JEMS

Benedikt Stufler

Local convergence of random planar graphs

Abstract. The present work describes the asymptotic local shape of a graph drawn uniformly at random from all connected simple planar graphs with n labelled vertices. We establish a novel uniform infinite planar graph (UIPG) as quenched limit in the local topology as $n \to \infty$. We also establish such limits for random 2-connected planar graphs and maps as their number of edges tends to infinity. Our approach encompasses a new probabilistic view on the Tutte decomposition. This allows us to follow the path along the decomposition of connectivity from planar maps to planar graphs in a uniform way, basing each step on condensation phenomena for random walks under subexponentiality and Gibbs partitions. Using large deviation results, we recover the asymptotic formula by Giménez and Noy (2009) for the number of planar graphs.

Keywords. Planar graphs, local convergence

1. Introduction

1.1. Main results

A graph is planar if it may be drawn in the plane so that edges intersect only at endpoints. The reader may consult the book [52] for details of graph embeddings on surfaces. We are interested in properties of the graph P_n selected uniformly at random among all simple connected planar graphs with vertices labelled from 1 to n. Here the term *simple* refers to the absence of loops and multiple edges.

Properties of the random graph P_n have received considerable attention in recent literature [6, 12, 21, 22, 31, 57]. We refer the reader to the comprehensive survey [54] for a detailed account. Our main theorem shows that P_n admits a local limit.

Theorem 1.1. *The uniform n-vertex connected planar graph P_n rooted at a uniformly selected vertex v_n admits a distributional limit $\hat{\mathsf{P}}$ in the local topology. We call $\hat{\mathsf{P}}$ the uniform infinite planar graph (UIPG). The regular conditional law $\mathfrak{L}((\mathsf{P}_n, v_n) \mid \mathsf{P}_n)$ satisfies*

$$\mathfrak{L}((\mathsf{P}_n, v_n) \mid \mathsf{P}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{P}}). \tag{1.1}$$

Benedikt Stufler: Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstr. 8–10, 1040 Wien, Austria; benedikt.stufler@tuwien.ac.at

The *quenched* convergence in (1.1) implies the *annealed* convergence $\mathsf{P}_n \xrightarrow{d} \hat{\mathsf{P}}$. See Section 2 for details on these forms of convergence. The root degree of $\hat{\mathsf{P}}$ follows the asymptotic degree distribution of P_n established in [21] and [57]. We also prove a version of this theorem (with a different limit object) where v_n is chosen according to the stationary distribution instead. By a celebrated result of [33, Thm. 1.1], this implies that the limit $\hat{\mathsf{P}}$ is almost surely recurrent. Milestones in the proof of our main result include local limits for 2-connected planar structures:

Theorem 1.2. *Let $v_n^{\mathscr{B}}$ denote a uniformly selected vertex of the uniform 2-connected planar graph B_n with n edges. There is a uniform infinite planar graph $\hat{\mathsf{B}}$ with*

$$\mathfrak{L}((\mathsf{B}_n, v_n^{\mathscr{B}}) \mid \mathsf{B}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{B}}). \tag{1.2}$$

We call $\hat{\mathsf{B}}$ the uniform infinite 2-connected planar graph (UI2PG).

In fact, we prove a more general vertex-weighted version (see Theorem 9.13). There is a natural coupling where $\hat{\mathsf{P}}$ is obtained from $\hat{\mathsf{B}}$ by attaching i.i.d. Boltzmann distributed connected vertex-marked planar graphs at non-root vertices of $\hat{\mathsf{B}}$, and a Boltzmann distributed doubly vertex-marked connected planar graph at the root of $\hat{\mathsf{B}}$. (See Section 4.1 for the definition of the Boltzmann distribution of a class of structures.)

Theorem 1.3. *Let $v_n^{\mathscr{V}}$ denote a uniformly selected corner of the random non-separable planar map V_n with n edges. There is a uniform infinite planar map $\hat{\mathsf{V}}$ with*

$$\mathfrak{L}((\mathsf{V}_n, v_n^{\mathscr{V}}) \mid \mathsf{V}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{V}}). \tag{1.3}$$

We call $\hat{\mathsf{V}}$ the uniform infinite 2-connected planar map (UI2PM).

Again, we actually prove a more general version with vertex weights (see Theorem 9.11). The degree distribution of the non-separable case has been studied in [23]. The well-known uniform infinite planar map has received considerable attention in the literature [9, 16, 50, 61] (see also [4, 40]). It may be obtained from $\hat{\mathsf{V}}$ by attaching i.i.d. Boltzmann distributed planar maps at each non-root corner, and a Boltzmann distributed doubly corner-rooted planar map at its root corner.

The methods we develop in this paper yield a novel probabilistic view on the Tutte decomposition [70] of these objects (see Sections 6 and 7). We do not re-prove or build upon local convergence of uniform 3-connected planar maps and graphs with n edges. This highly relevant result was established in [2] using a different approach. As a further major application we recover a celebrated result in enumerative combinatorics by Giménez and Noy:

Theorem 1.4 ([31, Thm. 1]). *The number p_n of labelled simple planar graphs with n vertices satisfies the asymptotic*

$$p_n \sim c_{\mathscr{G}} \rho_{\mathcal{C}}^{-n} n^{-7/2} n!, \tag{1.4}$$

with the constants $c_{\mathscr{G}}$ and $\rho_{\mathcal{C}}$ admitting analytic expressions given in (8.8) *and* (8.10).

See Section 8, in particular Subsection 8.1, for a detailed proof. Giménez and Noy obtained this result (concluding a sequence of rougher estimates [6, 18, 30, 56]) by performing analytic integration and employing results of [6] on the number of 2-connected graphs. An approach employing "combinatorial integration" was given in [13]. We re-prove (1.4) by different methods, without any integration step at all, deducing the asymptotic number of connected graphs from the number of 2-connected graphs using results for the big-jump domain [19, Cor. 2.1] and properties of subexponential probability distributions [28]. We emphasize that the approach of Giménez and Noy additionally yields singular expansions for the generating series involved, and our proof does not. Hence the methods of [31] yield stronger results, and the methods employed here work under weaker assumptions.

Theorem 1.1 has applications concerning subgraph count asymptotics. By a general result of [41, Lem. 4.3] and using the asymptotic degree distribution of P_n established in [21], one gets:

Corollary 1.5. *For any finite connected graph H the number* $\mathrm{emb}(H, \mathsf{P}_n)$ *of occurrences of H in P_n as a subgraph satisfies*

$$\frac{\mathrm{emb}(H, \mathsf{P}_n)}{n} \xrightarrow{p} \mathbb{E}[\mathrm{emb}^{\bullet}(H^{\bullet}, \hat{\mathsf{P}})]. \tag{1.5}$$

Here $H^{\bullet}$ denotes any fixed vertex-rooted version of H, and $\mathrm{emb}^{\bullet}(H^{\bullet}, \hat{\mathsf{P}})$ counts the number of root-preserving embeddings of $H^{\bullet}$ into $\hat{\mathsf{P}}$.

The study of the number of *pendant* copies (or *appearances*) of a fixed graph in P_n was initiated in [49], and a normal central limit theorem was established in [32, Sec. 4.3]. The difficulty of studying $\mathrm{emb}(H, \mathsf{P}_n)$ stems from the fact that it requires looking inside the giant 2-connected component of P_n, whereas pendant copies lie with high probability in the components attached to it. It is natural to conjecture convergence to a normal limit law for the fluctuations of $\mathrm{emb}(H, \mathsf{P}_n)$ around $n\mathbb{E}[\mathrm{emb}^{\bullet}(H^{\bullet}, \hat{\mathsf{P}})]$ at the scale $\sqrt{n}$. Such a result has recently been established for the number of triangles in random cubic planar graphs [55], and for the number of double triangles in random planar maps [25]. In light of [41], it would be interesting to know whether such a central limit theorem may be established in a way that applies to general sequences of random graphs that are locally convergent in some strengthened sense.

Apart from the asymptotic local shape of random planar graphs studied in the present work, it is desirable to describe their asymptotic global shape. Several models of random planar maps are known to admit the Brownian map as Gromov–Hausdorff scaling limit [43, 51] as their size tends to infinity. One of the key techniques is that planar maps admit bijective encodings in terms of well-labelled trees. These tree-like encodings facilitate the asymptotic enumeration of planar maps, and convergence of these well-labelled trees towards a Brownian tree equipped with a label process facilitates the description of the asymptotic global shape of random planar maps. However, no such direct encodings are known for planar graphs, and this is one of the main difficulties in enumerating planar graphs and studying their shape.

1.2. Summary of the main theorem's proof

A quenched local limit for the random planar map M_n^t with n edges and weight $t > 0$ at vertices was established in [66]. We pass this convergence down to a quenched limit for the non-separable core $\mathcal{V}(\mathsf{M}_n^t)$. For this, we employ a quenched version of an inductive argument discovered in [64, Thm. 6.59]. The idea is that we have full information about the components attached to the core. The neighbourhood of a uniformly selected corner of M_n^t gets patched together from a connected component containing it, a neighbourhood in the core, and neighbourhoods in components attached to the core neighbourhood. Expressing this yields a recursive equation, which by an inductive argument allows us to prove convergence of $\mathcal{V}(\mathsf{M}_n^t)$. It is important to note that $\mathcal{V}(\mathsf{M}_n^t)$ has a random size, hence a priori properties of $\mathcal{V}(\mathsf{M}_n^t)$ do not carry over automatically to properties of the uniform non-separable planar map V_n^t with n edges and weight t at vertices.

We reduce the study of the non-separable core to the study of non-serial networks using a Gibbs partition result of [62]. We proceed to establish a novel fully recursive tree-like combinatorial encoding for non-serial networks in terms of a complex construct that we call $\bar{\mathcal{R}}$-networks. This allows us to generate a non-serial network by starting with a random network $\bar{\mathsf{R}}$ where one edge is marked as "terminal". The process proceeds recursively by substituting non-terminal edges by independent copies of $\bar{\mathsf{R}}$ until only terminal edges are left. This allows us to apply recent results of [65] on subcritical branching processes, yielding a local limit theorem for the number of edges in a giant $\bar{\mathcal{R}}$-core $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ of the $\mathcal{V}$-core $\mathcal{V}(\mathsf{M}_n^t)$, and implies that the network $\mathcal{V}(\mathsf{M}_n^t)$ behaves like a network obtained from the $\bar{\mathcal{R}}$-core $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ by substituting all but a negligible number of edges by independent copies of the Boltzmann distributed $\bar{\mathcal{R}}$-network $\bar{\mathsf{R}}$. If we choose any fixed number of corners independently and uniformly at random, the corresponding $\bar{\mathcal{R}}$-components containing them will follow size-biased distributions by the famous waiting time paradox. This gives us full information on the $\bar{\mathcal{R}}$-components in the vicinity of these components. Since we substitute at edges, the resulting recursive equations for the probability that neighbourhoods in $\mathcal{V}(\mathsf{M}_n^t)$ have a fixed shape do *not* allow for the same inductive argument as before. The reason is that the event that a radius r neighbourhood in $\mathcal{V}(\mathsf{M}_n^t)$ has a fixed shape with k edges may correspond to configurations with more than k edges in an r-neighbourhood in $\bar{\mathcal{R}}(\mathsf{M}_n^t)$, since components of edges between vertices at distance r from the centre do not always contribute to the r-neighbourhood in $\mathcal{V}(\mathsf{M}_n^t)$. We solve this problem by abstraction, working with a more general convergence-determining family of events (instead of shapes of neighbourhoods we look at shapes of what we call *communities*, see the proof of Lemma 9.8) that allows the induction step to work.

Having arrived at a quenched local limit for the $\bar{\mathcal{R}}$-core, we again apply the Gibbs partition result of [62] to deduce convergence of what we call the $\bar{\mathcal{O}}$-core $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ and what is a random sized map obtained from a 3-connected planar map by blowing up edges into paths. As we have at hand a local limit theorem for the number of edges of $\bar{\mathcal{O}}(\mathsf{M}_n^t)$, we may transfer properties of $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ to other random sized $\bar{\mathcal{O}}$-networks satisfying a similar local limit theorem (but with possibly different constants). For example, we may define similarly the $\bar{\mathcal{O}}$-core $\bar{\mathcal{O}}(\mathsf{V}_n^t)$ of V_n^t. The quenched convergence of $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ transfers

to quenched local convergence of $\bar{\mathcal{O}}(\mathsf{V}_n^t)$. The arguments we used to pass convergence from $\mathcal{V}(\mathsf{M}_n^t)$ to $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ may be reversed to pass convergence from $\bar{\mathcal{O}}(\mathsf{V}_n^t)$ to V_n^t, yielding a quenched local limit for V_n^t.

Whitney's theorem ensures that we may group $\bar{\mathcal{O}}$-maps into pairs such that each pair corresponds to a unique graph. We call such graphs $\mathcal{O}$-graphs. $\bar{\mathcal{O}}$-networks and $\mathcal{O}$-graphs form the link between planar maps and planar graphs in our proof. We could have proceeded further to work with 3-connected planar maps and graphs instead, but this is not necessary for our arguments.

Networks that encode 2-connected planar graphs differ from the $\mathbb{P}$ networks that encode 2-connected planar maps, since we do not allow multiple edges and do not care about the order in parallel compositions. But guided by the fully recursive decomposition for non-serial networks encoding maps, we establish a somewhat more technical novel fully recursive decomposition for non-serial networks encoding graphs. The price we have to pay is that this decomposition is no longer isomorphism-preserving. We stress this point in §6.3.2. This is not an issue or downside for the present work, which concerns exclusively *labelled* planar graphs. However, future applications to random *unlabelled* planar graphs will require careful consideration and further study of how this step affects the symmetries. The decomposition allows us to argue analogously as for planar maps (again using repeated application of the authors' results on Gibbs partitions [62] and subcritical branching processes [65]). Hence quenched local convergence of the random 2-connected planar graph B_n^t with n edges and weight t at vertices follows from the corresponding convergence of a giant $\mathcal{O}$-core $\mathcal{O}(\mathsf{B}_n^t)$, obtained via a transfer from the core $\bar{\mathcal{O}}(\mathsf{M}_n^t)$.

If we condition the 2-connected core of the random planar graph P_n to have a fixed number m of edges, we do not obtain the uniform distribution on the 2-connected planar graphs with m edges. This effect does *not* go away as n tends to infinity. Instead, a result of [32] shows that the 2-connected core has a vanishing total variational distance from a mixture of $(\mathsf{B}_n^t)_{n \geq 1}$ for the special case of vertex weight $t = \rho_\mathcal{B}$, the radius of convergence of the generating series for 2-connected planar graphs. This allows us to deduce quenched local convergence of the 2-connected core $\mathcal{B}(\mathsf{P}_n)$ of the random connected planar graph P_n. A quenched extension of [64, Thm. 6.39] then yields quenched convergence of P_n, completing the proof of Theorem 1.1.

1.3. Notation

All unspecified limits are as $n \to \infty$. For any Polish space S we let $\mathbb{M}_1(S)$ denote the collection of probability measures on the Borel σ-algebra $\mathfrak{B}(S)$. We equip $\mathbb{M}_1(S)$ with the weak convergence topology, making it a Polish space. Given an S-valued random variable X defined on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, we let $\mathfrak{L}(X) \in \mathbb{M}_1(S)$ denote its law. If $Y : \Omega \to T$ is a random variable with values in some Polish space T, we let $\mathfrak{L}(X \mid Y)$ denote the conditional law of X given Y.

For two sequences $(X_n)_n$ and $(Y_n)_n$ of random variables with values in S we write $X_n \overset{d}{\approx} Y_n$ if their total variation distance $d_{\mathrm{TV}}(X_n, Y_n) = \sup_{A \in \mathfrak{B}(S)} |\mathbb{P}(X_n \in A) - \mathbb{P}(Y_n \in A)|$

tends to zero. We say an event holds *with high probability* if its probability tends to 1 as n becomes large. Convergences in probability and in distribution are denoted by $\xrightarrow{p}$ and $\xrightarrow{d}$. For any sequence $a_n > 0$ we let $o_p(a_n)$ denote an unspecified random variable Z_n such that $Z_n/a_n \xrightarrow{p} 0$. Likewise $O_p(a_n)$ is a random variable Z_n such that Z_n/a_n is stochastically bounded.

2. Local convergence

2.1. *The local topology*

The local topology quantifies how similar two rooted graphs are in the vicinity of the root vertices. We briefly recall the relevant notions and refer the reader to the elegant presentation in [17] for details.

Let $\mathfrak{G}$ denote the collection of (representatives of) vertex-rooted locally finite connected simple graphs viewed up to root-preserving graph isomorphism. For any integer $k \geq 0$ we may consider the subset $\mathfrak{G}_k \subset \mathfrak{G}$ of graphs with radius at most k, equipped with the discrete topology. The projection $U_k : \mathfrak{G} \to \mathfrak{G}_k$ maps a rooted graph $G^\bullet$ to the k-neighbourhood $U_k(G^\bullet)$ of its root vertex. The *local topology* on $\mathfrak{G}$ is the coarsest topology that makes these projections continuous. This projective limit topology is metrized by

$$d_{\mathrm{loc}}(G^\bullet, H^\bullet) = \frac{1}{1 + \sup\{k \geq 0 \mid U_k(G^\bullet) = U_k(H^\bullet)\}}, \quad G^\bullet, H^\bullet \in \mathfrak{G},$$

making $(\mathfrak{G}, d_{\mathrm{loc}})$ a Polish space. Analogously, the collection $\mathfrak{M}$ of corner-rooted locally finite planar maps may be endowed with a local metric. We will always implicitly mean convergence in $\mathfrak{M}$ when talking about local convergence of random maps, so that the limit preserves the embedding. Distributional convergence of a sequence $(\mathsf{G}_n^\bullet)_{n \geq 0}$ of random elements of $\mathfrak{G}$ or $\mathfrak{M}$ is equivalent to distributional convergence of each neighbourhood $U_k(\mathsf{G}_n^\bullet)$, $k \geq 0$, as n tends to infinity.

Given a finite graph or a finite planar map G there are two natural ways to select a random root vertex: either uniformly or with probability proportional to the vertex degree. The latter is called the *stationary distribution*, and also corresponds to selecting a uniform corner of a planar map.

2.2. *Convergence of random probability measures*

The set $\mathbb{M}_1(\mathfrak{G})$ of probability measures on the Borel σ-algebra of $\mathfrak{G}$ is a Polish space with respect to the weak convergence topology. Given a random finite connected simple graph G, we may view the law of the rooted graph obtained via the uniform distribution or stationary distribution on the vertex set of G as a random probability measure. That is, it is a random element of the space $\mathbb{M}_1(\mathfrak{G})$. Convergence in probability of such random probability measures may be characterized as follows.

Proposition 2.1. *Let $\mu, \mu_1, \mu_2, \ldots$ be random Borel probability measures on $\mathfrak{G}$, defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The following statements are equivalent.*

(1) *$\mu_n \xrightarrow{p} \mu$, that is, $d_{\mathrm{P}}(\mu_n, \mu) \xrightarrow{p} 0$ for the Prokhorov distance d_{P}.*

(2) *$\mathbb{E}_{\mu_n}[f] \xrightarrow{p} \mathbb{E}_{\mu}[f]$ for each bounded continuous function $f : \mathfrak{G} \to \mathbb{R}$.*

(3) *Each subsequence (n') has a subsequence (n'') with $\mu_{n''}(\omega) \to \mu(\omega)$ for almost all $\omega \in \Omega$.*

(4) *$\mu_n(U_k(\cdot) = H^{\bullet}) \xrightarrow{p} \mu(U_k(\cdot) = H^{\bullet})$ for each $k \geq 0$ and every finite simple rooted graph $H^{\bullet}$.*

An analogous statement holds for random planar maps and random measures on $\mathfrak{M}$.

Proof. The equivalence of the first three conditions is a classical property of general random measures, and the fourth condition is a special case of the second. It remains to show that the fourth condition already implies one of the others.

Any open set in $\mathfrak{G}$ is the countable union of pre-images $U_k^{-1}(H^{\bullet})$ with $k \geq 0$ and $H^{\bullet} \in \mathfrak{G}$ a graph with radius at most k. Hence the indicator random variables $(\mathbb{1}_{U_k(\cdot) = H^{\bullet}})_{k, H^{\bullet}}$ form a countable convergence-determining family by [8, Thm. 2.2]. Letting $(f_i)_{i \geq 1}$ denote a fixed ordering of this family, it follows that the metric

$$d_1(\nu, \nu') = \sum_{i \geq 1} \frac{1}{2^i} |\mathbb{E}_{\nu}[f_i] - \mathbb{E}_{\nu'}[f_i]|, \quad \nu, \nu' \in \mathbb{M}_1(\mathfrak{G}),$$

induces the weak convergence topology on $\mathbb{M}_1(\mathfrak{G})$ [60, Ex. B.9].

Suppose that the fourth condition holds. It follows by a diagonalizing argument that any subsequence (n') has a subsequence (n'') such that almost all $\omega \in \Omega$ have the property that $\mathbb{E}_{\mu_{n''}(\omega)}[f_i] \to \mathbb{E}_{\mu(\omega)}[f_i]$ for all $i \geq 1$. Thus $d_1(\mu_{n''}(\omega), \mu(\omega)) \to 0$ and consequently $\mu_{n''}(\omega) \to \mu(\omega)$ for almost all $\omega \in \Omega$. ∎

For example, if μ_n is the uniform rooting of a random simple finite connected graph, then $\mu_n(U_k(\cdot) = H^{\bullet})$ equals the percentage of vertices whose k-neighbourhood equals $H^{\bullet}$. If μ_n is the stationary distribution, it equals the percentage of oriented edges with the k-neighbourhood of the origin being equal to $H^{\bullet}$.

Proposition 2.2. *Let G_n be a sequence of random finite simple connected graphs. Let $v_n, v_n^{(1)}, v_n^{(2)}$ denote i.i.d. random vertices of G_n, and let $\hat{\mathsf{G}}, \hat{\mathsf{G}}^{(1)}, \hat{\mathsf{G}}^{(2)}$ be i.i.d. random elements of $\mathfrak{G}$. The following statements are equivalent.*

(1) *$\mathfrak{L}((\mathsf{G}_n, v_n) \mid \mathsf{G}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{G}})$.*

(2) *$((\mathsf{G}_n, v_n^{(1)}), (\mathsf{G}_n, v_n^{(2)})) \xrightarrow{d} (\hat{\mathsf{G}}^{(1)}, \hat{\mathsf{G}}^{(2)})$.*

(3) *For any integer $k \geq 0$ and any two graphs $H_1^{\bullet}, H_2^{\bullet} \in \mathfrak{G}$ with radius at most k,*
$$\mathbb{P}(U_k(\mathsf{G}_n, v_n^{(1)}) = H_1^{\bullet}, U_k(\mathsf{G}_n, v_n^{(2)}) = H_2^{\bullet}) \to \mathbb{P}(U_k(\hat{\mathsf{G}}) = H_1^{\bullet})\mathbb{P}(U_k(\hat{\mathsf{G}}) = H_2^{\bullet}).$$

An analogous statement holds for random planar maps and random measures on $\mathfrak{M}$.

Proof. The equivalence of the first and second condition follows from general results by [11, Lem. 2.3]. In detail, consider the bounded Lipschitz metric

$$d_{\mathrm{BL}}(v, v') = \sup_f |\mathbb{E}_v[f] - \mathbb{E}_{v'}[f]|, \quad v, v' \in \mathbb{M}_1(\mathscr{G}),$$

with f ranging over all functions $f : \mathscr{G} \to [-1, 1]$ that are Lipschitz continuous with Lipschitz constant 1. By [11, Lem. 2.3],

$$\mathfrak{L}((\mathsf{G}_n, v_n^{(1)}), (\mathsf{G}_n, v_n^{(2)})) \to \mathfrak{L}(\hat{\mathsf{G}}) \otimes \mathfrak{L}(\hat{\mathsf{G}}) \tag{2.1}$$

in the weak convergence topology is equivalent to

$$d_{\mathrm{BL}}(\mathfrak{L}((\mathsf{G}_n, v_n) \mid \mathsf{G}_n), \mathfrak{L}(\hat{\mathsf{G}})) \xrightarrow{p} 0. \tag{2.2}$$

As d_{BL} induces the weak convergence topology on $\mathbb{M}_1(\mathscr{G})$ by [11, Lem. 2.4], it follows by Proposition 2.2 that (2.2) is equivalent to

$$\mathfrak{L}((\mathsf{G}_n, v_n) \mid \mathsf{G}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{G}}). \tag{2.3}$$

Hence condition (1) and (2) of Proposition 2.2 are equivalent. Condition (3) is a special case of (2). It remains to show that (3) already implies (2). Since $U_i \circ U_j = U_i$ for all integers $j \geq i \geq 0$, it follows from (3) that

$$(U_k(\mathsf{G}_n, v_n^{(1)}), U_\ell(\mathsf{G}_n, v_n^{(2)})) \xrightarrow{d} (U_k(\hat{\mathsf{G}}^{(1)}), U_\ell(\hat{\mathsf{G}}^{(2)})) \tag{2.4}$$

for all $k, \ell \geq 0$. Hence

$$((\mathsf{G}_n, v_n^{(1)}), (\mathsf{G}_n, v_n^{(2)})) \xrightarrow{d} (\hat{\mathsf{G}}^{(1)}, \hat{\mathsf{G}}^{(2)}). \qquad \blacksquare$$

Using the language of statistical physics, we may call $(\mathsf{G}_n, v_n) \xrightarrow{d} \hat{\mathsf{G}}$ the annealed version of the limit, and $\mathfrak{L}((\mathsf{G}_n, v_n) \mid \mathsf{G}_n) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{G}})$ the quenched version. See also [36, Sec. 7] for similar terminology regarding fringe subtree count asymptotics. [41] used the quenched version of convergence in the context of the local topology for applications concerning subgraph counts.

The following observation is an easy exercise, which will be useful later on.

Proposition 2.3. *Let (G_n, u_n) be a sequence of random finite vertex-pointed connected graphs. Let v_n be chosen according to the stationary or uniform distribution on the vertex set. Suppose that (G_n, v_n) has a distributional limit in the local topology. If the number $|\mathsf{G}_n|$ of vertices satisfies $|\mathsf{G}_n| \xrightarrow{d} \infty$, then $d_{\mathsf{G}_n}(v_n, u_n) \xrightarrow{d} \infty$.*

Proof. Let v_n' be drawn independently of v_n and according to the same law. The convergence of (G_n, v_n) implies that for any integer $r \geq 1$ the numbers of edges and vertices in the neighbourhood $U_r(\mathsf{G}_n, v_n)$ are stochastically bounded. Since $|\mathsf{G}_n| \xrightarrow{d} \infty$, this implies that v_n' lies outside of $U_r(\mathsf{G}_n, v_n)$ with high probability. As $r \geq 1$ was arbitrary, this

yields $d_{\mathsf{G}_n}(v_n, v_n') \xrightarrow{d} \infty$. Now suppose that $d_{\mathsf{G}_n}(v_n, u_n)$ does not converge in distribution to infinity. Then there is an $\varepsilon > 0$, a constant $r \geq 1$, and a subsequence (n') such that along that subsequence $\mathbb{P}(d_{\mathsf{G}_n}(v_{n'}, u_{n'}) \leq r) \geq \varepsilon$. By the triangle inequality, it follows that $\mathbb{P}(d_{\mathsf{G}_n}(v_{n'}, v_{n'}') \leq 2r) \geq \varepsilon^2$, contradicting $d_{\mathsf{G}_n}(v_n, v_n') \xrightarrow{d} \infty$. Hence $d_{\mathsf{G}_n}(v_n, u_n) \xrightarrow{d} \infty$. ∎

All statements for random graphs in the present section hold analogously for random planar maps and random measures on the Borel σ-algebra of $\mathfrak{M}$.

3. Condensation in simply generated trees

3.1. Trees and random walk

A detailed exposition of simply generated trees may be found in the comprehensive survey [36]. A *planted plane tree* is a rooted unlabelled tree T, where the offspring of any vertex v is endowed with a linear order. We let $d_T^+(v)$ denote the outdegree of v. The number of vertices of T is denoted by $|T|$. Given a weight sequence $\mathbf{w} = (\omega_k)_{k \geq 0}$ with $\omega_0 > 0$ and $\omega_k > 0$ for at least one $k \geq 2$, we define the weight $\omega(T) = \prod_{v \in T} \omega_{d_T^+(v)}$. The corresponding generating series for the class Z of finite planted plane trees satisfies

$$\mathsf{Z}(z) = z\phi(\mathsf{Z}(z)). \tag{3.1}$$

with $\phi(z) := \sum_{k \geq 0} \omega_k z^k$. Letting ρ_ϕ denote the radius of $\phi(z)$, we define the parameter

$$\nu = \lim_{t \nearrow \rho_\phi} \frac{\phi'(t)t}{\phi(t)}. \tag{3.2}$$

For $0 < \nu \leq 1$ we set $\tau = \rho_\phi$. For $\nu > 1$ we let τ denote the unique positive real number with $\tau\phi'(\tau) = \phi(\tau)$. We let ξ denote a random non-negative integer with probability generating function $\mathbb{E}[z^\xi] = \phi(\tau z)/\phi(\tau)$. The following result is given in [36, Secs. 3, 7] and [15, Cor. 18.17].

Lemma 3.1. *Suppose that $\nu > 0$. The simply generated tree T_n is distributed like a ξ-Galton–Watson tree T conditioned on having n vertices. We have $\mathbb{E}[\xi] = \min(1, \nu)$ and $\mathsf{Z}(z)$ evaluated at its radius of convergence $\rho_{\mathsf{Z}} = \tau/\phi(\tau) < \infty$ equals $\mathsf{Z}(\rho_{\mathsf{Z}}) = \tau < \infty$. Moreover,*

$$[z^n]\mathsf{Z}(z) = \left(\frac{\tau}{\phi(\tau)}\right)^{-n} \frac{\tau}{n} \mathbb{P}(\xi_1 + \cdots + \xi_n = n - 1), \tag{3.3}$$

with $(\xi_i)_{i \geq 1}$ denoting independent copies of ξ.

It follows from the results for the big-jump domain in random walk, established in [19, Cor. 2.1] in a more general context, that if

$$\mathbb{P}(\xi = n) = f(n)n^{-1-\alpha} \tag{3.4}$$

for some constant $\alpha > 1$ and a slowly varying function f, then for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \sup_{x\geq\varepsilon+\mathbb{E}[\xi]} \left| \frac{\mathbb{P}(\xi_1 + \cdots + \xi_n = nx)}{\mathbb{P}(\xi = \lfloor n(x - \mathbb{E}[\xi])\rfloor)} - 1 \right| = 0. \tag{3.5}$$

Hence if $\mathbb{E}[\xi] < 1$, then

$$\frac{1}{n}\mathbb{P}\left(\sum_{i=1}^{n}\xi_i = n-1\right) \sim \mathbb{P}(\xi = \lfloor n(1 - \mathbb{E}[\xi])\rfloor) \sim \frac{f(n)}{(n(1 - \mathbb{E}[\xi]))^{1+\alpha}}. \tag{3.6}$$

Thus in this setting, (3.3) simplifies to

$$[z^n]\mathcal{Z}(z) \sim \left(\frac{\rho_\phi}{\phi(\rho_\phi)}\right)^{-n} n^{-1-\alpha} \frac{\rho_\phi f(n)}{\left(1 - \frac{\phi'(\rho_\phi)\rho_\phi}{\phi(\rho_\phi)}\right)^{1+\alpha}}. \tag{3.7}$$

3.2. Condensation

Setting $\theta = \min(\alpha, 2)$, we let X denote the θ-stable random variable with Laplace exponent $\mathbb{E}[\exp(-\lambda X)] = \exp(\lambda^\theta)$. Let h be the density of X. In [36, Thm. 19.34] and [39, Thm. 1] limits concerning the maximum degree $\Delta(\mathsf{T}_n)$ of the simply generated tree T_n in the condensation regime were established. The following extension was recently given in [65, Thm. 1.1].

Lemma 3.2. *Suppose that $\mathbb{E}[\xi] < 1$ and that (3.4) holds. Then there is a slowly varying function g such that*

$$\mathbb{P}(\Delta(\mathsf{T}_n) = \ell) = \frac{1}{g(n)n^{1/\theta}}\left(h\left(\frac{(1 - \mathbb{E}[\xi])n - \ell}{g(n)n^{1/\theta}}\right) + o(1)\right) \tag{3.8}$$

uniformly for all $\ell \in \mathbb{Z}$. If f converges to a constant, then g may be chosen to be constant.

A *marked plane tree* is a (planted) plane tree with a distinguished vertex. For $\mathbb{E}[\xi] < 1$ we have $\mathbb{E}[|\mathsf{T}|] < \infty$. This allows us to define the size-biased tree $\mathsf{T}^\bullet$ with distribution

$$\mathbb{P}(\mathsf{T}^\bullet = (T, v)) = \frac{\mathbb{P}(\mathsf{T} = T)}{\mathbb{E}[|\mathsf{T}|]}. \tag{3.9}$$

We let T° denote a random marked tree that is distributed like $\mathsf{T}^\bullet$ conditioned on having a marked leaf.

The *fringe subtree* of a plane tree at a vertex is the subtree consisting of the vertex and all its descendants. The simply generated tree T_n may be completely described by the ordered list $(F_i(\mathsf{T}_n))_{1\leq i\leq\Delta(\mathsf{T}_n)}$ of fringe subtrees dangling from the lexicographically first vertex v_Δ with maximum outdegree $\Delta(\mathsf{T}_n)$, and the marked tree $F_0(\mathsf{T}_n)$ obtained from T_n by marking the vertex v_Δ and cutting away all its descendants.

In [37], [36, Thm. 20.2], [1, Thm. 1.3], and [39, Cor. 2.7, Thm. 2] limits concerning $F_0(\mathsf{T}_n)$ and the list $(F_i(\mathsf{T}_n))_{1\leq i\leq\Delta(\mathsf{T}_n)}$ are proved. The following extension is given in [65, Thm. 1.2].

Lemma 3.3. *Suppose that $\mathbb{E}[\xi] < 1$ and that* (3.4) *holds. Setting $C := 2\mathbb{E}[|\mathsf{T}|]$, for any sequence $(t_n)_{n\geq 1}$ of integers with $t_n \to \infty$ we have*

$$\left(F_0(\mathsf{T}_n), (F_i(\mathsf{T}_n))_{1\leq i\leq\Delta(\mathsf{T}_n)-t_n}, \mathbb{1}_{\sum_{i=\Delta(\mathsf{T}_n)-t_n}^{\Delta(\mathsf{T}_n)}|F_i(\mathsf{T}_n)|\geq Ct_n}\right) \overset{d}{\approx} (\mathsf{T}^\circ, (\mathsf{T}^i)_{1\leq i\leq\Delta_{\langle n\rangle}-t_n}, 0)$$

for

$$\Delta_{\langle n\rangle} := \sup\left\{d \geq 1 \,\middle|\, |\mathsf{T}^\circ| + \sum_{i=1}^{d}|\mathsf{T}^i| \leq n\right\}.$$

The dependence of $\Delta_{\langle n\rangle}$ on $(\mathsf{T}^\circ, (\mathsf{T}^i)_{i\geq 1})$ may be dealt with using the following corollary from the proof of [65, Cor. 1.5]:

Lemma 3.4. *Suppose that $\mathbb{E}[\xi] < 1$ and that* (3.4) *holds. Let $\Delta_{[n]}$ denote an identically distributed copy of $\Delta(\mathsf{T}_n)$ that is independent of T° and $(\mathsf{T}^i)_{i\geq 1}$. Let $0 < \delta < 1 - \mathbb{E}[\xi]$ and $\varepsilon > 0$ be given. There are constants $0 < c < C$, $N_0 > 0$ and events $(\mathscr{E}_n)_{n\geq 1}$ (each consisting of a collection of finite sequences of finite trees, with the first tree carrying a marked leaf) such that for all $n > N_0$,*

$$\mathbb{P}\left((F_0(\mathsf{T}_n), (F_i(\mathsf{T}_n))_{1\leq i\leq\Delta(\mathsf{T}_n)-\delta n}) \in \mathscr{E}_n\right) > 1 - \varepsilon,$$
$$\mathbb{P}\left((\mathsf{T}^\circ, (\mathsf{T}^i)_{1\leq i\leq\Delta_{[n]}-\delta n}) \in \mathscr{E}_n\right) > 1 - \varepsilon$$

and uniformly for all sequences $S \in \mathscr{E}_n$,

$$c < \frac{\mathbb{P}((F_0(\mathsf{T}_n), (F_i(\mathsf{T}_n))_{1\leq i\leq\Delta(\mathsf{T}_n)-\delta n}) = S)}{\mathbb{P}((\mathsf{T}^\circ, (\mathsf{T}^i)_{1\leq i\leq\Delta_{[n]}-\delta n}) = S)} < C.$$

We are also going to require knowledge on the vicinity of some fixed number $k \geq 1$ of uniformly and independently selected vertices of T_n. For $k = 1$ this question was answered in [63] in a very general setting. In the specific setting of Lemmas 3.3 and 3.4, we may study an idealized version $\tilde{\mathsf{T}}_n$ of T_n instead, where $F_0(\mathsf{T}_n)$ is replaced by T° and $F_i(\mathsf{T}_n)$ is replaced by an independent copy of T for each $1 \leq i \leq \Delta(\mathsf{T}_n)$. Here we assume that T° and the independent copies of T are independent of $\Delta(\mathsf{T}_n)$. We let $\tilde{\mathsf{T}}_n^{\bullet(k)}$ denote the result of marking k uniformly and independently selected vertices of $\tilde{\mathsf{T}}_n$, and let $\mathsf{T}^{\bullet,1}, \mathsf{T}^{\bullet,2}, \ldots$ denote independent copies of $\mathsf{T}^\bullet$.

Corollary 3.5. *Suppose that $\mathbb{E}[\xi] < 1$ and that condition* (3.4) *is satisfied. Then*

$$\left(F_0(\tilde{\mathsf{T}}_n^{\bullet(k)}), (F_i(\tilde{\mathsf{T}}_n^{\bullet(k)}))_{1\leq i\leq\Delta(\mathsf{T}_n)}\right) \overset{d}{\approx} (\mathsf{T}^\circ, (\tilde{\mathsf{T}}^i)_{1\leq i\leq\Delta(\mathsf{T}_n)}). \tag{3.10}$$

Here $\tilde{\mathsf{T}}_i$ is defined by letting $j_1, \ldots, j_k$ be a sequence of distinct integers between 1 and $\Delta(\mathsf{T}_n)$, selected uniformly at random, and setting

$$\tilde{\mathsf{T}}^i := \begin{cases} \mathsf{T}^i, & i \notin \{j_1, \ldots, j_k\}, \\ \mathsf{T}^{\bullet,r}, & i = j_r \text{ with } 1 \leq r \leq k. \end{cases}$$

Proof. It is clear that with high probability all k marked vertices are going to be part of distinct fringe subtrees dangling from the vertex with maximal outdegree in $\tilde{\mathsf{T}}_n$.

We set $\bar{\mathsf{T}}^i = (\mathsf{T}^i, v_j)$ when $v_j \in \mathsf{T}^i$ for some $1 \leq j \leq k$, and $\bar{\mathsf{T}}^i = \mathsf{T}^i$ otherwise. For any sequence $(T_1, \ldots, T_\ell)$ of planted plane trees among which precisely k selected trees $T_{\ell_1}, \ldots, T_{\ell_k}$ have a marked vertex, with $\ell := \Delta(\mathsf{T}_n)$ we have

$$\frac{\mathbb{P}((\bar{\mathsf{T}}^i)_{1 \leq i \leq \ell} = (T_1, \ldots, T_\ell) \mid \ell)}{\mathbb{P}((\tilde{\mathsf{T}}^i)_{1 \leq i \leq \ell} = (T_1, \ldots, T_\ell) \mid \ell)} = \prod_{j=1}^{k} \frac{(\ell - j + 1)\mathbb{E}[|\mathsf{T}|]}{\sum_{i=1}^{\ell} |T_i| - \sum_{r=1}^{j-1} |T_{\ell_r}|}.$$

By the law of large numbers, it follows that $(\bar{\mathsf{T}}^i)_{1 \leq i \leq \ell} \overset{d}{\approx} (\tilde{\mathsf{T}}^i)_{1 \leq i \leq \ell}$. ∎

4. Enriched tree encodings and sampling procedures

Combinatorial bijections may serve to reduce the study of complex random graphs to the study of random structures that are more tractable by probabilistic methods. The notion of enriched trees in the framework of combinatorial species additionally allows for a unified view on a large class of (random) structures.

4.1. Combinatorial species of structures

A detailed account of combinatorial species may be found in the pioneering work [38] and the comprehensive book [7]. The present section only aims to give a brief informal recap.

Informally speaking, a weighted combinatorial species $\mathcal{F}$ is a collection of labelled combinatorial structures (such as trees or graphs) which is closed under isomorphism. Any structure is assigned a weight, which for our purposes will always be a non-negative real number. The default case is where each structure receives weight 1, and we will explicitly state whenever we deviate from this.

Each structure also has an underlying finite set (such as the corners of a map or the vertices of a graph), and weight-preserving isomorphisms are obtained by relabelling along bijections between underlying sets. We refer to the elements of the set as *labels* or *atoms*.

The collection of structures over some fixed set U is denoted by $\mathcal{F}[U]$, and we use the notation $\mathcal{F}_n$ for $U = \{1, \ldots, n\}$. The size $|F|$ of an $\mathcal{F}$-structure F is given by the number of elements of its underlying set. We may form the exponential generating series $\mathcal{F}(z)$ where the coefficient $[z^n]\mathcal{F}(z)$ of z^n is given by $1/n!$ times the sum of all weights of structures from $\mathcal{F}_n$. We require this number to be finite for all n.

There are special examples of species, such as the species SET with $\text{SET}[U] = \{U\}$ for all finite sets U, yielding $\text{SET}(z) = \exp(z)$. The species SEQ of sequences sends a set U to the collection of all linear orderings of U, yielding $\text{SEQ}(z) = 1/(1 - z)$.

The radius of convergence of $\mathcal{F}(z)$ is denoted by $\rho_{\mathcal{F}}$. The weight of an $\mathcal{F}$-object F is denoted by $\omega(F)$. Given a parameter $0 < t \leq \rho_{\mathcal{F}}$ with $\mathcal{F}(t) < \infty$, the *Boltzmann distribution* with parameter t selects an object F from $\bigcup_{n \geq 0} \mathcal{F}_n$ with probability $\omega(F)t^{|F|}/\mathcal{F}(t)$.

Unless we explicitly state a parameter, we will always refer to the case $t = \rho_{\mathcal{F}}$ as *the Boltzmann distribution* of $\mathcal{F}$.

An unlabelled $\mathcal{F}$-object is given by an isomorphism class. We may form the collection $\tilde{\mathcal{F}}_n$ of all n-sized unlabelled $\mathcal{F}$-objects. For asymmetric classes of structures (such as corner-rooted planar maps or planted plane trees), where each structure over an n-sized set has precisely $n!$ different labellings, the notions of labelled and unlabelled structures are equivalent.

Given weighted species $\mathcal{F}$ and $\mathcal{G}$ there are numerous ways to create new species such as the sum $\mathcal{F} + \mathcal{G}$, the product $\mathcal{F} \cdot \mathcal{G}$, and the substitution $\mathcal{F}(\mathcal{G})$. There are also unary operations such as the derivative $\mathcal{F}'$ and the pointing $\mathcal{F}^{\bullet}$. It is also possible to restrict a species, for example for any integer $k \geq 0$ the species $\mathcal{F}_{\geq k}$ is the restriction of $\mathcal{F}$ to all object of size at least k. We refer the reader to the cited sources for details on these operations.

The species $\mathcal{F}$ and $\mathcal{G}$ are termed isomorphic, denoted by $\mathcal{F} \simeq \mathcal{G}$ or $\mathcal{F} = \mathcal{G}$, if for any finite set U there is a weight-preserving bijection $\mathcal{F}[U] \to \mathcal{G}[U]$. This family of bijections is required to be compatible with the relabelling bijections.

The concept of species may be generalized to k-type species (with $k \geq 1$ fixed) in a straightforward way. Rather than sets one uses k-type sets, which are tuples $(U_1, \ldots, U_k)$ of finite sets. Multivariate generating series $\mathcal{F}(z_1, \ldots, z_k)$ and partial derivatives $\frac{\partial \mathcal{F}}{\partial z_i}$ are defined in an analogous manner to the monotype case.

4.2. Enriched trees

The concept of enriched tree was introduced in [42]. Given a weighted class $\mathcal{R}$ an *$\mathcal{R}$-enriched tree* consists of a rooted unordered labelled tree T together with a function α that assigns to vertex $v \in T$ with offspring set M_v an element $\alpha(v) \in \mathcal{R}[M_v]$ as decoration. In the algebra of species the class $\mathcal{A}_{\mathcal{R}}$ of $\mathcal{R}$-enriched trees may be specified by the decomposition

$$\mathcal{A}_{\mathcal{R}} = \mathcal{X} \cdot \mathcal{R}(\mathcal{A}_{\mathcal{R}}), \tag{4.1}$$

because any such element consists of a root vertex (accounting for the factor $\mathcal{X}$) together with an $\mathcal{R}$-structure whose labels may be identified with the enriched fringe subtrees dangling from the root. There are two key facts that we are going to use in our proofs:

Lemma 4.1. Any *class of structures satisfying a decomposition of the form $\mathcal{F} \simeq \mathcal{X} \cdot \mathcal{R}(\mathcal{F})$ admits a canonical (weight-preserving) isomorphism $\mathcal{F} \simeq \mathcal{A}_{\mathcal{R}}$.*

In particular, if we want to sample a labelled structure from $\mathcal{F}_n$ with probability proportional to its weight, we may sample an element from $(\mathcal{A}_{\mathcal{R}})_n$ in this way, and apply the combinatorial bijection. Sampling a random enriched tree may be done by adorning a simply generated plane tree with independent decorations:

Lemma 4.2. *The following sampling procedure draws an n-sized enriched tree from $(\mathcal{A}_{\mathcal{R}})_n$ with probability proportional to its weight.*

(1) *Select a random simply generated plane tree* T_n *with weight sequence* $([z^k]\mathcal{R}(z))_{k \geq 0}$.

(2) *For each vertex* $v \in \mathsf{T}_n$ *choose an* $\mathcal{R}$-*structure* $\beta_n(v)$ *from the set* $\mathcal{R}_{d_{\mathcal{T}_n}^+(v)}$ *with probability proportional to its weight.*

(3) *Relabel this enriched plane tree* (T_n, β_n) *by choosing a bijection from its vertex set to* $\{1, \ldots, n\}$ *uniformly at random.*

We are going to refer to the way of choosing the random decorations in (2) as the *canonical way* of decorating the plane tree T_n. The random decoration β_n is also referred to as the *canonical decoration*.

Lemma 4.1 is a special case of Joyal's Implicit Species [38, Thm. 6]. The idea is to *unroll* the isomorphism via a recursive procedure Γ: Suppose we are given an $\mathcal{F}$-object $F \in \mathcal{F}[U]$ over some finite non-empty set U. Applying the bijection from the isomorphism to F yields a single atom $a \in U$ (corresponding to the factor $\mathcal{X}$), and an $\mathcal{R}$-object R whose atoms correspond to some collection M of $\mathcal{F}$-objects. The underlying sets of structures in M partition the set $U \setminus \{a\}$. If $M = \emptyset$ we stop and return $\Gamma(F)$ as output a tree consisting of a single root vertex with label a. Otherwise we apply the procedure Γ recursively to each element from the collection M, resulting in a collection M' of enriched trees. The output $\Gamma(F)$ is then formed by an enriched tree with root labelled with the atom a, with M being the collection of branches dangling from a, and the $\mathcal{R}$-structure of a formed by relabelling R canonically with the labels of the root vertices of the enriched trees from M'.

As for the second key fact, Lemma 4.2, such a sampling procedure was given in [64, Lem. 6.1] and, in a less general setting, in [58, Prop. 3.6]. If the species $\mathcal{R}$ is asymmetric, so is the species $\mathcal{A}_{\mathcal{R}}$. In this case, we may actually work with unlabelled $\mathcal{A}_{\mathcal{R}}$-objects that correspond bijectively to pairs (T, β) of a planted plane tree T and a function β that assigns to each vertex $v \in T$ with outdegree $d_T^+(v)$ an unlabelled $\mathcal{R}$-object $\beta(v) \in \tilde{\mathcal{R}}_{d_T^+(v)}$.

5. Giant components in random compound structures

5.1. Gibbs partitions

Let $\mathcal{F}$ and $\mathcal{G}$ be weighted species with $\mathcal{G}(0) = 0$. For any integer $n > 0$ with $[z^n]\mathcal{F}(\mathcal{G}(z)) > 0$ we may draw an element S_n from $(\mathcal{F}(\mathcal{G}))_n$ with probability proportional to its weight. This random compound structure comes with a partition of the underlying set $[n]$, which is called a Gibbs partition. The term was coined in the comprehensive work [59].

The convergent case. If we remove one of the largest components of S_n and replace it by a placeholder, we are left with an $\mathcal{F}'(\mathcal{G})$-object. We let S_n' denote the corresponding unlabelled object.

Suppose that $0 < \rho_{\mathcal{G}} < \infty$ and $\mathcal{F}'(\mathcal{G}(\rho_{\mathcal{G}})) < \infty$ (with $\rho_{\mathcal{G}}$ denoting the radius of convergence of $\mathcal{G}(z)$) and that $[z^n]\mathcal{F}(\mathcal{G}(z)) > 0$ for infinitely many n. We say $\mathcal{F} \circ \mathcal{G}$

has *convergent type* if S'_n converges weakly to *the* Boltzmann distribution of $\mathcal{F}'(\mathcal{G})$. This implies that the size $\Delta(S_n)$ of the largest component of S_n equals $n - O_p(1)$.

As shown in [62, Thm. 3.1], a sufficient condition for this behaviour is that we have $\rho_\mathcal{F} > \mathcal{G}(\rho_\mathcal{G})$ and the coefficients $g_n := [z^n]\mathcal{G}(z)$ satisfy

$$\frac{g_n}{g_{n+1}} \to \rho_\mathcal{G} \quad \text{and} \quad \frac{1}{g_n} \sum_{i+j=n} g_i g_j \to 2\mathcal{G}(\rho_\mathcal{G}) < \infty. \tag{5.1}$$

Condition (5.1) means that the size of a Boltzmann distributed $\mathcal{G}$-object has a subexponential density [28]. This entails (see [14, Thm. 1], [26, Thm. C])

$$[z^n]\mathcal{F}(\mathcal{G}(z)) \sim \mathcal{F}'(\mathcal{G}(\rho_\mathcal{G}))[z^n]\mathcal{G}(z). \tag{5.2}$$

Condition (5.1) is satisfied when $g_n \rho_\mathcal{G}^n$ varies regularly with an index smaller than -1. Another sufficient condition was given in [62, Lem. 3.3]:

Lemma 5.1. *Suppose that there is a power series $\phi(z) = \sum_{k \geq 0} \omega_k z^k$ with non-negative coefficients and positive radius of convergence such that*

$$\mathcal{G}(z) = z\phi(\mathcal{G}(z)),$$

and $\omega_0 > 0$, $\omega_k > 0$ for at least one $k \geq 2$, and $\gcd\{k \mid \omega_k > 0\} = 1$. Then (5.2) holds and S_n is convergent.

The difference $n - \Delta(S_n)$ admits a limit distribution if $\mathcal{F} \circ \mathcal{G}$ has convergent type. We will also require the following bound. The proof is by the same arguments as for the bound in [65, (3.16)].

Proposition 5.2. *Suppose that $\rho_\mathcal{F} > \mathcal{G}(\rho_\mathcal{G})$ and (5.1) holds. Let X denote the size of a Boltzmann distributed $\mathcal{G}$-object. Then there are constants $C, c > 0$ such that for all $k \geq 0$,*

$$\mathbb{P}(n - \Delta(S_n) = k) \leq C \frac{\mathbb{P}(X = n - k)\mathbb{P}(X = k)}{\mathbb{P}(X = n)} \exp\left(-c \frac{k}{n - k}\right). \tag{5.3}$$

Convergence in the superexponential case. Suppose that $\rho_\mathcal{G} = 0$ and $\rho_\mathcal{F} > 0$. Furthermore, suppose that $[z^1]\mathcal{F}(z) > 0$ and $g_n := [z^n]\mathcal{G}(z) > 0$ for sufficiently large integers n. It follows from [64, Cor. 6.19, (6.30)] that if

$$\sum_{i=1}^{n-1} g_i g_{n-i} = o(g_n), \tag{5.4}$$

then with high probability, S_n consists of a single $\mathcal{G}$-object. In this case we say S_n is *superconvergent*. A sufficient condition was given in [64, Lem. 6.17]:

Lemma 5.3. *Suppose that $\mathcal{G}(z) = z\phi(\mathcal{G}(z))$ for a non-analytic series $\phi(z) = \sum_{k \geq 0} \omega_k z^k$ such that $\omega_0 > 0$, $\omega_k > 0$ for at least one $k \geq 2$, and $\gcd\{k \mid \omega_k > 0\} = 1$. Then (5.4) holds and S_n is superconvergent.*

5.2. *Random product structures*

Let $\mathcal{F}$ and $\mathcal{G}$ denote weighted species with $f_n := [z^n]\mathcal{F}(z) > 0$ and $g_n := [z^n]\mathcal{G}(z) > 0$ for all sufficiently large n. As before, we let $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{G}}$ denote the radii of convergence of the generating series $\mathcal{F}(z)$ and $\mathcal{G}(z)$. We may draw a random pair S_n with probability proportional to its weight from all n-sized $\mathcal{F}\mathcal{G}$-objects, and look at its $\mathcal{F}$- and $\mathcal{G}$-components $\mathcal{F}(\mathsf{S}_n)$ and $\mathcal{G}(\mathsf{S}_n)$. We describe two observations where it is unlikely for both components to be large at the same time. Both are immediate consequences of standard properties of random variables with subexponential densities [28].

Proposition 5.4. *Suppose that the coefficients* $(g_n)_{n \geq 0}$ *satisfy* (5.1) *with* $\rho_{\mathcal{G}} > 0$. *If* $f_n = o(g_n)$, *then up to relabelling, the* $\mathcal{F}$-*component* $\mathcal{F}(\mathsf{S}_n)$ *converges in distribution to a Boltzmann distributed* $\mathcal{F}$-*object with parameter* $\rho_{\mathcal{G}}$. *Moreover,*

$$[z^n]\mathcal{F}(z)\mathcal{G}(z) \sim \mathcal{F}(\rho_{\mathcal{G}})g_n. \tag{5.5}$$

Proof. Let X and Y denote the sizes of Boltzmann-distributed $\mathcal{F}$-objects and $\mathcal{G}$-objects with parameter $\rho_{\mathcal{G}}$. In order to verify the first claim, it suffices to show that $|\mathcal{F}(\mathsf{S}_n)| \overset{d}{\to} X$. It is elementary that

$$(|\mathcal{F}(\mathsf{S}_n)|, |\mathcal{G}(\mathsf{S}_n)|) \overset{d}{=} ((X, Y) \mid X + Y = n). \tag{5.6}$$

Condition (5.1) ensures that $\mathbb{P}(Y = n) \sim \mathbb{P}(Y = n + 1)$. By [28, Thm. 4.23] it follows that

$$\mathbb{P}(X + Y = n) \sim \mathbb{P}(Y = n). \tag{5.7}$$

Hence for any constant $k \geq 0$ we get

$$\mathbb{P}(X = k \mid X + Y = n) = \frac{\mathbb{P}(X = k)\mathbb{P}(Y = n - k)}{\mathbb{P}(X + Y = n)} \to \mathbb{P}(X = k). \tag{5.8}$$

This shows weak convergence of $\mathcal{F}(\mathsf{S}_n)$ (up to relabelling). Moreover, (5.5) follows directly from (5.7). ∎

The following proposition describes the asymptotic behaviour of random $\mathcal{F}\mathcal{G}$-structure where either the $\mathcal{F}$-component or the $\mathcal{G}$-component has macroscopic size, but not both at the same time.

Proposition 5.5. *Suppose that the coefficients* $(g_n)_{n \geq 0}$ *satisfy* (5.1) *with* $\rho := \rho_{\mathcal{G}} > 0$, *and* $f_n/g_n \to \lambda$ *for some constant* $0 < \lambda < \infty$. *Then*

$$[z^n]\mathcal{F}(z)\mathcal{G}(z) \sim \mathcal{F}(\rho)g_n + \mathcal{G}(\rho)f_n. \tag{5.9}$$

Moreover,

$$d_{\mathrm{TV}}(\mathsf{S}_n, \hat{\mathsf{S}}_n) \to 0 \tag{5.10}$$

for a random object $\hat{\mathsf{S}}_n$ *constructed by the following procedure:*

(1) *Set* $p = \mathcal{F}(\rho)/(\mathcal{F}(\rho) + \lambda\mathcal{G}(\rho))$, *and flip a biased coin that shows head with probability* p.

(2) *If it shows head, sample a Boltzmann-distributed $\mathcal{F}$-object F with parameter ρ. If $|\mathsf{F}| \le n$ and $g_{n-|\mathsf{F}|} > 0$, let $\hat{\mathsf{S}}_n$ be the $\mathcal{FG}$-structure with $\mathcal{F}$-component F and an $(n-|\mathsf{F}|)$-sized $\mathcal{G}$-component drawn with probability proportional to its weight. If $|\mathsf{F}| > n$ or $g_{n-|\mathsf{F}|} = 0$, we set $\hat{\mathsf{S}}_n$ to some placeholder value $\diamond$.*

(3) *If the coin flip shows tails, we sample a Boltzmann-distributed $\mathcal{G}$-object G with parameter ρ. If $|\mathsf{G}| \le n$ and $f_{n-|\mathsf{F}|} > 0$, we let $\hat{\mathsf{S}}_n$ be the $\mathcal{FG}$-structure with $\mathcal{G}$-component G and an $(n-|\mathsf{G}|)$-sized $\mathcal{F}$-component drawn with probability proportional to its weight. If $|\mathsf{G}| > n$ or $f_{n-|\mathsf{G}|} = 0$, we set $\hat{\mathsf{S}}_n$ to some placeholder value $\diamond$.*

Proof. We let X and Y denote the sizes of independent Boltzmann-distributed $\mathcal{F}$-objects and $\mathcal{G}$-objects with parameter ρ. Condition (5.6) reduces the entire problem to comparing the sizes

$$(X_n, Y_n) := (|\mathcal{F}(\mathsf{S}_n)|, |\mathcal{G}(\mathsf{S}_n)|) \overset{d}{=} ((X, Y) \mid X + Y = n)$$

with the sizes

$$(\hat{X}_n, \hat{Y}_n) := (|\mathcal{F}(\hat{\mathsf{S}}_n)|, |\mathcal{G}(\hat{\mathsf{S}}_n)|).$$

Note that

$$\mathbb{P}(X = n) \sim \lambda \frac{\mathcal{G}(\rho)}{\mathcal{F}(\rho)} \mathbb{P}(Y = n).$$

By [28, Thm. 4.23] it follows that

$$\mathbb{P}(X + Y = n) \sim \mathbb{P}(X = n) + \mathbb{P}(Y = n).$$

This verifies (5.9). Condition (5.1) ensures that $\mathbb{P}(Y = n) \sim \mathbb{P}(Y = n + 1)$. Hence for any constant $k \ge 0$ we get

$$\mathbb{P}(X = k \mid X + Y = n) = \frac{\mathbb{P}(X = k)\mathbb{P}(Y = n - k)}{\mathbb{P}(X + Y = n)}$$
$$\sim \mathbb{P}(X = k)p \sim \mathbb{P}(\hat{X}_n = k). \qquad (5.11)$$

Likewise, $\mathbb{P}(X = n) \sim \mathbb{P}(X = n + 1)$ and

$$\mathbb{P}(Y = k \mid X + Y = n) = \frac{\mathbb{P}(Y = k)\mathbb{P}(X = n - k)}{\mathbb{P}(X + Y = n)}$$
$$\sim \mathbb{P}(Y = k)(1 - p) \sim \mathbb{P}(\hat{Y}_n = k). \qquad (5.12)$$

This entails that there is a sequence $(t_n)_n$ of integers that tends to infinity sufficiently slowly such that

$$\mathbb{P}((X_n, Y_n) = (k, n - k)) = (1 + o(1))\mathbb{P}((\hat{X}_n, \hat{Y}_n) = (k, n - k)) \qquad (5.13)$$

uniformly for all integers k with $0 \le k \le t_n$ or $n - t_n \le k \le n$. As $\mathbb{P}(t_n \le \hat{X}_n \le n - t_n) \to 0$ and

$$\mathbb{P}(t_n \le X_n \le n - t_n) = O(1) \sum_{k=t_n}^{n-t_n} \frac{\mathbb{P}(X = k)\mathbb{P}(X = n - k)}{\mathbb{P}(X = n)} \to 0$$

by subexponentiality, this implies (5.10). ∎

6. Tree-like graph decompositions and convergent Gibbs partitions

Given a class $\mathcal{G}$ of simple graphs, we may form the subclass $\mathcal{C}$ of connected graphs in $\mathcal{G}$. Graphs in $\mathcal{C}$ must have at least one vertex. We let $\mathcal{B}$ denote the subclass of 2-connected graphs in $\mathcal{C}$. Here we consider the graph K_2 consisting of two vertices joined by a single edge to be 2-connected, so that 2-connected graphs must have at least two vertices. We let $\mathcal{F}$ denote the subclass of 3-connected graphs in $\mathcal{B}$. We require 3-connected graphs to have at least four vertices.

Throughout this section we assume that $\mathcal{G}$ is stable under Tutte's decomposition [70] (see also [15, 29, 35, 45, 68, 70, 71]). That is, a simple graph lies in $\mathcal{G}$ if and only if all 3-connected components lie in $\mathcal{F}$. Note that this implies that K_2 belongs to $\mathcal{G}$. We also consider $\mathcal{G} = \mathcal{G}(x, y)$ and its subclasses as 2-sort species, with x counting vertices and y edges.

6.1. Decomposition into connected components

A graph consists of a collection of connected components. Hence the species $\mathcal{G}$ and $\mathcal{C}$ are related by the well-known decomposition

$$\mathcal{G} = \text{SET}(\mathcal{C}). \tag{6.1}$$

Since $\mathcal{G}$ is stable under Tutte's decomposition, it is also stable under taking 2-connected components in the block decomposition. Hence by [62, Thm. 4.1] and [64, Cor. 6.33] the Gibbs partition obtained by taking a uniform random graph from $\mathcal{G}$ with n vertices is convergent or superconvergent. This behaviour was established earlier for the class of planar graphs, general minor-closed addable classes, and many related classes [46–48].

6.2. The block decomposition

If we root a connected graph at a vertex, then this marked vertex is contained in some set of *blocks*, that is, maximal 2-connected subgraphs. The entire graph may be decomposed into this collection of vertex-marked blocks and rooted graphs attached to the non-marked vertices. This block decomposition [7, 34, 38] is expressed as follows:

$$x\frac{\partial \mathcal{C}}{\partial x} = x\text{SET}\left(\frac{\partial \mathcal{B}}{\partial x}\left(x\frac{\partial \mathcal{C}}{\partial x}, y\right)\right). \tag{6.2}$$

Hence Lemma 4.1 applies, implying that the class $x\frac{\partial \mathcal{C}}{\partial x}$ may be identified with the $\text{SET}(\frac{\partial \mathcal{B}}{\partial x})$-enriched trees.

6.3. Tutte's decomposition

6.3.1. The 3-decomposition for simple graphs. We recall the 3-decomposition grammar of simple graphs following [29].

A *network* is a graph with two distinguished unlabelled vertices $*_0$ and $*_1$, such that adding the edge $*_0 *_1$ (if absent) yields a 2-connected graph. Here we do not exclude the case that the graph was already 2-connected without this edge. For ease of reference we call $*_0$ the *south pole* and $*_1$ the *north pole*. We let $\mathcal{N}$ denote the class of all networks whose 3-connected components in the Tutte decomposition lie in $\mathcal{F}$, and that are not equal to the trivial network consisting of $*_0$ and $*_1$ without an edge between them. (However, $\mathcal{N}$ does contain the graph consisting of $*_0$ and $*_1$ joined by an edge.). Hence

$$\mathcal{N} = (1+y)\frac{2}{x^2}\frac{\partial \mathcal{B}}{\partial y} - 1. \tag{6.3}$$

There are three types of networks in $\mathcal{N}$, yielding

$$\mathcal{N} = \mathcal{S} + \mathcal{P} + \mathcal{H} \tag{6.4}$$

for the following subclasses of $\mathcal{N}$: Networks from $\mathcal{S} \subset \mathcal{N}$ are *series networks*, obtained by identifying the north pole of one network with the south pole of another, and labelling this vertex and the remaining non-pole vertices. Any series network may be decomposed uniquely into a sequence of non-series networks, yielding

$$\mathcal{S} = (\mathcal{H} + \mathcal{P}) \cdot \mathrm{SEQ}_{\geq 1}(x(\mathcal{H} + \mathcal{P})). \tag{6.5}$$

Networks from $\mathcal{P} \subset \mathcal{N}$ are *parallel networks*, obtained by identifying the south poles of two networks with each other, as well as their north poles. We additionally declare the network consisting of $*_0$ and $*_1$ joined by a single edge to be a parallel network. Hence parallel networks with non-adjacent poles correspond in a unique way to an unordered collection of at least two non-parallel networks. Parallel networks with adjacent poles correspond to a possibly empty unordered collection of non-parallel networks. Hence

$$\mathcal{P} = \mathrm{SET}_{\geq 2}(\mathcal{H} + \mathcal{S}) + y\,\mathrm{SET}(\mathcal{H} + \mathcal{S}). \tag{6.6}$$

Networks from $\mathcal{H} \subset \mathcal{N}$, called *h-networks*, are obtained as follows. The class $\mathcal{F}_{0,1}$ is formed by taking a graph from $\mathcal{F}$, removing an edge, and making its endpoints poles. Thus

$$\mathcal{F}_{0,1} = \frac{2}{x^2}\frac{\partial \mathcal{F}}{\partial y}. \tag{6.7}$$

The class $\mathcal{H}$ is obtained by taking networks from $\mathcal{F}_{0,1}$ and replacing their edges by arbitrary networks from $\mathcal{N}$ (in any possible way). Hence

$$\mathcal{H} = \mathcal{F}_{0,1}(x, \mathcal{N}). \tag{6.8}$$

This leads to

$$\mathcal{N} = (1+y)\mathrm{SET}(\mathcal{F}_{0,1}(x, \mathcal{N}) + \mathcal{S}) - 1. \tag{6.9}$$

6.3.2. An enriched tree encoding. We want to obtain an enriched tree encoding for $\mathcal{N}$. The class $\mathcal{S}$ could be expressed by $\mathcal{N}$ by overcounting and subtracting:

$$\mathcal{S} = \frac{x\mathcal{N}^2}{1 + x\mathcal{N}}. \tag{6.10}$$

Applied to (6.9), this yields a recursive equation for $\mathcal{N}$ involving subtraction operations. However, we require a *subtraction-free* decomposition. For this reason, we take a different path, and define

$$\mathcal{K} := \mathcal{H} + \mathcal{P}. \tag{6.11}$$

Combining (6.4) and (6.5) yields

$$\mathcal{N} = \mathcal{K}\operatorname{SEQ}(x\mathcal{K}) \quad \text{and} \quad x\mathcal{N} = \operatorname{SEQ}_{\geq 1}(x\mathcal{K}). \tag{6.12}$$

By Lemmas 5.1 and 5.3, and condition (6.18) below, the corresponding Gibbs partition obtained by taking a random $\mathcal{N}$-object with n edges and weight $x > 0$ at vertices is convergent or superconvergent. This reduces the study of $\mathcal{N}$ to the study of $\mathcal{K}$.

We proceed to show that $\mathcal{K}$ admits an enriched tree encoding. Condtion (6.5) may be rewritten by

$$\mathcal{S} = \mathcal{K}\operatorname{SEQ}_{\geq 1}(x\mathcal{K}) \quad \text{and} \quad x\mathcal{S} = \operatorname{SEQ}_{\geq 2}(x\mathcal{K}). \tag{6.13}$$

Combining (6.8) and (6.12) yields

$$\mathcal{H} = \mathcal{F}_{0,1}(x, \mathcal{K}\operatorname{SEQ}(x\mathcal{K})). \tag{6.14}$$

Combining (6.11), (6.6), (6.13), and (6.14) yields

$$\begin{aligned}
\mathcal{K} = {} & \mathcal{F}_{0,1}(x, \mathcal{K}\operatorname{SEQ}(x\mathcal{K})) \\
& + \operatorname{SET}_{\geq 2}\big(\mathcal{F}_{0,1}(x, \mathcal{K}\operatorname{SEQ}(x\mathcal{K})) + \mathcal{K}\operatorname{SEQ}_{\geq 1}(x\mathcal{K})\big) \\
& + y\operatorname{SET}\big(\mathcal{F}_{0,1}(x, \mathcal{K}\operatorname{SEQ}(x\mathcal{K})) + \mathcal{K}\operatorname{SEQ}_{\geq 1}(x\mathcal{K})\big).
\end{aligned} \tag{6.15}$$

We may write this as

$$\mathcal{K} = \mathcal{I}(x, \mathcal{K}) + y\mathcal{J}(x, \mathcal{K}), \tag{6.16}$$

with $\mathcal{I}$ and $\mathcal{J}$ representing combinatorial species.

We are not done yet, but the final step is more delicate than it appears. Isomorphisms or identities of combinatorial species are *always* required to be compatible with relabelling. This is what allows us to deduce equations for cycle index sums and ordinary generating series from a single isomorphism between two combinatorial species. It is clear that, for example, an unordered collection of at least two $\mathcal{K}$-objects (that is, an element of $\operatorname{SET}_{\geq 2}(\mathcal{K})$), has no canonically distinguished $\mathcal{K}$-object. Hence there is no species $\mathcal{D}$ with an isomorphism $\operatorname{SET}_{\geq 2}(\mathcal{K}) = \mathcal{K}\mathcal{D}(\mathcal{K})$. That being said, we may still define a weighted species $\mathcal{D}$ that for all $k \geq 1$ has a single object of size k and weight

$1/(k + 1)$. This way, the product $\mathcal{K}\mathcal{D}(\mathcal{K})$ of weighted species has the same exponential generating series as $\mathrm{SET}_{\geq 2}(\mathcal{K})$. Moreover, for each finite set U, the objects from $(\mathcal{K}\mathcal{D}(\mathcal{K}))[U]$ may be grouped into disjoint subsets such that each of these classes has sum of weights 1 and corresponds bijectively to an element of $\mathrm{SET}_{\geq 2}(\mathcal{K})[U]$. We denote this fact by $\mathrm{SET}_{\geq 2}(\mathcal{K}) \equiv \mathcal{K}\mathcal{D}(\mathcal{K})$. In the same way we may form a weighted species $\mathcal{I}^*$ and an analogous correspondence

$$\mathcal{I}(x, \mathcal{K}) \equiv \mathcal{K}\mathcal{I}^*(x, \mathcal{K}). \tag{6.17}$$

We stress that in order to study random unlabelled objects we cannot use this equation directly. Instead we would have to put additional effort into understanding the identity of cycle index sums derived from (6.16). Since the present work concerns exclusively random labelled graphs, this will not be an issue at all, but we want to stress this point due to ongoing research on random unlabelled planar graphs.

Equation (6.17) allows us to unwind (6.16), yielding by induction

$$\begin{aligned}
\mathcal{K} &\equiv y\mathcal{I}(x, \mathcal{K}) + \mathcal{K}\mathcal{I}^*(x, \mathcal{K}) \\
&= y\mathcal{I}(x, \mathcal{K}) + (y\mathcal{I}(x, \mathcal{K}) + \mathcal{K}\mathcal{I}^*(x, \mathcal{K}))\mathcal{I}^*(x, \mathcal{K}) \\
&= y\mathcal{I}(x, \mathcal{K}) + y\mathcal{I}(x, \mathcal{K})\mathcal{I}^*(x, \mathcal{K}) + \mathcal{K}\mathcal{I}^*(x, \mathcal{K})^2 \\
&= \cdots \\
&= y\mathcal{I}(x, \mathcal{K}) \sum_{k \geq 0} \mathcal{I}^*(x, \mathcal{K})^k.
\end{aligned}$$

Thus

$$\mathcal{K} \equiv y\mathcal{R}(x, \mathcal{K}) \tag{6.18}$$

with

$$\mathcal{R}(x, y) = \mathcal{I}(x, y)\mathrm{SEQ}(\mathcal{I}^*(x, y)). \tag{6.19}$$

Hence Lemma 4.1 yields a correspondence between the class $\mathcal{K}$ and the class of $\mathcal{R}$-enriched trees.

7. Tree-like planar map decompositions and convergent Gibbs partitions

We consider planar maps that are rooted at a corner, or equivalently at a half-edge. An exception is made only for the map consisting of a single vertex and no edges. This map has no corners to be rooted at, but we count it as corner-rooted nevertheless.

Throughout this section we let $\mathcal{M}$ denote a class of planar maps that is closed under re-rooting and corners, and stable under Tutte's decomposition. We write $\mathcal{M} = \mathcal{M}(x, z)$ with x marking *non-root* vertices and z marking corners.

7.1. From connected to non-separable

A planar map is termed *separable* if its edge set may be partitioned into two disjoint subsets E_1 and E_2 such that there is precisely one vertex that is incident to both a member of E_1 and a member of E_2. Note that the only non-separable map containing a loop is the map consisting of a single vertex with a loop. The map consisting of a single vertex with no edges is also non-separable.

In [69, Sec. 6] a corner-rooted map was described as consisting of a non-separable map with arbitrary maps inserted at each corner. This may be expressed in combinatorial language. Let $\mathcal{V}$ denote the subclass of all non-separable maps in $\mathcal{M}$. Then

$$z\mathcal{M} = z\mathcal{V}(x, z\mathcal{M}), \tag{7.1}$$

implying by Lemma 4.1 that the class $z\mathcal{M}$ is isomorphic to that of $\mathcal{V}$-enriched trees.

7.2. From non-separable to 3-connected

7.2.1. The 3-decomposition for maps. Let $\mathcal{D}$ denote the class of all plane networks obtained by taking a non-separable map from $\mathcal{V}$ with at least two vertices, removing the root edge, and distinguishing its origin and destination as the south pole and north pole of the network. This way we make the original root edge "invisible". We additionally forbid the network consisting of two poles and no edges between them. We write $\mathcal{D} = \mathcal{D}(x, y)$ with x marking *non-pole* vertices and y marking edges, and likewise for any subclass of $\mathcal{D}$. We let $\bar{\mathcal{F}}_{0,1}$ denote the subclass of $\mathcal{D}$ obtained in the same way from all maps in $\mathcal{V}$ that are 3-connected (and in particular simple, as the definition of k-connectedness for multigraphs additionally requires this). The classes $\mathcal{V}$ and $\mathcal{D}$ are related by

$$\mathcal{V}(x, z) = 1 + z^2 + z^2 x \mathcal{D}(x, z^2) \tag{7.2}$$

with 1 representing the map consisting of a single vertex, and z^2 representing the map with a single vertex and a loop edge. The class $\mathcal{D}$ has a known decomposition (see [15, 35, 45, 68, 70, 71]) into parallel networks $\bar{\mathcal{P}}$, series networks $\bar{\mathcal{S}}$, and h-networks $\bar{\mathcal{H}}$:

$$\mathcal{D} = \bar{\mathcal{S}} + \bar{\mathcal{P}} + \bar{\mathcal{H}}, \tag{7.3}$$

$$\bar{\mathcal{S}} = (\bar{\mathcal{P}} + \bar{\mathcal{H}})\mathrm{SEQ}_{\geq 1}(x(\bar{\mathcal{P}} + \bar{\mathcal{H}})), \tag{7.4}$$

$$\bar{\mathcal{P}} = y + (y + \bar{\mathcal{H}} + \bar{\mathcal{S}})\mathcal{D}, \tag{7.5}$$

$$\bar{\mathcal{H}} = \bar{\mathcal{F}}_{0,1}(x, \mathcal{D}). \tag{7.6}$$

7.2.2. An enriched tree encoding. We proceed in much the same way as for the enriched tree encoding of 2-connected graphs. Setting

$$\bar{\mathcal{K}} = \bar{\mathcal{H}} + \bar{\mathcal{P}}, \tag{7.7}$$

we find from (7.3) and (7.4) that

$$\mathcal{D} = \bar{\mathcal{K}}\,\mathrm{SEQ}(x\bar{\mathcal{K}}) \quad \text{and} \quad x\mathcal{D} = \mathrm{SEQ}_{\geq 1}(x\bar{\mathcal{K}}). \tag{7.8}$$

Lemmas 5.1 and 5.3, and equation (7.14) below, imply that the corresponding Gibbs partition obtained by taking a random $\mathcal{D}$-object with n edges and weight $x > 0$ at vertices is convergent or superconvergent. This reduces the study of $\mathcal{D}$ to the study of $\bar{\mathcal{K}}$.

We are going to show that $\bar{\mathcal{K}}$ admits an enriched tree encoding. Combining (7.4) and (7.6)–(7.8) yields

$$\bar{\mathcal{H}} = \bar{\mathcal{F}}_{0,1}(x, \bar{\mathcal{K}}\text{SEQ}(x\bar{\mathcal{K}})), \tag{7.9}$$

$$\bar{\mathcal{S}} = \bar{\mathcal{K}}\text{SEQ}_{\geq 1}(x\bar{\mathcal{K}}). \tag{7.10}$$

Equations (7.7) and (7.5) yield

$$\bar{\mathcal{K}} = \bar{\mathcal{S}}\mathcal{D} + \bar{\mathcal{H}}(1 + \mathcal{D}) + y(1 + \mathcal{D}). \tag{7.11}$$

Substituting $\mathcal{D}$, $\bar{\mathcal{S}}$, and $\bar{\mathcal{H}}$ by their expressions in terms of $\bar{\mathcal{K}}$ ((7.8)–(7.10)) yields

$$\bar{\mathcal{K}} = \bar{\mathcal{K}}^2\text{SEQ}_{\geq 1}(x\bar{\mathcal{K}})\text{SEQ}(x\bar{\mathcal{K}}) + \bar{\mathcal{F}}_{0,1}(x, \bar{\mathcal{K}}\text{SEQ}(x\bar{\mathcal{K}}))(1 + \bar{\mathcal{K}}\text{SEQ}(x\bar{\mathcal{K}}))$$
$$+ y(1 + \bar{\mathcal{K}}\text{SEQ}(x\bar{\mathcal{K}})). \tag{7.12}$$

We may write this as

$$\bar{\mathcal{K}} = \bar{\mathcal{K}}\bar{\mathcal{J}}^*(x, \bar{\mathcal{K}}) + y\bar{\mathcal{J}}(x, \bar{\mathcal{K}}),$$

with $\bar{\mathcal{J}}^*$ and $\bar{\mathcal{J}}$ representing combinatorial species. We may unroll this identity using induction, yielding

$$\bar{\mathcal{K}} = y\bar{\mathcal{J}}(x, \bar{\mathcal{K}})\text{SEQ}(\bar{\mathcal{J}}^*(x, \bar{\mathcal{K}})). \tag{7.13}$$

and hence

$$\bar{\mathcal{K}} = y\bar{\mathcal{R}}(x, \bar{\mathcal{K}}) \tag{7.14}$$

for

$$\bar{\mathcal{R}} = \bar{\mathcal{J}}(x, y)\text{SEQ}(\bar{\mathcal{J}}^*(x, y)). \tag{7.15}$$

Hence the class $\bar{\mathcal{K}}$ may be identified with the $\bar{\mathcal{R}}$-enriched trees by Lemma 4.1.

8. Asymptotic enumeration using random walks with negative drift

From now on we let $\mathcal{M}$ denote the class of all planar maps, and define the subclasses considered in Section 7 accordingly. Likewise $\mathcal{G}$ denotes the class of all planar graphs and the subclasses of Section 6 are defined accordingly. Hence by Whitney's theorem, given in [72],

$$\mathcal{F}_{0,1} = \frac{1}{2}\bar{\mathcal{F}}_{0,1} = \frac{2}{x^2}\frac{\partial\mathcal{F}}{\partial y}. \tag{8.1}$$

Hence the complete grammar from the previous two sections may be summarized as follows:

$$\mathcal{G} = \text{SET}(\mathcal{C})$$

$$x\frac{\partial \mathcal{C}}{\partial x} = x\,\text{SET}\left(\frac{\partial \mathcal{B}}{\partial x}\left(x\frac{\partial \mathcal{C}}{\partial x}, y\right)\right) \qquad\qquad z\mathcal{M} = z\mathcal{V}(x, z\mathcal{M})$$

$$\mathcal{N} = (1+y)\frac{2}{x^2}\frac{\partial \mathcal{B}}{\partial y} - 1 \qquad\qquad\qquad \mathcal{V} = 1 + z^2 + z^2 x\mathcal{D}(x, z^2)$$

$$\mathcal{N} = \mathcal{K}\,\text{SEQ}(x\mathcal{K}) \qquad\qquad\qquad\qquad \mathcal{D} = \bar{\mathcal{K}}\,\text{SEQ}(x\bar{\mathcal{K}})$$

$$\mathcal{K} \equiv y\mathcal{R}(x, \mathcal{K}) \qquad\qquad\qquad\qquad\quad \bar{\mathcal{K}} = y\bar{\mathcal{R}}(x, \bar{\mathcal{K}})$$

$$\mathcal{R} = \mathcal{J}\,\text{SEQ}(\mathcal{J}^*) \qquad\qquad\qquad\qquad\quad \bar{\mathcal{R}} = \bar{\mathcal{J}}\,\text{SEQ}(\bar{\mathcal{J}}^*)$$

$$\mathcal{J}^* = \frac{1}{y}\mathcal{F}_{0,1}(x, y\,\text{SEQ}(xy)) \qquad\qquad\quad \bar{\mathcal{J}}^* = y\,\text{SEQ}_{\geq 1}(xy)\,\text{SEQ}(xy)$$

$$+ \frac{1}{y}\text{SET}_{\geq 2}(\mathcal{F}_{0,1}(x, y\,\text{SEQ}(xy)) + y\,\text{SEQ}_{\geq 1}(xy)) \qquad + \frac{1}{y}\bar{\mathcal{F}}_{0,1}(x, y\,\text{SEQ}(xy))(1 + y\,\text{SEQ}(xy))$$

$$\mathcal{J} = \text{SET}(\mathcal{F}_{0,1}(x, y\,\text{SEQ}(xy)) + y\,\text{SEQ}_{\geq 1}(xy)) \qquad \bar{\mathcal{J}} = 1 + y\,\text{SEQ}(xy)$$

$$\mathcal{F}_{0,1} = \frac{1}{2}\bar{\mathcal{F}}_{0,1} = \frac{2}{x^2}\frac{\partial \mathcal{F}}{\partial y}$$

8.1. Planar graphs

In [6] it is proved that

$$[x^n]\mathcal{B}(x, 1) \sim c_{\mathcal{B}}\rho_{\mathcal{B}}^{-n}n^{-7/2}, \tag{8.2}$$

with $c_{\mathcal{B}} \approx 0.37042 \cdot 10^{-5}$ and $\rho_{\mathcal{B}} \approx 0.03819$. Setting $\phi_{\mathcal{C}}(x) := \exp(\frac{\partial \mathcal{B}}{\partial x}(x, 1))$ we have

$$\nu_{\mathcal{C}} := \lim_{t \nearrow \rho_{\mathcal{B}}} \phi'_{\mathcal{C}}(x)x/\phi_{\mathcal{C}}(x) = \rho_{\mathcal{B}}\frac{\partial^2 \mathcal{B}}{\partial x^2}(\rho_{\mathcal{B}}, 1). \tag{8.3}$$

Any connected graph with n vertices has at least $n - 1$ edges. Using this it is elementary that

$$\nu_{\mathcal{C}} < \frac{\partial^2 \mathcal{B}}{\partial x \partial y}(\rho_{\mathcal{B}}, 1). \tag{8.4}$$

The proof in [6] also gives a singular expansion

$$\mathcal{N}(x, 1) = D_0 + D_2 X^2 + D_3 X^3 + O(X^4), \quad X = \sqrt{1 - x/\rho_{\mathcal{B}}},$$

with analytic expressions for the constants $D_0 \approx 1.09417$ and $D_2 \approx -0.13749$.[1] Together with (6.3) this allows us to evaluate the upper bound in (8.4), yielding

$$\nu_{\mathcal{E}} < \rho_{\mathcal{B}} \left(\frac{1 + D_0}{2} - 1 \right) - \frac{D_2 \rho_{\mathcal{B}}}{4} \approx 0.041302 < 1. \tag{8.5}$$

By (5.2) it follows that

$$[x^n]\phi_{\mathcal{E}}(x) \sim \exp\left(\frac{\partial \mathcal{B}}{\partial x}(\rho_{\mathcal{B}}, 1) \right) [x^n]\frac{\partial \mathcal{B}}{\partial x}(x, 1) \sim \rho_{\mathcal{B}}^{-1} \exp\left(\frac{\partial \mathcal{B}}{\partial x}(\rho_{\mathcal{B}}, 1) \right) c_{\mathcal{B}} \rho_{\mathcal{B}}^{-n} n^{-5/2}. \tag{8.6}$$

Since $\nu_{\mathcal{E}} < 1$ and since the coefficients of $\phi_{\mathcal{E}}(\rho_{\mathcal{B}} x)$ are regularly varying with index $-5/2$, we may apply (3.7) to obtain

$$[x^n]\mathcal{C}(x, 1) \sim c_{\mathcal{E}} \rho_{\mathcal{E}}^{-n} n^{-7/2} \tag{8.7}$$

with

$$\rho_{\mathcal{E}} = \rho_{\mathcal{B}} / \phi_{\mathcal{E}}(\rho_{\mathcal{B}}) \tag{8.8}$$

and $c_{\mathcal{E}} = c_{\mathcal{B}} (1 - \nu_{\mathcal{E}})^{-5/2}$. Condition (5.2) implies

$$[x^n]\mathcal{G}(x, 1) \sim c_{\mathcal{G}} \rho_{\mathcal{E}}^{-n} n^{-7/2} \tag{8.9}$$

with

$$c_{\mathcal{G}} = c_{\mathcal{E}} \exp(\mathcal{C}(\rho_{\mathcal{E}}, 1)). \tag{8.10}$$

Equations (8.7)–(8.10) were obtained in [31, Thm. 1] using analytic methods, which additionally yield singular expansions of all the generating series involved.

8.2. 3-connected planar networks

In [53] the expression

$$\bar{\mathcal{F}}_{0,1}(x, y) = y \left(\frac{1}{1 + xy} + \frac{1}{1 + y} - 1 - \frac{(1 + u)^2(1 + v)^2}{(1 + u + v)^3} \right) \tag{8.11}$$

is obtained, where $u = u(x, y)$ and $v = (x, y)$ are specified by the system

$$u = xy(1 + v)^2 \quad \text{and} \quad v = y(1 + u))^2. \tag{8.12}$$

This system yields the asymptotic enumeration of 3-connected planar networks. We follow the presentation in [20, Thm. 9.13]. Let $t > 0$ be a constant. Applying [20, Thm. 2.19]

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[1] When verifying these approximations, note that the expression for D_2 in [6] contains a small typo: it lacks a factor t. See [31].

yields square root singular expansions for $y \mapsto u(t, y)$ and $y \mapsto v(t, y)$. This leads to the representation

$$\frac{(1+u)^2(1+v)^2}{(1+u+v)^3} = E_0(t)+E_2(t)Z^2+E_3(t)Z^3+O(Z^4) \quad \text{with} \quad Z = \sqrt{1-\frac{y}{\rho_{\mathcal{F}}(t)}}. \tag{8.13}$$

Here $E_0(t)$, $E_2(t)$, $E_3(t)$, $\rho_{\mathcal{F}}(t)$ denote non-zero constants that depend only on t and admit explicit expressions. Specifically, if u_0 is the positive solution of

$$t = \frac{(1 + u_0)(3u_0 - 1)^3}{16u_0}, \tag{8.14}$$

then

$$\rho_{\mathcal{F}}(t) = \frac{1}{(u_0 + 1)(3u_0 - 1)}, \quad E_0(t) = \frac{16(3u_0 - 1)}{27u_0(u_0 + 1)},$$
$$E_2(t) = \frac{16(3u_0^2 + 1)(3u_0 - 1)}{81u_0^2(u_0 + 1)^2}. \tag{8.15}$$

Hence

$$\bar{\mathcal{F}}_{0,1}(t, y) = F_0(t) + F_2(t)Z^2 + F_3(t)Z^3 + O(Z^4) \tag{8.16}$$

for some non-zero constants $F_0(t)$, $F_2(t)$, $F_3(t)$, $\rho_{\mathcal{F}}(t)$ that depend only on t. Hence, by transfer theorems in [27, Ch. 6],

$$[y^n]\bar{\mathcal{F}}_{0,1}(t, y) \sim c_{\mathcal{F}}(t)\rho_{\mathcal{F}}(t)^{-n}n^{-5/2} \tag{8.17}$$

with $c_{\mathcal{F}}(t) = F_3(t)\frac{3}{4\sqrt{\pi}}$.

8.3. Non-separable planar maps

The functions $y \mapsto \mathcal{R}(t, y)$ and $y \mapsto \bar{\mathcal{R}}(t, y)$ both have radius of convergence

$$\rho_{\mathcal{R}}(t) = \frac{\rho_{\mathcal{F}}(t)}{1 + t\rho_{\mathcal{F}}(t)}. \tag{8.18}$$

Simplifying (7.15) yields

$$\bar{\mathcal{R}}(t, y) = \frac{(1 - ty)(1 + \tilde{u} + \tilde{v})^3}{(1 + \tilde{u})^2(1 + \tilde{v})^2} \quad \text{with} \quad \tilde{u} = u(t, \frac{y}{1 - ty}), \tilde{v} = v\left(t, \frac{y}{1 - ty}\right). \tag{8.19}$$

Expansion (8.13) allows us also to deduce a singular expansion of $\bar{\mathcal{R}}(t, y)$, yielding

$$[y^n]\bar{\mathcal{R}}(t, y) \sim c_{\bar{\mathcal{R}}}(t)\rho_{\mathcal{R}}(t)^{-n}n^{-5/2} \tag{8.20}$$

for some constant $c_{\bar{\mathcal{R}}}(t) > 0$. The ν-parameter from (3.2) corresponding to $\bar{\mathcal{R}}(t, y)$ is given by

$$\nu_{\bar{\mathcal{K}}}(t) := \frac{\rho_{\mathcal{R}}(t)\frac{\partial \bar{\mathcal{R}}}{\partial y}(t, \rho_{\mathcal{R}}(t))}{\bar{\mathcal{R}}(t, \rho_{\mathcal{R}}(t))} = \rho_{\mathcal{R}}(t)\frac{\partial}{\partial y}\log(\bar{\mathcal{R}}(t, y))\Big|_{y=\rho_{\mathcal{R}}(t)}. \tag{8.21}$$

Using expansion (8.13) it follows that

$$v_{\bar{\mathcal{K}}}(t) = \frac{E_2}{E_0}(1 + t\rho_{\mathcal{F}}(t)) - t\rho_{\mathcal{F}}(t) = \frac{21u_0^2 + 6u_0 + 1}{48u_0^2}. \tag{8.22}$$

The rational function $r \mapsto \frac{(1+r)(3r-1)^3}{16r}$ is strictly increasing on $\mathbb{R}_{>0}$ and assumes the value 0 at $r = 1/3$. As $t > 0$, it follows from (8.14) that

$$u_0 = u_0(t) > 1/3. \tag{8.23}$$

The function $s \mapsto \frac{21s^2+6s+1}{48s^2}$ is strictly decreasing on $\mathbb{R}_{>0}$, and assumes the value 1 at $s = 1/3$. Hence

$$v_{\bar{\mathcal{K}}}(t) < 1 \tag{8.24}$$

for all $t > 0$. This allows us to apply (3.7), yielding

$$[y^n]\bar{\mathcal{K}}(t, y) \sim c_{\bar{\mathcal{K}}}(t)\rho_{\bar{\mathcal{K}}}(t)^{-n}n^{-5/2} \tag{8.25}$$

with $\rho_{\bar{\mathcal{K}}}(t) = \rho_{\mathcal{R}}(t)/\bar{\mathcal{R}}(t, \rho_{\mathcal{R}})$ and $c_{\bar{\mathcal{K}}}(t) = \rho_{\mathcal{R}}(t)c_{\bar{\mathcal{R}}}(t)(1 - v_{\bar{\mathcal{K}}}(t))^{-5/2}$. Condition (5.2) implies

$$[y^n]\mathcal{D}(t, y) \sim c_{\mathcal{D}}(t)\rho_{\bar{\mathcal{K}}}(t)^{-n}n^{-5/2} \tag{8.26}$$

with $c_{\mathcal{D}}(t) = c_{\bar{\mathcal{K}}}(t)/(1 - t\bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}(t)))$. Hence we may deduce the known asymptotic formula

$$[z^{2n}]\mathcal{V}(t, z) \sim c_{\mathcal{V}}(t)\rho_{\bar{\mathcal{K}}}(t)^{-n}n^{-5/2} \tag{8.27}$$

with $c_{\mathcal{V}}(t) = tc_{\mathcal{D}}(t)\rho_{\bar{\mathcal{K}}}(t)$.

8.4. Planar maps

By (8.27), the function $z \mapsto \mathcal{V}(t, z)$ has radius of convergence $\rho_{\mathcal{V}}(t) = \sqrt{\rho_{\bar{\mathcal{K}}}(t)}$. Hence the v-parameter from (3.2) corresponding to $\mathcal{V}(t, z)$ is given by

$$v_{\mathcal{M}}(t) := \frac{\rho_{\mathcal{V}}(t)\frac{\partial \mathcal{V}}{\partial y}(t, \rho_{\mathcal{V}}(t))}{\mathcal{V}(t, \rho_{\mathcal{V}}(t))} = \frac{2\rho_{\bar{\mathcal{K}}}(t)(1 + t\mathcal{D}(t, \rho_{\bar{\mathcal{K}}}(t)) + t\rho_{\bar{\mathcal{K}}}(t)\frac{\partial \mathcal{D}}{\partial y}(t, \rho_{\bar{\mathcal{K}}}(t)))}{1 + \rho_{\bar{\mathcal{K}}}(t)(1 + t\mathcal{D}(t, \rho_{\bar{\mathcal{K}}}(t)))}. \tag{8.28}$$

We have $\bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}(t)) = \rho_{\mathcal{R}}(t)$ by Lemma 3.1. Hence, in view of (8.18),

$$\mathcal{D}(t, \rho_{\bar{\mathcal{K}}}(t)) = \frac{\bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}(t))}{1 - t\bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}(t))} = \frac{\rho_{\mathcal{R}}(t)}{1 - t\rho_{\mathcal{R}}(t)} = \rho_{\mathcal{F}}(t). \tag{8.29}$$

Using $\rho_{\bar{\mathcal{K}}}(t) = \rho_{\mathcal{R}}(t)/\bar{\mathcal{R}}(t, \rho_{\mathcal{R}}(t))$, $\bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}(t)) = \rho_{\mathcal{R}}(t)$, and (8.18), we obtain

$$\rho_{\bar{\mathcal{K}}}(t)\frac{\partial \mathcal{D}}{\partial y}(t, \rho_{\bar{\mathcal{K}}}(t)) = \frac{\rho_{\bar{\mathcal{K}}}(t)\frac{\partial \bar{\mathcal{K}}}{\partial y}(t, \rho_{\bar{\mathcal{K}}}(t))}{1 - t\bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}(t))} = \rho_{\mathcal{F}}(t)\frac{\frac{\partial \bar{\mathcal{K}}}{\partial y}(t, \rho_{\bar{\mathcal{K}}}(t))}{\bar{\mathcal{R}}(t, \rho_{\mathcal{R}})}. \tag{8.30}$$

Differentiating $\bar{\mathcal{K}}(t, y) = y\bar{\mathcal{R}}(t, \bar{\mathcal{K}}(t, y))$ yields

$$\frac{\partial \bar{\mathcal{K}}}{\partial y}(t, y) = \frac{\bar{\mathcal{R}}(t, \bar{\mathcal{K}}(t, y))}{1 - y\frac{\partial \bar{\mathcal{R}}}{\partial y}(t, \bar{\mathcal{K}}(t, y))}. \tag{8.31}$$

Using the definition (8.21) of $\nu_{\bar{\mathcal{K}}}(t)$, one can simplify (8.30) to

$$\rho_{\bar{\mathcal{K}}}(t)\frac{\partial \mathcal{D}}{\partial y}(t, \rho_{\bar{\mathcal{K}}}(t)) = \frac{\rho_{\mathcal{F}}(t)}{1 - \nu_{\bar{\mathcal{K}}}(t)}. \tag{8.32}$$

Using (8.29), we simplify (8.28) to

$$\nu_{\mathcal{M}}(t) = \frac{2\rho_{\bar{\mathcal{K}}}(t)(1 + t\rho_{\mathcal{F}}(t)(1 + 1/(1 - \nu_{\bar{\mathcal{K}}}(t))))}{1 + \rho_{\bar{\mathcal{K}}}(t)(1 + t\rho_{\mathcal{F}}(t))}. \tag{8.33}$$

Note that by (8.19), (8.18) and (8.13),

$$\rho_{\bar{\mathcal{K}}}(t) = \frac{\rho_{\mathcal{R}}(t)}{\bar{\mathcal{R}}(t, \rho_{\mathcal{R}}(t))} = \rho_{\mathcal{F}}(t)\frac{(1 + u(1, \rho_{\mathcal{F}}(t)))^2(1 + v(\rho_{\mathcal{F}}(t)))^2}{(1 + u(t, \rho_{\mathcal{F}}(t)) + v(t, \rho_{\mathcal{F}}(t)))^3} = \rho_{\mathcal{F}}(t)E_0(t). \tag{8.34}$$

Plugging this into (8.33) yields an expression of $\nu_{\mathcal{M}}(t)$ in terms of t, $\rho_{\mathcal{F}}(t)$, $E_0(t)$, and $\nu_{\bar{\mathcal{K}}}(t)$. Equations (8.15) and (8.22) yield expressions of $\rho_{\mathcal{F}}(t)$, $E_0(t)$, and $\nu_{\bar{\mathcal{K}}}(t)$ in terms of u_0. Hence we obtain an expression of $\nu_{\mathcal{M}}(t)$ in terms of t and u_0. Using (8.14) we may simplify this to an expression in terms of u_0 alone:

$$\nu_{\mathcal{M}}(t) = \frac{2(1 + 19u_0 + 51u_0^2 + 225u_0^3)}{(1 + u_0)(1 + 3u_0)^3(1 + 9u_0)}. \tag{8.35}$$

This rational expression is strictly decreasing in u_0, and assumes the value 1 at $1/3$. As $u_0 = u_0(t) > 1/3$ by (8.23), this implies

$$\nu_{\mathcal{M}}(t) < 1. \tag{8.36}$$

From (3.3) and (3.5), it follows that

$$[z^n]\mathcal{M}(t, z) \sim c_{\mathcal{M}}(t)n^{-5/2}\rho_{\mathcal{M}}(t)^{-n} \tag{8.37}$$

for $n \in 2\mathbb{N}_0$ large, with $\rho_{\mathcal{M}}(t) = \frac{\rho_{\mathcal{V}}(t)}{\mathcal{V}(t, \rho_{\mathcal{V}}(t))}$ and $c_{\mathcal{M}}(t) = \frac{c_{\mathcal{V}}(t)\rho_{\bar{\mathcal{K}}}(t)}{\mathcal{V}(t, \rho_{\mathcal{V}}(t))(1 - \nu_{\mathcal{M}}(t))^{5/2}}$.

8.5. Planar networks

Simplifying (6.19) yields

$$\mathcal{R}(t, y) = \frac{e^{\mathcal{F}(t, \frac{y}{1-ty}) + \frac{ty^2}{1-ty}}y(1 - ty)}{1 + y - ty - e^{\mathcal{F}(t, \frac{y}{1-ty}) + \frac{ty^2}{1-ty}}(1 - ty)}. \tag{8.38}$$

The ν-parameter from (3.2) corresponding to $\mathcal{R}(t, y)$ may be expressed by

$$\nu_{\mathcal{K}}(t) := \frac{\rho_{\mathcal{R}}(t)\frac{\partial \mathcal{R}}{\partial y}(t, \rho_{\mathcal{R}}(t))}{\mathcal{R}(t, \rho_{\mathcal{R}}(t))} = \rho_{\mathcal{R}}(t)\frac{\partial}{\partial y}\log(\mathcal{R}(t, y))\Big|_{y=\rho_{\mathcal{R}}(t)}. \tag{8.39}$$

Setting

$$F(y) := \mathcal{F}\left(t, \frac{y}{1 - ty}\right) + \frac{ty^2}{1 - ty} \tag{8.40}$$

we obtain

$$\frac{\partial}{\partial y}\log(\mathcal{R}(t, y)) = F'(y) + \frac{1}{y} - \frac{t}{1 - ty} - \frac{1 - t + \exp(F(y))(t - F'(y)(1 - ty))}{1 + y(1 - t) - \exp(F(y))(1 - ty)}. \tag{8.41}$$

Much as for planar maps, this allows us to express $\nu_K(t)$ in terms of u_0, and using (8.23) we get

$$\nu_{\mathcal{K}}(t) < 1. \tag{8.42}$$

Expansion (8.13) implies a singular expansion of $\mathcal{R}(t, y)$, yielding

$$[y^n]\mathcal{R}(t, y) \sim c_{\mathcal{R}}(t)\rho_{\mathcal{R}}(t)^{-n}n^{-5/2} \tag{8.43}$$

for some constant $c_{\mathcal{R}}(t) > 0$. By (3.7) it follows that

$$[y^n]\mathcal{K}(t, y) \sim c_{\mathcal{K}}(t)\rho_{\mathcal{K}}(t)^{-n}n^{-5/2} \tag{8.44}$$

with $\rho_{\mathcal{K}}(t) = \rho_{\mathcal{R}}(t)/\mathcal{R}(t, \rho_{\mathcal{R}})$ and $c_{\mathcal{K}}(t) = \rho_{\mathcal{R}}(t)c_{\mathcal{R}}(t)(1 - \nu_{\mathcal{K}}(t))^{-5/2}$. Hence by (5.2),

$$[y^n]\mathcal{N}(t, y) \sim c_{\mathcal{N}}(t)\rho_{\mathcal{K}}(t)^{-n}n^{-5/2} \tag{8.45}$$

with $c_{\mathcal{N}}(t) = c_{\mathcal{K}}(t)/(1 - t\mathcal{K}(t, \rho_{\mathcal{K}}(t)))$.

9. Quenched local convergence

9.1. Weighted non-separable maps

We write $v(\cdot)$, $e(\cdot)$, and $c(\cdot)$ for the numbers of vertices, edges, and corners. Let $t > 0$ be a constant. We let M_n^t denote a random planar map with n edges that is drawn with probability proportional to $t^{v(M)}$.

As corner-rooted planar maps are asymmetric, it follows from (7.1) and Section 4.2 that any planar map M with n corners corresponds bijectively to a pair (T, β) of a (planted) plane tree T with $2n + 1$ vertices and a function β that assigns to each vertex $v \in T$ a non-separable map with $d_T^+(v)$ corners. Here the decoration $\beta(o)$ of the root vertex o of T corresponds to the non-separable component of M containing the root edge. The non-root corners of $\beta(o)$ correspond to the offspring vertices of o in T. The decorated

fringe subtree at such an offspring vertex represents the map inserted at the correspond-
ing corner of $\beta(o)$. Here *corresponding* means according to a fixed canonical ordering of
the non-root corners of $\beta(o)$. We choose this ordering according to a breadth-first-search
exploration, so that the distance to the root corner is non-decreasing in the ordering.

The enriched tree $(\mathsf{T}_n^{\mathcal{M}}, \beta_n^{\mathcal{M}})$ corresponding to the random map M_n^t admits an easy
description. By Lemma 4.2 and subsequent remarks on asymmetric species in Section 4.2,
the random plane tree $\mathsf{T}_n^{\mathcal{M}}$ is a simply generated tree with weight sequence $(\omega_k^{\mathcal{M}})_{k \geq 0}$ given
by

$$\omega_k^{\mathcal{M}} = [z^k]\mathcal{V}(t, z), \quad k \geq 0. \tag{9.1}$$

Given $\mathsf{T}_n^{\mathcal{M}}$, each decoration $\beta_n^{\mathcal{M}}(v)$, $v \in \mathsf{T}_n^{\mathcal{M}}$, is drawn with probability proportional to its
weight from all non-separable maps with $d_{\mathsf{T}_n^{\mathcal{M}}}^+(v)$ corners, independently of the remaining
decorations. Here the weight of such a map V is $t^{v(V)-1}$.

By inequality (8.36), the asymptotic expression (8.27), and Lemma 3.1 it follows that
$\mathsf{T}_n^{\mathcal{M}}$ is distributed like a $\xi^{\mathcal{M}}$-Galton–Watson tree $\mathsf{T}^{\mathcal{M}}$ conditioned on having $2n + 1$ ver-
tices, with offspring distribution $\xi^{\mathcal{M}}$ satisfying

$$\mathbb{E}[\xi^{\mathcal{M}}] < 1 \quad \text{and} \quad \mathbb{P}(\xi^{\mathcal{M}} = 2n) \sim \frac{c_{\mathcal{V}}(t)}{\mathcal{V}(t, \rho_{\bar{\mathcal{K}}}(t))} n^{-5/2}. \tag{9.2}$$

This offspring distribution $\xi^{\mathcal{M}}$ is a random even integer. Hence $\xi^{\mathcal{M}}/2$ satisfies condition
(3.4), but $\xi^{\mathcal{M}}$ does not. However, it is easy to see that Lemma 3.2 may be extended to
this setting. The reason is that the proof of Lemma 3.2 given in [65, Thm. 1.1] uses the
well-known fact that $\Delta(\mathsf{T}_n^{\mathcal{M}})$ corresponds to the largest jump in an n-step random walk
with step-distribution $\xi^{\mathcal{M}}$ conditioned on arriving at $n - 1$ after n steps. Hence the proof
of Lemma 3.2 may be adapted to this setting by rescaling by the factor $\frac{1}{2}$. Keeping in
mind that $\mathsf{T}_n^{\mathcal{M}}$ has $2n + 1$ vertices, it follows that the largest non-separable component
$\mathcal{V}(\mathsf{M}_n^t)$ (unique with high probability) satisfies

$$\mathbb{P}(e(\mathcal{V}(\mathsf{M}_n^t)) = \ell) = \frac{1}{g_{\mathcal{M}}(t)n^{2/3}} \left(h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{g_{\mathcal{M}}(t)n^{2/3}} \right) + o(1) \right) \tag{9.3}$$

uniformly for all $\ell \in \mathbb{Z}$ with $g_{\mathcal{M}}(t) > 0$ a constant. A similar probabilistic approach to
the block sizes in random planar maps was used in [3], and the local limit theorem itself
is a celebrated result of [5]. Likewise, Lemma 3.3 also holds in this setting despite the
periodicity, as its proof given in [65, Thm. 1.2] may be adapted analogously.

The following quenched limit was shown recently in [66] using quenched limits
for extended fringe subtrees of re-rooted multi-type Galton–Watson trees from [67] and
applying the Bouttier–Di Francesco–Guitter bijection [10, Sec. 2]. An annealed version
may be deduced by applying planar duality to the earlier annealed convergence of face-
weighted random planar maps by Stephenson [61], who established local convergence of
such multi-type trees close to the fixed root.

Lemma 9.1 ([66, Thm. 1]). *Let c_n denote a uniformly selected corner of M_n^t. There is a random infinite planar map $\hat{\mathsf{M}}^t$ with*

$$\mathfrak{L}((\mathsf{M}_n^t, c_n) \mid \mathsf{M}_n^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{M}}^t). \tag{9.4}$$

Recall that we may enumerate the corners of the non-separable core $\mathcal{V}(\mathsf{M}_n^t)$ in a canonical order, for example by using a breadth-first-search starting with the root corner. The map M_n^t consists of the core $\mathcal{V}(\mathsf{M}_n^t)$ together with maps $(\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))}$ attached to its corners. The map $\mathcal{M}_1(\mathsf{M}_n^t)$ has an additional marked corner, corresponding to the root corner of M_n^t. For each $2 \leq i \leq \Delta(\mathsf{T}_n^{\mathcal{M}})$ the map $\mathcal{M}_i(\mathsf{M}_n^t)$ attached to the ith corner is determined by the fringe subtree $F_i(\mathsf{T}_n^{\mathcal{M}})$ and the restriction $\beta_n^{\mathcal{M}}|_{F_i(\mathsf{T}_n^{\mathcal{M}})}$ to its vertex set. The root corner of $\mathcal{V}(\mathsf{M}_n^t)$ (corresponding to $i = 1$) differs, as there are two maps attached to it: the map determined by $(F_1(\mathsf{T}_n^{\mathcal{M}}), \beta_n^{\mathcal{M}}|_{F_1(\mathsf{T}_n^{\mathcal{M}})})$, and the bi-corner-rooted map determined by the marked plane tree $F_0(\mathsf{T}_n^{\mathcal{M}})$ and the restriction of $\beta_n^{\mathcal{M}}$ to its non-marked vertices.

We let M^t denote the random finite planar map corresponding to the tree $\mathsf{T}^{\mathcal{M}}$ with canonical random decorations $\beta^{\mathcal{M}}$. That is, for each vertex $v \in \mathsf{T}^{\mathcal{M}}$ the decoration $\beta^{\mathcal{M}}(v)$ is drawn with probability proportional to its weight from all non-separable maps with $d_{\mathsf{T}^{\mathcal{M}}}^+(v)$ corners, independently of the remaining decorations. For each $i \geq 2$ let $\mathsf{M}(i)$ be an independent copy of the planar map M^t. Let $\mathsf{T}^{\circ \mathcal{M}}$ be defined for the offspring distribution $\xi^{\mathcal{M}}$ analogously to T° defined for ξ in Section 3.2. We define $\mathsf{M}(1)$ as an independent random map with a second marked corner, so that $\mathsf{M}(1)$ corresponds to canonically decorated versions of $\mathsf{T}^{\circ \mathcal{M}}$ and $\mathsf{T}^{\mathcal{M}}$ in the same way as $\mathcal{M}_1(\mathsf{M}_n^t)$ does to decorated versions of $F_0(\mathsf{T}_n^{\mathcal{M}})$ and $F_1(\mathsf{T}_n^{\mathcal{M}})$.

Consider the random planar map $\mathsf{M}_{[n]}^t$ constructed from the core $\mathcal{V}(\mathsf{M}_n^t)$ by attaching for each integer $1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))$ the independent random map $\mathsf{M}(i)$ instead of $\mathcal{M}_i(\mathsf{M}_n^t)$ at the ith corner of $\mathcal{V}(\mathsf{M}_n^t)$. Thus, $\mathsf{M}_{[n]}^t$ and M_n^t are different models of random maps. For example, the number of edges in $\mathsf{M}_{[n]}^t$ is $n + O_p(n^{2/3})$ and does not have to equal precisely n. Nevertheless, local convergence of M_n^t is equivalent to local convergence of $\mathsf{M}_{[n]}^t$, with the same limit map in both cases:

Lemma 9.2. *Let $c_{[n]}$ denote a uniformly selected corner of $\mathsf{M}_{[n]}^t$. Then*

$$\mathfrak{L}((\mathsf{M}_{[n]}^t, c_{[n]}) \mid \mathsf{M}_{[n]}^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{M}}^t). \tag{9.5}$$

Proof. Keeping in mind that $\mathsf{T}_n^{\mathcal{M}}$ has $2n + 1$ vertices, we set

$$\Delta_{\langle n \rangle}^{\mathcal{M}} := \sup \left\{ d \geq 1 \;\middle|\; c(\mathsf{M}(1)) + \sum_{i=2}^{d} (1 + c(\mathsf{M}(i))) \leq 2n \right\}.$$

By (our adaption of) Lemma 3.3 it follows that for any sequence $(t_n)_n$ of integers with $t_n \to \infty$ and $t_n = o(n)$,

$$(\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t)) - t_n} \overset{d}{\approx} (\mathsf{M}(i))_{1 \leq i \leq \Delta_{\langle n \rangle}^{\mathcal{M}} - t_n} \tag{9.6}$$

and with high probability,

$$\sum_{i=c(\mathcal{V}(\mathsf{M}_n^t))-t_n}^{c(\mathcal{V}(\mathsf{M}_n^t))} c(\mathcal{M}_i(\mathsf{M}_n^t)) \leq 2\mathbb{E}[c(\mathsf{M}^t)]t_n. \tag{9.7}$$

Let $0 < \delta < 2(1 - \mathbb{E}[\xi^{\mathcal{M}}])$ and $\varepsilon > 0$ be given. In view of Lemma 3.4 there are constants $0 < c < C$, $N_0 > 0$ and collections $(\mathcal{E}_n)_{n \geq 1}$ (each consisting of a set of finite sequences of planar maps, with the first map having an additional marked corner) such that for all $n > N_0$,

$$\mathbb{P}((\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor} \in \mathcal{E}_n) > 1 - \varepsilon,$$
$$\mathbb{P}((\mathsf{M}(i))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor} \in \mathcal{E}_n) > 1 - \varepsilon \tag{9.8}$$

and uniformly for all sequences $S \in \mathcal{E}_n$,

$$c < \frac{\mathbb{P}((\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor} = S)}{\mathbb{P}((\mathsf{M}(i))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor} = S)} < C. \tag{9.9}$$

Note that knowing $(\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor}$ implies knowing $c(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor$ and hence $c(\mathcal{V}(\mathsf{M}_n^t))$. Furthermore, conditionally on $c(\mathcal{V}(\mathsf{M}_n^t))$ the core $\mathcal{V}(\mathsf{M}_n^t)$ assumes an $c(\mathcal{V}(\mathsf{M}_n^t))$-sized non-separable map with probability proportional to its weight. Let $\mathcal{E}_n^{\mathcal{M}}$ denote the collection of all pairs (V, S) such that S is a tuple in $\mathcal{E}_n$ and V is a non-separable map whose number of corners agrees with the core size corresponding to S. That is, the number of corners of V equals the dimension of S plus $\lfloor \delta n \rfloor$. It follows from (9.8) and (9.9) that

$$\mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}\big) > 1 - \varepsilon, \tag{9.10}$$

$$\mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t), (\mathsf{M}(i))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}\big) > 1 - \varepsilon, \tag{9.11}$$

and uniformly for all $(V, S) \in \mathcal{E}_n^{\mathcal{M}}$,

$$c < \frac{\mathbb{P}((\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor}) = (V, S))}{\mathbb{P}((\mathcal{V}(\mathsf{M}_n^t), (\mathsf{M}(i))_{1 \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor}) = (V, S))} < C. \tag{9.12}$$

From (9.7) it follows that with high probability

$$\sum_{i=c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor \delta n \rfloor+1}^{c(\mathcal{V}(\mathsf{M}_n^t))} c(\mathcal{M}_i(\mathsf{M}_n^t)) \leq 2\mathbb{E}[c(\mathsf{M}^t)]\lfloor \delta n \rfloor. \tag{9.13}$$

Let $O_{n,\delta}$ denote the subset of vertices in M_n^t corresponding to the components $\mathcal{M}_i(\mathsf{M}_n^t)$ for $c(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor \leq i \leq c(\mathcal{V}(\mathsf{M}_n^t))$. For each $k \geq 0$ we let $U_k(\mathsf{M}_n^t, O_{n,\delta})$ denote the collection of vertices of M_n^t with graph distance at most k from $O_{n,\delta}$. Using only the local convergence of M_n^t and the fact that $O_{n,\delta}$ becomes small as $\delta \to \infty$ we are going to show that for each $k \geq 0$ and each $\varepsilon' > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}(|U_k(\mathsf{M}_n^t, O_{n,\delta})| > \varepsilon' n) = 0. \tag{9.14}$$

The verification of (9.14) is by induction on k. For $k = 0$ the statement is clear, because $|U_0(\mathsf{M}_n^t, O_{n,\delta})| = |O_{n,\delta}|$ and (9.13) entails that with high probability, $|O_{n,\delta}| \le 2\mathbb{E}[\mathsf{c}(\mathsf{M}^t)]\delta n$. For the induction step, suppose that $k \ge 1$ and (9.14) holds for $k-1$ but fails for k. Then there is an $\varepsilon_0 > 0$, sequences $\varepsilon_\ell', \delta_\ell \downarrow 0$, and a subsequence n_ℓ such that

$$\mathbb{P}\big(|U_k(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})| > \varepsilon' n_\ell, \ |U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})| < \varepsilon_\ell' n_\ell\big) > \varepsilon_0$$

for all $\ell \ge 1$. This means that with probability bounded away from 0 there are at least $n_\ell(\varepsilon' - \varepsilon_\ell')$ vertices in $U_k(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell}) \setminus U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$, each connected by at least one edge to a vertex from $U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$. Let Ω_ℓ be a sequence satisfying $\Omega_\ell \to \infty$ and $\Omega_\ell \varepsilon_\ell' \to 0$. The subset of vertices from $U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$ with degree less than Ω_ℓ is linked to at most $\varepsilon_\ell' n_\ell \Omega_\ell = o(n_\ell)$ vertices from $U_k(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell}) \setminus U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$. Hence at least $(\varepsilon' - o(1))n_\ell$ vertices from $U_k(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell}) \setminus U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$ have a neighbour from $U_{k-1}(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$ with degree at least Ω_ℓ. This implies that with probability bounded away from 0 the uniform corner c_n of M_n^t has a neighbour with degree at least Ω_ℓ. But this is impossible, since $\Omega_\ell \to \infty$ and neighbourhoods of c_n must remain stochastically bounded by the local convergence in Lemma 9.1. This completes the induction, proving that (9.14) holds.

Let us write $U_k^c(\mathsf{M}_n^t, O_{n,\delta})$ for the collection of corners in M_n^t that are incident to vertices of $U_k(\mathsf{M}_n^t, O_{n,\delta})$. It follows from (9.14) and the local convergence in Lemma 9.1 that for any $\varepsilon' > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}(|U_k^c(\mathsf{M}_n^t, O_{n,\delta})| > \varepsilon' n) = 0. \tag{9.15}$$

Indeed, if (9.15) did not hold, we could use (9.14) to find an $\varepsilon_0 > 0$, a subsequence n_ℓ, and a sequence $\delta_\ell \to 0$ such that for all $\ell \ge 1$,

$$\mathbb{P}\big(|U_k^c(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})| > \varepsilon' n_\ell, \ |U_k(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})| < \delta_\ell n_\ell\big) \ge \varepsilon_0.$$

Let Ω_ℓ be a sequence such that $\Omega_\ell \to \infty$ and $\delta_\ell \Omega_\ell \to 0$. Then with probability bounded away from zero (as $\ell \to \infty$) at most $\delta_\ell \Omega_\ell n_\ell = o(n_\ell)$ corners from $U_k^c(\mathsf{M}_n^t, O_{n_\ell,\delta_\ell})$ are incident to vertices with degree at most Ω_ℓ. Hence, again with probability bounded away from zero at least $\varepsilon' n_\ell - \delta_\ell \Omega_\ell n_\ell = (\varepsilon' - o(1))n_\ell$ corners are incident to vertices with degree at least Ω_ℓ. But this is impossible since $\Omega_\ell \to \infty$ and the degree of the vertex incident to a uniformly selected corner in M_n^t must remain stochastically bounded by Lemma 9.1. This completes the verification of (9.15).

We now have all ingredients for proving the convergence in (9.5). Let $\varepsilon > 0$ be given. With foresight, we use (9.15) for $\varepsilon' := \varepsilon/(2 + 4\mathbb{E}[\mathsf{c}(\mathsf{M}^t)])$ to pick a $\delta > 0$ that is small enough so that for all large enough n,

$$\mathbb{P}\big(|U_k^c(\mathsf{M}_n^t, O_{n,\delta})| > \varepsilon n/(2 + 4\mathbb{E}[\mathsf{c}(\mathsf{M}^t)])\big) < \varepsilon. \tag{9.16}$$

For this δ and ε we may choose constants $c, C > 0$ and a collection $\mathcal{E}_n^{\mathcal{M}}$ such that (9.10)–(9.12) hold for all sufficiently large n. Let $\mathcal{E}_n^{\mathcal{M}}(1) \subset \mathcal{E}_n^{\mathcal{M}}$ denote the subset of all families $(V, M_1, M_2, \ldots)$ with the property that at most $\varepsilon n/(2 + 4\mathbb{E}[\mathsf{c}(\mathsf{M}^t)])$ corners of V are

incident to vertices with graph distance at most k from the last $\lfloor \delta n \rfloor$ corners of V (in the canonical ordering of corners). From (9.10), (9.11) and (9.16) it follows that

$$\mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\big) > 1 - 2\varepsilon, \qquad (9.17)$$

$$\mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t), (\mathsf{M}(i))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\big) > 1 - 2\varepsilon. \qquad (9.18)$$

Let H be a corner-rooted planar map with radius k. We let $N_H(\cdot)$ denote the function that takes a planar map as input and counts the number of corners with k-neighbourhood H. By Lemma 9.1, if $p_H := \mathbb{P}(U_k(\hat{\mathsf{M}}^t) = H)$ then

$$\frac{N_H(\mathsf{M}_n^t)}{2n} \xrightarrow{p} p_H. \qquad (9.19)$$

We may write

$$N_H(\mathsf{M}_n^t) = N_H'(\mathsf{M}_n^t) + N_H''(\mathsf{M}_n^t) \qquad (9.20)$$

with $N_H'(\mathsf{M}_n^t)$ counting only the corners whose k-neighbourhood is disjoint from the vertices incident to the last $\lfloor \delta n \rfloor$ corners of $\mathcal{V}(\mathsf{M}_n^t)$ (in the canonical ordering of corners). Thus, $N_H'(\mathsf{M}_n^t)$ is fully determined by $(\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor})$. We may hence define the subset $\mathcal{E}_n^{\mathcal{M}}(2) \subset \mathcal{E}_n^{\mathcal{M}}(1)$ of all sequences in $\mathcal{E}_n^{\mathcal{M}}(1)$ for which $|N_H'(\cdot)/(2n) - p_H| > 2\varepsilon$. (Not all such sequences have to lie in $\mathcal{E}_n^{\mathcal{M}}(1)$, but we only consider those which do.) Note that if $|N_H'(\cdot)/(2n) - p_H| > 2\varepsilon$ holds, then $|N_H(\cdot)/(2n) - p_H| > \varepsilon$ or $N_H''(\cdot)/(2n) > \varepsilon$. By inequality (9.12) it follows that

$$\mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t), (\mathsf{M}(i))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(2)\big)$$
$$\leq c^{-1} \mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(2)\big)$$
$$\leq c^{-1} \mathbb{P}\big(|N_H(\mathsf{M}_n^t)/(2n) - p_H| > \varepsilon\big)$$
$$+ c^{-1} \mathbb{P}\big(N_H''(\mathsf{M}_n^t)/(2n) > \varepsilon, (\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\big). \qquad (9.21)$$

From (9.19) it follows that

$$\mathbb{P}(|N_H(\mathsf{M}_n^t)/(2n) - p_H| > \varepsilon) \to 0. \qquad (9.22)$$

Furthermore, we defined $\mathcal{E}_n^{\mathcal{M}}(1)$ so that at most $\varepsilon n / (2 + 4\mathbb{E}[\mathrm{c}(\mathsf{M}^t)])$ corners of the non-separable core are incident to vertices with graph distance at most k from the last $\lfloor \delta n \rfloor$ of its corners. Let I' denote the collection of indices of these corners, and set $I := I' \cap \{1, \dots, \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor\}$. This way, if $(\mathcal{V}(\mathsf{M}_n^t), (\mathcal{M}_i(\mathsf{M}_n^t))_{1 \leq i \leq \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)$, then

$$|I| + \lfloor \delta n \rfloor \leq \varepsilon n / (2 + 4\mathbb{E}[\mathrm{c}(\mathsf{M}^t)]). \qquad (9.23)$$

Furthermore, if $N_H''(\mathsf{M}_n^t)/(2n) > \varepsilon$, then

$$\sum_{i \in I}(1 + \mathrm{c}(\mathcal{M}_i(\mathsf{M}_n^t))) + \sum_{i = \mathrm{c}(\mathcal{V}(\mathsf{M}_n^t)) - \lfloor \delta n \rfloor + 1}^{\mathrm{c}(\mathcal{V}(\mathsf{M}_n^t))}(1 + \mathrm{c}(\mathcal{M}_i(\mathsf{M}_n^t))) > 2n\varepsilon. \qquad (9.24)$$

We know from inequality (9.13) that with high probability,

$$\sum_{i=c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor+1}^{c(\mathcal{V}(\mathsf{M}_n^t))} (1+c(\mathcal{M}_i(\mathsf{M}_n^t))) \le (1+2\mathbb{E}[c(\mathsf{M}^t)])\lfloor\delta n\rfloor \le \varepsilon n/2. \qquad (9.25)$$

By inequalities (9.12) and (9.23), and the law of large numbers,

$$\mathbb{P}\Big(\sum_{i\in I}(1+c(\mathcal{M}_i(\mathsf{M}_n^t))) > \varepsilon n/2,\ (\mathcal{V}(\mathsf{M}_n^t),(\mathcal{M}_i(\mathsf{M}_n^t))_{1\le i\le c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\Big)$$

$$\le C\,\mathbb{P}\Big(\sum_{i\in I}(1+c(\mathsf{M}(i))) > \varepsilon n/2\Big) \to 0. \qquad (9.26)$$

Combining this with inequality (9.24) gives

$$\mathbb{P}\big(N_H''(\mathsf{M}_n^t)/(2n) > \varepsilon,\ (\mathcal{V}(\mathsf{M}_n^t),(\mathcal{M}_i(\mathsf{M}_n^t))_{1\le i\le c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\big) \to 0. \qquad (9.27)$$

Thus, the upper bound in (9.21) tends to zero as $n \to \infty$, yielding

$$\mathbb{P}\big((\mathcal{V}(\mathsf{M}_n^t),(\mathsf{M}(i))_{1\le i\le c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(2)\big) \to 0. \qquad (9.28)$$

Arguing as for (9.24) and (9.23), it follows by the law of large numbers that

$$\mathbb{P}\big(N_H''(\mathsf{M}_{[n]}^t)/(2n) > \varepsilon,\ (\mathcal{V}(\mathsf{M}_n^t),(\mathsf{M}(i))_{1\le i\le c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\big)$$

$$\le \mathbb{P}\Big(\sum_{i\in I}(1+c(\mathsf{M}(i))) + \sum_{i=c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor+1}^{c(\mathcal{V}(\mathsf{M}_n^t))} (1+c(\mathsf{M}(i))) > 2n\varepsilon\Big) \to 0. \qquad (9.29)$$

Combining (9.28) and (9.29), we obtain

$$\mathbb{P}\big(|N_H(\mathsf{M}_{[n]}^t)/(2n) - p_H| > 3\varepsilon,\ (\mathcal{V}(\mathsf{M}_n^t),(\mathsf{M}(i))_{1\le i\le c(\mathcal{V}(\mathsf{M}_n^t))-\lfloor\delta n\rfloor}) \in \mathcal{E}_n^{\mathcal{M}}(1)\big) \to 0. \qquad (9.30)$$

By (9.18), it follows that

$$\mathbb{P}\big(|N_H(\mathsf{M}_{[n]}^t)/(2n) - p_H| > 3\varepsilon\big) \le 2\varepsilon + o(1). \qquad (9.31)$$

Since H, k, and $\varepsilon > 0$ were arbitrary, it follows that

$$\mathfrak{L}((\mathsf{M}_{[n]}^t, c_{[n]}) \mid \mathsf{M}_{[n]}^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{M}}^t). \qquad (9.32)$$

This completes the proof. ∎

Using this convergence we deduce the following lemma, which is a quenched version of a more general argument on random block-weighted planar maps with no tail assumptions [64, Thm. 6.59].

Lemma 9.3. *The maximal non-separable component* $\mathcal{V}(\mathsf{M}_n^t)$ *admits a distributional limit* $\hat{\mathsf{V}}^t$ *in the local topology. If* c_n *denotes a uniformly selected corner of* $\mathcal{V}(\mathsf{M}_n^t)$, *then*

$$\mathfrak{L}((\mathcal{V}(\mathsf{M}_n^t), c_n) \mid \mathcal{V}(\mathsf{M}_n^t)) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{V}}^t). \tag{9.33}$$

Proof. Select two corners v_1, v_2 of $\mathsf{M}_{[n]}^t$ uniformly at random. It will be notationally convenient to refer to v_1 as a *red corner*, and v_2 as a *blue corner*. If v_1 or v_2 corresponds to a corner in a component $\mathcal{M}_i(\mathsf{M}_{[n]}^t)$ or to the corner of $\mathcal{V}(\mathsf{M}_n^t)$ where $\mathcal{M}_i(\mathsf{M}_n^t)$ is attached, then we let $\bar{\mathcal{M}}_i(\mathsf{M}_n^t)$ denote $\mathcal{M}_i(\mathsf{M}_n^t)$ together with the location(s) and colour(s) of v_1 and/or v_2. Otherwise we just set $\bar{\mathcal{M}}_i(\mathsf{M}_n^t) = \mathcal{M}_i(\mathsf{M}_n^t)$. Note that this / these location(s) may either be a corner of $\mathcal{M}_i(\mathsf{M}_n^t)$ or an additional placeholder corner, referring to the corner of $\mathcal{V}(\mathsf{M}_n^t)$ where $\mathcal{M}_i(\mathsf{M}_n^t)$ is attached.

Let $\mathsf{T}^{\bullet,\mathcal{M}}$ be defined for the offspring distribution $\xi^{\mathcal{M}}$ analogously to the definition of $\mathsf{T}^\bullet$ for ξ in Section 3.2. Let $\mathsf{M}^\bullet$ denote the random bi-corner-rooted map corresponding to a canonical decoration of the tree $\mathsf{T}^{\bullet,\mathcal{M}}$. Let $\mathsf{M}_1^\bullet$ and $\mathsf{M}_2^\bullet$ denote independent copies of $\mathsf{M}^\bullet$. We colour the marked corner of $\mathsf{M}_1^\bullet$ red, and the marked corner of $\mathsf{M}_2^\bullet$ blue. Note that $|\mathsf{T}^{\bullet,\mathcal{M}}| = c(\mathsf{M}^\bullet) + 1$. If the marked vertex of $\mathsf{T}^{\bullet,\mathcal{M}}$ coincides with its root-vertex, then we view $\mathsf{M}^\bullet$ as marked at a placeholder location.

Let j_1, j_2 be a uniformly selected pair of distinct integers between 2 and $\Delta(\mathsf{T}_n^{\mathcal{M}}) - t_n$. Furthermore, for each $i \geq 1$ set $\tilde{\mathsf{M}}(i) = \mathsf{M}_k^\bullet$ if $i = j_k$, $k \in \{1, 2\}$, and $\tilde{\mathsf{M}}(i) = \mathsf{M}(i)$ otherwise. It follows by Corollary 3.5 that

$$(\bar{\mathcal{M}}_i(\mathsf{M}_{[n]}^t))_{1 \leq i \leq \Delta(\mathsf{T}_n^{\mathcal{M}})} \overset{d}{\approx} (\tilde{\mathsf{M}}(i))_{1 \leq i \leq \Delta(\mathsf{T}_n^{\mathcal{M}})}. \tag{9.34}$$

Let r_1 and r_2 denote fixed non-negative integers. By Proposition 2.3 and Lemma 9.2 the neighbourhoods $U_{r_1}(\mathsf{M}_{[n]}^t, v_1)$ and $U_{r_2}(\mathsf{M}_{[n]}^t, v_2)$ are disjoint with high probability. Let M_1 and M_2 be finite corner-rooted planar maps with radii r_1 and r_2. For $k = 1, 2$ let v_k' be the corner of $\mathcal{V}(\mathsf{M}_n^t)$ where the component containing v_k is attached. If the distance between v_k and v_k' is at least r_k, then $U_{r_k}(\mathsf{M}_{[n]}^t, v_k)$ is fully contained in the component containing v_k. If the distance equals some $h < r_k$, then $U_{r_k}(\mathsf{M}_{[n]}^t, v_k)$ may be patched together from the r_k-neighbourhood of v_k in that component (with additional knowledge of the location of v_k' within that neighbourhood), the neighbourhood $U_{r_k - h}(\mathcal{V}(\mathsf{M}_n^t), v_k')$, and neighbourhoods in the components attached to corners $c \in U_{r_k - h}(\mathcal{V}(\mathsf{M}_n^t), v_k') \setminus \{v_k'\}$ with distance less than $r_k - h$ from v_k'. By (9.34) and the observations in the penultimate paragraph, we know that jointly and asymptotically the component containing v_k behaves like $\mathsf{M}^\bullet$, and the components attached to the corners c behave like independent copies of a map M corresponding to a canonical decoration of $\mathsf{T}^{\mathcal{M}}$. This allows us to write

$$\mathbb{P}(U_{r_k}(\mathsf{M}_{[n]}^t, v_k) = M_k)$$
$$= o(1) + C_{r_k}(M_k) + \sum_{h=0}^{r_k - 1} p_h \sum_H c_{h,H} \mathbb{P}(U_{r_k - h}(\mathcal{V}(\mathsf{M}_n^t), v_k') = H). \tag{9.35}$$

Here $C_{r_k}(M_k)$ denotes the probability that jointly the distance between the root corner and the marked corner in $\mathsf{M}^\bullet$ is at least r_k, and that the r_k-neighbourhood of the marked corner in $\mathsf{M}^\bullet$ equals M_k. The constant $p_h > 0$ denotes the probability that solely this distance in $\mathsf{M}^\bullet$ equals h. The constant $c_{h,H}$ represents the sum of the product probabilities for the finitely many ways of patching M_k together from a rooted r_k-neighbourhood in $\mathsf{M}^\bullet$ (conditioned on having distance h between the roots), and neighbourhoods in independent M-distributed components attached to non-root corners c of the sum index map H. Note that H ranges over specific submaps of M_k, and the case $H = M_k$ occurs only once and for $h = 0$. Specifically,

$$c_{0,M_k} = \mathbb{P}(e(\mathsf{M}^\bullet) = 0)\mathbb{P}(e(\mathsf{M}) = 0)^{s(M_k)-1} > 0, \tag{9.36}$$

with $s(M_k)$ denoting the number of corners in M_k with distance less than r_k from the root corner. The case $H = M_k$ and $h = 0$ is the only summand on the right-hand side of (9.35) where $|H| + (r_k - h)$ attains its maximum. As the left-hand side of (9.35) converges by Lemma 9.1, it follows by induction on $r_k + |M_k|$ that the probability $\mathbb{P}(U_{r_k}(\mathcal{V}(\mathsf{M}_n^t), v_k') = M_k)$ converges to some constant p_{r_k,M_k}. (The base case $r_k = 0$ is trivial.)

In order to deduce distributional convergence of the neighbourhood it remains to verify $\sum_{M_k} p_{r_k,M_k} = 1$. Suppose that $1 - \sum_{M_k} p_{r_k,M_k} =: \varepsilon > 0$. Then for any $s > 0$,

$$\mathbb{P}(c(U_{r_k}(\mathcal{V}(\mathsf{M}_n^t), v_k')) > s) = 1 - \sum_{M_k, c(M_k) \leq s} \mathbb{P}(U_{r_k}(\mathcal{V}(\mathsf{M}_n^t), v_k') = M_k)$$

$$\to 1 - \sum_{M_k, c(M_k) \leq s} p_{r_k,M_k} \geq \varepsilon.$$

This implies that there is a sequence $s_n \to \infty$ with $\mathbb{P}(c(U_{r_k}(\mathcal{V}(\mathsf{M}_n^t), v_k')) > s_n) \geq \varepsilon/2$ for all n. As the distance $d_{\mathsf{M}_{[n]}^t}(v_k, v_k')$ between the corners v_k and v_k' admits the limit distribution $(p_h)_{h \geq 0}$, this implies

$$\mathbb{P}(c(U_{r_k}(\mathsf{M}_{[n]}^t), v_k) > s_n) \geq \mathbb{P}(d_{\mathsf{M}_{[n]}^t}(v_k, v_k') = 0)\mathbb{P}(c(U_{r_k}(\mathcal{V}(\mathsf{M}_n^t), v_k')) > s_n)$$

$$\geq p_0 \varepsilon + o(1).$$

As $p_0 > 0$, this contradicts the distributional convergence of $U_{r_k}(\mathsf{M}_n^t, v_k)$. Consequently, one has $\sum_{M_k} p_{r_k,M_k} = 1$.

Summing up, there is a random infinite graph $\hat{V}^t$ which is the distributional limit of $\mathcal{V}(\mathsf{M}_n^t)$ rooted according to the stationary distribution, and satisfies

$$\mathbb{P}(U_{r_k}(\hat{\mathsf{M}}^t) = M_k) = C_{r_k}(M_k) + \sum_{h=0}^{r_k-1} p_h \sum_H c_{h,H} \mathbb{P}(U_{r_k-h}(\hat{V}^t) = H). \tag{9.37}$$

Since with high probability $U_{r_1}(\mathsf{M}^t_{[n]}, v_1)$ and $U_{r_2}(\mathsf{M}^t_{[n]}, v_2)$ do not intersect, analogously to (9.35) we obtain

$$
\mathbb{P}\big(U_{r_1}(\mathsf{M}^t_{[n]}, v_1) = M_1,\ U_{r_2}(\mathsf{M}^t_{[n]}, v_2) = M_2\big) = o(1) + C_{r_1}(M_1)C_{r_2}(M_2)
$$

$$
+ C_{r_1}(M_1) \sum_{h=0}^{r_2-1} p_h \sum_{H_2} c_{h,H_2} \mathbb{P}(U_{r_2-h}(\mathcal{V}(\mathsf{M}^t_n), v'_2) = H_2)
$$

$$
+ C_{r_2}(M_2) \sum_{h=0}^{r_1-1} p_h \sum_{H_1} c_{h,H_1} \mathbb{P}(U_{r_1-h}(\mathcal{V}(\mathsf{M}^t_n), v'_1) = H_1)
$$

$$
+ \sum_{\substack{0 \le h_1 < r_1 \\ 0 \le h_2 < r_2}} p_{h_1} p_{h_2}
$$

$$
\times \sum_{H_1,H_2} c_{h_1,H_1} c_{h_2,H_2} \mathbb{P}\big(U_{r_1-h_1}(\mathcal{V}(\mathsf{M}^t_n), v'_1) = H_1,\ U_{r_2-h_2}(\mathcal{V}(\mathsf{M}^t_n), v'_2) = H_2\big).
$$

$$(9.38)$$

By Lemma 9.1 and Proposition 2.2 the left-hand side satisfies

$$
\mathbb{P}\big(U_{r_1}(\mathsf{M}^t_{[n]}, v_1) = M_1,\ U_{r_2}(\mathsf{M}^t_{[n]}, v_2) = M_2\big)
$$
$$
\to \mathbb{P}(U_{r_1}(\hat{\mathsf{M}}^t) = M_1)\mathbb{P}(U_{r_2}(\hat{\mathsf{M}}^t) = M_2). \quad (9.39)
$$

Using convergence of the marginals $\mathbb{P}(U_{r_k-h}(\mathcal{V}(\mathsf{M}^t_n), v'_k) = H_k)$ and an inductive argument as before it follows that the joint probability

$$
\mathbb{P}\big(U_{r_1}(\mathcal{V}(\mathsf{M}^t_n), v'_1) = M_1,\ U_{r_2}(\mathcal{V}(\mathsf{M}^t_n), v'_2) = M_2\big)
$$

converges to some constant p_{r_1,r_2,M_1,M_2}. From (9.37) and (9.39) it follows that

$$
\sum_{\substack{0 \le h_1 < r_1 \\ 0 \le h_2 < r_2}} p_{h_1} p_{h_2} \sum_{H_1,H_2} c_{h_1,H_1} c_{h_2,H_2} \mathbb{P}(U_{r_1-h_1}(\hat{\mathsf{V}}^t) = H_1)\mathbb{P}(U_{r_2-h_2}(\hat{\mathsf{V}}^t) = H_2)
$$
$$
= \sum_{\substack{0 \le h_1 < r_1 \\ 0 \le h_2 < r_2}} p_{h_1} p_{h_2} \sum_{H_1,H_2} c_{h_1,H_1} c_{h_2,H_2} p_{r_1,r_2,M_1,M_2}.
$$

Hence, again by induction (on $r_1 + r_2 + c(M_1) + c(M_2)$, with the base case being trivial),

$$
p_{r_1,r_2,M_1,M_2} = \mathbb{P}(U_{r_1}(\hat{\mathsf{V}}^t) = M_1)\mathbb{P}(U_{r_2}(\hat{\mathsf{V}}^t) = M_2). \quad (9.40)
$$

This verifies that if $c_n^{(1)}$ and $c_n^{(2)}$ are uniform independent corners of $\mathcal{V}(\mathsf{M}^t_n)$, then

$$
((\mathcal{V}(\mathsf{M}^t_n), c_n^{(1)}), (\mathcal{V}(\mathsf{M}^t_n), c_n^{(2)})) \xrightarrow{d} (\hat{\mathsf{V}}^{t,(1)}, \hat{\mathsf{V}}^{t,(2)}), \quad (9.41)
$$

with $\hat{\mathsf{V}}^{t,(1)}, \hat{\mathsf{V}}^{t,(2)}$ denoting independent copies of $\hat{\mathsf{V}}^t$. Hence by Proposition 2.2,

$$
\mathfrak{L}((\mathcal{V}(\mathsf{M}^t_n), c_n) \mid \mathcal{V}(\mathsf{M}^t_n)) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{V}}^t). \qquad \blacksquare
$$

9.2. $\bar{\mathcal{K}}$-networks

Let D_n^t and $\bar{\mathsf{K}}_n^t$ denote random $\mathcal{D}$- and $\bar{\mathcal{K}}$-networks with n edges, drawn with probability proportional to their weight (given by $t^{\mathrm{v}(\cdot)}$). The relation $\mathcal{V}(x,z) = 1 + z^2 + z^2 x \mathcal{D}(x, z^2)$ from (7.2) entails that, for $n \geq 2$, V_n may be sampled by converting the "invisible" root edge of D_{n-1} into a regular one.

The Gibbs partition $\mathcal{D} = \bar{\mathcal{K}} \mathrm{SEQ}(x\bar{\mathcal{K}})$ from (7.8) represents the fact that any $\mathcal{D}$-network consists of a series composition of a positive number of $\bar{\mathcal{K}}$-networks. The discussion in Section 6.3.2 entails that $\mathcal{D} = \bar{\mathcal{K}} \mathrm{SEQ}(x\bar{\mathcal{K}})$ has convergent type. Hence, identifying $\mathcal{D}$-networks with sequences of $\bar{\mathcal{K}}$-networks, we find that

$$\mathsf{D}_n^t \stackrel{d}{\approx} (\bar{\mathsf{K}}(1), \ldots, \bar{\mathsf{K}}(N), \bar{\mathsf{K}}_{n-A}^t, \bar{\mathsf{K}}'(1), \ldots, \bar{\mathsf{K}}'(N')) \tag{9.42}$$

with

$$A := \sum_{i=1}^{N} \mathrm{e}(\bar{\mathsf{K}}(i)) + \sum_{i=1}^{N'} \mathrm{e}(\bar{\mathsf{K}}'(i)). \tag{9.43}$$

Here N and N' denote independent identically distributed non-negative integers with geometric distribution

$$\mathbb{P}(N = k) = \bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}})^k (1 - \bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}})), \quad k \geq 0. \tag{9.44}$$

The networks $\bar{\mathsf{K}}(i)$ and $\bar{\mathsf{K}}'(i)$, $i \geq 1$, denote independent copies of a random $\bar{\mathcal{K}}$-network $\bar{\mathsf{K}}$ with distribution given by

$$\mathbb{P}(\bar{\mathsf{K}} = \bar{K}) = t^{\mathrm{v}(\bar{K})} \rho_{\bar{\mathcal{K}}}^{\mathrm{e}(\bar{K})} / \bar{\mathcal{K}}(t, \rho_{\bar{\mathcal{K}}}). \tag{9.45}$$

Condition (9.42) tells us that a large $\mathcal{D}$-network has with high probability a giant $\bar{\mathcal{K}}$-component. We let $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ denote the unique (with high probability) $\bar{\mathcal{K}}$-core of the $\mathcal{D}$-network $\mathcal{D}(\mathsf{M}_n^t)$ corresponding to the non-separable core $\mathcal{V}(\mathsf{M}_n^t)$.

Corollary 9.4. *Lemma 9.3 holds analogously for $\bar{\mathcal{K}}(\mathsf{M}_n^t)$, when we treat the "invisible" root edge of $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ as a real one.*

Proof. Lemma 9.3 and the fourth characterization of quenched local convergence in Proposition 2.1 tell us that for any $r \geq 0$ the percentage of corners whose r-neighbourhoods have a fixed shape M in $\mathcal{V}(\mathsf{M}_n^t)$ concentrates around the limit probability $\mathbb{P}(U_r(\hat{\mathsf{V}}^t) = M)$. The asymptotic (9.42) and Proposition 2.3 entail that all but a stochastically bounded number of these corners have their M-shaped r-neighbourhood lying entirely in $\bar{\mathcal{K}}(\mathsf{M}_n^t)$. Again by Proposition 2.1, $\hat{\mathsf{V}}^t$ is also the quenched local limit of $\bar{\mathcal{K}}(\mathsf{M}_n^t)$. ∎

Corollary 9.5. *Uniformly for $\ell \in \mathbb{Z}$,*

$$\mathbb{P}(\mathrm{e}(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = \ell) = \frac{1}{g_{\mathcal{M}}(t) n^{2/3}} \left(h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{g_{\mathcal{M}}(t) n^{2/3}} \right) + o(1) \right). \tag{9.46}$$

Proof. As $\lim_{x\to\infty} h(x) = 0$, it suffices to show this for integers $\ell \geq 0$. The density function h is bounded and uniformly continuous. Hence for any $k \geq 0$, by (9.3) and (9.42),

$$g_{\mathcal{M}}(t)n^{2/3}\mathbb{P}\big(\mathrm{e}(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = \ell, \, \mathrm{e}(\bar{\mathcal{D}}(\mathsf{M}_n^t)) = \ell + k\big)$$
$$= h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{g_{\mathcal{M}}(t)n^{2/3}}\right)\mathbb{P}(A = k) + o(1),$$

with the $o(1)$ term being uniform in ℓ. Hence for any sequence t_n tending sufficiently slowly to infinity,

$$g_{\mathcal{M}}(t)n^{2/3}\sum_{k=0}^{t_n}\mathbb{P}\big(\mathrm{e}(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = \ell, \mathrm{e}(\mathcal{D}(\mathsf{M}_n^t)) = \ell + k\big)$$
$$= h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{g_{\mathcal{M}}(t)n^{2/3}}\right) + o(1), \quad (9.47)$$

with a uniform $o(1)$ term. Condition (8.25) implies that

$$\mathbb{P}(\mathrm{e}(\bar{\mathsf{K}}) = n) \sim c_{\bar{\mathcal{K}}}(t)n^{-5/2}. \quad (9.48)$$

Using (9.3), Proposition 5.2, and the fact that h is bounded, it follows that there are constants $C, c > 0$ such that

$$g_{\mathcal{M}}(t)n^{2/3}\sum_{t_n \leq k \leq n}\mathbb{P}\big(\mathrm{e}(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = \ell, \, \mathrm{e}(\mathcal{D}(\mathsf{M}_n^t)) = \ell + k\big)$$
$$\leq C\sum_{t_n \leq k \leq n}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell - k}{g_{\mathcal{M}}(t)n^{2/3}}\right) + o(1)\right)\frac{\mathbb{P}(\mathrm{e}(\bar{\mathsf{K}}) = \ell)\mathbb{P}(\mathrm{e}(\bar{\mathsf{K}}) = k)}{\mathbb{P}(\mathrm{e}(\bar{\mathsf{K}}) = \ell + k)}\exp(-ck/\ell)$$
$$\leq O(1)\sum_{t_n \leq k \leq n}\mathbb{P}(\mathrm{e}(\bar{\mathsf{K}}) = k)(1 + k/\ell)^{5/2}\exp(-ck/\ell)$$
$$\leq O(\mathbb{P}(\mathrm{e}(\bar{\mathsf{K}}) \geq t_n)). \quad (9.49)$$

Together with (9.47) this verifies claim (9.46). ∎

9.3. $\bar{\mathcal{R}}$-networks

Equation (7.14), that is, $\bar{\mathcal{K}} = y\bar{\mathcal{R}}(x, \bar{\mathcal{K}})$, tells us that a $\bar{\mathcal{K}}$-network may be recursively described as an $\bar{\mathcal{R}}$-network where we insert an additional edge (corresponding to the factor y) at a specified location, and substitute all other regular edges (if any) by $\bar{\mathcal{K}}$-networks.

It follows from (7.14) and the general results of Section 4.2 that any $\bar{\mathcal{K}}$-network with n edges corresponds bijectively to a pair (T, β) of a planted plane tree T with n vertices and a function β that assigns to each vertex $v \in T$ an $\bar{\mathcal{R}}$-network with $d_T^+(v)$ regular edges.

The bijection may be recursively described as follows: The decoration $\beta(o)$ of the root vertex o of T is a network with an "invisible" edge connecting the poles, $d_T^+(o)$ regular

edges and an additional "terminal" edge that we label with o. The regular edges may be enumerated in some canonical way (say in a breadth-first-search manner), so that each corresponds bijectively to one of the $d_T^+(o)$ offspring vertices v of o. The decorated fringe subtree rooted at such a vertex v corresponds bijectively to a network, and we replace the edge of $\beta(o)$ that corresponds to v by the "invisible" root edge of that network, hence creating the final network by a glueing operation.

The enriched tree $(\mathsf{T}_n^{\bar{\mathcal{K}}}, \beta_n^{\bar{\mathcal{K}}})$ corresponding to a random $\bar{\mathcal{K}}$-network $\bar{\mathsf{K}}_n^t$ with n edges (drawn with probability proportional to its weight $t^{v(\cdot)}$) is a canonically decorated simply generated tree with weight sequence $(\omega_k^{\bar{\mathcal{K}}})_{k \geq 0}$ given by

$$\omega_k^{\bar{\mathcal{K}}} = [y^k]\bar{\mathcal{R}}(t, y). \tag{9.50}$$

It follows from (8.24), (8.20), and Lemma 3.1 that $\mathsf{T}_n^{\bar{\mathcal{K}}}$ is distributed like a Galton–Watson tree $\mathsf{T}^{\bar{\mathcal{K}}}$ conditioned on having n vertices, with offspring distribution $\xi^{\bar{\mathcal{K}}}$ satisfying

$$\mathbb{E}[\xi^{\bar{\mathcal{K}}}] < 1 \quad \text{and} \quad \mathbb{P}(\xi^{\bar{\mathcal{K}}} = n) \sim \frac{c_{\bar{\mathcal{R}}}(t)}{\bar{\mathcal{R}}(t, \rho_{\mathcal{R}}(t))}n^{-5/2}. \tag{9.51}$$

By Lemma 3.2 it follows that the (unique with high probability) largest $\bar{\mathcal{R}}$-component $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ satisfies

$$\mathbb{P}(\mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) = \ell) = \frac{1}{g_{\bar{\mathcal{K}}}(t)n^{2/3}}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])n - \ell}{g_{\bar{\mathcal{K}}}(t)n^{2/3}}\right) + o(1)\right) \tag{9.52}$$

uniformly for all $\ell \in \mathbb{Z}$ with $g_{\bar{\mathcal{K}}}(t) > 0$ a constant.

We let $\bar{\mathcal{R}}(\mathsf{M}_n^t) := \bar{\mathcal{R}}(\bar{\mathcal{K}}(\mathsf{M}_n^t))$ denote the largest $\bar{\mathcal{R}}$-component within the largest $\bar{\mathcal{K}}$-component of $\mathcal{V}(\mathsf{M}_n^t)$. Note the subtlety of this definition. It is clear that with probability tending to 1 the component $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ actually equals the largest $\bar{\mathcal{R}}$-component within the entire map M_n^t. However, determining the speed of convergence would require additional work. For this reason, getting a local limit theorem for the size $\mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}_n^t))$ requires a little less effort than getting a local limit theorem for the size of the largest $\bar{\mathcal{R}}$-component in M_n^t.

A local limit theorem for the size of the largest 3-connected component of M_n^t was established in [32, Thm. 6.4], and we are going to argue analogously.

Corollary 9.6. *Uniformly for all $\ell \in \mathbb{Z}$,*

$$\mathbb{P}(\mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \ell) = \frac{1}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}}\right) + o(1)\right) \tag{9.53}$$

with

$$\tilde{g}_{\bar{\mathcal{K}}}(t) = \left((g_{\mathcal{M}}(t)(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}]))^{3/2} + g_{\bar{\mathcal{K}}}(t)^{3/2}(1 - \mathbb{E}[\xi^{\mathcal{M}}])\right)^{2/3}. \tag{9.54}$$

Proof. Since $\lim_{x\to\infty} h(x) = 0$ it suffices to show this for integers $\ell \geq 0$. It follows from the definition of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ that

$$\mathbb{P}(e(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \ell) = \sum_{N=1}^{n} \mathbb{P}(e(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = N)\mathbb{P}(e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_N^t)) = \ell). \tag{9.55}$$

Recall that

$$\bar{\mathcal{K}}(\mathsf{M}_n^t) \leq \mathcal{D}(\mathsf{M}_n^t) \overset{d}{=} \tfrac{1}{2}\Delta(\mathsf{T}_n^{\mathcal{M}}) - 1 \tag{9.56}$$

(and $\mathsf{T}_n^{\mathcal{M}}$ has $2n + 1$ vertices). It follows from [65, (2.10)] that there is a constant $\varepsilon_1 > 0$ such that

$$\mathbb{P}(\Delta(\mathsf{T}_n^{\mathcal{M}}) \leq \varepsilon_1 n/\log n) = o(n^{-2/3}). \tag{9.57}$$

By inequality (9.49), for any $N, s \geq 1$,

$$\mathbb{P}(e(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = N, e(\mathcal{D}(\mathsf{M}_n^t)) \geq N + s) \leq O(1)n^{-2/3}s^{-3/2}. \tag{9.58}$$

By [65, (2.11), (2.12)] it follows that for any $0 < \varepsilon_2 < 1 - \mathbb{E}[\xi^{\mathcal{M}}]$,

$$\mathbb{P}(\varepsilon_1 n/\log n \leq \Delta(\mathsf{T}_n^{\mathcal{M}}) \leq \varepsilon_2 n) = O(n^{-2/3}). \tag{9.59}$$

Condition (9.52) implies that $\mathbb{P}(e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_N^t)) = \ell) = O(N^{-2/3})$, hence using (9.57), (9.58) and (9.59) we obtain

$$\sum_{N=1}^{\lfloor \varepsilon_2 n \rfloor} \mathbb{P}(e(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = N)\mathbb{P}(e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_N^t)) = \ell)$$

$$= \sum_{N=1}^{\lfloor \varepsilon_2 n \rfloor} \mathbb{P}\big(e(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = N, \, e(\mathcal{D}(\mathsf{M}_n^t)) \leq N + n/\log^2 n\big)\mathbb{P}(e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_N^t)) = \ell)$$

$$+ O(1)n^{-2/3}(n/\log^2 n)^{-3/2} \sum_{N=1}^{\lfloor \varepsilon_2 n \rfloor} N^{-2/3}$$

$$= o(n^{-2/3}).$$

Hence by Corollary 9.5 it follows that for each $\varepsilon > 0$ we may select a constant $M_1 > 0$ large enough such that the interval I_n with endpoints $(1 - \mathbb{E}[\xi^{\mathcal{M}}])n \pm M_1 n^{2/3}$ satisfies

$$\limsup_{n\to\infty} \sup_{\ell \in \mathbb{Z}} n^{2/3} \sum_{N \notin I_n} \mathbb{P}(e(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = N)\mathbb{P}(e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_N^t)) = \ell) \leq \varepsilon, \tag{9.60}$$

Using again (9.52), it follows that there is a constant M_2 (depending only on ε) such that the interval J_n with endpoints $(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])n \pm M_2 n^{2/3}$ satisfies

$$\limsup_{n\to\infty} \sup_{\ell \notin J_n} n^{2/3}\mathbb{P}(e(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \ell) \leq 2\varepsilon. \tag{9.61}$$

Hence it suffices to verify (9.53) uniformly for $\ell \in J_n$ as n becomes large. We may write $N \in I_n$ as $N = (1 - \mathbb{E}[\xi^{\mathcal{M}}])n + x_N n^{2/3}$ with $|x_N| \leq M_1$. Likewise, we may write $\ell \in J_n$ as $\ell = (1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])n + y_\ell n^{2/3}$ with $|y_\ell| \leq M_2$. Conditions (9.46), (9.52), and the fact that h is bounded and uniformly continuous imply that uniformly for $\ell \in J_n$,

$$n^{2/3} \sum_{N \in I_n} \mathbb{P}(e(\bar{\mathcal{K}}(\mathsf{M}_n^t)) = N)\mathbb{P}(e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_N^t)) = \ell)$$

$$= \frac{1 + o(1)}{g_{\mathcal{M}}(t)g_{\bar{\mathcal{K}}}(t)(1 - \mathbb{E}[\xi^{\mathcal{M}}])^{2/3}n^{2/3}} \sum_{N \in I_n} h\left(\frac{-x_N}{g_{\mathcal{M}}(t)}\right) h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])x_N - y_\ell}{g_{\bar{\mathcal{K}}}(t)(1 - \mathbb{E}[\xi^{\mathcal{M}}])^{2/3}}\right).$$

Taking M_1 large enough and using $|y_\ell| < M_2$, it follows that this expression lies in the interval with endpoints

$$\pm\varepsilon + o(1) + \frac{1}{g_{\mathcal{M}}(t)g_{\bar{\mathcal{K}}}(t)(1 - \mathbb{E}[\xi^{\mathcal{M}}])^{2/3}} \int_{-\infty}^{\infty} h\left(\frac{-z}{g_{\mathcal{M}}(t)}\right) h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])z - y_\ell}{g_{\bar{\mathcal{K}}}(t)(1 - \mathbb{E}[\xi^{\mathcal{M}}])^{2/3}}\right) dz.$$

Setting $a = g_{\mathcal{M}}(t)(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])$ and $b = g_{\bar{\mathcal{K}}}(t)(1 - \mathbb{E}[\xi^{\mathcal{M}}])^{2/3}$, and making a linear variable transform, we may rewrite the last summand by $\int_{-\infty}^{\infty} \frac{1}{a}h(-\frac{z}{a})\frac{1}{b}h(\frac{z-y_\ell}{b}) dz$. Recall that h is the density of a $3/2$-stable random variable X with Laplace transform $\mathbb{E}[e^{-\lambda X}] = e^{\lambda^{3/2}}$. Hence the integral is the density of a sum $-aX_1 - bX_2$ evaluated at the point y_ℓ, with X_1 and X_2 denoting independent copies of X. By comparing Laplace transforms we find $-aX_1 - bX_2 \stackrel{d}{=} -cX$ with $c = (a^{3/2} + b^{3/2})^{2/3}$. Thus

$$\int_{-\infty}^{\infty} \frac{1}{a}h\left(-\frac{z}{a}\right)\frac{1}{b}h\left(\frac{z-y_\ell}{b}\right) dz = \frac{1}{c}h\left(-\frac{y_\ell}{c}\right) = \frac{1}{c}h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{cn^{2/3}}\right).$$

$\blacksquare$

The (regular and "invisible") edges of the core $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ (that is, the canonically selected largest $\bar{\mathcal{R}}$-component of $\bar{\mathsf{K}}_n^t$) may be enumerated from 0 to $e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$ in a canonical way, starting with the invisible edge. We give each edge an orientation according to a fair independent coin flip. The map $\bar{\mathsf{K}}_n^t$ is constructed from the core $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ by replacing the ith (oriented) edge by a network $\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t)$ for all $0 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$. Here *replacing* means deleting the edge and identifying its start vertex with the south pole and its end vertex with the north pole of the network.

The network $\bar{\mathcal{K}}_0(\bar{\mathsf{K}}_n^t)$ inserted at the "invisible" edge of the core $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ carries a second pair of poles that correspond to the poles of $\bar{\mathsf{K}}_n^t$. For each $1 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$ the network $\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t)$ is fully described by the fringe subtree $F_i(\mathsf{T}_n^{\bar{\mathcal{K}}})$ and the restriction $\beta_n^{\bar{\mathcal{K}}}|_{F_i(\mathsf{T}_n^{\bar{\mathcal{K}}})}$. The network $\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t)$ (and its second pair of poles) is fully described by the marked tree $F_0(\mathsf{T}_n^{\bar{\mathcal{K}}})$ and the restriction of $\beta_n^{\bar{\mathcal{K}}}$ to all unmarked vertices of $F_0(\mathsf{T}_n^{\bar{\mathcal{K}}})$.

We define $\mathsf{T}^{\bar{\mathcal{K}}}$, $\mathsf{T}^{\bullet\bar{\mathcal{K}}}$ and $\mathsf{T}^{\circ\bar{\mathcal{K}}}$ for the offspring distribution $\xi^{\bar{\mathcal{K}}}$ analogously to T, $\mathsf{T}^{\bullet}$ and T° defined for ξ in Section 3.2. For each integer $i \geq 1$ we let $\bar{\mathsf{K}}(i)$ denote an independent copy of the network $\bar{\mathsf{K}}^t$ corresponding to $\mathsf{T}^{\bar{\mathcal{K}}}$ with canonical random $\bar{\mathcal{R}}$-decorations

(as defined in Section 4.2). We let $\bar{\mathsf{K}}(0)$ denote the network (with two sets of poles) corresponding to a canonical decoration of the random marked tree $\mathsf{T}^{\circ\bar{\mathcal{K}}}$.

We let $\bar{\mathsf{K}}_{[n]}^t$ denote the network constructed from the core $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ by substituting for all $0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$ the ith edge of $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ by $\bar{\mathsf{K}}(i)$ instead of $\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t)$. This way, $\bar{\mathsf{K}}_{[n]}^t$ and $\bar{\mathsf{K}}_n^t$ are different models. For example, the number of edges of $\bar{\mathsf{K}}_{[n]}^t$ fluctuates around n, but need not be precisely equal to n. We let $\bar{\mathsf{K}}_{\langle n \rangle}^t$ be constructed analogously from the core $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ instead of $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$. Since the core $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ is a mixture of $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$, it follows that $\bar{\mathsf{K}}_{\langle n \rangle}^t$ is a mixture of $\bar{\mathsf{K}}_{[n]}^t$.

We will show that local convergence of $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ is equivalent to local convergence of $\bar{\mathsf{K}}_{\langle n \rangle}^t$, with the same limit map in both cases. This is analogous to Lemma 9.2.

Lemma 9.7. *Let $c_{\langle n \rangle}$ denote a uniformly selected corner of $\bar{\mathsf{K}}_{\langle n \rangle}^t$. Then*

$$\mathfrak{L}((\bar{\mathsf{K}}_{\langle n \rangle}^t, c_{\langle n \rangle}) \mid \bar{\mathsf{K}}_{\langle n \rangle}^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{V}}^t). \tag{9.62}$$

Proof. We set

$$\Delta_{\langle n \rangle}^{\bar{\mathcal{K}}} := \sup \left\{ d \ge 0 \,\middle|\, \mathrm{e}(\bar{\mathsf{K}}(0)) + \sum_{i=1}^d \mathrm{e}(\bar{\mathsf{K}}(i)) \le n \right\}.$$

It follows from Lemma 3.3 that for any sequence $(t_n)_n$ of integers with $t_n \to \infty$ and $t_n = o(n)$,

$$(\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - t_n} \stackrel{d}{\approx} (\bar{\mathsf{K}}(i))_{0 \le i \le \Delta_{\langle n \rangle}^{\bar{\mathcal{K}}} - t_n} \tag{9.63}$$

and with probability tending to 1 as n becomes large,

$$\sum_{i = \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - t_n}^{\mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))} \mathrm{e}(\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t)) \le 2\mathbb{E}[\mathrm{e}(\bar{\mathsf{K}}^t)]t_n. \tag{9.64}$$

Let $0 < \delta < 1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}]$ and $\varepsilon > 0$ be given. By Lemma 3.4 there are constants $0 < c < C$, $N_0 > 0$ and collections $(\mathcal{E}_n)_{n \ge 1}$ (each consisting of a set of finite sequences of $\mathcal{K}$-networks, with the first having an additional marked corner) such that for all $n > N_0$,

$$\begin{aligned} \mathbb{P}((\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor} \in \mathcal{E}_n) &> 1 - \varepsilon, \\ \mathbb{P}((\bar{\mathsf{K}}(i))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor} \in \mathcal{E}_n) &> 1 - \varepsilon, \end{aligned} \tag{9.65}$$

and uniformly for all sequences $S \in \mathcal{E}_n$,

$$c < \frac{\mathbb{P}((\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor} = S)}{\mathbb{P}((\bar{\mathsf{K}}(i))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor} = S)} < C. \tag{9.66}$$

Note that knowledge of $(\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor}$ implies knowledge of $\mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$. Moreover, conditionally on $\mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$ the core $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ admits an $\mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))$-sized $\bar{\mathcal{R}}$-network with probability proportional to its weight. Let $\mathcal{E}_n^{\bar{\mathcal{K}}}$ denote the collection of all

pairs (R, S) such that $S \in \mathcal{E}_n$ and R is an $\bar{\mathcal{R}}$-network whose number of (visible) edges agrees with the core size corresponding to S. That is, the number of edges of R equals the dimension of S plus $\lfloor \delta n \rfloor - 1$. It follows from (9.65) and (9.66) that

$$\mathbb{P}\big((\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t), (\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t))_{0 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\bar{\mathcal{K}}}\big) > 1 - \varepsilon, \tag{9.67}$$

$$\mathbb{P}\big((\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t), (\bar{\mathsf{K}}(i))_{0 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor}) \in \mathcal{E}_n^{\bar{\mathcal{K}}}\big) > 1 - \varepsilon, \tag{9.68}$$

and uniformly for all $(R, S) \in \mathcal{E}_n^{\bar{\mathcal{K}}}$,

$$c < \frac{\mathbb{P}((\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t), (\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t))_{0 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor}) = (R, S))}{\mathbb{P}((\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t), (\bar{\mathsf{K}}(i))_{0 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor}) = (R, S))} < C. \tag{9.69}$$

From (9.7) it follows that with high probability,

$$\sum_{i = e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)) - \lfloor \delta n \rfloor + 1}^{e(\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t))} e(\bar{\mathcal{K}}_i(\bar{\mathsf{K}}_n^t)) \leq 2\mathbb{E}[e(\bar{\mathsf{K}}^t)] \lfloor \delta n \rfloor. \tag{9.70}$$

The core $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ is a mixture of $\bar{\mathsf{K}}_n^t$, with a random size $e(\bar{\mathcal{K}}(\mathsf{M}_n^t))$ that we determined in Corollary 9.46. Hence (9.67)–(9.70) imply the corresponding equalities and inequalities for $\bar{\mathcal{K}}(\mathsf{M}_n^t)$.

Let $O_{n,\delta}$ denote the subset of vertices in $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ corresponding to the components $\bar{\mathcal{K}}_i(\bar{\mathcal{K}}(\mathsf{M}_n^t))$ for $e(\bar{\mathcal{R}}(\bar{\mathsf{M}}_n^t)) - \lfloor \delta n \rfloor + 1 \leq i \leq e(\bar{\mathcal{R}}(\bar{\mathsf{M}}_n^t))$. For each $k \geq 0$ we let $U_k(\bar{\mathcal{K}}(\mathsf{M}_n^t), O_{n,\delta})$ denote the collection of vertices of $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ with graph distance at most k from $O_{n,\delta}$.

Using only the local convergence of $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ from Corollary 9.4 and the fact that $O_{n,\delta}$ becomes small as $\delta \to \infty$ we may argue as for (9.14) to deduce that for each $k \geq 0$ and each $\varepsilon' > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}(|U_k(\bar{\mathcal{K}}(\mathsf{M}_n^t), O_{n,\delta})| > \varepsilon' n) = 0. \tag{9.71}$$

We write $U_k^c(\bar{\mathcal{K}}(\mathsf{M}_n^t), O_{n,\delta})$ for the collection of corners in $\bar{\mathcal{K}}(\mathsf{M}_n^t)$ that are incident to vertices of $U_k(\bar{\mathcal{K}}(\mathsf{M}_n^t), O_{n,\delta})$. Arguing as for (9.15), we obtain

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}(|U_k^c(\bar{\mathcal{K}}(\mathsf{M}_n^t), O_{n,\delta})| > \varepsilon' n) = 0. \tag{9.72}$$

Having (9.72) as well as (9.67)–(9.70) at hand, the proof may be completed by copying almost word for word the corresponding part of the proof of Lemma 9.2. ∎

Lemma 9.8. *The $\bar{\mathcal{R}}$-core $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ admits a distributional limit $\hat{\bar{\mathsf{R}}}^t$ in the local topology. If c_n denotes a uniformly selected corner of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$, then*

$$\mathfrak{L}((\bar{\mathcal{R}}(\mathsf{M}_n^t), c_n) \mid \bar{\mathcal{R}}(\mathsf{M}_n^t)) \xrightarrow{p} \mathfrak{L}(\hat{\bar{\mathsf{R}}}^t). \tag{9.73}$$

Proof. The corners of $\bar{\mathsf{K}}^t_{\langle n \rangle}$ (counting the "invisible" edge between the poles as a real edge) correspond bijectively to the corners of the collection $(\bar{\mathcal{K}}_i(\bar{\mathsf{K}}^t_{\langle n \rangle}))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\bar{\mathsf{K}}^t_{\langle n \rangle}))} = (\bar{\mathsf{K}}(i))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}^t_n))}$ (treating the "invisible" edge between the poles of $\bar{\mathcal{K}}_0(\mathsf{M}^t_n)$, that correspond to the poles of $\bar{\mathsf{K}}^t_{\langle n \rangle}$, like a real edge). Let us select a corner v_1 (which we colour red) and a corner v_2 (which we colour blue) of $\bar{\mathsf{K}}^t_{\langle n \rangle}$ uniformly and independently at random. This may be done by uniformly selecting two independent edges, and flipping fair coins for each to determine which of the corresponding half-edges to use. For each $0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}^t_n))$ we set $\tilde{\bar{\mathcal{K}}}_i(\bar{\mathsf{K}}^t_{\langle n \rangle}) = \bar{\mathcal{K}}_i(\bar{\mathsf{K}}^t_{\langle n \rangle})$ if neither v_1 nor v_2 lies in this component, and otherwise we let $\tilde{\bar{\mathcal{K}}}_i(\bar{\mathsf{K}}^t_{\langle n \rangle})$ be given by $\bar{\mathcal{K}}_i(\bar{\mathsf{K}}^t_{\langle n \rangle})$ with an additional marked red and/or blue corner corresponding to the location(s) of v_1 and/or v_2.

We let $\bar{\mathsf{K}}^\bullet$ denote the network with a marked corner obtained by taking the network corresponding to a canonical decoration of $\mathsf{T}^{\bullet, \bar{\mathcal{K}}}$ and flipping a fair coin on which of the two half-edges corresponding to the marked edge to distinguish. We let $\bar{\mathsf{K}}^\bullet_1$ and $\bar{\mathsf{K}}^\bullet_2$ denote independent copies of $\bar{\mathsf{K}}^\bullet$ where we colour the corners red and blue, respectively. Let j_1, j_2 denote a uniformly selected pair of distinct integers between 1 and $\mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}^t_n))$. For each $i \ge 0$ we set $\tilde{\bar{\mathsf{K}}}(i) = \bar{\mathsf{K}}(i)$ if $i \notin \{j_1, j_2\}$, and $\tilde{\bar{\mathsf{K}}}(i) = \bar{\mathsf{K}}^\bullet_k$ if $i = j_k$ for $k \in \{1, 2\}$. By Corollary 3.5 it follows that

$$(\tilde{\bar{\mathcal{K}}}_i(\bar{\mathsf{K}}^t_{\langle n \rangle}))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}^t_n))} \overset{d}{\approx} (\tilde{\bar{\mathsf{K}}}(i))_{0 \le i \le \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}^t_n))}. \tag{9.74}$$

Let $r \ge 1$ be a constant. For each $k \in \{1, 2\}$ let e_k be the *oriented* edge of $\bar{\mathcal{R}}(\mathsf{M}^t_n)$ where the component containing v_k is inserted. (Recall that we substitute edges of $\bar{\mathcal{R}}(\mathsf{M}^t_n)$ by networks, and each edge of $\bar{\mathcal{R}}(\mathsf{M}^t_n)$ was given a direction so that there is no ambiguity which of the end vertices gets identified with which pole.)

If the distance between v_k and (both endpoints of) e_k is at least r, the neighbourhood $U_r(\bar{\mathsf{K}}^t_{\langle n \rangle}), v_k)$ is fully contained in the $\bar{\mathcal{R}}$-component containing the corner v_k. If the distance a_k from v_k to the origin $e_k(1)$ and the distance b_k from v_k to the destination $e_k(2)$ satisfy $\min(a_k, b_k) < r$, then $U_r(\bar{\mathsf{K}}^t_{\langle n \rangle}, v_k)$ may be assembled canonically from the following parts:

(1) The r-neighbourhood of the corner v_k in the component containing it, with additional knowledge of the location of e_k relative to that neighbourhood.

(2) The (connected) submap $U(k)$ of $\bar{\mathcal{R}}(\mathsf{M}^t_n)$ (rooted at e_k) induced by all edges of $\bar{\mathcal{R}}(\mathsf{M}^t_n)$ whose components contain edges from $U_r(\bar{\mathsf{K}}^t_{\langle n \rangle}), v_k)$.

(3) Neighbourhood(s) of one or both poles (possibly with different radii) within the components inserted at edges $e \ne e_k$ in $U(k)$.

Let us define a *semi-network* analogously to a planar network, with the only different requirement being that adding the "invisible" oriented root edge must make the semi-network connected (instead of non-separable, as in the case of networks). The necessity for this notion stems from the fact that in the case of $\min(a_k, b_k) < r$, if a network K gets inserted at an edge $e \ne e_k$ of $U(k)$, then its contribution to (or intersection with) the r-neighbourhood of v_k in $\bar{\mathsf{K}}^t_{\langle n \rangle}$ may have two different shapes: Either it consists of

a neighbourhood in K of only one of the poles (and the other is too far away), or it consists of the union of neighbourhoods with possibly different radii of the south pole and the north pole. These neighbourhoods may or may not overlap. Hence the need for this terminology, to describe how the r-neighbourhood of v_k in $\check{\mathcal{K}}(\mathsf{M}_n^t)$ gets assembled by inserting semi-networks at edges of $U(k)$.

It follows from Proposition 2.3 and the local convergence of $\bar{\mathsf{K}}_{\langle n \rangle}^t$ ensured by Lemma 9.7 that the neighbourhoods $U_r(\bar{\mathsf{K}}_{\langle n \rangle}^t, v_1)$ and $U_r(\bar{\mathsf{K}}_{\langle n \rangle}^t, v_2)$ are with high probability disjoint. Applying Proposition 2.3 repeatedly also entails that we may choose the sequence $(t_n)_n$ so that it converges sufficiently slowly to infinity so that with high probability neither of these neighbourhoods contains the "invisible" edge between the poles of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ or any of the last t_n edges of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ (with respect to the canonical ordering of those edges). (If there existed a subsequence (n') along which the probability that v_1 has distance less than r from the "invisible" edge or some ith last edge of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ is bounded away from zero, then so is the probability that this happens jointly for v_1 and v_2 (as they are i.i.d.), contradicting $d_{\bar{\mathcal{K}}(\mathsf{M}_n^t)}(v_1, v_2) \xrightarrow{d} \infty$.)

By (9.74) and the discussion of the previous paragraph it follows that jointly and asymptotically the components containing v_1 and v_2 behave like independent copies of $\bar{\mathsf{K}}^\bullet$, and the components inserted at edges $\neq e_1, e_2$ of $U(1)$ and $U(2)$ behave jointly like independent copies of $\bar{\mathsf{K}}$. It also follows that $U(1) \cap U(2) = \emptyset$ with high probability. (We are going to use this fact below when studying joint probabilities.)

We would like to establish an analogue of (9.35) and perform a similar proof by induction as we did in the proof of Lemma 9.3. That is, we would like to use convergence of the r-neighbourhoods of v_1 and v_2 in $\bar{\mathsf{K}}_{\langle n \rangle}^t$ (ensured by Lemma 9.7) to deduce convergence of the r-neighbourhoods of e_1 and e_2 in $\bar{\mathcal{R}}(\mathsf{M}_n^t)$. However, there is a big problem with this approach if we work directly with probabilities that neighbourhoods of fixed radii have given shapes: The map $U_r(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)$ may have more edges than $U(k)$, hence breaking the induction step.

For this reason, we are going to perform the induction with a different family of convergence-determining events. For any planar maps R and H, any subset A of vertices from H, and any half-edge v of R we let $\mathcal{E}(H, A, R, v)$ denote the indicator variable (and, by abuse of notation, also the corresponding event when R is random) that H may be embedded as a submap of R with the root edge corresponding to v, so that R has no additional edges that are incident to A. We are going to refer to the pair (H, A) as a *community*, and to the subset A as the *conservative* members of the community. (The analogy is that some community members are open to form new connections to others within and outside of their community, whereas members of the subset A are more conservative.)

Let (M_1, A_1) and (M_2, A_2) be given finite communities, with M_1 and M_2 having radii $r_1, r_2 < r$. In the event $\mathcal{E}(M_k, A_k, \bar{\mathsf{K}}_{\langle n \rangle}^t, v_k)$ there are finitely many possible shapes H_k of the submap of the core $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ induced by edges whose components contain edges of the embedding of M_k in $\bar{\mathsf{K}}_{\langle n \rangle}^t$. (Here we consider H_k as rooted at the oriented edge corresponding to e_k.) For example, if H_k consists of a single oriented edge, then the image of M_k lies entirely in the component inserted at e_k. The marked corner in that compon-

ent corresponds to the root edge of M_k. Furthermore, the poles of this component may not correspond to conservative members. As the component inserted at e_k asymptotically behaves like an independent copy of $\bar{\mathsf{K}}^\bullet$, it follows that the limiting probability of this subevent is a constant $C(M_k, A_k)$ determined by the probability of some event for $\bar{\mathsf{K}}^\bullet$. If H_k consists of more than its root edge, then M_k gets assembled by substituting each edge e of H_k by a semi-network $K_e(k)$. The semi-network inserted at the root edge of H_k has a marked corner, which corresponds to the root corner of M_k. There may be multiple (but only finitely many) choices for such families $(K_e(k))_e$ to assemble M_k in this way. If we know the shape H_k then we know the subset B_k of vertices of H_k that correspond to conservative members of M_k, and if we additionally know the family $(K_e(k))_e$ then for each e we know the subset of non-pole vertices of the inserted network K_e that correspond to conservative members. Hence, this subevent is characterized by requiring the event $\mathcal{E}(H_k, B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)$ to take place, and additionally for each edge e of H_k the corresponding component needs to have the semi-network $K_e(k)$ as sub-semi-network and no further edges incident to conservative members of $K_e(k)$. We know that jointly and asymptotically the component corresponding to the root edge of H_k behaves like an independent copy of $\bar{\mathsf{K}}^\bullet$, and the components corresponding to all other edges behave like independent copies of $\bar{\mathsf{K}}^t$. Hence the probability of the entire subcase corresponding to H_k may be expressed by

$$o(1) + C_{(H_k, B_k)} \mathbb{P}(\mathcal{E}(H_k, B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)).$$

Here $C_{(H_k, B_k)} \geq 0$ denotes a constant that corresponds to a sum (over all choices for the family $(K_e(k))_e$) of product probabilities for events of independent copies of $\bar{\mathsf{K}}^t$ (and one copy of $\bar{\mathsf{K}}^\bullet$). As there are only finitely many choices for H_k, this allows us to write

$$\mathbb{P}(\mathcal{E}(M_k, A_k, \bar{\mathsf{K}}^t_{\langle n \rangle}, v_k)) = o(1) + C(M_k, A_k)$$
$$+ \sum_{\substack{(H_k, B_k) \\ e(H_k) \geq 2}} C_{(H_k, B_k)} \mathbb{P}(\mathcal{E}(H_k, B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)). \tag{9.75}$$

Here the sum index (H_k, B_k) ranges over some finite set of communities, each having the properties $e(H_k) \geq 2$ and $v(H_k) \leq v(M_k)$ and $e(H_k) \leq e(M_k)$. There are at most two cases where jointly $v(H_k) = v(M_k)$ and $e(H_k) = e(M_k)$, namely $H_k = M_k$ and $H_k = \mathrm{Inv}(M_k)$, the map obtained by reversing the orientation of the root edge of M_k. (It can happen that $M_k = \mathrm{Inv}(M_k)$, entailing that these cases coincide.)

Note that both conditions $v(H_k) = v(M_k)$ and $e(H_k) = e(M_k)$ are necessary to nail the cases of H_k down to M_k / $\mathrm{Inv}(M_k)$. There are potentially many more cases of H_k with $e(H_k) = e(M_k)$, but all of them satisfy $v(H_k) < v(M_k)$. For example, consider the case where M_k is a path of length 4 with the root edge incident to and pointing away from the middle vertex. We could assemble M_k by replacing the edges of a 4-cycle by semi-networks – by replacing two of the square edges with a network consisting of two poles joined by a single regular edge (plus the "invisible" edge that we discard when substituting), and the other two by a semi-network where there is a single regular edge

incident to one of the poles but not to the other. There are also potentially many more cases for H_k with $v(H_k) = v(M_k)$, but all of them satisfy $e(H_k) < e(M_k)$, for example when M_k consists of two vertices joined by three edges.

Let us focus on the special case $H_k = M_k$. We may assume that M_k has at least two edges. For $H_k = M_k$ the semi-network $K_e(k)$ inserted at an edge e of H_k to form M_k must have precisely one regular edge (plus the "invisible" edge that we discard when substituting). Hence there are three possible choices for $K_e(k)$: either the regular edge connects the two poles (we denote this by $*_S - *_N$), or it is only incident to the south pole ($*_S-$) or only to the north pole ($-*_N$).

(1) If e is an edge of H_k with both ends having degree at least 2, then $K_e(k)$ has to equal $*_S - *_N$. If e is the root edge, then the edge of $K_e(k)$ has to be oriented to point from the south pole to the north pole.

(2) If e points from $e(1)$ to $e(2)$ such that $e(2)$ has degree 1 (and hence $e(1)$ does not, since M_k was assumed to have at least two edges), then $K_e(k)$ may be either $*_S - *_N$ or $*_S-$. If e is the root edge, then the edge of $K_e(k)$ is oriented to point away from the south pole.

(3) If $e(1)$ has degree 1 (and hence $e(2)$ does not), then $K_e(k)$ may be either $*_S - *_N$ or $-*_N$. If e is the root edge, then the edge of $K_e(k)$ is oriented and needs to point towards the north pole.

This entails that

$$C_{(M_k, B_k)} = 0 \quad \text{for } B_k \neq A_k.$$

Indeed, if B_k is a strict subset of A_k, then at least one of the networks $(K_e(k))_e$ has a conservative member. But the probability is zero that $\bar{K}^t$ or $\bar{K}^\bullet$ has one of the three described shapes with an additional conservative member that may not be incident to further edges. It also follows that

$$C_{(M_k, A_k)} > 0. \tag{9.76}$$

Let $\mathrm{Inv}(e_k)$ denote the result of reversing the direction of e_k. By symmetry,

$$C_{(\mathrm{Inv}(M_k), B_k)} \mathbb{P}(\mathcal{E}(\mathrm{Inv}(M_k), B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k))$$
$$= C_{(M_k, B_k)} \mathbb{P}(\mathcal{E}(M_k, B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), \mathrm{Inv}(e_k))) = C_{(M_k, B_k)} \mathbb{P}(\mathcal{E}(M_k, B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)).$$

This allows us to express (9.75) as

$$\mathbb{P}(\mathcal{E}(M_k, A_k, \bar{\mathsf{K}}_{\langle n \rangle}^t, v_k)) = o(1) + \sum_{(H_k, B_k) \in \mathfrak{C}(M_k, A_k)} C_{(H_k, B_k)} \mathbb{P}(\mathcal{E}(H_k, B_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k))$$
$$+ C(M_k, A_k) + D_{(M_k, A_k)} \mathbb{P}(\mathcal{E}(M_k, A_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)) \tag{9.77}$$

with

$$D_{(M_k, A_k)} = C_{(M_k, A_k)}(1 + \mathbb{1}_{M_k \neq \mathrm{Inv}(M_k)}).$$

and $\mathfrak{C}(M_k, A_k)$ denoting a finite collection of communities (H_k, B_k), all with $e(H_k) \geq 2$, and $v(H_k) < v(M_k)$ or $e(H_k) < e(M_k)$.

The left-hand side of (9.77) converges by Corollary 9.4. As $D_{(M_k,A_k)} > 0$ by (9.76), it follows by induction on $v(M_k) + e(M_k)$ (with the base case being trivial) that there is a constant $p_{M_k,A_k} \geq 0$ with

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{E}(M_k, A_k, \bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)) = p_{M_k,A_k}. \tag{9.78}$$

Given a planar map M with a specified corner c, we define the *edge neighbourhood* $E_r(M, c)$ as the planar map (rooted at c) induced by all edges where at least one endpoint has distance at most $r - 1$ from c. Hence $E_r(M, c)$ may be obtained from $U_r(M, c)$ by removing all edges where both endpoints have distance r from c. It is clear that for any sequence $(\mathsf{X}_n)_{n\geq 1}$ of random corner-rooted maps, weak convergence of $E_{r+1}(\mathsf{X}_n)$ implies weak convergence of $U_r(\mathsf{X}_n)$. Conversely, weak convergence of $U_r(\mathsf{X}_n)$ implies weak convergence of $E_r(\mathsf{X}_n)$.

Given a planar map H, the event $E_r(M, c) = H$ is equivalent to $\mathcal{E}(H, U_{r-1}(H), M, c)$. Recalling that we assumed M_k to have radius r_k, it follows from (9.78) that

$$\lim_{n\to\infty} \mathbb{P}(E_{r_k}(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k) = M_k) = p_{M_k, U_{r_k-1}(M_k)} =: p_{r_k, M_k}. \tag{9.79}$$

In order to deduce weak convergence of $E_{r_k}(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)$, we need to show that $\sum_{M_k} p_{r_k,M_k} = 1$. To reach a contradiction, suppose that $1 - \sum_{M_k} p_{r_k,M_k} =: \varepsilon > 0$. Then for any constant $s > 0$,

$$\mathbb{P}(e(E_{r_k}(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)) > s) = 1 - \sum_{M_k, e(M_k)\leq s} \mathbb{P}(U_{r_k}(E_{r_k}(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)) = M_k)$$

$$\to 1 - \sum_{M_k, e(M_k)\leq s} p_{r_k,M_k} \geq \varepsilon.$$

It follows that there is a sequence $s_n \to \infty$ with $\mathbb{P}(e(E_{r_k}(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)) > s_n) \geq \varepsilon/2$ for all n. The component containing v_k (inserted at e_k) admits $\bar{\mathsf{K}}^\bullet$ as weak limit, hence the probability that v_k corresponds to e_k converges to a constant $p > 0$. It follows that

$$\mathbb{P}(e(E_{r_k}(\bar{\mathsf{K}}_{\langle n\rangle}^t, v_k)) > s_n) \geq (p + o(1))\mathbb{P}(e(E_{r_k}(\bar{\mathcal{R}}(\mathsf{M}_n^t), e_k)) > s_n) = p\varepsilon + o(1).$$

But this contradicts the distributional convergence of $E_{r_k}(\bar{\mathsf{K}}_{\langle n\rangle}^t, v_k)$ ensured by Corollary 9.4. It follows that

$$\sum_{M_k} p_{r_k,M_k} = 1. \tag{9.80}$$

As this holds for all r_k, there is a random infinite graph $\hat{\bar{\mathsf{R}}}^t$ (with a root corner $\hat{e}_{\hat{\bar{\mathsf{R}}}^t}$) that is the distributional limit of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ rooted according to the stationary distribution. Recall that $\hat{\mathsf{V}}^t$ is the local limit of $\bar{\mathsf{K}}_{\langle n\rangle}^t$, and let $\hat{e}_{\hat{\mathsf{V}}^t}$ be its root corner. It follows from (9.77) that

$$\mathbb{P}(\mathcal{E}(M_k, A_k, \hat{\mathsf{V}}^t, \hat{e}_{\hat{\mathsf{K}}}^t)) = \sum_{(H_k,B_k)\in\mathfrak{C}(M_k,A_k)} C_{(H_k,B_k)}\mathbb{P}(\mathcal{E}(H_k, B_k, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t}))$$

$$+ C_{(M_k,A_k)} + D_{(M_k,A_k)}\mathbb{P}(\mathcal{E}(M_k, A_k, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t})). \tag{9.81}$$

As stated above, for any $r \geq 1$ the neighbourhoods $U_r(\bar{\mathsf{K}}^t_{\langle n \rangle}, v_1)$ and $U_r(\bar{\mathsf{K}}^t_{\langle n \rangle}, v_2)$ do not intersect with high probability. Hence jointly and asymptotically the components inserted at e_1 and e_2 behave like independent copies of $\bar{\mathsf{K}}^\bullet$, and the components inserted at the remaining edges of $U(1)$ and $U(2)$ like independent copies of $\bar{\mathsf{K}}^t$. Hence, analogously to (9.77), we obtain

$$\mathbb{P}\big(\mathcal{E}(M_1, A_1, \bar{\mathcal{K}}(\mathsf{M}^t_n), v_1) \text{ and } \mathcal{E}(M_2, A_2, \bar{\mathcal{K}}(\mathsf{M}^t_n), v_2)\big) = o(1) + C(M_1, A_1)C(M_2, A_2)$$
$$+ C(M_1, A_1)\Big(\sum_{(H_2, B_2) \in \mathfrak{C}(M_2, A_2)} C_{(H_2, B_2)} \mathbb{P}(\mathcal{E}(H_2, B_2, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_2))$$
$$+ D_{(M_2, A_2)} \mathbb{P}(\mathcal{E}(M_2, A_2, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_2))\Big)$$
$$+ C(M_2, A_2)\Big(\sum_{(H_1, B_1) \in \mathfrak{C}(M_1, A_1)} C_{(H_1, B_1)} \mathbb{P}(\mathcal{E}(H_1, B_1, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_1))$$
$$+ D_{(M_1, A_1)} \mathbb{P}(\mathcal{E}(M_1, A_1, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_1))\Big)$$
$$+ \sum_{\substack{(H_1, B_1) \in \mathfrak{C}(M_1, A_1) \\ (H_2, B_2) \in \mathfrak{C}(M_2, A_2)}} C_{(H_1, B_1)} C_{(H_2, B_2)} \mathbb{P}\big(\mathcal{E}(H_1, B_1, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_1) \text{ and } \mathcal{E}(H_2, B_2, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_2)\big)$$
$$+ D_{(M_1, A_1)} D_{(M_2, A_2)} \mathbb{P}\big(\mathcal{E}(M_1, A_1, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_1) \text{ and } \mathcal{E}(M_2, A_2, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_2)\big). \qquad (9.82)$$

Corollary 9.4 and Proposition 2.2 entail that the left-hand side satisfies

$$\mathbb{P}\big(\mathcal{E}(M_1, A_1, \bar{\mathsf{K}}^t_{\langle n \rangle}, v_1) \text{ and } \mathcal{E}(M_2, A_2, \bar{\mathsf{K}}^t_{\langle n \rangle}, v_2)\big)$$
$$\to \mathbb{P}(\mathcal{E}(M_1, A_1, \hat{\mathsf{V}}^t, \hat{e}_{\hat{\mathsf{V}}^t})) \mathbb{P}(\mathcal{E}(M_2, A_2, \hat{\mathsf{V}}^t, \hat{e}_{\hat{\mathsf{K}}})). \qquad (9.83)$$

Since the marginal probabilities $\mathbb{P}(\mathcal{E}(H_k, B_k, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_k))$ and $\mathbb{P}(\mathcal{E}(M_k, A_k, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_k))$ converge, and since $C_{(M_1, A_1)} C_{(M_2, A_2)} > 0$, it follows by induction on $v(M_1) + e(M_1) + v(M_2) + e(M_2)$ (with the base case being trivial) that there is a constant $p_{M_1, A_1, M_2, A_2} \geq 0$ with

$$\lim_{n \to \infty} \mathbb{P}\big(\mathcal{E}(M_1, A_1, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_1) \text{ and } \mathcal{E}(M_2, A_2, \bar{\mathcal{R}}(\mathsf{M}^t_n), e_2)\big) = p_{M_1, A_1, M_2, A_2}.$$

It follows from (9.81)–(9.83) that

$$\sum_{\substack{(H_1, B_1) \in \mathfrak{C}(M_1, A_1) \\ (H_2, B_2) \in \mathfrak{C}(M_2, A_2)}} C_{(H_1, B_1)} C_{(H_2, B_2)} p_{H_1, B_1, H_2, B_2} + D_{(M_1, A_1)} D_{(M_2, A_2)} p_{M_1, A_1, M_2, A_2}$$
$$= \sum_{\substack{(H_1, B_1) \in \mathfrak{C}(M_1, A_1) \\ (H_2, B_2) \in \mathfrak{C}(M_2, A_2)}} C_{(H_1, B_1)} C_{(H_2, B_2)} \mathbb{P}(\mathcal{E}(H_1, B_1, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t})) \mathbb{P}(\mathcal{E}(H_2, B_2, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t}))$$
$$+ D_{(M_1, A_1)} D_{(M_2, A_2)} \mathbb{P}(\mathcal{E}(M_1, A_1, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t})) \mathbb{P}(\mathcal{E}(M_2, A_2, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t})).$$

By induction on $v(M_1) + e(M_1) + v(M_2) + e(M_2)$ it follows that

$$p_{M_1, A_1, M_2, A_2} = \mathbb{P}(\mathcal{E}(M_1, A_1, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t})) \mathbb{P}(\mathcal{E}(M_2, A_2, \hat{\bar{\mathsf{R}}}^t, \hat{e}_{\hat{\bar{\mathsf{R}}}^t})).$$

Thus, if $c_n^{(1)}$ and $c_n^{(2)}$ are uniform independent corners of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$, then

$$((\bar{\mathcal{R}}(\mathsf{M}_n^t), c_n^{(1)}), (\bar{\mathcal{R}}(\mathsf{M}_n^t), c_n^{(2)})) \xrightarrow{d} (\hat{\bar{\mathsf{R}}}^{t,(1)}, \hat{\bar{\mathsf{R}}}^{t,(2)}), \qquad (9.84)$$

with $\hat{\bar{\mathsf{R}}}^{t,(1)}, \hat{\bar{\mathsf{R}}}^{t,(2)}$ denoting independent copies of $\hat{\bar{\mathsf{R}}}^t$. It follows by Proposition 2.2 that

$$\mathfrak{L}((\bar{\mathcal{R}}(\mathsf{M}_n^t), c_n) \mid \bar{\mathcal{R}}(\mathsf{M}_n^t)) \xrightarrow{p} \mathfrak{L}(\hat{\bar{\mathsf{R}}}^t). \qquad \blacksquare$$

9.4. $\bar{\mathcal{O}}$-networks

We define the class of networks $\bar{\mathcal{O}}$ by

$$\bar{\mathcal{O}} := \bar{\mathcal{F}}_{0,1}(x, y\,\mathrm{SEQ}(xy)), \qquad (9.85)$$

with x marking vertices (not counting the poles) and y marking regular edges (not counting the "invisible" edge between the poles). That is, it is obtained from a 3-connected map by declaring the oriented root edge "invisible", its origin the south pole, its destination the north pole, and substituting all remaining edges by paths of positive length.

We let $\bar{\mathcal{O}}^*$ denote the class obtained by making a canonical choice of an edge (with a canonical orientation) in $\bar{\mathcal{O}}$, and declaring it invisible. We may think of the ends of this edge as the *second* pair of poles of the network. That is, the species $\bar{\mathcal{O}}$ and $\bar{\mathcal{O}}^*$ are related by

$$\bar{\mathcal{O}} = y\bar{\mathcal{O}}^*. \qquad (9.86)$$

Let us recall the decomposition of $\bar{\mathcal{R}}$:

$$\begin{aligned}
\bar{\mathcal{R}} &= \bar{\mathcal{J}}\,\mathrm{SEQ}(\bar{\mathcal{J}}^*), \\
\bar{\mathcal{J}}^* &= y\,\mathrm{SEQ}_{\geq 1}(xy)\mathrm{SEQ}(xy) + \bar{\mathcal{O}}^*(x, y)(1 + y\,\mathrm{SEQ}(xy)), \\
\bar{\mathcal{J}} &= 1 + y\,\mathrm{SEQ}(xy).
\end{aligned}$$

That is, an $\bar{\mathcal{R}}$-network consists of a $\bar{\mathcal{J}}$-component and a possibly empty ordered sequence of $\bar{\mathcal{J}}^*$-components. We are going to describe this decomposition in detail. A network from the species $\bar{\mathcal{J}}$ may have two different shapes:

(1) It may be the trivial network consisting of a south pole, a north pole, and only the "invisible" edge between them. This accounts for the summand 1.

(2) It may consist of two poles joined by a single path of positive length (and, in parallel, the "invisible" edge between the poles). This accounts for the summand $y\,\mathrm{SEQ}(xy)$.

Recall that equation (7.14), that is, $\bar{\mathcal{K}} = y\bar{\mathcal{R}}(x, \bar{\mathcal{K}})$, may be interpreted as a recursive description of $\bar{\mathcal{K}}$-networks. It tells us that a $\bar{\mathcal{K}}$-network consists of an $\bar{\mathcal{R}}$-network where we insert an additional edge (corresponding to the factor y) between the poles of its $\bar{\mathcal{J}}$-component, and substitute all other regular edges (if any) by $\bar{\mathcal{K}}$-networks. In particular, when we interpret $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ as a planar map, we have to replace the "invisible" edge

between its poles by a regular edge, and add an additional edge between the poles of its $\bar{\mathcal{J}}$-component.

An element from $\bar{\mathcal{J}}^*$ is a network with a second pair of poles joined by a second "invisible" edge. It may have the following shapes:

(1) It may be an $\bar{\mathcal{O}}^*$-network or the parallel composition of an $\bar{\mathcal{O}}^*$-network with a path of positive length. These cases account for the summand $\bar{\mathcal{O}}^*(x, y)(1 + y\,\mathrm{SEQ}(xy))$.

(2) It may be constructed as follows: Take the parallel composition of a path of length at least 2 with a path of positive length. Declare the first edge of the first path as "invisible" and its ends as the second pair of poles. This accounts for the summand $y\,\mathrm{SEQ}_{\geq 1}(xy)\mathrm{SEQ}(xy)$.

Finally, a network from

$$\bar{\mathcal{R}} = \bar{\mathcal{J}}\,\mathrm{SEQ}(\bar{\mathcal{J}}^*) = \bar{\mathcal{J}} + \bar{\mathcal{J}}\bar{\mathcal{J}}^* + \bar{\mathcal{J}}(\bar{\mathcal{J}}^*)^2 + \bar{\mathcal{J}}(\bar{\mathcal{J}}^*)^3 + \cdots$$

may have the following shapes:

(1) It may consist of a $\bar{\mathcal{J}}$-network. This accounts for the summand $\bar{\mathcal{J}}$.

(2) It may be constructed as follows. Take an integer $k \geq 1$. Choose arbitrary $\bar{\mathcal{J}}^*$-networks $I_1, \ldots, I_k$ and a $\bar{\mathcal{J}}^*$-network J. We substitute the second pair of poles of I_1 by I_2, then the second pair of poles of I_2 by I_3, and so on. Finally, we substitute the second pair of poles of I_k by J. This accounts for the summand $\bar{\mathcal{J}}(\bar{\mathcal{J}}^*)^k$.

We let $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ denote the largest $\bar{\mathcal{O}}$-component in the decomposition of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$.

Lemma 9.9. (1) *The $\bar{\mathcal{O}}$-component $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ admits a distributional limit $\hat{\mathsf{O}}^t$ in the local topology. If we let c_n denote a uniformly selected corner of $\bar{\mathcal{O}}(\mathsf{M}_n^t)$, then*

$$\mathfrak{L}((\bar{\mathcal{O}}(\mathsf{M}_n^t), c_n) \mid \bar{\mathcal{O}}(\mathsf{M}_n^t)) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{O}}^t). \tag{9.87}$$

(2) *Uniformly for all $\ell \in \mathbb{Z}$,*

$$\mathbb{P}(\mathrm{e}(\bar{\mathcal{O}}(\mathsf{M}_n^t)) = \ell) = \frac{1}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}}\right) + o(1)\right). \tag{9.88}$$

Proof. The singular expansion (8.16) entails that

$$[y^n]\bar{\mathcal{O}}^*(t, y) \sim c_{\bar{\mathcal{O}}}(t)\rho_{\mathcal{R}}(t)^{-n}n^{-5/2} \tag{9.89}$$

for some constant $c_{\bar{\mathcal{O}}}(t) > 0$. The constant

$$\rho_{\mathcal{R}}(t) < 1/t \tag{9.90}$$

is given in (8.18). The summand $y\,\mathrm{SEQ}_{\geq 1}(ty)\mathrm{SEQ}(ty)$ has radius of convergence strictly larger than $\rho_{\mathcal{R}}(t)$. Using (5.5) it follows that

$$[y^n]\bar{\mathcal{J}}^*(t, y) \sim c_{\bar{\mathcal{O}}}(t)(1 + \rho_{\mathcal{R}}(t)/(1 - t\rho_{\mathcal{R}}(t)))\rho_{\mathcal{R}}(t)^{-n}n^{-5/2}. \tag{9.91}$$

Hence by (5.2),

$$[y^n]\mathrm{SEQ}(\bar{\mathcal{J}}^*(t, y)) \sim (1 - \bar{\mathcal{J}}^*(t, \rho_{\mathcal{R}}(t)))^{-2}[y^n]\bar{\mathcal{J}}^*(t, y). \tag{9.92}$$

The factor $\bar{\mathcal{J}}(t, y)$ has radius of convergence strictly larger than $\rho_{\mathcal{R}}(t)$. By Proposition 5.4 it follows that the $\bar{\mathcal{J}}$-component $\bar{\mathcal{J}}(\mathsf{M}_n^t)$ of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ converges to a random $\bar{\mathcal{J}}(t, y)$-object following a Boltzmann distribution with parameter $\rho_{\mathcal{R}}(t)$. By Lemma 5.1 the $\mathrm{SEQ}(\bar{\mathcal{J}}^*)$-component $\mathrm{SEQ}(\bar{\mathcal{J}}^*)(\mathsf{M}_n^t)$ of $\bar{\mathcal{R}}(\mathsf{M}_n^t)$ has a giant component and the small fragments converge (as in (9.42)) to a Boltzmann-distributed $\mathrm{SEQ}(\bar{\mathcal{J}}^*(t, y))^2$-object with parameter $\rho_{\mathcal{R}}(t)$. The generating series $y\,\mathrm{SEQ}_{\geq 1}(ty)\mathrm{SEQ}(ty)$ has radius of convergence strictly larger than $\rho_{\mathcal{R}}(t)$. Hence the (canonically selected) maximal $\bar{\mathcal{J}}^*$-component $\bar{\mathcal{J}}^*(\mathsf{M}_n^t)$ belongs to $\bar{\mathcal{O}}^*(t, y)(1 + y\,\mathrm{SEQ}(ty))$ with probability tending exponentially fast to 1 as n becomes large. It follows from Proposition 5.4 that the $(1 + y\,\mathrm{SEQ}(ty))$-component of $\bar{\mathcal{J}}^*(\mathsf{M}_n^t)$ admits a Boltzmann limit distribution with parameter $\rho_{\mathcal{R}}(t)$. Summing up,

$$\mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \mathrm{e}(\bar{\mathcal{O}}(\mathsf{M}_n^t)) + O_p(1). \tag{9.93}$$

The convergence (9.87) now follows from Lemma 9.8, by arguments entirely analogous to those in the proof of Corollary 9.4.

It remains to verify the local limit theorem. To this end, it suffices to verify (9.88) for all $\ell \geq 1$. We set $\lambda := (1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])$. Let X_1 denote the size of a Boltzmann distributed $\bar{\mathcal{J}}(t, y)$-object with parameter $\rho_{\mathcal{R}}(t)$. Note that X_1 has finite exponential moments. Using Corollary 9.6 and the fact that h is bounded and uniformly continuous, we see that for each constant $k \geq 0$,

$$\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}\mathbb{P}\big(\mathrm{e}(\mathrm{SEQ}(\bar{\mathcal{J}}^*)(\mathsf{M}_n^t)) = \ell,\ \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \ell + k\big)$$
$$= h\left(\frac{\lambda n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}}\right)\mathbb{P}(X_1 = k) + o(1).$$

Here the $o(1)$-term is uniform in ℓ. Hence for any sequence $(t_n)_n$ of integers that tends to infinity sufficiently slowly,

$$\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}\sum_{k=0}^{t_n}\mathbb{P}\big(\mathrm{e}(\mathrm{SEQ}(\bar{\mathcal{J}}^*)(\mathsf{M}_n^t)) = \ell,\ \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \ell + k\big)$$
$$= h\left(\frac{\lambda n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}}\right) + o(1). \tag{9.94}$$

Again, the $o(1)$-term is uniform in ℓ. Let Y_1 denote the size of a Boltzmann-distributed $\mathrm{SEQ}(\bar{\mathcal{J}}^*)$-object with parameter $\rho_{\mathcal{R}}(t)$. Note that by (9.91) and (9.92),

$$\mathbb{P}(Y_1 = n) \sim c_1 n^{-5/2}$$

for some constant $c_1 > 0$. Using Corollary 9.6, (5.6) and (5.8), and the fact that h is bounded, we see that

$$\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3} \sum_{t_n \leq k \leq n} \mathbb{P}\big(\mathrm{e}(\mathrm{SEQ}(\bar{\mathcal{J}}^*)(\mathsf{M}_n^t)) = \ell, \; \mathrm{e}(\bar{\mathcal{R}}(\mathsf{M}_n^t)) = \ell + k\big)$$

$$= \sum_{t_n \leq k \leq n} \left(h\left(\frac{\lambda n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \right) + o(1) \right) \frac{\mathbb{P}(X_1 = k)\mathbb{P}(Y_1 = \ell)}{\mathbb{P}(X_1 + Y_1 = \ell + k)}$$

$$\leq O(1) \sum_{t_n \leq k \leq n} \mathbb{P}(X_1 = k)\left(1 + \frac{k}{\ell} \right)^{5/2}. \tag{9.95}$$

This bound tends to zero uniformly for all $\ell \geq 1$, since X_1 has finite exponential moments. Combining (9.94) and (9.95) yields

$$\mathbb{P}(\mathrm{e}(\mathrm{SEQ}(\bar{\mathcal{J}}^*)(\mathsf{M}_n^t)) = \ell) = \frac{1}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \left(h\left(\frac{\lambda n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \right) + o(1) \right). \tag{9.96}$$

By arguments as in the proof of Corollary 9.5, it follows that

$$\mathbb{P}(\mathrm{e}(\bar{\mathcal{J}}^*(\mathsf{M}_n^t)) = \ell) = \frac{1}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \left(h\left(\frac{\lambda n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \right) + o(1) \right). \tag{9.97}$$

Using identical arguments as for (9.94)–(9.96), it follows that

$$\mathbb{P}(\mathrm{e}(\bar{\mathcal{O}}(\mathsf{M}_n^t)) = \ell) = \frac{1}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \left(h\left(\frac{\lambda n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \right) + o(1) \right). \qquad \blacksquare$$

9.5. Transfer between different mixtures

In the preceding arguments, we transferred properties of M_n^t to different cores: $\mathcal{V}(\mathsf{M}_n^t)$, $\bar{\mathcal{K}}(\mathsf{M}_n^t)$, $\bar{\mathcal{R}}(\mathsf{M}_n^t)$, and $\bar{\mathcal{O}}(\mathsf{M}_n^t)$. It is an important subtlety of these arguments that each of these cores has a *random* number of edges, for which we deduced a local limit theorem with a $3/2$-stable limit law. Conditioned on having a fixed number k of edges, each core is drawn with probability proportional to its weight (defined by putting weight t at vertices) from all k-edge elements of the corresponding class. That is, each core is a mixture of random weighted objects.

9.5.1. An absolute continuity relation.
Observe that we have a certain degree of freedom in changing these mixtures. To this end, suppose that S is a space and let $\mathfrak{B}(S)$ denote its Borel σ-algebra. Let $(\mathsf{S}_n)_n$ denote a sequence of S-valued random variables. Let X_n and Y_n denote random integers, each independent of $(\mathsf{S}_n)_n$. Suppose that there are constants $\mu_X, \mu_Y, g_X, g_Y > 0$ such that

$$\mathbb{P}(X_n = \ell) = \frac{1}{g_X n^{2/3}} \left(h\left(\frac{\mu_X n - \ell}{g_X n^{2/3}} \right) + o(1) \right) \tag{9.98}$$

and

$$\mathbb{P}(Y_n = \ell) = \frac{1}{g_Y n^{2/3}} \left(h\left(\frac{\mu_Y n - \ell}{g_Y n^{2/3}} \right) + o(1) \right) \tag{9.99}$$

uniformly for all $\ell \in \mathbb{Z}$.

Lemma 9.10. *Let* $s_n = \lfloor n\mu_Y/\mu_X \rfloor$.

(1) *For each* $\varepsilon > 0$ *there are constants* $0 < c < C$ *and* $N_0 > 0$ *such that for all* $n \geq N_0$ *and all events* $\mathcal{E} \in \mathfrak{B}(S)$,

$$c\mathbb{P}(\mathsf{S}_{X_{s_n}} \in \mathcal{E}) - \varepsilon \leq \mathbb{P}(\mathsf{S}_{Y_n} \in \mathcal{E}) \leq C\mathbb{P}(\mathsf{S}_{X_{s_n}} \in \mathcal{E}) + \varepsilon. \tag{9.100}$$

(2) *If* $(\mathsf{S}_{X_n})_{n \geq 1}$ *is uniformly tight, then so is* $(\mathsf{S}_{Y_n})_{n \geq 1}$.

(3) *If* $\mathsf{S}_{X_n} \in \mathcal{E}$ *holds with high probability, then so does* $\mathsf{S}_{Y_n} \in \mathcal{E}$.

(4) *When* $S = \mathfrak{G}$ *or* $S = \mathfrak{M}$ *(defined in Section 2), quenched local convergence of* S_{X_n} *to a deterministic law* μ *implies quenched local convergence of* S_{Y_n} *to* μ.

Proof. Using (9.98) and (9.99) (and the fact that the density function h is bounded, uniformly continuous and positive) we deduce that for each constant $M > 0$ there are constants $0 < c_M < C_M$ such that uniformly for all integers $k = \mu_Y n + x n^{2/3}$ with $|x| \leq M$,

$$\frac{\mathbb{P}(X_{s_n} = k)}{\mathbb{P}(Y_n = k)} = \frac{g_Y h\left(\frac{x}{g_X(\mu_Y/\mu_X)^{2/3}} \right) + o(1)}{g_X \left(\frac{\mu_Y}{\mu_X} \right)^{2/3} h\left(\frac{x}{g_Y} \right) + o(1)} \in [c_M + o(1), C_M + o(1)]. \tag{9.101}$$

This yields

$$\mathbb{P}(\mathsf{S}_{Y_n} \in \mathcal{E}) \leq \mathbb{P}(|Y_n - \mu_Y n| \geq M n^{2/3}) + \sum_{|k - n\mu_Y| \leq M n^{2/3}} \mathbb{P}(Y_n = k)\mathbb{P}(\mathsf{S}_k \in \mathcal{E})$$

$$\leq \mathbb{P}(|Y_n - \mu_Y n| \geq M n^{2/3})$$
$$+ (C_M + o(1)) \sum_{|k - n\mu_Y| \leq M n^{2/3}} \mathbb{P}(X_{s_n} = k)\mathbb{P}(\mathsf{S}_k \in \mathcal{E})$$
$$\leq o(1) + \mathbb{P}(|Y_n - \mu_Y n| \geq M n^{2/3}) + C_M \mathbb{P}(\mathsf{S}_{X_{s_n}} \in \mathcal{E}),$$

with an $o(1)$-term that only depends on M and n. Likewise

$$\mathbb{P}(\mathsf{S}_{Y_n} \in \mathcal{E}) \geq o(1) - \mathbb{P}(|X_{s_n} - \mu_Y n| \geq M n^{2/3}) + c_M \mathbb{P}(\mathsf{S}_{X_{s_n}} \in \mathcal{E}).$$

Given $\varepsilon > 0$, it follows from (9.98) and (9.99) that we may select M sufficiently large such that

$$\mathbb{P}(|Y_n - \mu_Y| \geq M n^{2/3}) \leq \varepsilon/2 \quad \text{and} \quad \mathbb{P}(|X_{s_n} - \mu_X| \geq M n^{2/3}) \leq \varepsilon/2.$$

This proves (9.100).

As for the second claim, suppose that $(S_{X_n})_{n \geq 1}$ is uniformly tight. Let $\varepsilon > 0$ be given. Then there is a compact subset $K_0 \subset X$ with $\mathbb{P}(S_{X_n} \notin K_0) \leq \varepsilon/C$ for all n. From (9.100) it follows that there is a constant N_0 with $\mathbb{P}(S_{Y_n} \notin K_0) \leq 2\varepsilon$ for all $n \geq N_0$.

The third claim follows directly from (9.100).

As for the fourth, Proposition 2.1 implies that quenched local convergence corresponds to convergence in probability of the percentage of vertices / corners with an (arbitrary but fixed) radius $r \geq 1$ neighbourhood having an (arbitrary but fixed) shape M. Hence if S_{X_n} converges in the quenched sense to a deterministic limit law μ, then this percentage of specified points in S_{X_n} converges in probability to a constant $p_{r,M}$ given by the corresponding μ-probability. The third claim now implies that the same holds for S_{Y_n}, yielding quenched convergence of S_{Y_n}. ∎

9.6. An application to random 2-connected planar maps

Let V_n^t denote the random non-separable planar map with n edges drawn with probability proportional to the weight $t^{v(\cdot)}$. Recall that in Lemma 9.3 we established a quenched local limit $\hat{\mathsf{V}}^t$ of the core $\mathcal{V}(\mathsf{M}_n^t)$.

Theorem 9.11. *If we let c_n denote a uniformly selected corner of V_n^t, then*

$$\mathfrak{L}((\mathsf{V}_n^t, c_n) \mid \mathsf{V}_n^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{V}}^t). \tag{9.102}$$

We call $\hat{\mathsf{V}}^t$ the uniform infinite non-separable planar map *with weight t at vertices.*

The limit $\hat{\mathsf{M}}^t$ may be constructed from the uniform infinite non-separable map $\hat{\mathsf{V}}^t$ by inserting independent random planar maps (with explicit distributions) at each corner of the uniform infinite non-separable planar map [64, Thm. 6.59]. The asymptotic degree distribution of V_n^1 was established in [23].

Proof of Theorem 9.11. By Lemma 9.9, the $\bar{\mathcal{O}}$-core $\bar{\mathcal{O}}(\mathsf{M}_n^t)$ admits a quenched limit $\hat{\bar{\mathsf{O}}}^t$ in the local topology, and

$$\mathbb{P}(e(\bar{\mathcal{O}}(\mathsf{M}_n^t)) = \ell) = \frac{1}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \left(h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])(1 - \mathbb{E}[\xi^{\mathcal{M}}])n - \ell}{\tilde{g}_{\bar{\mathcal{K}}}(t)n^{2/3}} \right) + o(1) \right)$$

uniformly in $\ell \in \mathbb{Z}$. We may define the cores $\bar{\mathcal{K}}(\mathsf{V}_n^t)$, $\bar{\mathcal{R}}(\mathsf{V}_n^t)$ and $\bar{\mathcal{O}}(\mathsf{V}_n^t)$ as for M_n, and by analogous arguments it follows that

$$\mathbb{P}(e(\bar{\mathcal{O}}(\mathsf{V}_n^t)) = \ell) = \frac{1}{g(t)n^{2/3}} \left(h\left(\frac{(1 - \mathbb{E}[\xi^{\bar{\mathcal{K}}}])n - \ell}{g(t)n^{2/3}} \right) + o(1) \right) \tag{9.103}$$

for some constant $g(t) > 0$. By Lemma 9.10 it follows that $\hat{\bar{\mathsf{O}}}^t$ is also the quenched local limit of $\bar{\mathcal{O}}(\mathsf{V}_n^t)$. The arguments in the proof of Lemma 9.9 that pass quenched local convergence of a large random $\bar{\mathcal{R}}$-structure down to its giant $\bar{\mathcal{O}}$-core also entail, conversely, that convergence of such an $\bar{\mathcal{O}}$-core entails convergence of the $\bar{\mathcal{R}}$-structure. Hence $\hat{\bar{\mathsf{R}}}^t$ is also

the quenched local limit of $\bar{\mathcal{R}}(\mathsf{V}_n^t)$. Likewise, the arguments in the proofs of Lemmas 9.7 and 9.8 that pass convergence from a large random $\bar{\mathcal{K}}$-structure down to its $\bar{\mathcal{R}}$-core easily imply that convergence of the $\bar{\mathcal{R}}$-core implies convergence of the $\bar{\mathcal{K}}$-structure. Hence $\bar{\mathcal{K}}(\mathsf{V}_n^t)$ admits $\hat{V}^t$ as quenched local limit. The same holds for Corollary 9.4, showing that $\hat{V}^t$ is the quenched local limit of V_n^t. ∎

9.7. $\mathcal{K}$-networks

We let K_n^t denote a random $\mathcal{K}$-network, drawn with probability proportional to its weight given by $t^{v(\cdot)}$. Equation (6.18), that is, $\mathcal{K} \equiv y\mathcal{R}(x, \mathcal{K})$, and the discussion in Section 4.2 imply that $\mathcal{R}$-enriched plane trees may be transformed into $\mathcal{K}$-networks. The network corresponding to such an enriched tree (T, β) with n vertices has n edges and gets constructed as follows. The $\mathcal{R}$-structure $\beta(o)$ corresponding to the root vertex o of T is a network with $d_T^+(o)$ regular edges and an additional "terminal" edge. The regular edges correspond bijectively to the fringe subtrees dangling from o. The total network gets constructed recursively by replacing each regular edge by the network corresponding to its fringe subtree. The terminal edge corresponds to the factor y in $\mathcal{K} \equiv y\mathcal{R}(x, \mathcal{K})$.

We let $\mathsf{T}_n^{\mathcal{K}}$ denote the simply generated tree with weight sequence $(\omega_k^{\mathcal{K}})_{k \geq 0}$ given by

$$\omega_k^{\mathcal{K}} = [y^k]\mathcal{R}(t, y). \tag{9.104}$$

For each vertex v of $\mathsf{T}_n^{\mathcal{K}}$ we draw a $d_{\mathsf{T}_n^{\mathcal{K}}}^+$-sized $\mathcal{R}$-structure $\beta_n^{\mathcal{K}}(v)$ with probability proportional to its weight. Lemma 4.2 implies that the random $\mathcal{K}$-structure corresponding to the random enriched plane tree $(\mathsf{T}_n^{\mathcal{K}}, \beta_n^{\mathcal{K}})$ is distributed like K_n^t.

Inequality (8.42), condition (8.43), and Lemma 3.1 imply that the simply generated tree $\mathsf{T}_n^{\mathcal{K}}$ is distributed like a Galton–Watson tree $\mathsf{T}^{\mathcal{K}}$ conditioned on having n vertices, with offspring distribution $\xi^{\mathcal{K}}$ satisfying

$$\mathbb{E}[\xi^{\mathcal{K}}] < 1 \quad \text{and} \quad \mathbb{P}(\xi^{\mathcal{K}} = n) \sim \frac{c_{\mathcal{R}}(t)}{\mathcal{R}(t, \rho_{\mathcal{R}}(t))} n^{-5/2}. \tag{9.105}$$

Lemma 3.2 entails that there is a constant $g_{\mathcal{K}}(t) > 0$ such that the largest $\mathcal{R}$-component $\mathcal{R}(\mathsf{K}_n^t)$ satisfies

$$\mathbb{P}(e(\mathcal{R}(\mathsf{K}_n^t)) = \ell) = \frac{1}{g_{\mathcal{K}}(t)n^{2/3}}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{K}}])n - \ell}{g_{\mathcal{K}}(t)n^{2/3}}\right) + o(1)\right) \tag{9.106}$$

uniformly for all $\ell \in \mathbb{Z}$.

Much as we defined the class $\bar{\mathcal{O}}$ of networks, we let

$$\mathcal{O} := \mathcal{F}_{0,1}(x, y\,\mathrm{SEQ}(xy)) \tag{9.107}$$

denote the pendant of networks obtained by blowing up regular edges of $\mathcal{F}_{0,1}$-networks into paths. Whitney's theorem [72] yields a $1 : 2$ correspondence between $\mathcal{O}$-networks and $\bar{\mathcal{O}}$-networks, because up to reflection any 3-connected graph has a unique embedding

into the 2-sphere (and any 3-connected map has at least four vertices and differs from its mirror image). Thus

$$\mathcal{O} = \tfrac{1}{2}\bar{\mathcal{O}}. \qquad (9.108)$$

Recall that $\mathcal{R}$ admits the decomposition

$$\mathcal{R} = \mathcal{J}\,\mathrm{SEQ}(\mathcal{I}^*), \quad y\mathcal{I}^* = \mathcal{O} + \mathrm{SET}_{\geq 2}(\mathcal{O} + \mathcal{L}), \quad \mathcal{J} = \mathrm{SET}(\mathcal{O} + \mathcal{L})$$

with

$$\mathcal{L} := y\,\mathrm{SEQ}_{\geq 1}(xy). \qquad (9.109)$$

This means an $\mathcal{R}$-network consists of a $\mathcal{J}$-component and a possibly empty sequence of $\mathcal{I}^*$-components. We explain this in detail:

The $\mathcal{J}$-component is a network consisting of the parallel composition of a (possibly empty) unordered collection of networks that are either $\mathcal{O}$-networks or paths of length at least 2 (corresponding to $\mathcal{L}$). If the collection is empty, we interpret this as the network consisting of two poles and no regular edges.

A $y\mathcal{I}^*$-network is either an $\mathcal{O}$-network, or the parallel composition of an unordered collection of at least two networks, each being either an $\mathcal{O}$-network or a path of length at least 2. An $\mathcal{I}^*$-network is a weighted network (weighted by both the fact that we have weight t at vertices, and that we divided by y) that, in addition to the "invisible" edge between the poles, has a second distinguished "invisible" edge.

A network from

$$\mathcal{R} = \mathcal{J}\,\mathrm{SEQ}(\mathcal{I}^*) = \mathcal{J} + \mathcal{J}\mathcal{I}^* + \mathcal{J}(\mathcal{I}^*)^2 + \mathcal{J}(\mathcal{I}^*)^3 + \cdots$$

either is a $\mathcal{J}$-network, or belongs to $\mathcal{J}(\mathcal{I}^*)^k$ for some $k \geq 1$. That is, it is constructed as follows. Take k $\mathcal{I}^*$-networks $I_1, \ldots, I_k$ and a $\mathcal{J}$-network J. Substitute the second "invisible" edge of I_1 by I_2, then the second "invisible" edge of I_2 by I_3, and so on. Finally, substitute the second "invisible" edge of I_k by J.

We let $\mathcal{O}(\mathsf{K}_n^t)$ denote the largest $\mathcal{O}$-component in the decomposition of $\mathcal{R}(\mathsf{K}_n^t)$.

Lemma 9.12. (1) *Uniformly for all $\ell \in \mathbb{Z}$,*

$$\mathbb{P}(e(\mathcal{O}(\mathsf{K}_n^t)) = \ell) = \frac{1}{g_{\mathcal{K}}(t)n^{2/3}}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{K}}])n - \ell}{g_{\mathcal{K}}(t)n^{2/3}}\right) + o(1)\right). \qquad (9.110)$$

(2) *If $c_n^{\mathcal{R}}$ denotes a uniformly selected corner of $\mathcal{R}(\mathsf{K}_n^t)$, then*

$$\mathfrak{L}((\mathcal{R}(\mathsf{K}_n^t), c_n^{\mathcal{R}}) \mid \mathcal{R}(\mathsf{K}_n^t)) \xrightarrow{p} \mathfrak{L}(\hat{\bar{\mathsf{O}}}^t). \qquad (9.111)$$

(3) *There is a random infinite planar map $\hat{\mathsf{K}}^t$ such that*

$$\mathfrak{L}((\mathsf{K}_n^t, c_n^{\mathcal{K}}) \mid \mathsf{K}_n^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{K}}^t) \qquad (9.112)$$

with $c_n^{\mathcal{K}}$ denoting a uniformly selected corner of K_n^t. There is also a random infinite planar graph $\hat{\mathsf{K}}^{u,t}$ such that

$$\mathfrak{L}((\mathsf{K}_n^t, v_n^{\mathcal{K}}) \mid \mathsf{K}_n^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{K}}^{u,t}) \tag{9.113}$$

with $v_n^{\mathcal{K}}$ denoting a uniformly selected vertex of K_n^t.

Proof. We start with the first claim. Equation (9.108) and the singular expansion (8.16) entail

$$[y^n]\mathcal{O}(t, y) \sim c_{\mathcal{O}}(t)\rho_{\mathcal{R}}(t)^{-n} n^{-5/2} \tag{9.114}$$

for some constant $c_{\mathcal{O}}(t) > 0$. Recall that $\rho_{\mathcal{R}}(t) < 1/t$ by (8.18), so

$$[y^n](\mathcal{O}(t, y) + \mathfrak{L}(t, y)) = [y^n]\mathcal{O}(t, y)(1 + o(1))$$

with the $o(1)$ term tending exponentially fast to zero. By Proposition 5.4 it follows that

$$[y^n]\mathcal{J}(t, y) \sim c_{\mathcal{J}}(t)[y^n]\mathcal{O}(t, y), \tag{9.115}$$

for $c_{\mathcal{J}}(t) := \exp(\mathcal{O}(t, \rho_{\mathcal{R}}(t)) + t\rho_{\mathcal{R}}(t)^2/(1 - \rho_{\mathcal{R}}(t)))$. It also follows that large $\mathcal{J}$-objects have a giant $\mathcal{O}$-component with a stochastically bounded remainder that admits a limit distribution. Likewise, Propositions 5.4 and 5.5 entail that

$$[y^n]\mathcal{J}^*(t, y) \sim c_{\mathcal{J}^*}(t)[y^n]\mathcal{O}(t, y) \tag{9.116}$$

for some constant $c_{\mathcal{J}^*}(t) > 0$. This yields

$$[y^n]\mathrm{SEQ}(\mathcal{J}^*(t, y)) \sim (1 - \mathcal{J}^*(t, \rho_{\mathcal{R}}(t)))^{-2} c_{\mathcal{J}^*}(t)[y^n]\mathcal{O}(t, y). \tag{9.117}$$

It also follows that large $\mathcal{J}^*$-objects have a giant $\frac{1}{y}\mathcal{O}(t, y)$-component (corresponding canonically to an $\mathcal{O}$-structure), with a stochastically bounded remainder that admits a limit distribution. By Proposition 5.5 and (9.106), $\mathcal{R}(\mathsf{K}_n^t)$ has a giant $\mathcal{O}$-component and a stochastically bounded remainder admitting a limit distribution. In particular,

$$e(\mathcal{R}(\mathsf{K}_n^t)) = e(\mathcal{O}(\mathsf{K}_n^t)) + O_p(1). \tag{9.118}$$

We let $\mathcal{J}(\mathsf{K}_n^t)$ denote the $\mathcal{J}$-component of $\mathcal{R}(\mathsf{K}_n^t)$, and set $X_n = e(\mathcal{J}(\mathsf{K}_n^t))$. Likewise, we let $\mathrm{SEQ}(\mathcal{J}^*)(\mathsf{K}_n^t)$ be the $\mathrm{SEQ}(\mathcal{J}^*)$-component and let Y_n denote its size. For any integer $m \geq 0$,

$$((X_n, Y_n) \mid e(\mathcal{R}(\mathsf{K}_n^t)) = m) \overset{d}{=} ((X, Y) \mid X + Y = m) \tag{9.119}$$

with X and Y denoting the sizes of Boltzmann-distributed $\mathcal{J}(t, y)$ and $\mathrm{SEQ}(\mathcal{J}^*(t, y))$ objects with parameter $\rho_{\mathcal{R}}(t)$. It follows from [65, (2.10)] that there is a constant $\varepsilon > 0$ such that

$$\mathbb{P}(e(\mathcal{R}(\mathsf{K}_n^t)) \leq \varepsilon n/\log n) = \mathbb{P}(\Delta(\mathsf{T}_n^{\mathcal{K}}) \leq \varepsilon n/\log n) = o(n^{-2/3}). \tag{9.120}$$

With foresight we set

$$t_n := \lfloor n^\delta \rfloor$$

for some constant δ satisfying $4/9 < \delta < 1$. Using (9.119) and (9.120) as well as the asymptotics (9.115) and (9.117) we find

$$n^{2/3} \mathbb{P}(t_n \leq X_n \leq \mathrm{e}(\mathcal{R}(\mathsf{K}_n^t)) - t_n)$$

$$= o(1) + n^{2/3} \sum_{r=\varepsilon n/\log n}^{n} \mathbb{P}(\mathrm{e}(\mathcal{R}(\mathsf{K}_n^t)) = r) \sum_{s=t_n}^{r-t_n} \frac{\mathbb{P}(X = s)\mathbb{P}(Y = r - s)}{\mathbb{P}(X + Y = r)}$$

$$= o(1) + O(n^{2/3}) \sum_{r=\varepsilon n/\log n}^{n} \mathbb{P}(\mathrm{e}(\mathcal{R}(\mathsf{K}_n^t)) = r) \sum_{s=t_n}^{r-t_n} \left(s \frac{r-s}{r} \right)^{-5/2}$$

$$= o(1) + O(n^{2/3}) t_n^{-3/2} = o(1).$$

This entails that $\mathbb{P}(\mathrm{e}(\mathcal{O}(\mathsf{K}_n^t)) = \ell)$ may be written as

$$o(n^{-2/3}) + \sum_{r=\varepsilon n/\log n}^{n} \frac{\mathbb{P}(\mathrm{e}(\mathcal{R}(\mathsf{K}_n^t)) = r)}{\mathbb{P}(X + Y = r)}$$

$$\times \sum_{s=0}^{t_n} (\mathbb{P}(X = r - s)\mathbb{P}(Y = s)A + \mathbb{P}(X = s)\mathbb{P}(Y = r - s)B).$$

Here A denotes the probability that the size of the largest $\mathcal{O}$-component found in the decomposition of a pair of a random s-sized $\mathrm{SEQ}(\mathcal{I}^*(t, y))$-structure and a random $(r - s)$-sized $\mathcal{J}(t, y)$-structure equals precisely ℓ. The probability B is defined analogously for a random $(r - s)$-sized $\mathrm{SEQ}(\mathcal{I}^*(t, y))$-structure and a random s-sized $\mathcal{J}(t, y)$-structure.

We would like to replace A and B by the corresponding probabilities that involve only the $(r - s)$-sized components. To this end, let δ' be a fixed constant satisfying $\delta < \delta' < 1$. Proposition 5.2 implies that the probability that the size of the largest $\mathcal{O}$-component in an $r - s \geq \Theta(n/\log n)$ sized random $\mathcal{J}(t, y)$-structure is smaller than $n^{\delta'}$ decays faster than any power of $1/n$. The same holds for the largest $\mathcal{I}^*(t, y)$-structure in a random $(r - s)$-sized $\mathrm{SEQ}(\mathcal{I}^*(t, y))$-structure. As $\mathcal{L}(t, y)$ has radius of convergence strictly larger than $\rho_{\mathcal{R}}(t)$, the total variation distance between a random m-sized $y\mathcal{I}^*$-structure and a random m-sized

$$\mathcal{L} + y\mathcal{I}^* = \mathrm{SET}_{\geq 1}(\mathcal{O} + \mathcal{L}) \tag{9.121}$$

is exponentially small in m as $m \to \infty$. This allows us to apply Proposition 5.2 again, implying that the probability that the largest $\mathcal{O}$-component within the largest $\mathcal{I}^*(t, y)$ in a random $(r - s)$-sized $\mathrm{SEQ}(\mathcal{I}^*(t, y))$-structure is less than $n^{\delta'}$ tends to zero faster than any power of $1/n$.

Summing up, we may assume that

$$\ell \geq n^{\delta'} \tag{9.122}$$

and replace the constants A and B in the previous expression for $\mathbb{P}(e(\mathcal{O}(\mathsf{K}_n^t)) = \ell)$ by constants $\tilde{A}$ and $\tilde{B}$, with $\tilde{A}$ the probability that the size of the largest $\mathcal{O}$-component in a random $(r - s)$-sized $\mathcal{J}(t, y)$-structure is ℓ, and $\tilde{B}$ analogously the probability that the size of the largest $\mathcal{O}$-component in a random $(r - s)$-sized SEQ($\mathcal{J}^*(t, y)$)-structure is ℓ.

There exists a constant $0 < p < 1$ such that uniformly for all $r \geq \varepsilon n / \log n$ and all $0 \leq s \leq t_n$,

$$\frac{\mathbb{P}(X = r - s)}{\mathbb{P}(X + Y = r)} \sim p \quad \text{and} \quad \frac{\mathbb{P}(Y = r - s)}{\mathbb{P}(X + Y = r)} \sim 1 - p.$$

Hence

$$n^{2/3}\mathbb{P}(e(\mathcal{O}(\mathsf{K}_n^t)) = \ell)$$

$$= o(1) + (p + o(1))n^{2/3} \sum_{r = \varepsilon n / \log n}^{n} \mathbb{P}(e(\mathcal{R}(\mathsf{K}_n^t)) = r) \sum_{s=0}^{t_n} \mathbb{P}(Y = s)\tilde{A}$$

$$+ (1 - p + o(1))n^{2/3} \sum_{r = \varepsilon n / \log n}^{n} \mathbb{P}(e(\mathcal{R}(\mathsf{K}_n^t)) = r) \sum_{s=0}^{t_n} \mathbb{P}(X = s)\tilde{B}. \quad (9.123)$$

Let $(s_n)_n$ denote an arbitrary sequence of positive integers satisfying $s_n \leq t_n$ and $s_n \to \infty$. We are going to argue that in (9.123) only summands with $0 \leq s \leq s_n$ contribute, regardless of how slowly s_n tends to infinity. We start with the sum involving $\tilde{A}$. Let Z denote the size of a random Boltzmann-distributed $\mathcal{O}(t, y)$-object. By Proposition 5.2 and the fact that $\mathcal{L}(t, y)$ has radius of convergence strictly larger than $\rho_{\mathcal{R}}(t)$ it follows that uniformly for $\varepsilon n / \log n \leq r \leq n$ and $s_n \leq s \leq t_n$,

$$\tilde{A} \leq E_n + C \frac{\mathbb{P}(Z = \ell)\mathbb{P}(Z = r - s - \ell)}{\mathbb{P}(Z = r - s)} \exp\left(-\frac{r - s - \ell}{\ell}\right) \mathbb{1}_{\ell \leq r - s} \quad (9.124)$$

for some constant $C > 0$ and an error term E_n that depends only on n and tends to zero faster than any power of $1/n$. (Here is a detailed justification: E_n is bounded by the probability that the largest $\mathcal{O} + \mathcal{L}$-component in the random $(r - s)$-sized $\mathcal{J}(t, y) =$ SET($\mathcal{O} + \mathcal{L}$) structure is an $\mathcal{L}$-structure. Proposition 5.2 ensures that the size of this structure is at least $n^{\delta'}$ with probability that tends to zero faster than any power of $1/n$. As $\mathcal{L}(t, y)$ has radius of convergence strictly larger than $\rho_{\mathcal{R}}(t)$, the same holds for the decay of E_n.)

Continuing the argument, we may consider the cases $\ell \geq (r - s)/2$ and $\ell < (r - s)/2$ separately to obtain

$$\frac{\mathbb{P}(Z = \ell)\mathbb{P}(Z = r - s - \ell)}{\mathbb{P}(Z = r - s)}$$

$$= O(1)\big(\mathbb{P}(Z = \ell)\mathbb{1}_{\ell < (r - s)/2} + \mathbb{P}(Z = r - s - \ell)\mathbb{1}_{\ell \geq (r - s)/2}\big). \quad (9.125)$$

Using (9.106) and (9.122), we find that

$$n^{2/3} \sum_{r=\varepsilon n/\log n}^{n} \mathbb{P}(e(\mathcal{R}(K_n^t)) = r) \sum_{s=s_n}^{t_n} \mathbb{P}(Y = s)\tilde{A}$$

$$= o(1) + O(1) \sum_{s=s_n}^{t_n} \mathbb{P}(Y = s)$$

$$\times \sum_{r=\max(\varepsilon n/\log n, \ell+s)}^{n} \left((r - s - \ell)^{-5/2} + \mathbb{P}(Z = \ell) \exp\left(-\frac{r - s - \ell}{\ell} \right) \right)$$

$$= o(1) + O(1)\mathbb{P}(s_n \le Y \le t_n).$$

This bound clearly tends to zero. Hence we have tight control of the size of the components; the next step is to control the deviation of ℓ from $r - s$. For any sequence $(u_n)_n$ of positive integers that tends to infinity it follows by the same exact bounds that

$$n^{2/3} \sum_{r=\varepsilon n/\log n}^{n} \mathbb{P}(e(\mathcal{R}(K_n^t)) = r) \sum_{s=0}^{s_n} \mathbb{P}(Y = s)\tilde{A} \mathbb{1}_{r-s-\ell \ge u_n} = o(1).$$

The Gibbs partition $\mathcal{J} = \mathrm{SET}(\mathcal{O} + \mathcal{L})$ is convergent. Hence for any constant integer $u \ge 0$ the quantity $\tilde{A}$ (which depends on $r - s$ and ℓ) converges to a limiting probability a_u (with $\sum_{u \ge 0} a_u = 1$) uniformly for all r, s, ℓ with $r - s = \ell + u$. Moreover, the local limit theorem in (9.106) entails that

$$\mathbb{P}(e(\mathcal{R}(K_n^t)) = \ell + x) \sim \mathbb{P}(e(\mathcal{R}(K_n^t)) = \ell + x)$$

uniformly for all integers x with $|x| = o(n^{2/3})$ and all integers ℓ satisfying (9.122). Hence we may choose the sequences $(t_n)_n$ and $(u_n)_n$ to tend to infinity sufficiently slowly so that

$$n^{2/3} \sum_{r=\varepsilon n/\log n}^{n} \sum_{s=0}^{s_n} \mathbb{P}(Y = s)\tilde{A} \mathbb{1}_{r-s-\ell \le u_n}$$

$$= n^{2/3} \sum_{s=0}^{s_n} \sum_{u=0}^{u_n} \mathbb{P}(e(\mathcal{R}(K_n^t)) = \ell + u + s)\mathbb{P}(Y = s)(a_u + o(1))$$

$$= n^{2/3} \mathbb{P}(e(\mathcal{R}(K_n^t)) = \ell).$$

Hence (9.123) simplifies to

$$n^{2/3}\mathbb{P}(e(\mathcal{O}(K_n^t)) = \ell) = o(1) + p n^{2/3}\mathbb{P}(e(\mathcal{R}(K_n^t)) = \ell)$$

$$+ (1 - p + o(1))n^{2/3} \sum_{r=\varepsilon n/\log n}^{n} \mathbb{P}(e(\mathcal{R}(K_n^t)) = r) \sum_{s=0}^{t_n} \mathbb{P}(X = s)\tilde{B}, \quad (9.126)$$

with the $o(1)$ terms being uniform in n and all ℓ satisfying (9.122). We have also established above that $n^{2/3}\mathbb{P}(e(\mathcal{O}(K_n^t)) = \ell)$ tends to zero uniformly for all ℓ that do not satisfy

(9.122), as is the case for the right-hand side of (9.110). Hence the restriction on ℓ is not a real restriction at all.

The double sum involving $\tilde{B}$ may be treated using analogous arguments: First argue as before that we may discard all summands for which $s_n \leq s \leq t_n$. Then expand $\tilde{B}$ and discard the summands for which the largest $\mathcal{I}^*$-component in the $(r - s)$-sized component is not close to $r - s$. Then use (9.121) and again the same arguments to discard all summands for which the largest $\mathcal{O}$-component within that largest $\mathcal{I}^*$-component is not close to the size of the $\mathcal{I}^*$-component. It is clear how to carry out each of these tedious but not difficult steps, hence we leave the details to the inclined reader.

Having taken care of the double sum involving $\tilde{B}$, we see that (9.126) reduces to

$$\mathbb{P}(\mathrm{e}(\mathcal{O}(\mathsf{K}_n^t)) = \ell) = o(n^{-2/3}) + \mathbb{P}(\mathrm{e}(\mathcal{R}(\mathsf{K}_n^t)) = \ell). \tag{9.127}$$

By (9.106) this readily implies (9.110).

Relation (9.110) and Lemma 9.9 state local limit theorems for the sizes of the cores $\bar{\mathcal{O}}(\mathsf{K}_n^t)$ and $\mathcal{O}(\mathsf{K}_n^t)$. Equation (9.108) ensures that conditioned on having a common fixed size, the cores follow the same conditional distribution. This allows us to apply Lemma 9.10 to transfer the quenched local limit of $\bar{\mathcal{O}}(\mathsf{K}_n^t)$ (stated in Lemma 9.9) to a quenched local limit theorem for $\mathcal{O}(\mathsf{K}_n^t)$. That is,

$$\mathfrak{L}((\mathcal{O}(\mathsf{K}_n^t), v_n) \mid \mathcal{O}(\mathsf{K}_n^t)) \xrightarrow{p} \mathfrak{L}(\hat{\bar{\mathsf{O}}}^t), \tag{9.128}$$

with v_n denoting a uniformly selected corner of $\mathcal{O}(\mathsf{K}_n^t)$. Condition (9.118) now allows us to argue as in Corollary 9.4 to deduce the local limit (9.111).

It remains to show the statements directly concerning K_n^t. We may copy the proof of Lemma 9.7 almost word for word (replacing all occurrences of $\bar{\mathcal{K}}$ and $\bar{\mathcal{R}}$ by $\mathcal{K}$ and $\mathcal{R}$) to show that quenched local convergence of K_n^t is equivalent to quenched local convergence of a random $\mathcal{K}$-network $\mathsf{K}_{[n]}^t$ that is constructed from the core $\mathcal{R}(\mathsf{K}_n^t)$ by substituting edges with independent random $\mathcal{K}$-networks analogously to the way $\bar{\mathsf{K}}_{[n]}^t$ was constructed from the core $\bar{\mathcal{R}}(\bar{\mathsf{K}}_n^t)$ by substituting edges with independent random $\bar{\mathcal{K}}$-networks. Moreover, the proof of Lemma 9.8 may also be copied almost word for word to show that an analogue of (9.74) holds, meaning that if we select two corners of $\mathsf{K}_{[n]}^t$ uniformly and independently at random, then their components asymptotically behave like independent copies of a size-biased version $\mathsf{K}^\bullet$ of K^t, and the edges corresponding to those components are asymptotically uniform edges of $\mathcal{R}(\mathsf{K}_n^t)$. The subsequent arguments in the proof of Lemma 9.8 that pass convergence from a large random $\bar{\mathcal{K}}$-structure down to its $\bar{\mathcal{R}}$-core also imply that conversely, convergence of the $\bar{\mathcal{R}}$-core implies convergence of the $\bar{\mathcal{K}}$-structure. As we may copy the proof of Lemma 9.8 word for word (replacing all occurrences of $\bar{\mathcal{K}}$ and $\bar{\mathcal{R}}$ by $\mathcal{K}$ and $\mathcal{R}$), this means that the local convergence (9.111) implies the local limit (9.112).

It remains to prove the local convergence (9.113) with respect to the uniform distribution. We will use a transfer argument from [24], based on an extension of a formula in [44, (2.3.1)]. Given an integer $r \geq 1$ and a finite rooted graph G, the limit (9.112) implies

that the number Y_n of half-edges in K_n^t having r-neighbourhood isomorphic to G satisfies

$$Y_n/(2n) \xrightarrow{p} \mathbb{P}(U_r(\hat{\mathsf{K}}^t) = G).$$

Letting $d(G)$ denote the root-degree of G, the number X_n of vertices in K_n^t having r-neighbourhood isomorphic to G is given by

$$X_n = Y_n/d(G).$$

By the limit (9.133) below it follows that

$$X_n/\mathrm{v}(\mathsf{K}_n^t) = Y_n/(d(G)\mathrm{v}(\mathsf{K}_n^t)) \xrightarrow{d} 2\mathbb{P}(U_r(\hat{\mathsf{K}}^t) = G)/(d(G)q_1) =: p_{r,G}.$$

We may deduce $\sum_G p_{r,G} = 1$ as in the proof of [24, Eq. (6)], hence verifying the existence of a random infinite planar graph $\hat{\mathsf{K}}^{\mathrm{u},t}$ with

$$\mathfrak{L}((\mathsf{K}_n^t, v_n^{\mathcal{K}}) \mid \mathsf{K}_n^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{K}}^{\mathrm{u},t}). \qquad \blacksquare$$

9.8. 2-connected planar graphs

Let N_n^t denote a random $\mathcal{N}$-network with n regular edges and weight t at non-pole vertices. The convergent Gibbs partition

$$\mathcal{N}(t, y) = \mathcal{K}(t, y)\mathrm{SEQ}(t\mathcal{K}(t, y))$$

expresses that any $\mathcal{N}$-network consists of a series composition of a positive number of $\mathcal{K}$-networks. This is completely analogous to the fact that $\mathcal{D}$-networks are series compositions of a positive number of $\mathcal{N}$-networks. Hence (9.42)–(9.45) still hold if we replace D_n^t by N_n^t and any occurrence of $\bar{\mathsf{K}}$ and $\bar{\mathcal{K}}$ by K and $\mathcal{K}$. That is, identifying $\mathcal{N}$-networks with sequences of $\mathcal{K}$-networks, we have

$$\mathsf{N}_n^t \overset{d}{\approx} \big(\mathsf{K}(1), \ldots, \mathsf{K}(F), \mathsf{K}_{n-E}^t, \mathsf{K}'(1), \ldots, \mathsf{K}'(F')\big) \qquad (9.129)$$

with a random integer

$$E := \sum_{i=1}^{F} \mathrm{e}(\mathsf{K}(i)) + \sum_{i=1}^{F'} \mathrm{e}(\mathsf{K}'(i)).$$

Here we let F and F' denote i.i.d. geometric variables with distribution

$$\mathbb{P}(F = k) = \mathcal{K}(t, \rho_{\mathcal{K}})^k (1 - \mathcal{K}(t, \rho_{\mathcal{K}})), \quad k \geq 0. \qquad (9.130)$$

The networks $\mathsf{K}(i)$ and $\mathsf{K}'(i)$, $i \geq 1$, denote independent copies of a Boltzmann-distributed $\mathcal{K}(t, y)$-network K with distribution given by

$$\mathbb{P}(\mathsf{K} = K) = t^{\mathrm{v}(K)} \rho_{\mathcal{K}}^{\mathrm{e}(K)} / \mathcal{K}(t, \rho_{\mathcal{K}}). \qquad (9.131)$$

The relation $\mathcal{N} = (1 + y)\frac{2}{x^2}\frac{\partial \mathcal{B}}{\partial y} - 1$ entails that the result B_n^t of conditioning N_n^t on having an edge between its poles is the random 2-connected planar graph with n edges and weight t at vertices. The enumerative results of [6] entail that there is a number $0 < q_1 < 1$ such that

$$v(\mathsf{B}_n^t)n^{-1} \xrightarrow{p} q_1. \tag{9.132}$$

By (9.129), it readily follows that

$$v(\mathsf{K}_n^t)n^{-1} \xrightarrow{p} q_1. \tag{9.133}$$

This concentration phenomenon is used in the proof of Lemma 9.12 in order to pass from the stationary distribution to the uniform distribution. Hence Lemma 9.12 is now fully verified and having it at hand, we deduce from (9.129) (by arguments as in the proof of Corollary 9.4)

Theorem 9.13. *Let $v_n^{\mathcal{B}}$ denote a vertex selected according to the uniform distribution $\mu_n^{\mathcal{B},t}$ on the vertex set of B_n^t. Then*

$$\mathfrak{L}((\mathsf{B}_n^t, v_n^{\mathcal{B}}) \mid \mathsf{B}_n^t) \xrightarrow{p} \mathfrak{L}(\hat{\mathsf{K}}^{\mathrm{u},t}). \tag{9.134}$$

Likewise, (9.112) implies such a limit when v_n^{e} is chosen according to the stationary distribution instead. We let $\hat{\mathsf{B}}$ denote the limit in the case $t = 1$.

Letting $\mathcal{O}(\mathsf{B}_n^t)$ denote the $\mathcal{O}$-core of the largest $\mathcal{K}$-component of the $\mathcal{N}$-network N_n^t (from which we constructed B_n^t), we see by arguments as in the proof of Corollary 9.5 that (9.110) implies

$$\mathbb{P}(\mathrm{e}(\mathcal{O}(\mathsf{B}_n^t)) = \ell) = \frac{1}{g_{\mathcal{K}}(t)n^{2/3}}\left(h\left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{K}}])n - \ell}{g_{\mathcal{K}}(t)n^{2/3}}\right) + o(1)\right) \tag{9.135}$$

uniformly for all $\ell \in \mathbb{Z}$.

9.9. Bundles of 2-connected planar graphs

Equation (6.2) entails that planar graphs (with vertices as atoms) are $\mathcal{W}$-enriched trees for the class $\mathcal{W}$ given by

$$\mathcal{W}(x) = \mathrm{SET}\left(\frac{\partial \mathcal{B}}{\partial x}(x, 1)\right). \tag{9.136}$$

That is, a $\mathcal{W}$-object is an unordered collection (or bundle) of derived (that is, rooted at a vertex without a label) 2-connected planar graphs that are glued together at their distinguished vertices. The resulting vertex becomes the root of the $\mathcal{W}$-object. By the discussion in Section 4.2, this entails that the random planar graph P_n may be generated as follows (see also [58, Prop. 3.6]):

(1) Generate a simply generated tree $\mathsf{T}_n^{\mathcal{P}}$ with weight sequence $(\omega_k^{\mathcal{P}})_{k \geq 0}$ given by $\omega_k^{\mathcal{P}} = [x^k]\mathcal{W}(x)$.

(2) For each vertex $v \in \mathsf{T}_n^{\mathcal{P}}$ let $\beta_n^{\mathcal{P}}(v)$ denote a uniformly selected $\mathcal{W}$-structure with $d_{\mathsf{T}_n^{\mathcal{P}}}^+(v)$ labelled non-root vertices.

(3) Assemble P_n from the $\mathcal{W}$-enriched tree $(\mathsf{T}_n^{\mathcal{P}}, \beta_n^{\mathcal{P}})$ by applying the correspondence between rooted planar graphs and $\mathcal{W}$-enriched trees, and forgetting about the root vertex.

The last step means that we start with the $\mathcal{W}$-object $\beta^{\mathcal{P}}(o)$ of the root o of $\mathsf{T}_n^{\mathcal{P}}$ and identify its non-marked vertices in a canonical way with the offspring vertices of o. The graph is then constructed recursively by identifying each non-marked vertex of $\beta^{\mathcal{P}}(o)$ with the rooted graph corresponding to the enriched fringe subtree of $(\mathsf{T}_n^{\mathcal{P}}, \beta_n^{\mathcal{P}})$ at the corresponding offspring of o.

Inequality (8.5), the asymptotic expression (8.6), and Lemma 3.1 entail that $\mathsf{T}_n^{\mathcal{P}}$ follows the distribution of a Galton–Watson tree $\mathsf{T}^{\mathcal{P}}$ conditioned on having n vertices, with offspring law $\xi^{\mathcal{P}}$ satisfying

$$\mathbb{E}[\xi^{\mathcal{P}}] = \rho_{\mathcal{B}} \frac{\partial^2 \mathcal{B}}{\partial x^2}(\rho_{\mathcal{B}}, 1) < 1 \quad \text{and} \quad \mathbb{P}(\xi^{\mathcal{P}} = n) \sim c_{\mathcal{P}} \rho_{\mathcal{B}}^{-n} n^{-5/2} \tag{9.137}$$

for some constant $c_{\mathcal{P}} > 0$. We let $\mathcal{W}(\mathsf{P}_n)$ denote the $\mathcal{W}$-structure corresponding to the lexicographically first vertex of $\mathsf{T}_n^{\mathcal{P}}$ with maximal outdegree. By Lemma 3.2, the unique (with high probability) largest $\mathcal{W}$-component $\mathcal{W}(\mathsf{P}_n)$ satisfies

$$\mathbb{P}(\mathrm{v}(\mathcal{W}(\mathsf{P}_n)) = \ell) = \frac{1}{g_{\mathcal{P}} n^{2/3}} \left(h \left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{P}}])n - \ell}{g_{\mathcal{P}} n^{2/3}} \right) + o(1) \right) \tag{9.138}$$

uniformly for all $\ell \in \mathbb{Z}$ with $g_{\mathcal{P}} > 0$ a constant. The second largest $\mathcal{W}$-component has order $O_p(n^{2/3})$: this follows for example from [36, Thm. 19.34]. The Gibbs partition (9.136) is convergent, that is, $\mathcal{W}(\mathsf{P}_n)$ exhibits a giant $\mathcal{B}$-component denoted by $\mathcal{B}(\mathcal{W}(\mathsf{P}_n))$, and the small fragments admit a limit distribution. Hence the $\mathcal{B}$-core $\mathcal{B}(\mathsf{P}_n) := \mathcal{B}(\mathcal{W}(\mathsf{P}_n))$ corresponds with high probability to the largest 2-connected block of P_n.

Remark 9.14. As in the proof of Corollary 9.5, condition (9.138) implies that uniformly for all $\ell \in \mathbb{Z}$,

$$\mathbb{P}(\mathrm{v}(\mathcal{B}(\mathsf{P}_n)) = \ell) = \frac{1}{g_{\mathcal{P}} n^{2/3}} \left(h \left(\frac{(1 - \mathbb{E}[\xi^{\mathcal{P}}])n - \ell}{g_{\mathcal{P}} n^{2/3}} \right) + o(1) \right). \tag{9.139}$$

A local limit law for the number of vertices L_n of the largest block in P_n was proven by [32, Thm. 5.4]. Note that (9.139) is a slightly different statement. Clearly $L_n \overset{d}{\approx} \mathrm{v}(\mathcal{B}(\mathsf{P}_n))$, but in order to deduce a local limit theorem for L_n we would additionally have to verify that the probability of the event that simultaneously $\mathrm{v}(\mathcal{B}(\mathsf{P}_n)) = \ell$ and $L_n > \mathrm{v}(\mathcal{B}(\mathsf{P}_n))$ is $o(n^{-2/3})$ uniformly in ℓ. The proof is similar to arguments used in the proof of [65, Thm. 1.1], specifically the step that shows that the bound in [65, (3.23)] tends to zero. We leave the details to the reader, since this subtle difference between $\mathcal{B}(\mathsf{P}_n)$ and the largest 2-connected block is not relevant for the arguments in the present work.

Note that conditioning the core $\mathcal{B}(\mathsf{P}_n)$ on having a certain number of edges does not yield the uniform distribution on the 2-connected planar graphs with that number of edges. This effect does *not* go away as n becomes large. In fact, letting E_n denote the number of edges in the largest 2-connected block of P_n, it was shown by [32, Lem. 6.6] that

$$\mathcal{B}(\mathsf{P}_n) \overset{d}{\approx} \mathsf{B}_{E_n}^{\rho_{\mathcal{B}}} \tag{9.140}$$

as n tends to infinity. That is, we have to introduce weight $t = \rho_{\mathcal{B}}$ at vertices. Here we assume E_n is independent of $(\mathsf{B}_k^{\rho_{\mathcal{B}}})_{k \geq 0}$. In [32, Thm. 6.5] it was shown that the number E_n has order $\alpha_0 n$ with an analytically given constant

$$\alpha_0 \approx 2.17 \tag{9.141}$$

and a fluctuation of order $n^{2/3}$ that admits a local limit theorem of Airy type. Letting v_n denote a uniformly selected vertex of $\mathcal{B}(\mathsf{P}_n)$, we infer from (9.140) and Theorem 9.13 that

$$\mathfrak{L}((\mathcal{B}(\mathsf{P}_n), v_n) \mid \mathcal{B}(\mathsf{P}_n)) \overset{p}{\to} \mathfrak{L}(\hat{\mathsf{B}}). \tag{9.142}$$

As the Gibbs partition (9.136) is convergent, it follows from (9.142) that

$$\mathfrak{L}((\mathcal{W}(\mathsf{P}_n), v_n) \mid \mathcal{B}(\mathsf{P}_n)) \overset{p}{\to} \mathfrak{L}(\hat{\mathsf{B}}). \tag{9.143}$$

9.10. Connected planar graphs

A result of [64, Thm. 6.39] states that for block-weighted random graphs in a certain condensation regime (encompassing P_n), annealed local convergence of the random connected graph is equivalent to annealed local convergence of its 2-connected core. By (9.142) the 2-connected core $\mathcal{B}(\mathsf{P}_n)$ of the uniform connected planar graph P_n with n labelled vertices admits a distributional limit $\hat{\mathsf{B}}$ in the local topology. Hence P_n admits an annealed local limit $\hat{\mathsf{P}}$. As stated in [64, Thm. 6.39], the UIPG $\hat{\mathsf{P}}$ may be constructed from the uniform infinite 2-connected planar graph $\hat{\mathsf{B}}$ by inserting an independent copy of a Boltzmann-distributed rooted connected planar graph at each non-root vertex of the uniform infinite random 2-connected graph $\hat{\mathsf{B}}$, and a Boltzmann-distributed doubly rooted connected planar graph at the root of $\hat{\mathsf{B}}$.

We are now going to prove quenched convergence of P_n.

Proof of Theorem 1.1. Let $\mathsf{T}^{\circ, \mathcal{P}}$ and $\mathsf{T}^{\bullet, \mathcal{P}}$ be defined for the offspring distribution $\xi^{\mathcal{P}}$ just as T° and $\mathsf{T}^\bullet$ were defined for ξ in Section 3.2. Let P° and $\mathsf{P}^\bullet$ denote the random bi-rooted planar graphs corresponding to canonical decorations of $\mathsf{T}^{\circ, \mathcal{P}}$ and $\mathsf{T}^{\bullet, \mathcal{P}}$. Likewise, let P denote the Boltzmann map corresponding to a canonical decoration of $\mathsf{T}^{\mathcal{P}}$.

The graph P_n consists of its $\mathcal{W}$-core $\mathcal{W}(\mathsf{P}_n)$ together with planar graphs $(\mathcal{P}_i(\mathsf{P}_n))_{0 \leq i \leq v(\mathcal{W}(\mathsf{P}_n))}$ attached to each of its vertices. We assume that the index $i = 0$ corresponds to the component attached to the root of $\mathcal{W}(\mathsf{P}_n)$. Let $\mathsf{P}(0)$ denote an independent copy of P°, and for all $i \geq 1$ let $\mathsf{P}(i)$ denote an independent copy of P.

We let $\mathsf{P}_{[n]}$ denote the graph obtained from the core $\mathcal{W}(\mathsf{P}_n)$ by attaching for all $0 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))$ the graph $\mathsf{P}(i)$ instead of $\mathcal{P}_i(\mathsf{P}_n)$ to the ith vertex of $\mathcal{W}(\mathsf{P}_n)$. We may copy the proof of Lemma 9.2 almost word for word to show that quenched local convergence of P_n is equivalent to quenched local convergence of $\mathsf{P}_{[n]}$, and the limit graphs agree in that case.

It remains to verify quenched local convergence of $\mathsf{P}_{[n]}$. To this end, we select two vertices v_1 and v_2 of $\mathsf{P}_{[n]}$ uniformly and independently at random. We refer to v_1 as the *red vertex*, and v_2 as the *blue vertex*. The vertices of $\mathsf{P}_{[n]}$ correspond bijectively to the vertices of $(\mathcal{P}_i(\mathsf{P}_{[n]}))_{0 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))} = (\mathsf{P}(i))_{0 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))}$. (The edges of $\mathsf{P}_{[n]}$ correspond bijectively to the edges of $(\mathcal{P}_i(\mathsf{P}_{[n]}))_{0 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))}$ plus the edges of $\mathcal{W}(\mathsf{P}_n)$.) For all $0 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))$ we let $\bar{\mathcal{P}}_i(\mathsf{P}_{[n]})$ denote the vertex-rooted connected planar graph $\mathcal{P}_i(\mathsf{P}_{[n]})$ with the additional information if and where it contains a marked red or blue vertex. We let $\mathsf{P}_1^{\bullet}$ and $\mathsf{P}_2^{\bullet}$ denote independent copies of a Boltzmann-distributed doubly vertex-rooted planar graph. We colour the second root of $\mathsf{P}_1^{\bullet}$ red and the second root of $\mathsf{P}_2^{\bullet}$ blue. We let j_1 and j_2 denote a pair of uniformly selected distinct integers between 1 and $\mathrm{v}(\mathcal{W}(\mathsf{P}_n))$. For each $0 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))$ we set $\bar{\mathsf{P}}(i) = \mathsf{P}(i)$ if $i \neq j_1$ and $i \neq j_2$. If $i = j_k$ (for $k = 1$ or $k = 2$) we set $\bar{\mathsf{P}}(i) = \mathsf{P}_k^{\bullet}$. It follows by Corollary 3.5 that

$$(\bar{\mathcal{P}}_i(\mathsf{P}_{[n]}))_{1 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))} \overset{d}{\approx} (\bar{\mathsf{P}}(i))_{1 \leq i \leq \mathrm{v}(\mathcal{W}(\mathsf{P}_n))}. \tag{9.144}$$

We let $r \geq 0$ denote a fixed arbitrary integer. By Proposition 2.3 and the local convergence (9.143) the neighbourhoods $U_r(\mathsf{P}_{[n]}, v_1)$ and $U_r(\mathsf{P}_{[n]}, v_2)$ are with high probability disjoint.

For $k \in \{1, 2\}$ we let v_k' denote the vertex of $\mathcal{W}(\mathsf{P}_n)$ corresponding to the component containing v_k. If $d_{\mathsf{P}_{[n]}}(v_k, v_k') \geq r$ then $U_r(\mathsf{P}_{[n]}, v_k)$ is fully contained in the $\mathcal{P}$-component containing v_k. If the distance equals some $h < r$, then $U_r(\mathsf{P}_{[n]}, v_k)$ is glued together from the r-neighbourhood of v_k in that component (with additional information on the location of v_k' within that neighbourhood), the neighbourhood $U_{r-h}(\mathcal{W}(\mathsf{P}_n), v_k')$, and neighbourhoods in the $\mathcal{P}$-components corresponding to vertices from $U_{r-h-1}(\mathcal{W}(\mathsf{P}_n), v_k') \setminus \{v_k'\}$.

The asymptotic (9.144), the local convergence (9.143) and the observations made in the penultimate paragraph entail that asymptotically and jointly the components corresponding to v_1' and v_2' behave like independent copies of $\mathsf{P}^{\bullet}$, and the components corresponding to vertices from $U_r(\mathcal{W}(\mathsf{P}_n), v_1') \setminus \{v_1'\}$ and $U_r(\mathcal{W}(\mathsf{P}_n), v_2') \setminus \{v_2'\}$ behave like independent copies of P, and the neighbourhoods $U_r(\mathcal{W}(\mathsf{P}_n), v_1')$ and $U_r(\mathcal{W}(\mathsf{P}_n), v_1')$ behave like independent copies of the neighbourhood $U_r(\hat{\mathsf{B}})$. It follows that the pair of neighbourhoods $(U_r(\mathsf{P}_n, v_1), U_r(\mathsf{P}_n, v_2))$ converges in distribution to a pair of independent copies of a certain random rooted graph with radius r. As this is true for arbitrary r, Proposition 2.2 shows that there exists an infinite random rooted planar graph $\hat{\mathsf{P}}$ with

$$\mathfrak{L}((\mathsf{P}_n, v_n) \mid \mathsf{P}_n) \overset{p}{\to} \mathfrak{L}(\hat{\mathsf{P}}). \qquad \blacksquare$$

The number of edges $\mathrm{e}(\mathsf{P}_n)$ is known to satisfy a normal central limit theorem [32, Thm. 4.1]. Arguing as in the proof of [24, Thm. 2.1], we find that the convergence of

Theorem 1.1 also entails a local limit (following a *different* distribution) for P_n marked according to the stationary distribution. We close the section with the following remark on the small blocks in P_n.

Remark 9.15. The Gibbs partition (9.136) is convergent. This entails that the $\mathcal{W}$-core $\mathcal{W}(\mathsf{P}_n)$ consists of a giant 2-connected component and a remainder that asymptotically behaves like a Boltzmann-distributed $\mathcal{W}$-object, that is, a $\mathrm{Poisson}(\mathcal{B}(\rho_{\mathcal{B}}, 1))$ number of independent copies of a Boltzmann-distributed 2-connected $\mathcal{B}(x, 1)$-graph B. It follows that the collection $\mathrm{frag}(\mathsf{P}_n)$ of all blocks with non-maximal size satisfies

$$\mathrm{frag}(\mathsf{P}_n) \stackrel{d}{\approx} (\mathsf{B}(i))_{1 \le i \le N_n} \tag{9.145}$$

with $(\mathsf{B}(i))_{i \ge 1}$ denoting independent copies of B, and N an independent random integer with Poisson distribution

$$N_n \stackrel{d}{=} \mathrm{Poisson}(n\mathcal{B}(\rho_{\mathcal{B}}, 1)). \tag{9.146}$$

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