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On the C^∞-well-posedness of Goursat Problems

Dedicated to Professor S. Mizohata on the occasion of his 60th birthday

By

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§ 0. Introduction

Many authors have investigated about null-solutions of characteristic Cauchy problems. When the coefficients are real-analytic, many systematic results have been obtained. When the coefficients are only C^∞, however, few results are known. In this paper, as one of the cases when we can get well-parametrized null-solutions, we consider Goursat problems on $\boldsymbol{R}^{n+1}(n\geqq 1)$.

To give more explanation, we introduce some notations as follows;

$(t, x, y)=(t, x, y_1, \ldots, y_{n-1})$ are variables in $\boldsymbol{R}^{n+1}$,

$\partial_t=\partial/\partial t$, $D_t=-i\partial_t$ etc, $D_y^\alpha=D_{y_1}^{\alpha_1}\ldots D_{y_{n-1}}^{\alpha_{n-1}}$ where

$\alpha=(\alpha_1, \ldots, \alpha_{n-1})$ is a multi-index,

for a polynomial $p(t, x, y; \tau, \xi, \eta)$ of (τ, ξ, η) with $C^\infty(\boldsymbol{R}^{n+1})$-coefficients, we denote the homogeneous part of degree h by $p_h(t, x, y; \tau, \xi, \eta)$.

For a differential operator $P=p(t, x, y; D_t, D_x, D_y)$ of order m and an integer r such that $0<r<m$, consider the following *Goursat problem*:

$$\text{(G. P.)}\qquad \begin{cases} Pu=f(t, x, y) \text{ on } \Omega=[0, T]\times\boldsymbol{R}^n \quad (T>0), \\ \partial_t^j u_{|t=0}=g_j(x, y) \text{ on } \boldsymbol{R}^n \quad (0\leqq j<m-r), \\ \partial_x^k u_{|x=0}=h_k(t, y) \text{ on } \Omega_0=[0, T]\times\boldsymbol{R}^{n-1} \quad (0\leqq k<r), \end{cases}$$

where f, g_j, h_k are given C^∞-functions and satisfy the *compatibility-*

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condition:

(C) $\partial_x^k g_j(0,y)=\partial_t^j h_k(0,y)$ on $\mathbb{R}^{n-1}$ $(0\leqq j<m-r, 0\leqq k<r)$.

For simplicity, we assume all the coefficients of P belong to $\mathscr{B}^\infty(\mathbb{R}^{n+1}) =\{f\in C^\infty(\mathbb{R}^{n+1}); \partial_t^j\partial_x^k\partial_y^\alpha f$ is bounded for any $(j,k,\alpha)\}$. In the case of constant coefficients, Y. Hasegawa [1], T. Nishitani [5] have investigated the C^∞-well-posedness of Goursat problems. Following them, we assume that the operator P has the following structure throughout this paper.

$$P=\sum_{j=0}^{m-r} c_j(t,x,y;D_x,D_y)D_t^{m-r-j}, \text{ where ord. } c_j\leqq r+j$$
$$(0\leqq j\leqq m-r) \text{ and } c_{0,r}(t,x,y;1,0)=1.$$

Under this assumption, T. Nishitani [5] found a necessary and sufficient condition for C^∞-well-posedness in the case of constant coefficients. His condition is the following.

$$\begin{cases} \text{There exists a positive constant } \varepsilon \text{ such that the polynomial} \\ p(\tau,\xi,\eta) \text{ is hyperbolic with respect to } (\tau,\xi,\eta)=(1,\delta,0) \\ \text{for every } \delta \text{ with } 0<|\delta|\leqq\varepsilon. \end{cases}$$

This condition implies the following two conditions.

$$\begin{cases} \text{(i) } p_m(\tau,\xi,\eta) \text{ is divisible by } c_{0,r}(\xi,\eta)\text{, that is, there exists a poly-} \\ \text{nomial } q_{m-r}(\tau,\xi,\eta) \text{ such that } p_m(\tau,\xi,\eta)=c_{0,r}(\xi,\eta)q_{m-r}(\tau,\xi,\eta). \\ \text{Further, } c_{0,r} \text{ is hyperbolic with respect to } (\xi,\eta)=(1,0) \text{ and} \\ q_{m-r} \text{ is hyperbolic with respect to } (\tau,\xi,\eta)=(1,0,0). \\ \text{(ii) } p_{m-1}(\tau,\xi,\eta)-c_{0,r-1}(\xi,\eta)q_{m-r}(\tau,\xi,\eta) \text{ is divisible by } c_{0,r}(\xi,\eta). \end{cases}$$

Taking these results into account, we want to know what kind of conditions should be imposed in the case of variable coefficients.

§ 1. Statement of Results

We set the following assumption.

$$\text{(A-1)}\quad \begin{cases} p_m(t,x,y;\tau,\xi,\eta)=c_{0,r}(t,x,y;\xi,\eta)q_{m-r}(t,x,y;\tau,\xi,\eta), \text{ where} \\ \text{(i) the equation } q_{m-r}(t,x,y;\tau,\xi,\eta)=0 \text{ with respect to } \tau \text{ has} \\ \text{only real distinct roots for any } (t,x,y;\xi,\eta)\in\Omega\times(\mathbb{R}^n\setminus\{0\}), \\ \text{(ii) either } n=1 \text{ or the equation } c_{0,r}(t,x,y;\xi,\eta)=0 \text{ with re-} \\ \text{spect to } \xi \text{ has only real distinct roots for any } (t,x,y;\eta)\in \\ \Omega\times(\mathbb{R}^{n-1}\setminus\{0\}). \end{cases}$$

Definition 1.1. 1) Put $\Sigma_2'=\{(t, 'x, y; \tau, \xi, \eta)\in\Omega\times(R^{n+1}\setminus\{0\}); c_{0,r}=q_{m-r}=0\}$.

2) For a positive number M, put

$$\begin{cases} \Gamma_1(t_0, x_1, y_1)=\{(t, x, y)\in\Omega;\ (|x-x_1|^2+|y-y_1|^2)^{1/2}\leqq M(t_0-t)\}, \\ \Gamma_2(x_0, y_0)=\begin{cases}\{(x, y)\in R^n;\ |y-y_0|\leqq M(x_0-x),\ x\geqq 0\}\ (\text{if } x_0\geqq 0), \\ \{(x, y)\in R^n;\ |y-y_0|\leqq M(x-x_0),\ x\leqq 0\}\ (\text{if } x_0\leqq 0),\end{cases} \\ \Gamma(t_0, x_0, y_0)=\cup\{\Gamma_1(t_0, x_1, y_1);\ (x_1, y_1)\in\Gamma_2(x_0, y_0)\}. \end{cases}$$

Theorem 1. *Assume* (A-1) *and*

(A-2) $\sigma_{m-1}(P-q_{m-r}(t, x, y; D_t, D_x, D_y)\circ c_{0,r}(t, x, y; D_x, D_y))=0$ *on* Σ_2', *where* $\sigma_{m-1}(Q)$ *denotes the principal symbol of* Q *as an operator of order* $m-1$ *and* $\circ$ *denotes the composition of two operators.*

Then, for any $f\in C^\infty(\Omega)$, *any* $g_j\in C^\infty(R^n)$ $(0\leqq j<m-r)$ *and any* $h_k\in C^\infty(\Omega_0)$ $(0\leqq k<r)$ *with* (C), *there exists a unique solution* $u\in C^\infty(\Omega)$ *of* (G. P.). *Further, there exists a positive constant* M *such that for any* $(t_0, x_0, y_0)\in\Omega$, *the set* $\Gamma(t_0, x_0, y_0)$ *is a dependence domain of* (t_0, x_0, y_0), *that is, if* $f=0$ *on* $\Gamma(t_0, x_0, y_0)$, $g_j=0$ *on* $\Gamma(t_0, x_0, y_0)\cap\{t=0\}$ $(0\leqq j<m-r)$ *and* $h_k=0$ *on* $\Gamma(t_0, x_0, y_0)\cap\{x=0\}$ $(0\leqq k<r)$, *then* $u=0$ *on* $\Gamma(t_0, x_0, y_0)$.

If the conclusion of the theorem is satisfied, we say that the Goursat problem is *C^∞-well-posed with good dependence domains.* This conception make it easy to get necessary conditions, like the conception of the existence of a finite propagation speed in the non-characteristic Cauchy problems. (Cf. T. Nishitani [6].)

Remark 1.2. (i) If $r=1$, this theorem has been essentially proved by Y. Hasegawa ([1]).
(ii) If n=1, then $\Sigma_2'=\phi$, hence (A-2) is satisfied.

Theorem 2. *Assume* (A-1) *and*

(A-3) $\{q_{m-r}, c_{0,r}\}=0$ *on* Σ_2', *where* $\{,\}$ *denotes the Poisson bracket with respect to* $(t, x, y; \tau, \xi, \eta)$.

If the Goursat problem is C^∞-well-posed with good dependence domains, then (A-2) *is satisfied.*

Remark 1.3. Under the condition (A-3), the condition (A-2) is equivalent to

(A-2)′ $\begin{cases} p^s_{m-1}=0 \text{ on } \Sigma'_2, \text{ where } p^s_{m-1} \text{ denotes the subprincipal symbol} \\ \text{of } P. \end{cases}$

It is natural to ask what kind of conditions on lower order terms are necessary, if (A-3) is not satisfied. As to this question, the author believes that without any conditions on lower order terms, the Goursat problem is C^∞-well-posed with good dependence domains, if (A-1) and the following (A-4) is satisfied.

(A-4) $$\{q_{m-r}, c_{0,r}\} \neq 0 \text{ on } \Sigma'_2.$$

We can prove, however, only the following.

Theorem 3. *Assume* $m=n=2$, $r=1$ *and that* (A-1) *is satisfied. Further, assume* (A-4) *and*

(A-5) $$q_{m-r}=q_1(t, x, y; \tau, \xi, \eta) \text{ is independent of } \xi.$$

Then, the Goursat problem is C^∞*-well-posed with good dependence domains.*

Example 1.4. Consider $P=\partial_t\partial_x - x\partial_x\partial_y +$ (lower order terms) on $\boldsymbol{R}^3$. The Goursat problem for P is C^∞-well-posed with good dependence domains for any lower order terms.

§ 2. Proof of Theorem 1

The idea of our proof is the same as that of Hasegawa's ([1]).

By the assumption of (A-1), the following Cauchy problem is C^∞-well-posed with a finite propagation speed.

$$\begin{cases} c_0(0, x, y; D_x, D_y)v=w(x, y) \text{ on } \boldsymbol{R}^n, \\ \partial_x^k v_{|x=0}=v_k(y) \text{ on } \boldsymbol{R}^{n-1} \ (0\leqq k<r). \end{cases}$$

Solving suitable equations of this type, we can determine $\partial_t^j u_{|t=0}$ $(j\geqq m-r)$ uniquely from (G. P.), hence we may assume $g_j(x, y)=0$ $(0\leqq j<m-r)$ and f, h_k $(0\leqq k<r)$ are *flat* at $t=0$, that is, all the derivatives vanish at $t=0$. In this case, the solution u of (G. P.) is also flat at $t=0$.

Let Q be an arbitrary differential operator whose principal symbol is q_{m-r} and put $C_0(t_0)=c_0(t_0, x, y; D_x, D_y)$. To avoid ambiguity, we fix some terminologies.

Definition 2.1. 1) The set $\Gamma \subset \Omega$ (resp. $\gamma \subset \boldsymbol{R}^n$) is called a *uniqueness domain* of the Cauchy problem for Q (resp. $C_0(t_0)$), if $u \in C^\infty(\boldsymbol{R}^{n+1})$ (resp. $v \in C^\infty(\boldsymbol{R}^n)$) satisfies $Qu=0$ on Γ and $\partial_t^j u_{|t=0}=0$ on $\Gamma \cap \{t=0\}$ $(0 \leqq j < m-r)$ (resp. $C_0(t_0)v=0$ on γ and $\partial_x^k v_{|x=0}=0$ on $\gamma \cap \{x=0\}$ $(0 \leqq k < r)$), then $u=0$ on Γ (resp. $v=0$ on γ).

2) A *dependence domain* of (t_0, x_0, y_0) (resp. (x_0, y_0)) of the Cauchy problem for Q (resp. $C_0(t_0)$) is a uniqueness domain including (t_0, x_0, y_0) (resp. (x_0, y_0)).

Let M and T_0 be positive numbers such that $T_0 \geqq 2T$. Put $\gamma_t = \{(x, y) \in \boldsymbol{R}^n;\ |y| \leqq M(T_0-t)(2MT_0-|x|),\ |x| \leqq M(T_0-t)\}$ for $t \in [0, T]$. The following lemma is easy, hence the proof is omitted.

Lemma 2.2. *For sufficiently large M, the set γ_{t_0} (resp. $\Gamma_{t_0} = \bigcup_{0 \leqq t \leqq t_0} \{t\} \times \gamma_t$) is a uniqueness domain of the Cauchy problem for $C_0(t_0)$ (resp. Q) for any $t_0 \in [0, T]$.*

The following energy inequalities are the main points of the proof. (See Lemma 6.1 and 6.2 in [1].)

Lemma 2.3. *For any non-negative integers p, q, s, there exists a constant C such that the following two inequalities hold.*

(2-1)
$$\sum_{j=0}^{p+m-r-1} |||\partial_t^j v(t, \cdot, \cdot)|||_{s+m-r-1-j, t} \leqq C \sum_{j=0}^{p} \int_0^t |||\partial_t^j Qv(t', \cdot, \cdot)|||_{s-j, t'} dt' \quad (0 \leqq t \leqq T),$$

for any $v \in C_+^\infty(\boldsymbol{R}^{n+1}) = \{v \in C_0^\infty(\boldsymbol{R}^{n+1});\ v=0 \text{ on } t<0\}$.

(2-2)
$$\sum_{j=0}^{q} \sum_{k=0}^{p+r-1-j} ||\partial_t^j \partial_x^k u(t, x, \cdot)||_{p+r-1-k-j, t, x} \leqq C\Big\{\sum_{j=0}^{q} |||\partial_t^j C_0(t) u(t, \cdot, \cdot)|||_{p-j, t} + \sum_{j=0}^{q} \sum_{k=0}^{r-1} ||\partial_t^j \partial_x^k u(t, 0, \cdot)||_{p+r-1-k-j, t, 0}\Big\}$$
$$(0 \leqq t \leqq T,\ |x| \leqq M(T_0-t)), \text{ for any } u \in C_+^\infty(\boldsymbol{R}^{n+1}).$$

Here, $||| \cdot |||_{s,t}$ denotes the Sobolev norm of order s on the domain γ_t and

$||\cdot||_{s,t,x_1}$ *denotes the Sobolev norm of order* s *on the domain* $\gamma_t \cap \{x=x_1\}$.

Proof. Let $|||\cdot|||_s$ (resp. $||\cdot||_s$) denote the Sobolev norm of order s on $\boldsymbol{R}^n_{(x,y)}$ (resp. $\boldsymbol{R}^{n-1}_y$).

(1) It is well-known that there holds the following energy inequality for Q on an arbitrary compact set K.

$$\sum_{j=0}^{p+m-r-1} |||\partial_t^j v(t, \cdot, \cdot)|||_{s+m-r-1-j}$$
$$\leqq C\sum_{j=0}^{p}\int_0^t |||\partial_t^j Qv(t', \cdot, \cdot)|||_{s-j}dt' \quad (0\leqq t\leqq T),$$

for any $v\in C_+^\infty(\boldsymbol{R}^{n+1})$ with $\operatorname{supp} v\subset K$.

Since Γ_t is a uniqueness domain of the Cauchy problem for Q, the inequality (2-1) follows from this.

(2) It is also well-known that there holds the following energy inequality for $C_0(t)$.

$$\sum_{j=0}^{q}\sum_{k=0}^{p+r-1-j} ||\partial_t^j\partial_x^k u(t, x, \cdot)||_{p+r-1-k-j}$$
$$\leqq C\sum_{j=0}^{q}\{\sum_{k=0}^{p-j}\left|\int_0^x ||\partial_t^j\partial_x^k C_0(t)u(t, x', \cdot)||_{p-k-j}dx'\right|$$
$$+\sum_{k=0}^{p+r-1-j}||\partial_t^j\partial_x^k u(t, 0, \cdot)||_{p+r-1-k-j}\}$$
$$(0\leqq t\leqq T,\ |x|\leqq MT_0), \text{ for any } u\in C_0^\infty(K).$$

If $x_1\geqq 0$ (resp. $x_1\leqq 0$), then $\gamma_t\cap\{0\leqq x\leqq x_1\}$ (resp. $\gamma_t\cap\{x_1\leqq x\leqq 0\}$) is a uniqueness domain of the Cauchy problem for $C_0(t)$. Therefore, the following holds.

$$\sum_{j=0}^{q}\sum_{k=0}^{p+r-1-j} ||\partial_t^j\partial_x^k u(t, x, \cdot)||_{p+r-1-k-j,t,x}$$
$$\leqq C\sum_{j=0}^{q}\{\sum_{k=0}^{p-j}\left|\int_0^x ||\partial_t^j\partial_x^k C_0(t)u(t, x', \cdot)||_{p-k-j,t,x'}dx'\right|$$
$$+\sum_{k=0}^{p+r-1-j}||\partial_t^j\partial_x^k u(t, 0, \cdot)||_{p+r-1-k-j,t,0}\}.$$

We can easily obtain

$$\sum_{j=0}^{q}\sum_{k=0}^{p+r-1-j} ||\partial_t^j\partial_x^k u(t, 0, \cdot)||_{p+r-1-k-j,t,0}$$
$$\leqq C\sum_{j=0}^{q}\{\sum_{l=0}^{p-1-j}||\partial_t^j\partial_x^l C_0(t)u(t, 0, \cdot)||_{p-1-l-j,t,0}$$
$$+\sum_{k=0}^{r-1}||\partial_t^j\partial_x^k u(t, 0, \cdot)||_{p+r-1-k-j,t,0}\}$$

$$\leqq C\sum_{j=0}^{q}\{|||\partial_t^j C_0(t)u(t,\cdot,\cdot)|||_{p-j,t}$$
$$+\sum_{k=0}^{r-1}||\partial_t^j\partial_x^k u(t,0,\cdot)||_{p+r-1-k-j,t,0}\}.$$

Further, we have

$$\sum_{l=0}^{p-j}\left|\int_0^x ||\partial_t^j\partial_x^l C_0(t)u(t,x',\cdot)||_{p-l-j,t,x'}dx'\right|$$
$$\leqq C|||\partial_t^j C_0(t)u(t,\cdot,\cdot)|||_{p-j,t},\ \text{if}\ |x|\leqq M(T_0-t).$$

Thus, we get (2-2). □

Corollary 2.4. *For any non-negative integers p and s, there exists a constant C such that the following inequality holds.*

$$\sum_{j=0}^{p+m-r-1}\sum_{k=0}^{s+m-2-j}||\partial_t^j\partial_x^k u(t,x,\cdot)||_{s+m-2-j-k,t,x}$$
$$\leqq C\{\sum_{j=0}^{p}\int_0^t|||\partial_t^j Q\circ C_0(t')u(t',\cdot,\cdot)|||_{s-j,t'}dt'$$
$$+\sum_{k=0}^{r-1}\sum_{j=0}^{p+m-r-1}||\partial_t^j\partial_x^k u(t,0,\cdot)||_{s+m-2-j-k,t,0}\}$$
$$(0\leqq t\leqq T,\ |x|\leqq M(T_0-t)),\ \textit{for any}\ u\in C_+^\infty(\mathbf{R}^{n+1}).$$

Lemma 2.5. *Assume* (A-1) *and* (A-2). *Then, the operator P can be decomposed as follows.*

$$P=Q\circ C_0(t)+R.$$

Here, Q is a differential operator with the principal symbol q_{m-r} and R is a differential operator whose total order is less than $m-1$ and whose order with respect to ∂_t is less than $m-r$.

Proof. Put $r(t,x,y;\tau,\xi,\eta)=\sigma_{m-1}(P-q_{m-r}(t,x,y;D_t,D_x,D_y)\circ c_{0,r}(t,x,y;D_x,D_y))$. We can write

$$r(t,x,y;\tau,\xi,\eta)=c_{0,r-1}(t,x,y;\xi,\eta)q_{m-r}(t,x,y;\tau,\xi,\eta)$$
$$+d(t,x,y;\tau,\xi,\eta),$$

where the degree of d with respect to τ is less than $m-r$. By the assumption (A-2), we have $d=0$ on Σ_2'. Let $\tau=\tau_j(t,x,y;\xi,\eta)$ $(1\leqq j\leqq m-r)$ be the roots of $q_{m-r}=0$. We have $d(t,x,y;\tau_j(t,x,y;\xi,\eta),\xi,\eta)=0$ $(1\leqq j\leqq m-r)$ if $c_{0,r}(t,x,y;\xi,\eta)=0$. Put $d(t,x,y;\tau,\xi,\eta)=\sum_{j=0}^{m-r-1}d_j(t,x,y;\xi,\eta)\tau^{m-r-1-j}$. Since τ_j $(1\leqq j\leqq m-r)$ are distinct for $(\xi, \eta)\neq(0, 0)$, we have $d_j(t, x, y; \xi, \eta)=0$ $(0\leqq j<m-r)$ if $c_{0,r}(t, x, y; \xi, \eta)=0$ and $(\xi, \eta)\neq(0, 0)$. Since $c_{0,r}=0$ has only real distinct roots ξ for $\eta\neq 0$, there follows that d_j $(0\leqq j<m-r)$ are divisible by $c_{0,r}$. Put $d=c_{0,r}q_{m-r-1}$. Put $Q=(q_{m-r}+q_{m-r-1})$ $(t, x, y; D_t, D_x, D_y)$ and $R=P-Q\circ C_0(t)$. Then, the operator R has the desired properties. □

Now, we shall construct the solution of (G. P.). Solving Cauchy problems for Q and $C_0(t)$, we can determine u_l $(l\geqq 0)$ by the following iteration.

$$\begin{cases} Q\circ C_0(t)u_0=f(t,x,y) \text{ on } \Omega, \\ \partial_t^j(C_0(t)u_0)_{|t=0}=0 \text{ on } \boldsymbol{R}^n \quad (0\leqq j<m-r), \\ \partial_x^k u_{0|x=0}=h_k(t,y) \text{ on } \Omega_0 \quad (0\leqq k<r). \end{cases}$$

$$\begin{cases} Q\circ C_0(t)u_{l+1}=-Ru_l \text{ on } \Omega, \\ \partial_t^j(C_0(t)u_{l+1})_{|t=0}=0 \text{ on } R^n \quad (0\leqq j<m-r), \\ \partial_x^k u_{l+1|x=0}=0 \text{ on } \Omega_0 \quad (0\leqq k<r), \quad (l\geqq 0). \end{cases}$$

By Corollary 2. 4, there holds

$$\text{(2-3)} \qquad \begin{aligned} &\sum_{j=0}^{p}\{|||\partial_t^j Ru(t, \cdot, \cdot)|||_{s-j,t}+|||\partial_t^j u(t, \cdot, \cdot)|||_{s+m-2-j,t}\} \\ &\leqq C\{\sum_{j=0}^{p}\int_0^t|||\partial_t^j Q\circ C_0(t')u(t', \cdot, \cdot)|||_{s-j,t'}dt' \\ &\quad+\sum_{k=0}^{r-1}\sum_{j=0}^{p+m-r-1}||\partial_t^j\partial_x^k u(t, 0, \cdot)||_{s+m-2-j-k,t,0}\}. \end{aligned}$$

Therefore, the infinite series $\sum_{l=0}^{\infty}u_l$ converges in $C^\infty(\Omega)$ and gives a solution u of (G. P.). The uniqueness of the solution also follows from (2-3). Since for sufficiently large M, the set $\Gamma_1(t_0, x_0, y_0)$ (resp. $\Gamma_2(x_0, y_0)$) is a dependence domain of (t_0, x_0, y_0) (resp. (x_0, y_0)) of the Cauchy problem for Q (resp. $C_0(t_0)$), it follows from the construction of solutions that $\Gamma(t_0, x_0, y_0)$ is a dependence domain of (t_0, x_0, y_0) of the Goursat problem for P.

§ 3. Proof of Theorem 2

For a positive constant δ, consider the following coordinate transformation: $s=t\pm\delta x$, $z=x$, $w=y$. Let $P_\pm^\sim$ be the transformed operator from P, that is, $P_\pm^\sim=p(s\mp\delta z, z, w; D_s, D_z\pm\delta D_s, D_w)$. From the assumption that the Goursat problem is C^∞-well-posed with good dependence domains, it follows that for any $(t_0, x_0, y_0) \in \Omega$, there exist a neighborhood U of (t_0, x_0, y_0) and constants N, C such that

$$|u(t_1, x_1, y_1)| \leqq C \sum_{j+k+|\alpha| \leqq N} \sup_{\Gamma(t_1, x_1, y_1)} |\partial_t^j \partial_x^k \partial_y^\alpha Pu|$$

for any $u \in C_0^\infty(U)$ and any $(t_1, x_1, y_1) \in U$.

From this inequality, there holds the following inequality for sufficiently small positive number δ: For any (t_0, x_0, y_0) with $\pm x_0 > 0$, there exist a neighborhood $U^\sim$ of $(s_0, z_0, w_0) = (t_0 \pm \delta x_0, x_0, y_0)$ and constants N, C, $M^\sim$ such that

$$|u^\sim(s_1, z_1, w_1)| \leqq C \sum_{j+k+|\alpha| \leqq N} \sup_{\Gamma^\sim(s_1, z_1, w_1)} |\partial_s^j \partial_z^k \partial_w^\alpha P_\pm^\sim u^\sim|$$

for any $u^\sim \in C_0^\infty(U^\sim)$ and any $(s_1, z_1, w_1) \in U^\sim$,

where $\Gamma^\sim(s_1, z_1, w_1) = \{(s, z, w);\ (|z-z_1|^2 + |w-w_1|^2)^{1/2} \leqq M^\sim(s_1 - s)\}$. Now, put $q_{m-r}^\sim(s, z, w; \sigma, \zeta, \omega) = q_{m-r}(s \pm \delta z, z, w; \sigma, \zeta \pm \delta\sigma, \omega)$ and $c_{0,r}^\sim(s, z, w; \sigma, \zeta, \omega) = c_{0,r}(s \pm \delta z, z, w; \zeta \pm \delta\sigma, \omega)$. Then, we have $p_{\pm,m}^\sim(s, z, w; \sigma, \zeta, \omega) = p_m(s \pm \delta z, z, w; \sigma, \zeta \pm \delta\sigma, \omega) = q_{m-r}^\sim(s, z, w; \sigma, \zeta, \omega) c_{0,r}^\sim(s, z, w; \sigma, \zeta, \omega)$. By the assumption (A-3) and the invariance of the Poisson bracket with respect to coordinate transformations, we have $\{q_{m-r}^\sim, c_{0,r}^\sim\}^\sim = 0$ on $\Sigma_2'^\sim = \{(s, z, w; \sigma, \zeta, \omega);\ q_{m-r}^\sim = c_{0,r}^\sim = 0\}$, where $\{,\}^\sim$ is the Poisson bracket with respect to $(s, z, w; \sigma, \zeta, \omega)$. Since we can apply Theorem 2 of Ivrii-Petkov [2], we have $p_{m-1}^{\sim s} = 0$ on $\Sigma_2'^\sim$, where $p_{m-1}^{\sim s}$ is the subprincipal symbol of $P^\sim$. By the invariance of the subprincipal symbol, we have (A-2)$'$. It is an easy calculation to show that (A-2)$'$ is equivalent to (A-2) under the assumption (A-3).

§ 4. Proof of Theorem 3

Assume that $n = m = 2$ and $p_2 = (\tau - a(t, x, y)\eta)(\xi - b(t, x, y)\eta)$, where $\{\tau - a\eta, \xi - b\eta\} \neq 0$ on $\Sigma_2' = \{(t, x, y; \tau, \xi, \eta) \in \Omega \times R^3;\ \tau = a\eta, \xi = b\eta, \eta \neq 0\}$. Note that Σ_2' is the *critical set* of p_2, that is, $\Sigma_2' = \{(t, x, y; \tau, \xi, \eta) \in \Omega \times R^3;\ p_2 = 0,\ \nabla_{(t,x,y;\tau,\xi,\eta)} p_2 = 0,\ \eta \neq 0\}$. As shown in §2, we can reduce the Goursat problem to the case where $g_0(x, y) = 0$ and f, h_0 are flat at $t = 0$. Similarly, by the assumption (A-5), we can further reduce to the case where $g_0 = h_0 = 0$ and f is flat on $\{t = 0$ or $x = 0\}$. Thus, we have only to solve the followihg reduced problem.

$$(\mathrm{R.G.P.})_{\pm} \qquad \begin{cases} Pu_{\pm}=f_{\pm}(t,x,y) \text{ on } \Omega, \\ u_{\pm|t=0}=0 \text{ on } R^2, \\ u_{\pm|x=0}=0 \text{ on } \Omega_0, \end{cases}$$

where $f_{\pm}\in C^{\infty}(R^3)$ and $f_{\pm}=0$ on $S_{\pm}=\{(t, x, y);\ t\leqq 0, \text{ or } \pm x\leqq 0\}$.

Consider the coordinate transformation; $s=t\pm x$, $z=x$, $w=y$. Let $P_{\pm}^{\sim}$ be the transformed operator from P. The principal symbol $p_{\pm,2}^{\sim}=\pm(\sigma-a\omega)(\sigma\pm\zeta\mp b\omega)$ is *effectively hyperbolic* on the critical set of $p_{\pm,2}^{\sim}$. Therefore, by Theorem 2 of V. Ya. Ivrii [3] (cf. N. Iwasaki [4]), we can solve the following Cauchy problem for $P_{\pm}^{\sim}$;

$$(\mathrm{C.P.})_{\pm} \begin{cases} P_{\pm}^{\sim}u_{\pm}^{\sim}=f_{\pm}^{\sim}(s,z,w)=f_{\pm}(s\mp z,z,w) \text{ on } R^3, \\ u_{\pm|s=0}^{\sim}=\partial_s u_{\pm|s=0}^{\sim}=0 \text{ on } R^2. \end{cases}$$

Note that $f_{\pm}^{\sim}=0$ on $S_{\pm}^{\sim}=\{(s, z, w);\ s\mp z\leqq 0,\ \pm z\leqq 0\}$. By the well-known sweep-out method, we can prove that

$$\Gamma_{\pm}^{\sim}(s_0,z_0,w_0)=\{(s,z,w);\ (|z-z_0|^2+|w-w_0|^2)^{1/2} \leqq M(s_0-s),\ s-s_0\leqq\pm(z-z_0)\leqq 0\}$$

is a dependence domain of (s_0, z_0, w_0) of the Cauchy problem for $P_{\pm}^{\sim}$. Hence, the solution $u_{\pm}^{\sim}$ of $(\mathrm{C.P.})_{\pm}$ satisfies $u_{\pm}^{\sim}=0$ on $S_{\pm}^{\sim}$. This means that $u_{\pm}(t, x, y)=u_{\pm}^{\sim}(t\pm x, x, y)=0$ on $S_{\pm}$. Thus, we can solve $(\mathrm{R.G.P.})_{\pm}$. We can also prove the existence of good dependence domains from the fact that $\Gamma_{\pm}^{\sim}(s_0, z_0, w_0)$ is a dependence domain of (s_0, z_0, w_0) of the Cauchy problem for $P_{\pm}^{\sim}$.

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