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Maxim Kontsevich · Vasily Pestun · Yuri Tschinkel

Equivariant birational geometry and modular symbols

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Abstract. We introduce new invariants in equivariant birational geometry and study their relation to modular symbols and cohomology of arithmetic groups.

Keywords. Equivariant birational geometry, birational invariants, cohomology of arithmetic groups

1. Introduction

Let G be a finite abelian group and $A = G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$ the group of characters of G. Fix an integer $n \geq 2$. Consider the $\mathbb{Z}$-module $\mathcal{B}_n(G)$ generated by symbols

$$[a_1, \dots, a_n], \quad a_i \in A,$$

such that $a_1, \dots, a_n$ generate A, i.e., $\sum_i \mathbb{Z}a_i = A$, and subject to relations:

(S) for all permutations $\sigma \in \mathfrak{S}_n$ and all $a_1, \dots, a_n \in A$ we have

$$[a_{\sigma(1)}, \dots, a_{\sigma(n)}] = [a_1, \dots, a_n],$$

(B) for all $2 \leq k \leq n$, all $a_1, \dots, a_k \in A$, and all $b_1, \dots, b_{n-k} \in A$ such that

$$\sum_i \mathbb{Z}a_i + \sum_j \mathbb{Z}b_j = A$$

we have

$$[a_1, \dots, a_k, b_1, \dots, b_{n-k}] = \sum_{1 \leq i \leq k,\, a_i \neq a_{i'},\, \forall i' < i} [a_1 - a_i, \dots, a_i \text{ (on } i\text{-th place)}, \dots, a_k - a_i, b_1, \dots, b_{n-k}].$$

Maxim Kontsevich: Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France; maxim@ihes.fr

Vasily Pestun: Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France; vasily.pestun@ihes.fr

Yuri Tschinkel: Courant Institute, New York University, New York, NY 10012, USA; tschinkel@cims.nyu.edu

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We have

$$\mathcal{B}_1(G) = \begin{cases} \mathbb{Z}^{\phi(N)} & \text{if } G = \mathbb{Z}/N\mathbb{Z}, N \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $n = 4$ and $k = 3$ and $a_1 = a_2 = a$ and $a_3 = a' \neq a$ and $b_1 = b$, the relation translates to

$$[a, a, a', b] = [a, 0, a' - a, b] + [a - a', a - a', a', b]. \tag{1.1}$$

When $n = 2$ there is only one possibility for k, namely, $k = 2$.

Example 1. The group $\mathcal{B}_2(G)$ is generated by symbols $[a_1, a_2]$ such that

$$a_1, a_2 \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a_1, a_2, N) = 1,$$

and subject to relations

- $[a_1, a_2] = [a_2, a_1]$,
- $[a_1, a_2] = [a_1, a_2 - a_1] + [a_1 - a_2, a_2]$, where $a_1 \neq a_2$,
- $[a, a] = [a, 0]$ for all $a \in \mathbb{Z}/N\mathbb{Z}$ with $\gcd(a, N) = 1$.

For $p \geq 5$ a prime, the $\mathbb{Q}$-rank of $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$ equals

$$\frac{p^2 + 23}{24}. \tag{1.2}$$

For us, this was the first sign that automorphic forms play a role in this theory. We will discuss the connection to modular symbols in Section 11.

Remark 2. The group $\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z})$ can have torsion: e.g., for $p = 37$, there is ℓ-torsion for $\ell = 3$ and 19.

For $n \geq 3$, the system of relations in $\mathcal{B}_n(G)$ is highly overdetermined. Nevertheless, computer experiments show that nontrivial solutions exist: e.g., for $G = \mathbb{Z}/27\mathbb{Z}$ or $\mathbb{Z}/43\mathbb{Z}$, the $\mathbb{Q}$-rank of $\mathcal{B}_4(G)$ equals 1.

Let X be a smooth irreducible projective algebraic variety of dimension $n \geq 2$, over a fixed algebraically closed field of characteristic zero (e.g., $\mathbb{C}$), equipped with a birational, generically free action of G. After G-equivariant resolution of singularities, we may assume that the action of G is regular. To such an X we associate an element of $\mathcal{B}_n(G)$ as follows: Let

$$X^G = \coprod_{\alpha \in \mathcal{A}} F_\alpha \tag{1.3}$$

be the G-fixed point locus; it is a disjoint union of closed smooth irreducible subvarieties of X. Put

$$\dim(F_\alpha) = n_\alpha \leq n - 1.$$

On each irreducible component F_α we fix a point $x_\alpha \in F_\alpha$ and consider the action of G in its tangent space $\mathcal{T}_{x_\alpha} X$ in X; it decomposes into eigenspaces of characters $a_{1,\alpha}, \ldots, a_{n,\alpha}$, defined up to permutation of indices (here we identify algebraic characters of G with $\mathbb{C}^\times$-valued characters). By the assumption that the action of G is generically free, we have

$$\sum_i \mathbb{Z} a_{i,\alpha} = A.$$

This does not depend on the choice of $x_\alpha \in F_\alpha$. The dimension $\dim(F_\alpha)$ equals the number of zeros among the $a_{i,\alpha}$. Thus we have a symbol, for each α,

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}] \in \mathcal{B}_n(G).$$

Put

$$\beta(X) := \sum_\alpha [a_{1,\alpha}, \ldots, a_{n,\alpha}]. \tag{1.4}$$

One of our main results is that expression (1.4), considered as an element in $\mathcal{B}_n(G)$, is invariant under G-equivariant blowups.

Theorem 3. *The class $\beta(X) \in \mathcal{B}_n(G)$ is a G-equivariant birational invariant.*

Now we introduce another $\mathbb{Z}$-module $\mathcal{M}_n(G)$, generated by symbols

$$\langle a_1, \ldots, a_n \rangle,$$

such that $a_1, \ldots, a_n$ generate A, and subject to relations which are almost identical to those for $\mathcal{B}_n(G)$:

(S) for all $\sigma \in \mathfrak{S}_n$ and all $a_1, \ldots, a_n \in A$ we have

$$\langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle = \langle a_1, \ldots, a_n \rangle,$$

(M) for all $2 \le k \le n$, all $a_1, \ldots, a_k \in A$ and all $b_1, \ldots, b_{n-k} \in A$ such that

$$\sum_i \mathbb{Z} a_i + \sum_j \mathbb{Z} b_j = A,$$

we have

$$\begin{aligned}&\langle a_1, \ldots, a_k, b_1, \ldots, b_{n-k} \rangle \\ &\quad = \sum_{1 \le i \le k} \langle a_1 - a_i, \ldots, a_i \text{ (on } i\text{-th place)}, \ldots, a_k - a_i, b_1, \ldots, b_{n-k} \rangle.\end{aligned}$$

Note that we eliminated the constraint $a_i \neq a_{i'}$ for $i' < i$ from the sum. Clearly,

$$\mathcal{M}_1(G) = \begin{cases} \mathbb{Z}^{\phi(N)} & \text{if } G = \mathbb{Z}/N\mathbb{Z}, N \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 4$ and $k = 3$ and $a_1 = a_2 = a$ and $a_3 = a' \neq a$ and $b_1 = b$ relation (M) translates to

$$\langle a, a, a', b \rangle = \langle a, 0, a' - a, b \rangle + \langle 0, a, a' - a, b \rangle + \langle a - a', a - a', a', b \rangle. \tag{1.5}$$

The right side equals

$$2\langle a, 0, a' - a, b\rangle + \langle a - a', a - a', a', b\rangle,$$

by symmetry relations. Notice the difference between (1.5) and (1.1).

In Section 6, we show that relation (M) follows from the subcase $k = 2$.

These groups carry naturally defined, commuting, linear operators

$$T_{\ell,r} : \mathcal{M}_n(G) \to \mathcal{M}_n(G)$$

for all primes ℓ coprime to the order of G and all $1 \le r \le n$. We call these *Hecke operators*. One can consider their spectrum for

$$\mathcal{M}_n(G) \otimes \bar{\mathbb{Q}} \quad \text{or} \quad \mathcal{M}_n(G) \otimes \bar{\mathbb{F}}_p,$$

where p is any prime not dividing $\#G$, the order of the group G. We expect that the joint spectrum of $T_{\ell,r}$ is related to automorphic forms and present evidence for this in Sections 9 and 11.

Consider the map $\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G)$ defined on symbols as follows:

($\boldsymbol{\mu}_0$) $[a_1, \dots, a_n] \mapsto \langle a_1, \dots, a_n\rangle$ if all $a_1, \dots, a_n \neq 0$,

($\boldsymbol{\mu}_1$) $[0, a_2, \dots, a_n] \mapsto 2\langle 0, a_2, \dots, a_n\rangle$ if all $a_2, \dots, a_n \neq 0$,

($\boldsymbol{\mu}_2$) $[0, 0, a_3, \dots, a_n] \mapsto 0$ for all $a_3, \dots, a_n$,

and extended by $\mathbb{Z}$-linearity.

Theorem 4. *The map μ is a well-defined homomorphism, which is a surjection modulo 2-torsion.*

Note that

$$\langle 0, 0, a_3, \dots, a_n\rangle = 0 \in \mathcal{M}_n(G),$$

which follows from the relations by putting $k = 2$, $a_1 = a_2 = 0$, and $b_i = a_{i+2}$ for all $i = 1, \dots, n-2$.

We expect that μ is an isomorphism modulo torsion (see Conjectures 8 and 9).[1]

Our notation $\mathcal{B}_n(G)$ and $\mathcal{M}_n(G)$ stands for

birational vs. **motivic/modular.**

This paper consists of two parts: in Part I, we present proofs of Theorems 3 and 4. We recast the definition of $\mathcal{M}_n(G)$ in terms of scissor type relations on lattices with cones. We introduce a certain quotient $\mathcal{M}_n^-(G)$ of $\mathcal{M}_n(G)$ and define multiplication and co-multiplication on these groups. We formulate a series of conjectures reducing the structure of $\mathcal{M}_n(G) \otimes \mathbb{Q}$ to certain primitive pieces. We define Hecke operators on $\mathcal{M}_n(G)$, which are compatible with the hypothetical decomposition. We present results of computer experiments with equations for new invariants.

[1] This has been established in [6, Theorem 1.2].

In Part II, we introduce various generalizations of $\mathcal{B}_n(G)$ and $\mathcal{M}_n(G)$, not necessarily related to each other, reflecting a certain divergence of birational and automorphic sides. Our considerations led us to a new question (see Question 20 in Section 9), and a potentially new viewpoint on the Langlands program, based on higher-dimensional generalizations of modular symbols. We identify $\mathcal{M}_n^-(G)$ with cohomology of an arithmetic group, with coefficients in a 1-dimensional representation. We also explore, in the case $n = 2$, the relation between our groups of symbols and classical Manin symbols for modular forms of weight 2.

During the preparation of this paper we discovered the work of Borisov–Gunnels [2], who studied constructions related to the modular picture in the case $n = 2$ and raised the question of generalizations to $n \geq 3$ in [3, Remark 7.15].

Part I

2. Invariance under blowups

We use notation and conventions from the Introduction. Let X be a smooth irreducible projective n-dimensional variety equipped with a generically free regular action of a finite abelian group G, and $W \subset X$ a closed smooth irreducible G-stable subvariety with

$$0 \leq \dim(W) \leq n-2.$$

Let $\pi : \tilde{X} = \mathrm{Bl}_W(X) \to X$ be the blowup of X in W. By the G-equivariant Weak Factorization theorem, smooth projective G-birational models of X are connected by iterated blowups of such type.

In order to prove Theorem 3, it suffices to show that

$$\beta(\tilde{X}) = \beta(X) \in \mathcal{B}_n(G).$$

Choose an irreducible component $Z \subseteq W^G$. It suffices to consider the structure of the fixed locus of exceptional divisors in the neighborhood of Z. Let $F = F(Z) \subseteq X^G$ be the unique irreducible component containing Z; it equals one of the F_α in (1.3). Let $z \in Z$ be a point and

$$\mathcal{T}_z X = T_1 \oplus T_2 \oplus R_1 \oplus R_2$$

the decomposition of the tangent bundle at z, where T_i stand for trivial representations, and R_1, R_2 have only nontrivial characters, with

$$\mathcal{T}_z X^G = T_z F = T_1 \oplus T_2, \quad \mathcal{T}_z W = T_2 \oplus R_1.$$

Let

$$d_1 := \dim(T_1), \quad d_2 = \dim(T_2), \quad d_3 = \dim(R_1), \quad d_4 = \dim(R_2).$$

The spectrum of the action of G in $\mathcal{T}_z$ takes the form

$$\underbrace{0,\ldots,0}_{d_1} \mid \underbrace{0,\ldots,0}_{d_2} \mid b_1,\ldots,b_{d_3} \mid \underbrace{a^1,\ldots,a^1}_{\kappa_1},\ldots,\underbrace{a^m,\ldots,a^m}_{\kappa_m},$$

where $b_j \in A \setminus 0$, and $a^1, \dots, a^m \in A \setminus 0$ are pairwise distinct, with

$$\kappa_1 + \cdots + \kappa_m = d_4, \quad \kappa_i \geq 1,\ m \geq 0.$$

We have

- $d_2 = \dim(Z)$,
- $d_1 + d_2 + d_3 + d_4 = n$,
- $1 \leq d_3 + d_4$, since $\mathrm{codim}(X^G) \geq 1$,
- $2 \leq d_1 + d_4$, since $\mathrm{codim}(W) \geq 2$.

We consider several cases, with corresponding geometric configurations:

(I) $d_1 = 0, d_4 \geq 2$. Geometrically, this means that W contains a component Z of X^G. Blowing up W we obtain new contributions to formula (1.4). The new fixed locus, with m irreducible components, consists of subvarieties of the exceptional divisor, a projective bundle over W. These subvarieties, in turn, are total spaces of projective bundles over Z, with fibers

$$\mathbb{P}^{\kappa_i - 1}, \quad i = 1, \dots, m.$$

The corresponding contribution to $\beta(\tilde{X})$ is given by

$$\sum_{i=1}^{m} [\underbrace{0, \dots}_{d_2}, b_1, \dots, b_{d_3}, \underbrace{a^1 - a^i, \dots}_{\kappa_1}, \dots, a_i, \underbrace{0, \dots}_{\kappa_i - 1}, \dots, \underbrace{a^m - a^i, \dots}_{\kappa_m}].$$

Putting

$$a_1, \dots, a_k = \underbrace{a^1, \dots}_{\kappa_1}, \dots, \underbrace{a^m, \dots}_{\kappa_m}$$

and

$$b_1, \dots, b_{n-k} = b_1, \dots, b_{d_3}, \underbrace{0, \dots}_{d_2}$$

we find that the formula matches relation (B) when the sequence $\bar{a} = a_1, \dots, a_k$ does not contain zeros.

(II) $d_1, d_4 \geq 1$. Geometrically, this means that the tangent spaces of the fixed locus and W do not span the whole tangent space and, near Z, the component F is not contained in W. In the blowup, we will have a component of the fixed locus which is birational to F and new components which are projective bundles

$$\mathbb{P}^{\kappa_1 - 1}, \dots, \mathbb{P}^{\kappa_m - 1}$$

over Z. We need to show that the contribution of these m terms vanishes in $\mathcal{B}_n(G)$. Let

$$\bar{b} = b_1, \dots, b_{n-k} = b_1, \dots, b_{d_3}, \underbrace{0, \dots}_{d_2}.$$

The new components contribute

$$\sum_{i=1}^{m}[\underbrace{-a^i,\ldots,}_{d_1}\underbrace{a^1-a^i,\ldots,}_{\kappa_1}\ldots,a_i,\underbrace{0,\ldots,}_{\kappa_i-1}\ldots,\underbrace{a^m-a^i,\ldots,}_{\kappa_m}\bar{b}].$$

We claim that this sum vanishes in $\mathcal{B}_n(G)$. Indeed, consider relation (B) for the sequences

$$\bar{a}=a_1,\ldots,a_k=\underbrace{0,\ldots,}_{d_1}\underbrace{a^1,\ldots,}_{\kappa_1}\ldots,\underbrace{a^m,\ldots,}_{\kappa_m}$$

and $\bar{b}$. The left side of (B) equals

$$[\bar{a},\bar{b}]=[a_1,\ldots,a_k,\bar{b}]=[\underbrace{0,\ldots,}_{d_1}\underbrace{a^1,\ldots,}_{\kappa_1}\ldots,\underbrace{a^m,\ldots,}_{\kappa_m}\bar{b}].$$

The right side is the sum of $m+1$ terms. The first summand, corresponding to $a_i=a_1=0$ coincides with the left side. The remaining terms are the same as above.

(III) $d_1\geq 2, d_3\geq 1, d_4=0$. In this case, no new contributors to formula (1.4) arise.

This concludes the proof of Theorem 3.

Remark 5. There is a refinement of $\mathcal{B}_n(G)$, connecting it to the Burnside group of varieties considered in [5]. Let K be an algebraically closed field of characteristic zero. Let

$$\mathrm{Bir}_{n-1,m}(K),\qquad 0\leq m\leq n-1,$$

be the set of equivalence classes of $(n-1)$-dimensional irreducible varieties over K, modulo K-birational equivalence, which are K-birational to products $W\times\mathbb{A}^m$, and not to $W'\times\mathbb{A}^{m+1}$ for any W'. Let

$$\mathcal{B}_n(G,K):=\bigoplus_{m=0}^{n-1}\bigoplus_{[Y]\in\mathrm{Bir}_{n-1,m}(K)}\mathcal{B}_{m+1}(G)$$

with

$$\mathcal{B}_1(G)=\begin{cases}\bigoplus_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\mathbb{Z} & \text{if } G=\mathbb{Z}/N\mathbb{Z},\ N\geq 2,\\ 0 & \text{if } G \text{ is not cyclic.}\end{cases}$$

Let X be an irreducible K-variety with a generically free action of G. As in Section 1, we may assume that G acts regularly; let $X^G=\bigsqcup_\alpha F_\alpha$ be the decomposition of the fixed point locus into irreducible, disjoint components. The spectrum for the G-action in the tangent space to X at any point $x_\alpha\in F_\alpha$ is given by

$$a_1,\ldots,a_{n-\dim(F_\alpha)},\underbrace{0,\ldots,}_{\dim(F_\alpha)}\qquad a_i\neq 0.$$

Define $\beta_K(X)\in\mathcal{B}_n(G,K)$ by taking into account the birational types of fixed loci under G, as follows: write

$$Y_\alpha:=F_\alpha\times\mathbb{A}^{n-1-\dim(F_\alpha)}$$

and let $m_\alpha \in \mathbb{Z}_{>0}$ be the maximal integer such that $Y_\alpha \sim Z_\alpha \times \mathbb{A}^{m_\alpha}$; clearly,

$$m_\alpha \geq n - 1 - \dim(F_\alpha).$$

Then $\beta_K(X) = \sum_\alpha \beta_\alpha(X)$, where

$$\beta_\alpha(X) = [a_1, \ldots, a_{n-\dim(F_\alpha)}, \underbrace{0, \ldots}_{m_\alpha+1-n+\dim(F_\alpha)}] \in \text{copy of } \mathcal{B}_{m_\alpha+1}(G),$$

labeled by the birational type of Y_α.

The invariance under blowups follows from the fact that all $(n-1)$-dimensional birational types arising as labels in each particular subcase of the proof of Theorem 3 coincide with each other.

Remark 6. In a similar vein, one could introduce birational invariants for actions of algebraic tori, but we have not explored this direction.

3. Comparison

In this section we study the map

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G) \tag{3.1}$$

defined in Section 1. The proof that this is a well-defined homomorphism is a long chain of essentially trivial steps.

First we record several corollaries of defining relations for $\mathcal{M}_n(G)$:

(1) $\langle 0, 0, \ldots \rangle = 0$,

(2) $\langle a, a, \ldots \rangle = 2\langle a, 0, \ldots \rangle$,

(3) $\langle a, a, 0, \ldots \rangle = 0$,

(4) $\langle a, a, a', a', \ldots \rangle = 0$,

(5) $\langle a, a, a, \ldots \rangle = 0$,

(6) $\langle a, -a, \ldots \rangle = 0$;

here $\ldots$ stands for arbitrary sequences of elements in A such that the set of all elements of the symbol spans the whole A.

In the proofs below we freely use the symmetry relation (S).

(1) We use (M) for $k = 2$ and $a_1 = a_2 = 0$:

$$\langle 0, 0, \ldots \rangle = \langle 0, 0, \ldots \rangle + \langle 0, 0, \ldots \rangle.$$

(2) We use (M) for $k = 2, a_1 = a_2 = a$.

(3) We use (2) and (1):

$$\langle a, a, 0, \ldots \rangle \overset{(2)}{=} 2\langle a, 0, 0, \ldots \rangle + \langle 0, 0, \ldots \rangle \overset{(1)}{=} 0.$$

(4) We use again (2) and (1):

$$\langle a, a, a', a', \dots \rangle \overset{(2)}{=} 4\langle a, 0, a', 0, \dots \rangle \overset{(1)}{=} 0.$$

(5) We use (M) for $k = 3$ and $a_1 = a_2 = a_3 = a$, and then (1):

$$\langle a, a, a, \dots \rangle = 3\langle a, 0, 0, \dots \rangle \overset{(1)}{=} 0,$$

(6) We use (M) for $k = 2, a_1 = a, a_2 = 0$:

$$\langle a, 0, \dots \rangle = \langle a, -a, \dots \rangle + \langle a, 0, \dots \rangle.$$

We proceed to the proof of Theorem 4. The main point is to check the following compatibility equation:

$$\mu([a_1, \dots, a_k, b_1, \dots, b_{n-k}]) = \sum_{i, a_i \neq a_{i'} \text{ for } i < i'} \mu([a_1 - a_i, \dots, a_i, \dots, a_k - a_i, b_1, \dots, b_{n-k}]). \qquad (3.2)$$

For convenience, we sometimes write

$$[a_1, \dots, a_k \mid b_1, \dots, b_{n-k}] = [a_1, \dots, a_k, b_1, \dots, b_{n-k}] \in \mathcal{B}_n(G),$$

and similarly for the symbol in $\mathcal{M}_n(G)$, indicating the position of the separation of a and b variables in subsequent relations.

There are three cases, distinguished by the number of zeros in the sequence

$$\bar{b} := b_1, \dots, b_{n-k} :$$

(C0) $\bar{b}$ does not contain zeros.

(C1) $\bar{b}$ contains exactly one zero.

(C2) $\bar{b}$ contains at least two zeros.

Case (C2) is obvious, by relation (1), since all terms vanish, by the definition ($\boldsymbol{\mu}_2$) (in Section 1).

Case (C1) splits into subcases:

(C10) The sequence

$$\bar{a} := a_1, \dots, a_k$$

contains no zeros,

(C11) $\bar{a}$ contains at least one zero.

In case (C11), the left hand side maps to 0, by ($\boldsymbol{\mu}_2$):

$$\mu([0, \dots \mid 0, \dots]) = 0.$$

The terms of the right hand side in relation (B) are of two types, corresponding to $a_i = 0$ or $a_i = a \neq 0$. If $a_i = 0$, then the term has the form

$$[\underline{0}, \dots \mid 0, \dots],$$

mapping to zero, by $(\boldsymbol{\mu}_2)$. The underlined 0 indicates that a_i is left in its place in relation (B). If $a_i = a \neq 0$, then the corresponding term on the right hand side of (B) has the form

$$[-a, \dots, \underline{a}, \dots \mid 0, \dots],$$

mapping to

$$c \cdot \langle -a, \dots, a, \dots 0, \dots \rangle,$$

where $c = 0$ or 2, and the symbol in $\mathcal{M}_n(G)$ equals 0, by (6).

Case (C10) splits into two cases:

(C10$\neq$) all terms in $\bar{a}$ are pairwise distinct,

(C10$=$) there exist at least two equal terms in $\bar{a}$.

In case (C10$\neq$), on the left and on the right hand side of the relation (B), all symbols contain exactly one zero. Thus, they are mapped to similar symbols in $\mathcal{M}_n(G)$, but multiplied by 2, by $(\boldsymbol{\mu}_1)$. Since every element in $\bar{a}$ occurs only once, the expressions on the right side of (B) and (M) consist of matching terms.

In case (C10$=$), the left hand side of (B) equals

$$[a, a, \dots \mid 0, \dots] \in \mathcal{B}_n(G).$$

Its image under μ equals

$$2\langle a, a, \dots, 0, \dots \rangle \in \mathcal{M}_n(G),$$

which vanishes, by (3). We claim that all terms on the right side of (B) map to zero as well. Indeed, they are of the form either

$$[\underline{a}, 0, \dots \mid 0, \dots] \quad \text{or} \quad [a - a', a - a', \dots, \underline{a'}, \dots \mid 0, \dots], \quad a' \neq a.$$

The image of this symbol is proportional to

$$\langle a, 0, \dots, 0, \dots \rangle \quad \text{or} \quad \langle a - a', a - a', \dots, a', \dots, 0, \dots \rangle,$$

vanishing by (1) or (3), respectively.

Case (C0) splits into three cases:

(C00) $\bar{a}$ does not contain zeros,

(C01) $\bar{a}$ contains exactly one zero,

(C02) $\bar{a}$ contains at least two zeros.

Recall that $\bar{b}$ does not contain zeros in case (C0). We start with (C02). The left hand side in (B) has the form

$$[0, 0, \dots \mid \dots],$$

hence maps to 0, by $(\boldsymbol{\mu}_2)$. We check that all terms on the right hand side of (B) map to 0 as well. These symbols have the form

$$[\underline{0}, 0, \dots \mid \dots] \quad \text{or} \quad [-a, -a, \dots, \underline{a}, \dots \mid \dots], \quad a \neq 0,$$

mapping to elements in $\mathcal{M}_n(G)$ which are proportional to either

$$\langle 0,0,\ldots\rangle \quad \text{or} \quad \langle -a,-a,\ldots,a,\ldots\rangle,$$

vanishing by (1) or (6), respectively.

Case (C01) splits into two cases:

(C01$\neq$) all terms in $\bar{a}$ are pairwise distinct,

(C01$=$) there exist at least two equal terms in $\bar{a}$.

In case (C01$=$), the left side in (B) has the form

$$[0,a,a,\ldots \mid \ldots] \quad \text{for } a \neq 0,$$

mapping to 0, by relation (3). The right side contains terms of the form

$$[\underline{0},a,a,\ldots \mid \ldots] \quad \text{or} \quad [-a,\underline{a},0,\ldots \mid \ldots],$$

or

$$[-a',a-a',a-a',\ldots,\underline{a'},\ldots \mid \ldots], \quad a'\neq a,0.$$

Their images under μ are proportional to

$$\langle 0,a,a,\ldots\rangle, \quad \langle -a,-a,0,\ldots\rangle, \quad \text{or} \quad \langle -a',a-a',a-a',\ldots,a',\ldots\rangle,$$

which vanish by (3), (6), and (6), respectively.

Consider case (C01$\neq$). The left side of (B) has the form

$$[0,a_2,\ldots,a_k \mid \ldots] \quad \text{for } a_i\neq 0, i\geq 2, \text{ pairwise distinct}, b_j\neq 0.$$

Its image under μ equals, by $(\boldsymbol{\mu}_1)$,

$$2\langle 0,a_2,\ldots,a_k,\ldots\rangle.$$

The right side of (B) is the sum

$$[\underline{0},a_2,\ldots,a_k \mid \ldots]+[-a_2,\underline{a_2},\ldots,a_k-a_2 \mid \ldots]+[-a_3,a_2-a_3,\underline{a_3},\ldots \mid \ldots]+\cdots$$

where the first summand maps, by $(\boldsymbol{\mu}_1)$, to

$$2\langle 0,a_2,\ldots,a_k,\ldots\rangle$$

and all the other terms map to 0, by relation (6). This proves (C01$\neq$).

We are left with case (C00), i.e., all elements of the sequences $\bar{a}$ and $\bar{b}$ are nonzero. We have two cases:

(C00$\neq$) all terms in $\bar{a}$ are pairwise distinct,

(C00$=$) at least two terms in $\bar{a}$ are equal.

In case (C00$\neq$), the left and right sides of (B) do not contain symbols with zeros, hence we use $(\boldsymbol{\mu}_0)$, and (B) is mapped precisely to the corresponding relation (M).

Case (C00$=$) splits into three subcases:

(C00= 2) $\bar{a}$ has only one pair of equal terms, i.e.,

$$\bar{a} = a, a, a_3, \dots, a_k,$$

where $a_3, \dots, a_k$ are pairwise distinct and different from a,

(C00= 2, 2) $\bar{a}$ has the form

$$\bar{a} = a, a, a', a', a_5, \dots, a_k,$$

where $a \neq a'$ and $a_5, \dots, a_k$ are pairwise distinct and different from a, a',

(C00= 3) $\bar{a}$ has the form

$$\bar{a} = a, a, a, \dots.$$

We start with (C00= 3). The left side is mapped to 0, by relation (5). The right side has terms of the form

$$[\underline{a}, 0, 0, \dots \mid \dots] \quad \text{or} \quad [a - a', a - a', a - a', \dots, \underline{a'}, \dots \mid \dots], \qquad a \neq a'.$$

They are mapped to terms proportional to

$$\langle a, 0, 0, \dots \rangle \qquad \text{or} \qquad \langle a - a', a - a', a - a', \dots \rangle,$$

vanishing by (1) or (5), respectively.

We consider (C00= 2, 2). The left side is mapped to $\langle a, a, a', a', \dots \rangle$, which vanishes by relation (4). The right side has terms of three shapes,

$$[a, 0, a' - a, a' - a, \dots \mid \dots] \quad \text{or} \quad [a - a', a - a', \underline{a'}, 0, \dots \mid \dots], \qquad a \neq a',$$

or

$$[a - a'', a - a'', a' - a'', a' - a'', \dots, \underline{a''}, \dots \mid \dots], \qquad a, a', a'' \text{ pairwise distinct.}$$

Their images are proportional to

$$\langle a, 0, a' - a, a' - a, \dots \rangle \quad \text{or} \quad \langle a - a', a - a', \underline{a'}, 0, \dots \rangle, \quad a \neq a',$$

or

$$\langle a - a'', a - a'', a' - a'', a' - a'', \dots, \underline{a''}, \dots \rangle, \qquad a, a', a'' \text{ pairwise distinct,}$$

which vanish by (3), (3), and (4), respectively.

In the last case (C00= 2), relation (B) has the form

$$[a, a, a_3, \dots, a_k \mid \dots] = [\underline{a}, 0, a_3 - a, \dots, a_k - a \mid \dots]$$
$$+ [a - a_3, a - a_3, \underline{a_3}, \dots, a_k - a_3 \mid \dots] + [a - a_4, a - a_4, a_3 - a_4, \underline{a_4}, \dots \mid \dots] + \cdots.$$

The left side maps to $\langle a, a, a_3, \dots \rangle$ and the right side to

$$2\langle \underline{a}, 0, a_3 - a, \dots, a_k - a \mid \dots \rangle + \langle a - a_3, a - a_3, \underline{a_3}, \dots, a_k - a_3 \mid \dots \rangle + \cdots.$$

Here the first summand is obtained by $(\boldsymbol{\mu}_1)$ and the other summands by $(\boldsymbol{\mu}_0)$. We see that, modulo relation (S), the image of the right hand side of (B) coincides with the right hand side of (M) in $\mathcal{M}_n(G)$.

This concludes the proof of Theorem 4.

Proposition 7. *The homomorphism*

$$\mu : \mathcal{B}_2(G) \to \mathcal{M}_2(G) \tag{3.3}$$

is injective, with cokernel annihilated by $(\mathbb{Z}/2\mathbb{Z})^{\phi(N)}$ *if* $G \simeq \mathbb{Z}/N\mathbb{Z}$ *is a cyclic group, and is an isomorphism otherwise.*

Proof. Write the generators and relations for $\mathcal{B}_2(G)$ and $\mathcal{M}_2(G)$:

- **Generators**:
  - ("nondegenerate") symbols $[a_1, a_2]$ (resp. $\langle a_1, a_2\rangle$), where $a_1, a_2 \in A \setminus 0$ are such that $\mathbb{Z}a_1 + \mathbb{Z}a_2 = A$, and
  - ("degenerate") symbols $[a, 0]$ (resp. $\langle a, 0\rangle$), where $a \in A \setminus 0$ is such that $\mathbb{Z}a = A$.
- **Relations:**
  (**1**) $[a_1, a_2] = [a_2, a_1]$ (resp. $\langle a_1, a_2\rangle = \langle a_2, a_1\rangle$) for $a_1, a_2 \in A \setminus 0$,
  (**2**) $[a_1, a_2] = [a_1, a_2 - a_1] + [a_1 - a_2, a_2]$ (and correspondingly $\langle a_1, a_2\rangle = \langle a_1, a_2 - a_1\rangle + \langle a_1 - a_2, a_2\rangle$) for $a_1, a_2 \in A \setminus 0$ and $a_1 \neq a_2$,
  (**3**) $[a, a] = [a, 0]$ (resp. $\langle a, a\rangle = 2\langle a, 0\rangle$) for $a \neq 0$.

The first two relations are identical and deal only with nondegenerate symbols $[a_1, a_2]$ (resp. $\langle a_1, a_2\rangle$), when both a_1, a_2 are nonzero. In the case of $\mathcal{B}_2(G)$, relation (3) just identifies the degenerate symbol $[a, 0]$ via the nondegenerate symbol $[a, a]$, whereas in the case of $\mathcal{M}_2(G)$ it adds one half of the nondegenerate symbol $\langle a, a\rangle$. Obviously, if we add to any abelian group an extra generator which is one half of any given element of this group, then the new group contains the initial one, and the quotient is annihilated by $\mathbb{Z}/2\mathbb{Z}$. The statement of the proposition immediately follows, as the Euler function $\phi(N)$ is the number of degenerate elements $[a, 0]$ when $G \simeq A \simeq \mathbb{Z}/N\mathbb{Z}$. ∎

Conjecture 8. For $n \geq 3$ the homomorphism

$$\mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G) \tag{3.4}$$

is an isomorphism modulo torsion.

This statement reduces to the following: For any integer $N \geq 2$,

$$[0, 0, 1] \in \mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})$$

is a torsion element. Indeed, if this were the case, then any symbol $[0, 0, \ldots]$ would vanish modulo torsion, and then one could repeat the steps in the proof of Theorem 4 and construct an inverse morphism from $\mathcal{M}_n(G) \otimes \mathbb{Q}$ to $\mathcal{B}_n(G) \otimes \mathbb{Q}$.

Computer experiments for $N \leq 23$ support the following:

Conjecture 9. For $N \geq 2$, the element $[0,0,1] \in \mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})$ has order 1, i.e., $[0,0,1] = 0 \in \mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})$ if N is composite or $N = 2, 3, 5$, and is annihilated by

$$\frac{p^2-1}{24} \quad \text{if } N = p \geq 7 \text{ is a prime.}$$[2]

4. On generators and relations in $\mathcal{M}_n(G)$

In this section, G is a finite abelian group, with character group $A = \mathrm{Hom}(G, \mathbb{C}^\times)$, and $n \geq 2$ is an integer. We give a geometric reformulation of generators and relations of $\mathcal{M}_n(G)$.

We start with the following data:

- a (torsion-free) lattice $\mathbf{L} \simeq \mathbb{Z}^n$ of rank n,
- an element $\chi \in \mathbf{L} \otimes A$ such that the induced homomorphism $\mathbf{L}^\vee \to A$ is a surjection,
- a *basic simplicial cone*, i.e., a strictly convex cone $\Lambda \subset \mathbf{L}_\mathbb{R}$ spanned by a basis of $\mathbf{L}$. It is isomorphic to the standard octant $\mathbb{R}^n_{\geq 0}$ for $\mathbf{L} = \mathbb{Z}^n \subset \mathbb{R}^n$.

For every equivalence class of triples $(\mathbf{L}, \chi, \Lambda)$, up to isomorphism, we define a symbol

$$\psi(\mathbf{L}, \chi, \Lambda) \in \mathcal{M}_n(G)$$

as follows: choose a basis $e_1, \dots, e_n$ of $\mathbf{L}$, spanning Λ, express

$$\chi = \sum_{i=1}^{n} e_i \otimes a_i, \tag{4.1}$$

and put

$$\psi(\mathbf{L}, \chi, \Lambda) = \langle a_1, \dots, a_n \rangle \in \mathcal{M}_n(G).$$

The ambiguity in the choices is reflected in the action of the symmetric group $\mathfrak{S}_n$ on the basis elements, hence accounted for by condition (S). Relation (M) has the following geometric meaning. Let $e_1, \dots, e_n$ be an ordered basis of $\mathbf{L}$ spanning Λ:

$$\Lambda := \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} e_n. \tag{4.2}$$

Fix an integer $2 \leq k \leq n$. Then

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k, \tag{4.3}$$

where

$$\Lambda_i := \mathbb{R}_{\geq 0} e_1 + \cdots + \underbrace{\mathbb{R}_{\geq 0}(e_1 + \cdots + e_k)}_{i\text{-th place}} + \cdots + \mathbb{R}_{\geq 0} e_n,$$

[2] This has been established in [6].

i.e., we replace the i-th generator e_i by $e_1 + \cdots + e_k$; this is the set of maximal cones in the stellar subdivision of the face spanned by $e_1, \ldots, e_k$. The cones Λ_i are also basic simplicial and their interiors are disjoint. Decompose

$$\chi = e_1 \otimes a_1 + \cdots + e_k \otimes a_k + e_{k+1} \otimes b_1 + \cdots + e_n \otimes b_{n-k}$$

as in (4.1), i.e., $a_{k+i} = b_i$ for all $i = 1, \ldots, n-k$. Then, in the basis of Λ_i, χ decomposes as

$$e_1 \otimes (a_1 - a_i) + \cdots + (e_1 + \cdots + e_k) \otimes a_i + \cdots e_k \otimes (a_k - a_i) + \sum_{j=1}^{n-k} e_{k+j} \otimes b_j .$$

We see that relation (M) can be expressed as the identity

$$\psi(\mathbf{L}, \chi, \Lambda) = \sum_{i=1}^{k} \psi(\mathbf{L}, \chi, \Lambda_i), \tag{4.4}$$

which we can view as an analog of scissor relations. Our next result is that this relation follows from the special subcase $k = 2$. This is a corollary of a general result concerning simplicial subdivisions of basic simplicial cones. Namely, consider the $\mathbb{Z}$-module $\mathcal{F}_{\mathbf{L},\mathbb{Z}}$ generated by symbols $[\Lambda]$, where Λ is a basic simplicial cone, modulo relations (R_k), $k \geq 2$:

(R_k) $[\Lambda] = [\Lambda_1] + \cdots + [\Lambda_k]$,

where Λ and Λ_i are as above, with $e_1, \ldots, e_n$ an arbitrary basis of Λ.

Lemma 10. *Relations* (R_k) *for* $k \geq 3$ *follow from relations* (R_2).

Proof. We proceed by induction, assuming the claim for $k-1$. We want to prove the claim for $k \geq 3$, i.e.,

$$[\Lambda_1] + \cdots + [\Lambda_k] = [\Lambda].$$

By induction,

$$[\Lambda_k] = [\Lambda'_1] + \cdots + [\Lambda'_{k-1}],$$

where Λ'_i are the cones

$$\mathbb{R}_{\geq 0} e_1 + \cdots + \underbrace{\mathbb{R}_{\geq 0}(e_1 + \cdots + e_{k-1})}_{i\text{-th place}} + \cdots + \underbrace{\mathbb{R}_{\geq 0}(e_1 + \cdots + e_k)}_{k\text{-th place}} + \cdots + \mathbb{R}_{\geq 0} e_n .$$

Indeed, this is relation (R_{k-1}) in the basis

$$e_1, \ldots, e_{k-1}, e_1 + \cdots + e_k, e_{k+1}, \ldots, e_n .$$

Therefore,

$$[\Lambda_1] + \cdots + [\Lambda_k] = ([\Lambda_1] + [\Lambda'_1]) + \cdots + ([\Lambda_{k-1}] + [\Lambda'_{k-1}]).$$

For every $i = 1, \dots, k-1$, we have relation (R_2),

$$[\Lambda_i] + [\Lambda_i'] = [\Lambda_i''],$$

in an appropriate basis, where

$$\Lambda_i'' := \mathbb{R}_{\geq 0} e_1 + \cdots + \underbrace{\mathbb{R}_{\geq 0}(e_1 + \cdots + e_{k-1})}_{i\text{-th place}} + \cdots + \mathbb{R}_{\geq 0} e_n.$$

Finally, (R_{k-1}) in the basis $e_1, \dots, e_n$ says that

$$[\Lambda_1''] + \cdots + [\Lambda_{k-1}''] = [\Lambda],$$

which proves the claim. ∎

Now we can consider an a priori different group generated by symbols $[\Lambda]$, where Λ is any full-dimensional strictly convex rational polyhedral cone, subject to relations

$$[\Lambda] = [\Lambda]_1 + \cdots + [\Lambda_k],$$

where Λ is the union of cones Λ_i with disjoint interiors (here k can be any integer ≥ 2). The toric analog of Weak Factorization implies that the natural homomorphism from $\mathcal{F}_{\mathbf{L},\mathbb{Z}}$ to this group is an isomorphism. In these terms, Lemma 10 says that it suffices to consider blowups with centers in codimension 2.

In consequence, $\mathcal{M}_n(G)$ admits an alternative description: as the group generated by symbols

$$\psi(\mathbf{L}, \chi, \Lambda),$$

depending only on the isomorphism classes of triples, where $\mathbf{L}$ and χ are as above, and Λ is a finitely generated convex rational polyhedral cone, of full dimension, subject to the relations (4.4) whenever there is a decomposition

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$$

as above. This clearly extends to nonconvex cones.

We introduce a variant of the previous constructions: instead of

$$\chi \in \mathbf{L} \otimes A = \mathrm{Hom}(\mathbf{L}^\vee, A)$$

we can consider

$$\chi^* \in \mathrm{Hom}(\mathbf{L}, A),$$

again assuming that χ^* is surjective. In a similar fashion, we can introduce the group $\mathcal{M}_n^*(G)$, which we call the *co-vector* version of (the *vector* version) $\mathcal{M}_n(G)$. This group is generated by symbols

$$\langle a_1, \dots, a_n \rangle^*,$$

subject to relations

(S*) for all $\sigma \in \mathfrak{S}_n$ and all $a_1, \dots, a_n \in A$ we have

$$\langle a_{\sigma(1)}, \dots, a_{\sigma(n)} \rangle^* = \langle a_1, \dots, a_n \rangle^*,$$

(M*) for all $2 \le k \le n$, all $a_1, \dots, a_k \in A$ and all $b_1, \dots, b_{n-k} \in A$ such that

$$\sum_i \mathbb{Z} a_i + \sum_j \mathbb{Z} b_j = A$$

we have

$$\langle a_1, \dots, a_k, b_1, \dots, b_{n-k} \rangle^* = \sum_{1 \le i \le k} \Big\langle a_1, \dots, \sum_{j=1}^k a_j \text{ (on } i\text{-th place)}, \dots, a_k, b_1, \dots, b_{n-k} \Big\rangle^*.$$

As above, the relations for $k = 2$ imply all others.

It is not hard to show that the $\mathbb{Q}$-ranks of $\mathcal{M}_n(G)$ and $\mathcal{M}_n^*(G)$ are the same. Indeed, by a Möbius-type inversion formula, one can reduce the question to the extended versions of the groups $\mathcal{M}_n(G)$ and $\mathcal{M}_n^*(G)$ omitting the condition that the map

$$\chi : \mathbf{L}^\vee \to A, \quad \text{resp.} \quad \chi^* : \mathbf{L} \to A,$$

is surjective. Then the finite Fourier transform (after a choice of an identification $G \simeq A$) identifies the two complex vector spaces consisting of homomorphisms from the two extended groups to $\mathbb{C}$.

5. Multiplication and co-multiplication

In this section, we work with the vector version; the co-vector version is analogous. We consider $\mathcal{M}_n(G)$ in *both* variables $n \ge 1$ and G. We define multiplication and co-multiplication maps and study their properties. An important role will be played by $\mathcal{M}_n^-(G)$, which is defined *only for nontrivial groups* G, as the quotient of $\mathcal{M}_n(G)$ by the relation

$$\langle -a_1, \dots, a_n \rangle = -\langle a_1, \dots, a_n \rangle. \tag{5.1}$$

We denote by

$$\langle a_1, \dots, a_n \rangle^- \in \mathcal{M}_n^-(G)$$

the image of $\langle a_1, \dots, a_n \rangle$ under the natural projection

$$\mu^- : \mathcal{M}_n(G) \to \mathcal{M}_n^-(G). \tag{5.2}$$

We consider short exact sequences of finite abelian groups

$$0 \to G' \to G \to G'' \to 0$$

and the corresponding short exact sequences of character groups

$$0 \to A'' \to A \to A' \to 0.$$

Let

$$n = n' + n'', \quad n', n'' \geq 1.$$

We define a $\mathbb{Z}$-bilinear "multiplication" map

$$\nabla : \mathcal{M}_{n'}(G') \otimes \mathcal{M}_{n''}(G'') \to \mathcal{M}_{n'+n''}(G),$$

which on generators is given by the formula

$$\langle a'_1, \dots, a'_{n'} \rangle \otimes \langle a''_1, \dots, a''_{n''} \rangle \mapsto \sum \langle a_1, \dots, a_{n'}, a''_1, \dots, a''_{n''} \rangle, \tag{5.3}$$

where the sum runs over all lifts $a_i \in A$ of $a'_i \in A'$, and the elements a''_i are understood as elements of A, via the embedding $A'' \hookrightarrow A$.

The compatibility with defining relations (S) and (M) is obvious. The condition that the elements in each summand on the right span A follows from the corresponding condition on the left for the groups A', A''. Note that ∇ descends to a $\mathbb{Z}$-bilinear map of the corresponding quotient groups

$$\nabla^- : \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G'') \to \mathcal{M}^-_{n'+n''}(G),$$

where both G' and G'' are nontrivial.

Next we define a "co-multiplication" map

$$\Delta : \mathcal{M}_{n'+n''}(G) \to \mathcal{M}_{n'}(G') \otimes \mathcal{M}^-_{n''}(G''),$$

where G'' is nontrivial, and which on generators is given by the formula

$$\langle a_1, \dots, a_n \rangle \mapsto \sum \ \langle a_{I'} \bmod A'' \rangle \otimes \langle a_{I''} \rangle^-. \tag{5.4}$$

Here we put

$$\langle a_{I'} \bmod A'' \rangle = \langle a_{i_1} \bmod A'', \dots, a_{i_{n'}} \bmod A'' \rangle, \quad I' := \{i_1, \dots, i_{n'}\},$$

and similarly for $\langle a_{I''} \rangle$, using the symmetry relation (S). The sum is over all subdivisions

$$\{1, \dots, n\} = I' \sqcup I'' \quad \text{with } \#I' = n',\ \#I'' = n'',$$

such that

- for all $j \in I''$, we have $a_j \in A'' \subset A$, and, in the first term on the right, the elements $a_i, i \in I'$, are replaced by their images in $A' = A/A''$;
- (generation condition) the elements a_j, $j \in I''$, span A''.

Note that, given the generation condition in each term of the right side of the formula, the expression $\langle a_{I'} \bmod A'' \rangle^-$ is a symbol, since the condition $\sum \mathbb{Z} a_i = A$ implies that $\sum_{i \in I'} (a_i \bmod A'') = A'$. Therefore, the generation condition for the first term is automatic.

Proposition 11. *The map Δ extends to a well-defined $\mathbb{Z}$-linear homomorphism.*

Proof. By Lemma 10, it suffices to check the 2-term relations (R_2). We need to show that the image of the relation

$$\langle a_1, a_2, \ldots \rangle = \langle a_1 - a_2, a_2, \ldots \rangle + \langle a_1, a_2 - a_1, \ldots \rangle$$

on the left is a relation on the right, and that the terms on the right satisfy the generation condition (the linear combinations of elements span the corresponding group). The only interesting part is when the first two arguments are distributed over the different factors in (5.4), so that

$$\begin{aligned}\langle a_1, a_2, \ldots \rangle \mapsto \delta^{\mathrm{gen}}_{a_1 \in A''} \cdot \langle a_2 \bmod A'', \ldots \rangle \otimes \langle a_1, \ldots \rangle^- \\ + \delta^{\mathrm{gen}}_{a_2 \in A''} \cdot \langle a_2 \bmod A'', \ldots \rangle \otimes \langle a_2, \ldots \rangle^-, \end{aligned} \tag{5.5}$$

where, for $a \in A$,

$$\delta^{\mathrm{gen}}_{a \in A''} := \begin{cases} 1 & \text{if } a \in A'' \text{ and } \mathbb{Z}a + \sum_{j \in J''} \mathbb{Z}a_j = A'', \\ 0 & \text{otherwise.} \end{cases}$$

There are four cases:

(1) $a_1 \in A'', a_2 \in A''$,

(2) $a_1 \in A'', a_2 \notin A''$,

(3) $a_1 \notin A'', a_2 \in A''$,

(4) $a_1 \notin A'', a_2 \notin A''$.

We fix disjoint subsets

$$J' := I' \cap \{3, \ldots, n\}, \qquad J'' := I'' \cap \{3, \ldots, n\}$$

of cardinality $n' - 1$, respectively $n'' - 1$. For each symbol on the left of (5.4) there are at most two nonzero terms on the right (depending on the generation condition) corresponding to the cases $a_1 \in I', a_2 \in I''$ or $a_1 \in I'', a_2 \in I'$.

In case (1), we have

$$\langle a_1, a_2, \ldots \rangle \mapsto \delta^{\mathrm{gen}}_{a_1 \in A''} \cdot \langle 0, \ldots \rangle \otimes \langle a_1, \ldots \rangle^- + \delta^{\mathrm{gen}}_{a_2 \in A''} \cdot \langle 0, \ldots \rangle \otimes \langle a_2, \ldots \rangle^-$$

and

$$\begin{aligned}&\langle a_1 - a_2, a_2, \ldots \rangle + \langle a_1, a_2 - a_1, \ldots \rangle \\ &\quad \mapsto \delta^{\mathrm{gen}}_{a_1 - a_2 \in A''} \cdot \langle 0, \ldots \rangle \otimes \langle a_1 - a_2, \ldots \rangle^- \\ &\qquad + \delta^{\mathrm{gen}}_{a_2 \in A''} \cdot \langle 0, \ldots \rangle \otimes \langle a_2, \ldots \rangle^- + \delta^{\mathrm{gen}}_{a_1 \in A''} \cdot \langle 0, \ldots \rangle \otimes \langle a_1, \ldots \rangle^- \\ &\qquad + \delta^{\mathrm{gen}}_{a_2 - a_1 \in A''} \cdot \langle 0, \ldots \rangle \otimes \langle a_2 - a_1, \ldots \rangle^-.\end{aligned}$$

The first and the last term on the right cancel by relation (5.1), and the sum of the second and the third terms is the image of $\langle a_1, a_2, \dots\rangle$.

In case (2), we have

$$\langle a_1, a_2, \dots\rangle \mapsto \delta^{\mathrm{gen}}_{a_1 \in A''} \cdot \langle a_2 \bmod A'', \dots\rangle \otimes \langle a_1, \dots\rangle^-$$

and

$$\langle a_1 - a_2, a_2, \dots\rangle + \langle a_1, a_2 - a_1, \dots\rangle \\ \mapsto \delta^{\mathrm{gen}}_{a_1 \in A''} \cdot \langle a_2 - a_1 \bmod A'', \dots\rangle \otimes \langle a_1 - a_2, \dots\rangle^-.$$

The right sides of both expressions coincide, since $a_2 = a_2 - a_1 \bmod A''$.

Case (3) is similar to case (2).

In case (4), we have $\langle a_1, a_2, \dots\rangle \mapsto 0$ and

$$\langle a_1 - a_2, a_2, \dots\rangle + \langle a_1, a_2 - a_1, \dots\rangle \\ \mapsto \delta^{\mathrm{gen}}_{a_1 - a_2 \in A''} \cdot \langle a_2 \bmod A'', \dots\rangle \otimes \langle a_1 - a_2, \dots\rangle^- \\ + \delta^{\mathrm{gen}}_{a_2 - a_1 \in A''} \cdot \langle a_1 \bmod A'', \dots\rangle \otimes \langle a_2 - a_1, \dots\rangle^-,$$

and the terms on the right cancel by (5.1). ■

An easy check shows that Δ descends to a $\mathbb{Z}$-linear homomorphism

$$\Delta^- : \mathcal{M}^-_{n'+n''}(G) \to \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G''). \tag{5.6}$$

These constructions give rise to a natural complex: denote by $\mathcal{G}_\bullet$ a flag of subgroups

$$0 = G_{\le 0} \subsetneq G_{\le 1} \subsetneq \cdots \subsetneq G_{\le r} = G,$$

and let r be its length. Consider the diagram of homomorphisms

$$\mathcal{M}^-_n(G) \rightleftarrows \bigoplus_{\substack{n_1+n_2=n \\ \mathcal{G}_\bullet \text{ of length } 2}} \mathcal{M}^-_{n_1}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}^-_{n_2}(\mathrm{gr}_2(\mathcal{G}_\bullet)) \\ \rightleftarrows \bigoplus_{\substack{n_1+n_2+n_3=n \\ \mathcal{G}_\bullet \text{ of length } 3}} \mathcal{M}^-_{n_1}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}^-_{n_2}(\mathrm{gr}_2(\mathcal{G}_\bullet)) \otimes \mathcal{M}^-_{n_3}(\mathrm{gr}_3(\mathcal{G}_\bullet)) \rightleftarrows \cdots$$

where the right arrows are the natural simplicial extensions of the co-multiplication Δ^- (given by alternating sums) and the left arrows are the corresponding extensions of the multiplication maps. We obtain two complexes

$$\mathcal{C}^{\bullet,-}(G,n), \qquad \mathcal{C}^-_\bullet(G,n),$$

with differentials d_{Δ^-} and d_{∇^-} of degree $+1$ and -1, respectively.

Theorem 12. *Let G be a finite cyclic group. Then the cohomology of both complexes*

$$\mathcal{C}^{\bullet,-}(G,n), \qquad \mathcal{C}_\bullet^-(G,n),$$

after tensoring by $\mathbb{Q}$, is concentrated in degree 0.

Proof. The assumption that G is cyclic will only be used at the last step of the proof. Let $\mathcal{M}_n^\sim(G)$ be the $\mathbb{Q}$-vector space generated by symbols

$$\langle a_1,\dots,a_n\rangle^\sim$$

satisfying the symmetry condition (S) such that $a_1,\dots,a_n$ generate A, and $a_j \neq 0$ for all j. There is a natural map of $\mathbb{Q}$-vector spaces

$$\mathcal{M}_n^\sim(G) \to \mathcal{M}_n^-(G)\otimes\mathbb{Q},$$

given by

$$\langle a_1,\dots,a_n\rangle^\sim \mapsto \langle a_1,\dots,a_n\rangle^-. \tag{5.7}$$

Consider the co-multiplication

$$\Delta^\sim : \mathcal{M}_{n'+n''}^\sim(G) \to \mathcal{M}_{n'}^\sim(G')\otimes\mathcal{M}_{n''}^\sim(G''),$$

defined by

$$\langle a_1,\dots,a_n\rangle^\sim \mapsto \sum \langle a_{I'} \bmod A''\rangle^\sim \otimes \langle a_{I''}\rangle^\sim, \tag{5.8}$$

where $I', I'' \subsetneq I$ are nonempty subsets such that

- $I' \sqcup I'' = \{1,\dots,n\}$,
- $I'' = \{i \mid a_i \in A''$ with $\sum_{i\in I''}\mathbb{Z}a_i = A''\}$.

Similarly, we have a multiplication map

$$\nabla^\sim : \mathcal{M}_{n'}^\sim(G')\otimes\mathcal{M}_{n''}^\sim(G'') \to \mathcal{M}_{n'+n''}^\sim(G')$$

defined by formulas similar to (5.3). We obtain two complexes, as above:

$$\mathcal{C}^{\bullet,\sim}(G,n), \qquad \mathcal{C}_\bullet^\sim(G,n),$$

with the corresponding differentials $d_{\nabla^\sim}$ and $d_{\Delta^\sim}$. We have natural surjective homomorphisms of complexes

$$\mathcal{C}^{\bullet,\sim}(G,n) \twoheadrightarrow \mathcal{C}^{\bullet,-}(G,n)\otimes\mathbb{Q}, \qquad \mathcal{C}_\bullet^\sim(G,n) \twoheadrightarrow \mathcal{C}_\bullet^-(G,n)\otimes\mathbb{Q},$$

induced by the maps

$$\langle a_1,\dots,a_{n_i}\rangle^\sim \mapsto \langle a_1,\dots,a_{n_i}\rangle^-.$$

Clearly, these maps are compatible with the respective differentials; here we use the fact that the symbol $\langle a_1,\dots,a_{n_i}\rangle^-$ vanishes, modulo torsion, if at least one a_j is zero.

Consider the following statements:

(1) $H^{>0}(\mathcal{C}^{\bullet,\sim}(G,n)) = 0$.

(2) The operator

$$\mathbf{\Delta}^{\sim} = d_{\Delta^{\sim}} \circ d_{\nabla^{\sim}} + d_{\nabla^{\sim}} \circ d_{\Delta^{\sim}}$$

is invertible in degree > 0.

(3) The operator

$$\mathbf{\Delta}^{-} = d_{\Delta^{-}} \circ d_{\nabla^{-}} + d_{\nabla^{-}} \circ d_{\Delta^{-}}$$

is invertible in degree > 0.

(4) $H^{>0}(\mathcal{C}^{\bullet,-}(G,n)) = 0$, $H_{>0}(\mathcal{C}^{-}_{\bullet}(G,n)) = 0$.

We have a sequence of implications

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

Indeed:

- (1) and (2) are equivalent, because $d_{\nabla^{\sim}}$ and $d_{\Delta^{\sim}}$ are adjoint with respect to a positive-definite quadratic form, given by the identity matrix in the natural basis.
- (2)$\Rightarrow$(3), since we have a surjective homomorphism of complexes.
- (3)$\Rightarrow$(4), since the Laplacian $\mathbf{\Delta}^{-}$ is an endomorphism of both complexes

$$\mathcal{C}^{\bullet,-}(G,n) \otimes \mathbb{Q}, \quad \mathcal{C}^{-}_{\bullet}(G,n) \otimes \mathbb{Q},$$

  which is homotopic to zero for both complexes. The invertibility of this endomorphism in degrees > 0 implies invertibility in cohomology, in degrees > 0, and hence implies vanishing of cohomology in those degrees.

Hence it suffices to prove statement (1). For this, we will construct a homotopy

$$h : C^{\sim}_{j}(G,n) \to C^{\sim}_{j-1}(G,n), \quad j = 1, 2, \dots,$$

such that

$$\mathbf{\Delta}^{\sim}_{h} := h \circ d_{\Delta^{\sim}} + d_{\Delta^{\sim}} \circ h \tag{5.9}$$

is invertible in degrees > 0.

Recall that

$$C^{\sim}_{j}(G,n), \quad j \geq 0,$$

is a direct sum of terms labeled by flags of subgroups

$$0 = G_{\leq 0} \subsetneq G_{\leq 1} \subsetneq \cdots \subsetneq G_{r} = G, \quad r = j + 1.$$

Passing to characters, we obtain a chain of surjective homomorphisms

$$0 = A_{\leq 0} \overset{\neq}{\twoheadleftarrow} A_{\leq 1} \overset{\neq}{\twoheadleftarrow} \cdots \overset{\neq}{\twoheadleftarrow} A_{\leq r} = A.$$

We define h as follows:

$$\mathcal{M}^{\sim}_{n_1}(A_{\le 1}) \otimes \mathcal{M}^{\sim}_{n_2}(\mathrm{Ker}(A_{\le 2} \twoheadrightarrow A_{\le 1})) \otimes \cdots \to \mathcal{M}^{\sim}_{n_1+n_2}(A_{\le 2}) \otimes \cdots$$

acting as the identity on the omitted factors, and as

$$\langle a_1, \dots, a_{n_1}\rangle^{\sim} \otimes \langle b_1, \dots, b_{n_2}\rangle^{\sim} \mapsto \langle \psi(a_1), \dots, \psi(a_{n_1}), b_1, \dots, b_{n_2}\rangle^{\sim},$$

on the first two terms, where $\psi : A_{\le 1} \to A_{\le 2}$ is a section of the natural surjection, defined below.

We now use the assumption that G, and hence all $A_{\le j}$, are *cyclic*. Write

$$G = \mathbb{Z}/N\mathbb{Z} = \prod_i \mathbb{Z}/p_i^{k_i}\mathbb{Z},$$

and identify

$$\mathbb{Z}/p_i^{k_i}\mathbb{Z} = \{0, \dots, p_i - 1\}^{k_i},$$

by regarding the sequence of digits in base p_i. In this setup, there is a natural lift

$$\psi : A_{\le 1} \to A_{\le 2}$$

by *adding* zeros to the corresponding sequences of digits, for all p_i. Note that the differential $d_{\Delta^{\sim}}$ is given by *removing* digits in this presentation. The operator (see (5.9)) $\mathbf{\Delta}_h^{\sim} - \mathrm{Id}$ acting on $C^{j,\sim}(G, n)$ for $j \ge 1$ is *nilpotent*, since it strictly increases the number of zeros in our collection of digit sequences. Therefore, $\mathbf{\Delta}_h^{\sim}$ is invertible in degrees ≥ 1. ■

Remark 13. For *noncyclic* G, the structure of cohomology of $\mathcal{C}^{\bullet,-}$ is more complicated. Let $G = (\mathbb{Z}/p\mathbb{Z})^2$. In this case, the complex is

$$\mathcal{M}_2^-(G) \to \bigoplus_{p+1 \text{ copies}} \mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z}) \otimes \mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z}).$$

We claim that this map fails to be surjective for $p \ge 3$. Indeed, it suffices to produce a nontrivial functional on the right side, vanishing on the image of the differential d_{Δ^-}. We can describe

$$\mathrm{Coker}(d_{\Delta^-}) \otimes \mathbb{Q}$$

as the space of $\mathbb{Q}$-valued functions f on pairs of linearly independent vectors $a_1, a_2 \in (\mathbb{Z}/p\mathbb{Z})^2$ such that

- $f(a_1, a_2) = -f(-a_1, a_2) = -f(a_1, -a_2) = f(a_1, a_2 + \lambda a_1)$ for all $\lambda \in \mathbb{Z}/p\mathbb{Z}$,
- $f(a_1, a_2) + f(a_2, a_1) = 0$.

The first property describes functionals on $C^{1,-}(G, 2)$ and the second condition means that f is in $\mathrm{Ker}(d_{\Delta^-})$. Here we do not use the defining relation (M) for $\mathcal{M}_2(G)$. Solutions of this system of functional equations are given by maps

$$f(a_1, a_2) = g(a_1 \wedge a_2),$$

where g is any map

$$g : (\mathbb{Z}/p\mathbb{Z})^\times = \bigwedge^2(\mathbb{Z}/p\mathbb{Z})^2 \setminus 0 \to \mathbb{Q},$$

which is odd, i.e., $g(-\lambda) = -g(\lambda)$ for all λ. Hence

$$H^1(\mathcal{C}^{\bullet,-}(G,2)) \otimes \mathbb{Q} \simeq \mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q} = \mathbb{Q}^{(p-1)/2}.$$

We define

$$\mathcal{M}^-_{n,\mathrm{prim}}(G) := \mathrm{Ker}\Bigg(\mathcal{M}_n^-(G) \to \bigoplus_{\substack{n'+n''=n\\ n',n''\geq 1\\ 0\subsetneq G'\subsetneq G}} \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G/G')\Bigg); \tag{5.10}$$

this is the cohomology of the complex $\mathcal{C}^{\bullet,-}(G,n)$ in degree 0, with differential d_Δ. We define

$$\mathcal{M}^-_{n,\mathrm{coprim}}(G) := \mathrm{Coker}\Bigg(\mathcal{M}_n^-(G) \leftarrow \bigoplus_{\substack{n'+n''=n\\ n',n''\geq 1\\ 0\subsetneq G'\subsetneq G}} \mathcal{M}^-_{n'}(G') \otimes \mathcal{M}^-_{n''}(G/G')\Bigg); \tag{5.11}$$

this is the cohomology of the complex $\mathcal{C}^-_\bullet(G,n)$ in degree 0, with differential d_∇. Theorem 12 implies that, for G cyclic, we have

$$\dim(\mathcal{M}^-_{n,\mathrm{prim}}(G) \otimes \mathbb{Q}) = \dim(\mathcal{M}^-_{n,\mathrm{coprim}}(G) \otimes \mathbb{Q}) \tag{5.12}$$

and

$$\dim(\mathcal{M}^-_n(G) \otimes \mathbb{Q}) = \sum_r \sum_{\substack{n_1+\cdots+n_r=n\\ \mathcal{G}_\bullet \text{ of length } r}} \prod_{i=1}^r \dim(\mathcal{M}^-_{n_i,\mathrm{prim}}(\mathrm{gr}_i(\mathcal{G}_\bullet)) \otimes \mathbb{Q}). \tag{5.13}$$

Using ∇^-, we can obtain a homomorphism of vector spaces

$$\mathcal{M}^-_{n_1,\mathrm{prim}}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \cdots \otimes \mathcal{M}^-_{n_r,\mathrm{prim}}(\mathrm{gr}_r(\mathcal{G}_\bullet)) \otimes \mathbb{Q} \to \mathcal{M}^-_n(G) \otimes \mathbb{Q}.$$

Similarly, using Δ^-, we obtain a homomorphism of $\mathbb{Q}$-vector spaces

$$\mathcal{M}^-_{n_1,\mathrm{coprim}}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \cdots \otimes \mathcal{M}^-_{n_r,\mathrm{coprim}}(\mathrm{gr}_r(\mathcal{G}_\bullet)) \otimes \mathbb{Q} \leftarrow \mathcal{M}^-_n(G) \otimes \mathbb{Q}.$$

In view of the numerical identities (5.12) and (5.13) it is tempting to guess that the above maps are isomorphisms of $\mathbb{Q}$-vector spaces.

Now consider the diagram of homomorphisms

$$\mathcal{M}_n(G) \to \bigoplus_{\substack{n_1+n_2=n\\ \mathcal{G}_\bullet \text{ of length } 2}} \mathcal{M}_{n_1}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}^-_{n_2}(\mathrm{gr}_2(\mathcal{G}_\bullet))$$

$$\to \bigoplus_{\substack{n_1+n_2+n_3=n\\ \mathcal{G}_\bullet \text{ of length } 3}} \mathcal{M}_{n_1}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \mathcal{M}^-_{n_2}(\mathrm{gr}_2(\mathcal{G}_\bullet)) \otimes \mathcal{M}^-_{n_3}(\mathrm{gr}_3(\mathcal{G}_\bullet)) \to \cdots$$

where

- $\mathcal{G}_\bullet$ is a flag of subgroups of type

$$0 = G_{\leq 0} \subseteq G_{\leq 1} \subsetneq \cdots \subsetneq G_{\leq r} = G, \quad r \geq 1,$$

  with strict inclusions, except in the first step;
- in each term, the leftmost factor is the full group, and not the quotient by the relation (5.1).

Here the differential uses *both* maps Δ and Δ^-. Again, this is a complex, which we denote by $\mathcal{C}^\bullet(G,n)$; notice that here we do not have a dual differential in the other direction.

Theorem 14. *Let G be a finite cyclic group. Then the cohomology of the complex $\mathcal{C}^\bullet(G,n)$, after tensoring by $\mathbb{Q}$, is concentrated in degree* 0.

Proof. The proof is similar to the one given for Theorem 12. The key observation is that for finite cyclic groups, the projection μ^- defined in (5.2) admits a section

$$\nu : \mathcal{M}_n^-(G) \to \mathcal{M}_n(G), \tag{5.14}$$

which on symbols is given by the formula

$$\langle a_1, \ldots, a_n \rangle^- \mapsto \sum_{\varepsilon_1,\ldots,\varepsilon_n} (-1)^{\varepsilon_1\cdots\varepsilon_n} \langle \varepsilon_1 a_1, \ldots, \varepsilon_n a_n \rangle, \tag{5.15}$$

where $\varepsilon_i \in \{+1, -1\}$, and the sum is over all possibilities.

For $n = 1$, this is clearly compatible. To check the defining relations in general, it suffices to consider the case $n = 2$. For

$$a, b \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a, b, N) = 1,$$

(5.15) translates to

$$\langle a, b \rangle^- \mapsto \langle a, b \rangle + \langle -a, -b \rangle - \langle -a, b \rangle - \langle a, -b \rangle. \tag{5.16}$$

We need to verify that the relation

$$\langle a, b \rangle^- = \langle a, b - a \rangle^- + \langle a - b, b \rangle^-$$

is mapped to a relation in $\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})$. We write out the relations for each term in (5.16):

$$\begin{aligned}\langle a, b \rangle + \langle -a, -b \rangle - \langle -a, b \rangle - \langle a, -b \rangle \qquad & \\ \stackrel{?}{=} \langle a, b-a \rangle + \langle -a, a-b \rangle - \langle -a, b-a \rangle - \langle a, a-b \rangle & \\ + \langle a-b, b \rangle + \langle b-a, -b \rangle - \langle b-a, b \rangle - \langle a-b, -b \rangle. &\end{aligned}$$

The first terms on each line (and the second terms, considered separately) give a relation in $\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})$. It suffices to check

$$-\langle -a, b \rangle - \langle a, -b \rangle \stackrel{?}{=} -\langle -a, b-a \rangle - \langle a, a-b \rangle - \langle b-a, b \rangle - \langle a-b, -b \rangle.$$

Replacing $a \mapsto -a$, we have to show that

$$\langle a, b\rangle + \langle -a, -b\rangle \stackrel{?}{=} \langle a, b+a\rangle + \langle -a, -a-b\rangle + \langle b+a, b\rangle + \langle -a-b, -b\rangle.$$

Using the relations

$$\langle a, b+a\rangle = \langle a, b\rangle + \langle -b, b+a\rangle, \quad \langle -a, -b-a\rangle = \langle -a, -b\rangle + \langle b, -b-a\rangle,$$

we are reduced to showing

$$\begin{aligned}\boldsymbol{\delta}(a+b, b) &:= \langle a+b, b\rangle + \langle -(a+b), b\rangle + \langle a+b, -b\rangle + \langle -(a+b), -b\rangle \\ &\stackrel{?}{=} 0 \in \mathcal{M}_2(\mathbb{Z}/N\mathbb{Z}),\end{aligned}$$

i.e.,

$$\boldsymbol{\delta}(a, b) \stackrel{?}{=} 0 \in \mathcal{M}_2(\mathbb{Z}/N\mathbb{Z}).$$

Note that

$$\boldsymbol{\delta}(a+b, b) = \boldsymbol{\delta}(a+b, a), \qquad \boldsymbol{\delta}(a, b) = \boldsymbol{\delta}(-a, b) = \boldsymbol{\delta}(b, a).$$

It follows that $\boldsymbol{\delta}$ is invariant under the matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which generate $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, so that $\boldsymbol{\delta}(a, b)$ is constant. Considering the average and applying the defining relation to each term we obtain

$$\boldsymbol{S} := \sum_{a,b} \boldsymbol{\delta}(a, b) = 2\boldsymbol{S}, \quad \text{thus} \quad \boldsymbol{S} = 0.$$

To prove Theorem 14 we need to show that

$$\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \to \bigoplus_{N=N'N''} \mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}^-_{n''}(\mathbb{Z}/N''\mathbb{Z}), \quad n = n' + n'',$$

is surjective, where the sum is over all exact sequences

$$0 \to \mathbb{Z}/N''\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N'\mathbb{Z} \to 0, \quad N = N'N'',\ N \geq 2,$$

of finite cyclic groups. We now use the *inverse* (after tensoring by $\mathbb{Q}$), as discussed above:

$$\tilde{\nabla} : \mathcal{M}_{n'}(\mathbb{Z}/N'\mathbb{Z}) \otimes \mathcal{M}^-_{n''}(\mathbb{Z}/N''\mathbb{Z}) \to \mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}), \quad n = n' + n'',$$

which on generators is given by

$$\langle a'_1, \ldots, a'_{n'}\rangle \otimes \langle b_1, \ldots, b_{n''}\rangle^- \mapsto \sum_{\substack{\text{all lifts} \\ \varepsilon_1,\ldots,\varepsilon_{n''}}} (-1)^{\varepsilon_1 \cdots \varepsilon_{n''}} \langle a_1, \ldots a_{n'}, \varepsilon_1 b_1, \ldots, \varepsilon_{n''} b_{n''}\rangle,$$

where the sum is over all lifts a_i to $\mathbb{Z}/N\mathbb{Z}$ of $a'_i \in \mathbb{Z}/N'\mathbb{Z}$ and all possibilities for $\varepsilon_j \in \{+1, -1\}$ (see the definition of ν, (5.14)). This is compatible with the defining equations. ∎

We now define

$$\mathcal{M}_{n,\mathrm{prim}}(G) = \mathrm{Ker}\Big(\mathcal{M}_n(G) \to \bigoplus_{\substack{n'+n''=n\\ n',n''\geq 1\\ 0\subseteq G' \subsetneq G}} \mathcal{M}_{n'}(G') \otimes \mathcal{M}^-_{n''}(G/G')\Big); \tag{5.17}$$

this is the cohomology of the complex in degree 0; note that the inclusion G' could be trivial. We have

$$\mathcal{M}_1(G) = \mathcal{M}_{1,\mathrm{prim}}(G)$$

for all G; when $G = 1 = \mathbb{Z}/1\mathbb{Z}$ we have

$$\mathcal{M}_1(1) = \mathbb{Z}, \quad \mathcal{M}_n(1) = \mathcal{M}_{n,\mathrm{prim}}(1) = 0 \quad \text{for } n \geq 2.$$

Theorem 14 implies that there is a *noncanonical* isomorphism

$$\mathcal{M}_n(G) \otimes \mathbb{Q} \simeq \bigoplus_r \bigoplus_{\substack{n_1+\cdots+n_r=n\\ \mathcal{G}_\bullet \text{ of length } r}} \mathcal{M}_{n_1,\mathrm{prim}}(\mathrm{gr}_1(\mathcal{G}_\bullet)) \otimes \cdots \otimes \mathcal{M}^-_{n_r,\mathrm{prim}}(\mathrm{gr}_r(\mathcal{G}_\bullet)) \otimes \mathbb{Q}.$$

Computer experiments (see Section 8) suggest that, for all $N \geq 1$:

- $\mathcal{M}_{2,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \mathcal{M}^-_{2,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$, and this is equal to the dimension of the space of cusp forms of weight 2 for $\Gamma_1(N)$ – we will discuss this in Section 11;
- $\mathcal{M}_{3,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \mathcal{M}^-_{3,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$, and this is equal to the number of certain cuspidal automorphic representations for a congruence subgroup of $\mathrm{GL}_3(\mathbb{Z})$, generated by a vector invariant under a congruence subgroup,
- $\mathcal{M}_{n,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = \mathcal{M}^-_{n,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} = 0$ for $n \geq 4$.

Theorems 12 and 14 allow us to compute $\mathbb{Q}$-ranks of $\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$ using

- the Euler function:

$$\dim(\mathcal{M}_{1,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \phi(N), \quad N \geq 1,$$

$$\dim(\mathcal{M}^-_{1,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \begin{cases} 0, & N = 2,\\ \phi(N)/2, & N \geq 3.\end{cases}$$

- the well-known dimensions of the spaces of cusp forms for $\Gamma_1(N)$, which are given by closed formulas in N, e.g.,

N	…	11	12	13	14	15	16	17	18	19	20	…	180	181
	0	1	0	2	1	1	2	5	2	7	3	…	705	1276

- the somewhat mysterious dimensions in the case $n = 3$, e.g.

N	43	51	52	59	63	67	68	72	73	75	…	239	240
	1	1	1	1	2	2	1	1	8	4	…	3	22

Example 15. Theorem 14 implies that

$$\dim(\mathcal{M}_n(\mathbb{Z}/3^{n-1}\mathbb{Z}) \otimes \mathbb{Q}) = 1, \qquad n \geq 1,$$

coming from the term

$$\mathcal{M}_{1,\mathrm{prim}}(\mathbb{Z}/1\mathbb{Z}) \otimes \underbrace{\mathcal{M}^-_{1,\mathrm{prim}}(\mathbb{Z}/3\mathbb{Z}) \otimes \cdots \otimes \mathcal{M}^-_{1,\mathrm{prim}}(\mathbb{Z}/3\mathbb{Z})}_{n-1 \text{ times}}.$$

Directly, we see that the co-multiplications Δ, Δ^- give homomorphisms

$$\mathrm{Hom}(\mathcal{M}^{(-)}_{n_1}(G), \mathbb{Q}) \otimes \mathrm{Hom}(\mathcal{M}^{(-)}_{n_2}(G), \mathbb{Q}) \to \mathrm{Hom}(\mathcal{M}^{(-)}_{n}(G), \mathbb{Q}).$$

Using explicit nonzero elements

$$(\langle 0 \rangle \mapsto 1) \in \mathrm{Hom}(\mathcal{M}_1(\mathbb{Z}/1\mathbb{Z}), \mathbb{Q}),$$

$$(\langle \pm 1 \bmod 3 \rangle^- \mapsto \pm 1) \in \mathrm{Hom}(\mathcal{M}^-_1(\mathbb{Z}/3\mathbb{Z}), \mathbb{Q}),$$

we obtain an explicit functional on $\mathcal{M}_n(\mathbb{Z}/3^{n-1}\mathbb{Z})$ which maps

$$\langle 1 \bmod 3^{n-1}, 3 \bmod 3^{n-1}, \ldots, 3^{n-1} \bmod 3^{n-1} \rangle \mapsto 1,$$

hence is nonzero. In particular, we have

$$\dim(\mathcal{M}_n(\mathbb{Z}/3^{n-1}\mathbb{Z}) \otimes \mathbb{Q}) \geq 1.$$

Similarly, one can show that

$$\dim(\mathcal{M}_n(\mathbb{Z}/2^{n-1}\mathbb{Z}) \otimes \mathbb{F}_2) \geq 1,$$

Thus we obtain explicit nontrivial invariants for equivariant birational actions of $G = \mathbb{Z}/3^{n-1}\mathbb{Z}$ on n-dimensional varieties. Surprisingly, experiments show that the nontrivial invariant in $\mathrm{Hom}(\mathcal{M}_n(\mathbb{Z}/2^{n-1}\mathbb{Z}), \mathbb{F}_2)$ lifts to the trivial element in $\mathrm{Hom}(\mathcal{B}_n(\mathbb{Z}/2^{n-1}\mathbb{Z}), \mathbb{F}_2)$ for $n = 2, 3, 4, 5$.

Experiments suggest that

$$\dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = 0 \quad \text{for all } N < 3^{n-1}, \tag{5.18}$$

$$\dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2) = 0 \quad \text{for all } N < 2^{n-1}. \tag{5.19}$$

Moreover,

$$\dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2) = 0 \quad \text{for all } N < \begin{cases} 2^n - 1, & n = 2, 3, \\ 2^{n-1}, & n \geq 4. \end{cases}$$

6. Hecke operators

In this section, we define analogs of Hecke operators on $\mathcal{M}_n(G)$. Fix a prime ℓ not dividing $\#G$, and an integer $1 \le r \le n-1$. Put

$$T_{\ell,r}(\psi(\mathbf{L},\chi,\Lambda)) := \sum_{\mathbf{L}\subset\mathbf{L}'\subset\mathbf{L}\otimes\mathbb{R},\, \mathbf{L}'/\mathbf{L}\simeq(\mathbb{Z}/\ell\mathbb{Z})^r} \psi(\mathbf{L}',\chi,\Lambda), \tag{6.1}$$

where χ is now interpreted as an element of $\mathbf{L}'\otimes A$, via the inclusion $\mathbf{L}\otimes A\subset\mathbf{L}'\otimes A$; the surjectivity property for $\chi\in\mathbf{L}'\otimes A$ follows from the surjectivity of $\chi\in\mathbf{L}\otimes A$ and the assumption on coprimality of ℓ and the order of G.

Proposition 16. *The Hecke operators $T_{\ell,r}$ are well-defined on $\mathcal{M}_n(G)$, and commute with each other.*

Proof. Follows from the additivity of (4.4) and (6.1). ■

Example 17. We consider the case $n=2$ and $G=\mathbb{Z}/N\mathbb{Z}\simeq A$. Then $\mathcal{M}_n(G)$ is generated by

$$\langle a_1,a_2\rangle,\qquad a_1,a_2\in\mathbb{Z}/N\mathbb{Z},\qquad \gcd(a_1,a_2,N)=1,$$

such that

- $\langle a_1,a_2\rangle=\langle a_2,a_1\rangle$,
- $\langle a_1,a_2\rangle=\langle a_1,a_2-a_1\rangle+\langle a_1-a_2,a_2\rangle$ for all a_1,a_2.

We write down an example of a Hecke operator on $\mathcal{M}_2(G)$. For each ℓ coprime to N we have only one Hecke operator $T_\ell=T_{\ell,1}$.

Assume that N is odd and $\ell=2$. Let

$$\mathbf{L}=\mathbb{Z}^2,\qquad \chi=(1,0)\otimes a_1+(0,1)\otimes a_2,\qquad \Lambda=\mathbb{R}^2_{\ge 0},$$

the standard octant. There are three overlattices of $\mathbf{L}$ of index 2, corresponding to the three elements of $\mathbb{P}^1(\mathbb{F}_2)$:

- $\mathbf{L}'_0 := \mathbb{Z}\cdot\left(\frac{1}{2},0\right)+\mathbb{Z}\cdot(0,1)$,
- $\mathbf{L}'_1 := \mathbb{Z}\cdot\left(\frac{1}{2},\frac{1}{2}\right)+\mathbb{Z}\cdot(0,1)=\mathbb{Z}\cdot\left(\frac{1}{2},\frac{1}{2}\right)+\mathbb{Z}\cdot(1,0)$,
- $\mathbf{L}'_\infty := \mathbb{Z}\cdot\left(0,\frac{1}{2}\right)+\mathbb{Z}\cdot(1,0)$.

The corresponding cones in the first and third cases are basic simplicial, whereas in the second case the cone is not basic and can be decomposed in the union of two basic simplicial cones with respect to $\mathbf{L}'_1$:

$$\Lambda=\Lambda_1\cup\Lambda_2,$$
$$\Lambda_1=\mathbb{R}_{\ge 0}\cdot(1,0)+\mathbb{R}_{\ge 0}\cdot(1,1),\qquad \Lambda_2=\mathbb{R}_{\ge 0}\cdot(1,1)+\mathbb{R}_{\ge 0}\cdot(0,1).$$

Therefore,

$$T_2(\langle a_1,a_2\rangle)=\langle 2a_1,a_2\rangle+(\langle a_1-a_2,2a_2\rangle+\langle 2a_1,a_2-a_1\rangle)+\langle a_1,2a_2\rangle.$$

The middle term follows from the equalities

$$e_1 \otimes a_1 + e_2 \otimes a_2 = e_1 \otimes (a_1 - a_2) + \frac{e_1 + e_2}{2} \otimes 2a_2 = \frac{e_1 + e_2}{2} \otimes 2a_1 + e_2 \otimes (a_2 - a_1).$$

We leave it as an exercise to write down a similar formula for the action of T_3 on $\mathcal{M}_2(G)$ and T_2 on $\mathcal{M}_3(G)$.

To define Hecke operators $T^*_{\ell,r}$ in the co-vector version, we consider sublattices $\mathbf{L}' \subset \mathbf{L}$ of index ℓ^r such that the quotient is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^r$. In particular, $T^*_2 = T^*_{2,1}$ on $\mathcal{M}^*_2(G)$ is given by

$$T^*_2([a_1, a_2]^*) = [2a_1, a_2]^* + [2a_1, a_1 + a_2]^* + [a_1 + a_2, 2a_2]^* + [a_1, 2a_2]^*,$$

and $T^*_{2,1}$ on $\mathcal{M}_3(G)$ by

$$\begin{aligned} T^*_{2,1}([a_1, a_2, a_3]^*) = {} & [2a_1, a_2, a_3]^* + [a_1, 2a_2, a_3]^* + [a_1, a_2, 2a_3]^* \\ & + [2a_1, a_1 + a_2, a_3]^* + [a_1 + a_2, 2a_2, a_3]^* + [a_1, 2a_2, a_2 + a_3]^* \\ & + [a_1, a_2 + a_3, 2a_3]^* + [2a_1, a_2, a_1 + a_3]^* + [a_1 + a_3, a_2, a_3]^* \\ & + [2a_1, a_1 + a_2, a_1 + a_3]^* + [a_1 + a_2, 2a_2, a_2 + a_3]^* \\ & + [a_1 + a_3, a_2 + a_3, 2a_3]^* + [a_1 + a_2, a_2 + a_3, a_1 + a_3]^*. \end{aligned}$$

Remark 18. The homomorphisms Δ and ∇ are compatible with the action of Hecke operators; in particular, the groups $\mathcal{M}_{n,\mathrm{prim}}(G)^-$ defined in (5.11) are preserved by Hecke operators.

7. Variants

Consider an irreducible algebraic representation $\rho_\lambda : \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{Aut}(\mathsf{V}_\lambda)$ with highest weight

$$\lambda = (\lambda_1 \le \cdots \le \lambda_n), \quad \lambda_i \in \mathbb{Z}.$$

The representation ρ_λ defines a functor from the groupoid of n-dimensional $\mathbb{Q}$-vector spaces to the category $\mathrm{Vect}_\mathbb{Q}$ of all $\mathbb{Q}$-vector spaces, which we denote by the same letter. In particular, for any lattice $\mathbf{L}$ of rank n we can speak of

$$\rho_\lambda(\mathbf{L} \otimes \mathbb{Q}) \in \mathrm{Vect}_\mathbb{Q}.$$

For example, if ρ_λ is the m-th symmetric power $\mathrm{Sym}^m(V)$ of the standard representation, i.e., $\lambda = (0, \dots, 0, m)$, then

$$\rho_\lambda(\mathbf{L} \otimes \mathbb{Q}) = \mathrm{Sym}^m(\mathbf{L} \otimes \mathbb{Q}).$$

Consider the $\mathbb{Q}$-vector space $\mathcal{M}_n(G, \rho_\lambda)$ generated by symbols

$$\psi(\mathbf{L}, \chi, \Lambda, v),$$

for isomorphism classes of quadruples, where $\mathbf{L}$, χ, Λ are as in Section 6 and

$$v \in \rho_\lambda(\mathbf{L} \otimes \mathbb{Q}),$$

subject to relations

- $\psi(\mathbf{L}, \chi, \Lambda, v_1 + v_2) = \psi(\mathbf{L}, \chi, \Lambda, v_1) + \psi(\mathbf{L}, \chi, \Lambda, v_2)$,
- $\psi(\mathbf{L}, \chi, \Lambda, v) = \sum_{i=1}^k \psi(\mathbf{L}, \chi, \Lambda_i, v)$ for any decomposition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$.

Here, one can assume that the subcones Λ_i are basic simplicial and that the decomposition is standard, as in Section 6, or simply that Λ_i are finitely generated rational subcones of full dimension, with disjoint interiors. The action of Hecke operators on $\mathcal{M}_n(G, \rho_\lambda)$ is defined as in (6.1).

The co-vector version of this construction is straightforward.

Remark 19. We expect that for $n = 2$, $G = \mathbb{Z}/N\mathbb{Z}$, and ρ_λ given by the m-th symmetric power, the $\mathbb{Q}$-vector spaces $\mathcal{M}_n(G, \rho_\lambda)$, endowed with the action of the Hecke operators $T_{\ell,r}$, are related to modular forms of weight $m + 2$ for the congruence subgroup $\Gamma_1(N)$.

8. Experiments

Here we present results of numerical experiments, performed using a fast linear algebra solver [7]. We computed the dimensions of $\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z})$ and $\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$ over $\mathbb{Q}$ and various finite fields. The size of the (very sparse) matrices grows as $\sim N^n$. For example, for $n = 5$ and $N = 81$, the part of the constraints corresponding to $k = 2$ in (B) or (M) gives $\sim 3 \cdot 10^8$ equations in $\sim 3 \cdot 10^7$ variables, with $\sim 10^9$ nonzero coefficients. This overdetermined system has a unique (up to scalar) nontrivial solution in $\mathbb{Q}$. The calculation took about four hours.

Numerically, we found:

- For p a prime,

$$\dim(\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = \frac{p^2 - 1}{24} + 1 = \frac{p^2 + 23}{24},$$

  while the difference

$$\Delta_{2,\ell}(\mathbb{Z}/p\mathbb{Z}) := \dim(\mathcal{B}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{F}_\ell) - \frac{p^2 + 23}{24}$$

  varies significantly; there are frequent jumps when $\ell \mid (p \pm 1)$, e.g.,

$$\Delta_{2,31}(\mathbb{Z}/61\mathbb{Z}) = 1.$$

- For p a prime,

$$\Delta_{3,\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) := \dim(\mathcal{B}_3(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) - \frac{(p-5)(p-7)}{24} = 0$$

  for all primes up to 41, but

$$\Delta_{3,\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) = 1 \quad \text{for } p = 43, 59, \ldots.$$

- The difference

$$\Delta_{3,\ell}(\mathbb{Z}/p\mathbb{Z}) := \dim(\mathcal{B}_3(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{F}_\ell) - \frac{(p-5)(p-7)}{24}$$

also jumps for many $\ell \mid (p \pm 1)$.

- For all primes p up to 41 we have $\dim(\mathcal{B}_4(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = 0$, but

$$\dim(\mathcal{B}_4(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = 1 \quad \text{for } p = 43, 59, \ldots.$$

Next we present a more systematic table of dimensions. All dimensions, for $\mathbb{Q}$-coefficients, are compatible with the conjectures in Section 5. The items in bold indicate the smallest N for which the rank is positive.

- $\dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q})$ for $n = 2, 3$:

N	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$n=2$	0	**1**	1	2	2	3	3	5	4	6	7	8	7	13	10	13	12
$n=3$	0	0	0	0	0	0	0	**1**	0	1	2	2	1	5	3	5	5

N	19	20	21	22	23	24	25	26	27	28	29	...	180	181
$n=2$	16	17	23	16	23	23	30	22	34	31	36	...	989	1366
$n=3$	7	7	11	7	12	13	16	12	21	17	22	...	1740	1276

- $\dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q})$ for $n = 4$:

N	27	28	29	30	31	32	33	34	35	36	...	105	106	107
n=4	**1**	0	0	0	0	0	2	0	0	3	...	114	0	3

- $\dim(\mathcal{M}^-_{4,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = 0$ for $N \le 242$.
- $\dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q})$ for $n = 5$:

N	$\cdots \le 80$	81	82
n=5	0	**1**	0

- $\dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2)$ and $\dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2)$ for $n = 2, 3, 4, 5$:

N	2	3	4	5	6	7	8	...	16	...	32
$\mathcal{B}_2$	0	**1**	1	2	3	4	4	...	13	...	44
$\mathcal{M}_2$	**1**	2	3	5	5	8	8	...	21	...	60
$\mathcal{B}_3$	0	0	0	0	0	**1**	1	...	8	...	43
$\mathcal{M}_3$	0	0	**1**	1	3	2	5	...	21	...	87
$\mathcal{B}_4$	0	0	0	0	0	0	0	...	**1**	...	12
$\mathcal{M}_4$	0	0	0	0	0	0	**1**	...	9	...	55
$\mathcal{B}_5$	0	0	0	0	0	0	0	...	0	...	**1**
$\mathcal{M}_5$	0	0	0	0	0	0	0	...	**1**	...	13

Equations (B) in Section 1 are labeled by pairs of positive integers n, k, where n is the dimension and $2 \leq k \leq n$. Computer experiments show a remarkable property of our equations: for given n *and* k, the highly overdetermined subsystem of linear equations (B) or (M) (and assuming implicitly (S), the symmetry property) has a very large space of solutions, usually much larger than the whole system for given n, which is the conjunction of subsystems for $k = 2, \dots, n$ (or just the subsystem for $k = 2$, see Lemma 10 in Section 4). We have no explanation for this striking fact. There are no obvious actions of Hecke operators on the solution spaces n, k individually, for $k > 2$, and it is very surprising that the highly overdetermined systems admit any nontrivial solution at all.

- $\mathbb{Q}$-ranks of partial systems $\mathcal{B}_{n,k}$ and $\mathcal{M}_{n,k}$ for $k \geq 3$, and for some primes and composite numbers N:

N	2	3	5	7	11	13	17	19	23	9	12	27	36
$\mathcal{B}_{3,3}$	**1**	2	4	6	12	15	22	27	35	11	36	87	468
$\mathcal{M}_{3,3}$	0	**1**	3	3	7	10	15	18	24	9	40	78	480
$\mathcal{B}_{4,3}$	0	0	0	0	0	0	0	0	0	0	**1**	5	63
$\mathcal{M}_{4,3}$	0	0	0	0	1	2	5	7	12	**1**	5	24	121
$\mathcal{B}_{4,4}$	0	**3**	6	9	17	20	29	35	45	42	101	620	2515
$\mathcal{M}_{4,4}$	0	**3**	2	3	7	8	13	17	23	45	123	649	2716
$\mathcal{B}_{5,3}$	0	0	0	0	0	0	0	0	0	0	0	0	**1**
$\mathcal{M}_{5,3}$	0	0	0	0	0	0	0	0	0	0	0	**1**	7
$\mathcal{B}_{5,4}$	0	0	0	0	0	0	0	0	0	**3**	4	55	267
$\mathcal{M}_{5,4}$	0	0	0	0	1	2	5	7	12	**5**	12	122	?
$\mathcal{B}_{5,5}$	**1**	3	9	12	22	26	37	44	56	30	161	572	?
$\mathcal{M}_{5,5}$	0	**1**	3	3	7	8	13	17	23	17	212	?	?

Part II

9. Algebraic versions of automorphic forms

A generalization of constructions in Section 7 takes place in the following context. Let G be a connected reductive group over $\mathbb{Q}$. There is a notion of admissible Harish-Chandra modules $\mathcal{E}$ for $\mathsf{G}(\mathbb{R})$: these are $\mathbb{C}$-vector spaces of countable dimension, endowed with an action of the maximal compact subgroup $\mathsf{K} \subset \mathsf{G}(\mathbb{R})$ and a compatible action of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes \mathbb{C}$. The action of K decomposes $\mathcal{E}$ into a countable sum of finite-dimensional representations of K, each appearing with finite multiplicity. We assume that the center $\mathfrak{z} \subset \mathfrak{U}(\mathfrak{g})$ acts by scalars, called the central character of $\mathcal{E}$. The group $\mathsf{G}(\mathbb{R})$ acts on the Schwartz completion of $\mathcal{S}(\mathcal{E})$. Let $\mathcal{S}(\mathcal{E})'$ be the *continuous* dual space, which is a subspace of the *algebraic* dual space $\mathcal{E}^{\vee}$. The congruence subgroups of $\mathsf{G}(\mathbb{Q})$ have finite-dimensional invariants in $\mathcal{S}(\mathcal{E})'$. One can view the theory of automorphic forms as the study of these finite-dimensional spaces of invariants, together with the action of a Hecke algebra. Note that in the last step we consider $\mathcal{S}(\mathcal{E})'$ only as a $\mathsf{G}(\mathbb{Q})$-module, and not as a $\mathsf{G}(\mathbb{R})$-module.

Almost all automorphic forms are unrelated to motives or Galois representations; the part relevant for number theory (called *algebraic* automorphic forms) is specified by a certain integrality constraint on the central character.

Returning to the considerations above, we see that we can imitate the theory of automorphic forms, with representations of $\mathsf{G}(\mathbb{Q})$ in $\mathcal{S}(\mathcal{E})'$, by taking a *different* class of representations of $\mathsf{G}(\mathbb{Q})$, defined over $\mathbb{Q}$. Assume that $\mathsf{G} = \mathrm{GL}_n$, over $\mathbb{Q}$. Let

$$\mathcal{F}_n = \langle \mathcal{X}_\Lambda \rangle_{\otimes\mathbb{Q}} = \mathcal{F}_{\mathbf{L},\mathbb{Z}} \otimes \mathbb{Q} \quad \text{for } \mathbf{L} = \mathbb{Z}^n, \tag{9.1}$$

be the $\mathbb{Q}$-vector space generated by characteristic functions $\mathcal{X}_\Lambda$ of convex finitely generated rational polyhedral cones $\Lambda \subset \mathbb{R}^n$, modulo functions with support of dimension $\leq n-1$. Note that

$$\mathcal{F}_n \subset \mathrm{L}_\infty(\mathbb{R}^n),$$

the space of bounded measurable functions. Clearly, $\mathsf{G}(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{Q})$ acts on $\mathcal{F}_n$. Let $\rho = \rho_\lambda : \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{Aut}(\mathsf{V}_\lambda)$ be a finite-dimensional irreducible representation as above. Let $\Gamma \subset \mathrm{GL}_n(\mathbb{Q})$ be an arithmetic subgroup. The spaces of invariants, respectively coinvariants,

$$H^0(\Gamma, \mathcal{F}_n^\vee \otimes \mathsf{V}_\lambda^\vee) = (\mathcal{F}_n^\vee \otimes \mathsf{V}_\lambda^\vee)^\Gamma, \quad H_0(\Gamma, \mathcal{F}_n \otimes \mathsf{V}_\lambda) = (\mathcal{F}_n \otimes \mathsf{V}_\lambda)_\Gamma, \tag{9.2}$$

are finite-dimensional spaces dual to each other, since the module of characteristic functions is finitely generated over the group ring of the arithmetic subgroup Γ.

For example, for $n \geq 2$, if ρ is the trivial representation, and $\Gamma \subset \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(\mathbf{L})$ is the stabilizer of the vector

$$\chi = (1, 0, 0, \ldots) \in \mathbf{L} \otimes \mathbb{Z}/N\mathbb{Z},$$

then the group of coinvariants is (up to torsion) our group $\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z})$. Similarly, by taking the stabilizer of the coordinate co-vector modulo N, we obtain the co-vector version $\mathcal{M}_n^*(\mathbb{Z}/N\mathbb{Z})$.

More generally, for any finite abelian group G with character group A such that G can be generated by at least n elements let us choose an element

$$\chi \in \mathbf{L} \otimes A, \quad \mathbf{L} = \mathbb{Z}^n,$$

such that the induced homomorphism $\mathbf{L}^\vee \to A$ is surjective. We define $\Gamma(G,n) \subset \mathrm{GL}_n(\mathbb{Z})$ as the stabilizer of χ. Note that the conjugacy class of the stabilizer does not depend on the choice of χ. Then, for $n \geq 2$ such that G is generated by at most n elements, we have

$$\mathcal{M}_n(G) \otimes \mathbb{Q} = H_0(\Gamma(G,n), \mathcal{F}_n). \tag{9.3}$$

A key observation is that $\mathcal{F}_n$ is a $\mathrm{GL}_n(\mathbb{Q})$-module which is *finitely generated* as a $\mathrm{GL}_n(\mathbb{Z})$-module; moreover,

$$\mathrm{Res}^{\mathrm{GL}_n(\mathbb{Q})}_{\mathrm{GL}_n(\mathbb{Z})}(\mathcal{F}_n) \in \mathrm{Perf}(\mathbb{Q}[\mathrm{GL}_n(\mathbb{Z})]\text{-mod}), \tag{9.4}$$

i.e., $\mathcal{F}_n$, considered as a $\mathrm{GL}_n(\mathbb{Z})$-module, admits a finite-length resolution by finitely generated projective modules over the group ring of $\mathrm{GL}_n(\mathbb{Z})$ (see Proposition 21).

Question 20. Are there other interesting $\mathrm{GL}_n(\mathbb{Q})$-modules which are finitely generated as $\mathrm{GL}_n(\mathbb{Z})$-modules, or more strongly, belong to

$$\mathrm{Perf}(\mathbb{Q}[\mathrm{GL}_n(\mathbb{Z})]\text{-mod})?$$

An even more general question would be to find a bounded from above complex of representations of $\mathsf{G}(\mathbb{Q})$ which, after restriction to $\mathsf{G}(\mathbb{Z})$, is quasi-isomorphic to a complex of finitely generated projective modules over the group ring.

Both $\mathbb{Q}$-vector spaces in (9.2) carry actions of Hecke operators, which have algebraic eigenvalues in these spaces. By (9.4),

$$\dim(H_i(\Gamma, \mathcal{F}_n \otimes \mathsf{V}_\lambda)) < \infty \quad \text{for all } i \geq 0,$$

and the spaces, for $i \geq 1$, also carry actions of Hecke operators with algebraic eigenvalues.

We will see below that our representation $\mathcal{F}_n$ falls into a well-studied subclass of *cohomological* automorphic forms, i.e., those realized in cohomology of arithmetic groups with coefficients in finite-dimensional representations ρ.

Recall the definition of *Steinberg* modules: Let $V/\mathbb{Q}$ be a $\mathbb{Q}$-vector space of dimension $n \geq 0$, and $\mathcal{T}_n$ the simplicial complex of flags of $\mathbb{Q}$-vector subspaces of V, i.e., the geometric realization of the poset of nontrivial subspaces in V. Put

$$\mathrm{St}(V) := \begin{cases} H_{n-2}(\mathcal{T}_n, \mathbb{Z}), & n \geq 3, \\ \mathbb{Z}\text{-combinations of lines in } V \text{ with total weight } 0, & n = 2, \\ \mathbb{Z}, & n = 0, 1. \end{cases}$$

This is a representation of $\mathrm{Aut}(V)$, which we denote by St_n for $V = \mathbb{Q}^n$. One of the roles of the Steinberg module is a dualizing module, in the sense that

$$H_i(\mathrm{SL}_n(\mathbb{Z}), \mathrm{St}_n \otimes M) = H^{n(n-1)/2-i}(\mathrm{SL}_n(\mathbb{Z}), M)$$

for any representation M of $\mathrm{SL}_n(\mathbb{Z})$ with coefficients in $\mathbb{Q}$.

Let $\mathcal{F}(V) = \mathcal{F}_n$, as in (9.1), where the identification depends on the choice of a basis of V; different choices are related by the action of $\mathrm{G}_n(\mathbb{Q})$ on $\mathcal{F}_n$. It has a filtration by submodules

$$0 \subset \mathcal{F}^{\leq 0}(V) \subset \mathcal{F}^{\leq 1}(V) \subset \cdots \subset \mathcal{F}^{\leq n}(V) = \mathcal{F}(V),$$

where $\mathcal{F}^{\leq i}(V)$ are generated by functions pulled back from quotient spaces of dimension i. In particular,

$$\mathcal{F}^{\leq 0}(V) = \mathbb{Z} = \{\text{constant } \mathbb{Z}\text{-valued functions on } V\}.$$

The following fact is presumably well-known:

Proposition 21.

$$\mathrm{gr}^i(\mathcal{F}(V)) = \bigoplus_{V \twoheadrightarrow V',\, \dim(V')=i} \mathrm{St}(V') \otimes \mathrm{or}(V'),$$

where $\mathrm{or}(V')$ *is the* 1*-dimensional* $\mathbb{Z}$*-module of orientations of* V'*, i.e.,* $\mathrm{GL}(V')$ *acts via the sign of the determinant.*

Proof. Let us first prove that

$$\mathrm{gr}^n(\mathcal{F}(V)) = \mathcal{F}(V)/\mathcal{F}^{\leq n-1}(V) \quad \text{is isomorphic to} \quad \mathrm{St}(V) \otimes \mathrm{or}(V).$$

We apply the Fourier transform to elements of $\mathcal{F}(V)$ viewed as distributions with moderate growth on $V \otimes \mathbb{R} \simeq \mathbb{R}^n$.

For example, the Fourier transform of the characteristic function of the standard coordinate octant $(\mathbb{R}_{\geq 0})^n$ is equal to the distribution

$$\prod_{i=1}^n \big(\sqrt{-1}\,\mathrm{v.p.}(1/x_i) + \pi\delta(x_i)\big) \prod_{i=1}^n |dx_i|$$

with values in volume forms, where $\mathrm{v.p.}(1/x)$ is the unique odd distribution of homogeneity degree -1 on $\mathbb{R}^1$ equal to $1/x$ on $\mathbb{R} \setminus 0$.

The image of $\mathcal{F}^{\leq n-1}(V)$ is characterized by the property that the support of the distribution is contained in a finite union of hyperplanes. Therefore, the quotient group $\mathcal{F}(V)/\mathcal{F}^{\leq n-1}(V)$ is identified with the abelian group generated by volume elements on the dual space $(V \otimes \mathbb{R})^\vee$, of the form

$$(\sqrt{-1})^n |dx_1 \wedge \cdots \wedge dx_n|/(x_1 \cdots x_n),$$

where $x_1, \ldots, x_n$ are coordinates in $(V \otimes \mathbb{R})^\vee$ in a rational basis. Choosing an orientation of V (or equivalently of $V^\vee$) and dividing by $(\sqrt{-1})^n$, we identify the latter space with the top-degree meromorphic differential forms on the vector space $V^\vee$ considered as an *algebraic variety* $\mathbb{A}^n_{\mathbb{Q}}$ over $\mathbb{Q}$ spanned by forms of type $\wedge_{i=1}^n (dx_i/x_i)$ for coordinates in a rational basis. This is an alternative description of the Steinberg module. The case of deeper terms of the dimension filtration is similar. ∎

This implies that the computation of cohomology with coefficients in $\mathcal{F}(V)$, tensored with finite-dimensional modules, and in particular of coinvariants, would reduce to the computation of cohomology for St-modules and their pullbacks from parabolic subgroups. There is extensive literature on the cohomology of St-modules (see, e.g., [1] and the references therein), but these computations do not capture the potentially interesting extension data in $\mathcal{F}(V)$.

To summarize, we have a surjective homomorphism

$$\mathcal{F}_n \twoheadrightarrow \mathrm{St}_n \otimes \mathrm{or}_n, \quad \text{where} \quad \mathrm{or}_n : \mathrm{GL}_n(\mathbb{Q}) \to \mathbb{Q}^\times,\ \gamma \mapsto \mathrm{sgn}(\det(\gamma)).$$

It gives rise to a surjective homomorphism

$$H_0(\Gamma(G,n), \mathcal{F}_n) \twoheadrightarrow H_0(\Gamma(G,n), \mathrm{St}_n \otimes \mathrm{or}_n).$$

Proposition 22. *There exists a commutative diagram*

$$\begin{array}{ccc} H_0(\Gamma(G,n),\mathcal{F}_n) & \twoheadrightarrow & H_0(\Gamma(G,n),\mathrm{St}_n\otimes\mathrm{or}_n) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathcal{M}_n(G)\otimes\mathbb{Q} & \xrightarrow{\mu^-} & \mathcal{M}_n^-(G)\otimes\mathbb{Q} \end{array}$$

where the horizontal arrows are the natural surjections, the left vertical arrow is the isomorphism (9.3) *and the right vertical arrow is an isomorphism as well.*

Proof. The proof of the commutativity is straightforward; we explain only the right vertical isomorphism. Recall that the Steinberg representation St_n restricted to $\mathrm{GL}_n(\mathbb{Z})$ is generated by the set of $\mathbb{Z}$-bases $\{(e_1,\dots,e_n)\}$ modulo relations

- $(e_{\sigma(1)},\dots,e_{\sigma(n)}) = (-1)^n(e_1,\dots,e_n)$, $\sigma\in\mathfrak{S}_n$,
- $(e_1,e_2,e_3,\dots,e_n) = (e_1+e_2,e_2,e_3,\dots,e_n)+(e_1,e_1+e_2,\dots,e_n)$,
- $(e_1,\dots,e_n) = (-e_1,e_2,\dots,e_n)$

(see, e.g., [4, Theorem B] and the references therein). Therefore, $\mathrm{St}_n\otimes\mathrm{or}_n$ restricted to $\mathrm{GL}_n(\mathbb{Z})$ is again generated by the set of $\mathbb{Z}$-bases $\{(e_1,\dots,e_n)\}$, but subject to new relations

- $(e_{\sigma(1)},\dots,e_{\sigma(n)}) = (e_1,\dots,e_n)$, $\sigma\in\mathfrak{S}_n$,
- $(e_1,e_2,e_3,\dots,e_n) = (e_1+e_2,e_2,e_3,\dots,e_n)+(e_1,e_1+e_2,\dots,e_n)$,
- $(e_1,\dots,e_n) = -(-e_1,e_2,\dots,e_n)$.

We see that the first relation is the symmetry (S), and the last one the anti-symmetry relation (5.1); the second relation translates to relation (M) for $k=2$. ∎

Put

$$\mathbb{H}_n := \mathrm{GL}_n(\mathbb{R})/\mathbb{R}^\times_{>0}\cdot\mathrm{O}_n(\mathbb{R});$$

for $n\ge 2$, and G generated by at most n elements, we have

$$\begin{aligned}\mathcal{M}_n^-(G)\otimes\mathbb{Q} &= H_0(\Gamma(G,n),\mathrm{St}_n\otimes\mathrm{or}_n) = H_{n-1}^{\mathrm{BM}}(\Gamma(G,n)\backslash\mathbb{H}_n,\mathrm{or}_n)\\ &= H^{n(n-1)/2}(\Gamma(G,n)\backslash\mathbb{H}_n,\mathrm{or}_n^{\otimes n}) = H^{n(n-1)/2}(\Gamma(G,n),\mathrm{or}_n^{\otimes n}).\end{aligned}$$

Indeed, the generator $(e_1,\dots,e_n)$ of St_n, where $e_1,\dots,e_n$ is the standard basis of $\mathbb{Z}^n$, maps to the homology class of the Borel–Moore chain

$$(\mathbb{R}^\times_{>0})^{n-1}\simeq\mathrm{Diag}_{>0,n}(\mathbb{R})/\mathbb{R}^\times_{>0}\subset\mathbb{H}_n.$$

The third isomorphism is Poincaré duality.

Let $\Gamma\subset\mathrm{GL}_n(\mathbb{Z})$ be an arithmetic group. The cuspidal part of cohomology, with coefficients in a finite-dimensional representation ρ of $\mathrm{GL}_n(\mathbb{Q})$, is

$$H^*_{\mathrm{cusp}}(\Gamma,\rho) := \mathrm{Image}(H^*_c(\Gamma\backslash\mathbb{H}_n,\rho)\to H^*(\Gamma\backslash\mathbb{H}_n,\rho)).$$

Notice that or_n restricted to $\mathrm{GL}_n(\mathbb{Z})$ coincides with the algebraic representation $\det_n : \gamma \mapsto \det(\gamma)$.

It is known that $H^i_{\mathrm{cusp}}(\Gamma, \rho) \neq 0$ only for

$$\frac{n(n+1)/2-1}{2} - \frac{[(n-1)/2]}{2} \leq i \leq \frac{n(n+1)/2-1}{2} + \frac{[(n-1)/2]}{2}.$$

The upper bound coincides with $n(n-1)/2$ for $n = 1, 2, 3$ and is strictly smaller for $n \geq 4$. Our experiments (see Section 8) suggest that

$$\mathcal{M}^-_{n,\mathrm{prim}}(G) = H^{n(n-1)/2}_{\mathrm{cusp}}(\Gamma(G,n), \mathrm{or}_n^{\otimes n}),$$

hence vanishes for $n \geq 4$.

In the following section, we will see that, for $n = 2$, the main actors are modular forms of weight 2, and sums of two Tate motives twisted by characters.

Other variants of the definition of $\mathcal{F}$ are possible:

- using $\mathbb{Z}$ or finite fields as coefficients, instead of $\mathbb{Q}$-coefficients, one can study torsion effects;
- one can omit the condition of factoring by characteristic functions with support in dimension $\leq n-1$;
- when the representation ρ is on the space of degree-d polynomials, one can consider *polynomial splines*, with respect to some complete rational fan Σ on $\mathbb{R}^n$, i.e., functions on $\mathbb{R}^n$ which are piecewise polynomial on the cones of Σ, with $\mathbb{Q}$-coefficients, and with continuous derivatives up to some fixed $d' < d$.

The last example is especially interesting as such representations are realized as submodules of extensions of Steinberg modules, and coinvariants with values in such modules could, potentially, capture higher homology groups of Steinberg modules, thus making them computationally much more accessible.

We finish this section with a challenge concerning the possibility, in the framework of Question 20, to go beyond the realm of cohomological (but still algebraic) automorphic forms.

Question 23. Can one find a representation of $\mathrm{SL}_2(\mathbb{Q})$ whose restriction to $\mathrm{SL}_2(\mathbb{Z})$ is finitely generated, and whose Hecke spectrum captures modular forms of weight 1 and Maass forms with Laplace eigenvalue $1/4$?

Morally, such modules should be realized in a class of odd/even distributions on $\mathbb{R}^2$ of homogeneity degree -1.

10. Lattice-theoretic approach to multiplication and co-multiplication

In this section, we reinterpret the multiplication and co-multiplication on $\mathcal{M}^-_n(G)$, defined in Section 5, in terms of lattices.

For any $n \geq 1$ and any nontrivial finite abelian group G we define

$$\mathcal{E}_n(G) := \mathbb{Q}^{\{\text{epi } \mathbb{Z}^n \twoheadrightarrow G\}};$$

it is a finite-dimensional permutation module for $\mathrm{GL}_n(\mathbb{Z})$. Define the stack (with finite stabilizers)

$$\mathbb{X}_n := \mathrm{GL}_n(\mathbb{Z}) \backslash \mathrm{GL}_n(\mathbb{R}) / \mathrm{O}_n(\mathbb{R}).$$

This stack parametrizes rank n Arakelov bundles on $\widehat{\mathrm{Spec}(\mathbb{Z})}$, i.e., pairs $(\mathbf{L}, h)$, where $\mathbf{L}$ is a lattice of rank n and h is a positive-definite quadratic form on $\mathbf{L} \otimes \mathbb{R}$. Let $\mathcal{L}_{n,G}$ be a $\mathbb{Q}$-local system on $\mathbb{X}_n$ associated with the representation $\mathcal{E}_n(G) \otimes \mathrm{or}_n$. Then we have

$$\mathcal{M}_n^-(G) \otimes \mathbb{Q} = H_n^{\mathrm{BM}}(\mathbb{X}_n, \mathcal{L}_{n,G}). \tag{10.1}$$

The multiplication ∇^-, defined in Section 5, admits the following reformulation in this language. Consider flags $\mathcal{G}_\bullet$ of subgroups

$$0 = G_{\leq 0} \subsetneq G_{\leq 1} \subsetneq \cdots \subsetneq G_{\leq r} = G, \qquad r \geq 1,$$

and sequences of positive integers $n_1, \ldots, n_r$ such that $n_1 + \cdots + n_r = n$. We have a homomorphism

$$\bigotimes_{i=1}^{r} H_{n_i}^{\mathrm{BM}}(\mathbb{X}_{n_i}, \mathcal{L}_{n_i, \mathrm{gr}_i(\mathcal{G}_\bullet)}) \to H_n^{\mathrm{BM}}(\mathbb{X}_n, \mathcal{L}_{n,G}), \tag{10.2}$$

defined as follows. Consider the graph

$$\mathbb{Y}^{\nabla}_{n_1,\ldots,n_r} \subset (\mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r}) \times \mathbb{X}_n$$

of the closed embedding (hence proper map) $\mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r} \to \mathbb{X}_n$, given by

$$(\mathbf{L}_1, h_1), \ldots, (\mathbf{L}_r, h_r) \mapsto (\mathbf{L} = \mathbf{L}_1 \oplus \cdots \oplus \mathbf{L}_r, h = h_1 \boxplus \cdots \boxplus h_r).$$

We have a diagram

$$\begin{array}{ccc} & \mathbb{Y}^{\nabla}_{n_1,\ldots,n} & \\ {\scriptstyle \pi_{n_1,\ldots,n_r}} \swarrow & & \searrow {\scriptstyle \pi_n} \\ \mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r} & & \mathbb{X}_n \end{array}$$

Here, $\pi_{n_1,\ldots,n_r}$ is an isomorphism. The morphism of local systems

$$\pi^*_{n_1,\ldots,n_r}(\mathcal{L}_{n_1,\mathrm{gr}_1(\mathcal{G}_\bullet)} \boxtimes \cdots \boxtimes \mathcal{L}_{n_r,\mathrm{gr}_r(\mathcal{G}_\bullet)}) \to \pi_n^* \mathcal{L}_{n,G}$$

is given, at any point, by

- a canonical identification of orientation bundles

$$\mathrm{or}(\mathbf{L}_1) \otimes \cdots \otimes \mathrm{or}(\mathbf{L}_r) \xrightarrow{\sim} \mathrm{or}(\mathbf{L}),$$

- a morphism of fibers of local systems associated to the permutation modules

$$\mathbb{Q}^{\{\mathrm{epi}\,\mathbf{L}_1^\vee \twoheadrightarrow A_1\}} \otimes \cdots \otimes \mathbb{Q}^{\{\mathrm{epi}\,\mathbf{L}_r^\vee \twoheadrightarrow A_r\}} \to \mathbb{Q}^{\{\mathrm{epi}\,\mathbf{L}^\vee \twoheadrightarrow A\}}, \tag{10.3}$$

  defined as follows. Consider $\chi \in \mathbf{L} \otimes A := \mathrm{Hom}(\mathbf{L}^\vee, A)$ such that the restriction of χ to $\mathbf{L}_i^\vee \subset \mathbf{L}^\vee$ takes values in characters of G vanishing on $G_{\le i-1}$, for all i; such characters induce characters of $\mathrm{gr}_i(\mathscr{G}_\bullet)$, and homomorphisms

$$\chi_i : \mathbf{L}_i^\vee \to A_i := \mathrm{Hom}(\mathrm{gr}_i(\mathscr{G}_\bullet), \mathbb{C}^\times);$$

  we insist that χ_i are surjective for all i (this implies that χ is surjective as well). Such a χ defines a morphism of permutation modules of rank 1, given by an elementary matrix with indices

$$(\chi_1, \ldots, \chi_r), \chi.$$

  Taking the sum over all such elementary matrices defines the desired homomorphism (10.3).

The co-multiplication Δ^-, defined in Section 5, also admits a geometric reformulation. We have a homomorphism

$$H_n^{\mathrm{BM}}(\mathbb{X}_n, \mathscr{L}_{n,G}) \to \bigotimes_{i=1}^r H_{n_i}^{\mathrm{BM}}(\mathbb{X}_{n_i}, \mathscr{L}_{n_i, \mathrm{gr}_i(\mathscr{G}_\bullet)}) \tag{10.4}$$

defined similarly to (10.2), but instead of the graph $\mathbb{Y}^\nabla_{n_1,\ldots n_r}$ of a map, we consider the *correspondence*

$$\mathbb{Y}^\Delta_{n_1,\ldots,n_r} \subset \mathbb{X}_n \times (\mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r}),$$

which is étale over $\mathbb{X}_n$ and *proper* over $\mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r}$, and which can be viewed as the graph of a multi-valued map. In detail, an element of $\mathbb{Y}_{n_1,\ldots,n_r}$ is given by these data:

- $(\mathbf{L}, h)$, a lattice of rank n, with a metric, i.e., a positive quadratic form h on $\mathbf{L} \otimes \mathbb{R}$ as above,
- a flag $\mathbf{L}_\bullet$ of full sublattices

$$0 = \mathbf{L}_{\le 0} \subsetneq \mathbf{L}_{\le 1} \subsetneq \cdots \subsetneq \mathbf{L}_{\le r} = \mathbf{L},$$

- a choice of isomorphisms $\mathbf{L}_i \simeq \mathrm{gr}_i(\mathbf{L}_\bullet)$ such that the induced metrics on $\mathbf{L}_{n_i} \otimes \mathbb{R}$ coincide with h_i.

We have a diagram

$$\begin{array}{ccc} & \mathbb{Y}^\Delta_{n_1,\ldots,n} & \\ {\scriptstyle \pi_n}\swarrow & & \searrow{\scriptstyle \pi_{n_1,\ldots,n_r}} \\ \mathbb{X}_n & & \mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r} \end{array}$$

The morphism of local systems on $\mathbb{Y}_n$,

$$\pi_n^* \mathscr{L}_{n,G} \to \pi^*_{n_1,\ldots,n_r}(\mathscr{L}_{n_1,\mathrm{gr}_1(\mathscr{G}_\bullet)} \oplus \cdots \oplus \mathscr{L}_{n_r,\mathrm{gr}_r(\mathscr{G}_\bullet)}),$$

is given, at any point, by

- a natural isomorphism of orientation bundles

$$\mathrm{or}(\mathbf{L}) \simeq \mathrm{or}(\mathbf{L}_1) \otimes \cdots \otimes \mathrm{or}(\mathbf{L}_r),$$

- a morphism of fibers of local systems associated to the permutation modules

$$\mathbb{Q}^{\{\mathrm{epi}\, \mathbf{L}^\vee \twoheadrightarrow A\}} \to \mathbb{Q}^{\{\mathrm{epi}\, \mathbf{L}_1^\vee \twoheadrightarrow A_1\}} \otimes \cdots \otimes \mathbb{Q}^{\{\mathrm{epi}\, \mathbf{L}_r^\vee \twoheadrightarrow A_r\}} \tag{10.5}$$

defined as follows. Consider $\chi \in \mathbf{L} \otimes A := \mathrm{Hom}(\mathbf{L}^\vee, A)$ such that it induces a commutative diagram

$$\begin{array}{ccccccc} \mathbf{L}^\vee = \mathbf{L}^\perp_{\leq 0} & \supsetneq & \mathbf{L}^\perp_{\leq 1} & \supsetneq & \cdots & \supsetneq & \mathbf{L}^\perp_{\leq r} \\ \downarrow & & \downarrow & & & & \downarrow \\ A = G^\perp_{\leq 0} & \supsetneq & G^\perp_{\leq 1} & \supsetneq & \cdots & \supsetneq & G^\perp_{\leq r} \end{array}$$

i.e.,

$$G^\perp_{\leq i} = \chi(\mathbf{L}^\perp_{\leq i}), \qquad i = 0, \ldots, r-1.$$

Such a character χ is surjective (case $i = 0$) and induces surjective homomorphisms

$$\chi_i : \mathbf{L}_i^\vee \to A_i = \mathrm{Hom}(G_i, \mathbb{C}^\times), \qquad i = 1, \ldots, r,$$

where $\mathbf{L}_i = \mathbf{L}_{\leq i}/\mathbf{L}_{\leq i-1}$ and $G_i = G_{\leq i}/G_{\leq i-1}$. Again, χ defines an elementary matrix with indices $\chi, (\chi_1, \ldots, \chi_r)$; taking the sum over all such χ we obtain the desired homomorphism.

Proposition 24. *Using the identifications*

$$\mathcal{M}_n^-(G) \otimes \mathbb{Q} = H_n^{\mathrm{BM}}(\mathbb{X}_n, \mathcal{L}_{n,G})$$

and formulas (10.2) *and* (10.4) *we obtain homomorphisms*

$$\mathcal{M}_{n_1}^-(G_1) \otimes \cdots \otimes \mathcal{M}_{n_r}^-(G_r) \otimes \mathbb{Q} \rightleftarrows \mathcal{M}_n^-(G) \otimes \mathbb{Q}$$

which are the same as those induced from Δ *and* ∇ *in Section* 5.

Proof. The case of the product follows immediately from the definition: a basis $e_1, \ldots, e_n$ of $\mathbf{L}$ gives a closed Borel–Moore chain $\simeq \mathbb{R}^n_{>0}$, consisting of diagonal forms h in this basis.

To verify the co-product we need the following: let $\mathbf{L} \simeq \mathbb{Z}^n$ be the standard coordinate lattice, up to the action of $\mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ interchanging the coordinates and acting by sign on each coordinate. We have a canonical Borel–Moore closed chain

$$C_n \subset \mathrm{Chains}_n^{\mathrm{BM}}(\mathbb{X}_n, \mathbb{Z}), \qquad \partial(C_n) = 0,$$

given by the image of positive diagonal matrices. Given a flag $0 = \mathbf{L}_{\leq 0} \subsetneq \cdots \subsetneq \mathbf{L}_{\leq r} = \mathbf{L}$ and using the correspondence $\mathbb{Y}^\Delta_{n_1,\ldots,n_r}$ we obtain a closed Borel–Moore chain

$$C_{\mathbf{L}_\bullet} \subset \mathrm{Chains}_n^{\mathrm{BM}}(\mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r}, \mathbb{Z}),$$

and to any point h in C_n we associate a collection

$$(h_1, \dots, h_r) \in \mathbb{X}_{n_1} \times \cdots \times \mathbb{X}_{n_r}.$$

The main observation is that if the flag is not compatible with the chosen coordinate decomposition, then the corresponding chain is a boundary. From this it follows that only the coordinate flags contribute to the formula. ■

Following the reasoning in Section 5, specifically (5.17), we define

$$H^{\mathrm{BM}}_{n,\mathrm{prim}}(\mathbb{X}_n, \mathcal{L}_{n,G}) \subset H^{\mathrm{BM}}_n(\mathbb{X}_n, \mathcal{L}_{n,G})$$

as the common kernel of all nontrivial co-multiplication homomorphisms ($r \geq 2$). Evidently, we have

$$\mathcal{M}^-_{n,\mathrm{prim}}(G) \otimes \mathbb{Q} = H^{\mathrm{BM}}_{n,\mathrm{prim}}(\mathbb{X}_n, \mathcal{L}_{n,G})$$

under the above identifications.

We recall the topological definition of cuspidal cohomology:

$$H_{n,\mathrm{cusp}}(\mathbb{X}_n, \mathcal{L}_{n,G}) := \mathrm{Image}(H_n(\mathbb{X}_n, \mathcal{L}_{n,G}) \to H^{\mathrm{BM}}_n(\mathbb{X}_n, \mathcal{L}_{n,G})).$$

Conjecture 25. For every nontrivial finite abelian group G and every $n \geq 1$, we have

$$H^{\mathrm{BM}}_{n,\mathrm{prim}}(\mathbb{X}_n, \mathcal{L}_{n,G}) = H_{n,\mathrm{cusp}}(\mathbb{X}_n, \mathcal{L}_{n,G}) \subset H^{\mathrm{BM}}_n(\mathbb{X}_n, \mathcal{L}_{n,G}).$$

This conjecture is essentially our guess, stated implicitly in Section 5. Assuming this conjecture, we would obtain the following reformulation:

Conjecture 26. For every nontrivial finite abelian group G and $n \geq 1$, the natural homomorphism

$$\bigoplus_{r=1}^{n} \bigoplus_{\substack{n_1+\cdots+n_r=n \\ \mathcal{G}_\bullet \text{ of length } r}} H_{n_1,\mathrm{cusp}}(\mathbb{X}_{n_1}, \mathcal{L}_{n_1,\mathrm{gr}_1(\mathcal{G}_\bullet)}) \otimes \cdots \otimes H_{n_r,\mathrm{cusp}}(\mathbb{X}_{n_r}, \mathcal{L}_{n_r,\mathrm{gr}_r(\mathcal{G}_\bullet)}) \to H^{\mathrm{BM}}_n(\mathbb{X}_n, \mathcal{L}_{n,G})$$

is an isomorphism.

Representation theory gives a canonical splitting of cohomology of arithmetic groups into the sum of the cuspidal and the remaining (Eisenstein) parts, after tensoring by $\mathbb{C}$. Our considerations, for $\mathrm{GL}_n(\mathbb{Z})$, suggest that we have a splitting over $\mathbb{Q}$. Namely, define

$$H^{\mathrm{BM}}_{n,\mathrm{coprim}}(\mathbb{X}_n, \mathcal{L}_{n,G})$$

as the quotient by the sum of the images of all nontrivial product maps (10.2). It is tempting to make a companion conjecture:

Conjecture 27. For every nontrivial finite abelian group G and $n \geq 1$, the homomorphism

$$H_n^{\mathrm{BM}}(\mathbb{X}_n, \mathcal{L}_{n,G}) \to \bigoplus_{r=1}^{n} \bigoplus_{\substack{n_1+\cdots+n_r=n \\ \mathcal{G}_\bullet \text{ of length } r}} H_{n_1,\mathrm{coprim}}(\mathbb{X}_{n_1}, \mathcal{L}_{n_1,\mathrm{gr}_1(\mathcal{G}_\bullet)}) \otimes \cdots \otimes H_{n_r,\mathrm{coprim}}(\mathbb{X}_{n_r}, \mathcal{L}_{n_r,\mathrm{gr}_r(\mathcal{G}_\bullet)})$$

is an isomorphism.

Conjecture 28. The composition

$$H^{\mathrm{BM}}_{n,\mathrm{prim}}(\mathbb{X}_n, \mathcal{L}_{n,G}) \hookrightarrow H^{\mathrm{BM}}_{n}(\mathbb{X}_n, \mathcal{L}_{n,G}) \twoheadrightarrow H^{\mathrm{BM}}_{n,\mathrm{coprim}}(\mathbb{X}_n, \mathcal{L}_{n,G})$$

is an isomorphism.

The considerations above fit into a general framework. For $n \geq 1$, let R_n be the set of finite-dimensional irreducible representations of $\mathrm{GL}_n(\mathbb{Z})$ which appear as direct summands of tensor products of

- representations of
$$\mathrm{GL}_n(\hat{\mathbb{Z}}) = \prod_p \mathrm{GL}_n(\mathbb{Z}_p),$$
- irreducible algebraic representations $\rho_\lambda : \mathrm{GL}_n(\mathbb{Q}) \to \mathsf{V}_\lambda$ with highest weight λ.

Obviously, R_1 consists of two elements, and R_n are countable infinite sets for $n \geq 2$.

Given

$$\rho_1 \in R_{n_1},\ \rho_2 \in R_{n_2},\ \rho \in R_n, \quad \text{for } n = n_1 + n_2,$$

we can define the multiplicity space

$$\mathrm{mult}^{\rho}_{\rho_1,\rho_2} \in \mathrm{Vect}_{\mathbb{C}},$$

which is a finite-dimensional complex vector space, by

$$\mathrm{Hom}_{\mathrm{GL}_{n_1}(\mathbb{Z})\times\mathrm{GL}_{n_1}(\mathbb{Z})}\big(\rho_{n_1} \boxtimes \rho_{n_2}, \rho|_{\mathrm{GL}_{n_1}(\mathbb{Z})\times\mathrm{GL}_{n_2}(\mathbb{Z})}\big).$$

The correspondence $\mathbb{Y}^{\nabla}_{n_1,n_2}$ gives rise to a natural homomorphism

$$\mathrm{mult}^{\rho}_{\rho_1,\rho_2} \otimes H^{\mathrm{BM}}_{*}(\mathbb{X}_n, \rho_{n_1}) \otimes H^{\mathrm{BM}}_{*}(\mathbb{X}_{n_2}, \rho_{n_2}) \to H^{\mathrm{BM}}_{*}(\mathbb{X}_n, \rho).$$

The collection of these can be organized in the following way. Let $\mathcal{C}$ be a semisimple (in the countable sense) $\mathbb{C}$-linear tensor category, with countable sums and tensor products commuting with sums, and with simple objects ϵ_ρ, corresponding to $\rho \in \coprod_{n\geq 1} R_n$; the tensor product is given by

$$\epsilon_{\rho_1} \otimes \epsilon_{\rho_2} = \bigoplus_{\rho} \mathrm{mult}^{\rho}_{\rho_1,\rho_2} \otimes_{\mathbb{C}} \epsilon_\rho.$$

The expression on the right is infinite. Put

$$\mathcal{A}_\bullet := \bigoplus_{n\geq 1} \bigoplus_{\rho\in R_n} H_\bullet^{\mathrm{BM}}(\mathbb{X}_n, \rho\otimes\epsilon_\rho) \in \mathrm{Ob}(\mathcal{C}).$$

The object $\mathcal{A}_\bullet$ carries the structure of a supercommutative associated $\mathbb{Z}$-graded nonunital algebra in $\mathcal{C}$. Using chains instead of homology groups gives rise to a commutative differential $\mathbb{Z}$-graded nonunital algebra, which by Koszul duality can be identified with a differential graded Lie algebra (or L_∞-algebra). The next question is: what is this algebra, or its Koszul dual dg Lie algebra?

The category $\mathcal{C}$ itself seems to have a description as a category of representations of a certain type of an infinite-dimensional semigroup.

In the model example, consider R_n^{fin}, consisting of irreducible representations of the symmetric group $\mathfrak{S}_n$. Then the corresponding analog $\mathcal{C}^{\mathrm{fin}}$ of the category $\mathcal{C}$ is a subcategory of Deligne's category of representations of $\mathfrak{gl}_t$, where t is a parameter (fractional dimension).

In the second model example, more relevant to our considerations, let R_n^{alg} be the set of irreducible algebraic representations $\rho_\lambda : \mathrm{GL}_n(\mathbb{Q}) \to \mathsf{V}_\lambda$ with highest weights λ. Defining multiplicity spaces $\mathrm{mult}^{\rho}_{\rho_1,\rho_2}$ in a similar fashion, we obtain a category $\mathcal{C}^{\mathrm{alg}}$ of highest weight representations of the (well-known) central extension

$$1 \to \mathbb{C}^\times \to \mathsf{G} \to \mathrm{GL}_\infty(\mathbb{C})^\circ \to 1,$$

where $\mathrm{GL}_\infty(\mathbb{C})^\circ$ is the connected component of the identity of the group

$$\mathrm{Aut}_{\mathrm{cont},\mathbb{C}\text{-mod}}(\mathbb{C}^\infty), \qquad \text{where } \mathbb{C}^\infty := \mathbb{C}((t)).$$

The group G acts on a space of countable dimension

$$\mathsf{V} := \bigoplus_{i\in\mathbb{Z}} \bigwedge^{\frac{\infty}{2}+i}(\mathbb{C}^\infty).$$

An analog of Weyl–Schur duality says that, for all $n\geq 1$, $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathsf{V}^{\otimes n}$, commuting with the G-action, and identifying highest weight representations of G of level n (i.e., those where the central extension acts with character $z\mapsto z^n$) with algebraic irreducible representations of $\mathrm{GL}_n(\mathbb{C})$.

From our perspective, it would be important to explicitly identify the category $\mathcal{C}^{/p}$, whose simple objects correspond to irreducible finite-dimensional representations of the groups $\mathrm{GL}_n(\mathbb{F}_p)$, $n\geq 1$, and the category $\mathcal{C}^p$, whose simple objects correspond to irreducible finite-dimensional continuous representations of $\mathrm{GL}_n(\mathbb{Z}_p)$, $n\geq 1$.

We can develop a similar framework for co-multiplication. Given

$$\rho_1\in R_{n_1},\ \rho_2\in R_{n_2},\ \rho\in R_n, \qquad \text{for } n = n_1+n_2,$$

we can define the co-multiplicity space $\mathrm{comult}_{\rho}^{\rho_1,\rho_2}\in \mathrm{Vect}_{\mathbb{C}}$, a finite-dimensional complex vector space, as

$$\mathrm{Hom}_{P_{n_1,n_2}(\mathbb{Z})}(\rho|_{P_{n_1,n_2}(\mathbb{Z})}, \rho_{n_1}\boxtimes\rho_{n_2}),$$

where $P_{n_1,n_2} \subset \mathrm{GL}_{n_1}$ is the stabilizer of the flag $\mathbb{Z}^{n_1} \subset \mathbb{Z}^n$. The correspondence $\mathbb{Y}^{\Delta}_{n_1,n_2}$ gives rise to a natural homomorphism

$$\mathrm{comult}^{\rho}_{\rho_1,\rho_2} \otimes H^{\mathrm{BM}}_*(\mathbb{X}_n, \rho) \to H^{\mathrm{BM}}_*(\mathbb{X}_n, \rho_{n_1}) \otimes H^{\mathrm{BM}}_*(\mathbb{X}_{n_2}, \rho_{n_2}).$$

We obtain a co-associative co-algebra, without a unit, in a tensor category which is no longer symmetric, a priori.

Note that there might be nontrivial extensions between two representations from R_n, which suggests that the definition of the category $\mathcal{C}$ and algebra $\mathcal{A}_\bullet$ could be enhanced by considering extension data. Also, the category $\mathcal{C}$ is not rigid, and hence should be interpreted not as a category of representations of a group but rather of a semigroup.

Finally, all considerations above can be carried over to the number field case, but in this case, instead of lattices we should consider all nontrivial finitely generated torsion-free modules.

11. Case $n = 2$: modular symbols

We recall the definition of modular symbols of weight 2 for

$$\Gamma_1(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}, \quad N \in \mathbb{Z}_{\geq 2}.$$

Let $\mathbb{M}_2(\Gamma_1(N))$ be the $\mathbb{Q}$-vector space generated by pairs (c, d) with

$$c, d \in \mathbb{Z}/N, \quad \gcd(c, d, N) = 1,$$

and subject to relations

(1) $(c, d) = -(d, -c)$ (and hence $= (-c, -d) = -(-d, c)$),

(2) $(c, d) + (d, -c - d) + (-c - d, c) = 0$.

It is known that $\mathbb{M}_2(\Gamma_1(N))$ is naturally identified with the Borel–Moore homology group $H^{\mathrm{BM}}_1(X_1(N), \mathbb{Q})$ of the complex modular curve

$$X_1(N) := \Gamma_1(N) \backslash \mathcal{H},$$

where $\mathcal{H}$ is the upper half-plane. The symbol (c, d) corresponds to the image in $X_1(N)$ of the geodesic path from $\mathbf{a}/\mathbf{c}$ to $\mathbf{b}/\mathbf{d}$, where

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \in \Gamma_1(N)$$

is any element with $c, d = \mathbf{c}, \mathbf{d} \bmod \mathbf{N}$.

Using (1) we can rewrite (2) as

(2′) $(d, c) = (d, c - d) + (d - c, c)$.

Indeed, substituting $c \mapsto -c$ into (2), we obtain

$$\begin{aligned} 0 &\overset{(2)}{=} (-c, d) + (d, c-d) + (c-d, -c) \\ &\overset{(1)}{=} -(d, c) + (d, c-d) + (c-d, -c) \\ &\overset{(1)}{=} -(d, c) + (d, c-d) + (d-c, c), \end{aligned}$$

There is an involution on $\mathbb{M}_2(\Gamma_1(N))$,

$$\iota : (c, d) \mapsto (-c, d) \overset{(1)}{=} -(d, c).$$

Written in the form $(c, d) \mapsto -(d, c)$ it obviously preserves relations (2′) and cyclic anti-symmetry (1). It corresponds to the automorphism of the first homology group coming from the anti-holomorphic involution on $X_1(N)$ associated with the map $\tau \mapsto -\bar{\tau}$, $\tau \in \mathcal{H}$, on the universal cover. Denote by $\mathbb{M}_2^-(\Gamma_1(N))$ the (−)-eigenspace for the involution ι.

The dimensions are given by

$$\dim(\mathbb{M}_2(\Gamma_1(N))) = 2g + C(N) - 1, \quad \dim(\mathbb{M}_2^-(\Gamma_1(N))) = g + \frac{C(N) - C_2(N)}{2},$$

where

- $g = g(N)$ is the genus of $\overline{X_1(N)}$, which is the same as the dimension of the space of cusp forms of weight 2 for $\Gamma_1(N)$ (see the table in Section 5),
- $C(N)$ is the number of cusps (elements of $\mathbb{P}^1(\mathbb{Q})/\Gamma_1(N)$), and
- $C_2(N)$ is the number of cusps fixed by the anti-holomorphic involution described above.

For $N = 1, 2, 3, 4$, $C(N) = C_2(N) = 1, 2, 2, 3$, respectively; and for $N \geq 5$,

$$C(N) = \frac{1}{2} \sum_{d|N} \phi(d)\phi(N/d), \quad C_2(N) = \begin{cases} \phi(N) + \phi(N/2) & \text{if } N \text{ is even,} \\ \phi(N) & \text{if } N \text{ is odd.} \end{cases}$$

Now we will discuss the connection to our groups of symbols $\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})$ and $\mathcal{M}_2^-(\mathbb{Z}/N\mathbb{Z})$.

Proposition 29. *$\mathcal{M}_2^-(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}$ is isomorphic to $\mathbb{M}_2^-(\Gamma_1(N))$.*

Proof. Indeed, the subspace $\mathbb{M}_2^-(\Gamma_1(N))$ (or rather its quotient space) can be described in terms of generators and relations as

(R1) $(a_1, a_2)^- = (a_2, a_1)^-$,

(R2) $(a_1, a_2)^- = (a_1, a_2 - a_1)^- + (a_1 - a_2, a_2)^-$,

(R3) $(a_1, a_2)^- = -(a_2, -a_1)^-$.

Here (R3) is the same as (1), (R2) is as (2′), and (R1) is ι-invariance. Therefore, the natural map

$$\mathcal{M}_2^-(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{M}_2^-(\Gamma_1(N)), \quad \langle a_1, a_2 \rangle^- \mapsto (a_1, a_2)^-,$$

is an isomorphism, as relations (R1)–(R3) are exactly the defining relations for $\mathcal{M}_2^-(\mathbb{Z}/N\mathbb{Z})$. ∎

Note that

$$(a,0)^- = (0,a)^- = 0 \in \mathbb{M}_2^-(\Gamma_1(N)),$$

by (R1) and (R3). Incidentally, relation (R2) can be replaced by the co-vector version

(R2*) $\quad (a_1,a_2)^- = (a_1+a_2,a_2)^- + (a_1,a_1+a_2)^-.$

Indeed, substitute $a_1 \mapsto a_1, a_2 \mapsto a_1+a_2$ into (R2) and use dihedral symmetry by (R1) and (R3).

As a corollary of Theorems 12 and 14, together with the guesses

$$\dim(\mathcal{M}_{2,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z})\otimes\mathbb{Q}) = \dim(\mathcal{M}^-_{2,\mathrm{prim}}(\mathbb{Z}/N\mathbb{Z})\otimes\mathbb{Q}) = g(N),$$

we would obtain a formula which follows from Proposition 29:

$$\dim(\mathcal{M}_2^-(\mathbb{Z}/N\mathbb{Z})\otimes\mathbb{Q}) = g(N) + \frac{1}{4}\sum_{d|N,\,3\le d\le N/3}\phi(d)\phi(N/d)$$
$$\overset{\text{for } N\ge 1}{=} \dim(\mathbb{M}_2^-(\Gamma_1(N))) = g(N) + \frac{C(N)-C_2(N)}{2}$$

and a *hypothetical* formula

$$\dim(\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z})\otimes\mathbb{Q}) \overset{?}{=} g(N) + \frac{1}{2}\sum_{d|N,\,d\ge 3}\phi(d)\phi(N/d)$$
$$\overset{\text{for } N\ge 5}{=} g(N) + C(N) - C_2(N)/2.$$

Presumably, one can deduce the above formula using the relation between the Steinberg module and module $\mathcal{F}_2$ (see Proposition 21). The formulas for dimensions simplify when $N = p \ge 5$ is a prime:

$$g(p) = \frac{(p-5)(p-7)}{24}, \qquad C(p) = C_2(p) = p-1,$$
$$\dim(\mathcal{M}_2^-(\mathbb{Z}/p\mathbb{Z})\otimes\mathbb{Q}) = \dim(\mathbb{M}_2^-(\Gamma_1(p))) = g(p),$$
$$\dim(\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z})\otimes\mathbb{Q}) \overset{?}{=} \frac{p^2+23}{24} = g(p) + \frac{p-1}{2}. \tag{11.1}$$

The rest of the section will be devoted to a direct proof of (11.1).

We have two maps

$$\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \twoheadrightarrow \mathcal{M}_2^-(\mathbb{Z}/p\mathbb{Z}), \qquad \langle a,b\rangle \mapsto \langle a,b\rangle^-, \tag{11.2}$$
$$\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\Delta} \mathcal{M}_1(1)\otimes\mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z}) = \mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z}), \tag{11.3}$$

where (11.3) is the (only possible) co-product map given by

$$\langle a,b\rangle \mapsto (1-\delta_{a,0})\langle a\rangle^- + (1-\delta_{b,0})\langle b\rangle^-.$$

The map (11.2) is surjective by definition, and (11.3) is surjective up to 2-torsion: its right inverse after tensoring with $\mathbb{Q}$ is given by

$$\langle a\rangle^- \mapsto \tfrac{1}{2}(\langle a,0\rangle - \langle -a,0\rangle). \tag{11.4}$$

The validity of (11.1) follows from the following result:

Proposition 30. *The map given by the sum of* (11.2) *and* (11.3),

$$\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \to \mathcal{M}_2^-(\mathbb{Z}/p\mathbb{Z}) \oplus \mathcal{M}_1^-(\mathbb{Z}/p\mathbb{Z}),$$

is an isomorphism up to torsion.

Proof. We will check (after tensoring with $\mathbb{Q}$) that the kernel of (11.2) is generated by the image of (11.4). By definition (5.1), this kernel is spanned by the elements $\langle a,b\rangle + \langle a,-b\rangle \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z})$.

Lemma 31. *For all $a,b \in \mathbb{Z}/p\mathbb{Z}$ with $a \neq 0$, we have*

$$\langle a,b\rangle + \langle a,-b\rangle = 2\cdot\langle a,0\rangle \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}).$$

Proof. From (M) we have

$$\langle a,b\rangle = \langle a-b,b\rangle + \langle a,b-a\rangle, \qquad \langle a-b,a\rangle = \langle -b,a\rangle + \langle a-b,b\rangle.$$

Taking the difference between the two equations, we obtain

$$\langle a,b\rangle + \langle -b,a\rangle = \langle a,b-a\rangle + \langle a,a-b\rangle,$$

which we can write, by (S), as

$$\langle a,b\rangle + \langle a,-b\rangle = \langle a,b-a\rangle + \langle a,-b+a\rangle.$$

Iterating this, we get

$$\langle a,b\rangle + \langle a,-b\rangle = \langle a,b-ma\rangle + \langle a,-b+ma\rangle, \qquad m = 1,\dots,p.$$

For $a \neq 0 \pmod p$, there is an m solving the equation $ma = b \pmod p$, which implies the claimed identity

$$\langle a,b\rangle + \langle a,-b\rangle = 2\cdot\langle a,0\rangle. \tag{11.5}$$

∎

Lemma 32. *For all $a \in \mathbb{Z}/p\mathbb{Z}, a \neq 0$, we have*

$$\langle a,0\rangle + \langle -a,0\rangle = 0 \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z})\otimes\mathbb{Q}.$$

Proof. Replacing a by $-a$ in Lemma 31 and adding the equations, we obtain

$$(\langle a,b\rangle + \langle -a,b\rangle) + (\langle a,-b\rangle + \langle -a,-b\rangle) = 2\cdot(\langle a,0\rangle + \langle -a,0\rangle).$$

Using again Lemma 31, with a replaced by b, respectively $-b$, we find

$$2 \cdot (\langle b, 0\rangle + \langle -b, 0\rangle) = 2 \cdot (\langle a, 0\rangle + \langle -a, 0\rangle) \tag{11.6}$$

for all $a, b \neq 0$. To show the vanishing of

$$\delta := \langle 1, 0\rangle + \langle -1, 0\rangle \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}$$

consider the sum

$$\sum_{a,b\neq 0} (\langle a, b\rangle + \langle b, -a\rangle) = 2(p-1) \cdot \sum_{b\neq 0} \langle b, 0\rangle = (p-1)^2\delta,$$

where we substituted (11.5) and (11.6). Apply the blowup relation (M) to each term and relate to the original sum:

$$\begin{aligned}
&\overset{\text{(M)}}{=} \sum_{a,b\neq 0} \langle a-b, b\rangle + \sum_{a,b\neq 0} \langle a, b-a\rangle + \sum_{a,b\neq 0} \langle b+a, -a\rangle + \sum_{a,b\neq 0} \langle b, -a-b\rangle \\
&\qquad \overset{\text{(S)}}{=} 4 \sum_{b\neq 0, a\neq -b} \langle a, b\rangle = 4 \sum_{a,b\neq 0} \langle a, b\rangle + 4\sum_{a\neq 0} \langle a, 0\rangle - 4\sum_{a\neq 0} \langle a, -a\rangle \\
&\qquad = 2(p-1)^2\delta + 2(p-1)\delta = 2p(p-1)\delta.
\end{aligned}$$

After the blowup relation, we changed variables in the summation using the symmetry relation, then related to the original range of the summation with discrepancy terms, and used the relations

$$\sum_{a\neq 0} (\langle a, 0\rangle + \langle -a, 0\rangle) = (p-1)\delta$$

and

$$\langle a, -a\rangle = 0 \iff \langle a, 0\rangle \overset{\text{(M)}}{=} \langle a, 0\rangle + \langle a, -a\rangle.$$

Finally, we obtain

$$(p-1)^2\delta = 2p(p-1)\delta,$$

which implies

$$(p^2-1)\delta = 0 \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}). \tag{11.7}$$

It follows that for all $a \neq 0$ we have the claimed identity

$$\langle a, 0\rangle + \langle -a, 0\rangle = 0 \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}. \qquad \blacksquare$$

Now we are ready to finish the proof of Proposition 30. By Lemma 31, the kernel of (11.2) is spanned (up to torsion) by elements of the form $\langle a, 0\rangle$. It follows from Lemma 32 that these elements can be written as

$$\langle a, 0\rangle = \tfrac{1}{2}(\langle a, 0\rangle - \langle -a, 0\rangle) \in \mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}.$$

Therefore, we get exactly the image of the right inverse (11.4). ■

Remark 33. The factor $p^2 - 1$ in (11.7) gives a partial explanation for the experimentally observed jumping behavior of $\dim(\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{F}_\ell)$ for primes $\ell \mid (p \pm 1)$ (see Section 8).

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