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The sensitivity conjecture, induced subgraphs of cubes, and Clifford algebras

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Abstract. We give another version of Huang's proof that an induced subgraph of the n-dimensional cube graph containing over half the vertices has maximal degree at least $\sqrt{n}$, which implies the Sensitivity Conjecture. This argument uses Clifford algebras of positive definite signature in a natural way. We also prove a weighted version of the result.

Keywords. Sensitivity conjecture, Clifford algebras

1. Introduction

In [2], Huang proves a theorem about induced subgraphs of hypercube graphs, and uses it to prove the Sensitivity Conjecture. The main result of that paper is as follows. Let $n \geq 1$ be a integer, and let Q^n be the n-dimensional (hyper)cube graph, with 2^n vertices given by $\{0,1\}^n$ and edges connecting points which differ at exactly one coordinate. For a graph G, let $\Delta(G)$ denote the maximum degree of its vertices.

Theorem 1.1. *Let H be a subgraph of Q^n induced by a subset of $(2^{n-1}+1)$ vertices. Then $\Delta(H) \geq \sqrt{n}$.*

Huang's proof uses a sequence of matrices A_n, which have size $2^n \times 2^n$. In this short paper we observe that these matrices are the matrices of multiplication by a certain natural element S in positive definite Clifford algebras. From this we are able to reformulate the proof in terms of Clifford algebras.

We note that Karasev in [3] has related Huang's matrices A_n to exterior algebras; these are special cases of Clifford algebras when the quadratic form is zero. By using a nontrivial quadratic form for our Clifford algebras we are able to see a close connection between the multiplicative structure of the Clifford algebras, and the combinatorics of the cube graph and its subgraphs.

Our approach generalises to give a "weighted" version of the theorem as follows. Let $a_1, \ldots, a_n$ be nonnegative real numbers. Each edge of Q^n joins points whose coordinates

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differ in one place; for a vertex v denote by $v(i)$ the unique vertex which differs from v only in the ith place, so that the vertices adjacent to v in Q^n are precisely $v(1), \dots, v(n)$. Let edges joining points whose coordinates differ in the ith place (i.e. vertices of the form v and $v(i)$) have weight a_i. Then for a vertex v of a subgraph H of Q^n, its *weighted degree* is the sum of the weights on adjacent edges. Denote by $\Delta_a(H)$ the maximum weighted degree of the vertices of H.

Theorem 1.2. *Let H be a subgraph of Q^n induced by a subset of $(2^{n-1}+1)$ vertices, and let the weights $a_1, \dots, a_n$ be any nonnegative real numbers. Then*

$$\Delta_a(H) \geq \sqrt{a_1^2 + \cdots + a_n^2}.$$

Setting $a_1 = \cdots = a_n = 1$ in Theorem 1.2 recovers Theorem 1.1.

A connection between Huang's proof and Clifford algebras has also been given by Tao [6].

After the first version of this paper was posted, the author was informed that essentially identical observations (including Theorems 1.1 and 1.2) had independently been made by T. Mrowka some weeks earlier, although they had not been published [5].

2. Clifford algebras

2.1. Background

We recall some well-known notions about Clifford algebras, which can be found in any standard text or introductory article on the subject (e.g. [1, 4, 7]). Let V be a real vector space, equipped with a symmetric bilinear form $B\colon V \times V \to \mathbb{R}$, or equivalently, a quadratic form $Q\colon V \to \mathbb{R}$, related by

$$Q(v) = B(v, v) \quad \text{and} \quad B(v, w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w)).$$

The *Clifford algebra* $\mathrm{Cl}(V, Q)$ is the associative algebra freely generated by V, subject to the relations

$$v^2 = B(v, v) = Q(v) \quad \text{or equivalently} \quad vw + wv = 2B(v, w)$$

for all $v, w \in V$. Alternatively, $\mathrm{Cl}(V, Q)$ is given by the tensor algebra of V, modulo the ideal generated by elements of the form $v \otimes v - Q(v)$.

We are interested here in positive definite B and Q. If V has dimension n, and $e_1, \dots, e_n$ form an orthonormal basis of V with respect to B, then the relations imply that for all $1 \leq i, j \leq n$,

$$e_i^2 = 1 \quad \text{and} \quad e_i e_j = -e_j e_i.$$

As a vector space, $\mathrm{Cl}(V, Q)$ has dimension 2^n and a basis is given by $e_{i_1} \cdots e_{i_k}$, for each sequence $1 \leq i_1 < \cdots < i_k \leq n$. This includes the empty sequence, whose corresponding basis element is the identity 1. In the positive definite case $\mathrm{Cl}(V, Q)$ only depends on the dimension n and we write $\mathrm{Cl}(n)$ for $\mathrm{Cl}(V, Q)$. It is known that $\mathrm{Cl}(n)$ is given as a direct sum of one or two matrix algebras, over the real numbers, complex numbers or quaternions, depending on n modulo 8.

The involution $\phi: V \to V$ given by $\phi(v) = -v$ extends to an involution of $\mathrm{Cl}(n)$, which we also denote ϕ, such that for each $1 \leq i_1 < \cdots < i_k \leq n$,

$$\phi(e_{i_1} \cdots e_{i_k}) = (-1)^k e_{i_1} \cdots e_{i_k}.$$

2.2. *Clifford algebras and cubes*

We can identify the basis element $e_{i_1} \cdots e_{i_k}$ (where $1 \leq i_1 < \cdots < i_k \leq n$) of $\mathrm{Cl}(n)$ with the vertex of Q^n with ones in positions $i_1, \ldots, i_k$ and zeroes elsewhere. Thus we write e_v for the basis element corresponding to the vertex $v \in \{0,1\}^n$. A general element $x \in \mathrm{Cl}(n)$ can be written uniquely as $x = \sum_v x_v e_v$, where the sum is over vertices $v \in \{0,1\}^n$ of Q^n and each $x_v \in \mathbb{R}$.

Observe that multiplying a basis element e_v by e_i, for $1 \leq i \leq n$ sends e_v to $\pm e_{v(i)}$. In other words, multiplication by e_i, up to sign, permutes basis elements by translating them along edges of Q^n in the ith direction.

2.3. *A Clifford element for counting degrees*

Consider the element $S = e_1 + \cdots + e_n \in \mathrm{Cl}(n)$. Observe that $S^2 = n$.

For each vertex v then $e_v S$ is a signed sum of e_w over the vertices w adjacent to v in Q^n, as is $S e_v$ (but in general with different signs). That is, $e_v S = \sum_{i=1}^n \varepsilon_i e_{v(i)}$, where each $\varepsilon_i = \pm 1$.

Similarly, if $x \in \mathrm{Cl}(n)$ is a sum of basis elements $x = \sum_{w \in W} e_w$, over some subset $W \subseteq \{0,1\}^n$, then the coefficient of e_v in xS is given by a sum of ± 1s, one for each vertex of W adjacent to v. Thus the coefficient of e_v in xS is bounded above by the degree of v in the subgraph of Q^n induced by W. In this way, multiplication by S can be used to bound degrees of vertices.

We therefore consider the map $M_S: \mathrm{Cl}(n) \to \mathrm{Cl}(n)$ given by multiplication on the right by S, i.e. $M_S(v) = vS$. Observe that

$$(\sqrt{n} + S)S = \sqrt{n}(\sqrt{n} + S) \quad \text{and} \quad (-\sqrt{n} + S)S = -\sqrt{n}(-\sqrt{n} + S)$$

so $\sqrt{n} + S$ and $-\sqrt{n} + S$ are eigenvectors of M_S with eigenvalues $\sqrt{n}$ and $-\sqrt{n}$, respectively. For convenience write $\alpha_+ = \sqrt{n} + S$ and $\alpha_- = -\sqrt{n} + S$.

Indeed then the principal left ideals $\mathrm{Cl}(n)\alpha_+$ and $\mathrm{Cl}(n)\alpha_-$ are contained in the $\sqrt{n}$ and $-\sqrt{n}$ eigenspaces, respectively. Hence their intersection is zero, but as $\alpha_+ - \alpha_- = 2\sqrt{n}$, they span $\mathrm{Cl}(n)$. Hence they are the entire eigenspaces and we have

$$\mathrm{Cl}(n) = \mathrm{Cl}(n)\alpha_+ \oplus \mathrm{Cl}(n)\alpha_- = \ker(M_S - \sqrt{n}) \oplus \ker(M_S + \sqrt{n}).$$

Since $\alpha_+\alpha_- = \alpha_-\alpha_+ = 0$, this is in fact a product of rings. Indeed we have

$$\alpha_+^2 = 2\sqrt{n}\alpha_+ \quad \text{and} \quad \alpha_-^2 = -2\sqrt{n}\alpha_-,$$

so $\frac{1}{2\sqrt{n}}\alpha_+$ and $\frac{1}{2\sqrt{n}}\alpha_-$ are complementary orthogonal idempotents. One can check that the matrix of M_S with respect to the lexicographically ordered basis is the matrix A_n of [2].

The involution ϕ takes $\mathrm{Cl}(n)\alpha_+$ to $\mathrm{Cl}(n)\alpha_-$ and vice versa, hence both have dimension 2^{n-1} as vector spaces.

More generally, if we take real numbers $a_1, \dots, a_n$, we can define

$$S_a = a_1 e_1 + \cdots + a_n e_n.$$

If $x = \sum_{w \in W} e_w$, over some subset $W \subseteq \{0,1\}^n$, then the coefficient of e_v in xS is given by a sum of $\pm a_i$ terms, one for each vertex of W adjacent to v. That is, the coefficient of e_v in xS is $\sum_i \varepsilon_i a_i$, where each $\varepsilon_i = \pm 1$, and the sum is over $1 \le i \le n$ such that $v(i) \in W$. This coefficient is bounded above by the weighted degree of v in the subgraph of Q^n induced by w. We have $S_a^2 = a_1^2 + \cdots + a_n^2$; multiplication M_{S_a} by S_a on the right has eigenvectors

$$\alpha_\pm = \pm\sqrt{a_1^2 + \cdots + a_n^2} + S_a$$

with eigenvalues $\pm\sqrt{a_1^2 + \cdots + a_n^2}$. If not all a_i are zero, we obtain a direct sum of rings

$$\mathrm{Cl}(n)\alpha_+ \oplus \mathrm{Cl}(n)\alpha_- = \ker\Big(M_{S_a} - \sqrt{a_1^2 + \cdots + a_n^2}\Big) \oplus \ker\Big(M_{S_a} + \sqrt{a_1^2 + \cdots + a_n^2}\Big).$$

3. Proofs of theorems

Let $W \subset \{0,1\}^n$ be a set of $2^{n-1}+1$ vertices of Q^n, and let H be the subgraph of Q^n induced by W. For a vertex $v \in W$, denote by $N(v)$ its neighbours in H, i.e. those vertices in W adjacent to v. Thus $|N(v)|$ is the degree of v in W, and in Theorem 1.1 we want to show some $|N(v)| \ge \sqrt{n}$.

Let $C_H \subset \mathrm{Cl}(n)$ be the vector subspace spanned by e_w, over all $w \in W$. In other words, $C_H = \bigoplus_{w \in W} \mathbb{R} e_w$.

Proof of Theorem 1.1. As C_H has dimension $2^{n-1}+1$, it has nontrivial intersection with $\mathrm{Cl}(n)\alpha_+ = \ker(M_S - \sqrt{n})$. Let $0 \ne x = \sum_{w \in W} x_w e_w$ lie in the intersection. Then one has $x(S - \sqrt{n}) = 0$, so $xS = \sqrt{n}x$.

Now in xS, for any $v \in \{0,1\}^n$, the coefficient of e_v is given by a sum $\sum_{w \in N(v)} \varepsilon_w x_w$, where each $\varepsilon_w = \pm 1$, which is at most $\sum_{w \in N(v)} |x_w|$ in absolute value. On the other hand, in $\sqrt{n}\,x$ the coefficient of e_v is of course $\sqrt{n}x_v$. So $\sqrt{n}x_v = \sum_{w \in N(v)} \varepsilon_w x_w$ and hence $\sqrt{n}|x_v| \le \sum_{w \in N(v)} |x_w|$.

This implies that some vertex of H has degree at least $\sqrt{n}$. Indeed, let v_0 be a vertex such that the coefficient x_v is largest in absolute value, i.e. $|x_{v_0}| \ge |x_v|$ for all $v \in \{0,1\}^n$. Then we have

$$\sqrt{n}|x_{v_0}| \le \sum_{w \in N(v_0)} |x_w| \le \sum_{w \in N(v_0)} |x_{v_0}| = |N(v_0)||x_{v_0}|.$$

Thus $|N(v_0)| \ge \sqrt{n}$, i.e. v_0 has degree at least $\sqrt{n}$, so $\Delta(H) \ge \sqrt{n}$. ∎

Proof of Theorem 1.2. If all weights a_i are zero, the result is immediate, so assume that $a_1^2 + \cdots + a_n^2 > 0$. We again have a nontrivial intersection of C_H with

$$\mathrm{Cl}(n)\alpha_+ = \ker\Big(M_{S_a} - \sqrt{a_1^2 + \cdots + a_n^2}\Big),$$

and we take $0 \neq x = \sum_{w \in W} x_w e_w$ in the intersection. We have

$$x S_a = \sqrt{a_1^2 + \cdots + a_n^2}\, x,$$

and comparing coefficients of e_v gives

$$\sqrt{a_1^2 + \cdots + a_n^2}\, x_v = \sum_{i=1}^{n} \varepsilon_i a_i x_{v(i)};$$

here each $\varepsilon_i = \pm 1$ and the nonzero terms in this sum correspond precisely to the neighbours of v in the vertex set W of H. Taking v_0 such that x_{v_0} is largest in absolute value we then have

$$\sqrt{a_1^2 + \cdots + a_n^2}\, |x_{v_0}| = \left| \sum_{i=1}^{n} \varepsilon_i a_i x_{v_0(i)} \right| \leq \sum_{i=1}^{n} a_i |x_{v_0(i)}| \leq |x_{v_0}| \sum_{v_0(i) \in W} a_i .$$

Now $\sum_{v_0(i) \in W} a_i$ is the weighted degree of the point v_0 in the vertex set W, and we have shown it is at least $\sqrt{a_1^2 + \cdots + a_n^2}$. ∎

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