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Coulomb branches of quiver gauge theories with symmetrizers

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Abstract. We generalize the mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ SUSY quiver gauge theories due to Nakajima (2016) and Braverman et al. (2018, 2019) to the cases with *symmetrizers*. We obtain generalized affine Grassmannian slices of type $BCFG$ as examples of the construction, and their deformation quantizations via truncated shifted Yangians. Finally, we study modules over these quantizations and relate them to the lower triangular part of the quantized enveloping algebra of type ADE.

Keywords. Coulomb branches, quiver gauge theories, quivers with symmetrizers, shifted Yangian, affine Grassmannian, zastava spaces

1. Introduction

Let I be a finite set. Recall $(c_{ij})_{i,j\in I}$ is a *symmetrizable Cartan matrix* if

- $c_{ii} = 2$ for all $i \in I$, and $c_{ij} \in \mathbb{Z}_{\le 0}$ for all $i \neq j$,
- there is $(d_i) \in \mathbb{Z}^I_{>0}$ such that $d_i c_{ij} = d_j c_{ji}$ for all i, j.

When $d_i = 1$ for any $i \in I$, a mathematical definition of the Coulomb branch of a $3d$ $\mathcal{N} = 4$ quiver gauge theory associated with two I-graded vector spaces $V = \bigoplus V_i$, $W = \bigoplus W_i$ was given in [3, 33], and its properties were studied in [4]. In this note, we generalize the definition to more general symmetrizable cases. This new definition is motivated by works of Geiss, Leclerc and Schröer ([16] and the subsequent papers [15, 17–20]) which aim to generalize various results on relations between symmetric Kac–Moody Lie algebras and quivers to symmetrizable cases. They modify quiver representations by replacing vector spaces at vertices by free modules of truncated polynomial rings. They use different variables for each truncated polynomial ring, with the powers of these variables being related to each other according to the d_i. This modification allows

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them to relate quiver representations to symmetrizable Kac–Moody algebras. Their work, and ours, is also partly motivated by the theory of modulated graphs [8, 35], another approach to quivers in symmetrizable types.

In [3] we work with vector bundles over the formal disc $D = \operatorname{Spec}\mathbb{C}[[z]]$. Since we can take different variables z_i for each vertex $i \in I$, the definition has the same modification.

A similar construction was considered in the context of $4d$ $\mathcal{N} = 2$ quiver gauge theories by Kimura and Pestun [27] under the name of *fractional quiver gauge theories*.

Let us recall that we defined Coulomb branches of quiver gauge theories associated with a *symmetrizable* Cartan matrix in a different way in [4, §4]. There, we realize a symmetrizable Cartan matrix by a *folding* of a graph. This folding gives a finite group action on the Coulomb branch of the quiver gauge theory of the (unfolded) graph. Then we may define the Coulomb branch of the symmetrizable theory as the corresponding fixed point subscheme. This construction recovers the twisted monopole formula by Cremonesi, Ferlito, Hanany and Mekareeya [5] as the Hilbert series of the coordinate ring. This gives supporting evidence that the folding construction is a reasonable candidate for a mathematical definition of the Coulomb branch.

Our new construction also gives the twisted monopole formula. It is natural to believe that the folding construction and the new one give isomorphic varieties. However, various properties of the Coulomb branch are obvious in the new construction, while they are not in the old one. For example, the twisted monopole formula requires a proof in the old construction, while it is obvious in the new construction. We also do not know how to show the normality in the old construction, while the proof in [3] works for the new construction. Therefore we believe that the new construction is of independent interest. In addition, work in progress by the first-named author will identify the new definition with the symmetric bow varieties introduced by de Campos Affonso [7] for quiver gauge theories of non-symmetric affine Lie algebras of classical type. This identification is not clear for the old construction of the Coulomb branch as a fixed point subscheme.

In fact, in §C we will give a second potential definition for the Coulomb branch of a quiver gauge theory with symmetrizers. In many cases both definitions agree, and in particular this is true in finite *BCFG* types. However, in general type they are different. This alternative definition applies to more general data than quivers with symmetrizers, which may be of independent interest.

As a generalization of one of the main results in [4], we show that our Coulomb branches are generalized slices in the affine Grassmannian when the Cartan matrix is of type *BCFG* (Theorem 4.1). Therefore the geometric Satake correspondence, as modified in [29], says that the direct sum of hyperbolic stalks of the intersection cohomology complexes of our Coulomb branches has the structure of a finite-dimensional irreducible representation of the Langlands dual Lie algebra. We expect that the same should be true for arbitrary symmetrizable Kac–Moody Lie algebras, giving a symmetrizable generalization of the conjecture in [4, §3(viii)]. (See also [31] for a refinement of the conjecture.)

Also as a generalization of the main result in seven authors' ($\mathrm{BFK^2NW^2}$) appendices of [4] and also of [40], we show that the quantization of the Coulomb branch is a truncated shifted Yangian when the Cartan matrix is of type *BCFG* (see Theorem 5.8).

This algebra's modules can be analyzed by using techniques of the localization theorem in equivariant homology groups, even though we use infinite-dimensional varieties [34, 37, 39]. In §B we study the fixed point subvariety with respect to a $\mathbb{C}^\times$-action in an infinite-dimensional variety used in the definition of the Coulomb branch. It turns out that the fixed point subvariety is the same as one that appears in the Coulomb branches of type *ADE*, which is a disjoint union of varieties appearing in Lusztig's work on canonical bases of $\mathbf{U}_q^-$ of type *ADE* [30]. This implies that a certain category of modules of the truncated shifted Yangian of type *BCFG categorifies* $\mathbf{U}_q^-$ of type *ADE*. (See Theorem B.6. We only explain a parametrization of simple modules for simplicity.) It is interesting to understand the relation between this analysis and the geometric Satake correspondence explained above, as we obtain different Lie algebras, type *ADE* and *BCFG*.

Let us also remark that our construction can be applied to more general situations than considered here. For example, the first-named author originally introduced the Coulomb branch via cohomology with compact support of the moduli space of twisted maps from $\mathbb{P}^1$ to the Higgs branch $\mathcal{M}_H$ (viewed as a quotient stack) with coefficients in the sheaf of vanishing cycles [33]. This definition can be generalized to our setting, just changing the domain $\mathbb{P}^1$ for each vertex $i \in I$. This viewpoint might shed a new light on the Higgs branch $\mathcal{M}_H$ corresponding to our new construction: we cannot make sense of $\mathcal{M}_H$, but the space of maps to $\mathcal{M}_H$ *does* make sense. In particular, enumerative problems for $\mathcal{M}_H$, such as discussed in [36], are meaningful.

We also hope that our viewpoint is useful to make advance in the program of Geiss, Leclerc and Schröer. We may hope to use the above space of maps to $\mathcal{M}_H$ to realize representations of the Lie algebra, or its cousins the Yangian and the quantum loop algebra associated with the symmetrizable Cartan matrix (c_{ij}).

The paper is organized as follows. In §2 we give the definition of Coulomb branches for symmetrizable Cartan matrix (c_{ij}). Since it is a modification of the original one in [3], we only explain where we change the definition. In §3 we determine Coulomb branches in some cases when the Cartan matrix is 2×2, and there are no framed vector spaces W_i. In §4 we show that Coulomb branches are generalized slices in the affine Grassmannian when the Cartan matrix is of type *BCFG*. The proof is the same as in [4, §3], once examples in §3 have been studied. In §5 we discuss quantized Coulomb branches. We show that they are isomorphic to truncated shifted Yangians in type *BCFG*. In §A we give an explicit presentation of the coordinate ring of the zastava space of degree $\alpha_1 + \alpha_2$ of type G_2. This is used in §3. The contents of §B have already been explained above. In §C we present a second possible definition for the Coulomb branch associated to a quiver with symmetrizers, as mentioned above.

2. Definition

2(i). A valued quiver

Let $(c_{ij})_{i,j \in I}$ be a symmetrizable Cartan matrix. We assign to it a *valued graph* which has vertices $i \in I$ and unoriented edges between i, j for $c_{ij} < 0$ with values $(|c_{ij}|, |c_{ji}|)$.

A *valued quiver* is a valued graph together with a choice of an orientation of each edge. Following [16], we set $g_{ij} = \gcd(|c_{ij}|, |c_{ji}|)$, $f_{ij} = |c_{ij}|/g_{ij}$ when $c_{ij} < 0$. Note that these are independent of d_i.

We take the formal disc $D_i = \operatorname{Spec} \mathbb{C}[[z_i]]$ for each vertex $i \in I$. For a pair (i, j) with $c_{ij} < 0$ we take the formal disc $D = \operatorname{Spec} \mathbb{C}[[z]]$ and consider its branched coverings $\pi_{ji} \colon D_i \to D$, $\pi_{ij} \colon D_j \to D$ defined by the maps $\pi_{ji}^*(z) = z_i^{f_{ij}}$, $\pi_{ij}^*(z) = z_j^{f_{ji}}$ of coordinate rings. The disc D depends on (i, j), but we drop i, j from the notation. Let D_i^*, D_j^*, D^* denote the punctured formal disc for D_i, D_j, D respectively.

Remark 2.1. In [16] the relation (H2) $\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}$ is imposed, where ε_i, ε_j are edge loops at i and j respectively, and $\alpha_{ij}^{(g)}$ is the g-th arrow from j to i. It means that we have $z_i^{f_{ji}} = z = z_j^{f_{ij}}$. Thus it differs from our convention by $f_{ij} \leftrightarrow f_{ji}$. This is probably compatible with geometric Satake correspondence: We will obtain generalized slices in the affine Grassmannian for G for (c_{ij}) below, and hence representations of $G^\vee$, by the work of Krylov [29]. On the other hand, the space of constructible functions on modules over the quiver with the relation (H2) is the enveloping algebra of the upper triangular subalgebra $\mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ for (c_{ij}). Since we hope to compare representations of the same Lie algebra in Coulomb branches and [16], we need to take the Langlands dual relation of (H2).

Note also that the relation imposed in [22] for a cluster algebra related to the quantum loop algebra $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ is the same as ours. See [16, §1.7.1]. We believe that this is compatible with our results in §5, as the K-theoretic version of our construction in §5 should yield $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$-modules. However, the K-theoretic version of our construction does not immediately give a new approach to the results of [22]. Modules obtained in this way are infinite-dimensional, while [22] discussed Kirillov–Reshetikhin modules, which are finite-dimensional. Nevertheless, we expect that it gives a first step in that direction.

2(ii). *A moduli space*

Fix a valued quiver for $(c_{ij})_{i,j \in I}$. Let $V = \bigoplus V_i$, $W = \bigoplus W_i$ be finite-dimensional I-graded complex vector spaces. Let $\mathbf{v}_i = \dim V_i$ and $\mathbf{w}_i = \dim W_i$. We consider the moduli space $\mathcal{R}$ parametrizing the following objects:

- a rank $\mathbf{v}_i$ vector bundle $\mathcal{E}_i$ over D_i together with a trivialization
$$\varphi_i \colon \mathcal{E}_i|_{D_i^*} \to V_i \otimes_{\mathbb{C}} \mathcal{O}_{D_i^*} \quad \text{for } i \in I,$$
- a homomorphism $s_i \colon W_i \otimes_{\mathbb{C}} \mathcal{O}_{D_i} \to \mathcal{E}_i$ such that $\varphi_i \circ (s_i|_{D_i^*})$ extends to D_i for $i \in I$,
- a homomorphism $s_{ij} \in \mathbb{C}^{g_{ij}} \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}_D}(\pi_{ij*}\mathcal{E}_j, \pi_{ji*}\mathcal{E}_i)$ such that
$$(\pi_{ij*}\varphi_i) \circ (s_{ij}|_{D^*}) \circ (\pi_{ji*}\varphi_j)^{-1}$$
extends to D, where $c_{ij} < 0$ and there is an arrow $j \to i$ in the quiver.

The moduli space of pairs $(\mathcal{E}_i, \varphi_i)$ as above is the affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}(V_i)}$ for $\mathrm{GL}(V_i)$.

Dropping the extension conditions in the second and third bullets, we have a larger moduli space $\mathcal{T}$, which is an infinite rank vector bundle over $\prod_i \mathrm{Gr}_{\mathrm{GL}(V_i)}$. Then $\mathcal{R}$ is a closed subvariety in $\mathcal{T}$.

When $c_{ij} = c_{ji}$, $\mathcal{R}$ is nothing but the variety of triples introduced in [3, §2(i)].

Let $G = \prod_i \mathrm{GL}(V_i)$, $G_\mathcal{O} = \prod_i \mathrm{GL}(V_i)[[z_i]]$, $\mathrm{Gr}_G = \prod_i \mathrm{Gr}_{\mathrm{GL}(V_i)}$. We have a $G_\mathcal{O}$-action on $\mathcal{R}$ by change of trivializations φ_i, and we consider the $G_\mathcal{O}$-equivariant Borel–Moore homology group $H_*^{G_\mathcal{O}}(\mathcal{R})$ with complex coefficients. This is defined rigorously as a double limit as in [3, §2(ii)].

The spaces $\mathbf{N}_\mathcal{O}$, $\mathbf{N}_\mathcal{K}$ appear during the construction of the convolution product in [3, §3(i)]. They were the spaces of sections (resp. rational sections) of the vector bundle associated with the *trivial* G-bundle. In our setting, $\mathbf{N}_\mathcal{O}$ is defined as the direct sum of $\mathrm{Hom}_{\mathcal{O}_{D_i}}(W_i \otimes_\mathbb{C} \mathcal{O}_{D_i}, V_i \otimes_\mathbb{C} \mathcal{O}_{D_i})$ and $\mathbb{C}^{g_{ij}} \otimes_\mathbb{C} \mathrm{Hom}_{\mathcal{O}_D}(\pi_{ij*}(V_j \otimes_\mathbb{C} \mathcal{O}_{D_j}), \pi_{ji*}(V_i \otimes_\mathbb{C} \mathcal{O}_{D_i}))$. For $\mathbf{N}_\mathcal{K}$, we take homomorphisms over $\mathcal{O}_{D_i^*}$ and $\mathcal{O}_{D^*}$. We have maps $\Pi\colon \mathcal{R} \to \mathbf{N}_\mathcal{O}$ and $\mathcal{T} \to \mathbf{N}_\mathcal{K}$.

2(iii). *Twisted monopole formula*

Recall that the monopole formula for the Hilbert series of the Coulomb branch of a gauge theory [6] is interpreted as the Poincaré polynomial of $H_*^{G_\mathcal{O}}(\mathcal{R})$ with a suitable modification in the ordinary untwisted case [3, 33]. The twisted monopole formula is given in [5] to cover Coulomb branches of quiver gauge theories for certain symmetrizable Cartan matrices. It is of the same form $\sum_\lambda t^{2\Delta(\lambda)} P_G(t;\lambda)$ as the untwisted monopole formula, where the summation runs over the set of dominant coweights λ of the gauge group G, and $P_G(t;\lambda)$ is the Poincaré polynomial of the equivariant cohomology ring $H^*_{\mathrm{Stab}_G(\lambda)}(\mathrm{pt})$. Only $\Delta(\lambda)$ is changed from the untwisted monopole formula: if $i, j \in I$, the ordinary $2\Delta(\lambda)$ contains the contribution $|\lambda_i^a - \lambda_j^b|$, where $(\lambda_i^a)_{a=1,\dots,\mathrm{v}_i}$, $(\lambda_j^b)_{b=1,\dots,\mathrm{v}_j}$ are the components of λ for vertices i, j respectively. In the twisted monopole formula, this contribution is simply replaced by $|f_{ji}\lambda_i^a - f_{ij}\lambda_j^b|$.

Let us check that the Poincaré polynomial of our new $\mathcal{R}$ is given by the twisted monopole formula. The argument is a simple modification of [3, §2(iii)]. We do so under an additional assumption:

$$\text{For all } i, j \in I, \text{ if } c_{ij} < 0 \text{ then } f_{ij} = 1 \text{ or } f_{ji} = 1. \tag{2.2}$$

In particular, all finite types satisfy this assumption.

Let Gr_G^λ denote the $G_\mathcal{O}$-orbit in Gr_G corresponding to a dominant coweight λ of G. Let $\mathcal{R}_\lambda$ and $\mathcal{T}_\lambda$ denote the inverse images of Gr_G^λ under the projections $\pi\colon \mathcal{R} \to \mathrm{Gr}_G$ and $\pi\colon \mathcal{T} \to \mathrm{Gr}_G$ respectively. As in [3, Lem. 2.2], $\mathcal{T}_\lambda/\mathcal{R}_\lambda$, is a vector bundle over Gr_G^λ. The fiber of $\mathcal{T}_\lambda$ at λ is

$$\bigoplus_{j\to i} \mathbb{C}^{g_{ij}} \otimes z_i^{\lambda_i} z_j^{-\lambda_j} \,\mathrm{Hom}_{\mathbb{C}[[z]]}(V_j \otimes \mathbb{C}[[z_j]], V_i \otimes \mathbb{C}[[z_i]]),$$

while the fiber of $\mathcal{R}_\lambda$ is its intersection with $\bigoplus_{j\to i}\mathbb{C}^{g_{ij}}\otimes \operatorname{Hom}_{\mathbb{C}[[z]]}(V_j\otimes\mathbb{C}[[z_j]], V_i\otimes\mathbb{C}[[z_i]])$. Here $z=z_i^{f_{ij}}=z_j^{f_{ji}}$. Therefore the rank of $\mathcal{T}_\lambda/\mathcal{R}_\lambda$ is

$$d_\lambda := \sum_{i\to j} g_{ij}\sum_{a=1}^{\mathbf{v}_i}\sum_{b=1}^{\mathbf{v}_j}\max(f_{ij}\lambda_j^b - f_{ji}\lambda_i^a, 0).$$

Following the notation of [3, §2(iii)], we may formally write the Poincaré polynomial of $\mathcal{R}$. Let $\mathcal{R}_{\le\mu}$ denote the inverse image of the closure $\overline{\mathrm{Gr}_G^\mu}=\bigsqcup_{\lambda\le\mu}\mathrm{Gr}_G^\lambda$ in $\mathcal{R}$. As in [3, Prop. 2.7], we have

Proposition 2.3. *The Poincaré polynomial for $\mathcal{R}_{\le\mu}$ is given by*

$$P_t^{G_\mathcal{O}}(\mathcal{R}_{\le\mu}) = \sum_{\lambda\le\mu} t^{2d_\lambda - 4\langle\rho,\lambda\rangle}P_G(t;\lambda)$$

where the sum is over dominant coweights λ with $\lambda\le\mu$.

In particular, taking the limit over μ we formally obtain

$$P_t^{G_\mathcal{O}}(\mathcal{R}) = \sum_{\lambda} t^{2d_\lambda - 4\langle\rho,\lambda\rangle}P_G(t;\lambda). \tag{2.4}$$

However, we note that this expression may not converge even as a Laurent series.

The monopole formula is closely related to this Poincaré polynomial: the contribution $\Delta(\lambda)$ mentioned above is given by

$$\begin{aligned}\Delta(\lambda) &:= d_\lambda - 2\langle\rho,\lambda\rangle - \frac{1}{2}\sum_{i\to j} g_{ij}\sum_{a=1}^{\mathbf{v}_i}\sum_{b=1}^{\mathbf{v}_j}(f_{ij}\lambda_j^b - f_{ji}\lambda_i^a)\\ &= -\sum_i\sum_{1\le a<b\le\mathbf{v}_i}|\lambda_i^a-\lambda_i^b| + \frac{1}{2}\sum_{i\to j} g_{ij}\sum_{a=1}^{\mathbf{v}_i}\sum_{b=1}^{\mathbf{v}_j}|f_{ij}\lambda_j^b - f_{ji}\lambda_i^a|.\end{aligned}$$

The difference $\frac{1}{2}\sum_{i\to j} g_{ij}\sum_{a=1}^{\mathbf{v}_i}\sum_{b=1}^{\mathbf{v}_j}(f_{ij}\lambda_j^b - f_{ji}\lambda_i^a)$ depends only on the sums $\sum_a\lambda_i^a$ and $\sum_b\lambda_j^b$. In particular, it is possible to view the twisted monopole formula as the Poincaré polynomial of $\mathcal{R}$, but with respect to a *different* grading (i.e. different from the homological grading). See [3, Rem. 2.8].

Remark 2.5. If assumption (2.2) does not hold, then the ranks of $\mathcal{T}_\lambda/\mathcal{R}_\lambda$ are generally not given by such a simple formula. The corresponding Poincaré polynomial (and monopole formula) is thus more complicated. See Remark C.8.

2(iv). Convolution product

The definition of the convolution product on $H_*^{G_\mathcal{O}}(\mathcal{R})$ goes exactly as in [3, §3(iii)]. Moreover, we have an algebra embedding $\mathbf{z}^*\colon H_*^{G_\mathcal{O}}(\mathcal{R})\to H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ as in [3, §5(iv)], where $\mathbf{z}\colon \mathrm{Gr}_G\to\mathcal{R}$ is the embedding. The algebra $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is isomorphic to that which appears in the ordinary construction (where all z_i's are replaced by z). In particular, $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is commutative, and therefore $H_*^{G_\mathcal{O}}(\mathcal{R})$ is commutative as well. We define the Coulomb branch as

$$\mathcal{M}_C := \operatorname{Spec} H_*^{G_\mathcal{O}}(\mathcal{R}).$$

There is an algebra homomorphism $H_G^*(\mathrm{pt}) \to H_*^{G_\mathcal{O}}(\mathcal{R})$, as in [3, §3(vi)].

The algebra $H_*^{G_\mathcal{O}}(\mathcal{R})$ is filtered in the same way as in [3, §6(i)], and as in [3, Prop. 6.8] we can prove that $\mathcal{A}$ is finitely generated.

The proof of normality in [3, Prop. 6.12] was given by reduction to the cases when the gauge group is $\mathbb{C}^\times$, SL(2) or PGL(2). That argument is applicable in our situation, and we are reduced to the case of a quiver with a single vertex under assumption (2.2). Then our modification of the definition of the Coulomb branch is unnecessary and we return to the original situation. Therefore we see that $\mathcal{M}_C$ is normal under assumption (2.2). In fact, normality always holds in the second definition given in Appendix C. Assumption (2.2) guarantees that the two definitions coincide. See Theorem C.7. Therefore the details of the proof of normality will be given in Appendix C.

Finally, $\mathcal{M}_C$ has a natural deformation quantization $\mathcal{A}_\hbar$ defined below in §5. This endows $\mathcal{M}_C$ with a Poisson structure, which is generically symplectic as in [3, Prop. 6.15]. The subalgebra $H_G^*(\mathrm{pt})$ is Poisson commutative, and defines an integrable systems on $\mathcal{M}_C$ (see [3, §1(iii)]).

Theorem 2.6. *Assume* (2.2)*. Then $\mathcal{M}_C$ is an irreducible normal variety of finite type. It carries a Poisson structure which is generically symplectic, with an integrable system* $\mathcal{M}_C \to \operatorname{Spec} H_G^*(\mathrm{pt})$.

Remark 2.7. As in [4, Rem. 3.9(3)], one can also consider the K-theoretic Coulomb branch.

3. Examples

3(i)

We consider the case $I = \{1,2\}$, $c_{12} = -1$, $c_{21} = -m$ (where $m \in \mathbb{Z}_{>0}$), $\mathbf{w}_1 = \mathbf{w}_2 = 0$, and $\mathbf{v}_1 = \mathbf{v}_2 = 1$. We choose the orientation $1 \leftarrow 2$. Note that G is the two-dimensional torus. We consider the embedding $\mathbf{z}^*\colon H_*^{G_\mathcal{O}}(\mathcal{R}) \to H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ of [3, §5(iv)]. Let w_1, w_2 be generators of the equivariant cohomology ring of a point for the first and second factors of $G = (\mathbb{C}^\times)^2$. Let $\mathsf{u}_{a,b}$ denote the fundamental class of the point $(a,b) \in \mathrm{Gr}_G = \mathbb{Z}^2$. We have

$$H_*^{G_\mathcal{O}}(\mathrm{Gr}_G) = \bigoplus_{a,b} \mathbb{C}[w_1, w_2]\mathsf{u}_{a,b}$$

with $\mathsf{u}_{a,b}\mathsf{u}_{a',b'} = \mathsf{u}_{a+a',b+b'}$. Let $\mathsf{y}_{a,b}$ denote the fundamental class of the fiber of $\mathcal{R} \to \mathrm{Gr}_G$ over (a,b). Then we have

$$\mathbf{z}^*(w_1) = w_1, \quad \mathbf{z}^*(w_2) = w_2, \quad \mathbf{z}^*(\mathsf{y}_{a,b}) = (w_1 - w_2)^{\max(b-ma,0)}\mathsf{u}_{a,b}.$$

Note that $\mathsf{y}_{1,m} = \mathsf{u}_{1,m}$ is invertible, with inverse $\mathsf{y}_{-1,-m}$. Therefore

$$H_*^{G_\mathcal{O}}(\mathcal{R}) \cong \mathbb{C}[w_1, \mathsf{y}_{1,m}^{\pm}, \mathsf{y}_{0,1}, \mathsf{y}_{0,-1}]$$

Note, for example, $w_1 - w_2 = \mathsf{y}_{0,1}\mathsf{y}_{0,-1}$. Thus the Coulomb branch is $\mathcal{M}_C = \mathbb{A}^3 \times \mathbb{A}^\times$.

On the other hand, let us consider the folding of the Coulomb branch of the quiver gauge theory $I = \{1, 2_1, 2_2, \dots, 2_m\}$ with edges 1–2_j for all $j = 1, \dots, m$ with $\mathbf{w}_i = 0$, $\mathbf{v}_i = 1$ for all $i \in I$. We consider the $\mathbb{Z}/m$-action on the quiver given by $2_1 \to 2_2 \to \cdots \to 2_m \to 2_1$. See Figure 1 for $m = 3$. In order to distinguish the two groups for this theory and the former gauge theory, let us write $\hat{G} = \prod \mathrm{GL}(V_i)$. Note that the diagonal scalar $\mathbb{C}^\times$ in $\hat{G}$ acts trivially on $\mathbf{N}$, and so we have $\hat{G} \cong \mathbb{C}^\times \times (\mathbb{C}^\times)^m$, $\mathbf{N} = \mathbb{C}^m$ where the first $\mathbb{C}^\times$ acts trivially on $\mathbf{N}$, and $(\mathbb{C}^\times)^m$ acts on $\mathbf{N}$ in the standard way. Therefore the (usual) Coulomb branch for $(\hat{G}, \mathbf{N})$ is $\widehat{\mathcal{M}}_C = \mathbb{A} \times \mathbb{A}^\times \times (\mathbb{A}^2)^m$ and $\mathbb{Z}/m$ acts by cyclically permuting the factors of $(\mathbb{A}^2)^m$. Therefore the fixed point locus is also $\mathbb{A} \times \mathbb{A}^\times \times \mathbb{A}^2$. Thus the former Coulomb branch is isomorphic to the fixed point locus of the latter Coulomb branch:

Proposition 3.1. *For the above data, there is an isomorphism* $\mathcal{M}_C \cong (\widehat{\mathcal{M}}_C)^{\mathbb{Z}/m}$.

More concretely, w_1, w_2 are identified with equivariant variables for $\mathrm{GL}(V_1)$ and $\mathrm{GL}(V_{2_j})$, where the latter is independent of j on the $\mathbb{Z}/m$-fixed point locus. The function $\mathsf{y}_{a,b}$ is identified with the restriction of the function $\hat{\mathsf{y}}_{a,b_1,\dots,b_m}$ on $\widehat{\mathcal{M}}_C$ given by the fundamental class over $(a, b_1, \dots, b_m) \in \mathrm{Gr}_{\hat{G}} = \mathbb{Z}^{1+m}$ where $b_1 \ge \cdots \ge b_m \ge b_1 - 1$ and $b = b_1 + \cdots + b_m$ (cf. the proof of [4, Prop. 4.1]).

In the cases $m = 2, 3$, we can identify our modified Coulomb branch with an *open zastava space* $\mathring{Z}^\alpha$. Recall that $\mathring{Z}^\alpha$ is the moduli space of based maps from $\mathbb{P}^1$ to the flag variety, of degree α (see [1, §2], [4, §2(i)]).

Lemma 3.2. *For $m = 2$ (resp. $m = 3$), $\mathcal{M}_C$ is isomorphic to the open zastava space $\mathring{Z}^{\alpha_1+\alpha_2}$ of type B_2 (resp. type G_2).*

Proof. Explicitly, we identify our description for $m = 2$ with the B_2 type open zastava space [1, §5.7][1] by $w_1 = -A_2$, $w_2 = -A_1$, $\mathsf{y}_{1,2} = b_{03}$, $\mathsf{y}_{0,1} = b_{01}$, and $\mathsf{y}_{0,-1} = b_{02}b_{03}^{-1}$, noticing that b_{03} is invertible.

Similarly, in the $m = 3$ case we appeal to the description of the G_2 type open zastava in terms of coordinates, from §A. ∎

Another perpsective on this result is via folding. Recall that there is an étale rational coordinate system $(y_{i,r}, w_{i,r})_{i \in I, 1 \le r \le \mathbf{v}_i}$ on the open zastava space $\mathring{Z}^\alpha$ for finite type [1, 10]. We claim that it is compatible with folding of the same type described above, namely the coordinate system for B_2, G_2 is the restriction of the coordinate system for A_3, D_4 to the $\mathbb{Z}/m$-fixed point ($m = 2, 3$), respectively. For B_2 with the above choice of $\mathbf{v}$, this can be checked directly from [1, §5.7] as $y_i = b_{01} = \mathsf{y}_{0,1}$, $y_j = b_{12} = b_{02}^2 b_{03}^{-1} = \mathsf{y}_{1,0}$.

[1] d_i in [1] is the square length of the simple *coroot* for i, while it is of the simple *root* here. Therefore i (resp. j) in [1] is 2 (resp. 1) here.

In general, it is enough to check the assertion when $\mathbf{v}_i$ is one-dimensional for a single vertex i and 0 otherwise by the compatibility of the coordinate system and the factorization in [1, Th. 1.6(3)], as the factorization and folding are compatible. In that case, a based map factors through $\mathbb{P}^1$ via the embedding of $\mathbb{P}^1$ into the flag variety corresponding to the vertex i. Then the assertion is clear. Alternatively we use the description from §A for G_2 to argue as in the B_2 case.

Recall that the isomorphism between $\mathring{Z}^\alpha$ and the corresponding Coulomb branch was defined so that the coordinate system $(y_{i,r}, w_{i,r})$ is mapped to $(\mathsf{y}_{i,r}, w_{i,r})$, where the latter $w_{i,r}$ is an equivariant variable as above, and $\mathsf{y}_{i,r}$ is the fundamental class of the fiber over the point corresponding to $w_{i,r}$ [4, §3]. Since the coordinate system is compatible with the above folding, and $\mathsf{y}_{1,0}$, $\mathsf{y}_{0,1}$ for B_2, G_2 are restrictions of appropriate $\hat{\mathsf{y}}_{1,0,\dots,0}$, $\hat{\mathsf{y}}_{0,1,0,\dots,0}$ for A_3, D_4, the coordinate system $(y_{1,0}, y_{0,1}, w_1, w_2)$ for B_2, G_2 is identified with $(\mathsf{y}_{1,0}, \mathsf{y}_{0,1}, w_1, w_2)$.

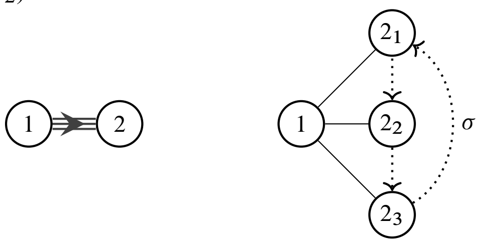


Fig. 1. G_2 and the folding of D_4.

3(ii)

As in [3, §3(ii)], we may consider the positive part Gr_G^+ of the affine Grassmannian Gr_G, the subvariety consisting of $(\mathcal{E}_i, \varphi_i)$ such that φ_i extends through the puncture as an embedding $\mathcal{E}_i \hookrightarrow V_i \otimes_{\mathbb{C}} \mathcal{O}_{D_i}$. We define $\mathcal{R}^+ \subset \mathcal{R}$ as the preimage. Then $H_*^{G_\mathcal{O}}(\mathcal{R}^+)$ forms a convolution subalgebra of $H_*^{G_\mathcal{O}}(\mathcal{R})$, equipped with an algebra homomorphism $H_*^{G_\mathcal{O}}(\mathcal{R}^+) \to H_G^*(\mathrm{pt})$.

Let us consider $H_*^{G_\mathcal{O}}(\mathcal{R}^+)$ for the example in §3(i). It is the subalgebra generated by w_1, w_2, $\mathsf{y}_{a,b}$ with $a, b \geq 0$. It is easy to check that it is, in fact, generated by w_1, w_2, $\mathsf{y}_{0,1}, \mathsf{y}_{1,0}, \mathsf{y}_{1,1}, \dots, \mathsf{y}_{1,m}$. We have $\mathsf{y}_{1,b}\mathsf{y}_{0,1} = (w_1 - w_2)\mathsf{y}_{1,b+1}$ for $0 \leq b \leq m-1$, and $\mathsf{y}_{1,b_1}\mathsf{y}_{1,b_2}\cdots = \mathsf{y}_{1,b_1'}\mathsf{y}_{1,b_2'}\cdots$ for $0 \leq b_i, b_i' \leq m$ with $b_1 + b_2 + \cdots = b_1' + b_2' + \cdots$.

If $m = 2$, the only non-trivial relation of the latter type is $\mathsf{y}_{1,0}\mathsf{y}_{1,2} = \mathsf{y}_{1,1}^2$. This coincides with the presentation of the B_2 type zastava space [1, §5.7] by $w_1 = -A_2$, $w_2 = -A_1$, $\mathsf{y}_{0,1} = b_{01}$, $\mathsf{y}_{1,0} = b_{12}$, $\mathsf{y}_{1,1} = b_{02}$, $\mathsf{y}_{1,2} = b_{03}$.

If $m = 3$, we have two more relations $\mathsf{y}_{1,0}\mathsf{y}_{1,3} = \mathsf{y}_{1,1}\mathsf{y}_{1,2}$, $\mathsf{y}_{1,1}\mathsf{y}_{1,3} = \mathsf{y}_{1,2}^2$. We could not find this presentation of the zastava space for G_2 for degree $\alpha_1 + \alpha_2$ in the literature. Therefore we include the proof in Appendix A.

Together, we obtain

Lemma 3.3. *For $m = 2$ (resp. $m = 3$), $\operatorname{Spec} H_*^{G_\mathcal{O}}(\mathcal{R}^+)$ is isomorphic to the zastava space $Z^{\alpha_1+\alpha_2}$ of type B_2 (resp. type G_2).*

Remark 3.4. For general m, a complete set of relations of the latter type is as follows: for all $1 \le a \le b < m$,

$$\mathsf{y}_{1,a}\mathsf{y}_{1,b} = \begin{cases} \mathsf{y}_{1,0}\mathsf{y}_{1,a+b} & \text{if } a+b \le m, \\ \mathsf{y}_{1,a+b-m}\mathsf{y}_{1,m} & \text{if } a+b > m. \end{cases}$$

4. Slices

Consider an adjoint group $\mathcal{G}$ of *BCFG* type, with fundamental coweights $\{\Lambda_i\}$ and simple coroots $\{\alpha_i\}$. Given a dominant coweight λ for $\mathcal{G}$, and a coweight μ such that $\lambda \ge \mu$, we define the corresponding *generalized affine Grassmannian slice* $\overline{\mathcal{W}}^{\lambda}_{\mu}$ as in [4, §2(ii)]. Recall that when μ is itself dominant, $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is isomorphic to an ordinary affine Grassmannian slice in $\mathrm{Gr}_{\mathcal{G}}$ as defined in [2, §2], [24, §2B].

The proofs of properties of $\overline{\mathcal{W}}^{\lambda}_{\mu}$, given in [4, §2], work for non-simply-laced types. In particular, $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is Cohen–Macaulay, normal, and affine. It has an integrable system $\overline{\mathcal{W}}^{\lambda}_{\mu} \to \mathbb{A}^{\alpha}$ where $\alpha = \lambda - \mu$, which satisfies factorization as in [4, §2(ix)].

Thanks to the analysis in the previous section, we can apply the argument in [4, §3] to symmetrizable cases:

Theorem 4.1. *Suppose that the valued quiver is of type BCFG, with adjoint group $\mathcal{G}$ as above.*

(1) *Suppose $W = 0$. Then $\mathcal{M}_C = \operatorname{Spec} H^{G_{\mathcal{O}}}_*(\mathcal{R})$ is isomorphic to the open zastava space $\mathring{Z}^{\alpha}$ for $\mathcal{G}$ of degree $\alpha = \sum_i \dim V_i \cdot \alpha_i$.*

(2) *Suppose $W \neq 0$. Then $\mathcal{M}_C = \operatorname{Spec} H^{G_{\mathcal{O}}}_*(\mathcal{R})$ is isomorphic to the generalized slice $\overline{\mathcal{W}}^{\lambda}_{\mu}$ for $\mathcal{G}$ where λ, μ are given by $\lambda = \sum_i \dim W_i \cdot \Lambda_i$ and $\mu = \lambda - \sum_i \dim V_i \cdot \alpha_i$.*

(3) *The spectrum $\operatorname{Spec} H^{G_{\mathcal{O}}}_*(\mathcal{R}^+)$ is isomorphic to the zastava space Z^{α} for $\mathcal{G}$ of degree $\alpha = \sum_i \dim V_i \cdot \alpha_i$.*

Proof. Since the proofs are essentially the same as in [4, §3], we just indicate the differences. To prove parts (1)–(3) we wish to appeal to [3, Thm. 5.26]: in a certain precise sense, it suffices to identify the varieties in codimension 1. This result generalizes to our present setting, with the same proof.

For (1), (2) we follow the proof of [4, Thms. 3.1, 3.10]. With the same notation, the only difference occurs when comparing the varieties in the case when t lies on a diagonal divisor $(w_{i,r} - w_{j,s})(t) = 0$ where $i \neq j$. In our present *BCFG* setting, we may meet factors of open zastava $\mathring{Z}^{\alpha_1+\alpha_2}$ of type B_2, G_2 in addition to the usual $A_1 \times A_1$ and A_2 types already discussed in [4, Rem. 2.2]. In these new cases we apply Lemma 3.2 to complete the proof.

For (3) we follow [4, Rem. 3.15], this time making use of Lemma 3.3. ■

Remark 4.2. Part (2) extends to relate the *flavor symmetry deformation* of $\mathcal{M}_C$ to a *BD* slice, generalizing [4, Thm. 3.20]. The same is true for line bundles and partial resolutions, generalizing [26, Thm. 5.5].

5. Quantization

In this section, we connect the deformed algebra $H_*^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times}(\mathcal{R})$ with truncated shifted Yangians in type *BCFG*, extending the results of [4, App. B].

5(i). Loop rotation

To discuss the deformation $H_*^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times}(\mathcal{R})$, we must first choose a $\mathbb{C}^\times$-action on $\mathcal{R}$. This action will depend on a choice of symmetrizers $(d_i)\in\mathbb{Z}_{>0}^I$ for our Cartan matrix $(c_{ij})_{i,j\in I}$. We define a $\mathbb{C}^\times$-action on $\mathbb{C}[[z_i]]$ by

$$z_i \mapsto z_i\tau^{d_i} \quad (\tau\in\mathbb{C}^\times).$$

Then the equality $z_i^{f_{ij}} = z_j^{f_{ji}}$ is preserved, as $d_i f_{ij} = d_j f_{ji}$. Therefore we have an induced $\mathbb{C}^\times$-action on $\mathcal{R}$.

5(ii). Embedding into the ring of difference operators

Consider a valued quiver along with vector spaces $V=\bigoplus V_i$ and $W=\bigoplus W_i$ as above. Consider the deformed algebra

$$\mathcal{A}_\hbar := H_*^{\widetilde{G}_{\mathcal{O}}\rtimes\mathbb{C}^\times}(\mathcal{R}),$$

where $\widetilde{G}=G\times T(W)$ with $T(W)\subset\prod_i \mathrm{GL}(W_i)$ the standard maximal torus, and where the $\mathbb{C}^\times$-action on $\widetilde{G}_{\mathcal{O}}$ and $\mathcal{R}$ is induced by the action on $\mathbb{C}[[z_i]]$ as in the previous section. We choose a basis $t_1,\dots,t_N$ of the character lattice of $T(W)$ compatible with the product decomposition $T(W)=\prod_i T(W_i)$. Thus $\mathcal{A}_\hbar$ is naturally an algebra over

$$H^*_{T(W)\times\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[\hbar,t_1,\dots,t_N],$$

which is a central subalgebra (see [3, §3(viii)]).

As in [4, App. A(i)–A(ii)], we can construct an embedding

$$\mathbf{z}^*(\iota_*)^{-1}\colon \mathcal{A}_\hbar \hookrightarrow \widetilde{\mathcal{A}}_\hbar$$

where we define an algebra

$$\widetilde{\mathcal{A}}_\hbar := \mathbb{C}[\hbar,t_1,\dots,t_N]\langle w_{i,r},\mathsf{u}_{i,r}^{\pm1},\hbar^{-1},(w_{i,r}-w_{i,s}+md_i\hbar)^{-1} : i\in Q_0, 1\le r\neq s\le \mathbf{v}_i, m\in\mathbb{Z}\rangle$$

by the relations $[\mathsf{u}_{i,r}^{\pm1},w_{j,s}] = \pm\delta_{i,j}\delta_{r,s}\hbar d_i\mathsf{u}_{i,r}^{\pm1}$ (all other elements commute). Note that $\widetilde{\mathcal{A}}_\hbar$ is a localization of $H_*^{T\times T(W)\times\mathbb{C}^\times}(\mathrm{Gr}_T)$.

For the homology classes of $\mathcal{R}$ associated to preimages $\mathcal{R}_\lambda$ of closed $G_{\mathcal{O}}$-orbits, we can explicitly write down the image under the map $\mathbf{z}^{-1}(\iota_*)^{-1}$, following [4, Prop. A.2]. Let λ be a miniscule dominant coweight, $W_\lambda\subset W$ its stabilizer, and $f\in\mathbb{C}[\mathfrak{t}]^{W_\lambda}$. Then

$$\mathbf{z}^*(\iota_*)^{-1} f[\mathcal{R}_\lambda] = \sum_{\lambda'=w\lambda\in W\lambda} \frac{wf\times e_{\lambda'}}{e(T_{\lambda'}\mathrm{Gr}_G^\lambda)}\mathsf{u}_{\lambda'}$$

where $e_{\lambda'}$ denotes the Euler class of the fiber of $\mathcal{T}$ over λ' modulo the fiber of $\mathcal{R}$ over λ'. Following [4, §A(ii)], we will compute these classes for the cocharacters $\varpi_{i,n}$ and $\varpi_{i,n}^*$ of $\mathrm{GL}(V_i)$. We find

$$\mathbf{z}^*(\iota_*)^{-1} f(\mathcal{Q}_i)[\mathcal{R}_{\varpi_{i,n}}] = \sum_{\substack{I\subset\{1,\dots,\mathbf{v}_i\}\\ \#I=n}} f(w_{i,I}) \times \frac{\prod_{\substack{h\in Q_1: o(h)=i\\ r\in I}} \prod_{s=1}^{\mathbf{v}_{i(h)}} \prod_{p=0}^{f_{i(h),i}-1} (-w_{i,r} + w_{i(h),s} + (-d_i f_{i,i(h)} + p d_{i(h)})\hbar)^{g_{i,i(h)}}}{\prod_{r\in I, s\notin I} (w_{i,r} - w_{i,s})} \prod_{r\in I} \mathsf{u}_{i,r} \tag{5.1}$$

and

$$\mathbf{z}^*(\iota_*)^{-1} f(\mathcal{S}_i)[\mathcal{R}_{\varpi_{i,n}^*}] = \sum_{\substack{I\subset\{1,\dots,\mathbf{v}_i\}\\ \#I=n}} f(w_{i,I} - d_i\hbar) \prod_{\substack{r\in I\\ k: i_k=i}} (w_{i,r} - t_k - d_i\hbar) \times \frac{\prod_{\substack{h\in Q_1: i(h)=i\\ r\in I}} \prod_{s=1}^{\mathbf{v}_{o(h)}} \prod_{p=0}^{f_{o(h),i}-1} (w_{i,r} - w_{o(h),s} - (d_i + p d_{o(h)})\hbar)^{g_{i,o(h)}}}{\prod_{r\in I, s\notin I} (-w_{i,r} + w_{i,s})} \prod_{r\in I} \mathsf{u}_{i,r}^{-1}. \tag{5.2}$$

5(iii). Shifted Yangians

The definition of shifted Yangians in [4, Def. B.2] extends naturally to all finite types. Thus in *BCFG* types, for any coweight μ there is a corresponding shifted Yangian Y_μ. It is a $\mathbb{C}$-algebra, with generators $E_i^{(q)}, F_i^{(q)}, H_i^{(p)}$ for $i \in Q_0$, $q > 0$ and $p > -\langle \alpha_i^\vee, \mu\rangle$. Here $\alpha_i^\vee$ denotes the simple *root* for $i \in I$.

The properties of Y_μ established in [9] have straightforward extensions to all finite types. In particular, Y_μ has a PBW basis, and for any coweights μ_1, μ_2 with $\mu = \mu_1 + \mu_2$ there is a filtration $F^\bullet_{\mu_1,\mu_2} Y_\mu$ of Y_μ. The associated graded $\mathrm{gr}^{F_{\mu_1,\mu_2}} Y_\mu$ is commutative, and the Rees algebras $\mathrm{Rees}^{F_{\mu_1,\mu_2}} Y_\mu$ are all canonically isomorphic as algebras (although not as graded algebras). For the purposes of this paper, we will choose μ_1, μ_2 as follows:

$$\langle \mu_1, \alpha_i^\vee\rangle = \langle \lambda, \alpha_i^\vee\rangle - \mathbf{v}_i + \sum_{h: i(h)=i} \mathbf{v}_{o(h)} c_{o(h),i}, \quad \langle \mu_2, \alpha_i^\vee\rangle = -\mathbf{v}_i + \sum_{h: o(h)=i} \mathbf{v}_{i(h)} c_{i(h),i}.$$

We write $\mathbf{Y}_\mu := \mathrm{Rees}^{F_{\mu_1,\mu_2}} Y_\mu$ for the corresponding Rees algebra, which we view as a graded algebra over $\mathbb{C}[\hbar]$ with $\deg \hbar = 1$.

Below, we work with the larger algebra $Y_\mu[t_1, \dots, t_N] = Y_\mu \otimes_{\mathbb{C}} \mathbb{C}[t_1, \dots, t_N]$, where $N = \sum_i \mathbf{w}_i$. The filtration F_{μ_1,μ_2} extends to $Y_\mu[t_1, \dots, t_N]$ by placing all t_i in degree 1. We denote the corresponding Rees algebra by $\mathbf{Y}_\mu[t_1, \dots, t_N]$.

Denote

$$T_i(t) = \prod_{k:i_k=i} (t - t_k - d_i\hbar),$$

and define elements $A_i^{(s)} \in Y_\mu[t_1, \ldots, t_N]$ for $s > 0$ according to

$$H_i(t) = T_i(t) \frac{\prod_{j \neq i} \prod_{p=1}^{-c_{ji}} (t - \frac{1}{2} d_i c_{ij} - p d_j)^{\mathrm{v}_j}}{t^{\mathrm{v}_i}(t - d_i)^{\mathrm{v}_i}} \frac{\prod_{j \neq i} \prod_{p=1}^{-c_{ji}} A_j(t - \frac{1}{2} d_i c_{ij} - p d_j)}{A_i(t) A_i(t - d_i)} \tag{5.3}$$

where

$$H_i(t) = t^{\mu_i} + \sum_{r > -\mu_i} H_i^{(r)} t^{-r}, \quad A_i(t) = 1 + \sum_{s>0} A_i^{(s)} t^{-p}.$$

5(iv). A representation using difference operators

Recall the $\mathbb{C}[\hbar, t_1, \ldots, t_N]$-algebra $\widetilde{\mathcal{A}}_\hbar$ defined in §5(ii). This algebra has a grading, defined by $\deg \hbar = \deg t_k = \deg w_{i,r} = 1$ and $\deg \mathsf{u}_{i,r}^{\pm 1} = 0$. Denote

$$W_i(t) = \prod_{s=1}^{\mathrm{v}_i} (t - w_{i,s}), \quad W_{i,r}(t) = \prod_{\substack{s=1 \\ s \neq r}}^{\mathrm{v}_i} (t - w_{i,s})$$

The following result is a common generalization of [4, Cor. B.17] and [24, Th. 4.5], which were in turn generalizations of work of Gerasimov–Kharchev–Lebedev–Oblezin [21].

Theorem 5.4. *There is a homomorphism of graded $\mathbb{C}[\hbar, t_1, \ldots, t_N]$-algebras*

$$\Phi_\mu^\lambda : \mathbf{Y}_\mu[t_1, \ldots, t_N] \to \widetilde{\mathcal{A}}_\hbar,$$

defined by

$$A_i(t) \mapsto t^{-\mathrm{v}_i} W_i(t),$$

$$E_i(t) \mapsto -d_i^{-1/2} \sum_{r=1}^{\mathrm{v}_i} T_i(w_{i,r}) \frac{\prod_{h \in Q_1 : i(h)=i} \prod_{p=1}^{-c_{o(h),i}} W_{o(h)}(w_{i,r} - (\frac{1}{2} d_i c_{i,o(h)} + p d_{o(h)})\hbar)}{(t - w_{i,r}) W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}^{-1},$$

$$F_i(t) \mapsto d_i^{-1/2} \sum_{r=1}^{\mathrm{v}_i} \frac{\prod_{h \in Q_1 : o(h)=i} \prod_{p=1}^{-c_{i(h),i}} W_{i(h)}(w_{i,r} - (\frac{1}{2} d_i c_{i,i(h)} - d_i + p d_{i(h)})\hbar)}{(t - w_{i,r} - d_i\hbar) W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}$$

In simply-laced type, a proof of this theorem was given in [4, §B(iii)–B(vii)]. In all finite types, a generalization of this theorem to shifted quantum affine algebras was proven in [12]. We thus omit the proof.

5(v). Relation to the quantized Coulomb branch

Consider the setup of §5(ii), restricted to *BCFG* type. Recall that in this case $g_{ij} = 1$ and thus $f_{ij} = |c_{ij}|$ whenever $c_{ij} < 0$. With this in mind, we see that the right-hand sides of (5.1), (5.2) for $n = 1$ are nearly identical to the images $\Phi^\lambda_\mu(F_i^{(r)})$, $\Phi^\lambda_\mu(E_i^{(r)})$ from the previous theorem, modulo shifts by $\hbar$ in the respective numerators.

Choose $\sigma_i \in \mathbb{Q}$ for each $i \in Q_0$, which solve the following system of equations: for each $h \in Q_1$, we require that

$$\tfrac{1}{2} d_{o(h)} c_{o(h),i(h)} = \sigma_{o(h)} - \sigma_{i(h)} - d_{o(h)} + d_{i(h)}. \tag{5.5}$$

Since (Q_0, Q_1) is an orientation of a tree, a solution exists and is unique up to an overall additive shift. Moreover, there is a solution with $\sigma_i \in \frac{1}{2}\mathbb{Z}$. However, in general these equations depend upon the choice of orientation of the Dynkin diagram.

Theorem 5.6. *Fix $\sigma_i \in \mathbb{Q}$ satisfying (5.5). Then there is a unique graded $\mathbb{C}[\hbar, t_1, \ldots, t_N]$-algebra homomorphism*

$$\overline{\Phi}^\lambda_\mu : \mathbf{Y}_\mu[t_1, \ldots, t_N] \to \mathcal{A}_\hbar$$

such that

$$\begin{aligned} A_i^{(r)} &\mapsto (-1)^p e_r(\{w_{i,r} - \sigma_i \hbar\}),\\ E_i^{(r)} &\mapsto (-1)^{\mathrm{v}_i} d_i^{-1/2} (c_1(\mathcal{S}_i) + (d_i - \sigma_i)\hbar)^{r-1} \cap [\mathcal{R}_{\varpi^*_{i,1}}],\\ F_i^{(r)} &\mapsto (-1)^{\sum_{h: o(h)=i} a_{i(h),i} \mathrm{v}_{i(h)}} d_i^{-1/2} (c_1(\mathcal{Q}_i) + (d_i - \sigma_i)\hbar)^{r-1} \cap [\mathcal{R}_{\varpi_{i,1}}]. \end{aligned}$$

Remark 5.7. The $\sigma_i \in \mathbb{Q}$ play the role of a “shift” in the action of the loop rotation from [3, §2(i)], where the loop $\mathbb{C}^\times$ also acts on $\mathbf{N}$ by weight $1/2$. Indeed, in our present setting we could modify the loop action of $\mathbb{C}^\times$ from §5(i), so that it also scales V_i, W_i with weight σ_i. Thus when acting on $\mathcal{R}$, in addition to rotating the discs D_i, $\tau \in \mathbb{C}^\times$ scales the morphism s_{ij} by $\tau^{\sigma_i - \sigma_j}$, and scales s_i by 1. (This may require taking a finite covering of $\mathbb{C}^\times$; a double covering will suffice, since we can take $\sigma_i \in \frac{1}{2}\mathbb{Z}$. However, we do not introduce new notation for this covering.) With this modified action, no shifts by σ_i would be needed in the statement of the theorem. Note that since this modified $\mathbb{C}^\times$-action factors through the usual action of $G \times \mathbb{C}^\times$, the modified algebra is isomorphic to the original one (cf. [3, Rem. 3.24(2)]).

Proof of Theorem 5.6. We may argue using the previous remark, and modify the loop $\mathbb{C}^\times$-action while preserving the algebra $\mathcal{A}_\hbar$ up to isomorphism. We give an equivalent elementary argument:

Consider the automorphism σ of $\widetilde{\mathcal{A}}_\hbar$ defined by $w_{i,r} \mapsto w_{i,r} + \sigma_i \hbar$ and $t_k \mapsto t_k + \sigma_{i_k} \hbar$, while fixing the generators $\hbar, \mathsf{u}_{i,r}^{\pm 1}$. We claim that in $\widetilde{\mathcal{A}}_\hbar$ we have the equalities

$$\Phi^\lambda_\mu(x) = \sigma \circ \mathbf{z}^*(\iota_*)^{-1}(y),$$

where $x \in \{A_i^{(r)}, E_i^{(r)}, F_i^{(r)}\}$, and where $y \in \mathcal{A}_\hbar$ is the claimed image $\overline{\Phi}^\lambda_\mu(x)$ from the statement of the theorem. For the elements $x = A_i^{(r)}$ this is obvious. For $x = E_i^{(r)}$, we are reduced to verifying that the shifts by $\hbar$ that appear in the numerators of $\Phi_\mu^\lambda(E_i^{(r)})$ and (5.2) agree. This is equivalent to (5.5) for those $h \in Q_1$ with $i(h) = i$. The case $x = F_i^{(r)}$ is similar, and is equivalent to those equalities where $o(h) = i$, proving the claim.

The elements $A_i^{(r)}, E_i^{(r)}, F_i^{(r)}$ generate $\mathbf{Y}_\mu[t_1, \ldots, t_N]$ as a Poisson algebra, under the Poisson bracket $\{a, b\} = \frac{1}{\hbar}(ab - ba)$. Since $\mathcal{A}_\hbar$ is almost commutative, it is closed under Poisson brackets. It follows that there is a containment of graded $\mathbb{C}[\hbar, t_1, \ldots, t_N]$-algebras

$$\Phi_\mu^\lambda(\mathbf{Y}_\mu[t_1, \ldots, t_N]) \subseteq \sigma\mathbf{z}^*(\iota_*)^{-1}(\mathcal{A}_\hbar).$$

Since $\sigma\mathbf{z}^*(\iota_*)^{-1} : \mathcal{A}_\hbar \hookrightarrow \widetilde{\mathcal{A}}_\hbar$ is an embedding, the homomorphism $\overline{\Phi}_\mu^\lambda$ exists as claimed. ■

The image of $\overline{\Phi}_\mu^\lambda$ is called the *truncated shifted Yangian*, and is denoted by $\mathbf{Y}_\mu^\lambda$.

We now give a generalization of [4, Cor. B.28] and [40, Th. A] to *BCFG* types:

Theorem 5.8. *For any $\lambda \geq \mu$ we have an isomorphism $\mathbf{Y}_\mu^\lambda = \mathcal{A}_\hbar$, and in particular $\mathbf{Y}_\mu^\lambda / \hbar\mathbf{Y}_\mu^\lambda \cong \overline{\mathcal{W}}_\mu^\lambda$.*

Proof. $\mathbf{Y}_\mu^\lambda \to \mathcal{A}_\hbar$ is injective by definition, so we must prove surjectivity. When μ is dominant, this follows exactly as in the proof of [4, Cor. B.28]. To extend it to general μ, we follow the same strategy as in the proof of [40, Th. 3.13]. First, we note that one can define shift homomorphisms for $\mathbf{Y}_\mu[t_1, \ldots, t_N]$ and $\mathcal{A}_\hbar$, which are compatible as in [40, Lem. 3.14]. Second, we claim that $\mathcal{A}_\hbar$ is generated by its subalgebras $\mathcal{A}_\hbar^\pm$ corresponding to the loci $\mathcal{R}^\pm$ lying over the positive and negative parts of the affine Grassmannian (cf. §3(ii)). Assuming this claim for the moment, the proof of [40, Th. 3.13] now goes through.

To prove the claim about generators, consider the semigroups of integral points in chambers of the generalized root hyperplane arrangement for $\mathcal{A}_\hbar$ (see [3, Def. 5.2]). The hyperplanes in our situation are of three types: (i) $w_{i,r} - w_{i,s} = 0$ for all $i \in I$ and $1 \leq r, s \leq \mathbf{v}_i$, (ii) $f_{ji} w_{i,r} - f_{ij} w_{j,s} = 0$ for any $c_{ij} \neq 0$ and $1 \leq r \leq \mathbf{v}_i, 1 \leq s \leq \mathbf{v}_j$, and (iii) $w_{i,r} = 0$ for any $W_i \neq 0$ and $1 \leq r \leq \mathbf{v}_i$. Even if $W_i = 0$, we can always refine our arrangement by adding all hyperplanes $w_{i,r}$. In this refined arrangement, any chamber is the product of its subcones of positive and negative elements. Thus we can choose generators for its semigroup of integral points which are each either positive or negative. Since the spherical Schubert variety through a positive (resp. negative) coweight lies inside Gr^+ (resp. Gr^-), we can lift the above semigroup generators to algebra generators for $\mathcal{A}_\hbar$, each lying in one of the $\mathcal{A}_\hbar^\pm$. This proves the claim. ■

Appendix A. A zastava space for G_2

We give an explicit presentation of the coordinate ring of the zastava $Z^{\alpha_1+\alpha_2}$ of type G_2, thought of as a variety over a field of characteristic zero (for simplicity, we will simply work over $\mathbb{C}$). This presentation is similar to those for other rank 2 types given in [1, §5.5–5.8].

Denote by $\mathfrak{g}$ the Lie algebra of type G_2, and write $V(\lambda)$ for its irreducible representation of highest weight λ. Following the notation of [14, Table 22.1], we pick a basis for the adjoint representation:

$$V(\varpi_2) \cong \mathfrak{g} = \operatorname{span}_{\mathbb{C}} \{H_1, H_2, X_i, Y_i : 1 \le i \le 6\},$$

Here X_i, H_i, Y_i with $i = 1, 2$ are the Chevalley generators with respect to the Cartan matrix $\left(\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right)$. Note this is the transpose of the convention taken in §3(ii). We define $X_3 = [X_1, X_2]$, $X_4 = \frac{1}{2}[X_1, X_3]$, $X_5 = -\frac{1}{3}[X_1, X_4]$, $X_6 = -[X_2, X_5]$ and similarly for the Y_i (but with opposite signs). In particular, X_6 is a highest weight vector. Following [14, p. 354], we also pick a basis for the first fundamental representation:

$$V(\varpi_1) = \operatorname{span}_{\mathbb{C}} \{V_4, V_3, V_1, U, W_1, W_3, W_4\},$$

where V_4 is a highest weight vector and $V_3 = Y_1 \cdot V_4$, $V_1 = -Y_2 \cdot V_3$, $U = Y_1 \cdot V_1$, $W_1 = \frac{1}{2} Y_1 \cdot U$, $W_3 = Y_2 \cdot W_1$, and $W_4 = -Y_1 \cdot W_3$.

Using the above notation, recall that $Z^{\alpha_1+\alpha_2}$ has a description as Plücker sections [11, §5]: it is the space of pairs $v_{\varpi_i} \in V(\varpi_i)[z]$ for $i = 1, 2$ such that (a) the coefficient of V_4 in v_{ϖ_1} (resp. X_6 in v_{ϖ_2}) is monic of degree 1, (b) the coefficients of all other basis vectors have degree zero, and (c) certain Plücker-type relations must hold (see the proof below for certain cases).

Proposition A.1. *Scheme-theoretically, $Z^{\alpha_1+\alpha_2}$ is the set of pairs*

$$v_{\varpi_1} = (z + A_1)V_4 + b_0 V_3 + b_2 V_1 + b_3 U + b_4 W_1,$$
$$v_{\varpi_2} = (z + A_2)X_6 + b_1 X_5 + b_2 X_4 + b_3 X_3 + b_4 X_2$$

whose coefficients satisfy

$$b_0 b_1 = (A_2 - A_1) b_2, \quad b_0 b_2 = (A_1 - A_2) b_3, \quad b_0 b_3 = (A_1 - A_2) b_4,$$
$$b_2^2 = -b_1 b_3, \quad b_2 b_3 = -b_1 b_4, \quad b_3^2 = b_2 b_4.$$

Proof. Fix a non-zero $\mathfrak{g}$-invariant element $\Omega_2 \in \mathfrak{g} \otimes \mathfrak{g}$.[2] This can be considered as an operator on any $V(\lambda) \otimes V(\mu)$, and it distinguishes the canonical summand $V(\lambda + \mu) \subset V(\lambda) \otimes V(\mu)$ as an eigenspace [23, §14.12].

Consider an arbitrary pair $v_{\varpi_i} \in V(\varpi_i)[z]$ for $i = 1, 2$ satisfying the degree requirements (a), (b) above. Using Sage, we compute the ideal defined by the above eigenvalue conditions for Ω_2 applied to $v_{\varpi_i} \otimes v_{\varpi_j}$ where $1 \le i \le j \le 2$. We find that this ideal has two primary components, which have dimensions 4 and 1, respectively. Since $Z^{\alpha_1+\alpha_2}$ is a four-dimensional irreducible closed subscheme living inside the vanishing locus of this ideal, it must correspond to the four-dimensional primary component. This yields the description claimed above. ■

[2]For the purposes of our Sage calculation, we chose Ω_2 corresponding to the trace form on $V(\varpi_1)$.

Remark A.2. Comparing the above with §3(ii) in the case $m = 3$, we can identify the above coordinates with the generators of the Coulomb branch as follows: $w_1 = -A_2$, $w_2 = -A_1$, $\mathsf{y}_{0,1} = b_0$, $\mathsf{y}_{1,0} = -b_1$, $\mathsf{y}_{1,1} = b_2$, $\mathsf{y}_{1,2} = b_3$ and $\mathsf{y}_{1,3} = b_4$.

Remark A.3. To match the proposition with the conventions of [1, §5.8], we take $\overline{w}_i = -A_1$, $\overline{w}_j = -A_2$, $\overline{y}_i = b_0$ and $\overline{y}_j = -b_1$ (we add overlines to avoid confusion with our notation for Coulomb branches). The equation of the boundary of $Z^{\alpha_1+\alpha_2}$ is then

$$-\frac{\overline{y}_i^3\overline{y}_j}{(\overline{w}_i - \overline{w}_j)^3} = -\frac{b_0^3 b_1}{(A_1 - A_2)^3} = b_4.$$

This is consistent with our comparison with the open zastava from §3(i): by the previous remark $b_4 = \mathsf{y}_{1,3}$, which is invertible in $H_*^{G_\mathcal{O}}(\mathcal{R})$. It is also easy to see that $H_*^{G_\mathcal{O}}(\mathcal{R})$ is generated by the inverse element $\mathsf{y}_{-1,-3}$ together with $H_*^{G_\mathcal{O}}(\mathcal{R}^+)$, as expected.

Appendix B. Fixed point sets

Consider the category $\mathcal{C}$ of finitely generated right modules of the quantized Coulomb branch $H_*^{G_\mathcal{O}\rtimes\mathbb{C}^\times}(\mathcal{R})$ such that (1) $\hbar \in H^*_{\mathbb{C}^\times}(\mathrm{pt})$ acts by a non-zero complex number, say 1, and (2) it is locally finite over $H^*_G(\mathrm{pt})$, hence it is a direct sum of generalized simultaneous eigenspaces of $H^*_G(\mathrm{pt})$. (When we include an additional flavor symmetry, we assume that the corresponding equivariant parameter acts by a complex number.) One can apply techniques of the localization theorem in equivariant (K)-homology groups of affine Steinberg varieties in [37] to study the category $\mathcal{C}$. This theory, for the ordinary Coulomb branch, will be explained elsewhere [34]. (See also [38,39] for another algebraic approach different from one in [37].) It also works in our current setting. As a consequence, we have for example

Theorem B.1. *Let $\lambda \in \mathfrak{t}$. There is a natural bijection between*

- *simple modules in $\mathcal{C}$ such that one of the eigenvalues above is given by evaluation $H^*_G(\mathrm{pt}) \cong \mathbb{C}[\mathfrak{t}]^{\mathbb{W}} \to \mathbb{C}$ at λ,*
- *simple perverse sheaves which appear, up to shift, in the direct image of constant sheaves on the fixed point subset $\mathcal{T}^{(\lambda,1)}$ under the projection $\mathcal{T}^{(\lambda,1)} \to \mathbf{N}_\mathcal{K}^{(\lambda,1)}$.*

Here $\mathfrak{t}$ is the Lie algebra of a maximal torus of G, $\mathbb{W}$ is the Weyl group of G, and $(\lambda, 1)$ is the element of the Lie algebra of $T \times \mathbb{C}^\times$, which acts on $\mathcal{T}$ and $\mathbf{N}_\mathcal{K}$, as a subgroup of $G_\mathcal{O} \rtimes \mathbb{C}^\times$. The fixed point subsets are written as $\mathcal{T}^{(\lambda,1)}$, $\mathbf{N}_\mathcal{K}^{(\lambda,1)}$, and the projection is the restriction of $\Pi\colon \mathcal{T} \to \mathbf{N}_\mathcal{K}$.

We study the fixed point sets $\mathcal{T}^{(\lambda,1)}$, $\mathbf{N}_\mathcal{K}^{(\lambda,1)}$ in this section. For simplicity, we assume λ is the differential of a cocharacter, denoted by the same symbol λ. (See Remark B.3 for the general case.) Therefore we study the fixed point set with respect to the one-parameter subgroup $\tau \mapsto (\lambda(\tau), \tau)$.

B(i)

Consider the affine Grassmannian Gr_G. We have an action of $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$ on Gr_G given by $(h(z), \tau) \cdot [g(z)] = [h(z)g(z\tau)]$. Take a cocharacter $\lambda\colon \mathbb{C}^\times \to T$ and consider the homomorphism $\tau \mapsto (\lambda(\tau), \tau) \in T \times \mathbb{C}^\times \subset G_{\mathcal{O}} \rtimes \mathbb{C}^\times$, where $\lambda(\tau)$ is regarded as a constant loop in $G_{\mathcal{O}}$. Let

$$\mathrm{Gr}_G^{(\lambda(\tau),\tau)} := \{[g(z)] \in \mathrm{Gr}_G \mid (\lambda(\tau), \tau) \cdot [g(z)] = [g(z)]\}$$

be the fixed point set of $\lambda \times \mathrm{id}$ in Gr_G. It consists of the equivalence classes $[g(z)]$ where

$$g(z) = \lambda(z)^{-1}\varphi(z) \quad \text{for a cocharacter } \varphi\colon \mathbb{C}^\times \to G.$$

To see this let us identify Gr_G with ΩG_c, the space of polynomial based maps $(S^1, 1) \to (G_c, 1)$, where G_c is a maximal compact subgroup of G. Then $g \in \Omega G_c$ is fixed if and only if $\lambda(\tau)g(z\tau)g(\tau)^{-1}\lambda(\tau)^{-1} = g(z)$. It means that $z \mapsto \lambda(z)g(z)$ is a group homomorphism.

Alternatively, the fixed point set can be identified as follows: Let

$$\begin{aligned} Z_{G_{\mathcal{K}}}(\lambda(\tau), \tau) &= \text{centralizer of } (\lambda(\tau), \tau) \text{ in } G_{\mathcal{K}} \\ &= \{g(z) \in G_{\mathcal{K}} \mid \lambda(\tau)g(z\tau)\lambda(\tau)^{-1} = g(z)\}. \end{aligned}$$

Then $g(z = 1)$ is well-defined and $g(z) = z^{-\lambda}g(z = 1)z^{\lambda}$, hence $Z_{G_{\mathcal{K}}}(\lambda(\tau), \tau) \cong G$ via $g(z) \mapsto g(1)$. (We switch the notation from $\lambda(z)$ to z^λ.) Then the fixed point set is

$$\bigsqcup_{\mu} Z_{G_{\mathcal{K}}}(\lambda(\tau), \tau) \cdot [z^{-\lambda+\mu}],$$

where μ is a dominant coweight of G, and $z^{-\lambda+\mu}$ is regarded as a point in Gr_G. The $Z_{G_{\mathcal{K}}}(\lambda(\tau), \tau)$-orbit through $z^{-\lambda+\mu}$ is the partial flag variety G/P_μ, where P_μ is the parabolic subgroup corresponding to μ.

B(ii)

More generally, consider a homomorphism $\tau \mapsto (\lambda(\tau), \tau^m)$ for $m \in \mathbb{Z}_{>0}$. We suppose $G = \mathrm{GL}(V)$ and decompose $V = \bigoplus V(k)$ so that $\lambda(\tau)$ acts on $V(k)$ by $\tau^k\, \mathrm{id}_{V(k)}$. We consider k modulo m and decompose V as

$$V = V\{1\} \oplus \cdots \oplus V\{m\}, \quad \text{where} \quad V\{k\} = \bigoplus_{l \equiv k \bmod m} V(l).$$

Let $G' := \mathrm{GL}(V\{1\}) \times \cdots \times \mathrm{GL}(V\{m\})$. Then $[g(z)]$ is fixed by $(\lambda(\tau), \tau^m)$ if and only if $[g(z)] = ([g_1(z)], \dots, [g_m(z)]) \in \mathrm{Gr}_{G'}$ is such that

$$g_k(z) = \lambda_k(z)^{-1}\varphi_k(z) \quad \text{for a cocharacter } \varphi_k\colon \mathbb{C}^\times \to \mathrm{GL}(V\{k\}) \quad (k = 1, \dots, m).$$

Here $\lambda_k(z)$ is defined so that it acts by $z^{(l-k)/m}$ on $V(l)$. This is proved as follows. Take a based loop model $g \in \Omega G_c$. It is fixed if and only if $\lambda(\tau)g(z\tau^m)g(\tau^m)^{-1}\lambda(\tau)^{-1} = g(z)$.

Taking $\tau = \omega$, a primitive m-th root of unity, we see that $g(z)$ preserves the decomposition $V = V\{1\} \oplus \cdots \oplus V\{m\}$, hence it is in $\mathrm{Gr}_{G'}$. Let $g_k(z)$ be the k-th component. Note that $\lambda(\tau)$ is $\tau^k \lambda_k(\tau^m)$ on $V\{k\}$. Therefore $\lambda_k(\tau^m) g_k(z\tau^m) g_k(\tau^m)^{-1} \lambda_k(\tau^m)^{-1} = g_k(z)$. Hence $\lambda_k(z) g_k(z)$ is a group homomorphism, which we denoted by $\varphi_k(z)$.

Let $\lambda' := \lambda_1 \oplus \cdots \oplus \lambda_m$ and $\varphi := \varphi_1 \oplus \cdots \oplus \varphi_m$. The connected component of $\mathrm{Gr}_G^{(\lambda(\tau),\tau^m)}$ containing $[g(z)] = [\lambda'(z)^{-1}\varphi(z)]$ is the partial flag manifold G'/P_φ where P_φ is the parabolic subgroup defined by $\{g \in G' \mid \exists \lim_{z\to 0} \varphi(z)^{-1} g \varphi(z)\}$.

Note that the decomposition $V = V\{1\} \oplus \cdots \oplus V\{m\}$ and the group G' depend on the choice of λ. If we take $\lambda = 1$ for example, we have $V = V\{m\}$ and $G' = G$.

An alternative description is as follows: Let

$$\begin{aligned} Z_{G_\mathcal{K}}(\lambda(\tau), \tau^m) &= \text{the centralizer of } (\lambda(\tau), \tau^m) \text{ in } G_\mathcal{K} \\ &= \{g(z) \in G_\mathcal{K} \mid g(z) = z^{-\lambda'} g(z=1) z^{\lambda'}, g(z=1) \in G'\}. \end{aligned}$$

It is isomorphic to G' by $g(z) \mapsto g(z=1) \in G'$. Then the fixed point set is

$$\bigsqcup_\mu Z_{G_\mathcal{K}}(\lambda(\tau), \tau^m) \cdot [z^{-\lambda'+\mu}],$$

where μ is a dominant cocharacter of G', and the orbit $Z_{G_\mathcal{K}}(\lambda(\tau), \tau^m) \cdot [z^{-\lambda'+\mu}]$ is isomorphic to the partial flag variety G'/P_μ.

Remark B.2. For general reductive groups G, the centralizer $Z_{G_\mathcal{K}}(\lambda(\tau), \tau^m)$ could be disconnected. Nevertheless, the description is still valid if we replace $Z_{G_\mathcal{K}}(\lambda(\tau), \tau^m)$ by its connected component $Z^0_{G_\mathcal{K}}(\lambda(\tau), \tau^m)$.

B(iii)

Let us consider the case $I = \{1, 2\}$, $c_{12} = -1$, $c_{21} = -m$ ($m \in \mathbb{Z}_{>0}$) as in §3(i). We have $z_1 = z = z_2^m$. We consider the variety $\mathcal{T}$, where we regard it as the space consisting of

- $[g_1(z_1)] \in \mathrm{GL}(V_1)((z_1))/\mathrm{GL}(V_1)[[z_1]]$,
- $[g_2(z_2)] \in \mathrm{GL}(V_2)((z_2))/\mathrm{GL}(V_2)[[z_2]]$,
- $B \in \mathrm{Hom}_{\mathbb{C}((z_1))}(V_1((z_1)), V_2((z_2)))$ such that $g_2(z_2)^{-1} B g_1(z_1)$ is regular at $z_1 = 0$.

Here $V_2((z_2))$ is regarded as a $\mathbb{C}((z_1))$-module via $z_1 = z_2^m$. By the projection formula, we identify it with an element in $\mathrm{Hom}_{\mathbb{C}((z_2))}(V_1((z_2)), V_2((z_2))) \cong \mathrm{Hom}_\mathbb{C}(V_1, V_2)((z_2))$, and denote it by $B(z_2)$. The action of $(\mathrm{GL}(V_1)[[z_1]] \times \mathrm{GL}(V_2)[[z_2]]) \rtimes \mathbb{C}^\times$ on the component $B(z_2)$ is given by

$$\begin{aligned} &B(z_2) \mapsto h_2(z_2) B(z_2\tau) h_1(z_1)^{-1}, \\ &\qquad\qquad (h_1(z_1), h_2(z_2), \tau) \in (\mathrm{GL}(V_1)[[z_1]] \times \mathrm{GL}(V_2)[[z_2]]) \rtimes \mathbb{C}^\times. \end{aligned}$$

Note that the loop rotation acts on $\mathrm{GL}(V_1)[[z_1]]$ by $h_1(z_1) \mapsto h_1(\tau^m z_1)$ as $z_1 = z_2^m$. Also $h_1(z_1)^{-1}$ is regarded as a function of z_2 via $z_1 = z_2^m$.

We take $\lambda_1\colon \mathbb{C}^\times \to T(V_1)$, $\lambda_2\colon \mathbb{C}^\times \to T(V_2)$ as above, and consider the fixed point set $\mathcal{T}^{(\lambda_1(\tau),\lambda_2(\tau),\tau)}$ in $\mathcal{T}$ with respect to $\lambda_1 \times \lambda_2 \times \mathrm{id}$. Then we have a decomposition

$$V_1 = V_1\{1\} \oplus \cdots \oplus V_1\{m\}$$

and $([g_1(z_1)], [g_2(z_2)]) \in \mathrm{Gr}_{\mathrm{GL}(V_1)} \times \mathrm{Gr}_{\mathrm{GL}(V_2)}$ is given as

$$g_1(z_1) = \lambda_1'(z_1)^{-1}\varphi_1(z_1), \quad g_2(z_2) = \lambda_2(z_2)^{-1}\varphi_2(z_2)$$

for cocharacters $\varphi_1\colon \mathbb{C}^\times \to \mathrm{GL}(V_1\{1\}) \times \cdots \times \mathrm{GL}(V_1\{m\})$ and $\varphi_2\colon \mathbb{C}^\times \to \mathrm{GL}(V_2)$. Here λ_1' is defined from λ_1 as above.

Remark B.3. More generally we could study the fixed point set with respect to a cocharacter $\tau \mapsto (\lambda_1(\tau), \lambda_2(\tau), \tau^d)$ for $d \in \mathbb{Z}_{>0}$. But the fixed point set will be just the union of d copies of the fixed point set below, hence it does not yield a new space. On the other hand, this modification yields a new space when a quiver has a loop (see [37]).

Let us consider the remaining component $B(z_2)$. It is fixed by the action if and only if

$$B(z_2) = \lambda_2(\tau)B(z_2\tau)\lambda_1(\tau)^{-1}.$$

If we expand $B(z_2)$ as

$$\cdots + B^{(-1)}z_2^{-1} + B^{(0)} + B^{(1)}z_2 + B^{(2)}z_2^2 + \cdots,$$

this equality is equivalent to

$$B^{(n)} = \tau^n\lambda_2(\tau)B^{(n)}\lambda_1(\tau)^{-1}.$$

When we decompose V_1, V_2 as $\bigoplus V_1(k)$, $\bigoplus V_2(k)$ into eigenspaces with respect to $\lambda_1(\tau)$, $\lambda_2(\tau)$ as before, this equality means that $B^{(n)}$ sends $V_1(i)$ to $V_2(i-n)$. In particular, $B^{(n)}$ must vanish if $|n|$ is sufficiently large, hence $B(z_2)$ is a Laurent polynomial. We see that the evaluation $B(z_2 = 1)$ at $z_2 = 1$ *does* make sense and is equal to $\cdots + B^{(-1)} + B^{(0)} + B^{(1)} + \cdots$. Then $B(z_2)$ is recovered from $B(z_2 = 1)$ by the formula

$$B(z_2) = \lambda_2(z_2)^{-1}B(z_2 = 1)\lambda_1(z_2).$$

Thus the fixed point set in $\mathrm{Hom}_{\mathbb{C}((z_2))}(V_1((z_2)), V_2((z_2)))$ is identified with the space $B(z_2 = 1) \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$.

Let us consider the condition that $g_2(z_2)^{-1}B(z_2)g_1(z_2^m)$ is regular at $z_2 = 0$ with $g_1(z_1) = \lambda_1'(z_1)^{-1}\varphi_1(z_1)$ and $g_2(z_2) = \lambda_2(z_2)^{-1}\varphi_2(z_2)$. It is equivalent to

$$\varphi_2(z_2)^{-1}B(z_2 = 1)\lambda_1(z_2)\lambda_1'(z_2^m)^{-1}\varphi_1(z_2^m) \tag{B.4}$$

being regular at $z_2 = 0$. Note that $\lambda_1(z_2)\lambda_1'(z_2^m)^{-1}$ is equal to z_2^k on the summand $V_1\{k\}$. We introduce a new grading on V_1, V_2 given by φ_1, φ_2. For V_2, we define $V_2^\varphi(k)$ as the τ^k eigenspace with respect to $\varphi_2(\tau)$ as above. For V_1, let us recall that φ_1 preserves the decomposition $V_1 = V_1\{1\} \oplus \cdots \oplus V_1\{m\}$. Then we define $V_1^\varphi(l)$ as the $\tau^{(l-k)/m}$ eigenspace with respect to $\varphi_1(\tau)$ in $V_1\{k\}$, where $1 \le k \le m$ is determined so that $l \equiv k \bmod m$. If $\varphi_1 = \lambda_1'$ (and hence $g_1(z_1) = \mathrm{id}$), it is just $V_1 = \bigoplus V_1(k)$. Then (B.4) is regular at z_2 if and only if

$$B(z_2 = 1)(V_1^{\varphi}(k)) \subset \bigoplus_{l \le k} V_2^{\varphi}(l). \tag{B.5}$$

Thus connected components of the fixed point sets $\mathcal{T}^{(\lambda(\tau),\tau)}$ (as well as their projections to $\mathbf{N}_{\mathcal{K}}^{(\lambda(\tau),\tau)}$) are almost the same as varieties appearing in Lusztig's construction of canonical bases from quivers [30, §1.5], where the quiver has vertices $1_1, \ldots, 1_m$, 2 and arrows $1_k \to 2$. See Figure 2 for $m = 3$. Note that this is different from the right quiver in Figure 1. The only differences from Lusztig's varieties are (1) the degree k subspace, i.e., $V_1^{\varphi}(k) \oplus V_2^{\varphi}(k)$, might not be concentrated at a single vertex, and (2) the degree l subspace at the vertex 1_k is only allowed when $k \equiv l \bmod m$. But these differences are *superficial*. If the flag types at vertices are the same and conditions (B.5) are the same, the grading is not relevant: we get isomorphic varieties.

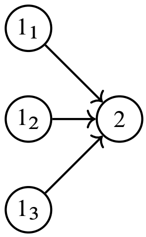


Fig. 2. The quiver appearing in the fixed point set.

B(iv)

The analysis of the fixed point set in the previous subsection can be applied to general cases. The final claim that components of the fixed point set $\mathcal{T}^{(\lambda(\tau),\tau)}$ are isomorphic to Lusztig's varieties remains true if the quiver (corresponding to Figure 2) has no loop, in particular, for type *BCFG*. Therefore we have

Theorem B.6. *Consider the quantized Coulomb branches $\mathcal{A}_\hbar$ of type BCFG with $W = 0$. Then we have a natural bijection between*

- *simple objects in the category $\mathcal{C}$ such that their eigenvalues are evaluations at cocharacters of T, and*
- *canonical base elements of weight $-\sum \dim(V_i\{k\})\alpha_{i_k}$ in the lower triangular part $\mathbf{U}_q^-$ of the quantized enveloping algebra of type ADE.*

Here i runs over the set of vertices of the original quiver, and k runs from 1 to d_i. Concretely, the correspondence between types is $B_n \mapsto A_{2n-1}$, $C_n \mapsto D_{n+1}$, $F_4 \mapsto E_6$, $G_2 \mapsto D_4$.

Remark B.7. (1) Note that the canonical base elements in the above theorem are in bijection also to simple objects in the category $\mathcal{C}$ (with the same constraint) of the quantized

Coulomb branch of type *ADE* by the same analysis of the fixed point set as above. Recall that the quantized Coulomb branch $\mathcal{A}_\hbar$ is a quotient of the shifted Yangian of type *BCFG* or *ADE*, the same type as the quiver. Therefore we have a bijective correspondence between simple modules in quotients of shifted Yangians of type *BCFG* and of *ADE*. This result resembles the result of Kashiwara, Kim and Oh [25], where a similar bijection was found between simple finite-dimensional modules of quantum affine algebras of types B_n and A_{2n-1}.

(2) Let us define the *q-character* of a simple module as the generating function of dimensions of simultaneous generalized eigenspaces of $H^*_G(\mathrm{pt})$. It is a natural generalization of one defined for a finite-dimensional simple module of the ordinary Yangian and the quantum affine algebra [13, 28]. As in [32], the q-character of a simple module is given by composition factor multiplicities of simple perverse sheaves in the direct images of constant sheaves of connected components of $\mathcal{T}^{(\lambda,1)}$, counted by ignoring degree shifts. Under the bijection in (1), the q-character of a simple module for *BCFG* is obtained from that of the corresponding module for *ADE* by imposing the condition $l \equiv k \bmod m$.

Appendix C. A second definition

In this section we present a second possible definition for a Coulomb branch associated to a quiver gauge theory with symmetrizers. When the Cartan matrix satisfies assumption (2.2), this definition agrees with that given in §2, but in general this is not the case. We note that this second definition applies to theories which are not of quiver type.

C(i). Covers of discs

For each $k \in \mathbb{Z}_{>0}$ consider the formal disc $D_k = \operatorname{Spec} \mathbb{C}[[x^k]]$. If $k \mid \ell$, there is a map

$$\rho_{k|\ell} : D_k \to D_\ell \tag{C.1}$$

corresponding to the inclusion of rings $\mathbb{C}[[x^\ell]] \hookrightarrow \mathbb{C}[[x^k]]$. Similarly, there are maps between the corresponding formal punctured discs, which we also denote $\rho_{k|\ell} : D_k^* \to D_\ell^*$ by abuse of notation. These maps are equivariant for the $\mathbb{C}^\times$-action by loop rotation, acting on D_k by $\tau : x^k \mapsto \tau^k x^k$.

C(ii). General definition

Fix a pair $(G_\bullet, \mathbf{N}_\bullet)$, consisting of a product $G_\bullet = \prod_{k=1}^d G_k$ of complex connected reductive groups, and a direct sum $\mathbf{N}_\bullet = \bigoplus_{k=1}^d \mathbf{N}_k$ of complex finite-dimensional representations of $G_\bullet$. In addition, we assume that G_k acts *trivially* on $\mathbf{N}_j$, unless $j \mid k$.

Given such a pair $(G_\bullet, \mathbf{N}_\bullet)$, we define $\mathcal{R}_{G_\bullet, \mathbf{N}_\bullet}$ to be the moduli space of triples $(\mathcal{P}_\bullet, \varphi_\bullet, s_\bullet)$, where $\mathcal{P}_\bullet = (\mathcal{P}_1, \ldots, \mathcal{P}_d)$, $\varphi_\bullet = (\varphi_1, \ldots, \varphi_d)$, and $s_\bullet = (s_1, \ldots, s_d)$ satisfy

(a) $\mathcal{P}_k$ is a principal G_k-bundle over D_k,

(b) φ_k is a trivialization of $\mathcal{P}_k$ over D_k^*,

(c) s_k is a section of the associated bundle

$$s_k \in \Gamma\Big(D_k, \Big(\prod_{k|\ell} \rho_{k|\ell}^* \mathcal{P}_\ell\Big) \times^{\prod_{k|\ell} G_\ell} \mathbf{N}_k\Big)$$

such that it is sent to a regular section of the trivial bundle under the trivialization $\prod_{k|\ell} \rho_{k|\ell}^* \varphi_\ell$ over D_k^*.

As usual we also define a larger moduli space $\mathcal{T}_{G_\bullet,\mathbf{N}_\bullet}$ by dropping the extension conditions in (c).

The group $G_{\bullet,\mathcal{O}} = \prod_{k=1}^d G_k[[x^k]]$ acts on $\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet}$ by changing $\varphi_\bullet$. There is also an action of $\mathbb{C}^\times$, acting by loop rotation of the discs D_k as in the previous section. We can define a convolution product on $H_*^{G_{\bullet,\mathcal{O}}}(\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet})$ just as in [3]. By the argument in 2(iv), it is a commutative ring, and we define the Coulomb branch

$$\mathcal{M}_C(G_\bullet, \mathbf{N}_\bullet) := \operatorname{Spec} H_*^{G_{\bullet,\mathcal{O}}}(\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet})$$

It has a deformation quantization defined by $H^{G_{\bullet,\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet})$, and in particular a Poisson structure.

The arguments from [3] apply with small modifications to $\mathcal{M}_C(G_\bullet, \mathbf{N}_\bullet)$. In particular, it is of finite type, integral, normal, and generically symplectic. One useful observation in modifying the proofs is the following:

Remark C.2. Suppose that $G_\bullet = G_\ell$ consists of a single factor, and define its representation $\mathbf{N}' = \bigoplus_{k|\ell} \mathbf{N}_k^{\oplus(\ell/k)}$. Then $\mathcal{M}_C(G_\bullet, \mathbf{N}_\bullet)$ is isomorphic to the usual Coulomb branch $\mathcal{M}_C(G_\ell, \mathbf{N}')$ as defined in [3]. This comes from the fact that there is an isomorphism

$$\mathbf{N}_k[[x^k]] = \bigoplus_{0 \le a < \ell/k} x^{ak} \mathbf{N}_k[[x^\ell]] \cong \mathbf{N}_k[[x^\ell]]^{\oplus(\ell/k)}$$

as representations of $G_\ell[[x^\ell]]$.

In particular, in order to prove that $\mathcal{M}_C(G_\bullet, \mathbf{N}_\bullet)$ is normal, we follow the same argument as in [3, Prop. 6.12]. This reduces to the case where $G_\bullet$ is a torus and all weights of $\mathbf{N}_\bullet$ lie on a line (called *type* (I)), or where $G_\bullet$ has semisimple rank 1 and the weights of $\mathbf{N}_\bullet$ lie on the line spanned by its roots (called *type* (II)). In the type (II) case, the above remark reduces the problem to an ordinary Coulomb branch, where [3, Prop. 6.12] applies. In the type (I) case, we have an analog of [3, Thm. 4.1], as is indicated in the computation in §C(v). It implies that $\mathcal{M}(G_\bullet, \mathbf{N}_\bullet)$ is the product of a cotangent bundle T^*T for some torus T times a simple type A singularity, and is thus normal. On the other hand, a useful analog of [3, Thm. 4.1] does not hold for the definition from §2(ii) (see Remark C.8).

C(iii). The quiver case

As in §2(i), consider a valued quiver associated to a symmetrizable Cartan matrix $(c_{ij})_{i,j\in I}$. Also choose symmetrizers $(d_i) \in \mathbb{Z}_{>0}^d$. Recall that we denote $g_{ij} = \gcd(|c_{ij}|, |c_{ji}|)$ and $f_{ij} = |c_{ij}|/g_{ij}$ when $c_{ij} < 0$. It is not hard to see that d_i must be a multiple of f_{ji} for any $c_{ij} < 0$, so we may define integers d_{ij} by the rule $d_i = d_{ji} f_{ji}$. They satisfy $d_{ij} = d_{ji}$.

Remark C.3. In fact, $\operatorname{lcm}(d_i, d_j) = d_i f_{ij} = d_j f_{ji}$ and $\gcd(d_i, d_j) = d_{ij} = d_{ji}$.

Choose vector spaces V_i and W_i for each $i \in I$. Given these choices, we define a pair $(G_\bullet, \mathbf{N}_\bullet)$ according to the following rules:

$$G_k = \prod_{\substack{i \in I,\\ d_i = k}} \mathrm{GL}(V_i), \tag{C.4}$$

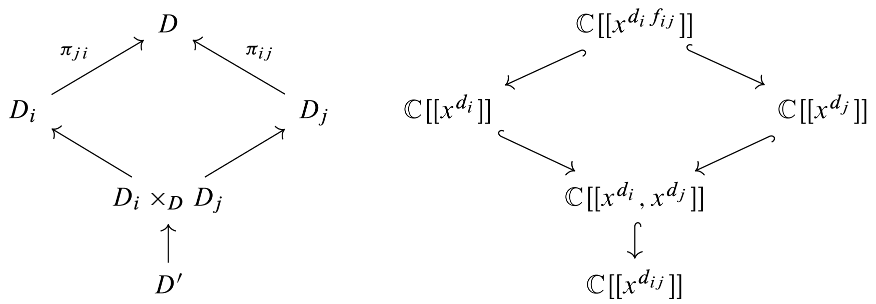
$$\mathbf{N}_k = \bigoplus_{\substack{i \in I,\\ d_i = k}} \operatorname{Hom}(W_i, V_i) \oplus \bigoplus_{\substack{j \to i,\\ d_{ij} = k}} \mathbb{C}^{g_{ij}} \otimes_{\mathbb{C}} \operatorname{Hom}(V_j, V_i). \tag{C.5}$$

Then $\mathbf{N}_\bullet$ is a representation of $G_\bullet$ in the natural way, and satisfies our assumption from the beginning of the previous section. By tracing through the definition one can see that the moduli space $\mathcal{R}_{G_\bullet, \mathbf{N}_\bullet}$ parametrizes

- a rank $\mathbf{v}_i$ vector bundle $\mathcal{E}_i$ over the disc D_{d_i} together with a trivialization $\varphi_i \colon \mathcal{E}_i|_{D^*_{d_i}} \to V_i \otimes_{\mathbb{C}} \mathcal{O}_{D^*_{d_i}}$ for $i \in I$,
- a homomorphism $s_i \colon W_i \otimes_{\mathbb{C}} \mathcal{O}_{D_{d_i}} \to \mathcal{E}_i$ such that $\varphi_i \circ (s_i|_{D^*_{d_i}})$ extends to D_{d_i} for $i \in I$,
- a homomorphism $s_{ij} \in \mathbb{C}^{g_{ij}} \otimes_{\mathbb{C}} \operatorname{Hom}_{\mathcal{O}_{D_{d_{ij}}}}(\rho^*_{d_{ij}|d_j} \mathcal{E}_j, \rho^*_{d_{ij}|d_i} \mathcal{E}_i)$ such that $(\rho^*_{d_{ij}|d_i} \varphi_i) \circ (s_{ij}|_{D^*_{d_{ij}}}) \circ (\rho^*_{d_{ij}|f_j} \varphi_j)^{-1}$ extends to $D_{d_{ij}}$, where $c_{ij} < 0$ and there is an arrow $j \to i$ in the quiver.

C(iv). Comparison

We now compare the above with the construction from §2(ii). For this it suffices to understand the case of a single edge $j \to i$. As explained in §5(i), we can $\mathbb{C}^\times$-equivariantly identify $D_i \cong D_{d_i}$ via $z_i \mapsto x^{d_i}$, $D_j \cong D_{d_j}$ via $z_j \mapsto x^{d_j}$, and $D \cong D_{d_i f_{ij}} = D_{d_j f_{ji}}$ via $z \mapsto x^{d_i f_{ij}}$. We also denote $D' = D_{d_{ij}} = D_{d_{ji}}$. Then there are commutative diagrams of discs and their corresponding rings, as in [20, §4.2]:[3]



[3] We thank the anonymous referee for pointing out this reference.

Both squares are Cartesian, while the inclusion $\mathbb{C}[[x^{d_i}, x^{d_j}]] \hookrightarrow \mathbb{C}[[x^{d_{ij}}]]$ is of finite codimension over $\mathbb{C}$. We also note that $\mathbb{C}[[x^{d_i f_{ij}}]] = \mathbb{C}[[x^{d_i}]] \cap \mathbb{C}[[x^{d_j}]]$.

For brevity, let us denote the covering maps $\rho_{ij} = \rho_{d_{ij}|d_j} : D' \to D_j$ and $\rho_{ji} = \rho_{d_{ji}|d_i} : D' \to D_i$. Then the difference between the two constructions from §2(ii) and §C(iii) is simply in the definition of the section s_{ij}: whether it lies in

$$\mathbb{C}^{g_{ij}} \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_D}(\pi_{ij*}\mathcal{E}_j, \pi_{ji*}\mathcal{E}_i) \quad \text{or} \quad \mathbb{C}^{g_{ij}} \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_{D'}}(\rho_{ij}^*\mathcal{E}_j, \rho_{ji}^*\mathcal{E}_i). \tag{C.6}$$

We now reformulate both sides in terms of the above power series rings, ignoring the tensor product with $\mathbb{C}^{g_{ij}}$ in each case. Denote by E_i the $\mathbb{C}[[x^{d_i}]]$-module corresponding to $\mathcal{E}_i$, and by E_j the $\mathbb{C}[[x^{d_j}]]$-module corresponding to $\mathcal{E}_j$. Then on the one hand, the left side of (C.6) corresponds to

$$\mathrm{Hom}_{\mathbb{C}[[x^{d_i f_{ij}}]]}(E_j, E_i) \;\cong\; \mathrm{Hom}_{\mathbb{C}[[x^{d_i}]]}(\mathbb{C}[[x^{d_i}, x^{d_j}]] \otimes_{\mathbb{C}[[x^{d_j}]]} E_j, E_i).$$

On the other hand, the right side of (C.6) corresponds to

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}[[x^{d_{ij}}]]}(\mathbb{C}[[x^{d_{ij}}]] \otimes_{\mathbb{C}[[x^{d_j}]]} E_j, \mathbb{C}[[x^{d_{ij}}]] \otimes_{\mathbb{C}[[x^{d_i}]]} E_i)&\\ \cong \mathrm{Hom}_{\mathbb{C}[[x^{d_i}]]}(\mathbb{C}[[x^{d_{ij}}]] \otimes_{\mathbb{C}[[x^{d_j}]]} E_j, E_i).&\end{aligned}$$

For this isomorphism we use the fact that induction and coinduction of modules between the rings $A = \mathbb{C}[[x^{d_i}]] \hookrightarrow B = \mathbb{C}[[x^{d_{ij}}]]$ are isomorphic as functors: there is an isomorphism of left B-modules $\mathrm{Hom}_A(B, A) \cong B$ (equivariant up to a grading shift, for the loop $\mathbb{C}^\times$-action).

Thus we see that the difference between the two sides of (C.6), and thus between our two constructions, is captured by the finite codimension inclusion of rings

$$\mathbb{C}[[x^{d_i}, x^{d_j}]] \hookrightarrow \mathbb{C}[[x^{d_{ij}}]].$$

Note that this map is an isomorphism if and only if $f_{ij} = 1$ or $f_{ji} = 1$.

Theorem C.7. *For a general valued quiver, if assumption* (2.2) *holds then our constructions from* §2(ii) *and* §C(iii) *are isomorphic. In particular, this is the case in all finite types.*

C(v). Twisted monopole formula

The previous section shows that the twisted monopole formula applies to $\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet}$ in the case when assumption (2.2) holds. But in fact it is not hard to see that the twisted monopole formula is valid for $\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet}$ even when this assumption does not hold. More precisely, Proposition 2.3 is valid for $\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet}$ in all types, with the same expression for d_λ from §2(iii).

The twisted monopole formula is related to the following generalization of the calculations from §3. We assume $\mathbf{v}_1 = \mathbf{v}_2 = 1$ and $\mathbf{w}_1 = \mathbf{w}_2 = 0$, with orientation $1 \leftarrow 2$. For an arbitrary rank 2 Cartan matrix we find that

$$\mathbf{z}^*(w_1) = w_1, \quad \mathbf{z}^*(w_2) = w_2, \quad \mathbf{z}^*(\mathsf{y}_{a,b}) = (w_1 - w_2)^{g_{12}\cdot\max(f_{12}b - f_{21}a, 0)}\mathsf{u}_{a,b}.$$

Indeed, the fiber of $\mathcal{T}_{G_\bullet,\mathbf{N}_\bullet}$ over $(a,b) \in \mathrm{Gr}_G = \mathbb{Z}^2$ is

$$\mathbb{C}^{g_{12}} \otimes_{\mathbb{C}} x^{ad_1-bd_2}\,\mathrm{Hom}_{\mathbb{C}}(V_2, V_1)[[x^{d_{12}}]],$$

while the fiber of $\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet}$ is its intersection with $\mathbb{C}^{g_{12}} \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V_2, V_1)[[x^{d_{12}}]]$. The contribution to $\mathbf{z}^*(\mathsf{y}_{a,b})$ above is the Euler class of the quotient, recalling that $d_1 = d_{21} f_{21}$ and $d_2 = d_{12} f_{12}$.

Take $(a_0, b_0) \in \mathbb{Z}^2$ such that $f_{12}b_0 - f_{21}a_0 = 1$. Then we have $\mathsf{y}_{f_{12},f_{21}}\mathsf{y}_{-f_{12},-f_{21}} = 1$ and $(w_1 - w_2)^{g_{12}} = \mathsf{y}_{a_0,b_0}\mathsf{y}_{-a_0,-b_0}$. Hence

$$H_*^{G_\bullet,\mathcal{O}}(\mathcal{R}_{G_\bullet,\mathbf{N}_\bullet}) \cong \mathbb{C}[w_1, \mathsf{y}^{\pm}_{f_{12},f_{21}}, \mathsf{y}_{a_0,b_0}, \mathsf{y}_{-a_0,-b_0}].$$

Therefore the Coulomb branch is $\mathbb{A} \times \mathbb{A}^\times \times \mathbb{A}^2/(\mathbb{Z}/g_{12}\mathbb{Z})$.

Remark C.8. Let us consider the definition of $\mathcal{M}_C$ from §2(ii). Let $d_1 = 2$ and $d_2 = 3$. Then the fiber of $\mathcal{T}/\mathcal{R}$ over (a,b) can be identified with $z^{2a-3b}\mathbb{C}[[z^2, z^3]]/\mathbb{C}[[z^2, z^3]] \cap z^{2a-3b}\mathbb{C}[[z^2, z^3]]$. The dimension of this space is $\max(0, -2a + 3b)$ if $2a - 3b \neq \pm 1$, but it is 2 if $2a - 3b = \pm 1$. We could easily check that this example gives a non-normal Coulomb branch $\mathcal{M}_C$, while the definition in §C(ii) is its normalization.

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