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Instabilities of invariant quasi-periodic tori

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Abstract. We prove the existence of real analytic Hamiltonians with topologically unstable quasi-periodic invariant tori. Using various versions of our examples, we solve the following problems in the stability theory of analytic quasi-periodic motion:

- (1) Show the existence of topologically unstable tori of arbitrary frequency. Moreover, the Birkhoff Normal Form at the invariant torus can be chosen to be convergent, equal to a planar or non-planar polynomial.
- (2) Show the optimality of the exponential stability for Diophantine tori.
- (3) Show the existence of real analytic Hamiltonians that are integrable on half of the phase space, and such that all orbits on the other half accumulate at infinity.
- (4) For sufficiently Liouville vectors, obtain invariant tori that are not accumulated by a positive measure set of quasi-periodic invariant tori.

Keywords. Hamiltonian systems, quasi-periodic invariant tori, stability, Birkhoff normal forms, Nekhoroshev theory, KAM theory

1. Introduction

Let H be a C^2 function defined on $\mathbb{T}^d \times \mathbb{R}^d$ and consider its Hamiltonian vector field $X_H(\theta, r) = (\partial_r H(\theta, r), -\partial_\theta H(\theta, r))$. If for some $\omega \in \mathbb{R}^d$, we have

$$H(\theta, r) = \langle \omega, r \rangle + \mathcal{O}(r^2), \quad (*)$$

then $\mathcal{T}_0 = \mathbb{T}^d \times \{0\}$ is invariant under the Hamiltonian flow Φ_H^t and the induced dynamics on this torus is the translation of frequency vector $\omega : \theta \mapsto \theta + t\omega$. Moreover, this torus is Lagrangian with respect to the canonical symplectic form $d\theta \wedge dr$ on $\mathbb{T}^d \times \mathbb{R}^d$.

In this work, we will mainly be interested in the non-resonant case, where the coordinates of ω are rationally independent, in which case the torus \mathcal{T}_0 can be seen as the closure of any orbit that starts on \mathcal{T}_0 . We call such an invariant torus a quasi-periodic torus of the Hamiltonian H , and for short, a QP torus.

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The study of the stability properties of a QP torus is an old problem of classical mechanics, especially in relation to the N-body problem of celestial mechanics. There exist three different notions of stability. The usual topological or Lyapunov stability, the stability in a measure theoretic or probabilistic sense (KAM stability), and the effective stability or quantitative stability in time.

In this paper, we will use variants of the approximation by conjugation method (AbC or Anosov–Katok method) to construct several examples with various instability properties of QP tori of a real analytic Hamiltonian, from all three points of view and in relation with the main known results and open questions in the field.

We show in particular the existence of real analytic Hamiltonians with topologically unstable quasi-periodic invariant tori with arbitrary frequencies. We also show sharpness of several results in Nekhoroshev and KAM theory.

In the AbC method, diffeomorphisms or flows of a manifold are constructed as limits of conjugates of diffeomorphisms or periodic flows. Volume preserving maps with various interesting, sometimes surprising, topological and ergodic properties can be obtained as limits of volume preserving periodic transformations that are conjugates via wild conjugacies to a simple periodic action on the manifold. For a general overview of the conjugation by approximation method we refer the reader to [12].

In the Hamiltonian setting, by periodic approximations we mean that the flow in some of the angle variables will be approached by periodic flows, which causes instabilities and drift in the action coordinates. More precisely, all our examples will be of the following form (see Section 5 for complete details):

$$H = \lim_{n \rightarrow \infty} H_n, \quad H_n(\theta, r) = \langle \omega(r_d), r \rangle - \sum_{j=2}^n \phi_j(r_d) \sin \left(2\pi \sum_{i=1}^{d-1} k_{j,i} \theta_i \right). \quad (1.1)$$

Here $\{k_j\}$ is a sequence of vectors in \mathbb{Z}^{d-1} , $\omega(\cdot)$ is a function from \mathbb{R} to \mathbb{R}^d , and the $\phi_j(\cdot)$ are functions from \mathbb{R} to \mathbb{R} that will be chosen adequately to guarantee the convergence of the sequence H_n and the various properties of our examples. Note that r_d is constant under the Hamiltonian flows we are considering and acts as a *parameter* in the constructions, in a way that we now explain. Note also that the flows are explicitly solvable, for r_d near the origin, and are conjugated to $H_0 = \langle \omega(r_d), r \rangle$. Namely, there exists an explicit canonical transformation Ψ_n such that $H_n = H_0 \circ \Psi_n$ (see Section 6.2 for the explicit form of Ψ_n). The wild behavior of the conjugacies Ψ_n can be caused by two possible scenarios:

- (i) The frequency vector $\omega(r_d) \equiv \tilde{\omega}$ is a constant Liouville vector (see Section 2 and (4.5) for the definitions) and the sequence $\{k_j\}$ is a sequence of almost resonant vectors with respect to the frequency $\tilde{\omega}$.
- (ii) As r_d goes to zero, the frequency vector $\omega(r_d)$ goes through resonances that correspond to the sequence $\{k_j\}$.

The AbC method was first applied in the Hamiltonian context by Katok in [22]. Recent applications of the AbC method in the Hamiltonian context that inspired this work can be found in [10, 16].

2. Notations

Let us introduce some notations that will be useful throughout the paper.

- For any vector $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ we will denote $\|v\| := \max_{1 \leq m \leq d} |v_m|$.
- We denote by \mathbb{T}_ρ^d the complex ρ -neighborhood of a standard real d dimensional torus

$$\mathbb{T}_\rho^d = \{z \in \mathbb{C}^d / \mathbb{Z}^d : |\operatorname{Im} z_i| < \rho, 1 \leq i \leq d\}.$$

We denote by $B_{\Delta, \rho}$ the complex ρ -neighborhood of the closed ball $B_\Delta \subset \mathbb{R}^d$ centered at the origin with radius $\Delta > 0$,

$$B_{\Delta, \rho} = \{z \in \mathbb{C}^d : \text{there exists } z' \in B_\Delta \text{ such that } |z - z'| < \rho\}.$$

We will also denote $D_{\Delta, \rho} = \mathbb{T}_\rho^d \times B_{\Delta, \rho}$.

- A holomorphic function f defined on $D_{\Delta, \rho}$ is said to be real if it gives real values to real arguments. We will denote by $C_{\Delta, \rho}^\omega$ the real and bounded holomorphic functions $f : D_{\Delta, \rho} \rightarrow \mathbb{C}$, which form a Banach space with the supremum norm

$$\|f\|_{\Delta, \rho} = \sup_{z \in D_{\Delta, \rho}} |f(z)|.$$

By $C_{0, \rho}^\omega$ we denote the subset of functions of $C_{\Delta, \rho}^\omega$ that depend only on θ . We will denote by C^ω the real holomorphic entire functions and $C_\rho^\omega := \bigcap_{\Delta > 0} C_{\Delta, \rho}^\omega$. Recall that with the compact-open topology both are a Fréchet spaces. In particular, we will use that convergence in $C_{\Delta, \rho}^\omega$ for all $\Delta, \rho > 0$ implies convergence in C^ω , and that convergence in $C_{\Delta, \rho}^\omega$ for all $\Delta > 0$ for a fixed $\rho > 0$ implies convergence in C_ρ^ω .

- *Formal power series.* Let $z = (z_1, \dots, z_d) \in \mathbb{C}^d$. An element

$$f \in \mathcal{C}^\omega(\mathbb{T}_\rho^d)[[z]]$$

is a formal power series

$$f = f(\theta, z) = \sum_{j \in \mathbb{N}^d} a_j(\theta) z^j$$

whose coefficients $a_j \in C_{0, \rho}^\omega$ (possibly vector valued).

- Given a vector $v = (v_1, \dots, v_d) \in \mathbb{R}^d$, denote by $\tilde{v} := (v_1, \dots, v_{d-1}) \in \mathbb{R}^{d-1}$ the new vector obtained by omitting the last component. Similarly for a map $f : \mathbb{R} \rightarrow \mathbb{R}^d$ we will denote by $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}^{d-1}$ the corresponding map where the last component is omitted.
- We will usually denote the last component of $r = (r_1, \dots, r_d) \in \mathbb{R}^d$ by $s := r_d$ to distinguish it from the rest of the components. We do so to stress the fact that in our constructive methods s plays the role of a parameter, it does not change with time. This happens because all the Hamiltonians we consider will not depend on θ_d , and thus satisfy $\dot{s} = -\frac{\partial H}{\partial \theta_d} = 0$ (see Section 5).

- We call ω a Diophantine vector of exponent $\tau > 0$ and constant $\gamma > 0$ if

$$|\langle \omega, k \rangle| \geq \frac{\gamma}{\|k\|^\tau} \quad \text{for all } k \in \mathbb{Z}^d \setminus \{0\}.$$

We denote by $\Omega_{\gamma,\tau}^d$ the set of all such vectors. Recall that for any $\tau > d - 1$, the set of all Diophantine vectors of exponent τ : $\Omega_\tau^d := \bigcup_\gamma \Omega_{\gamma,\tau}^d$ has full Lebesgue measure. A non-resonant vector that is not Diophantine is called a Liouville vector.

3. A brief reminder on Birkhoff normal forms and KAM stability

3.1. Birkhoff normal forms

We say that H as in (*) has a normal form N_H if N_H is a formal power series in r (possibly with 0 radius of convergence) and there exists a formal power series

$$f \in \mathcal{C}^\omega(\mathbb{T}_\rho^d)[[r]] \cap \mathcal{O}(r^2)$$

such that

$$H(\theta, r + \partial_\theta f(\theta, r)) = N_H(r).$$

If a normal form exists at a QP torus (non-resonant by our definition), it is unique. It is then called the Birkhoff normal form of H at the QP torus (we refer to [3] or [32] for more details on Birkhoff normal forms). A classical result is that when H is as in (*) and ω is Diophantine, the normal form exists and is unique.

3.2. Non-degenerate Birkhoff Normal Forms and KAM stability

A QP torus of a Hamiltonian system is said to be KAM stable if it is accumulated by a positive measure of QP tori, and if the set of these tori has Lebesgue density one at the original torus. We say that a formal power series N_H is *non-degenerate* or *non-planar* if there does not exist any vector γ such that for every r in some neighborhood of \mathcal{T}_0 , $\langle \nabla N_H(r), \gamma \rangle = 0$. The following was proven in [10].

Theorem A. *If N_H exists, is unique and is non-degenerate, then \mathcal{T}_0 is KAM stable. In particular, this is the case if ω is Diophantine and if N_H is non-degenerate.*

The condition that N_H is non-degenerate is essentially equivalent to Rüssmann’s non-degeneracy condition that guarantees the survival of a QP torus of an integrable system under small perturbations (see [30, 35]). In [10], it was shown to be a sufficient condition for KAM stability in the singular perturbation problem that appears in the study of the stability of a QP torus.

3.3. On the convergence of the BNF

We know that a convergent symplectic coordinate change that yields the BNF exists if and only if H is integrable [19] (see also [34, 37]). It was known to Poincaré that for “typical”

(in a sense we would call today generic) H , f_H will be divergent. Siegel [31] proved the same thing in a neighborhood of an elliptic equilibrium with another, and stronger, notion of “typical”. However, this does not solve the question of the convergence of the BNF itself, that is always defined when ω is Diophantine. When the radius of convergence of the formal power series $N_H(\cdot)$ is 0, we say that the BNF diverges.

For example, the following questions were asked by Eliasson [8–10]:

- (i) Can N_H be divergent?
- (ii) If H is non-integrable, can N_H be convergent?

A result of Pérez-Marco [27] states, for any fixed vector ω , that if N_H is divergent for some H as in (*), then N_H is divergent for “typical” (i.e. except for a pluri-polar set) H .

In [11], it was shown that for any $\omega \in \mathbb{R}^d$, $d \geq 4$ such that $\omega_1\omega_2 < 0$ there exists a real entire Hamiltonian $H : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ such that the origin is an elliptic equilibrium with frequency ω and such that the BNF of H at the origin is divergent. This construction can readily be extended to the case of QP tori as in (*). It follows from [27] that for any Diophantine $\omega \in \mathbb{R}^d$, $d \geq 4$, the BNF at a QP torus of frequency ω is generically divergent.

A contrario, one of the results that will be obtained here is an answer to (ii) with an example of a real entire Hamiltonian as in (*), with arbitrary non-resonant frequency $\omega \in \mathbb{R}^3$, such that the BNF at \mathcal{T}_0 exists and is convergent but \mathcal{T}_0 is Lyapunov unstable and thus H is non-integrable.

Extending this result to elliptic fixed points is unfortunately not readily available because the action angle coordinates are singular at the origin, and the extension of real analytic unstable constructions in this direction (from tori to points) is a challenging problem. For instance, it is not known how to adapt the Approximation by Conjugations construction method on the disc to the real analytic category (see [14] for a discussion on this topic).

4. Statement of the main results

4.1. Lyapunov stability

A closed invariant set of an autonomous Hamiltonian flow is said to be Lyapunov stable or topologically stable if all nearby orbits remain close to it for all forward time. Douady gave in [7] examples of smooth Hamiltonians having a Lyapunov unstable QP torus. Douady’s examples can have any chosen Birkhoff Normal Form at the origin provided its Hessian at the fixed point is non-degenerate. Douady’s examples are modelled on the Arnold diffusion mechanism through chains of heteroclinic intersections between lower dimensional partially hyperbolic invariant tori that accumulate towards the origin. The construction consists of a countable number of compactly supported perturbations of a completely integrable flow, and as such was carried out only in the C^∞ category. Examples of smooth Hamiltonians having a Lyapunov unstable QP torus with a degenerate Birkhoff normal form were obtained in [10, 16].

Topological instability of a QP torus is conjectured to hold for generic systems in three or more degrees of freedom. In fact, it was conjectured by Arnol'd that a “general” Hamiltonian should have a dense orbit on a “general” energy surface [1]. A great amount of work has been dedicated to proving this conjecture (giving a precise meaning to the word “general”), but the picture is not yet completely clear, especially when it comes to real analytic Hamiltonians (see for example [2] and references therein). For instance, not a single example was known up to now of a real analytic Hamiltonian that has a Lyapunov unstable QP torus. It was shown in [11] that for any $\omega \in \mathbb{R}^d, d \geq 4$, such that not all its coordinates are of the same sign, there exists a real entire Hamiltonian such that the origin is a Lyapunov unstable elliptic equilibrium with frequency ω . As we discussed earlier, the construction of [11] can readily be extended to the case of QP tori and the condition on the sign of the coordinates of ω can be dropped. However, all the examples that one obtains following the method of [11] would have a divergent BNF.

The constructions in this work are essentially different and their BNF will be convergent. Furthermore, we can choose the Birkhoff normal form to be either

$$\hat{N}(r) := \langle \hat{\omega}(r_d), r \rangle \quad \text{with } \hat{\omega}(s) := (\omega_1 + s, \omega_2, \dots, \omega_d), \quad \text{or} \quad (4.1)$$

$$\bar{N}(r) := \langle \bar{\omega}(r_d), r \rangle \quad \text{with } \bar{\omega}(s) := (\omega_1 + s, \omega_2 + s^2, \dots, \omega_{d-1} + s^{d-1}, \omega_d). \quad (4.2)$$

For sufficiently Liouville ω we will have some constructions with $N_H = N$, where

$$N(r) := \langle \omega, r \rangle.$$

Theorem B. *For any $\omega \in \mathbb{R}^d, d \geq 3$, there exists a real entire Hamiltonian H as in (*) such that the QP torus \mathcal{T}_0 is Lyapunov unstable.*

Moreover, the BNF of H at \mathcal{T}_0 can be chosen to be $\hat{N}(\cdot)$ or $\bar{N}(\cdot)$. In the latter case, if ω is Diophantine, then \mathcal{T}_0 is KAM stable.

While constructing these examples of Lyapunov unstable QP tori, we clarify several questions regarding the stability of QP motion in the analytic context. Namely,

- (i) Lyapunov instability of \mathcal{T}_0 can be obtained for arbitrary frequencies $\omega \in \mathbb{R}^d$.
- (ii) The examples of Theorem B have a convergent BNF, thus answering positively the question of Eliasson mentioned in Section 3(ii) (see [8–10]). The same question in the case of elliptic fixed points is still open (see [11, 14] for a discussion of this problem).
- (iii) The BNF can be chosen to be a very simple polynomial as in (4.1). This shows that Rüssmann’s local integrability result for Diophantine QP tori [29], that holds true when the BNF is completely degenerate (equal to a function of $\langle \omega, r \rangle$), does not hold for a simple highly degenerate form as \hat{N} .
- (iv) The Birkhoff normal form \bar{N} is non-degenerate in the sense of Rüssmann. Hence, Theorem A proves in this case the coexistence of diffusion and KAM stability.

Remark 1. Note that Herman conjectured that for Diophantine frequencies \mathcal{T}_0 is accumulated by a positive measure of QP tori in the analytic category (see Section 4.4 below).

If the conjecture is true, then even the examples with BNF \hat{N} should also have coexistence of Lyapunov instability and \mathcal{T}_0 being accumulated by a positive measure of QP tori.

4.2. Effective stability

An important question in classical mechanics is to estimate the escape rate of orbits starting in small neighborhoods of invariant objects such as fixed points or invariant tori. In our context we introduce, for a given H as in (*) and \mathcal{T}_0 ,

$$T(r) := \inf_{\theta \in \mathbb{T}^d, |r'| \leq r} \left\{ t > 0 : \text{dist}(\Phi_H^t(\theta, r'), \mathcal{T}_0) = \frac{1}{r} \right\}. \quad (4.3)$$

If $T(r)$ exists for all $r > 0$ sufficiently small,¹ then we say that \mathcal{T}_0 is *diffusive*. Based on the Diophantine exponent τ , exponential lower bounds for $T(r)$ can be derived from estimates on the remainder terms in the BNF reductions. It follows from [20, 25, 28] that for H as in (*), $\omega \in \Omega_{\gamma, \tau}^d$, there exist positive constants C, R such that for $r < R$,

$$T(r) \geq r^{-1} \exp(Cr^{-(\tau+1)^{-1}}). \quad (4.4)$$

One aim of this paper is to prove the optimality of the exponent in this bound for a certain class of Diophantine frequencies (see Corollary A). Many results on optimal Arnold diffusion times in smooth, Gevrey and analytic context exist in the literature, that sometimes relate the speed of diffusion to arithmetic conditions. We refer to [15, 21, 23, 24, 36] and references therein.

Remark 2. The usual definition of T in (4.3) requires diffusion up to distance $2r$ instead of r^{-1} , and this is the original context in which (4.4) was proved. In our case, the definition of diffusiveness with r^{-1} instead of $2r$ does not change the order of magnitude of the diffusion time and has the advantage of implying Lyapunov instability of a diffusive torus \mathcal{T}_0 .

Our main result on diffusion time is stated for vectors $\omega = (\tilde{\omega}, \omega_d)$, where $d \geq 3$ and $\tilde{\omega} \in \mathbb{R}^{d-1}$ does not belong to some Diophantine class.

Theorem C. For any $\tau > 0, C > 0$ and any $\omega = (\tilde{\omega}, \omega_d) \in \mathbb{R}^d, d \geq 3$, with $\tilde{\omega} \notin \Omega_{\tau}^{d-1}$, there is a real analytic Hamiltonian H as in equation (*) such that \mathcal{T}_0 is diffusive and $T(r_n) \leq \exp(Cr_n^{-(\tau+1)^{-1}})$ for a sequence $r_n \rightarrow 0$. Moreover, the BNF at \mathcal{T}_0 is given by $\hat{N}(\cdot)$.

The BNF of the Hamiltonians that we construct in Theorem C must be very special. Indeed, it was proven in [5, 26] that a QP torus with Diophantine frequency is generically and prevalently doubly exponentially stable. More precisely, it was shown that a point that starts at distance r from the torus remains within distance $2r$ close to it for an interval

¹We apologize for the double use of the notation r as a scalar in definition (4.3) and previously as a variable in \mathbb{R}^d .

of time which is larger than $\exp(\exp(Cr^{-(\tau+1)^{-1}}))$. The proof of double exponential stability is based on a combination of the estimates on the BNF and Nekhoroshev stability theory. It is worth mentioning that analogous results have been proved in the context of elliptic fixed points as well, both for exponential and double exponential stability (see [17] and [6]). To show how Theorem C allows to approach the known lower bound on the diffusion speed $T(r)$ for some Diophantine vectors, we will need the following simple arithmetic lemma.

Lemma 1. *For any $\tau > d - 1$ and $\tilde{\omega} \in \Omega_\tau^{d-1}$ a.e. $\omega_d \in \mathbb{R}$ satisfies $\omega := (\tilde{\omega}, \omega_d) \in \Omega_\tau^d$.*

Hence, if we pick $\tilde{\omega} \in \Omega_\tau^{d-1} \setminus \Omega_{\tau-\varepsilon}^{d-1}$, it is possible to “extend” it into $\omega = (\tilde{\omega}, \omega_d)$ for some ω_d such that $\omega \in \Omega_\tau^d$.

Corollary A. *For any $\tau > d - 1, \varepsilon > 0$ there is a real analytic Hamiltonian H as in (*) with $\omega \in \Omega_\tau^d$ such that \mathcal{T}_0 is diffusive and $T(r_n) \leq \exp(Cr_n^{-(\tau+1-\varepsilon)^{-1}})$ for a sequence $r_n \rightarrow 0$.*

Then due to Lemma 1 the proof of Corollary A becomes a direct application of Theorem C. The proof of Lemma 1 is elementary. We sketch it for completeness.

Proof of Lemma 1. Let I be an arbitrary bounded interval in \mathbb{R} . We denote by $D_{\tilde{\omega}, \tau, \gamma}$ the set of $\omega_d \in I$ satisfying $\omega = (\tilde{\omega}, \omega_d) \in \Omega_{\tau, \gamma}^d$. For any $k = (k_1, \dots, k_d) \in \mathbb{Z}^d, k \neq 0$, consider the set

$$A_{\tau, \gamma, k}^{\tilde{\omega}} = \left\{ \omega_d \in I : |\langle k, \omega \rangle| < \frac{\gamma}{\|k\|^\tau} \right\}.$$

Since $\tilde{\omega} \in \Omega_\tau^{d-1}$, we have

$$I \setminus D_{\tilde{\omega}, \tau, \gamma} \subset \bigcup_{k \in \mathbb{Z}^d, k_d \neq 0} A_{\tau, \gamma, k}^{\tilde{\omega}}.$$

Hence, for some constant $C_d > 0$ we get

$$\begin{aligned} \mu(I \setminus D_{\tilde{\omega}, \tau, \gamma}) &\leq \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{0 < |k_d| < \|\tilde{k}\|} \mu(A_{\tau, \gamma, k}^{\tilde{\omega}}) + \sum_{|k_d| > 0} \sum_{\|\tilde{k}\| \leq |k_d|} \mu(A_{\tau, \gamma, k}^{\tilde{\omega}}) \\ &\leq 2\gamma \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{0 < |k_d| < \|\tilde{k}\|} \frac{1}{|k_d| \|\tilde{k}\|^\tau} + 2\gamma \sum_{|k_d| > 0} \sum_{\|\tilde{k}\| \leq |k_d|} \frac{1}{|k_d|^{\tau+1}} \\ &\leq C_d \gamma \sum_{\tilde{k} \in \mathbb{Z}^{d-1} \setminus \{0\}} \frac{\ln \|\tilde{k}\|}{\|\tilde{k}\|^\tau} + C_d \gamma \sum_{|k_d| > 0} \frac{|k_d|^{d-1}}{|k_d|^{\tau+1}} = \mathcal{O}(\gamma). \end{aligned}$$

Therefore $\mu(I \setminus \bigcup_{\gamma > 0} D_{\tilde{\omega}, \tau, \gamma}) = 0$. ■

Liouville frequencies. For elliptic fixed points with non-resonant frequencies of smooth Hamiltonians, the existence of the BNF up to arbitrary order implies that the diffusion time from small r -neighborhoods of the origin cannot be faster than arbitrarily high powers in r^{-1} . For sufficiently Liouville frequencies, finite order BNFs may not be well defined at an invariant torus, even for real analytic Hamiltonians. In this case, diffusion

time may be much faster than in the case of elliptic equilibria. We will work with non-resonant frequencies $\omega = (\tilde{\omega}, \omega^d) \in \mathbb{R}^d$, $d \geq 3$, where $\tilde{\omega} \in \mathbb{R}^{d-1}$ is such that there is a sequence $\{\bar{k}_j\} \subset \mathbb{Z}^{d-1}$ satisfying

$$\lim_{j \rightarrow \infty} \frac{\ln|\langle \tilde{\omega}, \bar{k}_j \rangle|}{\|\bar{k}_j\|} = -\infty. \tag{4.5}$$

Theorem D. For any $\omega \in \mathbb{R}^d$ satisfying (4.5):

- (a) There exists a real entire Hamiltonian H as in (*) with the BNF of H at \mathcal{T}_0 given by $N(\cdot) = \langle \omega, \cdot \rangle$, and such that \mathcal{T}_0 is diffusive with $T(r_n) \leq r_n^{-n}$ for a sequence $r_n \rightarrow 0$.
- (b) There exists a real entire Hamiltonian H as in (*) such that \mathcal{T}_0 is diffusive and $T(r_n) \leq r_n^{-4}$ for a sequence $r_n \rightarrow 0$.

Remark 3. Notice that the difference between the two results in Theorem D is that having a faster diffusion in (b) comes with the price of not having a well defined BNF as we do have in (a). When, in Theorem 3, we will state the explicit constructions for both (a) and (b), we will explain in Remark 6 the reason behind the slowing down of the diffusion in (a).

Remark 4. It is easy to see from our proof that if we just ask to diffuse from an initial condition $\|z_n\| = r_n$ to n and not r_n^{-1} , then it is possible to replace the upper bound r_n^{-4} of case (b) by $r_n^{-2-\varepsilon}$, with $\varepsilon > 0$ arbitrarily small. Moreover, if we assume stronger Liouville conditions on $\tilde{\omega}$, we can even get diffusion times that are even closer to r_n^{-2} , which is clearly a lower bound for diffusion times for H as in (*).

4.3. Coexistence of diffusion and integrability

A natural question in Hamiltonian dynamics is whether a real analytic Hamiltonian system can be integrable on an open set of the phase space and not completely integrable.

One aim of this paper is to show that such examples do exist. We actually construct real analytic Hamiltonians that are analytically integrable on half of the phase space while all orbits on the other side accumulate at infinity. We will work with non-resonant frequencies satisfying (4.5). The main result is the following.

Theorem E. For any $\omega \in \mathbb{R}^d$ satisfying (4.5) there exists a real entire Hamiltonian H as in (*) such that:

- (i) There exists a real analytic symplectic diffeomorphism

$$\Psi : M^- = \mathbb{T}^d \times \mathbb{R}^{d-1} \times (-\infty, 0) \rightarrow M^- = \mathbb{T}^d \times \mathbb{R}^{d-1} \times (-\infty, 0)$$

such that on M^- we have $H \circ \Psi = H_0 := \langle \omega, r \rangle$.

- (ii) For any $(\theta, r) \in \mathbb{T}^d \times \mathbb{R}^{d-1} \times (0, \infty)$, we have

$$\limsup_{t \rightarrow \infty} |\Phi_H^t(\theta, r)| = \infty.$$

The BNF of H at \mathcal{T}_0 is given by $N(\cdot) = \langle \omega, \cdot \rangle$.

The question of coexistence of integrability and diffusion for analytic systems remains completely open if integrability is required to be non-degenerate (twist integrability for example). With a similar construction to that of Theorem E, we can obtain the following examples.

Theorem F. *For any $\omega \in \mathbb{R}^d$ satisfying (4.5), for any $l \in \mathbb{N}^*$, there exists a real entire Hamiltonian H as in (*) and a symplectic diffeomorphism Ψ on $\mathbb{T}^d \times \mathbb{R}^d$, that is of class C^l but not of class C^{l+1} , such that $H \circ \Psi = H_0 := \langle \omega, r \rangle$. The BNF of H at \mathcal{T}_0 is given by $N(\cdot) = \langle \omega, \cdot \rangle$.*

Observe that the main ingredient in the proof of Theorem F (see below the statement and proof of Theorem 5 that gives the explicit construction for Theorem F) is a fine tuning of the effect of the almost resonances of ω on the instabilities of a Hamiltonian as in (1.1). This fine tuning has the effect of maintaining linearizability in class C^l but destroying it in class C^{l+1} . An analogy can be seen with Sternberg’s linearization theorem near a hyperbolic fixed point that gives C^l regularity of the linearization provided a sufficient number, related to l , of non-resonance conditions hold [33].

4.4. KAM stability

It was conjectured by Herman (see [18]) that, without any non-degeneracy condition, a Diophantine KAM torus of an analytic Hamiltonian is accumulated by a set of positive measure of KAM tori. Herman’s conjecture is known to be true in two degrees of freedom [29], but remains open in general, with some progress being made in [10], where it is shown that an analytic invariant torus $\tilde{\mathcal{T}}_0$ with Diophantine frequency ω is never isolated from other KAM tori.

Herman’s conjecture on KAM stability of a Diophantine equilibrium or QP torus is known to be true in the smooth category for $d = 2$ due to Herman’s last geometric theorem (see [13]). Counter-examples to the conjecture in C^∞ and with arbitrary frequencies were built in [10] for $d \geq 4$, and later in [16] for $d = 3$.

One aim of this work is to build, starting from three degrees of freedom and for sufficiently Liouville frequencies ω , real analytic Hamiltonians that have QP tori with frequency ω that are not accumulated by a set of positive measure of KAM tori. This shows that some arithmetic condition in Herman’s conjecture is indeed necessary.

Theorem G. *For any $\omega \in \mathbb{R}^d$, $d \geq 3$, satisfying condition (4.5) there exists a real entire Hamiltonian H as in (*) such that for any $(\theta, r) \in \mathbb{T}^d \times \mathbb{R}^d$ with $r_d \neq 0$,*

$$\limsup_{t \rightarrow \infty} |\Phi_H^t(\theta, r)| = \infty.$$

The BNF of H at \mathcal{T}_0 is given by $N(\cdot) = \langle \omega, \cdot \rangle$.

Note that Bounemoura proved in [4] that an invariant quasi-periodic torus is KAM-stable under the hypothesis that the Hamiltonian is sufficiently smooth and has a non-degenerate Hessian matrix of its BNF of degree 2 (that part of the BNF is defined for all non-resonant frequencies). In our example, the entire BNF can be defined and is in

fact equal to $\langle \omega, r \rangle$. Theorem G thus shows that Rüssmann’s local integrability result of Diophantine QP tori with a degenerate BNF cannot be generalized to the case of sufficiently Liouville vectors.

Remark 5. In our construction \mathcal{T}_0 is not isolated, the hyperplane $r_d = 0$ is foliated by invariant tori with frequency ω . In [10] it was proved that Diophantine analytic QP tori are always accumulated by other QP tori. The question of the existence of Liouville QP tori that are completely isolated is still open, even for smooth Hamiltonians.

5. Constructions

Given $\omega \in \mathbb{R}^d$, all our examples will have the form

$$H = \lim_{n \rightarrow \infty} H_n, \quad H_n(\theta, r) = \langle \omega(s), r \rangle - \sum_{j=2}^n \phi_j(s) \sin(2\pi \langle k_j, \tilde{\theta} \rangle). \quad (5.1)$$

We can now give in Theorems 1–6 the specific forms of the Hamiltonians that will satisfy Theorems B–G. Theorem B can be rewritten as follows.

Theorem 1. *Let $\omega \in \mathbb{R}^d$, where $d \geq 3$. Choosing $\omega(\cdot)$ to be $\hat{\omega}(\cdot)$ (or $\bar{\omega}(\cdot)$), there exists a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that the Hamiltonian in (5.1) with $\phi_j(s) = s^j e^{-j \|k_j\|}$ satisfies the first (or second) conclusion of Theorem B.*

Although Theorem 1 holds for all frequencies, its proof depends on whether the frequency is resonant or not and also on the form of $\omega(\cdot)$. Different sequences must be constructed in the proof for the different cases.

Consider next $\omega = (\tilde{\omega}, \omega^d)$ with $\tilde{\omega} \notin \Omega_\tau^{d-1}$, $d \geq 3$. Then up to a permutation of indices for ω Theorem C can be without loss of generality restated as follows.

Theorem 2. *For any constants $C, \tau > 0$, there exists a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that for $\phi_j(s) = s^j e^{-\frac{C}{2} \|k_j\|}$ and $\omega(\cdot) = \hat{\omega}(\cdot)$ the Hamiltonian in (5.1) belongs to C_ρ^ω , where $\rho = \frac{C}{8\pi d}$, and satisfies the conclusion of Theorem C.*

We pass now to the purely Liouville constructions of Theorems D–G.

Theorem 3. *For any $\omega \in \mathbb{R}^d$, $d \geq 3$, satisfying condition (4.5), there exists a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that:*

- (a) *If $\phi_j(s) = s^j e^{-j \|k_j\|}$ and $\omega(\cdot) \equiv \omega$, then the Hamiltonian in (5.1) satisfies (a) of Theorem D.*
- (b) *If $\phi_j(s) = s^2 e^{-j \|k_j\|}$ and $\omega(\cdot) \equiv \omega$, then the Hamiltonian in (5.1) satisfies (b) of Theorem D.*

Remark 6. We will see in Section 6.2 why taking powers s^j in $\phi_j(s)$ as in (a) is required to guarantee that the BNF of H at \mathcal{T}_0 is given by $N(\cdot) = \langle \omega, \cdot \rangle$. Of course, this has the inconvenient of slowing down the diffusion compared to the definition of $\phi_j(s)$ with an s^2 as in (b).

Theorem 4. For any $\omega \in \mathbb{R}^d$, $d \geq 3$, satisfying (4.5), there exists a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that if $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j e^{\|k_j\|s}$ and $\omega(\cdot) \equiv \omega$, then the Hamiltonian in (5.1) satisfies the conclusion of Theorem E.

We also have:

Theorem 5. For any $\omega \in \mathbb{R}^d$, $d \geq 3$, satisfying (4.5), there exists a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that if $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j \|k_j\|^{-l-1} j^{-2}$ and $\omega(\cdot) \equiv \omega$, then the Hamiltonian in (5.1) satisfies the conclusion of Theorem F.

A simple modification of the construction in Theorem 4 gives a real entire Hamiltonian with a QP torus of Liouville frequency that is not accumulated by a positive measure set of KAM tori.

Theorem 6. For any $\omega \in \mathbb{R}^d$, $d \geq 3$, satisfying (4.5), there exists a sequence $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that if $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j e^{\|k_j\|s^2}$ and $\omega(\cdot) \equiv \omega$, then the Hamiltonian in (5.1) satisfies the conclusion of Theorem G.

6. Proofs

For convenience of the presentation we summarize the choices made in the various constructions of Theorems 1–6. Recall that H_n are constructed as in (5.1), with $\{k_j\} \subset \mathbb{Z}^{d-1}$ a strictly increasing sequence and the following possibilities for ϕ_j :

- (i) $\omega(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^d$ is either $\hat{\omega}$ or $\bar{\omega}$ and $\phi_j(s) = s^j e^{-j\|k_j\|}$,
- (ii) $\omega(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^d$ is $\hat{\omega}$ and $\phi_j(s) = s^j e^{-\frac{C}{2}\|k_j\|}$ for some $C > 0$,
- (iii) $\omega(\cdot) \equiv \omega$ and $\phi_j(s) = s^j e^{-j\|k_j\|}$,
- (iv) $\omega(\cdot) \equiv \omega$ and $\phi_j(s) = s^2 e^{-j\|k_j\|}$,
- (v) $\omega(\cdot) \equiv \omega$ and $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j e^{\|k_j\|s}$,
- (vi) $\omega(\cdot) \equiv \omega$ and $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j \|k_j\|^{-l-1} j^{-2}$,
- (vii) $\omega(\cdot) \equiv \omega$ and $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j e^{\|k_j\|s^2}$.

Let us now explain how the sequences $\{k_j\}$ will be chosen in the different cases. For cases (iii)–(vii), $\{k_j\}$ will be a fast growing subsequence of the sequence $\{\bar{k}_j\}$ satisfying (4.5). For cases (i) and (ii) we will use the following elementary fact.

Lemma 2. For any $\omega \in \mathbb{R}^d$, $d \geq 3$, assume $\omega(\cdot)$ satisfies either (4.1) or (4.2). There exists a sequence $\{s_j\} \subset \mathbb{R}$ and an increasing sequence in norm $\{k_j\} \subset \mathbb{Z}^{d-1}$ such that

- (a) $\lim_{j \rightarrow \infty} |s_j| = 0$,
- (b) $\lim_{j \rightarrow \infty} \|k_j\| = \infty$,
- (c) $\langle \tilde{\omega}(s_j), k_j \rangle = 0$.

In case $\tilde{\omega} \notin \Omega_\tau^{d-1}$ and if $\omega(\cdot)$ satisfies (4.1), we can assume without loss of generality that

$$\|k_j\| < |s_j|^{-(\tau+1)^{-1}}. \tag{6.1}$$

Proof. Let us denote $\omega' := (\omega_1, \omega_2)$ (we only consider the two first components of ω). We will divide the proof according to whether ω' is resonant or non-resonant. We will only treat the case where $\omega(\cdot)$ is as in (4.2), the case (4.1) being similar albeit easier.

- (a) Assume first that ω is such that ω' is non-resonant, $\omega(\cdot)$ as in (4.2). By Dirichlet's Theorem there exist a constant $C > 0$ and an increasing sequence in norm $\{k'_i\} \subset \mathbb{Z}^2$, $k'_i = (k_{i,1}, k_{i,2})$, such that

$$|\langle \omega', k'_i \rangle| < \frac{C}{\|k'_i\|}.$$

Consider $k_i := (k_{i,1}, k_{i,2}, 0, \dots, 0)$. Now $\langle \tilde{\omega}(s_i), k_i \rangle = 0$ is equivalent to

$$k_{i,2}s_i^2 + k_{i,1}s_i + \langle \omega', k'_i \rangle = 0, \tag{6.2}$$

which is easily seen to have a solution $s_i \rightarrow 0$ as required.

- (b) Assume now that ω is such that ω' is resonant, $\omega(\cdot)$ as in equation (4.2). There exists $m = (m_1, m_2)$ such that $\langle m, \omega' \rangle = 0$. Then for an increasing sequence $\{a_i\} \subset \mathbb{N}$ we define

$$k_i := (a_i m_1 + 1, a_i m_2, 0, \dots, 0) \in \mathbb{Z}^{d-1}.$$

The equation $\langle \tilde{\omega}(s_i), k_i \rangle = 0$ is then equivalent to

$$s_i k_{i,1} + s_i^2 k_{i,2} = -\langle \tilde{\omega}, k_i \rangle = -\omega_1,$$

which clearly has a solution $s_i \rightarrow 0$ as required.

In the non-resonant case, and $\omega(\cdot)$ as in (4.1), equation (6.2) becomes

$$k_{i,1}s_i + \langle \omega', k'_i \rangle = 0,$$

solved by $s_i := -\langle \tilde{\omega}, k_i \rangle / k_{i,1}$. If $\tilde{\omega} \notin \Omega_\tau^{d-1}$, we can assume without loss of generality that $|\langle \tilde{\omega}, k_i \rangle| < \|k_i\|^{-\tau}$ and $|k_{i,1}| = \|k_i\|$. Thus (6.1) holds. ■

6.1. Convergence

The following settles the convergence question in Theorems 1–6.

Proposition 1. *In cases (i), (iii)–(vii) the convergence $H_n \rightarrow H$ holds in the C_ρ^ω topology for any $\rho > 0$, hence the limit H is real entire. In case (ii), the convergence holds in $C_{\tilde{\rho}}^\omega$ for $\tilde{\rho} = \frac{C}{8\pi d}$, hence the limit $H \in C_{\tilde{\rho}}^\omega$.*

Proof. Cases (i)–(iv): We treat case (i), the other cases being similar. According to (5.1), we have that for any $\Delta, \rho > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$

$$\begin{aligned} \|H_m - H_n\|_{\Delta, \rho} &\leq \sum_{j=n+1}^m (\Delta + \rho)^j e^{-j\|k_j\|} \|\sin(2\pi \langle k_j, \tilde{\theta} \rangle)\|_\rho \\ &< \sum_{j=N}^\infty (\Delta + \rho)^j e^{-\|k_j\|(j-2\pi d\rho)}. \end{aligned}$$

Therefore $\{H_n\}$ is a Cauchy sequence in $C_{\Delta, \rho}^\omega$. Since $\Delta, \rho > 0$ are arbitrary, the limit H is a real entire function.

Cases (v)–(vii): We treat case (v), the other cases being similar. From condition (4.5), there exists a sequence $u_j \rightarrow \infty$ such that

$$\ln|\langle \tilde{\omega}, k_j \rangle| \leq -u_j \|k_j\|.$$

For any $\Delta, \rho > 0$, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$\|H_m - H_n\|_{\Delta, \rho} \leq \sum_{j=n+1}^m (\Delta + \rho)^j e^{\|k_j\|(\Delta + (2\pi d + 1)\rho - u_j)} < \varepsilon.$$

Therefore $\{H_n\}$ is a Cauchy sequence in $C_{\Delta, \rho}^\omega$. Since $\Delta, \rho > 0$ are arbitrary, the limit H is a real entire function. ■

6.2. Birkhoff normal forms

Proposition 2. *In Theorems 1–6, and except for Theorem 3 (b), the BNF at \mathcal{T}_0 is defined and equals $\langle \omega(r_d), r \rangle$.*

Proof. Define Ψ_n to be the canonical transformations obtained via the generating functions

$$S_n(\Theta, r) = \langle \Theta, r \rangle - \frac{1}{2\pi} \sum_{j=2}^n \langle \tilde{\omega}(s), k_j \rangle^{-1} \phi_j(s) \cos(2\pi \langle k_j, \tilde{\Theta} \rangle), \tag{6.3}$$

which is a real analytic function near the origin. More explicitly, for all $n \in \mathbb{N}$ we obtain the change of variables $(\Theta_n, R_n) = \Psi_n(\theta, r)$ given by the equations

$$\begin{aligned} \tilde{R}_n &= \frac{\partial S_n(\Theta_n, r)}{\partial \tilde{\Theta}_n} = \tilde{r} + \sum_{j=2}^n k_j \langle \tilde{\omega}(s), k_j \rangle^{-1} \phi_j(s) \sin(2\pi \langle k_j, \tilde{\Theta}_n \rangle), \\ R_{d,n} &= \frac{\partial S_n(\Theta_n, r)}{\partial \Theta_{d,n}} = s, \\ \tilde{\theta} &= \frac{\partial S_n(\Theta_n, r)}{\partial \tilde{r}} = \tilde{\Theta}_n, \\ \theta_d &= \frac{\partial S_n(\Theta_n, r)}{\partial s} = \Theta_{d,n} - \frac{1}{2\pi} \sum_{j=2}^n \partial_s (\langle \tilde{\omega}(s), k_j \rangle^{-1} \phi_j(s) \cos(2\pi \langle k_j, \tilde{\Theta}_n \rangle)). \end{aligned}$$

Thus

$$H_n = H_0 \circ \Psi_n,$$

where $H_0 = \langle \omega(r_d), r \rangle$. In fact, we can define in a formal way

$$S_\infty(\Theta, r) = \langle \Theta, r \rangle - \frac{1}{2\pi} \sum_{j=2}^\infty \langle \tilde{\omega}(s), k_j \rangle^{-1} \phi_j(s) \cos(2\pi \langle k_j, \tilde{\Theta} \rangle),$$

which formally conjugates the limit Hamiltonian H to H_0 . We only need to verify that

$$f = \frac{1}{2\pi} \sum_{j=2}^\infty \langle \tilde{\omega}(s), k_j \rangle^{-1} \phi_j(s) \cos(2\pi \langle k_j, \tilde{\Theta} \rangle) \in \mathcal{C}^\omega(\mathbb{T}_\rho^d)[[r]] \cap \mathcal{O}(r^2).$$

When $\phi_j(s) = c_j s^j$ as in Theorems 1–2, and $\omega(\cdot) = \hat{\omega}(\cdot)$, the coefficient of s^p in the power series of f is the trigonometric polynomial² given by

$$\frac{1}{2\pi} \sum_{\substack{l \geq 0, j \geq 2 \\ l+j=p}} \frac{c_j}{\langle k_j, \tilde{\omega} \rangle} \left(-\frac{k_{j,1}}{\langle k_j, \tilde{\omega} \rangle} \right)^l \cos(2\pi \langle k_j, \tilde{\Theta} \rangle).$$

In the other situations, for example when $\omega(\cdot) \equiv \omega$ and $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j e^{\|k_j\|s}$ as in Theorem 4, then the coefficient of s^p in the formal power series of $S_\infty(\Theta, r)$ is the trigonometric polynomial given by

$$\frac{1}{2\pi} \sum_{\substack{l \geq 0, j \geq 2 \\ l+j=p}} \|k_j\|^l \cos(2\pi \langle k_j, \tilde{\Theta} \rangle).$$

The other cases are similar. ■

We now consider the cases of Theorems 4 and 5 where the conjugacies to the degenerate BNF $\langle \omega, r \rangle$ do converge.

Proposition 3. *In case (v), the map $\Psi = \lim \Psi_n$ with Ψ_n as in (6.3) is well defined on $M^- = \mathbb{T}^d \times \mathbb{R}^{d-1} \times \mathbb{R}^-$ and is a real analytic symplectic diffeomorphism from M^- to itself.*

In case (vi), the map $\Psi = \lim \Psi_n$ with Ψ_n as in (6.3) is well defined on $M = \mathbb{T}^d \times \mathbb{R}^d$ and is a diffeomorphism from M to itself that is of class C^l but not of class C^{l+1} .

Proof. We start with case (v). From the definition (6.3) we have that Ψ_n is generated by

$$S_n(\Theta, r) = \langle \Theta, r \rangle - \frac{1}{2\pi} \sum_{2 \leq j \leq n} s^j e^{\|k_j\|s} \cos(2\pi \langle k_j, \tilde{\Theta} \rangle),$$

that preserves for every $\rho > 0$ the domain $M_\rho^- = \mathbb{T}^d \times \mathbb{R}^{d-1} \times (-\infty, -\rho)$. Moreover, S_n converges in $C_{\rho/(10d)}^\omega$ on M_ρ^- . Hence Ψ_n defines a real analytic symplectic diffeomorphism on every M_ρ^- , $\rho > 0$ (we assume k_2 is sufficiently large and $\{k_j\}$ is fast growing).

We treat now case (vi). In this case Ψ_n is generated by

$$S_n(\Theta, r) = \langle \Theta, r \rangle - \frac{1}{2\pi} \sum_{2 \leq j \leq n} s^j \|k_j\|^{-l-1} j^{-2} \cos(2\pi \langle k_j, \tilde{\Theta} \rangle),$$

and it is clear that the limit $\Psi = \lim \Psi_n$ is a diffeomorphism of M of class C^l but not of class C^{l+1} . ■

Remark 7. In principle, it should be possible to use our constructions to obtain examples that are Lyapunov stable but not KAM stable. A possible approach would be to replace the choice of ϕ_j in (vi) by $\phi_j(s) = \langle \tilde{\omega}, k_j \rangle s^j b_j$, with $|b_j| \leq 1$ chosen such that the resulting

²This is exactly where the increasing powers s^j in the definition of ϕ_j play a decisive role in controlling the Birkhoff Normal Form at \mathcal{T}_0 .

Ψ_n forms a sequence of diffeomorphisms of M such that $|\pi_2(\Psi_n(\theta, r))| \leq 10|r|$ for all n while Ψ_n diverges in the C^0 topology in a way that guarantees the absence of invariant tori besides the ones at $s = 0$.

6.3. Fast approximations

Let us denote by $\Phi_n^t(\cdot)$ the flow of H_n . It is clear that by choosing $\{k_n\}$ to grow sufficiently fast, one can guarantee that the flow of H will be very close to the flow of H_n during very long times. Thus, it is convenient to give finite time versions of all the properties required in Theorems 1–6 that we start by checking for the flow $\Phi_n^t(\cdot)$. For fixed $C, \tau > 0$, let us define the following conditions:

(\mathcal{P}_n^1) There exists $z \in \mathbb{T}^d \times \mathbb{R}^d$ with $\|z\| \leq \frac{1}{n}$ and $t > 0$ such that

$$\|\Phi^t(z)\| > n.$$

(\mathcal{P}_n^2) There exists $z \in \mathbb{T}^d \times \mathbb{R}^d$ with $\|z\| \leq \frac{1}{n}$ and $t \leq \exp(C\|z\|^{-(\tau+1)})^{-1}$ satisfying

$$\|\Phi^t(z)\| > \|z\|^{-1}.$$

(\mathcal{P}_n^3) There exists $z \in \mathbb{T}^d \times \mathbb{R}^d$ with $r_n := \|z\| \leq \frac{1}{n}$ and $t \leq r_n^{-2n}$ satisfying

$$\|\Phi^t(z)\| > \|z\|^{-1}.$$

(\mathcal{P}_n^4) There exists $z \in \mathbb{T}^d \times \mathbb{R}^d$ with $r_n := \|z\| \leq \frac{1}{n}$ and $t \leq r_n^{-4}$ satisfying

$$\|\Phi^t(z)\| > \|z\|^{-1}.$$

(\mathcal{P}_n^5) For all $z \in Q_n^+$ there exists $t > 0$ such that

$$\|\Phi^t(z)\| > n,$$

where $Q_n^+ := \mathbb{T}^d \times [-n, n]^{d-1} \times [n^{-1}, n]$.

(\mathcal{P}_n^6) For all $z \in Q_n$ there exists $t > 0$ such that

$$\|\Phi^t(z)\| > n,$$

where $Q_n := \mathbb{T}^d \times [-n, n]^{d-1} \times ([-n, -n^{-1}] \cup [n^{-1}, n])$.

We will write the previous conditions with $n = \infty$ to indicate that they hold for all n sufficiently large. All the proofs of Theorems 1–6 rely on the following lemma.

Proposition 4. *For any $i \in \{1, 2, 3, 4, 5, 6\}$, if k_2, \dots, k_n are chosen and condition (\mathcal{P}_n^i) is satisfied by the flow of H_n , then if k_{n+1} is chosen sufficiently large, the flow of H also satisfies condition (\mathcal{P}_n^i).*

Proof. It follows from the Grönwall inequalities that all conditions (\mathcal{P}_n^i) are open in the C^3 topology on the Hamiltonian. Hence, the lemma is a consequence of the fact that $\|H - H_n\|_{C^3} \rightarrow 0$ as $k_{n+1} \rightarrow \infty$. ■

6.4. Diffusion at finite scales

We now verify the diffusion properties (\mathcal{P}_n^i) for the flows $\Phi_n^t(\cdot)$ of H_n in the various cases.

Proposition 5. *There exists a sequence $\{k_n\} \subset \mathbb{Z}^{d-1}$ and $N > 0$ such that for all $n \geq N$, the following holds:*

- In case (i), Φ_n satisfies (\mathcal{P}_n^1) .
- In case (ii), Φ_n satisfies (\mathcal{P}_n^2) .
- In case (iii), Φ_n satisfies (\mathcal{P}_n^3) .
- In case (iv), Φ_n satisfies (\mathcal{P}_n^4) .
- In case (v), Φ_n satisfies (\mathcal{P}_n^5) .
- In case (vii), Φ_n satisfies (\mathcal{P}_n^6) .

Proof. We start with case (i). Consider the initial condition $z = (\theta, r)$ with

$$\theta = (0, \dots, 0, 0), \quad r = (0, \dots, 0, s_n), \tag{6.4}$$

where $\{s_n\}$ is the corresponding sequence for $\{k_n\}$ in Lemma 2. We can assume that $|s_n| \leq n^{-1}$, which implies $\|z\| \leq n^{-1}$. It follows from the expression of the Hamiltonian H_n that along the orbit of z we have $\dot{s} = -\frac{\partial H_n}{\partial \Theta_d} = 0$ and also $\dot{\theta} = \tilde{\omega}_n := \tilde{\omega}(s_n)$, hence from (c) in Lemma 2 we have $\langle k_n, \tilde{\theta}(t) \rangle \equiv 0$. Therefore the corresponding flow becomes $\Phi_n^t(z) = (\tilde{\omega}_n t, \theta_d(t), \tilde{r}(t), s_n)$ with

$$\tilde{r}(t) = A_n(t) + B_n(t),$$

where

$$A_n(t) = 2\pi k_n s_n^n e^{-n\|k_n\|t},$$

$$B_n(t) = \sum_{2 \leq j < n} s_n^j \frac{e^{-j\|k_j\|}}{\langle k_j, \tilde{\omega}_n \rangle} k_j \sin(2\pi \langle k_j, \tilde{\omega}_n \rangle t).$$

Then since $B_n(t)$ is bounded there exists $t > 0$ such that $\|\tilde{r}(t)\| > n$, which implies that Φ_n satisfies (\mathcal{P}_n^1) .

Consider now case (ii), where $\tilde{\omega} \notin \Omega_\tau^{d-1}$. For z as in (6.4), $\Phi_n^t(z)$ is as above with

$$\tilde{r}(t) = A_n(t) + B_n(t),$$

where now

$$A_n(t) = 2\pi k_n s_n^n e^{-C/2\|k_n\|t},$$

$$B_n(t) = \sum_{2 \leq j < n} s_n^j \frac{e^{-C/2\|k_j\|}}{\langle k_j, \tilde{\omega}_n \rangle} k_j \sin(2\pi \langle k_j, \tilde{\omega}_n \rangle t).$$

Then if $t := \exp(C|s_n|^{-(\tau+1)})^{-1}$, we get from estimate (6.1) that for n sufficiently large, $\|A_n(t)\| \geq 2|s_n|^{-1}$. By considering $\{s_j\}$ to decrease fast enough, we can assume that

$$|\langle k_j, \tilde{\omega}_n \rangle| = |s_j - s_n| |k_{j,1}| > |s_n|, \quad 2 \leq j < n,$$

and so $\|B_n(t)\| < 1$ as well. We conclude that $\|\tilde{r}(t)\| > |s_n|^{-1} = \|z\|^{-1}$, which implies that Φ_n satisfies (\mathcal{P}_n^2) .

Consider case (iii). We define z as in cases (i) and (ii) but with $s_n := e^{-n^2\|k_n\|}$. The flow becomes $\Phi_n^t(z) = (\tilde{\omega}t, \theta_d(t), \tilde{r}(t), s_n)$ with

$$\tilde{r}(t) = A_n(t) + B_n(t),$$

where now

$$A_n(t) = s_n^n \frac{e^{-n\|k_n\|}}{\langle k_n, \tilde{\omega} \rangle} k_n \sin(2\pi \langle k_n, \tilde{\omega} \rangle t),$$

$$B_n(t) = \sum_{2 \leq j < n} s_n^j \frac{e^{-j\|k_j\|}}{\langle k_j, \tilde{\omega} \rangle} k_j \sin(2\pi \langle k_j, \tilde{\omega} \rangle t).$$

We assume as before that $\|k_n\|$ grows sufficiently fast to guarantee that $\|B_n(t)\| < 1$. Also due to (4.5) we can assume that k_n are chosen in such a way that

$$|\langle \tilde{\omega}, k_n \rangle| \leq e^{-n^4\|k_n\|}.$$

Hence for $t := s_n^{-2n} = e^{2n^3\|k_n\|}$ and n big enough,

$$|\sin(2\pi \langle k_n, \tilde{\omega} \rangle t)| > |\langle k_n, \tilde{\omega} \rangle| t.$$

Therefore $\|\tilde{r}(t)\| > s_n^{-1}$, which implies that Φ_n satisfies (\mathcal{P}_n^3) .

The proof of case (iv) follows exactly in the same way, with the same choice of s_n and the initial condition z , but with s_n^2 in place of s_n^n and s_n^j in the expressions of A_n and B_n , which allows to take $t := s_n^{-4}$ and obtain (\mathcal{P}_n^4) instead of (\mathcal{P}_n^3) .

We consider now case (v). For any initial condition $z = (\theta, r)$, the flow of H_n satisfies $\Phi_n^t(z) = (\tilde{\theta} + \tilde{\omega}t, \theta_d(t), \tilde{r}(t), s)$, where

$$\tilde{r}(t) = \tilde{r} + \sum_{j=2}^n k_j s^j e^{\|k_j\|s} \sin(2\pi \langle k_j, \tilde{\theta} + \tilde{\omega}t \rangle) - \sum_{j=2}^n k_j s^j e^{\|k_j\|s} \sin(2\pi \langle k_j, \tilde{\theta} \rangle).$$

Notice that if we define $\tau_n := |\langle k_n, \tilde{\omega} \rangle|^{-1}$, then there exists $0 < t < \tau_n$ such that

$$|\sin(2\pi \langle k_n, \tilde{\theta} + \tilde{\omega}t \rangle) - \sin(2\pi \langle k_n, \tilde{\theta} \rangle)| = 1.$$

Therefore by choosing $\{k_n\}$ increasing sufficiently fast in norm we can impose that for all $z \in Q_n^+$ there exists $t > 0$ such that

$$\|\tilde{r}(t)\| \geq \left| \frac{\|k_n\|}{n^j} e^{\frac{\|k_n\|}{n}} - \sum_{j=2}^{n-1} n^j \|k_j\| e^{n\|k_j\|} - n \right| > n.$$

Case (vii) is similar to case (v) except for the fact that the exponent $\|k_j\|s^2$ is positive yields diffusion on all Q_n instead of Q_n^+ . ■

6.5. Concluding the proofs

We can now finish the proofs of Theorems 1–6.

Proof of Theorem 1. The convergence of H_n was proved in Proposition 1. The characterization of the BNF was proved in Proposition 2. The instability comes from the fact that the flow of H satisfies (\mathcal{P}_∞^1) , which follows from Propositions 4 and 5 (provided the sequence $\{k_j\}$ is chosen to grow sufficiently fast). It is left to verify that if $\omega(s) = \bar{\omega}(s)$ and ω is Diophantine, then \mathcal{T}_0 is KAM stable. From Theorem A, it suffices to see that \bar{N} satisfies the Rüssmann non-degeneracy condition, namely that there does not exist any vector $\gamma \neq 0$ such that for every r in some neighborhood of \mathcal{T}_0 ,

$$\langle \nabla \bar{N}(r), \gamma \rangle = 0. \quad (6.5)$$

In our case we have

$$\nabla \bar{N}(r) = \left(\omega_1 + s, \dots, \omega_{d-1} + s^{d-1}, \omega_d + \sum_{l=1}^{d-1} r_l l s^{l-1} \right),$$

and it is readily seen that (6.5) forces γ to be zero. Hence \bar{N} is Rüssmann non-degenerate. This finishes the proof of Theorem 1. ■

Proof of Theorem 2. The convergence of H_n was proved in Proposition 1. The characterization of the BNF was proved in Proposition 2. The upper bound on the diffusion times comes from the fact that the flow of H satisfies (\mathcal{P}_∞^2) , which follows from Propositions 4 and 5 (provided the sequence $\{k_j\}$ is chosen to grow sufficiently fast). ■

Proof of Theorem 3. The convergence of H_n was proved in Proposition 1. The characterization of the BNF for part (a) was proved in Proposition 2. The estimate on the diffusion times comes from the flow of H satisfying (\mathcal{P}_∞^3) or (\mathcal{P}_∞^4) , that follow as in the proof of Theorem 2 from Propositions 4 and 5. ■

Proof of Theorem 4. The convergence of H_n was proved in Proposition 1. The characterization of the BNF was proved in Proposition 2. The diffusion for $r_d > 0$ comes from the flow of H satisfying (\mathcal{P}_∞^5) , which holds again due to Propositions 4 and 5. The integrability on M^- was proved in Proposition 3. ■

Proof of Theorem 5. The convergence of H_n was proved in Proposition 1. The characterization of the BNF was proved in Proposition 2. The C^l and not C^{l+1} integrability was proved in Proposition 3. ■

Proof of Theorem 6. The convergence of H_n was proved in Proposition 1. The characterization of the BNF was proved in Proposition 2. The diffusion for $r_d \neq 0$ comes from the flow of H satisfying (\mathcal{P}_∞^6) , again by Propositions 4 and 5. ■

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