of the cones C_I for $I \subset \{1,\dots,n-2\}$ and the cones $\widetilde{C}_I$ for $I \subsetneq \{1,\dots,n-2\}$.

In order to evaluate them, denote by g_i the lattice length of the segment τ_i and notice that the lattice volume of τ_I equals $\sum_{i\in I} g_i$. Then the sum of the contributions

$$Z(\widetilde{C}_I)(s) = (-1)^{n-|I|-2}\frac{s}{s+1}\Big(\sum_{i\notin I} g_i\Big) J_{\widetilde{C}_I}(s)$$

of the cones $\widetilde{C}_I$ over $I \subsetneq \{1,\dots,n-2\}$ is equal to 0, because the functions $J_{\widetilde{C}_I}(s)$ for all such I are equal to each other by definition (cf. a similar calculation in Lemma 3.3).

In the same way we can show that the corresponding sum for the cones C_I over $I \subset \{1,\dots,n-2\}$ is 0, because the lattice volume of $\tilde{\tau}_I$ equals the lattice volume of τ_I plus the lattice length of the segment $\tilde{\tau}_\emptyset$, and all $J_{C_I}(s)$ are equal to each other as well (more specifically, they are p times smaller than $J_{\widetilde{C}_I}(s)$, where p is the coefficient from the decomposition of the primitive generator of the ray r into the linear combination $p\cdot(e_1+\cdots+e_{n-2}) + q\cdot$ (the primitive generator of τ°)). ■

Lemma 4.23. *The edges of a sprout or a delimiter sprout $S_{C,\tau}$ are either edges of the cone $(\partial C)\cap\sigma^\circ$ for some face $\sigma \prec \tau$, or coordinate rays.*

This directly follows from the definition of a sprout.

4.6. Critical cones

We now introduce and study the key notion of critical faces for a candidate pole $s_0 \neq -1$ of the zeta function $Z_{\mathrm{top},f}(s)$. Similarly to Lemma 3.4, we have the following lemma.

Lemma 4.24. *For $s_0 \neq -1$, any vertex P of a B-facet, and any hyperplane $\{v_i = 0\}$ which is either the base for this B-facet, or contains P, the equation $\langle u, P\rangle s_0 + \langle u, \mathbf{1}\rangle = 0$ is not satisfied by the coordinate vector $u = e_i$. In particular, this equation is non-trivial and thus defines a hyperplane L_P in $\mathbb{R}^n$.*

Proof. $\langle e_i, P\rangle s_0 + \langle e_i, \mathbf{1}\rangle$ equals 1 or $1 + s_0$ in this setting. ■

Starting from the following definition, the things depend on the choice of preferred bases and apices of B-faces, which we have fixed in Section 4.4 for the rest of the paper.

Definition 4.25. For a face F, we define the *critical set* $K_F \subset F^\circ$ to be the closure of $\bigcup_P(\text{rel.int } F^\circ \cap L_P)$, where in the union $\bigcup_P$ the vertex $P \prec \Gamma_+(f)$ ranges through the apices on the side of F for B-facets containing F (see Definition 4.14).

Lemma 4.26. (1) *If a face F is contained in a coordinate plane, then K_F is a finite union of hyperplanes in F°.*

(2) *If a face F is not contained in a coordinate plane, then K_F is given by one equation, $\{u \in F^\circ \mid \langle u, Q\rangle s_0 + \langle u, \mathbf{1}\rangle = 0\}$, where Q is an arbitrary point of F. In particular, either K_F is a hyperplane in F°, or $K_F = F^\circ$.*

Proof. (1) If F is in the coordinate plane $v_i = 0$, then $e_i \in F^\circ$. If F is in a B-facet τ contributing to s_0, then either F is in the preferred base of τ (so that $v_i = 0$ may be chosen to be this preferred base), or F contains the apex $P \in \tau$ on its side (so that P is in $v_i = 0$). In both cases, the set L_P does not contain $e_i \in F^\circ$ by Lemma 4.24, so no L_P can contain the whole F°.

(2) If F is not in a coordinate plane, then any P in the definition of the critical set is contained in F by Lemma 4.16, so K_F is given by the equation $\langle u, P\rangle s_0 + \langle u, \mathbf{1}\rangle = 0$. The set defined by this equation in F° does not change if we substitute P with any other point $Q \in F$, because $\langle Q - P, u\rangle = 0$ for $u \in F^\circ$. ∎

Definition 4.27. We say that a face $F \prec \Gamma_+(f)$ and its dual cone $F^\circ \in \Sigma_0$ are *critical* (for the candidate pole s_0) if $K_F = F^\circ$.

Lemma 4.28. *A critical face is not contained in a coordinate plane.*

This follows from Lemma 4.26 (1).

Proposition 4.29. *Assume that a candidate pole $s_0 \neq -1$ is contributed only by B-facets. In this case, for a face F the following conditions are equivalent:*

(1) *The face F is critical, that is, F is contained in a B-facet contributing s_0, and, for its apex P on the side of F, the plane L_P contains the cone F°.*

(2) *Every facet containing F is a B-facet contributing s_0, and, for every apex P of it on the side of F, the plane L_P contains the cone F°.*

(3) *Every facet containing F contributes the candidate pole s_0 of the topological ζ-function.*

The second condition will be mostly used in practice, and the third one is especially simple (and relates the first two).

Proof. (1)⇒(3) By Lemma 4.28 the critical face F is not contained in a coordinate hyperplane. Then by Lemma 4.26 (2), the equation of K_F in F° is given by

$$\langle u, Q\rangle s_0 + \langle u, \mathbf{1}\rangle = 0,$$

where Q is an arbitrary point of F. Since F is critical, i.e. $K_F = F^\circ$, the equation holds for any $u \in F^\circ$. In particular, for every facet τ containing F, its conormal vector $a(\tau) \in \tau^\circ \prec F^\circ$ satisfies the condition

$$\langle a(\tau), Q\rangle s_0 + \langle a(\tau), \mathbf{1}\rangle = 0.$$

Thus τ contributes the candidate pole s_0.

(3)⇒(2) By the assumptions of the proposition, every facet τ containing F is a B-facet. In particular, F is not contained in a coordinate hyperplane. By Lemma 4.16, every apex $P \in \tau$ (of every facet τ containing F) on the side of F is contained in F. Then by the condition (3), for the conormal vector $a(\tau) \in \tau^\circ \prec F^\circ$ we have

$$\langle a(\tau), P\rangle s_0 + \langle a(\tau), \mathbf{1}\rangle = 0.$$

Since such conormal vectors $a(\tau)$ generate the cone F°, for any $u \in F^\circ$ we have

$$\langle u, P\rangle s_0 + \langle u, \mathbf{1}\rangle = 0.$$

(2)⇒(1) This is evident. ∎

Corollary 4.30. *A face of a critical cone in the dual fan Σ_0 is a critical cone. Equivalently, a face containing a critical face is critical itself.*

Note that, however, for non-critical faces $F \subset G$, it is not in general true that $K_G \subset K_F$.

It is now a crucial observation that, under the assumptions of Theorem 4.3, every critical face F is a B_2-facet or B_1-pyramid (by Lemma 4.7 and Proposition 4.29). In the latter case, using the assumption of Theorem 4.3 we can choose apices of B_1-facets and B-facets of B_2-facets as in Section 4.4, so that all B-facets containing F have the same preferred base b_F and the same apex P_F on the side of F.

4.7. A tubular neighborhood of the critical subfan

We are now ready to prove Theorem 4.3. When referring to B-facets or critical faces in the course of the proof, we always mean only the faces contributing the candidate pole $s_0 \neq -1$, for which we are proving Theorem 4.3 (thus all the choices and objects that we introduce for the proof completely depend on the choice of s_0). Recall that up to now

- we have chosen once and for all a preferred base of every B_1-facet and every B_1-facet of every B_2-facet in the Newton polyhedron $\Gamma_+(f)$,
- depending on this choice, we have called some cones critical in the dual fan Σ_0,
- we have defined the delimiter hyperplane D_σ for every cone σ, dual to a B_2-facet (see Definition 4.11).

Choose once and for all an affine structure on the projectivization $\mathbb{P}\mathbb{R}^n_+$. For every cone C and fan Σ, denote their projectivizations by $\mathbb{P}C$ and $\mathbb{P}\Sigma$ respectively. In this subsection, we refer to cones and their projectivizations interchangeably whenever it causes no confusion.

According to Corollary 4.30, the set of projectivized critical cones in the dual fan Σ_0 is a polyhedral complex Σ_c, closed with respect to taking faces and intersections. We shall construct a generic piecewise linear tubular neighborhood of Σ_c in $\mathbb{P}\mathbb{R}^n_+$, whose boundary is transversal to the projectivized critical sets of non-critical cones.

In differential geometry, the standard way to construct a stratified tubular neighborhood starts with fixing a metric. We shall mimick the same approach in our PL setting.

Recall that two polyhedra in $\mathbb{R}^n$ are said to be *transversal* in $U \subset \mathbb{R}^n$ if they have no common points in U, or their union is not contained in an affine hyperplane. More generally, two piecewise linear sets in $\mathbb{R}^n$ are said to be transversal in $U \subset \mathbb{R}^n$ if they can be subdivided into relatively open polyhedra so that the polyhedra from the two subdivisions are pairwise transversal in U.

Recall that the *corner locus* of a continuous piecewise linear function is the (piecewise linear) set of all points at which the function is not smooth, and that for convex PL functions it has the natural structure of a polyhedral complex (defined by the projections of the faces of the subgraph of the function). When discussing the transversality to the corner locus, we always mean transversality in the sense of this polyhedral structure.

To every polyhedral complex M in $\mathbb{P}\mathbb{R}^n_+$, assign its tangent bundle TM: define the tangent plane T_xM at a point $x \in M$ as the maximal vector space L (lying in the $(n-1)$-dimensional vector space V, underlying the ambient affine space of $\mathbb{P}\mathbb{R}^n_+$) such that for every $\tilde{x} \in M$ sufficiently close to x the affine plane $\tilde{x} + L$ is contained in M in a small neighborhood of $\tilde{x}$. The tangent bundle TM is the (finite) set of all tangent spaces to M.

We shall say that $h : V \to \mathbb{R}$ is a *piecewise linear norm, transversal to a collection of subspaces* $V_1, \ldots, V_M \subset V$, if

(1) $h(w) = \max_{i=1}^N h_i(w)$, where $h_1, \ldots, h_N$ are linear functions,

(2) $h(w) > 0$ for $w \neq 0$,

(3) for every pair of subsets $I \subset \{1, \ldots, N\}$ and $J \subset \{1, \ldots, M\}$ and every cone $\sigma \in \Sigma_0$, the projections of the spaces $H_I := \{w \mid h_i(w) = h_k(w) \text{ for } i, k \in I\}$ and $\bigcap_{j \in J} V_j$ along the vector space parallel to $\mathbb{P}\sigma$ are transversal outside 0.

For $N \geq n$, condition (2) is satisfied for all tuples of linear functions $h_1, \ldots, h_N$ from a non-empty open cone in the space of all tuples. In this cone, condition (3) is satisfied for almost all tuples of linear functions h_i.

Thanks to this, we can choose once and for all a norm h transversal to the following collection of subspaces:

- the tangent bundle of the critical set, $T\mathbb{P}K_{\sigma'}$, for every non-critical cone $\sigma' \in \Sigma_0$;
- the tangent bundle of the delimiter set, $T\mathbb{P}(D_{\sigma''} \cap \sigma')$, for every non-critical cone $\sigma' \in \Sigma_0$ and its edge σ'' dual to a B_2-facet;
- the tangent bundle $T\mathbb{P}\mathbb{R}^n_+$.

For every projectivized closed cone $\sigma \in \Sigma_c$, consider the convex piecewise linear (in the sense of the selected affine structure) "distance function" $d_\sigma : \mathbb{P}\mathbb{R}^n_+ \to \mathbb{R}_+$ in the sense of the norm h, i.e. $d_\sigma(u) = \min_{u' \in \sigma} h(u - u')$. Properties (2) and (3) of the norm h above translate into the following properties of the distance function d_σ (in order to see how the property (3) of h translates into the properties (2)–(3) of the corner locus of d_σ, notice that the affine spans of the polyhedral cells of the corner locus of h are among the planes H_I from its property (3)):

(1) d_σ vanishes on σ and is strictly positive outside of it.

(2) For every non-critical cone $\sigma' \in \Sigma_0$, the corner locus of d_σ is transversal to the projectivized critical set $\mathbb{P}K_{\sigma'}$ in the complement to σ.

(3) For every non-critical cone $\sigma' \in \Sigma_0$ and its edge σ'' dual to a B_2-facet, the corner locus of d_σ is transversal to the projectivized delimiter $\mathbb{P}(D_{\sigma''} \cap \sigma')$ and the critical set $\mathbb{P}(D_{\sigma''} \cap K_{\sigma'})$ in the complement to σ.

In the sense of this distance, we shall consider the ε-neighborhoods $B_\sigma(\varepsilon) = \{u \mid d_\sigma(u) < \varepsilon\}$ and their boundaries $S_\sigma(\varepsilon)$, which are all piecewise-linear sets.

Associating positive numbers ε_σ to all $\sigma \in \Sigma_c$, introduce the following sets:

– the open neighborhood $U = \bigcup_{\sigma\in\Sigma_c} B_\sigma(\varepsilon_\sigma)$ of the critical complex Σ_c;
– for every $\sigma \in \Sigma_c$, the set $U_\sigma = B_\sigma(\varepsilon_\sigma) \setminus \bigcup_{\sigma'\subsetneq\sigma} B_{\sigma'}(\varepsilon_{\sigma'})$;
– for every $\sigma \in \Sigma_c$ dual to a B_2-facet, the *delimiter disk*

$$DD_\sigma = U_\sigma \cap \mathbb{P}D_\sigma.$$

Note that, by Lemmas 4.9 and 4.28, the cones $\sigma' \neq \sigma$ in DD_σ are not critical.

Lemma 4.31. *One can choose the numbers ε_σ so that*

(1) *for every non-critical projectivized cone $\sigma' \in \mathbb{P}\Sigma_0$, no vertex of the boundary of $U_\sigma \cap \sigma'$ or of the boundary of $DD_\sigma \cap \sigma'$ is contained in the critical set $\mathbb{P}K_{\sigma'} \cap \operatorname{rel.int}\sigma'$;*

(2) *U_σ is contained in the star of σ;*

(3) *$U_\sigma \cap U_\delta = \emptyset$ unless $\sigma = \delta$, and $\bar{U}_\sigma \cap \bar{U}_\delta = \emptyset$ unless $\sigma \subset \delta$ or vice versa;*

(4) *DD_σ divides U_σ into two connected components.*

Proof. Properties (2)–(4) are satisfied if the tuple (ε_σ) is chosen rapidly decreasing (i.e. $\varepsilon_\sigma \ll \varepsilon_\delta \ll 1$ for all $\delta \subset \sigma$), and even without genericity assumptions (2)–(3) on the distance function d_σ.

If the tuple is moreover chosen generically (i.e. avoiding finitely many hyperplanes in the space of all such tuples), then it satisfies (1) as well by the properties (2)–(3) of the corner locus of d_σ, because the vertices of $U_\sigma \cap \sigma'$ belong to $\sigma' \cap ((n + 1 - \dim\sigma')$-dimensional skeleton of the corner locus of d_σ), and the same for delimiter disks. ∎

Definition 4.32. Under the assumptions of Theorem 4.3, the *vertex function*

$$v : U \setminus (\text{delimiter disks}) \to (\text{vertices of } \Gamma_+(f))$$

is defined on every U_σ as follows:

– if $\sigma \in \Sigma_c$ is dual to a non-B_2 critical face F, then we define $v(\cdot)$ on U_σ as the apex P_F;
– if $\sigma \in \Sigma_c$ is dual to a B_2-facet τ, then we define $v(\cdot)$ on each of the two components of $U_\sigma \setminus DD_\sigma$ as the apex of τ on the side of this component.

Example 4.33. If σ is dual to the B_2-facet τ from Figure 2, then U_σ is split with the delimiter hyperplane (perpendicular to the segment XY) into two components, "left" and "right". Then, in the setting of Example 4.15, the vertex function v equals P on the left component and B on the right one.

Note that the vertex function is locally constant on its domain. We now use this observation to consistently substitute every piece of the neighborhood U by an appropriate sprout. Recall that we refer to cones and their projectivizations interchangeably, and, in particular, $S_{W,\tau}$ for the projectivization W of a cone C is another notation for the sprout $S_{C,\tau}$.

If $\sigma \in \Sigma_c$ is dual to a face F that is not a B_2-facet, then the vertex function v equals a constant P on it, so we define $V_{\sigma,0}$ as the sprout $S_{U_\sigma,\tau}$ (see Definition 4.17) for any B-facet $\tau \supset F$. Note that neither the sprout $V_{\sigma,0} := S_{U_\sigma,\tau}$ nor its root $R_{\sigma,0} := R_{U_\sigma,\tau} = P$ depend on the choice of τ.

If $\sigma \in \Sigma_c$ is dual to a B_2-facet τ with the apices $P_{\pm 1}$, then we define $V_{\sigma,\pm 1}$ as the sprout $S_{\{v=P_{\pm 1}\}\cap U_\sigma,\tau}$ and the root $R_{\sigma,\pm 1}$ as $R_{\{v=P_{\pm 1}\}\cap U_\sigma,\tau} = P_{\pm 1}$. Also, choosing a ray r in the dual cone σ' of the non-simplicial non-V-facet of τ outside of $K_{\sigma'}$, we define $V_{\sigma,0}$ as $S_{r,\tau}$ (see Lemma 4.22), leaving $R_{\sigma,0}$ undefined.

Define the *sprouting* $S_P \subset \mathbb{R}^n$ of a vertex P as the union of all $V_{\sigma,*}$ ($* = 0, \pm 1$) such that $R_{\sigma,*} = P$.

Lemma 4.34. (1) *The contribution $Z(S_P)(s)$ of the sprouting S_P to the zeta function $Z_{\mathrm{top},f}(s)$ is equal to*

$$\int_{S_P} \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du$$

modulo a function that has no pole at s_0.

(2) *No edge of the boundary of S_P is critical for (s_0, P).*

Proof. Part (1) follows from Lemmas 4.20 and 4.22. To deduce (2), it is enough (by Lemma 4.23) to show that every vertex of $(\partial U_\sigma) \cap \sigma'$ and $(\partial DD_\sigma) \cap \sigma'$ for every projectivized cone $\sigma' \supset \sigma$ is either not in $\mathbb{P}K_{\sigma'}$, or not a projectivized edge of ∂S_P. For non-critical σ', this follows from Lemma 4.31 (1), and every critical σ' is either in the projectivized interior of S_P, or disjoint from its closure, or intersects its boundary at an interior point of its facet $DD_{\sigma'}$ (thus no vertex in σ' can be a projectivized edge of ∂S_P). ■

Proof of Theorem 4.3. By the preceding lemma and Lemma 3.6, the contribution of every S_P to the topological zeta function of f has no pole at s_0. By Lemma 4.22, the same is true for $V_{\sigma,0}$ for every B_2-facet σ. Since

$$V = \bigcup_{\sigma,\, *=0,\pm 1} V_{\sigma,*}$$

contains all the critical cones in its interior, for every cone $\sigma' \in \Sigma_0$ the set $\sigma' \setminus V$ has edges of two types: edges of $(\partial V) \cap \sigma'$ or non-critical 1-dimensional cones of Σ_0. The edges of the first kind are not in the critical set of any cone by Lemma 4.34 (2), and for the second kind the same holds by definition. Thus the contribution of $\sigma' \setminus V$ to the topological zeta function has no pole at s_0 as well by Lemma 3.6. We have subdivided $\mathbb{R}^n_+$ into these pieces:

- S_P for some vertices P,
- $V_{\tau^\circ,0}$ for some B_2-facets τ ,
- $\sigma' \setminus V$ for some non-critical cones σ',

so that none of them contributes the pole s_0. ∎

5. Fake poles of the topological zeta function in dimension 4

Throughout this section we work in dimension $n = 4$. In the first subsection, we classify all non-contributing configurations of B-faces. The proof of the classification theorem occupies the subsequent two subsections and makes use of the tools introduced in the preceding two sections. In the last subsection, we show that every non-B-facet contains a non-B-simplex and discuss a possible general definition of B-facets in arbitrary dimension. Both of the above mentioned results will be used in the last section to prove the monodromy conjecture for all non-degenerate singularities of functions of four variables.

5.1. The main theorem

We shall prove that if a candidate pole of the topological zeta function $Z_{\mathrm{top},f}(s)$ is contributed only by B-facets, then it is fake, with one exception:

Definition 5.1. (1) A *border* is a triangular face of $\Gamma_+(f)$ such that, up to a reordering of coordinates, its vertices are of the form $A = (1, 1, *, *)$, $B = (0, 0, *, *)$ and $D = (0, 0, *, *)$, and the two facets containing it are B-facets with vertex A and bases in the coordinate hyperplanes $\{v_1 = 0\}$ and $\{v_2 = 0\}$ respectively.

In this definition we admit "infinite triangles" obtained by letting some of the starred coordinates tend to infinity. Namely, the notion of the border includes the Minkowski sum of the segment AD with the aforementioned coordinates and the 4th coordinate ray, as well as the Minkowski sum of the point $A = (1, 1, *, *)$ and the 3rd and 4th coordinate rays.

(2) The border ABD is a *B-border* unless (up to reordering B and D and the last two coordinates) we have $A = (1, 1, 0, a)$, $B = (0, 0, 0, b)$ and $D = (0, 0, 1, d)$. In the latter case (implying, in particular, that the edge BD is itself a B-edge in the coordinate plane O_{34}) the border ABD is said to be a *B^2-border* (see Figure 6). The vertex A is called the apex of the border, and the first two coordinates are called its bases.

In this definition we admit "infinite triangles" obtained by letting b tend to infinity, i.e. the Minkowski sum of the segment AD with the aforementioned coordinates and the 4th coordinate ray.

Let f be a non-degenerate polynomial on $\mathbb{C}^4$ with Newton polyhedron $\Gamma_+(f) \subset \mathbb{R}^4$ and dual fan Σ_0.

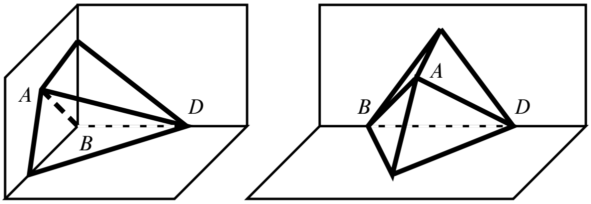

Fig. 6. A B^2-border and a B-border.

Theorem 5.2. *Assume that f is non-degenerate and does not have a Morse singularity at $0 \in \mathbb{C}^4$. Let a candidate pole $s_0 \neq -1$ of the topological zeta function $Z_{\mathrm{top},f}(s)$ be contributed only by B-facets, and no two of them contain the same B-border (although they may contain the same B^2-border). Then s_0 is fake, i.e. not an actual pole of $Z_{\mathrm{top},f}(s)$.*

The rest of this subsection is devoted to the proof of this theorem.

The first difference with the setting of Theorem 4.3 is that we have to prove that B^2-borders do not contribute candidate poles. For this purpose, we shall extend the notion of a sprout (Definition 4.17) to them. Let C be a (not necessarily convex) cone in the star of τ°, where τ is a border with bases i and j and apex P, adjacent to two B-facets contributing the pole s_0.

Note that we may assume throughout the rest of this section that τ is compact: indeed, if two B-facets sharing a common non-compact B^2-border contribute the same pole s_0, then $s_0 = -2$.

Definition 5.3. The cone C is said to be *border-convex* if every 3-dimensional plane through e_i and e_j intersects C in a convex cone.

Definition 5.4. The union of $C \cap \operatorname{rel.int} P^\circ$ and all 3-dimensional cones, generated by the coordinate vectors e_i, e_j and $u \in C \cap \partial P^\circ$, is denoted by $S_{C,\tau}$ and is called the *border sprout* of C. The vertex P is denoted by $R_{C,\tau}$ and is called the *root* of C.

The definition implies the following.

Lemma 5.5. *The edges of a border sprout of a cone C are either edges of $(\partial C) \cap \sigma^\circ$ for some faces σ of τ outside of coordinate planes, or coordinate rays.*

Lemma 5.6. *Let C be a border-convex cone in the star of the dual cone to a B^2-border τ, and assume that both of its adjacent B-facets contribute the same candidate pole $s_0 \neq -1$. Then the contribution $Z(S_{C,\tau})$ of the border sprout is equal to*

$$\int_{S_{C,\tau}} \exp(-\langle u, R_{C,\tau}\rangle s - \langle u, \mathbf{1}\rangle)\, du$$

modulo a function that has no pole at s_0.

Remark 5.7. Most borders do not have this property.

Proof of Lemma 5.6. Let $\tau = ABD$ be a B^2-border with coordinates $A = (1,1,0,a)$, $B = (0,0,0,b)$ and $D = (0,0,1,d)$. The fact that the adjacent B-facets τ_1 and τ_2 contribute the same candidate pole means that the line $s \cdot A + \mathbf{1}$ intersects the vector spans of $\tau_1 - A$ and $\tau_2 - A$ at the same point $s_0 \cdot A + \mathbf{1}$ (note that the dimension of the intersection of these two 3-dimensional linear subspaces is 2). Thus this point is a linear combination of the vectors $D - B$ and $A - B$:

$$D - B = \beta(s_0 \cdot A + \mathbf{1}) + \alpha \cdot (A - B).$$

Solving this system of equations for α, β and s_0, we find

$$s_0 = (a + d - 2b - 1)/b, \quad \alpha = -s_0 - 1, \quad \beta = 1.$$

At the same time, for the future reference, we interpret the latter equalities as the computation of the ratio of $D - B$ and $s_0 \cdot A + \mathbf{1}$ as linear functions on the dual space $(\mathbb{R}^4)^*$ restricted to the hyperplanes $(A - D)^\perp$ and $(A - B)^\perp$:

$$\frac{(D-B)|_{(A-B)^\perp}}{(s_0 \cdot A + \mathbf{1})|_{(A-B)^\perp}} = \beta = 1, \quad \frac{(D-B)|_{(A-D)^\perp}}{(s_0 \cdot A + \mathbf{1})|_{(A-D)^\perp}} = \beta/(1-\alpha) = 1/(2+s_0). \qquad (*)$$

We now first prove the lemma for a very special choice of the cone C.

Choose primitive vectors u_A, u_B and u_D in the interior of the cones ABD°, AB° and AD° respectively. Recall that the standard basis is denoted by e_1, e_2, e_3, e_4. Denote by N_{ABD} the union of the relatively open simplicial cones $\langle u_A, u_B, e_1, e_2\rangle$, $\langle u_A, u_D, e_1, e_2\rangle$ and $\langle u_A, e_1, e_2\rangle$.

Then the statement of the lemma is valid for $C = N_{ABD}$. To prove this, split $Z(N_{ABD})(s) - \int_{N_{ABD}} \exp(-s\langle u, A\rangle - \langle u, \mathbf{1}\rangle)\, du$ into

$$Z(\langle u_A, u_B, e_1, e_2\rangle) + Z(\langle u_A, u_D, e_1, e_2\rangle) + Z(\langle u_A, e_1, e_2\rangle)$$
$$- \int_{\langle u_A, u_B, e_1, e_2\rangle} \exp(-s\langle u, A\rangle - \langle u, \mathbf{1}\rangle)\, du - \int_{\langle u_A, u_D, e_1, e_2\rangle} \exp(-s\langle u, A\rangle - \langle u, \mathbf{1}\rangle)\, du,$$

and apply Lemma 2.2 to each of the five terms. The result consists of the five respective terms

$$\frac{|\det(e_1, e_2, u_A, u_D)|}{(s \cdot A + \mathbf{1})|_{u_A} \cdot (s \cdot A + \mathbf{1})|_{u_D}} + \frac{|\det(e_1, e_2, u_A, u_B)|}{(s \cdot A + \mathbf{1})|_{u_A} \cdot (s \cdot A + \mathbf{1})|_{u_B}} - \frac{s \cdot |e_1 \wedge e_2 \wedge u_A|}{(s+1) \cdot (s \cdot A + \mathbf{1})|_{u_A}}$$
$$- \frac{|\det(e_1, e_2, u_A, u_D)|}{(s+1)^2 \cdot (s \cdot A + \mathbf{1})|_{u_A} \cdot (s \cdot A + \mathbf{1})|_{u_D}} - \frac{|\det(e_1, e_2, u_A, u_B)|}{(s+1)^2 \cdot (s \cdot A + \mathbf{1})|_{u_A} \cdot (s \cdot A + \mathbf{1})|_{u_B}}, \qquad (**)$$

where $|\cdot|$ in the third term stands for the lattice length. Collecting similar terms and taking into account the identities $\det(e_1, e_2, u_A, x) = (e_1 \wedge e_2 \wedge u_A) \cdot x$ and $u_A \cdot (D - B) = 0$ (which in coordinates $u_A = (u_1, u_2, u_3, u_4)$ reads $u_3 = u_4 \cdot (b - d)$), we can rewrite $(**)$ as the uninteresting factor $\frac{s \cdot u_4}{(s+1)^2 \cdot (s \cdot A + \mathbf{1})|_{u_A}}$ times

$$\frac{(s+2)\big|(D-B)|_{u_D}\big|}{(s \cdot A + \mathbf{1})|_{u_D}} + \frac{(s+2)\big|(D-B)|_{u_B}\big|}{(s \cdot A + \mathbf{1})|_{u_B}} - (s+1). \qquad (\star)$$

Since u_B and u_D are chosen to be support vectors of the edges AB and AD respectively, we have $u_B \cdot A = u_B \cdot B < u_B \cdot D$ and $u_D \cdot A = u_D \cdot D < u_D \cdot B$, so $|(D - B)|_{u_B}| = (D - B)|_{u_B}$ and $|(D - B)|_{u_D}| = -(D - B)|_{u_D}$. Applying this and $(*)$, we conclude that $(\star)$ for $s = s_0$ equals $-(s_0 + 2)/(s_0 + 2) + (s_0 + 2) - (s_0 + 1) = 0$.

As a result, the sought difference $Z(N_{ABD})(s) - \int_{N_{ABD}} \exp(-s\langle u, A\rangle - \langle u, \mathbf{1}\rangle)\, du$ equals the product of the uninteresting factor that has a simple pole at s_0 and the rational (actually linear) function $(\star)$ that has a root at s_0. Thus the product is holomorphic at s_0, and we have proved the lemma for $C = N_{ABD}$.

Now, for an arbitrary border-convex C, the border sprout $S_{C,\tau}$ can be represented as the union (with disjoint interiors) of N_{ABD} and the sprouts of the form $S_{C^{(k)},\tau_k}$, $k = 1, 2$, for certain cones $C^{(k)}$. Thus the statement of the lemma is valid for it, applying the preceding computation to N_{ABD} and Lemma 4.20 to $S_{C^{(k)},\tau_k}$. ∎

We now comment on how one might arrive at considering the cone N_{ABD} in this proof. We do not know how to prove the lemma directly for an arbitrary sprout S of the border τ, but we already know similar identities for (non-border) sprouts S_i of the two adjacent B-facets $\tau_i \supset \tau$. So it is a natural idea to try to simplify the problem, subtracting from S non-overlapping sprouts S_i, trying to choose them so that the difference is as small and simple as possible. Now it is an easy exercise of spatial thinking to see that the smallest and simplest difference has the form N_{ABD}.

5.2. *Very critical cones*

In contrast to the setting of Theorem 4.3, we cannot in general choose preferred bases and apices of B_1-faces consistently, so we now choose them arbitrarily once and for all in this section. Recall that we fix the choice of apices P_τ and preferred bases b_τ for all B_1-facets τ and for all B_1-facets of B_2-facets τ, and then use them to define preferred bases and apices of B_2-facets on the side of a given face or cone; see Subsection 4.4 for details.

As a result, in contrast to the setting of Theorem 4.3, different B_1-facets, containing a given critical face, may have different preferred bases. Such critical faces are said to be very critical. They are studied in this subsection.

Definition 5.8. The *preferred base* $b_F = b_{F^\circ} \subset \{1, 2, 3, 4\}$ of a critical face $F \prec \Gamma_+(f)$ and its dual cone $F^\circ \in \Sigma_0$ is the set of preferred bases on the side of F for all B-facets containing F. We say that the face F and its dual cone $F^\circ \in \Sigma_0$ are *very critical* if $|b_F| \geq 2$.

Example 5.9. The projectivized pictures of the Newton polyhedron $\Gamma_+(f)$ on Figure 7 below give some examples of (2-dimensional) very critical faces (hatched), provided that all the facets on the pictures are B-facets contributing the same candidate pole s_0. The apices of B_1-faces are bold points, and the apices of the B_2-facet are the end points of the bold segment.

This subsection is devoted to the study of very critical faces depending on their dimension. First of all, by Lemma 4.24 there is no critical vertex, and a facet τ is critical if and

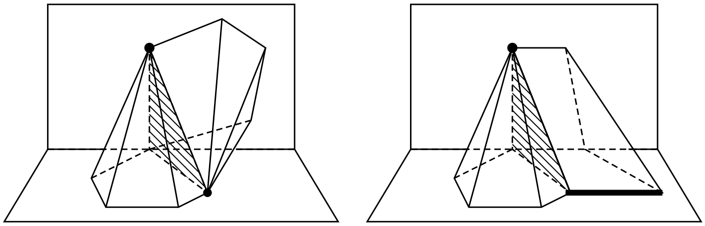

Fig. 7. Very critical faces.

only if the candidate pole s_0 is contributed by τ (and thus τ is a B-facet in the assumptions of Theorem 5.2). A 2-dimensional face $F \prec \Gamma_+(f)$ is critical if and only if it separates two B-facets τ_1 and τ_2, contributing s_0. This critical face F is very critical if moreover we have $b^F_{\tau_1} \neq b^F_{\tau_2}$.

Definition 5.10. Let F be a 2-dimensional very critical face separating two B-facets τ_1 and τ_2. Then we define the *F-delimiter* $D_F = D_{F^\circ} \subset \mathbb{R}^4$ to be the vector subspace in $\mathbb{R}^4$ generated by the $b^F_{\tau_1}$-th and $b^F_{\tau_2}$-th coordinate lines in $\mathbb{R}^4$ and a ray $R \subset \operatorname{rel.int} F^\circ$.

The delimiter is said to be *generic* if, for every non-critical face $G \subset F$, the critical set $D_F \cap K_G$ is nowhere dense in $D_F \cap \operatorname{rel.int} G^\circ$.

Lemma 5.11. *Under the assumptions of Theorem 5.2, any 2-dimensional very critical face F is a triangle. If it is a border, then the delimiter D_F is the affine span of the dual cone of the V-edge of F. Otherwise the affine span* $\operatorname{aff}(F^\circ)$ *is transversal to D_F. In particular, in both cases D_F is a hyperplane in $\mathbb{R}^4$.*

Proof. The face F is a triangle by Lemma 4.7. Assume that $\operatorname{aff}(F^\circ)$ is not transversal to the delimiter. Then for the coordinate plane $H = \{v_{b^F_{\tau_1}} = v_{b^F_{\tau_2}} = 0\} \subset \mathbb{R}^4$ we have $\dim(\operatorname{aff}(F) \cap H) \geq 1$. This implies that $\operatorname{aff}(F)$ is not transversal to H, and hence their intersection is a V-edge in both τ_1 and τ_2. Then F is a border. ∎

Corollary 5.12. *If F is a non-border very critical face, then the rays $R \subset \operatorname{rel.int} F^\circ$ in Definition 5.10 parameterize a 1-dimensional family of F-delimiters. Among them, all but finitely many delimiters are generic.*

An edge $F \prec \Gamma_+(f)$ is critical if and only if all the facets containing it are B-facets τ contributing the candidate pole s_0. Since $\dim F^\circ = 3$ in this case, for any apex P of such a τ on the side of F, the critical hyperplane L_P coincides with $\operatorname{aff}(F^\circ)$.

Proposition 5.13. *Under the assumption of Theorem 5.2, no edge of $\Gamma_+(f)$ is very critical.*

Proof. Assume that an edge P_1P_2 is very critical. Then, by Proposition 4.29 (2), there exist B-facets τ_1 and τ_2 containing it with apices Q_1 and Q_2 on the side of P_1P_2, and their critical hyperplanes $L_{Q_i} : \langle u, Q_i\rangle s_0 + \langle u, \mathbf{1}\rangle = 0$ both coincide with the hyperplane $\langle u, P_1 - P_2\rangle = 0$ generated by the dual cone $(P_1P_2)^\circ$. Moreover, since P_1P_2 is not in a coordinate plane, we have $P_1 = Q_1$ and $P_2 = Q_2$ or vice versa. Thus the vectors $P_1 s_0 + \mathbf{1}$ and $P_2 s_0 + \mathbf{1}$ are parallel.

On the other hand, it cannot happen that one of the points P_1 and P_2 is in the apex of τ_1 and in the base of τ_2, and the other one is in the apex of τ_2 and in the base of τ_1. Otherwise, reordering coordinates if necessary, we would have $P_1 = (1, 0, *, *)$, $P_2 = (0, 1, *, *)$, $P_1 s_0 + \mathbf{1} = (1 + s_0, 1, *, *)$, $P_2 s_0 + \mathbf{1} = (1, 1 + s_0, *, *)$. Since the last two vectors are parallel and $s_0 \neq 0$, we have $s_0 = -2$, $P_1 s_0 + \mathbf{1} = -(P_2 s_0 + \mathbf{1})$ and hence $P_1 + P_2 = \mathbf{1}$. Thus, up to reordering the coordinates and P_i's, we have $P_1 = (1, 0, 0, 0)$ (i.e. f has no singularity at the origin) or $P_1 = (1, 0, 1, 0)$ and $P_2 = (0, 1, 0, 1)$ and hence f has a Morse singularity at the origin by Lemma 4.8.

If one of P_i's is in the apex of all B-facets τ_j containing $P_1 P_2$, and one of τ_j's is B_2, then the other facet containing its quadrilateral face F also contains $P_1 P_2$, and thus is also B_2 (it cannot be B_1, because it has a quadrilateral face F outside the coordinate hyperplanes). Then we would have two B_2-facets with a common quadrilateral face F (see Lemma 4.7 for a contradiction).

Thus all the τ_j's are B_1-facets. As we have seen above, one of P_1 and P_2 is equal to the common apex of all τ_j's, and the other one is in the bases of all τ_j's. Then, among τ_j's, we can find at least two pairs of B_1-facets with a common apex, a common triangular face and different bases.

These two pairs surround two borders, and since there are no B-borders by the assumptions of Theorem 5.2, they are B^2-borders. Two B^2-borders with a common apex A, intersecting in a common edge AC in the interior of $\mathbb{R}^4_+$, by their definition have $C = (0, 0, 1, 1)$ and $A = (1, 1, 0, 0)$, which by Lemma 4.8 implies that the singularity f is Morse non-degenerate. ■

5.3. A tubular neighborhood of the critical subfan revisited

We are now ready to prove Theorem 5.2. When referring to B-facets or critical faces in the course of the proof, we always imply only the faces contributing the candidate pole $s_0 \neq -1$, for which we are proving Theorem 5.2 (thus all the choices and objects that we introduce for the proof completely depend on the choice of s_0). Recall that up to now

- we have chosen once and for all a preferred base of every B_1-facet and every B_1-facet of a B_2-facet in the Newton polyhedron $\Gamma_+(f)$,
- depending on this choice, we have called some cones critical in the dual fan Σ_0 (see Definitions 4.27 and 5.8 respectively),
- we have chosen once and for all a generic delimiter hyperplane D_σ for every 2-dimensional very critical cone σ, dual to a non-border (see Definition 5.10), and we have defined the delimiter hyperplane D_σ for every 1-dimensional very critical cone σ, dual to a B_2-facet (see Definition 4.11).

Definition 5.14. These two kinds of faces and their dual cones will be called *delimited*.

Recall that, similarly to Section 4.7, we choose once and for all an affine structure on the projectivization $\mathbb{P}\mathbb{R}^n_+$ and refer to cones and their projectivizations interchangeably whenever it causes no confusion.

According to Corollary 4.30, the set of projectivized critical cones in the dual fan Σ_0 is a closed polyhedral complex Σ_c. We shall construct a generic piecewise linear tubular neighborhood of Σ_c, whose boundary is transversal to the critical sets of non-critical cones and delimiters of very critical cones.

For every projectivized closed cone $\sigma \in \Sigma_c$, choose a convex piecewise linear (with respect to the selected affine structure) "distance function" $d_\sigma : \mathbb{P}\mathbb{R}^n_+ \to \mathbb{R}_+$, satisfying the following properties:

(1) d_σ vanishes on σ and is strictly positive outside of it.

(2) For every non-critical cone $\sigma' \in \Sigma_0$, the corner locus of d_σ is transversal to the projectivized critical set $\mathbb{P}K_{\sigma'}$ in the complement to σ.

(3) For every non-critical cone σ' and every delimited face $\sigma'' \subset \sigma'$, the corner locus of d_σ is transversal to the projectivized delimiter $\mathbb{P}D_{\sigma''} \cap \sigma'$ and to the critical set $\mathbb{P}D_{\sigma''} \cap K_{\sigma'}$ in the complement to σ.

Such a distance function can be constructed from a suitably generic piecewise-linear norm in the same way as in Section 4. For this distance, we shall consider the ε-neighborhoods $B_\sigma(\varepsilon) = \{u \mid d_\sigma(u) < \varepsilon\}$ and their boundaries $S_\sigma(\varepsilon)$, which are all piecewise-linear sets.

Associating positive numbers ε_σ to all $\sigma \in \Sigma_c$, introduce the following sets:

- the open neighborhood $U = \bigcup_{\sigma \in \Sigma_c} B_\sigma(\varepsilon_\sigma)$ of the critical complex Σ_c;
- for every $\sigma \in \Sigma_c$, the set $U_\sigma = B_\sigma(\varepsilon_\sigma) \setminus \bigcup_{\sigma' \subsetneq \sigma} B_{\sigma'}(\varepsilon_{\sigma'})$;
- for every delimited $\sigma \in \Sigma_c$, the *delimiter disk* $DD_\sigma = U_\sigma \cap \mathbb{P}D_\sigma$.

Lemma 5.15. *One can choose the numbers ε_σ so that*

(1) *for every projectivized non-critical cone $\sigma' \in \mathbb{P}\Sigma_0$, no vertex of the boundary of $U_\sigma \cap \sigma'$ or of the boundary of $DD_\sigma \cap \sigma'$ is contained in the critical set $\mathbb{P}K_{\sigma'} \cap \operatorname{rel.int} \sigma'$;*

(2) *U_σ is contained in the star of σ and is border-convex* (*Definition* 5.3) *if σ is dual to a border;*

(3) *$U_\sigma \cap U_\delta = \emptyset$ unless $\sigma = \delta$, and $\bar{U}_\sigma \cap \bar{U}_\delta = \emptyset$ unless $\sigma \subset \delta$ or vice versa;*

(4) *DD_σ divides U_σ into two connected components.*

Proof. All of these properties are satisfied if the tuple (ε_σ) is chosen generically (i.e. avoiding finitely many hyperplanes in the space of all such tuples) and rapidly decreasing (i.e. $\varepsilon_\sigma \ll \varepsilon_\delta \ll 1$ for all $\delta \subset \sigma$). ∎

Definition 5.16 (cf. Definition 4.32). The *vertex function*

$$v : U \setminus (\text{delimiter disks}) \to (\text{vertices of } \Gamma_+(f))$$

is defined on every U_σ as follows:

(1) If a projectivized cone $\sigma \in \Sigma_c$ is dual to a non-delimited face $F \subset \Gamma_+(f)$, then all facets containing F are B-facets, and their apices on the side of F are all equal to the same vertex P. Then we define $v(\cdot)$ on U_σ as P.

(2) If $\sigma \in \Sigma_c$ is dual to a delimited triangle F, separating two B-facets τ_1 and τ_2, then we define $v(\cdot)$ on each of the two components of $U_\sigma \setminus DD_\sigma$ as the apex of the corresponding facet τ_i on the side of this component.

(3) If $\sigma \in \Sigma_c$ is dual to a B_2-facet τ, then we define $v(\cdot)$ on each of the two components of $U_\sigma \setminus DD_\sigma$ as the apex of τ on the side of this component.

Note that the vertex function is locally constant on its domain (thanks to Proposition 5.13). We now use this observation to consistently substitute every piece of the neighborhood U by an appropriate sprout. Recall that we refer to cones and their projectivizations interchangeably, and, in particular, $S_{W,\tau}$ for the projectivization W of a cone C is another notation for the sprout $S_{C,\tau}$.

If $\sigma \in \Sigma_c$ is neither delimited nor dual to a border, then the vertex function v equals a constant P on σ, so we define $V_{\sigma,0}$ as the sprout $S_{U_\sigma,\tau}$ (see Definition 4.17) for any B-facet τ containing the dual face of σ. Note that neither the sprout $V_{\sigma,0} := S_{U_\sigma,\tau}$ nor its root $R_{\sigma,0} := R_{U_\sigma,\tau} = P$ depend on the choice of τ.

If $\sigma \in \Sigma_c$ is dual to a border, then the vertex function v equals a constant P on σ, and we define $V_{\sigma,0}$ as the border sprout $S_{U_\sigma,\sigma}$ (see Definition 5.4) and the root $R_{\sigma,0}$ as $R_{U_\sigma,\tau} = P$.

If $\sigma \in \Sigma_c$ is dual to a very critical triangle, separating two B-facets $\tau_{\pm 1}$ with vertices $P_{\pm 1}$ on the side of σ, then we define $V_{\sigma,\pm 1}$ as the sprout $S_{\{v=P_{\pm 1}\}\cap U_\sigma,\tau_{\pm 1}}$ and the root $R_{\sigma,\pm 1}$ as $R_{U_\sigma,\tau_{\pm 1}} = P_{\pm 1}$.

If $\sigma \in \Sigma_c$ is dual to a B_2-facet τ with apices $P_{\pm 1}$, then we define $V_{\sigma,\pm 1}$ as the sprout $S_{\{v=P_{\pm 1}\}\cap U_\sigma,\tau}$ and the root $R_{\sigma,\pm 1}$ as $R_{\{v=P_{\pm 1}\}\cap U_\sigma,\tau} = P_{\pm 1}$. Also, choosing a ray r in the dual cone σ' to the quadrilateral non-V-face of τ outside of $K_{\sigma'}$, we define $V_{\sigma,0}$ as the delimeter sprout $S_{r,\tau}$ (see Lemma 4.22), leaving $R_{\sigma,0}$ undefined.

Thanks to Proposition 5.13, we have no other cases to consider. Now define the *sprouting* S_P of a vertex P as the union of all the sprouts $V_{\sigma,*}$ ($* = 0, \pm 1$) such that $R_{\sigma,*} = P$.

We have introduced the same system of notation as in Section 4.7, giving it a meaning in a more general setting (most notably, admitting B^2-borders). With this wider meaning of notation at hand, the proof of Theorem 5.2 almost literally repeats the one for Theorem 4.3 (we repeat it for the convenience of the reader).

Lemma 5.17. (1) *The contribution of S_P equals*

$$\int_{S_P} \exp(-\langle u, P\rangle s - \langle u, \mathbf{1}\rangle)\, du$$

modulo a function that has no pole at s_0.

(2) *No edge of the boundary of S_P is critical for (s_0, P).*

Proof. Part (1) follows from Lemmas 4.20 and 5.6. To deduce (2), it is enough (by Lemma 4.23) to show that every vertex of $(\partial U_\sigma) \cap \sigma'$ and $(\partial DD_\sigma) \cap \sigma'$ for every projectivized cone $\sigma' \supset \sigma$ is either not in $\mathbb{P}K_{\sigma'}$, or not a projectivized edge of ∂S_P. For non-critical σ', this follows from Lemma 4.31 (1), and every critical σ' is either in the projectivized interior of S_P, or disjoint from its closure, or intersects its boundary at an interior point of its facet $DD_{\sigma'}$ (thus no vertex in σ' can be a projectivized edge of ∂S_P). ∎

Proof of Theorem 5.2. By the preceding lemma and Lemma 3.6, the contribution of every S_P to the topological ζ-function of f has no pole at s_0. By Lemma 4.22, the same is true for $V_{\sigma,0}$ for every B_2-facet σ. Since

$$V = \bigcup_{\sigma,\ *=0,\pm 1} V_{\sigma,*}$$

contains all the critical cones in its interior, for every cone $\sigma' \in \Sigma_0$ the set $\sigma' \setminus V$ has edges of two types: edges of $(\partial V) \cap \sigma'$ or non-critical 1-dimensional cones of Σ_0. The edges of the first kind are not in the critical set of any cone by Lemma 4.34 (2), and for the second kind the same holds by definition. Thus the contribution of $\sigma' \setminus V$ to the topological zeta function has no pole at s_0 as well by Lemma 3.6. We have subdivided $\mathbb{R}^n_+$ into these pieces:

- S_P for some vertices P,
- $V_{\tau^\circ,0}$ for some B_2-facets τ ,
- $\sigma' \setminus V$ for some non-critical cones σ',

so that none of them contributes the pole s_0. ∎

5.4. *Generalizing the notion of B-facets*

We have seen that a B_1- or B_2-facet alone never contributes its candidate pole. In Section 8 we shall prove a somewhat complementary fact:

For $n = 4$, *all other facets do contribute their* (*nearby*) *monodromy eigenvalues.* $(*)$

(See Section 8 for a precise statement.) This dichotomy is central for the proof of the monodromy conjecture for non-degenerate singularities.

However, for $n > 4$, in order to keep $(*)$ true, we should exclude from our consideration B-facets in a certain more general sense than the one assumed in Definition 3.10. What is the proper general notion of a B-facet in arbitrary dimension? A possible answer given in Definition 1.4 (2) is based on the following lemma that we need in order to prove $(*)$.

Lemma 5.18. *For* $n = 4$, *if a compact facet* $\tau \prec \Gamma_+(f)$ *is not a* B*-facet, then it splits into lattice simplices* (*with no new vertices*) *so that one of the simplices is not of type* B_1.

Equivalently: if any four affinely independent vertices of a compact facet $\tau \prec \Gamma_+(f)$ *form a* B_1*-simplex, then* τ *is a* B*-facet.*

Proof. The facet τ contains a face F not contained in a coordinate hyperplane. Note the following facts about every such F:

(1): F has at most four vertices. Otherwise it contains a triangle whose sides are not in coordinate hyperplanes, and the union of this triangle and any vertex of $\tau \setminus F$ gives a non-B_1-simplex in τ.

(2): If F is a quadrilateral, then some pair of its opposite edges are contained in coordinate hyperplanes, say, $\{v_1 = 0\}$ and $\{v_2 = 0\}$, otherwise we get the same contradiction as in (1). In this case, if a vertex of F at the hyperplane $\{v_1 = 0\}$ has $v_2 > 1$, then this vertex, the two vertices of $F \cap \{v_2 = 0\}$ and any other vertex of $\tau \setminus F$ form a non-B_1-simplex in τ. Thus both vertices of F in the hyperplane $\{v_1 = 0\}$ have $v_2 = 1$ and vice versa. Thus τ is a B_2-facet.

(3): If F is a triangle, then at least one of its edges is contained in a coordinate hyperplane, otherwise we get the same contradiction as in (1).

(3.1): If the triangle F has exactly one edge in a coordinate hyperplane, say, $\{v_1 = 0\}$, then the coordinate v_1 of the other vertex of F equals 1, otherwise F together with any vertex from $\tau \setminus F$ form a non-B_1-simplex in τ. Also in this case, all other vertices of τ should be in the hyperplane $\{v_1 = 0\}$, because otherwise a vertex together with F form a non-B_1-simplex in τ. Thus τ is a B_1-pyramid for v_1.

(3.2): If the triangle F has all three edges in coordinate hyperplanes, then denote by V the set of vertices of τ outside F. Note that every point of V is in a coordinate hyperplane, otherwise it would form a non-B_1-simplex together with F.

(3.2.1): If V is contained in a coordinate hyperplane L, containing one of the edges of F, then τ has one vertex v outside L. So, depending on the distance from v to L, the facet τ either contains a non-B_1-simplex, or is itself a B_1-pyramid with base L.

(3.2.2): If V has a point in each of the three coordinate 2-planes containing the vertices of F, then we may assume these three points are outside the common coordinate edge (say, $v_1 = v_2 = v_3 = 0$) of the three 2-planes (otherwise we would arrive at (3.2.1)). Since two vertices of a Newton diagram in a 2-plane cannot have the same coordinate, either the vertex of F or a point of V in the $(v_1 v_2)$-plane has $v_3 \neq 1$. This point, two similar points with respective non-unit coordinates in the other 2-planes, and one of the remaining vertices of τ form a non-B_1-simplex.

(3.2.3): If V has a point in a coordinate 2-plane (say, $v_1 = v_2 = 0$) containing a vertex of F, and a point in the coordinate 3-plane (say, $v_3 = 0$) containing the opposite edge of F, then we can assume these two points do not belong to smaller coordinate planes (otherwise we would arrive at (3.2.1)). Since two vertices of a Newton diagram in a 2-plane cannot have the same coordinate, either the vertex of F or a point of V in $v_1 = v_2 = 0$ has $v_3 \neq 1$. This point, together with two vertices of F and one point of V in $v_3 = 0$, form a non-B_1-simplex.

(3.2.4): It remains to consider the case when V has a point in each of at least two coordinate 3-planes (say, $v_1 = 0$ and $v_2 = 0$) containing the edges of F, and each of these two points is not contained in a smaller coordinate plane (otherwise we would arrive at (3.2.1–3)). Then the two above-mentioned points of V and the two vertices of F outside of $v_1 = v_2 = 0$ form a non-B_1-simplex (they cannot form a quadrilateral, otherwise we would arrive at (2)).

(3.3): The only remaining case is that F is a triangle, exactly two of whose faces are in coordinate hyperplanes. Since its third edge is not in a coordinate hyperplane, it should be an edge of another 2-dimensional face G of τ not contained in a coordinate hyperplane.

(3.3.1): If G is also a triangle, exactly two of whose faces are in coordinate hyperplanes, then the convex hull of $F \cup G$ has a triangular face F' whose edges are in three different coordinate hyperplanes. So this case can be handled in the same way as (3.2) (although F' is not necessarily a face of τ, we can still consider the set V of all vertices of τ outside F' and proceed as in (3.2)).

(3.3.2): Otherwise, G is of one of the types (2) or (3.1), and thus τ is B_1 or B_2 as shown in the corresponding paragraphs. ■

6. Eigenvalues of monodromy and corners

In the first subsection, we formulate the main result of this section: certain configurations of V-faces (so called corners) always contribute a non-zero multiplicity of the expected sign to the corresponding monodromy eigenvalue. As a corollary, we prove the monodromy conjecture for a large class of Newton-non-degenerate singularities in arbitrary dimension.

The rest of the section is devoted to the proof of the main result. In particular, in the second subsection we introduce the notion of a hypermodular function, which may be of independent interest for convex geometry and analysis.

6.1. Motivation and results

Let $f(x_1, \ldots, x_n)$ be a polynomial on $\mathbb{C}^n$ such that $f(0) = 0$. For lattice simplices τ contained in compact facets of $\Gamma_+(f)$ we define their V-faces and polynomials $\zeta_\tau(t) = (1 - t^{N(\tau)})^{\mathrm{Vol}_{\mathbb{Z}}(\tau)} \in \mathbb{C}[t]$ in the same way as for faces of $\Gamma_+(f)$.

Let us first observe the following fact.

Proposition 6.1. *Let $\tau \prec \Gamma_+(f)$ be a compact facet such that $\gamma = \tau \cap \{v_i = 0\}$ is one of its facets. Then $F_{\tau,\gamma}(t) := \zeta_\tau(t)/\zeta_\gamma(t) \in \mathbb{C}(t)$ is a polynomial of t. If we assume moreover that τ is not a B_1-pyramid for the variable v_i, then the complex number*

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau)}{N(\tau)}\right)$$

is a root of that polynomial.

Proof. By Lemma 9.4 we can easily prove that $F_{\tau,\gamma} = \zeta_\tau/\zeta_\gamma \in \mathbb{C}(t)$ is a polynomial. Let us prove the remaining assertion. If τ is not a pyramid over $\gamma = \tau \cap \{v_i = 0\}$, then we have $\mathrm{Vol}_{\mathbb{Z}}(\tau) > \mathrm{Vol}_{\mathbb{Z}}(\gamma)$ and the assertion is obvious. So it suffices to consider the case where τ is a pyramid over $\gamma = \tau \cap \{v_i = 0\}$ but its unique vertex $P \prec \tau$ such that $P \notin \gamma$ has height $h \geq 2$ over the hyperplane $\{v_i = 0\} \subset \mathbb{R}^n$. In this case, we define two hyperplanes H_τ and L_τ in $\mathbb{R}^n$ by

$$H_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = N(\tau)\},$$
$$L_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = \langle a(\tau), \mathbf{1}\rangle = \nu(\tau)\}.$$

Note that $P \in \tau \subset H_\tau$ and L_τ is the hyperplane passing through the point $(1, \dots, 1) \in \mathbb{R}^n_+$ and parallel to H_τ; that is, H_τ is the affine span $\mathrm{aff}(\tau) \simeq \mathbb{R}^{n-1}$. Moreover the affine subspace $L_\tau \cap \{v_i = 0\} \subset \mathbb{R}^n$ is parallel to the affine span $H_\tau \cap \{v_i = 0\} \subset \mathbb{R}^n$ of $\gamma = \tau \cap \{v_i = 0\}$. By Lemma 2.4, this implies that $\lambda = \exp(-2\pi i \nu(\tau)/N(\tau)) \in \mathbb{C}$ is a root of $\zeta_\gamma(t)$ if and only if $L_\tau \cap \{v_i = 0\}$ is rational, i.e. $L_\tau \cap \{v_i = 0\} \cap \mathbb{Z}^n \neq \emptyset$. On the other hand, it is easy to see that the affine subspace $H_\tau \cap \{v_i = h-1\} \subset \mathbb{R}^n$ is a parallel translation of $L_\tau \cap \{v_i = 0\}$ by a lattice vector. Hence if $L_\tau \cap \{v_i = 0\}$ is rational, then $H_\tau \cap \{v_i = h-1\} \cap \mathbb{Z}^n \neq \emptyset$ and the lattice height of the pyramid τ over its base $\gamma = \tau \cap \{v_i = 0\}$ is $h \geq 2$, i.e. $\mathrm{Vol}_{\mathbb{Z}}(\tau) \geq 2\,\mathrm{Vol}_{\mathbb{Z}}(\gamma)$. It follows that the polynomial $F_{\tau,\gamma}$ is divisible by $t - \lambda$. This completes the proof. ■

Motivated by this proposition, we introduce the following definitions.

Definition 6.2. Let τ be a k-dimensional lattice V-simplex contained in a compact facet of $\Gamma_+(f)$.

(1) We say that τ has a (possibly empty) *corner* $\tau_0 \prec \tau$ *of codimension* r if $\dim \tau - \dim \tau_0 = k - \dim \tau_0 = r$ and any face σ of τ containing τ_0 is a V-face.

(2) If τ has a (possibly empty) corner $\tau_0 \prec \tau$ of codimension r, then we set

$$F_{\tau,\tau_0}(t) = \prod_{\sigma:\, \tau_0 \prec \sigma \prec \tau,\, \sigma \neq \emptyset} \{\zeta_\sigma(t)\}^{(-1)^{k-\dim \sigma}} \in \mathbb{C}(t).$$

Remark 6.3. Every $(n-1)$-dimensional lattice simplex τ contained in a compact facet of $\Gamma_+(f)$ has a unique corner of maximal codimension, which we will denote by $\mathcal{C}_\tau$. We will also write briefly $F_\tau(t)$ for $F_{\tau,\mathcal{C}_\tau}(t)$.

In the next subsection we will prove the following result.

Theorem 6.4. (1) *Let τ be a k-dimensional lattice V-simplex contained in a compact facet of $\Gamma_+(f)$. Assume that for some $r \geq 1$ it has a non-empty corner τ_0 of codimension r. Then $F_{\tau,\tau_0}(t) \in \mathbb{C}(t)$ is a polynomial of t.*

(2) *If $k = n-1$ and τ is not a B_1-pyramid, then the complex number*

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau)}{N(\tau)}\right)$$

is a root of the polynomial $F_\tau(t)$.

We can generalize Theorem 6.4 slightly to allow also simplices with empty corners as follows. If an $(n-1)$-dimensional lattice simplex τ contained in a compact facet of $\Gamma_+(f)$ has an empty corner $\tau_0 = \emptyset$, then we have the expression $\tau = A_1 \cdots A_n$ such that for any $1 \leq i \leq n$ the vertex A_i is in the positive part of the i-th coordinate axis of $\mathbb{R}^n$.

Proposition 6.5. *Let τ be an $(n-1)$-dimensional lattice simplex contained in a compact facet of $\Gamma_+(f)$. Assume that it has an empty corner $\tau_0 = \emptyset$. Then the function*

$$F_\tau(t) \cdot (1-t)^{(-1)^n} \in \mathbb{C}(t)$$

is a polynomial. If moreover τ is not a B_1-simplex, then the complex number

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau)}{N(\tau)}\right)$$

is a root of that polynomial.

Proof. By the embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}$, $v \mapsto (v, 0)$, we regard τ as a lattice simplex in $\mathbb{R}^n \times \mathbb{R}$ and set $Q(0,1) \in \mathbb{R}^n \times \mathbb{R}$. Let τ' be the convex hull of $\{Q\} \cup \tau$ in $\mathbb{R}^n \times \mathbb{R}$. It is easy to see that

$$F_{\tau',Q}(t) = F_\tau(t) \cdot (1-t)^{(-1)^n},$$

from which the first assertion immediately follows by Theorem 6.4. Since $N(\tau') = N(\tau)$ and $\nu(\tau') = \nu(\tau) + N(\tau)$, the second assertion also follows from Theorem 6.4. ∎

Together with Theorem 4.3, following the strategy of Lemahieu–Van Proeyen [18] we can now confirm the monodromy conjecture for non-degenerate hypersurfaces in many cases also for $n \geq 4$. Let $\tau_1, \dots, \tau_k \prec \Gamma_+(f)$ be the compact facets of $\Gamma_+(f)$. Then we say that the Newton polytope of f has a *good pavement by lattice simplices* if for any $1 \leq i \leq k$ there exists a decomposition $\tau_i = \bigcup_{j=1}^{n_i} \tau_{ij}$ of τ_i into $(n-1)$-dimensional lattice simplices τ_{ij} for which the following conditions are satisfied:

(i) For any $1 \leq i \leq k$ and $1 \leq j \leq n_i$ the lattice simplex τ_{ij} has no V-face or it has a (non-empty) corner which is contained in any V-face of τ_{ij}.

(ii) If $\tau_{ij} \neq \tau_{i'j'}$ then they have no common V-face. Moreover, any V-face of $\Gamma_+(f)$ is decomposed into those of the lattice simplices τ_{ij}.

(iii) For any $1 \leq i \leq k$ the facet τ_i is a B-facet or there exists a lattice simplex τ_{ij} in it which is not a B_1-pyramid.

Theorem 6.6. *Assume that f is non-degenerate and the Newton polytope of f has a good pavement by lattice simplices. Let s_0 be a pole of $Z_{\mathrm{top},f}(s)$ which is contributed only by compact facets of $\Gamma_+(f)$, and those of them that are B-facets are consistent in the sense of Definition 4.2. Then $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy of f at $0 \in \mathbb{C}^n$.*

Proof. If $s_0 = -1$ the assertion is trivial. Otherwise, by Theorem 4.3 the pole $s_0 \neq -1$ is contributed also by a non-B-facet τ_i of $\Gamma_+(f)$. Moreover by Theorem 2.5 and conditions (i)–(ii) we have

$$\zeta_{f,0}(t) = \left\{ \prod_{i=1}^{k} \prod_{j=1}^{n_i} F_{\tau_{ij}}(t) \right\}^{(-1)^{n-1}}.$$

Then $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy of f at $0 \in \mathbb{C}^n$ by Theorem 6.4 applied to condition (iii). ■

Example 6.7. We consider the hypersurface

$$\begin{aligned} H : f(&x_1, x_2, x_3, x_4, x_5) \\ &= x_1^8 + x_2^6 + x_3^{10} + x_4^{12} + x_5^9 + x_1^2 x_3^3 + x_1 x_2 x_4 + x_1 x_2 x_5^3 + x_2^2 x_3 = 0, \end{aligned}$$

which is non-degenerate at the origin in $\mathbb{C}^5$. We denote the vertices of $\Gamma_+(f)$ by

$$\begin{aligned} &A = (8,0,0,0,0), \quad B = (0,6,0,0,0), \quad C = (0,0,10,0,0), \quad D = (0,0,0,12,0), \\ &E = (0,0,0,0,9), \quad F = (2,0,3,0,0), \quad G = (1,1,0,1,0), \quad H = (1,1,0,0,3), \\ &I = (0,2,1,0,0). \end{aligned}$$

The Newton polyhedron $\Gamma_+(f)$ has 10 compact facets

$$\begin{aligned} &\tau_1 = EFGHI, \quad \tau_2 = CEFGI, \quad \tau_3 = CDEFG, \quad \tau_4 = BEGHI, \quad \tau_5 = CDEGI, \\ &\tau_6 = BDEGI, \quad \tau_7 = ABGHI, \quad \tau_8 = AEFGH, \quad \tau_9 = ADEFG, \quad \tau_{10} = AFGHI. \end{aligned}$$

Then we find that

$$\zeta_{f,0}(t) = \prod_{i=1}^{10} F_{\tau_i}(t) F_{CDE}(t) F_{AEF}(t) F_{BEI}(t) F_{CEI,CI}(t) F_{CEF,CF}(t).$$

It follows immediately by Theorem 6.4 that the monodromy conjecture holds for H at the origin. Notice that we did not take $F_{CEI}(t) = \zeta_{CEI}(t)\zeta_C(t)/(\zeta_{CE}(t)\zeta_{CI}(t))$ nor $F_{CEF}(t) = \zeta_{CEF}(t)\zeta_C(t)/(\zeta_{CE}(t)\zeta_{CF}(t))$ as we already have the contributions of the V-faces CE and C in $F_{CDE}(t) = \zeta_{CDE}(t)\zeta_C(t)\zeta_D(t)\zeta_E(t)/(\zeta_{CD}(t)\zeta_{CE}(t)\zeta_{DE}(t))$. ■

6.2. *Hypermodular functions*

For the proof of Theorem 6.4 we shall introduce some new notions and their basic properties. Let S be a finite set and denote its power set by 2^S; elements of 2^S are subsets $I \subset S$ of S. Then for a function $\phi : 2^S \to \mathbb{Z}$ we define $\phi^{\downarrow}, \phi^{\uparrow} : 2^S \to \mathbb{Z}$ by

$$\phi^{\downarrow}(I) = \sum_{J \subset I} \phi(J), \qquad \phi^{\uparrow}(I) = \sum_{J \subset I} (-1)^{|I|-|J|} \phi(J).$$

We call $\phi^{\downarrow}$ (resp. $\phi^{\uparrow}$) the *antiderivative* (resp. *derivative*) of ϕ. Then we can easily check that $\phi^{\uparrow\downarrow} = \phi^{\downarrow\uparrow} = \phi$.

Definition 6.8. (1) We say that the function ϕ is *hypermodular* if $\phi^{\uparrow}(I) \geq 0$ for any subset $I \subset S$.

(2) The function ϕ is called *strictly hypermodular* if it is hypermodular and $\phi^{\uparrow}(S) > 0$.

Lemma 6.9. *The product of two hypermodular functions $\phi, \psi : 2^S \to \mathbb{Z}$ is hypermodular. Moreover, it is strictly hypermodular if and only if there exist $I, J \subset S$ such that $I \cup J = S$ and both $\phi^{\uparrow}(I)$ and $\psi^{\uparrow}(J)$ are strictly positive.*

Proof. For any subset $R \subset S$ of S we have

$$(\phi\psi)^{\uparrow}(R) = (\phi^{\uparrow\downarrow}\psi^{\uparrow\downarrow})^{\uparrow}(R) = \sum_{I\cup J\subset U\subset R} (-1)^{|R|-|U|}\phi^{\uparrow}(I)\psi^{\uparrow}(J)$$
$$= \sum_{I\cup J=R} \phi^{\uparrow}(I)\psi^{\uparrow}(J).$$

Hence the assertion follows immediately. ∎

6.3. *Reduction to the case $k = n-1$ and $r = n-1$*

We can obviously suppose that $k = n-1$ as the computations are made in the coordinate hyperplane containing τ. We now explain how to reduce the proof of Theorem 6.4 to the case $r = n-1$. For simplicity assume that the corner $\gamma \prec \tau$ of the simplex $\tau \subset \partial\Gamma_+(f)$ is defined by $\gamma = \tau \cap \{v_1 = \cdots = v_r = 0\}$. We set

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau)}{N(\tau)}\right) \in \mathbb{C}.$$

As in the proof of Proposition 6.1 we define two parallel affine hyperplanes H_τ and L_τ in $\mathbb{R}^n$ by

$$H_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = N(\tau)\},$$
$$L_\tau = \{v \in \mathbb{R}^n \mid \langle a(\tau), v\rangle = \langle a(\tau), \mathbf{1}\rangle = \nu(\tau)\}.$$

Let $W = \{v_1 = \cdots = v_r = 0\} \simeq \mathbb{R}^{n-r} \subset \mathbb{R}^n$ be the linear subspace of $\mathbb{R}^n$ spanned by γ. Similarly, for a face σ of τ containing γ let $W_\sigma \simeq \mathbb{R}^{\dim\sigma+1} \subset \mathbb{R}^n$ be the linear subspace of $\mathbb{R}^n$ spanned by σ. Then by Lemma 2.4, $\zeta_\sigma(\lambda) = 0$ if and only if the affine hyperplane $L_\tau \cap W_\sigma$ of W_σ is rational, i.e. $L_\tau \cap W_\sigma \cap \mathbb{Z}^n \neq \emptyset$. Let $\Phi_0 : W \xrightarrow{\sim} W$ be a unimodular transformation such that $\Phi_0(\gamma) \subset W \cap \{v_n = c\}$ for some $c \in \mathbb{Z}_{>0}$. Then we can easily extend Φ_0 to a unimodular transformation $\Phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ of $\mathbb{R}^n$ which preserves W_σ for any $\sigma \prec \tau$ containing γ and the point $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. We can choose Φ so that the heights of τ and $\Phi(\tau)$ over each coordinate hyperplane in $\mathbb{R}^n$ containing W are the same. Indeed, for the invertible matrix $A_0 \in \mathrm{GL}_{n-r}(\mathbb{Z})$ representing $\Phi_0 : W \xrightarrow{\sim} W$ it suffices to define $\Phi : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ by taking an invertible matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ of the form

$$A = \left(\begin{array}{c|c} I_r & 0 \\ \hline * & A_0 \end{array}\right) \in \mathrm{GL}_n(\mathbb{Z})$$

such that $A\mathbf{1} = \mathbf{1}$, where $I_r \in \mathrm{GL}_r(\mathbb{Z})$ stands for the identity matrix. By this construction of Φ, τ is a B_1-pyramid if and only if $\Phi(\tau)$ is. Set $\tau' = \Phi(\tau)$ and define two parallel affine hyperplanes $H_{\tau'}$ and $L_{\tau'}$ in $\mathbb{R}^n$ similarly to the case of τ so that $\Phi(H_\tau) = H_{\tau'}$.

Since $\Phi(L_\tau)$ is parallel to $\Phi(H_\tau) = H_{\tau'}$ and passes through $\Phi(\mathbf{1}) = \mathbf{1} \in \mathbb{R}^n$, we also have $\Phi(L_\tau) = L_{\tau'}$. Since the unimodular transformation Φ preserves lattice distances, we thus obtain $N(\tau) = N(\tau')$, $\nu(\tau) = \nu(\tau')$ and

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau')}{N(\tau')}\right).$$

Moreover, for any $\sigma \prec \tau$ containing γ we have $N(\sigma) = N(\Phi(\sigma))$ and hence $\zeta_\sigma(t) \equiv \zeta_{\Phi(\sigma)}(t)$. Then we obtain $F_\tau(t) = F_{\tau'}(t)$, where we slightly generalize Definition 6.2 in an obvious way to define $F_{\tau'}(t)$. Hence, to prove Theorem 6.4 we may assume that the corner γ of τ is contained in $W \cap \{v_n = c\}$ for some $c \in \mathbb{Z}_{>0}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{r+1}$, $v \mapsto (v_1, \ldots, v_r, v_n)$, be the projection. Then by the definition of normalized volumes, for any face σ of τ containing the corner $\gamma \subset W \cap \{v_n = c\}$ we have $\mathrm{Vol}_\mathbb{Z}(\sigma) = \mathrm{Vol}_\mathbb{Z}(\pi(\sigma)) \cdot \mathrm{Vol}_\mathbb{Z}(\gamma)$ and hence $\zeta_\sigma(t) = \{\zeta_{\pi(\sigma)}(t)\}^{\mathrm{Vol}_\mathbb{Z}(\gamma)}$. We thus obtain $F_\tau(t) = \{F_{\pi(\tau)}(t)\}^{\mathrm{Vol}_\mathbb{Z}(\gamma)}$. Moreover, $N(\tau) = N(\pi(\tau))$ and $\nu(\tau) = \nu(\pi(\tau))$. This implies that we have only to consider the case $r = n - 1$.

6.4. *The proof for $r = n - 1$*

We have reduced our proof to the case where $r = n - 1$, a vertex Q of our simplex $\tau = QA_1 \cdots A_{n-1}$ has the form $Q = (0, \ldots, 0, c)$ for some $c \in \mathbb{Z}_{>0}$ and its edges are given by

$$\overrightarrow{QA_1} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ b_1 \end{pmatrix}, \quad \overrightarrow{QA_2} = \begin{pmatrix} 0 \\ a_2 \\ \vdots \\ 0 \\ b_2 \end{pmatrix}, \ldots, \overrightarrow{QA_{n-1}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_{n-1} \\ b_{n-1} \end{pmatrix},$$

where $a_1, \ldots, a_{n-1} \in \mathbb{Z}_{>0}$ and $b_1, \ldots, b_{n-1} \in \mathbb{Z}$. We set

$$D = \prod_{i=1}^{n-1} a_i, \quad K_i = \frac{b_i}{a_i} \cdot D \quad (1 \le i \le n-1)$$

and $K = \sum_{i=1}^{n-1} K_i$. Note that $b_i, K_i < 0$. Moreover, for $I \subset S = \{1, \ldots, n-1\}$ we denote by $\tau_I \prec \tau$ the face of τ whose vertices are Q and A_i $(i \in I)$ and set

$$D_I = \prod_{i \in I} a_i, \quad \mathrm{gcd}_I = \mathrm{GCD}(D, K_i \ (i \in I)) > 0.$$

Lemma 6.10. *The $|I|$-dimensional normalized volume $\mathrm{Vol}_\mathbb{Z}(\tau_I)$ is given by*

$$\mathrm{Vol}_\mathbb{Z}(\tau_I) = \mathrm{gcd}_I \cdot \frac{D_I}{D}$$

and we have

$$N(\tau_I) = \frac{D}{\mathrm{gcd}_I} \cdot c = \frac{D_I}{\mathrm{Vol}_\mathbb{Z}(\tau_I)} \cdot c.$$

In particular,

$$N(\tau) = \frac{D}{\gcd_S} \cdot c.$$

Proof. We only treat the case $I = S$ and $\tau_I = \tau$; the general case can be treated similarly. First, note that the primitive conormal vector $a(\tau) \in \mathbb{Z}^n$ of the $(n-1)$-dimensional simplex τ is equal to

$$\frac{1}{\gcd_S} \begin{pmatrix} -K_1 \\ -K_2 \\ \vdots \\ -K_{n-1} \\ D \end{pmatrix}.$$

From this, the assertion for $N(\tau)$ immediately follows. Let $\widetilde{\tau} \subset \mathbb{R}^n$ be the n-dimensional simplex obtained by taking the convex hull of τ and the point

$$R = (0, \dots, 0, c+1).$$

Then by the above formula for $a(\tau)$, the lattice height of $\widetilde{\tau}$ over its base τ (i.e. the lattice distance from R to $\mathrm{aff}(\tau)$) is equal to $\frac{D}{\gcd_S}$. Since the n-dimensional normalized volume of $\widetilde{\tau}$ is D, we also get the remaining assertion $\mathrm{Vol}_{\mathbb{Z}}(\tau) = \gcd_S$. ∎

For $I \subset S = \{1, \dots, n-1\}$ we set

$$\zeta_I(t) = \{1 - t^{N(\tau_I)}\}^{\mathrm{Vol}_{\mathbb{Z}}(\tau_I)} \in \mathbb{C}[t],$$

so that

$$F_\tau(t) = \prod_{I \subset S} \{\zeta_I(t)\}^{(-1)^{n-1-|I|}}.$$

Lemma 6.11. *The complex number*

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau)}{N(\tau)}\right)$$

is a root of the polynomial $\zeta_I(t)$ if and only if $\gcd_I \mid K$.

Proof. By the formula for $a(\tau)$ in the proof of Lemma 6.10, we obtain

$$\nu(\tau) = \frac{D-K}{\gcd_S}$$

and

$$\frac{\nu(\tau)N(\tau_I)}{N(\tau)} = \frac{D-K}{\gcd_I}. \tag{6.1}$$

Note that λ is a root of $\zeta_I(t)$ if and only if $\nu(\tau)N(\tau_I)/N(\tau)$ is an integer. Then the assertion follows immediately from (6.1) and the fact that $\gcd_I \mid D$. ∎

By this lemma the multiplicity of $t-\lambda$ in the rational function $F_\tau(t)$ is equal to

$$\sum_{I:\,\mathrm{gcd}_I|K} (-1)^{n-1-|I|}\,\mathrm{gcd}_I\cdot\frac{D_I}{D}.$$

Similarly we obtain the following result.

Lemma 6.12. *For any* $m\in\mathbb{Z}$ *the complex number* $\exp(2\pi i m/N(\tau))$ *is a root of the polynomial* $\zeta_I(t)$ *if and only if* $\mathrm{gcd}_I \mid (m\cdot\mathrm{gcd}_S)$.

Proposition 6.13. *The function* $F_\tau(t)$ *is a polynomial in* t.

Proof. By Lemma 6.12 it suffices to show that for any $m\in\mathbb{Z}$ the alternating sum

$$G_m=\sum_{I:\,\mathrm{gcd}_I|(m\cdot\mathrm{gcd}_S)} (-1)^{n-1-|I|}\,\mathrm{gcd}_I\cdot\frac{D_I}{D}$$

is non-negative. Fix $m\in\mathbb{Z}$ and for a prime number p denote its multiplicities in the prime decompositions of a_i, b_i and m by $\alpha(p)_i, \beta(p)_i$ and $\delta(p)$ respectively. We set

$$\gamma(p)=\delta(p)+\min_{1\le i\le n-1}\{\beta(p)_i-\alpha(p)_i,0\}$$

and define a function $\phi_p: 2^S\to\mathbb{Z}$ by

$$\phi_p(I)=\begin{cases} p^{\min_{i\in I}\{\beta(p)_i-\alpha(p)_i,0\}+\sum_{i\in I}\alpha(p)_i} & \text{if } \min_{i\in I}\{\beta(p)_i-\alpha(p)_i,0\}\le\gamma(p),\\ 0 & \text{otherwise.}\end{cases}$$

Then it is easy to see that for the function $\phi=\prod_{p\text{ prime}}\phi_p: 2^S\to\mathbb{Z}$ we have

$$\phi^{\uparrow}(S)=G_m.$$

Indeed, this follows immediately from the fact that for $I\subset S$ the multiplicity of p in gcd_I is equal to

$$\min_{i\in I}\{\beta(p)_i-\alpha(p)_i,0\}+\sum_{1\le i\le n-1}\alpha(p)_i.$$

By Lemma 6.9 we have only to prove that for any prime number p the function $\phi_p: 2^S\to\mathbb{Z}$ is hypermodular. For this purpose, we reorder the pairs (a_i,b_i) $(1\le i\le n-1)$ so tha have

$$\beta(p)_1-\alpha(p)_1\le\beta(p)_2-\alpha(p)_2\le\cdots\le\beta(p)_{n-1}-\alpha(p)_{n-1}.$$

Fix a subset $I=\{i_1,i_2,\dots\}\subset S=\{1,\dots,n-1\}$ $(i_1<i_2<\cdots)$. We will show the non-negativity of the alternating sum

$$\phi_p^{\uparrow}(I)=\sum_{J\subset I}(-1)^{|I|-|J|}\phi_p(J). \tag{6.2}$$

We define $q \geq 0$ to be maximal such that $\beta(p)_{i_q} - \alpha(p)_{i_q} < 0$ (resp. $\beta(p)_{i_q} - \alpha(p)_{i_q} \leq \gamma(p)$) in the case $\gamma(p) \geq 0$ (resp. $\gamma(p) < 0$). First let us consider the case $\gamma(p) \geq 0$. Then for $1 \leq l \leq q$ the part of the alternating sum (6.2) over the subsets $J \subset I$ such that $\min J = i_l$ is equal to

$$(-1)^{l-1} p^{\beta(p)_{i_l}} \prod_{j>l} (p^{\alpha(p)_{i_j}} - 1).$$

Indeed, for instance the term in this alternating sum which corresponds to $J = \{i_l, i_{l+1}, \dots\} \subset I = \{i_1, i_2, \dots\}$ is equal to

$$(-1)^{l-1} p^{\beta(p)_{i_l} - \alpha(p)_{i_l} + \sum_{j \geq l} \alpha(p)_{i_j}} = (-1)^{l-1} p^{\beta(p)_{i_l}} \prod_{j>l} p^{\alpha(p)_{i_j}}.$$

Moreover, the remaining part of (6.2) is equal to

$$(-1)^q \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1).$$

We thus obtain

$$\begin{aligned} \phi_p^{\uparrow}(I) = {} & p^{\beta(p)_{i_1}} \prod_{j>1} (p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2} (p^{\alpha(p)_{i_j}} - 1) \\ & + \cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1) + (-1)^q \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1). \end{aligned}$$

Note that for any $1 \leq j \leq q$ we have $\beta(p)_{i_j} - \alpha(p)_{i_j} < 0$ and obtain

$$p^{\beta(p)_{i_{j-1}}} (p^{\alpha(p)_{i_j}} - 1) \geq p^{\alpha(p)_{i_j}} - 1 \geq p^{\beta(p)_{i_j}}. \tag{6.3}$$

Thus, subdividing the terms in the above expression of $\phi_p^{\uparrow}(I)$ into pairs, we get the desired non-negativity $\phi_p^{\uparrow}(I) \geq 0$. Finally, let us consider the case $\gamma(p) < 0$. We then have

$$\begin{aligned} \phi_p^{\uparrow}(I) = {} & p^{\beta(p)_{i_1}} \prod_{j>1} (p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2} (p^{\alpha(p)_{i_j}} - 1) \\ & + \cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q} (p^{\alpha(p)_{i_j}} - 1). \end{aligned}$$

Then by using the inequality (6.3) we can prove the non-negativity $\phi_p^{\uparrow}(I) \geq 0$ as in the case $\gamma(p) \geq 0$. This completes the proof. ■

Proposition 6.14. *Assume that τ is not a B_1-simplex. Then the complex number*

$$\lambda = \exp\left(-2\pi i \frac{\nu(\tau)}{N(\tau)}\right)$$

is a root of the polynomial $F_\tau(t)$.

Proof. By Lemma 6.11 it suffices to show that the alternating sum

$$G = \sum_{I:\,\mathrm{gcd}_I \mid K} (-1)^{n-1-|I|}\, \mathrm{gcd}_I \cdot \frac{D_I}{D}$$

is positive. For a prime number p denote its multiplicities in the prime decompositions of a_i, b_i and K by $\alpha(p)_i, \beta(p)_i$ and $\kappa(p)$ respectively. We set

$$\mu(p) = \kappa(p) - \sum_{i=1}^{n-1} \alpha(p)_i$$

and define a function $\psi_p : 2^S \to \mathbb{Z}$ by

$$\psi_p(I) = \begin{cases} p^{\min_{i\in I}\{\beta(p)_i - \alpha(p)_i, 0\} + \sum_{i\in I}\alpha(p)_i} & \text{if } \min_{i\in I}\{\beta(p)_i - \alpha(p)_i, 0\} \le \mu(p), \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see that for the function $\psi = \prod_{p \text{ prime}} \psi_p : 2^S \to \mathbb{Z}$ we have

$$\psi^{\uparrow}(S) = G.$$

Now set $S_p = \{1 \le i \le n-1 \mid \alpha(p)_i = 0\}$ and $I_p = S \setminus S_p = \{1 \le i \le n-1 \mid \alpha(p)_i > 0\}$. By our assumption we have $a_i > 1$ for any $1 \le i \le n-1$ and hence $\bigcup_{p \text{ prime}} I_p = S$. By Lemma 6.9, in order to show that $\psi^{\uparrow}(S) > 0$ it suffices to prove that for any prime p we have $\psi_p^{\uparrow}(I_p) > 0$. As in the proof of Proposition 6.13 we reorder the pairs (a_i, b_i) $(1 \le i \le n-1)$ so that

$$\beta(p)_1 - \alpha(p)_1 \le \beta(p)_2 - \alpha(p)_2 \le \cdots \le \beta(p)_{n-1} - \alpha(p)_{n-1}$$

and $\alpha(p)_i \ge \alpha(p)_{i+1}$ whenever $\beta(p)_i - \alpha(p)_i = \beta(p)_{i+1} - \alpha(p)_{i+1}$. Moreover, we set $I_p = \{i_1, i_2, \dots\}$ $(i_1 < i_2 < \cdots)$. We let $q \ge 0$ be maximal such that $\beta(p)_{i_q} - \alpha(p)_{i_q} < 0$ (resp. $\beta(p)_{i_q} - \alpha(p)_{i_q} \le \mu(p)$) in the case $\mu(p) \ge 0$ (resp. $\mu(p) < 0$). Then we have the same expressions of $\psi_p^{\uparrow}(I_p) > 0$ as in the proof of Proposition 6.13. In the case $\mu(p) \ge 0$ we have

$$\begin{aligned}\psi_p^{\uparrow}(I_p) = {} & p^{\beta(p)_{i_1}} \prod_{j>1}(p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2}(p^{\alpha(p)_{i_j}} - 1) \\ & + \cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1) + (-1)^q \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1). \qquad (6.4)\end{aligned}$$

In the case $\mu(p) < 0$ we have

$$\begin{aligned}\psi_p^{\uparrow}(I_p) = {} & p^{\beta(p)_{i_1}} \prod_{j>1}(p^{\alpha(p)_{i_j}} - 1) - p^{\beta(p)_{i_2}} \prod_{j>2}(p^{\alpha(p)_{i_j}} - 1) \\ & + \cdots + (-1)^{q-1} p^{\beta(p)_{i_q}} \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1). \qquad (6.5)\end{aligned}$$

By the definitions of $I_p = \{i_1, i_2, \dots\} \subset S$ and $q \geq 0$ we have $i \in I_p$ for any $i \leq i_q$. Eventually we find that $i_j = j$ for any $j \leq q$. First let us consider the case $I_p = \emptyset$. Then

$$\psi_p(I_p) = \begin{cases} p^0 = 1 & \text{if } \mu(p) \geq 0, \\ 0 & \text{if } \mu(p) < 0. \end{cases}$$

For $\mu(p) \geq 0$ we thus obtain $\psi_p^{\uparrow}(I_p) > 0$. But in the case $\mu(p) < 0$ the condition $I_p = \emptyset$ implies $q = 0$, which cannot occur by the following lemma.

Lemma 6.15. *The case $I_p = \emptyset$ and $\mu(p) < 0$ cannot occur.*

Proof. Assume that $I_p = \emptyset$ and $\mu(p) < 0$. By the definition of $\mu(p)$ we have

$$\operatorname{mult}_p(K) = \operatorname{mult}_p(D \cdot p^{\mu(p)}). \tag{6.6}$$

Moreover, for any $i \in S = S \setminus I_p$ we have

$$\operatorname{mult}_p(K_i) \geq \operatorname{mult}_p(D) > \operatorname{mult}_p(D \cdot p^{\mu(p)}),$$

where we have used the condition $\mu(p) < 0$ in the second inequality. We thus obtain

$$\operatorname{mult}_p(K) = \operatorname{mult}_p\Big(\sum_{i \in S} K_i\Big) > \operatorname{mult}_p(D \cdot p^{\mu(p)}),$$

which contradicts (6.6). ■

By this lemma, it remains to treat the case $I_p \neq \emptyset$, which we assume from now on. Note that inequality (6.3) becomes an equality only when $p = 2$, $\beta(p)_{i_{j-1}} = \beta(p)_{i_j} = 0$ and $\alpha(p)_{i_j} = 1$. By Lemma 6.15 this means that the sums (6.4) and (6.5) may be zero only in the following two cases:

Case 1: $p = 2$, $\mu(p) \geq 0$, $q = 2m + 1$ for $m \geq 0$ and $(\alpha(p)_1, \beta(p)_1) = (a, 0)$ for $a > 0$, $(\alpha(p)_2, \beta(p)_2) = \cdots = (\alpha(p)_q, \beta(p)_q) = (1, 0)$.

Case 2: $p = 2$, $\mu(p) < 0$, $q = 2m$ for $m \geq 1$ and $(\alpha(p)_1, \beta(p)_1) = (a, 0)$ for $a > 0$, $(\alpha(p)_2, \beta(p)_2) = \cdots = (\alpha(p)_q, \beta(p)_q) = (1, 0)$.

Indeed, in the case $\mu(p) \geq 0$ and $q = 2m$ for $m \geq 0$, if $q < |I_p|$ then the last term $(-1)^q \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1)$ of the alternating sum (6.4) is positive. Even if $q = |I_p|$ we still have the positivity

$$(-1)^q \prod_{j>q}(p^{\alpha(p)_{i_j}} - 1) = \psi_p(\emptyset) = 1 > 0.$$

Let us show that none of the above two cases can occur.

Case 1: Set $\alpha(p) = \sum_{i \in S} \alpha(p)_i$. Then $2^{\alpha(p)} \mid D$ and $2^{\alpha(p)} \mid K_i$ for any $i \in S_2$. Thus

$$\begin{aligned} K &\equiv 2^{\alpha(p)-a} \cdot \text{odd} + (q-1) \cdot 2^{\alpha(p)-1} \cdot \text{odd} + \sum_{j>q} 2^{\alpha(p)+\beta(p)_{i_j}-\alpha(p)_{i_j}} \\ &\equiv 2^{\alpha(p)-a} \cdot \text{odd} + (q-1) \cdot 2^{\alpha(p)-1} \cdot \text{odd} \equiv 2^{\alpha(p)-a} \cdot \text{odd} \bmod 2^{\alpha(p)}, \end{aligned}$$

where we have also used the fact that $\beta(p)_{i_j} - \alpha(p)_{i_j} \geq 0$ for any $j > q$. We conclude that $2^{\alpha(p)}$ does not divide K, which contradicts our assumption $\mu(p) \geq 0$.

Case 2: Since $q \geq 2$ we have $-1 = \beta(p)_{i_2} - \alpha(p)_{i_2} \leq \mu(p)$. Then from $\mu(p) < 0$ we obtain $\mu(p) = -1$. As in Case 1, by using the fact that $q - 1$ is odd and $\mu(p) = -1$, if $a = 1$ we obtain

$$K \equiv \sum_{j>q} 2^{\alpha(p)+\beta(p)_{i_j}-\alpha(p)_{i_j}} \equiv 0 \bmod 2^{\alpha(p)},$$

that is, $2^{\alpha(p)} \mid K$, which contradicts our assumption $\mu(p) < 0$. If $a > 1$ we obtain

$$K \equiv 2^{\alpha(p)-a} \cdot \text{odd} \bmod 2^{\alpha(p)-1},$$

which also contradicts $\mu(p) = -1$. This completes the proof. ■

7. On non-convenient Newton polyhedra

When dealing with a singularity $(f, 0)$ with non-convenient Newton polyhedron $\Gamma_+(f)$, it happens already in dimensions 2 and 3 that one has to search for the monodromy eigenvalue at some point of the hypersurface $f^{-1}(0)$ close to the origin.

Definition 7.1 (cf. [13]). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function. For all sufficiently small $x_0 \in \mathbb{C}^n$, the *nearby singularity* germ

$$f_{x_0} : (\mathbb{C}^n, x_0) \to (\mathbb{C}, 0), \quad f_{x_0}(x) = f(x_0 + x),$$

is well defined. We shall refer to the roots and poles of the monodromy ζ-function of the latter germ as *nearby monodromy eigenvalues* of f.

7.1. Nearby singularities at coordinate lines

Notice that the Newton polyhedron at a generic point of a k-dimensional coordinate plane is the product of the projection of the Newton polyhedron along that coordinate plane by $\mathbb{R}^k_+$. In this subsection, we prove the following generalization of [18, Lemma 9].

Proposition 7.2. *Assume that f is non-degenerate at $0 \in \mathbb{C}^n$. Then except for finitely many $c \in \mathbb{C}$ the polynomial $f_c(x) = f(x_1, x_2, \ldots, x_{n-1}, x_n + c)$ is non-degenerate at $0 \in \mathbb{C}^n$.*

Proof. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection along the last variable. Then except for finitely many $c \in \mathbb{C}$ the Newton polyhedron $\Gamma_+(f_c)$ is equal to $\pi(\Gamma_+(f)) \times \mathbb{R}_+$. Let τ' be a face of $\Gamma_+(f)$ which is non-compact for the variable v_n and denote by $\sigma \subset \mathbb{R}^{n-1}$ its image under the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$. Assume that σ is compact. Here we shall treat only the case where τ' is a facet and hence $\dim \sigma = n - 2$; the other cases can be treated similarly. By a unimodular transformation of $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ induced by that of its first factor $\mathbb{R}^{n-1}$. we regard τ' as a lattice polytope in its affine span $\mathrm{aff}(\tau') \simeq \mathbb{R}^{n-1}$, and the

τ'-part $f_{\tau'}$ of f as a Laurent polynomial on $T' = (\mathbb{C}^*)^{n-1}_{x_1,\dots,x_{n-1}}$, where the last variable x_n of $f_{\tau'}$ corresponds to the last one x_{n-1} of the Laurent polynomial. We denote the latter also by $f_{\tau'}$. Then by our assumption for any compact face τ of τ' the hypersurface $\{f_\tau = 0\} \subset T'$ is smooth. Moreover, the σ-part of the polynomial $f(x_1,\dots,x_{n-1},x_n+c)$ is naturally identified with the Laurent polynomial $f_{\tau'}(x_1,\dots,x_{n-2},c)$. Therefore, in order to prove the assertion, by our previous description of $\Gamma_+(f_c)$ it suffices to show that except for finitely many $c \in \mathbb{C}$ the hypersurface

$$W_c = \{(x_1,\dots,x_{n-2}) \mid f_{\tau'}(x_1,\dots,x_{n-2},c) = 0\}$$

in $T' \cap \{x_{n-1} = c\} \simeq (\mathbb{C}^*)^{n-2}$ is smooth. Let $h : T' = (\mathbb{C}^*)^{n-1} \to \mathbb{C}$ be defined by $h(x_1,\dots,x_{n-1}) = x_{n-1}$. Then the set of $c \in \mathbb{C}$ for which $W_c \subset (\mathbb{C}^*)^{n-2}$ is not smooth is contained in the discriminant variety of $h|_{\{f_{\tau'}=0\}} : \{f_{\tau'} = 0\} \to \mathbb{C}$. For $\varepsilon > 0$ let $B(0;\varepsilon)^* = \{c \in \mathbb{C} \mid 0 < |c| < \varepsilon\}$. Then there exists a sufficiently small $0 < \varepsilon \ll 1$ such that the hypersurface $W_c \subset (\mathbb{C}^*)^{n-2}$ is smooth for any $c \in B(0;\varepsilon)^*$. Indeed, let $\Delta = \tau' \cap \{v_n \leq l\} \subset \tau'$ ($l \gg 0$) be the truncation of τ'. Let Σ_0 be the dual fan of the $(n-1)$-dimensional polytope Δ in $\mathbb{R}^{n-1}$, and Σ its smooth subdivision. We denote by X_Σ the toric variety associated to Σ (see [14,25] etc.). Then X_Σ is a smooth compactification of $T' = (\mathbb{C}^*)^{n-1}$. Recall that $T' = (\mathbb{C}^*)^{n-1}$ acts naturally on X_Σ and the T'-orbits in it are parameterized by the cones in the smooth fan Σ. For a cone $C \in \Sigma$ denote by $T_C \simeq (\mathbb{C}^*)^{n-1-\dim C} \subset X_\Sigma$ the T'-orbit associated to C. By our assumption above, if $C \in \Sigma$ corresponds to a compact face τ of τ' then the hypersurface $W = \overline{\{f_{\tau'} = 0\}} \subset X_\Sigma$ intersects $T_C \subset X_\Sigma$ transversally. We denote the meromorphic extension of $h : T' = (\mathbb{C}^*)^{n-1} \to \mathbb{C}$ to X_Σ also by h. Note that h has no point of indeterminacy on the whole X_Σ (because it is a monomial). Then as $|c| \to 0$ the level set $h^{-1}(c) \subset X_\Sigma$ of h tends to the union of the T'-orbits which correspond to the compact faces of τ'. More precisely, if a cone $C \in \Sigma$ corresponds to a compact face of τ' then there exists an affine chart $\mathbb{C}^{n-1}_y$ of X_Σ on which

$$T_C = \{y = (y_1,\dots,y_{n-1}) \mid y_i = 0\ (1 \leq i \leq \dim C),\ y_i \neq 0\ (\dim C + 1 \leq i \leq n-1)\}$$

and $h(y) = y_1^{m_1} y_2^{m_2} \cdots y_k^{m_k}$ ($m_i \in \mathbb{Z}_{>0}$) for some $k \geq 1$. By this explicit description of h we see that for $0 < |c| \ll 1$ the hypersurface $h^{-1}(c)$ intersects W transversally. It follows that

$$W_c = W \cap h^{-1}(c) \cap T' \subset h^{-1}(c) \cap T' \simeq (\mathbb{C}^*)^{n-2}$$

is smooth for $0 < |c| \ll 1$. This completes the proof. ■

Note also that at almost all points on a coordinate axis contained in the hypersurface, the compact part of the Newton polyhedron there coincides with the compact part of the projection of $\Gamma_+(f)$ along that coordinate axis. Then the monodromy zeta function can be computed by the same Varchenko formula in one dimension less, since by Proposition 7.2 generic nearby singularity germs are still non-degenerate.

Example 7.3. (1) If $f(x_1,x_2) = x_1^{a_1} x_2^{a_2} g(x_1,x_2)$ with g not divisible by x_i, then we have the nearby eigenvalue $\sqrt[a_i]{1}$ on the i-th axis.

(2) If $f(x_1, x_2, x_3) = x_1^{a_1} x_2^{a_2} x_3^{a_3} g(x_1, x_2, x_3)$ with g not divisible by x_i, then we have the nearby eigenvalue $\sqrt[a_i]{1}$ at every point of the i-th coordinate plane except for the points of the coordinate lines and, most notably, of the surface $g = 0$.

7.2. *Nearby singularities outside coordinate lines*

The following example shows that from dimension 4 on, one might not always find the eigenvalue of monodromy corresponding to a pole of the topological zeta function at a point on a coordinate axis or even a generic point on a coordinate plane (a subtle shadow of this difference between generic and not so generic points of a coordinate plane can be seen already in dimension 3; see Example 7.3).

Example 7.4. Consider the polynomial $f(x_1, x_2, x_3, x_4) = x_3^6 + x_2^4 x_3^5 + x_1^2 x_2^{13} x_3^2 + x_2^{13} x_3^2 x_4^2$, which is non-degenerate at the origin. One finds that $-1/3$ is a pole of $Z_{\mathrm{top},f}(s)$, contributed by the only compact facet of $\Gamma_+(f)$. For the zeta function of monodromy at the origin one finds

$$\zeta_{f,0}(t) = \frac{1 - t^6}{1 - t^{24}},$$

and so $e^{-2i\pi/3}$ is not a zero or pole of this function. One can check that there does not exist a point $b = (c_1, c_2, c_3, c_4)$ in $\{f = 0\}$ such that the compact part of the Newton polyhedron $\Gamma_+(g)$ of $g(x_1, x_2, x_3, x_4) = f(x_1 - c_1, x_2 - c_2, x_3 - c_3, x_4 - c_4)$ is a projection of $\Gamma_+(f)$ along the minimal coordinate plane containing b, and $e^{-2i\pi/3}$ is a zero or pole of $\zeta_{f,b}(t)$. However, $e^{-2i\pi/3}$ is an eigenvalue of monodromy at the points of the curve $\mathcal{C} = \{(c, 0, 0, -ic), c \in \mathbb{C}^*\} \subset \{f = 0\}$. Note that these are exactly the points where $\Gamma_+(g)$ is strictly smaller than the projection of $\Gamma_+(f)$, due to a cancellation of two monomials in f.

In this particular example, one can check that the singularity is still non-degenerate at the points of the curve $\mathcal{C}$ where we found the eigenvalue of monodromy. However, this will not always be the case, as shown in the next example.

Example 7.5. We consider

$$g(x_1, x_2, x_3, x_4) = x_3^6 + x_2^4 x_3^5 + x_1^2 x_2^{13} x_3^2 + x_2^{13} x_3^2 x_4^2 + 2x_1^{100} x_2^7 x_3^4 x_4^{100} + x_1^{200} x_2^{14} x_3^2 x_4^{200},$$

which up to terms of higher order equals the polynomial f considered in Example 7.4. The polynomial g is non-degenerate at the origin, and its Newton polyhedron at the origin has one compact facet, spanned by the vertices $A(0, 13, 2, 2)$, $B(2, 13, 2, 0)$, $C(0, 4, 5, 0)$ and $D(0, 0, 6, 0)$. It contributes the candidate pole $-1/3$, which is a pole of $Z_{\mathrm{top},g}(s)$. For the zeta function of monodromy at the origin one finds

$$\zeta_{g,0}(t) = \frac{1 - t^6}{1 - t^{24}}.$$

In Example 7.4 we found the monodromy eigenvalue $e^{-2\pi i/3}$ at the points $(c, 0, 0, \pm ic)$, $c \in \mathbb{C}$. In the translated local coordinates at $(c, 0, 0, \pm ic)$, the principal part of g will be

$$x_3^6 + x_2^4 x_3^5 + 2c^{200} x_2^7 x_3^4 + c^{400} x_2^{14} x_3^2 + 2c x_1 x_2^{13} x_3^2 - 2ic x_2^{13} x_3^2 x_4,$$

having a degenerate edge $x_3^6 + 2c^{200} x_2^7 x_3^4 + c^{400} x_2^{14} x_3^2$.

This example shows we have to quit the non-degenerate setting to prove the monodromy conjecture for non-degenerate singularities in dimension 4 and higher. It also shows that the Newton polyhedron of a singularity at an adjacent singular point may depend not only on the Newton polyhedron of the initial singularity, but also on higher order terms.

This obstacle motivated the first author to introduce the notion of tropical monodromy eigenvalues (see [13]). The main result in [13] makes it possible to find some of the nearby monodromy eigenvalues outside the coordinate axes, given only the Newton polyhedron of a non-degenerate singularity at the origin. We recall this result and restrict to dimension 4 from now on.

Assume that $f(x) \in \mathbb{C}[x_1, \dots, x_4]$ is non-degenerate at the origin $0 \in \mathbb{C}^4$. Pick some pole s_0 of the topological zeta function and denote the corresponding candidate eigenvalue of the Milnor monodromy by $t_0 = \exp(2\pi i s_0)$.

We suppose that there is a V-vertex A contained in an unbounded face, contributing to the eigenvalue t_0. With no loss in generality, assume that A is on the coordinate axis O_1.

Let $I \subset \{1, \dots, 4\}$, $I \neq \emptyset$ and $I \neq \{1, \dots, 4\}$. We will denote by $\pi_I = \pi_{\{x_i\}_{i \in I}}$ the projection map

$$\mathbb{R}^4 \to \mathbb{R}^{4-|I|}, \quad (x_1, x_2, x_3, x_4) \mapsto (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4),$$

where $\hat{x}_i$ means that x_i is removed if and only if $i \in I$. When $I = \{x_i\}$ is a singleton, we will also write π_{x_i}.

Theorem 7.6 ([13, Cor. 6.15]). *Let $\pi_I : \mathbb{Z}^4 \to \mathbb{Z}^2$ be the projection map. Let N be the projection of the Newton polyhedron $\Gamma_+(f)$ under π_I. Let a compact edge E of the polygon N be the projection of a 2-dimensional compact face F of $\Gamma_+(f)$. Denote the lattice distance from E to the origin by d, and some root of the polynomial $t^d - 1$ by t_0'. If E is not a segment of lattice length 1 such that exactly one of its end points is a V-vertex of N contributing to the eigenvalue t_0' and such that another end point has a unique preimage in F, then t_0' is a monodromy eigenvalue of the germ of f at a non-zero point of $\{f = 0\}$.*

More specifically, if $I = \{x_1, x_2\}$, then there exists a curve $C_{E,N,t_0'}$ through the origin in the coordinate plane $x_3 = x_4 = 0$ (and outside the axes O_1 and O_2) such that t_0' is a monodromy eigenvalue of the germ of f at a generic point of $C_{E,N,t_0'}$.

Remark 7.7. (1) If the face F contains a V-vertex on a coordinate axis out of I contributing to t_0', then the condition in Theorem 7.6 is fulfilled.

(2) The nearby monodromy eigenvalues provided by this theorem are called *tropical*, because the proof of their existence in [13] is based on the calculus of so called tropical characteristic classes.

For instance, this theorem allows one to find the "complicated" nearby monodromy eigenvalue in Example 7.5.

8. The monodromy conjecture for $n = 4$

Assume that $f(x) \in \mathbb{C}[x_1, \dots, x_4]$ is non-degenerate at $0 \in \mathbb{C}^4$. Pick some pole s_0 of the topological zeta function and denote the corresponding candidate eigenvalue of the Milnor monodromy by $t_0 = \exp(2\pi i s_0)$. Our aim is to prove that, once Theorem 5.2 does not guarantee that s_0 is fake, t_0 is a root or pole of the monodromy ζ-function of a singularity of f at some point near the origin.

For the rest of the paper, we may assume the compactness of every B-border, adjacent to two B-facets contributing the pole s_0. Indeed, towards a contradiction, if at least one such border τ were non-compact, then we would have one of the following (up to reordering the coordinates):

- if τ is the Minkowski sum of the point $(1, 1, *, *)$ and the 3rd and 4th coordinate rays, then $s_0 = -1$;
- if τ is the Minkowski sum of the segment from $(1, 1, *, *)$ to $(0, 0, *, *)$ and the 4th coordinate ray, then its projection along the 4th coordinate is a segment from $(1, 1, *)$ to $(0, 0, *)$, adjacent to two B-faces contributing s_0 in the Newton polyhedron of $f(x_1, x_2, x_3, x_4 + \varepsilon)$ for a small constant $\varepsilon \neq 0$. Then [18] ensures that in this case $\exp(2\pi i s_0)$ is a monodromy eigenvalue of the singularity of f at $(0, 0, 0, \varepsilon)$.

In the second subsection we will see how [13] allows us to isolate many cases of the combinatorial structure of the Newton polyhedron $\Gamma_+(f)$, which ensure that t_0 is a nearby monodromy eigenvalue outside the origin.

In the third subsection, we continue this work in the presence of a triangulation of $\Gamma_+(f)$ (which can be constructed only if $\Gamma_+(f)$ does not fall within the scope of the second subsection). In the fourth subsection, we subdivide the V-pieces of this triangulation into groups such that each of them "contributes" a non-negative multiplicity to t_0 as an eigenvalue of the monodromy ζ-function at the origin, in the sense of the following definition.

Definition 8.1. Recall that a V-face F is said to *contribute to the eigenvalue* t_0 if $t_0^{N(F)} = 1$. The number $(-1)^{\operatorname{codim} F - 1} \operatorname{Vol}_{\mathbb{Z}}(F)$ for a contributing F and 0 for a non-contributing F is called *the contribution of* F. The sum of the contributions of all faces from some set S of faces is called *the contribution of* S.

Finally, in the last subsection, we show that once one of the aforementioned groups violates the assumptions of Theorem 5.2, its contribution is strictly positive. This proves the monodromy conjecture for non-degenerate singularities of four variables.

The aforementioned subdivision of facets into groups is based on the fact that every V-simplex F has a unique minimal corner (possibly equal to F), and the following notion.

Definition 8.2. For any triangulation of the union of the compact faces of $\Gamma_+(f)$, the *family of a V-simplex* F of this triangulation is the set of all faces of F, containing its minimal corner (note that all of them are V-faces by definition of the corner). A family is said to be *trivial* if it consists of one element. The dimension of the maximal simplex in a family is also referred to as the *dimension* of the family.

Example 8.3. We illustrate the notion of family of a V-simplex by an example in dimension 3. Let $P = (0, 0, a)$, $Q = (0, b, c)$, $R = (d, 0, e)$ with a, b, c, d and e non-zero and let S be a vertex outside the coordinate planes such that PQS and PRS are different V-triangles containing P. Then the minimal corner of PQS is PQ and the family of PQS is $\{PQS, PQ\}$. The minimal corner of PRS is PR. Hence the family of PRS is $\{PRS, PR\}$.

Recall that, by Theorem 6.4, the contribution of every family to every eigenvalue is non-negative or non-positive, depending on the dimension of the family.

Before implementing our general plan, we devote the first subsection to a sandbox 3-dimensional version of this story (first, in order to illustrate a significantly more complicated 4-dimensional case beforehand, and, secondly, because we shall need this 3-dimensional statement anyway). Although the 3-dimensional result is essentially covered by [18], the logic of our reasoning is different, and this difference becomes important in higher dimensions. Indeed, the result in [13] makes us approach the monodromy conjecture for non-isolated singularities by first searching for the monodromy eigenvalue outside the origin and then, if necessary, at the origin (having already excluded many combinatorial possibilities for the structure of the Newton polytope).

8.1. 2- and 3-dimensional cases

We intentionally formulate things in a more complicated way than we could for two or three variables, in order to keep all the wording consistent with the 4-dimensional case.

Theorem 8.4. *For every non-degenerate $f \in \mathbb{C}[x, y]$, if a family F has positive contribution to the candidate eigenvalue t_0, then t_0 is a nearby monodromy eigenvalue for f.*

Proof. Note that positive contribution implies that the family F is 1-dimensional (not 0-dimensional).

Step 1: Looking for the monodromy eigenvalue outside the origin (cf. Example 7.3). If $\Gamma_+(f)$ has a V-vertex that contributes to the eigenvalue t_0 and is not contained in a compact V-edge, then t_0 is obviously a nearby monodromy eigenvalue of f at a point of a coordinate axis.

Step 2: Splitting $\Gamma_+(f)$ into families otherwise. Define the *register* of $\Gamma_+(f)$ as the set of families of all V-edges. Neglecting the cases covered by Step 1, we notice that every V-simplex enters exactly one family in the register.

All families in the register have non-negative contribution, and the family F has positive contribution, thus the total contribution to the eigenvalue t_0 is positive. ∎

Theorem 8.5. *For every non-degenerate $f \in \mathbb{C}[x, y, z]$ and a triangulation of the compact faces of its Newton polyhedron, if a family F of this triangulation has positive contribution to the candidate eigenvalue t_0, then t_0 is a nearby monodromy eigenvalue for f, unless all compact V-simplices containing F are contained in the same* (*larger*) *family whose contribution is* 0.

Proof. **Step 1:** Looking for the monodromy eigenvalue outside the origin. If $\Gamma_+(f)$ has a V-edge G, whose family has non-zero contribution to the eigenvalue t_0, and which is not contained in a compact V-triangle, then such a face is contained in a non-compact face, parallel to a coordinate axis (say, O_1) and not contained in a coordinate plane. Then G is also a V-edge of the projection of $\Gamma_+(f)$ along O_1, satisfying the assumption of the preceding theorem (recall Proposition 7.2), so t_0 is a nearby monodromy eigenvalue of f at a point of axis O_1.

Step 2: Splitting $\Gamma_+(f)$ into families otherwise. Define the *register* of $\Gamma_+(f)$ as the set of

– families of all V-triangles,

– for all V-edges outside the aforementioned families, the families of these V-edges;

– (families of) all V-vertices outside the aforementioned families.

Every V-simplex enters exactly one family on the register. In particular, assume towards a contradiction that a V-vertex A is contained in several families. If one of them contains the others, then the others are not on the register; otherwise A is contained in two 1-dimensional families, but then none of them is on the register.

Neglecting the cases covered by Step 1, there are no families of V-edges that are not contained in V-triangles and have non-zero contribution. Thus the register contains only even-dimensional families and families having zero contribution.

By Theorem 6.4 the contribution of every even-dimensional family to the multiplicity of t_0 is non-negative. Thus all families on the register have non-negative contribution.

Moreover, at least one of them has positive contribution: if F is a 2-dimensional family, then it is on the register with positive contribution. If F is 0-dimensional, and all compact faces containing F are contained in a larger family, then the larger family is on the register with non-zero (i.e. positive) contribution. If F is 0-dimensional otherwise, then it is on the register with positive contribution. ∎

8.2. *Dimension 4: Looking for the eigenvalue t_0 outside the origin*

We suppose that there is a V-vertex A contributing to the eigenvalue t_0, and we assume that A is on the coordinate axis O_1.

In this subsection we start to exploit Proposition 7.2 and Theorem 7.6 as much as possible to derive properties of the combinatorial structure of the Newton polyhedron locally around A.

Definition 8.6. The *link* L_A is the subdivision of the triangle T_A defined by

$$x_1 = 0, \quad x_2 + x_3 + x_4 = \varepsilon, \quad x_2, x_3, x_4 \geq 0,$$

into the isomorphic images of faces of $\Gamma_+(f)$ intersected with the hyperplane $x_2 + x_3 + x_4 = \varepsilon$ under the projection π_{x_1}, for $\varepsilon > 0$ small enough. (The link does not depend on the choice of $\varepsilon > 0$ in the sense that the links for all $\varepsilon > 0$ small enough are affinely isomorphic to each other.)

The image of a face $F \subset \Gamma_+(f)$ containing A in the link L_A is referred to as the *link of* F in L_A.

For example, the link of a vertex of a Newton polyhedron is shown in bold in Figure 8.

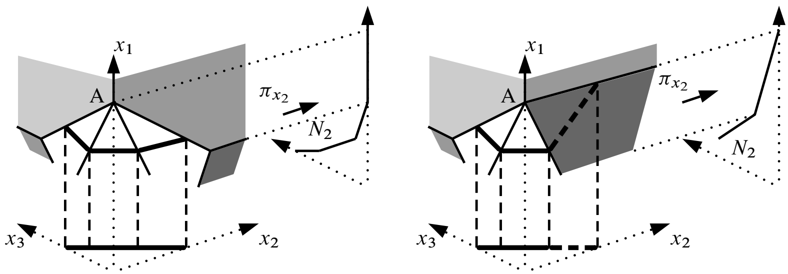

Fig. 8. Links of Newton polyhedra.

The following fact seems to be common knowledge, but we give a proof as we have not found an exact reference, and the fact is not entirely tautological.

Proposition 8.7. *The union of the relative interiors of the links of bounded faces in L_A is closed and contractible.*

Proof. We denote $A(a, 0, 0, 0)$. By taking a slice of the Newton polyhedron $\Gamma_+(f)$ with the hyperplane $x_1 = a - \epsilon$, it is sufficient to prove that the union of the compact faces $\mathcal{C}$ of every Newton polyhedron N is contractible.

Let N_ϵ be the Minkowski sum of N and a ball of radius ϵ. The union $\mathcal{C}_\epsilon$ of its compact faces (i.e. the set of those boundary points that do not belong to a ray of boundary points) is a topological disk, because the homeomorphism with the standard simplex is provided by the Gauss–Bonnet map (sending every boundary point to its unit exterior normal vector). Now the family of sets $\mathcal{C}_\epsilon$ is a family of vanishing neighborhoods of $\mathcal{C}$, so the contractibility of all $\mathcal{C}_\epsilon$ implies the contractibility of $\mathcal{C}$. ∎

Recall that a ray $r \in \mathbb{R}^n$ is said to belong to the *recession cone* of a set $S \subset \mathbb{R}^n$ if the Minkowski sum of r and S equals S.

Definition 8.8. A face of $\Gamma_+(f)$ is called an *at least I-face* (resp. *at most I-face)*, where $I \subset \{1, 2, 3, 4\}$, if its recession cone contains the positive coordinate axes O_i, $i \in I$ (resp. its recession cone does not contain any coordinate axis O_i with $i \notin I$), and is called an (exactly) *I-face* if it is an at least and at most I-face. This terminology passes to the corresponding pieces of the link L_A.

For example, in Figure 8, unbounded facets of the Newton polyhedron are grey, and a 2-piece of the link, corresponding to an unbounded 1-facet, is shown in bold dashes.

Definition 8.9. Let v_i be the vertex of the triangle T_A opposite to its edge $x_i = 0$. Let l be a line separating v_i from the other vertices of the pieces of L_A (i.e. passing close enough to v_i). Then the subdivision of the segment $l \cap T_A$ into its intersections with the pieces of L_A is independent of the choice of l (up to a projective transformation) and is called the *link of the vertex v_i in the link L_A*.

Remark 8.10. Let v_i be the vertex of the triangle T_A opposite to its edge $x_i = 0$. Denote by N_i the projection $\pi_{x_i}\Gamma_+(f)$ along O_i. Notice that for almost every $c \in \mathbb{C}$, the polynomial $f(x_1, \dots, x_{i-1}, x_i - c, x_{i+1}, \dots, x_n)$ has N_i as Newton polyhedron and is non-degenerate by Proposition 7.2.

(1) If the vertex v_i as a piece of the subdivision L_A corresponds to a bounded edge of $\Gamma_+(f)$, then no pieces of the link L_A correspond to at least i-faces. In this case the point $\pi_{x_i}(A)$ is not a vertex of N_i.
(2) Otherwise, the pieces of the link L_A, containing v_i, are exactly the pieces corresponding to at least i-faces. Then the point $\pi_{x_i}(A)$ is a vertex A_i, whose link is (projectively) isomorphic to a subdivision of the link of v_i in L_A.

See Figure 8 for 3-dimensional examples of both cases. The preceding proposition extends to I-faces as follows. We shall refer to the links of I-faces in the link L_A as *I-pieces* of the link L_A.

Corollary 8.11. *The union of the relative interiors for all exactly I-pieces of the link L_A is contractible, and the union of the relative interiors for all at most I-pieces of the link L_A is closed.*

We get the first result about the combinatorial configuration locally at the V-vertex A.

Lemma 8.12. *If A contributes to the monodromy eigenvalue t_0, and t_0 is not a nearby eigenvalue outside the origin, then there are two possibilities: either*

- *A is contained in a unique facet outside the coordinate planes, and this facet is an $\{i, j\}$-facet, or*
- *A is contained in no I-faces for $|I| > 1$, and in at most one i-facet for every $i \in \{2, 3, 4\}$.*

Proof. We discuss only faces containing A and only pieces of the link L_A.

(0) If there is an $\{i, j, k\}$-facet, then t_0 is a nearby monodromy eigenvalue at every point of the $\{i, j, k\}$-coordinate hyperplane.

(1) If there is more than one 2-dimensional at least i-piece in the link, then, by Remark 8.10, the (family of the) vertex $\pi_{x_i}(A)$ of the polyhedron N_i satisfies the assumption of Theorem 8.5 (for any triangulation of compact faces of N_i).

(2) Assume there is an $\{i, j\}$-piece in the link such that at least one of its edges is not contained in the boundary of T_A and is disjoint from its vertices. Then the 2-dimensional

face $F \subset \Gamma_+(f)$ corresponding to this edge satisfies the assumptions of Theorem 7.6 by Remark 7.7.

(3) Assume there is an $\{i, j\}$-piece in the link such that none of its edges satisfies the condition requested in (2). If it is the unique 2-dimensional piece in the link L_A, then we arrive at situation (0); otherwise we arrive at situation (1) (possibly with j instead of i). ■

8.3. Dimension 4: Triangulating

In this subsection we continue exploiting Proposition 7.2 to get further information on the combinatorial structure of the link L_A. We shall investigate in particular the cases when t_0 is a monodromy eigenvalue of the singularity of f at some point of the i-th coordinate axis (although not at the origin). The Newton polyhedron of such a singularity equals $\mathbb{R}_+ \times N_i$ (see Remark 8.10 for notation), so we could apply the 3-dimensional Theorem 8.5 to its analysis. However, for that theorem, we need triangulations of the bounded faces of the polyhedra N_i.

The preceding Lemma 8.12 will play a crucial role. Indeed, under its assumption, every triangulation $\mathcal{T}$ of the compact faces of the Newton polyhedron $\Gamma_+(f)$ "naturally" (see Remark 8.13) induces a triangulation of the link L_A of a V-vertex A, contributing to the eigenvalue t_0:

(1) Assuming $A = (*, 0, 0, 0)$, take the isomorphic images of simplices of $\mathcal{T}$ intersected with the hyperplane $x_2 + x_3 + x_4 = \varepsilon$ under the projection π_{x_1} for small ε.

(2) Subdivide every i-piece (Definition 8.8) of the link L_A by several segments from the vertex v_i, so that the resulting pieces together with the ones from (1) form a triangulation of T_A.

This triangulation will be denoted by $\tilde{L}_A$. (If the link L_A contains a unique 2-dimensional piece corresponding to an $\{i, j\}$-facet, we triangulate it trivially.)

Remark 8.13. (1) The triangulation $\tilde{L}_A$ is natural in the sense that, in the notation of Remark 8.10, it agrees with the corresponding triangulations of the projection polyhedra N_i. More specifically, every compact face of N_i is the projection of one compact face of $\Gamma_+(f)$, so the triangulation $\mathcal{T}$ of $\Gamma_+(f)$ induces a triangulation $\mathcal{T}_i$ of the compact faces of N_i. Assume that the link L_A contains an i-piece. Then the $\mathcal{T}_i$-triangulated link of A_i in N_i is affinely isomorphic to the triangulated link of v_i in $\tilde{L}_A$. This is an important refinement of Remark 8.10, as we shall see later in Lemma 8.16.

(2) No triangulation of the link L_A may be natural in the above sense in the presence of $\{i, j\}$-pieces with edges outside the boundary of T_A. So we really have to work under the assumptions of Lemma 8.12 in this subsection. In particular, we have no natural notion of a link triangulation $\tilde{L}_A$ associated to $\mathcal{T}$ for a V-vertex A that does not contribute to the eigenvalue t_0.

In what follows, we refer to the simplices in the triangulation $\mathcal{T}$ as V-simplices or just V-faces, because we shall not be interested in faces of $\Gamma_+(f)$ in the usual sense anymore.

We will now continue to study how the combinatorics of $\Gamma_+(f)$ ensures the existence of a nearby monodromy eigenvalue t_0 outside the origin, but, this time, taking into account the chosen triangulation $\mathcal{T}$.

We will choose once and for all a triangulation $\mathcal{T}$ and corresponding link triangulations $\tilde{L}_A$ according to the following lemma.

Lemma 8.14. *If s_0 is a pole of the topological zeta function, then there exists a triangulation $\mathcal{T}$ (with no new vertices) of the Newton polyhedron $\Gamma_+(f)$ such that either it has a non-B-simplex (see Definition 3.10) contributing to the eigenvalue $t_0 = \exp 2\pi i s_0$, or a B-border, whose V-edge contributes to the eigenvalue t_0.*

Proof. Since $\Gamma_+(f)$ does not satisfy the assumptions of Theorem 5.2 for s_0, it contains either a B-border contributing to the eigenvalue t_0, or a non-B-facet, contributing the pole s_0. In the first case, the V-edge of the border contributes to the sought eigenvalue (see the proof of Lemma 8.31). In the second case, triangulate the contributing non-B-facet by the use of Lemma 5.18 (so that one of the resulting non-B-simplices contributes to the sought eigenvalue) and extend this triangulation arbitrarily to the whole $\Gamma_+(f)$. ■

The absence of the nearby monodromy eigenvalue t_0 outside the origin imposes lots of restrictions on combinatorics of the Newton polyhedron $\Gamma_+(f)$ and the triangulation $\mathcal{T}$. Let us use these restrictions to make some crucial conclusions about the combinatorics of the triangulated links of the V-vertices.

Definition 8.15. The vertex v_i of the triangle T_A opposite to the edge $x_i = 0$ is said to be the *i-th corner* of the link L_A, if the link triangulation $\tilde{L}_A$ contains a piece of the form $v_i BC$, where B and C are points on the two edges of T_A containing v_i. Its star is the set of four pieces, $v_i BC, v_i B, v_i C, v_i$.

Lemma 8.16. *If A contributes to the eigenvalue t_0 and t_0 is not a nearby eigenvalue outside the origin, then for every $i \in \{2, 3, 4\}$, either the edge of $\Gamma_+(f)$ corresponding to the vertex $v_i \in T_A$ is bounded and is not a corner of a V-facet, or the link L_A has the i-th corner (which may correspond to an i-edge or to a bounded V-face of $\Gamma_+(f)$). In particular, there are six alternatives for the vertex A in this situation:*

- *The vertex A is a corner of a V-facet.*
- *The vertex A is a corner of a V-edge and is contained in a unique facet which is an $\{i, j\}$-facet.*
- *The vertex A is contained in a V-edge that is not a corner of a V-facet, and the link L_A has two corners.*
- *The vertex A is contained in two V-edges that are not corners of V-facets, and the link L_A has one corner.*
- *The vertex L_A is contained in three V-edges that are not corners of V-facets.*
- *The link L_A has three corners.*

Proof. If we exclude the first two cases from our consideration, then, by Lemma 8.12, there is at most one 2-dimensional i-piece in the link L_A for $i \in \{2, 3, 4\}$. However, it still could contain more than one 2-dimensional i-piece of the triangulated link $\tilde{L}_A$. Once we exclude this possibility, we prove the lemma.

So assume towards a contradiction that the unique i-piece of the link L_A (corresponding to an i-facet $\tau \ni A$) contains

(a) no (i, j)-pieces and

(b) at least two i-pieces of the triangulated link $\tilde{L}_A$ (corresponding to compact triangles $\tau_k \ni A$ in the boundary of τ).

From this, we shall conclude that Theorem 8.5 is applicable to the family $F = \{A_i\}$ in the polyhedron N_i with the triangulation $\mathcal{T}_i$ (in the notation of Remarks 8.10 and 8.13), because this family cannot fall within the "unless" case that we exclude in the statement of Theorem 8.5. Once we prove this, Theorem 8.5 ensures that t_0 is a nearby eigenvalue on the i-th coordinate axis outside the origin, which contradicts our assumption.

It remains to deduce from (a) and (b) above that the family F does not fall within the case that we exclude in the statement of Theorem 8.5. Indeed, (a) implies that all faces of N_i which contain the vertex A_i and are not contained in a coordinate plane, are compact. And (b) ensures that such faces contain at least two triangles with vertex A_i in the triangulation $\mathcal{T}_i$: these are the projections of τ_k along the i-th coordinate axis.

Thus all pieces of the triangulation $\mathcal{T}_i$ containing A_i cannot belong to the same family (because every family contains at most one triangle). ■

We will need the following combinatorial observation, applicable to the conclusion of Lemma 8.16. Let $\tilde{L}$ be a triangulation of a triangle T.

Definition 8.17. A triangle of $\tilde{L}$ is said to be *interior* if none of its edges is contained in the edges of T, and no one of its vertices is a vertex of T. An interior triangle of $\tilde{L}$ is said to be *inscribed* if its three vertices are in the interior of the three edges of T. A vertex of T is called a *corner* of the triangulation $\tilde{L}$ if it is a vertex of only one of the triangles in the triangulation.

Lemma 8.18. (1) *If the triangulation $\tilde{L}$ of a triangle T has three corners, then either it has an inscribed triangle, or it has at least three interior triangles.*

(2) *If the triangulation $\tilde{L}$ has a corner, and no edge of the triangulation connects a vertex of T to its opposite edge, then T has an interior triangle.*

Informally speaking, this means that, in the setting of Lemma 8.16, the link looks similarly to one of the six examples in Figure 9. The pieces of the link that may correspond

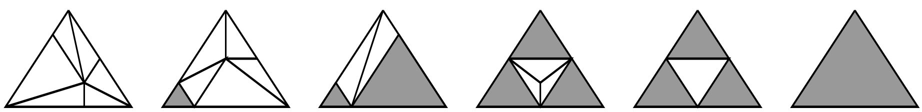

Fig. 9. Examples of links of vertices in the setting of Lemma 8.16.

to i-facets for some i are shown in grey; the white area may be subdivided into pieces in a more complicated way than shown in the picture.

Proof of Lemma 8.18. The proof of part (1) proceeds by induction on the number of triangles in the triangulation. If any corner triangle can be glued with its (unique) adjacent triangle into a larger triangle, then glue and apply the induction hypothesis. Otherwise each of the three corner triangles is adjacent to some interior triangle T_i. If $T_i = T_j$, then its vertices are in the interior of the three edges of T, and otherwise T_i are three different interior triangles.

Part (2) is proved in the same way. ∎

From Proposition 7.2 we can also deduce the following result.

Lemma 8.19. *If a V-triangle $F \subset \Gamma_+(f)$ is not contained in a V-facet, and its family* (*see Definition* 8.2) *contributes to the eigenvalue t_0, then t_0 is a nearby monodromy eigenvalue outside the origin.*

Proof. Under these assumptions, if F is in the i-th coordinate hyperplane, then $\pi_{x_i} F$ is a facet of $N_i = \pi_{x_i}\Gamma_+(f)$, whose family contributes to t_0, so t_0 is a nearby monodromy eigenvalue at a point of the i-th coordinate hyperplane by Theorem 8.5. ∎

8.4. *Monodromy conjecture for non-degenerate singularities of four variables*

We now prove the monodromy conjecture for non-degenerate singularities of four variables, similarly to Theorem 8.5. Recall our setting.

Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ be non-degenerate at the origin. Let s_0 be a pole of the topological zeta function of f and set $t_0 = \exp(2\pi i s_0)$. Analogously to the proof of Theorem 8.5, we choose once and for all a triangulation $\mathcal{T}$ of the Newton polyhedron $\Gamma_+(f)$ in accordance with Lemma 8.14, and we will define the register of $\Gamma_+(f)$ as a disjoint union of certain groups of families. It will be practical to work with what we call extended families.

Definition 8.20. The *extended family* of a V-simplex F is the set of all V-faces from the family of F and, in the case of $\dim F = 3$, also the V-vertex of F whenever it is a maximal (under inclusion) V-face of F. In this case, F has no other V-faces except for, maybe, the facet of F opposite to the V-vertex (notice that if A and PQ are V-faces in a facet $APQR$, then APQ is also a V-face).

Remark 8.21. A V-vertex A is in the extended family of a V-tetrahedron τ if and only if the image of τ in the triangulated link $\tilde{L}_A$ is an interior triangle or coincides with T_A.

Definition 8.22. The *register* $\mathcal{R}$ of $\Gamma_+(f)$ (depending on the chosen triangulation) is the set of the following extended families:

- extended families of all 3-dimensional V-simplices;
- extended families of all 2-dimensional V-simplices that do not enter the aforementioned extended families;

– extended families of all 1-dimensional V-simplices that do not enter the aforementioned extended families.

Remark 8.23. Notice that every positive-dimensional V-simplex enters exactly one extended family of the register. This is obvious for simplices of dimensions 2 and 3, and a V-edge E may enter families of two different V-triangles, but in this case, by the subsequent Lemma 8.24, both of these triangles are themselves in families of V-tetrahedra τ_1 and τ_2, so their own families are not in the register. Thus the extended family of E is itself in the register for $\tau_1 \neq \tau_2$ (because E is not a corner of any tetrahedron in this case), and E is in the extended family of $\tau_1 = \tau_2$ otherwise.

Lemma 8.24. *If a bounded $(n-3)$-dimensional V-face ν of a Newton polyhedron in $\mathbb{R}^n$ is contained in two bounded $(n-2)$-dimensional V-faces ν_1 and ν_2, then each of these faces is contained in a bounded V-facet.*

Proof. This is obvious for $n = 3$, and the general case reduces to $n = 3$ by taking the projection of the Newton polyhedron along the affine span of ν. ∎

We will now prove the monodromy conjecture for non-degenerate singularities of four variables, modulo several lemmas in the next subsection regarding certain exotic families.

Theorem 8.25. *Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$ be non-degenerate at the origin. Let $s_0 \neq -1$ be a pole of the topological zeta function of f and set $t_0 = \exp(2\pi i s_0)$. If t_0 is not a (tropical) nearby monodromy eigenvalue outside the origin, then t_0 is a root of the monodromy zeta function of f at the origin, and hence a monodromy eigenvalue.*

Proof. Let $\mathcal{T}$ be a triangulation of the Newton polyhedron $\Gamma_+(f)$, according to Lemma 8.14, and let us induce from $\mathcal{T}$ the corresponding link triangulations $\tilde{L}_v$ for every V-vertex v, contributing to the eigenvalue t_0 (see the beginning of the preceding subsection).

For every V-vertex v, denote by $r(v)$ the number of extended families containing v in the register $\mathcal{R}$.

By Remark 8.23, we represent the multiplicity of the candidate root $t_0 = \exp(2\pi i s_0)$ of the monodromy zeta function as

$$\sum_{\mathcal{F}} (\text{contribution of } \mathcal{F} \text{ to the multiplicity of } t_0) + \sum_{v} (r(v) - 1), \qquad (*)$$

where $\mathcal{F}$ runs over the register, and v runs over V-vertices, contributing to t_0.

We will prove that every term in every sum of $(*)$ is non-negative, and moreover at least one term in $(*)$ is strictly positive.

In our setting we have:

1. The contribution of every extended family $\mathcal{F} \in \mathcal{R}$ to the multiplicity of t_0 is non-negative. For 3-dimensional extended families, this follows from Theorem 6.4 and Lemma 8.28 in the next subsection. Note that there are no 2-dimensional families contributing to

t_0 by Lemma 8.19. As $s_0 \neq -1$, the contribution of every 1-dimensional family to t_0 is also non-negative (see for example [18, proof of Proposition 5]).

2. For every V-vertex A contributing to the eigenvalue t_0, we have $r(A) \geq 1$. This is because, by Lemmas 8.16 and 8.18 (1), the vertex A is contained in one of the following:

- a V-tetrahedron for which A is a corner;
- a V-edge whose extended family is in the register;
- a V-tetrahedron whose image in the triangulated link $\tilde{L}_A$ is interior, and whose extended family thus contains A.

3. The contribution of at least one odd-dimensional extended family $\mathcal{F} \in \mathcal{R}$ is positive, or $r(v) > 1$ for some V-vertex v. To see this, consider the following possible cases:

- If Lemma 8.14 provides a B-border AA_1A_2 contained in two V-tetrahedra, whose candidate pole of the topological zeta function equals s_0, then we have two subcases for its V-edge A_1A_2:
  - Assume for every A_i the following: if it is a V-vertex contributing to t_0, then the edge AA_i is in a coordinate plane. Under this assumption, the contribution of the family of the V-edge A_1A_2 of this border is positive by Lemma 8.31 below.
  - Assume that some A_i (say, A_1) fails the preceding assumption. Then, in the triangulated link $\tilde{L}_{A_1}$, exactly one interior segment contains the vertex v corresponding to the V-edge A_1A_2: this segment corresponds to the border triangle AA_1A_2 and does not split the link by our assumption. Now we have again two subcases:
    - $\tilde{L}_{A_1}$ has exactly two corners. Then Lemma 8.18 (2) applies to $\tilde{L}_{A_1}$ and provides an interior triangle. The V-vertex A_1 is then contained in the extended families of the V-edge A_1A_2 and the V-tetrahedron, corresponding to the interior triangle, so $r(A_1) > 1$ (both of these families are on the register, since A_1A_2 is not a corner).
    - $\tilde{L}_{A_1}$ has at most one corner. Then A_1 is contained by Lemma 8.16 in the extended families of two V-edges which are not corners, hence on the register, so $r(A_1) > 1$.
- If Lemma 8.14 provides a contributing V-tetrahedron whose extended family coincides with the usual family, then the contribution of this family is positive by Theorem 6.4.
- If Lemma 8.14 provides a V-tetrahedron F for which the candidate pole is s_0, and whose extended family consists of the family of F and one additional contributing V-vertex A, then by Lemma 8.16 we have two subcases for the triangulated link $\tilde{L}_A$:
  - One of the vertices of $\tilde{L}_A$ corresponds to a V-edge whose extended family contains A and is contained in the register. In this case $r(A) \geq 2$, because A is also in the extended family of F.
  - There are three corners in $\tilde{L}_A$. This case subdivides into the following subcases by Lemma 8.18 (1):
    - There are three interior triangles in $\tilde{L}_A$. Then $r(A) > 2 > 1$.
    - There is an inscribed triangle in $\tilde{L}_A$, and it is not the image of F. Then $r(A) > 1$.

– There is an inscribed triangle in $\tilde{L}_A$, and it is the image of F. Then Lemma 8.26 or 8.29 from the next subsection applies to F, so its extended family has a positive contribution to the multiplicity of t_0.

We conclude that t_0 is a root of the monodromy zeta function and, in particular, a monodromy eigenvalue of the singularity f at the origin. ∎

8.5. *Exotic families*

In the course of the proof of the monodromy conjecture for $n = 4$, we encountered certain exotic families of V-faces, whose contributions to the multiplicity of the corresponding monodromy eigenvalue ought to be non-zero. Their contributions are estimated in this subsection.

For the most part (Lemmas 8.26–8.29), we will study the extended family of a facet $\tau = ABCD$ if it does not coincide with the family of τ (i.e. consists of a V-vertex $A \prec \tau$ and possibly its opposite triangular face whenever it is a V-face), and often moreover assume that τ defines an inscribed triangle (Definition 8.17) in the link of A.

Lemma 8.26. *Let $\tau = ABCD$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ such that $v = A(0,0,0,\alpha)$ is its only proper V-face contributing to $t_0 = e^{-2\pi i \nu(\tau)/N(\tau)}$. If $B(\beta_1, \beta_2, \beta_3, \beta_4)$, $C(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and $D(\delta_1, \delta_2, \delta_3, \delta_4)$ with $\beta_2 = \gamma_3 = \delta_1 = 0$, then* $\mathrm{Vol}_{\mathbb{Z}}(\tau) \neq 1$.

Proof. Assuming to the contrary that $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 1$, the vector product of AB, AC, AD is a primitive vector (a, b, c, d), normal to τ:

$$\begin{aligned}
&\mathrm{aff}(\tau) : ax_1 + bx_2 + cx_3 + dx_4 = N, \quad \text{with} \\
&a = \gamma_2\delta_3(\alpha - \beta_4) + \delta_2\beta_3(\alpha - \gamma_4) - \beta_3\gamma_2(\alpha - \delta_4), \\
&b = \delta_3\beta_1(\alpha - \gamma_4) + \beta_3\gamma_1(\alpha - \delta_4) - \gamma_1\delta_3(\alpha - \beta_4), \\
&c = \beta_1\gamma_2(\alpha - \delta_4) + \gamma_1\delta_2(\alpha - \beta_4) - \delta_2\beta_1(\alpha - \gamma_4), \\
&d = \beta_1\gamma_2\delta_3 + \gamma_1\delta_2\beta_3 \\
&N = \alpha d.
\end{aligned}$$

Since A contributes to t_0, we have $d \mid (a + b + c + d)$. Let $k \in \mathbb{Z}$ be such that $a + b + c + d = (-k + 1)d$ and let M be the matrix

$$\begin{pmatrix} \beta_1 & 0 & \beta_3 & \beta_4 - \alpha \\ \gamma_1 & \gamma_2 & 0 & \gamma_4 - \alpha \\ 0 & \delta_2 & \delta_3 & \delta_4 - \alpha \\ 1 & 1 & 1 & k \end{pmatrix}.$$

Then $M^t(a, b, c, d) = {}^t0$. By Lemma 9.5 it follows that d divides the minors M_4^i, $1 \leq i \leq 4$, which are

$$M_4^1 = \gamma_1\delta_2 + \gamma_2\delta_3 - \gamma_1\delta_3,$$

$$M_4^2 = \delta_3\beta_1 + \delta_2\beta_3 - \delta_2\beta_1,$$
$$M_4^3 = \beta_1\gamma_2 + \beta_3\gamma_1 - \beta_3\gamma_2,$$
$$M_4^4 = d.$$

As A is a maximal (under inclusion) proper V-face of τ, none of the numbers β_1, β_3, $\gamma_1, \gamma_2, \delta_2, \delta_3$ equals 0, and then obviously M_4^1, M_4^2, and M_4^3 are strictly less than d. As

$$\frac{M_4^1}{\gamma_1\delta_3} + \frac{M_4^2}{\delta_2\beta_1} + \frac{M_4^3}{\beta_3\gamma_2} = \left(\frac{\delta_2}{\delta_3} + \frac{\delta_3}{\delta_2}\right) + \left(\frac{\gamma_2}{\gamma_1} + \frac{\gamma_1}{\gamma_2}\right) + \left(\frac{\beta_3}{\beta_1} + \frac{\beta_1}{\beta_3}\right) - 3 > 0,$$

it follows that at least one of the minors M_4^1, M_4^2, or M_4^3 is strictly positive, which contradicts the fact that d divides the minors M_4^i, $1 \le i \le 4$. ■

Lemma 8.27. *Let $\tau = APQR$ with $A(2,0,0,0)$, $P(0,0,p_2,p_3)$, $Q(0,q_1,0,q_3)$, $R(0,r_1,r_2,0)$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ contributing to $t_0 = e^{-2\pi i\nu(\tau)/N(\tau)} \neq 1$. Assume that $v = A$ and $\sigma = PQR$ are the only proper V-faces in τ contributing to t_0. Then* $(\mathrm{Vol}_{\mathbb{Z}}(\sigma), \mathrm{Vol}_{\mathbb{Z}}(\tau)) \neq (1, 2)$.

Proof. Suppose that $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 2$ and $\mathrm{Vol}_{\mathbb{Z}}(\sigma) = 1$. Let

$$\mathrm{aff}(\tau):\ ax_1 + bx_2 + cx_3 + dx_4 = N(\tau) = 2a$$

be the equation of $\mathrm{aff}(\tau)$ with $\mathrm{GCD}(a,b,c,d) = 1$. One has $\frac{N(\tau)}{\mathrm{GCD}(b,c,d)} = N(\sigma)$ and as σ contributes to t_0, we have

$$\frac{\nu(\tau)}{N(\tau)}N(\sigma) = \frac{a+b+c+d}{N(\tau)}N(\sigma) \in \mathbb{Z}.$$

This implies that $\mathrm{GCD}(b,c,d)$ divides a. As $\mathrm{GCD}(a,b,c,d) = 1$, we get $\mathrm{GCD}(b,c,d) = 1$. Since the V-face $v = A$ contributes to t_0, we also have $a \mid (b+c+d)$. Then by $N(\tau) = 2a$, we obtain $t_0 = 1$ (which is excluded) or $t_0 = -1$. We study what happens when $t_0 = -1$. This implies that $2a \mid (b+c+d)$. As $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 2$, by Proposition 9.2 (2) the even integers $2a, 2b, 2c$ and $2d$ are the 3×3 minors of the matrix

$$\begin{pmatrix} PA \\ QA \\ RA \end{pmatrix} = \begin{pmatrix} -2 & 0 & p_2 & p_3 \\ -2 & q_1 & 0 & q_3 \\ -2 & r_1 & r_2 & 0 \end{pmatrix},$$

and hence the expressions for a, b, c and d become

$$a = \frac{q_1r_2p_3 + r_1p_2q_3}{2}, \qquad b = r_2p_3 + p_2q_3 - q_3r_2,$$
$$c = p_3q_1 + q_3r_1 - r_1p_3, \quad d = q_1r_2 + r_1p_2 - p_2q_1.$$

For the integer $k = (b+c+d)/a$ the vector $(-k,1,1,1)$ is a rational linear combination of $\overrightarrow{AP}, \overrightarrow{AQ}$ and $\overrightarrow{AR}$, because $(-k,1,1,1)\cdot(a,b,c,d) = 0$, and $\overrightarrow{AP}, \overrightarrow{AQ}$ and $\overrightarrow{AR}$ generate the orthogonal complement to (a,b,c,d).

Let $H \simeq \mathbb{R}^3$ be the affine hyperplane in $\mathbb{R}^4$ containing τ and consider the lattice $L := H \cap \mathbb{Z}^4 \simeq \mathbb{Z}^3$ in it. For the integer $k = (b+c+d)/a$, we have $(-k, 1, 1, 1) \in L$. Since the normalized volume of τ is 2, the sublattice K of L generated by the three vectors $\overrightarrow{AP}$, $\overrightarrow{AQ}$ and $\overrightarrow{AR}$ is of index 2 in L, i.e. $[L : K] = 2$.

This means there exist integers x, y, z such that

$$\begin{pmatrix} -2 & -2 & -2 \\ 0 & q_1 & r_1 \\ p_2 & 0 & r_2 \\ p_3 & q_3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2k \\ 2 \\ 2 \\ 2 \end{pmatrix}. \tag{8.1}$$

We define

$$M = \begin{pmatrix} 0 & q_1 & r_1 \\ p_2 & 0 & r_2 \\ p_3 & q_3 & 0 \end{pmatrix}.$$

Then by Cramer's rule we find that

$$x = \frac{2(r_2q_1 + r_1q_3 - r_2q_3)}{\det(M)}, \quad y = \frac{2(r_2p_3 + r_1p_2 - r_1p_3)}{\det(M)},$$
$$z = \frac{2(q_1p_3 + p_2q_3 - p_2q_1)}{\det(M)}.$$

Now we study the possible signs of x, y and z. If $p_2 \geq p_3$ and $q_1 \geq q_3$, then $y, x > 0$. If $p_2 \geq p_3$ and $q_1 \leq q_3$, then $y, z > 0$. If $p_2 \leq p_3$ and $r_1 \leq r_2$, then $z, y > 0$ and so on. Thus we find that at least two of the integers x, y and z are always positive. By permuting them, we may assume that $x, y > 0$. As none of $p_2, p_3, q_1, q_3, r_1, r_2$ is equal to 0, the equation $p_3x + q_3y = 2$ obtained by (8.1) implies that $p_3 = q_3 = 1$ and $x = y = 1$. Consequently,

$$a = \frac{q_1r_2 + r_1p_2}{2}, \quad b = p_2, \quad c = q_1, \quad d = q_1r_2 + r_1p_2 - p_2q_1$$

and $\det(M) = q_1r_2 + r_1p_2$. As we have supposed that $2a \mid (b+c+d)$, we have

$$(q_1r_2 + r_1p_2) \mid (p_2 + q_1 + q_1r_2 + r_1p_2 - p_2q_1) \iff \det(M) | (p_2 + q_1 - p_2q_1)$$

and z is an even integer. Hence, again by (8.1) and since $x = y = 1$, we find that p_2 and q_1 should be even. Then

$$\mathrm{GCD}(b, c, d) = \mathrm{GCD}(p_2, q_1, q_1r_2 + r_1p_2 - p_2q_1) \geq 2,$$

which contradicts $\mathrm{GCD}(b, c, d) = 1$. ∎

Recall that, for a V-face τ, we define

$$\zeta_\tau(t) = (1 - t^{N(\tau)})^{\mathrm{Vol}_{\mathbb{Z}}(\tau)} \in \mathbb{C}[t].$$

Lemma 8.28. (1) *Let $\tau = APQR$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$ such that $v = A(\alpha, 0, 0, 0)$ and $\sigma = PQR$ are V-faces. Then*

$$\frac{\zeta_\tau(t) \cdot (1-t)}{\zeta_v(t) \cdot \zeta_\sigma(t)}$$

is a polynomial.

(2) *If $\mathrm{Vol}_{\mathbb{Z}}(\tau) = \mathrm{Vol}_{\mathbb{Z}}(\sigma)$, then 1 is the only common root of the polynomials in the denominator.*

Proof. If $\mathrm{Vol}_{\mathbb{Z}}(\tau) > \mathrm{Vol}_{\mathbb{Z}}(\sigma)$ then the assertion is obvious. So suppose that $\mathrm{Vol}_{\mathbb{Z}}(\tau) = \mathrm{Vol}_{\mathbb{Z}}(\sigma)$. Let

$$\mathrm{aff}(\tau) : \; ax_1 + bx_2 + cx_3 + dx_4 = N(\tau)$$

be the equation of $\mathrm{aff}(\tau)$ with $\mathrm{GCD}(a, b, c, d) = 1$. Since

$$N(v) = \alpha = \frac{N(\tau)}{a}, \quad N(\sigma) = \frac{N(\tau)}{\mathrm{GCD}(b, c, d)}$$

and $\mathrm{GCD}(a, b, c, d) = 1$, the equivalent (by Proposition 9.2 and Lemma 9.4) conditions

$$\mathrm{Vol}_{\mathbb{Z}}(\tau) = \mathrm{Vol}_{\mathbb{Z}}(\sigma) \iff \mathrm{GCD}(N(v), N(\sigma)) = 1$$

imply that $N(\tau) = a \cdot \mathrm{GCD}(b, c, d)$, $N(\sigma) = a$ and $N(v) = \alpha = \mathrm{GCD}(b, c, d)$. Hence

$$\frac{\zeta_\tau(t)}{\zeta_v(t) \cdot \zeta_\sigma(t)} = \frac{(1 - t^{a \cdot \alpha})^{\mathrm{Vol}_{\mathbb{Z}}(\sigma)}}{(1 - t^\alpha) \cdot (1 - t^a)^{\mathrm{Vol}_{\mathbb{Z}}(\sigma)}}.$$

The only common zero of $\zeta_v(t)$ and $\zeta_\sigma(t)$ is 1, because $\mathrm{GCD}(a, \alpha) = 1$. Thus the denominator divides the numerator. ∎

Lemma 8.29. *Let $\tau = APQR$ with $A(\alpha, 0, 0, 0)$, $P(0, 0, p_2, p_3)$, $Q(0, q_1, 0, q_3)$, $R(0, r_1, r_2, 0)$ be a 3-dimensional lattice simplex in a compact facet of $\Gamma_+(f)$. Assume that $v = A$ and $\sigma = PQR$ are its proper V-faces contributing to $t_0 = e^{-2\pi i \nu(\tau)/N(\tau)} \neq 1$. Then for the polynomial*

$$F(t) = \frac{\zeta_\tau(t) \cdot (1-t)}{\zeta_v(t) \cdot \zeta_\sigma(t)} \in \mathbb{C}[t]$$

(*see Lemma 8.28*) *we have $F(t_0) = 0$.*

Example 8.30. The only V-facet of the function $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^3 x_3^3 + x_2^3 x_4^3 + x_3^3 x_4^3$ has the extended family as in Lemma 8.29 and contributes the pole $s_0 = -3/4$. One easily checks that the multiplicity of the corresponding monodromy eigenvalue $t_0 = e^{-8\pi i/3}$ equals $18 - 9 + 1 = 10 \neq 0$.

Proof of Lemma 8.29. Writing $F(t)$ by the definition as

$$F(t) = \frac{(1 - t^{N(\tau)})^{\mathrm{Vol}_{\mathbb{Z}}(\tau)} \cdot (1-t)}{(1 - t^{N(v)})^1 \cdot (1 - t^{N(\sigma)})^{\mathrm{Vol}_{\mathbb{Z}}(\sigma)}},$$

the statement is obvious if $\mathrm{Vol}_{\mathbb{Z}}(\tau) > \mathrm{Vol}_{\mathbb{Z}}(\sigma) + 1$. Since $\mathrm{Vol}_{\mathbb{Z}}(\sigma)$ divides $\mathrm{Vol}_{\mathbb{Z}}(\tau)$ by Proposition 9.2, the only exceptions from the first inequality would be the cases

(1) $\mathrm{Vol}_{\mathbb{Z}}(\tau) = \mathrm{Vol}_{\mathbb{Z}}(\sigma)$ or

(2) $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 2$ and $\mathrm{Vol}_{\mathbb{Z}}(\sigma) = 1$.

In the first case the sought statement follows from Lemma 8.28 (2). Suppose now that $\mathrm{Vol}_{\mathbb{Z}}(\tau) = 2$ and $\mathrm{Vol}_{\mathbb{Z}}(\sigma) = 1$. Let

$$\mathrm{aff}(\tau): \; ax_1 + bx_2 + cx_3 + dx_4 = N(\tau) = a \cdot \alpha$$

with $\mathrm{GCD}(a, b, c, d) = 1$. If the V-face σ does not contribute to t_0, then clearly $F(t_0) = 0$. If it does, then

$$N(\sigma)\frac{a + b + c + d}{N(\tau)} \in \mathbb{Z}.$$

As $N(\sigma) = N(\tau)/\mathrm{GCD}(b, c, d)$, this would imply that

$$\frac{a + b + c + d}{\mathrm{GCD}(b, c, d)} \in \mathbb{Z}.$$

As $\mathrm{GCD}(a, b, c, d) = 1$, one gets $\mathrm{GCD}(b, c, d) = 1$ and hence $N(\sigma) = N(\tau)$. As

$$\mathrm{Vol}_{\mathbb{Z}}(\tau) = \frac{\alpha \cdot \det(M)}{N(\tau)} = 2 \quad \text{and} \quad \mathrm{Vol}_{\mathbb{Z}}(\sigma) = \frac{\det(M)}{N(\sigma)} = 1,$$

where

$$M = \begin{pmatrix} 0 & q_1 & r_1 \\ p_2 & 0 & r_2 \\ p_3 & q_3 & 0 \end{pmatrix},$$

we find $\alpha = 2$. This case is excluded by Lemma 8.27. ∎

It now remains to study the contributions of B-borders (Definition 5.1), generalizing the proof of [18, Theorem 15].

Lemma 8.31. *Assume that $f(x) \in \mathbb{C}[x_1, \ldots, x_4]$ is non-degenerate at $0 \in \mathbb{C}^4$. Let τ_1 and τ_2 be compact B-facets in $\Gamma_+(f)$ intersecting in a B-border AA_1A_2 with a V-edge A_1A_2, and contributing the same candidate pole $s_0 \neq 1$.*

Assume additionally for every A_i the following: if it is a V-vertex, contributing to the eigenvalue $t_0 = \exp(-2\pi i s_0)$, then the edge AA_i is in a coordinate plane.

Then the contribution of the family of the V-edge A_1A_2 of this border to the multiplicity of the corresponding eigenvalue $t_0 = \exp(-2\pi i s_0)$ is positive, i.e. for the polynomial

$$F(t) = \frac{\zeta_{A_1A_2}(t) \cdot (1 - t)}{\zeta_{A_1}(t) \cdot \zeta_{A_2}(t)} \in \mathbb{C}[t]$$

we have $F(t_0) = 0$.

Proof. Without loss of generality, we can assume that both τ_1 and τ_2 are compact simplicial B_1-facets: indeed, if τ_i is a B_2-facet, then it contains a B_1-tetrahedron with the same B-border, and we can consider this B_1-tetrahedron instead of τ_i.

We rename $\tau_1 = ABCD$ and $\tau_2 = ABCE$ so that $A = (1, 1, \alpha_3, \alpha_4)$, $B = (0, 0, \beta_3, \beta_4)$, $C = (0, 0, \gamma_3, \gamma_4)$, $D = (0, \delta_2, \delta_3, \delta_4)$ and $E = (\varepsilon_1, 0, \varepsilon_3, \varepsilon_4)$. By computing the equation of the affine space passing through A, B, C and D, we compute the candidate pole s_0 contributed by τ_1 and τ_2:

$$s_0 = \frac{(\gamma_4 - \beta_4)(\alpha_3 - \beta_3 - 1) + (\beta_3 - \gamma_3)(\alpha_4 - \beta_4 - 1)}{\beta_3(\gamma_4 - \beta_4) + \beta_4(\beta_3 - \gamma_3)}.$$

As the lattice index of the V-segment BC is equal to the absolute value of

$$\frac{\beta_3(\gamma_4 - \beta_4) + \beta_4(\beta_3 - \gamma_3)}{\mathrm{GCD}(\gamma_4 - \beta_4, \beta_3 - \gamma_3)},$$

the V-face BC contributes to t_0. If neither B nor C contributes to the eigenvalue t_0, then the statement is obvious by projecting along BC.

So first suppose that B is a contributing V-vertex and C is not. Say $\beta_3 = 0$. Then, by the additional assumption in the statement of the lemma, we have $A = (1, 1, 0, \alpha_4)$, so the affine space passing through the facet $ABCD$ has equation

$$\begin{aligned} x_1\,[\gamma_3(\delta_4 - \beta_4) - \delta_3(\gamma_4 - \beta_4) - \delta_2\gamma_3(\alpha_4 - \beta_4)] + x_2[\delta_3(\gamma_4 - \beta_4) - \gamma_3(\delta_4 - \beta_4)] \\ + x_3\,[\delta_2(\beta_4 - \gamma_4)] + x_4(\gamma_3\delta_2) = \beta_4\gamma_3\delta_2. \end{aligned}$$

The corresponding candidate pole is then

$$\frac{\gamma_3(\alpha_4 - \beta_4) + \gamma_4 - \beta_4 - \gamma_3}{\beta_4\gamma_3}.$$

As B also contributes to t_0, one sees that γ_3 divides $\beta_4 - \gamma_4$. Note that $\mathrm{Vol}_{\mathbb{Z}}(BC) = \gamma_3$ and $\gamma_3 > 1$ (otherwise we have a B^2-border).

We now suppose that both B and C are contributing V-vertices to t_0. Then, by the additional assumption in the statement of the lemma, we have $A = (1, 1, 0, 0)$, so as before one gets $\mathrm{Vol}_{\mathbb{Z}}(BC) > 1$. As C contributes to t_0, one would have a cancelation if $\mathrm{Vol}_{\mathbb{Z}}(BC) = 2$. Now

$$s_0 = \frac{-\gamma_3\beta_4 - \beta_4 - \gamma_3}{\beta_4\gamma_3}$$

and so β_4 should divide γ_3. We have $\mathrm{Vol}_{\mathbb{Z}}(BC) = \mathrm{GCD}(\beta_4, \gamma_3) = \gamma_3$, because B contributes to t_0. If $\mathrm{Vol}_{\mathbb{Z}}(BC) = 2$, then $\gamma_3 = 2$ and $\beta_4 = 2$ (otherwise we have a B^2-border), so $t_0 = 1$. ■

9. Appendix: some elements of lattice geometry

We recall some basic notions and facts about the geometry of $\mathbb{Z}^n$ that we use throughout the paper. A *latticed space* is a real affine space A with an integer lattice $L \subset A$ such that $\dim L = \dim A$. For instance:

– $\mathbb{R}^n$ is always considered a latticed space with lattice $\mathbb{Z}^n$;

– a rational affine subspace $A \subset \mathbb{R}^n$ is always considered a latticed space with lattice $A \cap \mathbb{Z}^n$;
– the quotient space of $\mathbb{R}^n$ along its rational affine subspace A (i.e. $\mathbb{R}^n/(A-a)$ for $a \in A$) is always considered a latticed space whose lattice is the image of $\mathbb{Z}^n$ under the quotient map.

A *lattice polytope* in a latticed space A is a polytope all of whose vertices belong to the lattice.

The *lattice volume form* on a latticed space A with lattice L is the volume form such that the volume of A/L equals (dim A)!, or equivalently the minimal positive volume form such that the volume of every lattice polytope in A is an integer.

A segment in a latticed space A is said to be *primitive* if its end points are the only lattice points that it contains. The *lattice distance* between two lattice points a and b is the number of primitive segments into which the lattice points subdivide the segment ab. In coordinates, the lattice distance between $a, b \in \mathbb{Z}^n$ is the GCD of the coordinates of the difference $b - a$.

The lattice distance from a lattice affine subspace $A \subset \mathbb{R}^n$ to the origin can be defined in one of the following equivalent ways:

(I) It is the lattice distance between 0 and the image of A under the projection p of $\mathbb{R}^n$ along A; in particular, if A is a hypersurface given by $a_1v_1 + \cdots + a_nv_n = q$ with coprime integer coefficients a_i and q, then the lattice distance from A to 0 equals $|q|$.

(II) It is the maximum of the lattice distances between the points of A and 0.

Remark 9.1. By the definition, the lattice distance from 0 to any point of A divides the distance from 0 to A. As a consequence, the distance from any affine subspace $A' \subset A$ divides the distance from 0 to A.

The lattice distance from a lattice polytope P in $\mathbb{R}^n$ to a lattice point a is defined as the lattice distance from the affine hull of the shifted polytope $P - a$ to the origin.

Metric computations with lattice length and distances naturally translate into the lattice setting. For instance, the following statements directly follow from their well known metric versions:

Proposition 9.2. (1) *The lattice volume of a lattice pyramid in $\mathbb{R}^n$ equals the lattice volume of its base times the lattice distance from the base to the apex.*

(2) *The lattice volume of an $(n-1)$-dimensional simplex in $\mathbb{Z}^n$ generated by vectors $v_1 \ldots, v_{n-1}$ is equal to the lattice length of the vector product $v_1 \wedge \cdots \wedge v_{n-1}$.*

Denote the lattice distance from a lattice polytope $P \in \mathbb{R}^n$ to the origin by $N(P)$.

Remark 9.3. (1) If π is the projection of $\mathbb{R}^n$ along an affine subspace of the affine hull of the polytope P, then $N(P) = N(\pi(P))$ by the definition of the lattice distance.

(2) If j is the embedding $\mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^m$, then $N(P) = N(j(P))$.

Lemma 9.4. (1) *For a lattice polytope $\tau \subset \mathbb{R}^n$ and its face γ, we have $N(\gamma) \mid N(\tau)$.*

(2) *For lattice polytopes $\gamma \subset \mathbb{R}^k$ and $\gamma' \subset \mathbb{R}^{n-k}$ and their affine hull $\tau \subset \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, we have $N(\tau) = \mathrm{LCM}(N(\gamma), N(\gamma'))$.*

(3) *If γ' is a point in the setting of* (2)*, then the lattice distance from γ' to γ equals $\mathrm{GCD}(N(\gamma), N(\gamma'))$.*

Proof. Part (1) rephrases Remark 9.1. Parts (2) and (3) are obvious if γ and γ' are points and $k = n - k = 1$. The general case reduces to this one by Remark 9.3 for the projections of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ along the affine hulls of γ and γ'. ∎

Besides these geometric observations, in the study of exotic families, we use the following observation from integer linear algebra. Let us first fix notation. For an $n \times n$ matrix A and $I, J \subset \{1, \ldots, n\}$ with the same cardinality, we write A_J^I for the minor of A obtained by removing the rows with index in I and the columns with index in J. When $I = \{x_i\}$ is a singleton, we will also write $A_J^{x_i}$ and idem for J.

Lemma 9.5. *Let $A \in M_n(\mathbb{Z})$. Let $P = (P_1, \ldots, P_n)^t$ be a primitive vector of size n such that $AP = 0$. Then P_n divides the minors A_n^i for $1 \le i \le n$.*

Proof. We proceed by induction on n. One easily verifies the statement when $n = 2$. We now take a matrix $A \in M_n(\mathbb{Z})$ with $n > 2$. By elementary row operations over $\mathbb{Z}$, we transform A into a matrix B with first column $(k, 0, \ldots, 0)^t$ with $k \in \mathbb{Z}$. Then we still have $BP = 0$. We set $d = \mathrm{GCD}(P_2, \ldots, P_n)$. As P is a primitive vector, we deduce that d divides k. By the induction hypothesis, P_n/d divides the minors $B_{\{1,n\}}^{\{1,i\}}$ for all $i \in \{2, \ldots, n\}$. Hence P_n divides $kB_{\{1,n\}}^{\{1,i\}} = B_n^i$ for all $i \in \{2, \ldots, n\}$. As $B_n^1 = 0$, we find that P_n divides B_n^i for all $i \in \{1, \ldots, n\}$ and so P_n also divides the original minors A_n^i for all $i \in \{1, \ldots, n\}$. ∎

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