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The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians

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Abstract. We compute the subgroup of the monodromy group of a generalized Kummer variety associated to equivalences of derived categories of abelian surfaces. The result was announced by Markman and Mehrotra (2017). Mongardi (2016) showed that the subgroup constructed here is in fact the whole monodromy group. As an application we prove the Hodge conjecture for the generic abelian fourfold of Weil type with complex multiplication by an arbitrary imaginary quadratic number field K, but with trivial discriminant invariant in $\mathbb{Q}^*/\mathrm{Nm}(K^*)$. The latter result is inspired by a recent observation of O'Grady that the third intermediate Jacobians of smooth projective varieties of generalized Kummer deformation type form complete families of abelian fourfolds of Weil type. Finally, we prove the surjectivity of the Abel–Jacobi map from the Chow group $\mathrm{CH}^2(Y)_0$ of codimension 2 algebraic cycles homologous to zero on every projective irreducible holomorphic symplectic manifold Y of Kummer type onto the third intermediate Jacobian of Y, as predicted by the generalized Hodge Conjecture.

Keywords. Abelian surfaces and fourfolds, hyperkähler varieties, Hodge Conjecture, derived categories

Contents

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1. Introduction

1.1. Monodromy of generalized Kummers

Let X be a complex projective abelian surface, $X^{(n)}$ its n-th symmetric product, and $X^{[n]}$ the Hilbert scheme of length n zero-dimensional subschemes of X. Let $\pi : X^{[n]} \to X$ be the composition of the Hilbert–Chow morphism $X^{[n]} \to X^{(n)}$ and the summation morphism $X^{(n)} \to X$. The generalized Kummer variety $K_X(n-1)$ is the fiber of π over $0 \in X$; it is a smooth, projective, simply connected variety of dimension $2n-2$. It admits a holomorphic symplectic form, unique up to a constant multiple [2]. The morphism π is an isotrivial family, with every fiber isomorphic to $K_X(n-1)$. The variety $K_X(1)$ is the Kummer $K3$ surface associated to X. Let $s_n \in H^{\text{even}}(X, \mathbb{Z})$ be the Chern character of the ideal sheaf of a length n subscheme of X. The moduli space $\mathcal{M}(s_n)$ of rank 1 torsion free sheaves on X with Chern character s_n is isomorphic to $X^{[n]} \times \text{Pic}^0(X)$.

Set

$$V := H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})^*.$$

V has a natural symmetric unimodular bilinear pairing, given by

$$((a_1, a_2), (b_1, b_2)) = b_2(a_1) + a_2(b_1), \tag{1.1}$$

and $H^*(X, \mathbb{Z})$ is the spin representation of the arithmetic group $\text{Spin}(V)$ recalled below in (4.6). The half-spin representations are $S^+ := H^{\text{even}}(X, \mathbb{Z})$ and $S^- := H^{\text{odd}}(X, \mathbb{Z})$. Each of the half-spin representations admits a symmetric integral and unimodular

$\mathrm{Spin}(V)$-invariant bilinear paring, recalled below in (4.15), and $\mathrm{Spin}(V)$, $\mathrm{Spin}(S^+)$, and $\mathrm{Spin}(S^-)$ all embed as the same subgroup of $SO(V) \times SO(S^+) \times SO(S^-)$. The latter identification of the three spin groups is a consequence of an integral version of triality for Spin(8) (Theorem 4.6). Let $\tilde{\tau}$ act on $H^i(X,\mathbb{Z})$ by $(-1)^{i(i-1)/2}$ and on V by $\tilde{\tau}(a_1,a_2) = (-a_1,a_2)$. Denote by $G(S^+)^{\mathrm{even}}$ the subgroup of $\mathrm{GL}(V \oplus S^+ \oplus S^-)$ generated by $\mathrm{Spin}(V)$ and $\tilde{\tau}$. The group $G(S^+)^{\mathrm{even}}$ arrises naturally as one of the Clifford groups (see (4.6)). We describe next a monodromy representation of $G(S^+)^{\mathrm{even}}$ on the cohomology ring of the moduli space $\mathcal{M}(s_n)$.

Definition 1.1. Let Y be a smooth projective variety. An automorphism g of the cohomology ring $H^*(Y,\mathbb{Z})$ is called a *monodromy operator* if there exists a family $\mathcal{Y} \to B$ (which may depend on g) of compact Kähler manifolds, having Y as a fiber over a point $b_0 \in B$, and such that g belongs to the image of $\pi_1(B,b_0)$ under the monodromy representation. The *monodromy group* $\mathrm{Mon}(Y)$ of Y is the subgroup of $\mathrm{GL}(H^*(Y,\mathbb{Z}))$ generated by all the monodromy operators.

Let $\mathcal{P}$ be an object in the bounded derived category $D^b(X \times X)$ of coherent sheaves on $X \times X$, which is the Fourier–Mukai kernel of an auto-equivalence $\Phi_{\mathcal{P}} : D^b(X) \to D^b(X)$ of the derived category of the abelian surface X. Then $\mathrm{ch}(\mathcal{P})$, considered as a correspondence, induces an automorphism of $H^*(X,\mathbb{Z})$, which is the image of an element g of $\mathrm{Spin}(V)$ in $\mathrm{GL}(H^*(X,\mathbb{Z}))$ via the spin representation, by results of Mukai and Orlov (see Section 7). Let $\mathcal{E}$ be a universal sheaf over $X \times \mathcal{M}(s_n)$. Let π_{ij} be the projection from $X \times \mathcal{M}(s_n) \times X \times \mathcal{M}(s_n)$ onto the product of the i-th and j-th factors. Set

$$\gamma_g := c_{2n+2}\big(\pi_{24,*}[\pi_{12}^*\mathcal{E}^* \otimes \pi_{34}^*\mathcal{E} \otimes \pi_{13}^*\mathcal{P}][1]\big),$$

where the pull-back, push-forward, dual $\mathcal{E}^*$, and tensor product are all taken in the derived category, and [1] is the shift. One can express the right hand side above in terms of the cohomology class $\mathrm{ch}(\mathcal{P})$, using the Grothendieck–Riemann–Roch theorem. We get the class γ_g in $H^{4n+4}(\mathcal{M}(s_n) \times \mathcal{M}(s_n),\mathbb{Z})$, associated to every element g of $\mathrm{Spin}(V)$, by replacing $\mathrm{ch}(\mathcal{P})$ by g in the latter cohomological expression. Considering also the analogue for compositions of equivalences of derived categories and dualization yields a class γ_g for every element $g \in G(S^+)^{\mathrm{even}}$ (see (6.3)). Let $G(S^+)^{\mathrm{even}}_{s_n}$ be the subgroup of $G(S^+)^{\mathrm{even}}$ stabilizing s_n. Define $\mathrm{Spin}(V)_{s_n}$ similarly. Assume $n \geq 3$.

Theorem 1.2 (Theorem 8.6). (1) *The correspondence γ_g induces a graded ring automorphism for every $g \in G(S^+)^{\mathrm{even}}_{s_n}$. The resulting map*

$$\mathrm{mon} : G(S^+)^{\mathrm{even}}_{s_n} \to \mathrm{Aut}[H^*(\mathcal{M}(s_n),\mathbb{Z})]$$

is a group homomorphism and its image is contained in the monodromy group $\mathrm{Mon}(\mathcal{M}(s_n))$.

(2) *For every $g \in \mathrm{Spin}(V)_{s_n}$ there exists a topological complex line bundle L_g over $\mathcal{M}(s_n)$ such that*

$$(g \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E})) = \mathrm{ch}(\mathcal{E})\pi_{\mathcal{M}}^*\,\mathrm{ch}(L_g),$$

where $\pi_{\mathcal{M}} : X \times \mathcal{M}(s_n) \to \mathcal{M}(s_n)$ is the projection.

Let Γ_X be the group of points of order n on X. The translation action of Γ_X on X induces a translation action on $X^{[n]}$. The morphism $\pi : X^{[n]} \to X$ is invariant with respect to the latter action and so Γ_X acts on $K_X(n-1)$. This induces an embedding of Γ_X in $\mathrm{Mon}(K_X(n-1))$. The action of Γ_X on $H^i(K_X(n-1), \mathbb{Z})$ is trivial, for $i = 2, 3$, by Lemma 10.1 (4).

Proposition 1.3 (Proposition 10.2). *There exists a unique injective homomorphism*

$$\overline{\mathrm{mon}} : G(S^+)^{\mathrm{even}}_{s_n} \to \mathrm{Mon}(K_X(n-1))/\Gamma_X \tag{1.2}$$

such that the restriction homomorphisms $H^i(\mathcal{M}(s_n), \mathbb{Z}) \to H^i(K_X(n-1), \mathbb{Z})$, $i = 2, 3$, are $G(S^+)^{\mathrm{even}}_{s_n}$-equivariant with respect to the homomorphisms mon *and* $\overline{\mathrm{mon}}$.

Let Y be a hyperkähler variety deformation equivalent to the generalized Kummer $K_X(m)$ of an abelian surface, $m \geq 2$. We determine the image $\mathrm{Mon}^2(Y)$ of the monodromy group in $\mathrm{Aut}[H^2(Y, \mathbb{Z})]$. The second cohomology $H^2(Y, \mathbb{Z})$ admits the symmetric bilinear Beauville–Bogomolov–Fujiki pairing [2]. It has signature $(3, -4)$. Given an element u of $H^2(Y, \mathbb{Z})$ with (u, u) equal to 2 or -2, let $R_u : H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be the reflection in u, $R_u(w) = w - 2\frac{(w,u)}{(u,u)}u$, and set $r_u := \frac{(u,u)}{-2} R_u$. Then r_u is the reflection in u when $(u, u) = -2$, and $-r_u$ is the reflection in u when $(u, u) = 2$. Set

$$\mathcal{W} := \langle r_u : u \in H^2(Y, \mathbb{Z}) \text{ and } (u, u) = \pm 2 \rangle \tag{1.3}$$

to be the subgroup of $O(H^2(Y, \mathbb{Z}))$ generated by the elements r_u. Then $\mathcal{W}$ is a normal subgroup of finite index in $O(H^2(Y, \mathbb{Z}))$.

The lattice $H^2(Y, \mathbb{Z})$ is not unimodular. The residual group $H^2(Y, \mathbb{Z})^*/H^2(Y, \mathbb{Z})$ is cyclic of order $\dim(Y) + 2$. The image of $\mathcal{W}$ in the automorphism group of $H^2(Y, \mathbb{Z})^*/H^2(Y, \mathbb{Z})$ has order 2 and is generated by multiplication by -1, by [26, Lemma 4.10]. We get a character

$$\chi : \mathcal{W} \to \{1, -1\} \subset \mathbb{C}^{\times}. \tag{1.4}$$

The character group $\mathrm{Hom}(\mathcal{W}, \mathbb{C}^{\times})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and is generated by χ and the determinant character det (the proof is identical to that of [26, Cor. 7.9]). Note that $\det(r_u) = (u, u)/2$ and $\chi(r_u) = -(u, u)/2$. Consequently, their product $\det \cdot \chi$ takes r_u to -1, for both $+2$ and -2 vectors u. Let $\mathcal{W}^{\det \cdot \chi}$ be the kernel of $\det \cdot \chi$.

Theorem 1.4. *The image* $\mathrm{Mon}^2(Y)$ *in* $O(H^2(Y, \mathbb{Z}))$ *of the monodromy group* $\mathrm{Mon}(Y)$ *is equal to* $\mathcal{W}^{\det \cdot \chi}$. *Consequently, the homomorphism*

$$\overline{\mathrm{mon}} : G(S^+)^{\mathrm{even}}_{s_n} \to \mathrm{Mon}(K_X(n-1))/\Gamma_X,$$

given in (1.2)*, is an isomorphism.*

The inclusion $\mathcal{W}^{\det \cdot \chi} \subset \mathrm{Mon}^2(Y)$ is proven in Section 10.1. The reverse inclusion was proven by Mongardi [33], using general results about the action of the monodromy of an irreducible holomorphic symplectic manifold[1] on classes of extremal curves, as well as an additional monodromy constraint proven in [31, Cor. 4.8]. Proposition 1.3 and Theorem 1.4 yield the short exact sequence

$$1 \to \Gamma_X \to \mathrm{Mon}(K_X(n-1)) \to G(S^+)^{\mathrm{even}}_{s_n} \to 1,$$

the extension class of which is yet to be determined.

Theorem 1.4 implies that the image of $\mathrm{Mon}^2(Y)$ in $O(H^2(Y,\mathbb{Z}))/(-1)$ has index $2^{\rho(n)}$, where $n = \frac{\dim Y+2}{2}$ and $\rho(n)$ is the Euler number of n (the number of distinct prime divisors of n) (see [27, Lemma 4.2]). Consequently, the Hodge-isometry class of $H^2(Y,\mathbb{Z})$ does not determine the bimeromorphic class of Y, for Y with a generic period. There are $2^{\rho(n)}$ distinct bimeromorphic classes of hyperkähler varieties, deformation equivalent to the generalized Kummer, for each generic weight 2 Hodge-isometry class, by Verbitsky's Torelli Theorem for irreducible holomorphic symplectic manifolds [28, Theorem 1.3]. In particular, the Kummers $K_X(n-1)$ and $K_{\widehat{X}}(n-1)$, of a generic complex torus X and its dual $\widehat{X}$, are not bimeromorphic. Namikawa proved this counterexample in the case of Kummer fourfold (where $n=3$) [41].

1.2. A monodromy representation for more general moduli spaces

Let $w \in H^{\mathrm{even}}(X,\mathbb{Z})$ be a primitive Hodge class. Denote by w_i the graded summand in $H^{2i}(X,\mathbb{Z})$. Set

$$(w,w) := \int_X (2w_0w_2 - w_1^2).$$

Assume that $w_0 \geq 0$ and $(w,w) \leq -6$. Choose a w-generic ample line bundle H on X (see Section 3 for the definition). Then the moduli space $\mathcal{M}_H(w)$ of H-stable sheaves with Chern character w is a projective, non-singular, connected, and holomorphic symplectic variety of dimension $2-(w,w)$, by results of Mukai and Yoshioka [56]. Yoshioka proved that the Albanese variety of $\mathcal{M}_H(w)$ is isomorphic to $X \times \mathrm{Pic}^0(X)$ and each fiber of the Albanese map $\mathrm{alb} : \mathcal{M}_H(w) \to X \times \mathrm{Pic}^0(X)$ is deformation equivalent to a generalized Kummer variety [56]. The analogue of Theorem 1.2 for $\mathcal{M}_H(w)$ is proved in Corollary 9.4 below, where a group homomorphism

$$\mathrm{mon} : G(S^+)^{\mathrm{even}}_w \to \mathrm{Mon}(\mathcal{M}_H(w))$$

is constructed. Proposition 1.3 is stated and proved in Proposition 10.2 for any fiber $K_a(w)$ of the Albanese map $\mathrm{alb} : \mathcal{M}_H(w) \to X \times \mathrm{Pic}^0(X)$, replacing Γ_X by the subgroup Γ_w of $\mathrm{Aut}(K_a(w))$ acting trivially on $H^i(K_a(w),\mathbb{Z})$, for $i=2,3$. The constructed homomorphism

$$\overline{\mathrm{mon}} : G(S^+)^{\mathrm{even}}_w \to \mathrm{Mon}(K_a(w))/\Gamma_w$$

is an isomorphism, by Theorem 1.4.

[1] An *irreducible holomorphic symplectic manifold* Y is a simply connected compact Kähler manifold such that $H^0(Y,\Omega^2_Y)$ is 1-dimensional spanned by a nowhere degenerate 2-form.

1.3. The Hodge conjecture for a generic abelian fourfold of Weil type of discriminant 1

A $2n$-dimensional abelian variety A is of *Weil type* if there exists an embedding $\eta : K \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of an imaginary quadratic number field $K := \mathbb{Q}[\sqrt{-d}]$, where d is a positive integer, such that the eigenspaces with eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ for the action of $\eta(\sqrt{-d})$ on $H^{1,0}(A)$ are both n-dimensional. A *polarized* $2n$*-dimensional abelian variety of Weil type* is a triple (A, K, h), with (A, K) as above and $h \in H^{1,1}(A, \mathbb{Z})$ an ample class, such that $\eta(\sqrt{-d})^* h = dh$. Any abelian variety of Weil type (A, K) admits such an ample class h [49, Lemma 5.2 (1)].

Polarized $2n$-dimensional abelian varieties of Weil type come in n^2-dimensional families [55], [49, Sec. 5.3]. The top exterior power $\bigwedge_K^{2n} H^1(A, \mathbb{Q})$ of $H^1(A, \mathbb{Q})$ as a K-vector space is naturally embedded as a subspace of $H^{n,n}(A, \mathbb{Q})$, which together with h^n spans a 3-dimensional subspace over $\mathbb{Q}$. The generic abelian variety of Weil type A has a cyclic Picard group but a 3-dimensional $H^{n,n}(A, \mathbb{Q})$ [55], [49, Th. 6.12]. If A is an abelian variety of dimension 4, which is not isogenous to a product of abelian varieties, and such that the cup product homomorphism $\mathrm{Sym}^2 H^{1,1}(A, \mathbb{Q}) \to H^{2,2}(A, \mathbb{Q})$ is not surjective, then A is of Weil type, by [34]. This reduced the proof of the Hodge conjecture for abelian fourfolds to those of Weil type, by [46, Theorem 4.11].

Let $\mathrm{Nm} : K^* \to \mathbb{Q}^*$ be the norm homomorphism, sending $a + b\sqrt{-d}$ to $a^2 + db^2$. Associated to the isogeny class of a polarized abelian variety of Weil type (A, K, h) is a discrete invariant in $\mathbb{Q}^*/\mathrm{Nm}(K^*)$, called its *discriminant* [49, Lemma 5.2 (3)]. Following is the second main result of this paper, which is proven in Section 13.

Theorem 1.5 (Theorem 13.4). *Let (A, K, h) be a polarized abelian fourfold of Weil type of discriminant* 1*. The* 3*-dimensional subspace of $H^{2,2}(A, \mathbb{Q})$ spanned by h^2 and $\bigwedge_K^4 H^1(A, \mathbb{Q})$ consists of algebraic classes. In particular, the Hodge conjecture holds for the generic such (A, K, h).*

The case $K = \mathbb{Q}[\sqrt{-3}]$ was previously proven by Schoen for arbitrary discriminant in [48] and for $K = \mathbb{Q}[\sqrt{-1}]$ and discriminant 1 in [47].

1.4. The modular sheaf over a universal deformation of $\mathcal{M}_H(w) \times \mathcal{M}_H(w)$

The proof of Theorem 1.5 involves the construction of a coherent sheaf over every 4-dimensional compact complex torus in a 5-dimensional family of such tori, which contains a representative of each isogeny class of abelian 4-folds of Weil type of discriminant 1. In Section 1.4.1 we describe this 5-dimensional family. In Section 1.4.2 we describe the coherent sheaf over $X \times \mathrm{Pic}^0(X)$. In Section 1.4.3 we briefly explain why this coherent sheaf deforms to one over each member of this 5-dimensional family of complex tori. In Section 1.4.4 we outline the proof of Theorem 1.5 about the algebraicity of the Hodge–Weil classes.

1.4.1. A period domain for two families. Let X be an abelian surface and set $V := H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})^*$, $S^+ := H^{\mathrm{even}}(X, \mathbb{Z})$ and $S^- := H^{\mathrm{odd}}(X, \mathbb{Z})$. We endow S^+

with the unimodular symmetric bilinear pairing (4.15) of signature $(4, 4)$ (minus the Mukai pairing). Let $w \in S^+$ be the Chern character of the ideal sheaf of a length $n + 1$ subscheme, $n \geq 2$. Let $w^\perp$ be the sublattice orthogonal to w in S^+. Then $w^\perp$ is naturally isometric to $H^2(K_X(n), \mathbb{Z})$ (with minus the Beauville–Bogomolov–Fujiki pairing), and the period domain of $2n$-dimensional irreducible holomorphic symplectic manifolds deformation equivalent to the generalized Kummer variety $K_X(n)$ is

$$\Omega_{w^\perp} := \{\ell \in \mathbb{P}(w^\perp \otimes_{\mathbb{Z}} \mathbb{C}) : (\ell, \ell) = 0, (\ell, \bar{\ell}) < 0\},$$

by [2]. Choose a basis $\{e_1, e_2, e_3, e_4\}$ of $H^1(X, \mathbb{Z})$, compatible with the orientation of X, and let $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ be the dual basis. Consider the following classes in $\bigwedge^4 V$:

$$\begin{aligned}
\alpha &:= \sum_{i=1}^{4} e_i \wedge e_i^*,\\
\beta &:= e_1 \wedge e_2 \wedge e_3 \wedge e_4,\\
\gamma &:= e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*,\\
c_w &:= -(n+1)^2\alpha^2 + 4(n+1)^3\beta + 4(n+1)\gamma.
\end{aligned}$$

We refer to c_w as the *Cayley class* due to part (1) of the following.

Proposition 1.6. (1) (Proposition 11.2) *The class c_w spans the 1-dimensional subspace of $\bigwedge^4_{\mathbb{Q}} V$ invariant under* $\mathrm{Spin}(V)_w$.

(2) (Lemmas 12.1, 12.2) *$\Omega_{w^\perp}$ is also the period domain of integral weight 1 Hodge structures (V, J), where $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a complex structure satisfying:*

(a) *$V^{1,0}$ and $V^{0,1}$ are maximal isotropic with respect to the symmetric bilinear pairing (1.1) on $V_{\mathbb{C}}$.*

(b) *Each of the two lifts of $J \in SO(V_{\mathbb{R}})$ to $\mathrm{Spin}(V_{\mathbb{R}})$ maps to an involution of $S^+_{\mathbb{R}}$, determined by J up to sign, one of whose eigenspaces is a negative definite 2-dimensional subspace of $w^\perp_{\mathbb{R}}$.*

(c) *The class c_w is of Hodge type $(2, 2)$.*

Note that $\Omega_{w^\perp}$ is an open analytic subset of the quadric of isotropic lines in $w^\perp_{\mathbb{C}} \subset S^+_{\mathbb{C}}$. The correspondence between periods $\ell \in \Omega_{w^\perp}$ and maximal isotropic subspaces $V^{1,0}$ in $V_{\mathbb{C}}$ is the restriction to $\Omega_{w^\perp}$ of the well known isomorphism between the quadric of isotropic lines in the half-spin representation $S^+_{\mathbb{C}}$ of $\mathrm{Spin}(V_{\mathbb{C}})$ and one of the connected components of the maximal isotropic Grassmannian of $V_{\mathbb{C}}$ (see [6, III.1.6]).

Proposition 1.6 gives rise to the universal torus

$$\mathcal{T} \to \Omega_{w^\perp} \tag{1.5}$$

over the 5-dimensional period domain $\Omega_{w^\perp}$, and c_w determines an integral class of Hodge type $(2, 2)$ in the cohomology $H^4(T_\ell, \mathbb{Z})$ of the fiber T_ℓ over each $\ell \in \Omega_{w^\perp}$.

Proposition 1.7 (Proposition 12.6 and Corollary 12.9). *Let $h \in S^+$ satisfy $(h, w) = 0$ and $(h, h) < 0$. Set $d := (w, w)(h, h)/4$. The restriction of the universal torus $\mathcal{T}$ to the 4-dimensional subspace $\Omega_{\{w,h\}^\perp} \subset \Omega_{w^\perp}$ consisting of periods ℓ orthogonal to h is a complete family of polarized 4-dimensional abelian fourfolds of Weil type of discriminant 1 with complex multiplication by $\mathbb{Q}[\sqrt{-d}]$.*

All possible imaginary quadratic number fields arise, since the lattice S^+ is unimodular. Complex multiplication by $\sqrt{-d}$ is best explained by an integral version of triality for Spin(8). The groups V and S^- are the two half-spin representations of $\mathrm{Spin}(S^+)$. The elements w, h of S^+ are elements of the Clifford algebra

$$C(S^+) := \bigoplus_{k=0}^{\infty} (S^+)^{\otimes k} / \langle w_1 \otimes w_2 + w_2 \otimes w_1 - (w_1, w_2) : w_1, w_2 \in S^+ \rangle.$$

The spin representation $V \oplus S^-$ is a $C(S^+)$-module, a fact which corresponds to an algebra isomorphism $m : C(S^+) \to \mathrm{End}(V \oplus S^-)$. Each of h and w maps each of the two direct summands of $V \oplus S^-$ to the other direct summand. Hence, the product $w \cdot h$ maps V to itself. We have $w \cdot h + h \cdot w = (h, w) = 0$, by the defining relation of the Clifford algebra, and so $(m_w \circ m_h)^2 = -m_w^2 \circ m_h^2 = -\frac{(w,w)(h,h)}{4} \mathrm{id}_{V \oplus S^-} = (-d)\, \mathrm{id}_{V \oplus S^-}$.

Proposition 1.7 was first proved by O'Grady in the following set-up. The complex torus T_ℓ, $\ell \in \Omega_{w^\perp}$, is isogenous to the third intermediate Jacobian of every marked $2n$-dimensional irreducible holomorphic symplectic manifold Y in $\mathfrak{M}^0_{w^\perp}$ with period ℓ, by Lemma 12.15. O'Grady observed that every ample class h on Y with Beauville–Bogomolov–Fujiki degree $(h, h) = 2k$ induces complex multiplication on the intermediate Jacobian with imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$, where $d := (n + 1)k$ [43].

1.4.2. The modular sheaf. Following is a second description of the class c_w. Let $\mathcal{M}(w)$ be the moduli space of rank 1 torsion free sheaves on X with Chern character w. Then $\mathcal{M}(w)$ is $2n + 4$-dimensional. Let $\mathcal{E}$ be a universal sheaf over $X \times \mathcal{M}(w)$. Let π_{ij} be the projection from $\mathcal{M}(w) \times X \times \mathcal{M}(w)$ onto the product of the i-th and j-th factors. Let

$$E := \mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E}) \tag{1.6}$$

be the relative extension sheaf over $\mathcal{M}(w) \times \mathcal{M}(w)$. Then E is a reflexive torsion free sheaf of rank $2n + 2$, which is locally free away from the diagonal [29, Rem. 4.6]. Given a sheaf $F \in \mathcal{M}(w)$, let E_F be the restriction of E to $\{F\} \times \mathcal{M}(w)$. Set $A := X \times \mathrm{Pic}^0(X)$. Then A acts faithfully on $\mathcal{M}(w)$, the first factor via push-forward by translation automorphisms of X and the second factor by tensorization. For a generic sheaf F' in $\mathcal{M}(w)$, the resulting morphism onto the A-orbit of F',

$$\iota_{F'} : A \to \mathcal{M}(w),$$

is an embedding. $\mathrm{Spin}(V)_w$ acts on $H^*(\mathcal{M}(w), \mathbb{Z})$ via a monodromy action (Theorem 1.2) and on $H^*(A, \mathbb{Z}) := \bigwedge^* V$ via the natural action on the exterior algebra of the fundamental representation V.

Theorem 1.8. (1) (Theorem 11.1) *The class $c_2(\mathcal{E}nd(E)) \in H^4(\mathcal{M}(w) \times \mathcal{M}(w), \mathbb{Z})$ is* $\mathrm{Spin}(V)_w$*-invariant with respect to the diagonal monodromy representation of Theorem* 1.2*. Similarly, $c_2(\mathcal{E}nd(E_F)) \in H^4(\mathcal{M}(w), \mathbb{Z})$ is* $\mathrm{Spin}(V)_w$*-invariant.*

(2) (Corollary 9.6) *The homomorphism $\iota_{F'}^* : H^*(\mathcal{M}(w), \mathbb{Z}) \to H^*(A, \mathbb{Z})$ is* $\mathrm{Spin}(V)_w$*-equivariant. Hence, the class $\iota_{F'}^* c_2(\mathcal{E}nd(E_F))$ is* $\mathrm{Spin}(V)_w$*-invariant.*

(3) (Proposition 11.2) *The equality $c_w = \iota_{F'}^* c_2(\mathcal{E}nd(E_F))$ holds.*

The $\mathrm{Spin}(V)_w$-invariance of $c_2(\mathcal{E}nd(E))$ follows from the automorphic property of the Chern character of the universal sheaf in Theorem 1.2 (2). The $\mathrm{Spin}(V)_w$-invariance of c_w follows from Theorem 1.8 (3).

1.4.3. Deforming the modular sheaf. We would like to deform the coherent sheaf $\iota_{F'}^* \mathcal{E}nd(E_F)$ on $A = X \times \mathrm{Pic}^0(X)$ to a coherent sheaf over every fiber T_ℓ, $\ell \in \Omega_{w^\perp}$, of the universal torus (1.5). This would prove the algebraicity of the $\mathrm{Spin}(V)_w$-invariant Hodge class c_w whenever T_ℓ is algebraic. For that purpose we deform the moduli space $\mathcal{M}(w)$ and the sheaf E_F. We first describe the deformation of $\mathcal{M}(w)$.

There exists a moduli space $\mathfrak{M}_{w^\perp}$ of marked irreducible holomorphic symplectic manifolds and a surjective and generically injective period map $\mathrm{Per} : \mathfrak{M}^0_{w^\perp} \to \Omega_{w^\perp}$ from each connected component $\mathfrak{M}^0_{w^\perp}$ of moduli [18, 52]. Choose a connected component $\mathfrak{M}^0_{w^\perp}$ containing a marked generalized Kummer. There exists a universal family $p : \mathcal{Y} \to \mathfrak{M}^0_{w^\perp}$ of irreducible holomorphic symplectic manifolds of generalized Kummer deformation type [30, Th. 1.1]. Pulling back to $\mathfrak{M}^0_{w^\perp}$ the universal torus $\mathcal{T} \to \Omega_{w^\perp}$ constructed in (1.5) via the period map we get a universal fiber product $\mathrm{Per}^*\mathcal{T} \times_{\mathfrak{M}^0_{w^\perp}} \mathcal{Y}$. The latter admits a diagonal action by a trivial group scheme $\underline{\Gamma_w}$ over $\mathfrak{M}^0_{w^\perp}$, whose quotient is a universal deformation

$$\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp} \tag{1.7}$$

of the moduli space $\mathcal{M}(w)$. The family Π is constructed in (12.12).

The above discussion is worked out for more general smooth and compact moduli space $\mathcal{M}(w)$ over X, for more general primitive classes $w \in S^+$. We choose $\mathcal{M}(w)$, so that the universal sheaf $\mathcal{E}$ is twisted, and the sheaf E_F is *maximally twisted*, i.e., its Brauer class in the analytic Brauer group $H^2_{\mathrm{an}}(\mathcal{M}(w), \mathcal{O}^*_{\mathcal{M}(w)})$ has order equal to the rank of E_F. It follows that the sheaf E_F does not have any proper non-trivial subsheaves and is thus slope-stable with respect to every Kähler class on $\mathcal{M}(w)$. The second Chern class of $\mathcal{E}nd(E_F)$ is $\mathrm{Spin}(V)_w$-invariant, hence it remains of Hodge type $(2,2)$ over the whole of $\mathfrak{M}^0_{w^\perp}$, by Lemma 12.14. A theorem of Verbitsky states that if G is a slope-stable reflexive sheaf and $c_2(\mathcal{E}nd(G))$ remains of Hodge type over the whole of $\mathfrak{M}^0_{w^\perp}$, then G deforms to a twisted sheaf over each fiber of the universal family Π (see Theorem 13.3). Such sheaves G are said to be *hyper-holomorphic*. E_F has these properties and we obtain the following result, which is the main cycle-theoretic construction of the paper.

Theorem 1.9 (Theorem 13.3). *The sheaf E_F deforms with $\mathcal{M}(w)$ to a reflexive sheaf, locally free on the complement of a point, over every fiber of the universal family Π given in* (1.7)*. Similarly, the sheaf E given in* (1.6) *deforms with $\mathcal{M}(w) \times \mathcal{M}(w)$ to a reflexive* *sheaf, locally free away from the diagonal, over the cartesian square of every fiber of the universal family* Π.

Verbitsky's theorem applies to slope-stable reflexive sheaves over hyperkähler manifolds. It applies in our set-up, since every hyperkähler structure on a marked irreducible holomorphic symplectic manifold Y with period ℓ determines a hyperkähler structure on the complex torus T_ℓ, $\ell \in \Omega_{w^\perp}$, and both correspond to the same twistor line in $\Omega_{w^\perp}$, by Proposition 12.6. Hence, the restriction of the universal family $\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp}$ to twistor lines in $\mathfrak{M}^0_{w^\perp}$ consists of twistor deformations of the fiber of Π as well. Note that Verbitsky's theorem has already been used to prove the algebraicity of monodromy invariant Hodge classes in [4, 5, 29].

1.4.4. Outline of the proof of Theorem 1.5. It remains to prove that the 3-dimensional subspace of Hodge–Weil classes in $H^{2,2}(T_\ell, \mathbb{Q})$ consists of algebraic classes when ℓ is a period in $\Omega_{\{w,h\}^\perp}$ as in Proposition 1.7. The embedding $\iota_{F'} : A \to \{F\} \times \mathcal{M}(w)$ deforms to an embedding $\iota : T_\ell \to \{F_\ell\} \times \mathcal{M}_\ell$ associated to a choice of a pair of points (F_ℓ, F'_ℓ) in the cartesian square $\mathcal{M}_\ell \times \mathcal{M}_\ell$ of the fiber over ℓ of the universal family Π given in (1.7). Hence, the 1-dimensional subspace of $\mathrm{Spin}(V)_w$-invariant classes in $H^{2,2}(T_\ell, \mathbb{Q})$ is spanned by the algebraic class c_w, by Theorems 1.8 (3) and 1.9. Now $\mathbb{Q}[\sqrt{-d}]$ acts on T_ℓ via rational correspondences, which are algebraic, and we show that the $\mathbb{Q}[\sqrt{-d}]$-translates of c_w and the square Θ_h^2 of the polarization of T_ℓ span the 3-dimensional space of Hodge–Weil classes (Theorem 13.4).

1.5. Surjectivity of the Abel–Jacobi map

Let $\Pi : \mathcal{M} \to \mathfrak{M}^0_{\omega^\perp}$ be the universal deformation of the moduli space $\mathcal{M}_H(w)$ of sheaves on an abelian surface X given in (1.7). Assume that the dimension of $\mathcal{M}_H(w)$ is ≥ 8. Given $b \in \mathfrak{M}^0_{\omega^\perp}$, let E_b be the deformation of the modular sheaf (1.6) constructed in Theorem 1.9 over the cartesian square $\mathcal{M}_b \times \mathcal{M}_b$ of the fiber $\mathcal{M}_b$ of Π. Let $e_b : Y_b \to \mathcal{M}_b$ be the inclusion of a fiber of the Albanese map $\mathrm{alb} : \mathcal{M}_b \to \mathrm{Alb}(\mathcal{M}_b)$. Let $J^2(Y_b) := H^3(Y_b, \mathbb{C})/[H^{2,1}(Y_b) + H^3(Y_b, \mathbb{Z})]$ be the intermediate Jacobian. Assume that Y_b is projective. Then $J^2(Y_b)$ is the codomain for the Abel–Jacobi map associated to any family of complex codimension 2 algebraic cycles on Y_b homologous to 0.

Given $F \in \mathcal{M}_b$, let E_F be the restriction of E_b to $\{F\} \times \mathcal{M}_b$. Fix a point $F_0 \in \mathcal{M}_b$ and consider the map

$$\mathrm{AJ}_b : \mathcal{M}_b \to J^2(Y_b)$$

sending $F \in \mathcal{M}_b$ to the Abel–Jacobi image of an algebraic cycle representing the Chow class

$$e_b^*[c_2(E_F^\vee \overset{L}{\otimes} E_F) - c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0})], \tag{1.8}$$

where $E_F^\vee$ is $R\mathcal{H}om(E_F, \mathcal{O}_{\mathcal{M}_b})$ and the tensor product is taken in the derived category. The morphism AJ_b factors through a morphism

$$\overline{\mathrm{AJ}}_b : \mathrm{Alb}(\mathcal{M}_b) \to J^2(Y_b), \tag{1.9}$$

since the fibers of alb are simply connected.

Theorem 1.10. *The morphism* AJ_b *is surjective for every* $b \in \mathfrak{M}^0_{\omega^\perp}$ *for which* Y_b *is projective.*

Claire Voisin suggested to the author to prove the above theorem as a corollary of Theorem 1.9. The proof of Theorem 1.10 is provided in Section 14.

1.6. Organization of the paper

In Section 3 we recall Yoshioka's result that the fibers of the Albanese map of a moduli space of stable sheaves on an abelian surface are irreducible holomorphic symplectic manifolds of generalized Kummer deformation type. We also recall the relationship between the second cohomology of the fibers and the Mukai lattice.

In Section 4 we recall the integral versions of the Clifford algebra and Clifford groups associated to the cohomology of an abelian surface X. We then recall the integral version of triality for $\mathrm{Spin}(V)$, where V is the rank 8 lattice $H^1(X,\mathbb{Z}) \oplus H^1(X,\mathbb{Z})^*$.

In Section 5 we identify a set of generators for the stabilizer $\mathrm{Spin}(V)_{s_n}$ of the Chern character s_n of the ideal sheaf of a length n subscheme of an abelian surface.

In Section 6 we construct a class γ_g in the cohomology of a product of two moduli spaces $\mathcal{M}_{H_1}(w_1) \times \mathcal{M}_{H_2}(w_2)$ of H_i-stable sheaves with Chern character w_i over an abelian surface X_i associated to a parity preserving isomorphism $g : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ satisfying $g(w_1) = w_2$. When g is a parallel transport operator,[2] or when g is induced by an equivalence of the derived categories of X_1 and X_2, then $g \otimes \gamma_g : H^*(X_1 \times \mathcal{M}_{H_1}(w_1)) \to H^*(X_2 \times \mathcal{M}_{H_2}(w_2))$ maps the class of a universal sheaf to a class of a universal sheaf and γ_g is a parallel transport operator. We show that the assignment $g \mapsto \gamma_g$ is multiplicative and we extend the construction to include the contravariant functor of dualization of sheaves.

In Section 7 we lift certain generators for the stabilizer $\mathrm{Spin}(V)_{s_n}$ found in Section 5 to auto-equivalences of the derived category of an abelian surface X.

In Section 8 we use results of Yoshioka to show that the auto-equivalences found in Section 7 map sheaves in $\mathcal{M}(s_n)$ to sheaves in $\mathcal{M}(s_n)$. This enables us to prove Theorem 1.2 about the monodromy representation $\mathrm{mon} : G(S^+)^{\mathrm{even}}_{s_n} \to \mathrm{Aut}(H^*(\mathcal{M}(s_n),\mathbb{Z}))$.

In Section 9 we extend the latter monodromy representation to an action of a groupoid $\mathcal{G}$ (a category all of whose morphisms are isomorphisms). Objects (X, w, H) of $\mathcal{G}$ consist of an abelian surface X, a primitive Chern character w, and a w-generic polarization H. Morphisms in $\mathrm{Hom}_{\mathcal{G}}((X_1,w_1,H_1),(X_2,w_2,H_2))$ are parallel transport operators $\gamma : H^*(\mathcal{M}_{H_1}(w_1),\mathbb{Z}) \to H^*(\mathcal{M}_{H_2}(w_2),\mathbb{Z})$. A result of Yoshioka implies the existence of such morphisms whenever the dimensions of the two moduli spaces are

[2] Let X_1, X_2 be compact Kähler manifolds. An isomorphism $g : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ is a *parallel transport operator* if there exists a family $\pi : \mathcal{X} \to B$ of compact Kähler manifolds, points $b_1, b_2 \in B$, isomorphisms $\psi_i : X_i \to \mathcal{X}_{b_i}$ with the fibers over b_i, and a continuous path γ from b_1 to b_2 in B, such that $\psi_{2,*} \circ g \circ \psi_1^*$ is induced by parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ.

equal. Yoshioka's result enables us to construct the monodromy representation mon : $G(S^+)^{\text{even}}_w \to \text{Mon}(\mathcal{M}(w))$ for all primitive Chern characters w. We also establish in Section 9 the $\text{Spin}(V)_w$-equivariance of the homomorphism $\iota^*_{F'} : H^*(\mathcal{M}_H(w), \mathbb{Z}) \to H^*(X \times \widehat{X}, \mathbb{Z})$ of Theorem 1.8.

In Section 10.1 we prove Theorem 1.4 about the monodromy group of generalized Kummer varieties. In Section 10.2 we identify the Lie algebra of the Zariski closure of the monodromy group as a subalgebra of the Looijenga–Lunts–Verbitsky Lie algebra. We then verify that monodromy invariant classes of an irreducible holomorphic symplectic manifold of generalized Kummer deformation type are Hodge classes.

In Section 11 we prove the $\text{Spin}(V)_w$-invariance of $c_2(\mathcal{E}nd(E))$ for the modular sheaf E over $\mathcal{M}(w) \times \mathcal{M}(w)$ in Theorem 1.8 (1). We then prove that the Cayley class c_w is the pull-back of $c_2(\mathcal{E}nd(E))$ (Theorem 1.8 (3)).

In Section 12.1 we construct the universal torus $\mathcal{T}$, given in (1.5), over the period domain $\Omega_{w^\perp}$ of irreducible holomorphic symplectic manifolds of generalized Kummer deformation type. In Section 12.2 we prove Proposition 1.7; we construct the polarization Θ_h and the complex multiplication for the complex tori with periods in the 4-dimensional subloci $\Omega_{\{w,h\}^\perp}$ in the 5-dimensional period domain $\Omega_{w^\perp}$. In Section 12.3 we construct a hyperkähler structure on the complex torus T_ℓ associated with a Kähler class on an irreducible holomorphic symplectic manifold with period ℓ (Proposition 12.6). In Section 12.4 we prove that the subloci $\Omega_{\{w,h\}^\perp}$ parametrize abelian fourfolds of Weil type of discriminant 1. In Section 12.5 we construct the universal deformation $\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp}$ of the moduli space of sheaves over the moduli space of marked irreducible holomorphic symplectic manifolds of generalized Kummer deformation type. In Section 12.6 we prove that the torus T_ℓ is isogenous to the third intermediate Jacobian of the irreducible holomorphic symplectic manifold of generalized Kummer deformation type with period ℓ.

In Section 13 we prove Theorem 1.5 about the algebraicity of the Hodge–Weil classes on abelian fourfolds of Weil type of discriminant 1.

In Section 14 we prove Theorem 1.10 verifying the generalized Hodge conjecture for codimension 2 algebraic cycles homologous to 0 on every projective irreducible holomorphic symplectic manifold of generalized Kummer deformation type.

2. Notation

Let X be a complex projective abelian surface, $\widehat{X} := \text{Pic}^0(X)$ its dual surface, and $X^{[n]}$ the Hilbert scheme of length n zero-dimensional subschemes of X. Let $X^{(n)}$ be the n-th symmetric product of X and $\text{hc} : X^{[n]} \to X^{(n)}$ the Hilbert–Chow morphism. Let $\sigma : X^{(n)} \to X$ be the morphism sending a cycle to its sum and let $\pi : X^{[n]} \to X$ be the composition $\sigma \circ \text{hc}$. The $(2n-2)$-dimensional generalized Kummer $K_X(n-1)$ is the fiber of π over $0 \in X$. It is a simply connected projective holomorphically symplectic variety. The algebra of holomorphic differential forms on $K_X(m)$ is generated by the holomorphic symplectic form, which is unique up to a constant scalar [2]. Consequently, $K_X(m)$ admits a hyperkähler structure. The first three Betti numbers $b_i(K_X(m))$ of $K_X(m)$, $m > 1$, are $b_1(K_X(m)) = 0$, $b_2(K_X(m)) = 7$, and $b_3(K_X(m)) = 8$, by [10, Th. 2.4.11].

Let H be an ample line bundle on an abelian surface X and let $\mathcal{M}_H(v)$ denote the moduli space of Gieseker–Simpson H-stable coherent sheaves on X with Chern character v. We always assume that v is primitive and H is v-generic (see Section 3), so that $\mathcal{M}_H(v)$ is smooth and projective. We denote by $K_a(v)$ the fiber of the Albanese morphism from $\mathcal{M}_H(v)$ to its Albanese variety over the point a. Then $K_0(s_n) = K_X(n-1)$, where s_n is the Chern character of the ideal sheaf of a length n subscheme.

A glossary of notation is included in Section 15.

3. The Mukai lattice and the second cohomology of a moduli space of sheaves

Let X be an abelian surface. Set $S^+ := \bigoplus_{i=0}^{2} H^{2i}(X,\mathbb{Z})$. The *Mukai pairing* on S^+ is given by

$$\langle x, y\rangle := \int_X (x_1 y_1 - x_0 y_2 - x_2 y_0), \tag{3.1}$$

where $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$, and x_i, y_i are the graded summands in $H^{2i}(X,\mathbb{Z})$. We refer to S^+ as the *Mukai lattice* of X and elements of S^+ will be called *Mukai vectors*. Following Mukai we endow $S^+_{\mathbb{C}} := S^+ \otimes_{\mathbb{Z}} \mathbb{C}$ with a weight 2 Hodge structure by setting $(S^+)^{2,0} := H^{2,0}(X)$, $(S^+)^{0,2} := H^{0,2}(X)$, and $(S^+)^{1,1} := H^0(X) \oplus H^{1,1}(X) + H^4(X)$.

Let $v = (r, c_1, \chi) \in S^+$ be a primitive Mukai vector, with $c_1 \in H^2(X,\mathbb{Z})$ of Hodge type $(1,1)$. There is a system of hyperplanes in the ample cone of X, called v-*walls*, that is countable but locally finite [19, Ch. 4C]. An ample class is called v-*generic* if it does not belong to any v-wall. Choose a v-generic ample class H. Assume that the moduli space $\mathcal{M}_H(v)$, of Gieseker–Simpson H-stable sheaves on X with Chern character v, is non-empty. Then $\mathcal{M}_H(v)$ is smooth, connected, projective, and holomorphic symplectic of dimension $\langle v, v\rangle + 2$ [36]. The Albanese variety of $\mathcal{M}_H(v)$ is isomorphic to $X \times \widehat{X}$ whenever the dimension of $\mathcal{M}_H(v)$ is at least 4, and each fiber of the Albanese map alb : $\mathcal{M}_H(v) \to X \times \widehat{X}$ is simply connected and deformation equivalent to the generalized Kummer variety of the same dimension [56, Th. 0.2].

Denote by $K_a(v)$ the fiber of alb over a point $a \in X \times \widehat{X}$. The second cohomology $H^2(K_a(v),\mathbb{Z})$ is endowed with a natural symmetric bilinear form, the Beauville–Bogomolov–Fujiki form [2]. Let $\mathcal{E}$ be a quasi-universal family over $X \times \mathcal{M}_H(v)$ of similitude σ, so that $\mathcal{E}$ restricts to $X \times \{t\}$, $t \in \mathcal{M}_H(v)$, as the direct sum $E_t^{\oplus\sigma}$ of σ copies of a sheaf E_t representing the isomorphism class t. Let $v^\perp$ be the sublattice of S^+ orthogonal to v with respect to the Mukai pairing. Denote by π_X and $\pi_{\mathcal{M}}$ the projections from $X \times \mathcal{M}_H(v)$ onto the corresponding factors. Given a Mukai vector $y = (y_0, y_1, y_2)$ set $y^\vee := (y_0, -y_1, y_2)$. Let

$$\theta' : v^\perp \to H^2(\mathcal{M}_H(v),\mathbb{Z})$$

be given by

$$\theta'(y) = -\frac{1}{\sigma}[\pi_{\mathcal{M},*}(\mathrm{ch}(\mathcal{E})\pi_X^* y^\vee)]_1,$$

where the subscript 1 denotes the graded summand in $H^2(\mathcal{M}_H(v), \mathbb{Z})$. We denote the composition of θ' with restriction to the Albanese fiber by

$$\theta : v^\perp \to H^2(K_a(v), \mathbb{Z}). \tag{3.2}$$

Theorem 3.1 ([56, Th. 0.2]). *If the dimension of $\mathcal{M}_H(v)$ is ≥ 8, then the homomorphism θ is a Hodge isometry with respect to the Mukai pairing and the Hodge structure on $v^\perp$ and the Beauville–Bogomolov–Fujiki pairing on $H^2(K_a(v), \mathbb{Z})$.*

4. Spin(8) and triality

Let X be an abelian surface. Set $S := H^*(X, \mathbb{Z})$, $S^+ := H^{\mathrm{even}}(X, \mathbb{Z})$, $S^- := H^{\mathrm{odd}}(X, \mathbb{Z})$, and $V := H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z})$. In Section 4.1 we review the basic facts about Clifford algebras and groups associated to the abelian surface X via the rank 8 lattice V. In Section 4.2 we review the bilinear operations $V \otimes S^+ \to S^-$, $V \otimes S^- \to S^+$, and $S^+ \otimes S^- \to V$ induced by the Clifford product. In Section 4.3 we review triality for the arithmetic subgroup $\mathrm{Spin}(V)$ of Spin(8).

4.1. The Clifford groups

Endow V with the symmetric bilinear form

$$(a, b)_V := b_2(a_1) + a_2(b_1), \tag{4.1}$$

where $a = (a_1, a_2)$, $b = (b_1, b_2)$, $a_1, b_1 \in H^1(X, \mathbb{Z})$, and $a_2, b_2 \in H^1(\hat{X}, \mathbb{Z})$, and we use the natural identification of $H^1(\hat{X}, \mathbb{Z})$ with $H^1(X, \mathbb{Z})^*$. Then V is an even unimodular lattice, the orthogonal direct sum of four copies of the hyperbolic plane, and $Q(a) := \frac{1}{2}(a, a)_V$ is an integral quadratic form. Let $C(V)$ be the Clifford algebra, i.e., the quotient of the tensor algebra of V by the relation

$$v \cdot w + w \cdot v = (v, w)_V. \tag{4.2}$$

As a general reference on Clifford algebras and the theory of spinors we recommend [6]. The free abelian group $H^*(X, \mathbb{Z})$ is isomorphic to $\bigwedge^* H^1(X, \mathbb{Z})$ and is a $C(V)$-module and $C(V)$ maps isomorphically onto $\mathrm{End}[H^*(X, \mathbb{Z})]$, by [9, Prop. 3.2.1 (e)]. The $C(V)$-module structure of $H^*(X, \mathbb{Z})$ is seen as follows. V embeds in $C(V)$, and also in $\mathrm{End}[H^*(X, \mathbb{Z})]$ by sending $w \in H^1(X, \mathbb{Z})$ to the wedge product

$$L_w := w \wedge (\bullet) \tag{4.3}$$

and $\theta \in H^1(\hat{X}, \mathbb{Z}) \cong H^1(X, \mathbb{Z})^*$ to the corresponding derivation D_θ of $H^*(X, \mathbb{Z})$ sending $H^i(X, \mathbb{Z})$ to $H^{i-1}(X, \mathbb{Z})$. The resulting homomorphism $m : V \to \mathrm{End}[H^*(X, \mathbb{Z})]$

satisfies the analogue of the relation (4.2), i.e., $m(v) \circ m(w) + m(w) \circ m(v) = (v, w)\mathrm{id}_{H^*(X,\mathbb{Z})}$. Hence m extends to an algebra homomorphism $m : C(V) \to \mathrm{End}[H^*(X, \mathbb{Z})]$, by the universal property of $C(V)$.

We let

$$\tau : C(V) \to C(V) \tag{4.4}$$

be the *main anti-automorphism*, sending $v_1 \cdot \cdots \cdot v_r$ to $v_r \cdot \cdots \cdot v_1$. Let

$$\alpha : C(V) \to C(V) \tag{4.5}$$

be the *main involution*, acting as multiplication by -1 on $C^{\mathrm{odd}}(V)$ and as the identity on $C^{\mathrm{even}}(V)$. The *conjugation* $x \mapsto x^*$ is the composition of τ and α. Set[3]

$$\begin{aligned}
\mathrm{Spin}(V) &:= \{x \in C(V)^{\mathrm{even}} : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\},\\
\mathrm{Pin}(V) &:= \{x \in C(V) : x \cdot x^* = 1 \text{ and } x \cdot V \cdot x^* \subset V\},\\
G_0(V) &:= \{x \in C(V) : x \cdot \tau(x) = 1 \text{ and } x \cdot V \cdot \tau(x) \subset V\},\\
G(V) &:= \{x \in C(V)^\times \ : \ x \cdot V \cdot x^{-1} \subset V\}, && (4.6)\\
G(V)^{\mathrm{even}} &:= G(V) \cap C(V)^{\mathrm{even}}. && (4.7)
\end{aligned}$$

The condition $x \cdot V \cdot x^{-1} \subset V$ in the definition of $G(V)$, combined with the fact that V has even rank, implies that x is either even or odd and $G(V)$ is contained in the union $C(V)^{\mathrm{even}} \cup C(V)^{\mathrm{odd}}$ [6, II.3.2]. Note that $x \in C(V)$ is invertible if and only if $x \cdot x^*$ is, i.e., if and only if $x \cdot x^* = \pm 1$. Thus, $\mathrm{Pin}(V)$ is an index 2 subgroup of $G(V)$. We get the extension

$$1 \to \mathrm{Spin}(V) \to G(V)^{\mathrm{even}} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Given a vector $v \in V$, we have $v^* = -v$ and $v \cdot v^* = -v \cdot v = -Q(v)$. If v_1, v_{-1} are vectors in V with $Q(v_1) = 1$ and $Q(v_{-1}) = -1$, then v_{-1} belongs to $\mathrm{Pin}(V)$, v_1 belongs to $G(V)$, and $(v_1 \cdot v_{-1})$ belongs to $G(V)^{\mathrm{even}}$.

The standard representation V is defined[4] by the homomorphism

$$\rho : G(V) \to O(V), \quad \rho(x)(v) = x \cdot v \cdot x^{-1}. \tag{4.8}$$

If $Q(v) = \pm 1$, then $-\rho(v)$ is reflection in v,

$$-\rho(v)(w) = w - \frac{2(v, w)_V}{(v, v)_V} \cdot v, \quad \forall w \in V. \tag{4.9}$$

[3]In reference [6] these groups are called *Clifford groups* of the orthogonal group of V with respect to the quadratic form Q. The group $G(V)$ is denoted in [6] by Γ, the group $G_0(V)$ by Γ_0, the group $G(V)^{\mathrm{even}}$ by Γ^+, and the group $\mathrm{Spin}(V)$ by Γ_0^+. The main anti-automorphism τ is denoted in [6, II.3.5] by α.

[4]Chevalley denotes the standard representation by $\chi : G(V) \to O(V)$ [6, Sec. 2.3]. The convention in [7, Sec. 2.7] is different and there $\rho(x)(v) = \alpha(x) \cdot v \cdot x^{-1}$. The two representations agree on $G(V)^{\mathrm{even}}$.

Let

$$m : C(V) \to \mathrm{End}(S) \tag{4.10}$$

be the Clifford algebra representation and denote by

$$m : G(V) \to \mathrm{GL}(S) \tag{4.11}$$

as well its restriction to the subset $G(V) \subset C(V)$.

The character group $\mathrm{Hom}(O(V), \mathbb{C})$ of $O(V)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and is generated by the determinant character det and the *orientation character*

$$\mathrm{ort} : O(V) \to \{\pm 1\}. \tag{4.12}$$

The latter is defined as follows. The positive cone in $V \otimes_{\mathbb{Z}} \mathbb{R}$ is homotopic to the 3-sphere. The character ort represents the action of isometries on the third singular cohomology of the positive cone [28, Sec. 4]. Denote by $O_+(V)$ the kernel of ort.

Lemma 4.1. *The homomorphism ρ is surjective and it maps* $\mathrm{Pin}(V)$ *onto* $O_+(V)$ *and* $\mathrm{Spin}(V)$ *onto* $SO_+(V)$.

Proof. The lattice V is the orthogonal direct sum of four copies of the even unimodular rank 2 hyperbolic lattice U. Hence, $O(V)$ is generated by the reflections $-\rho(v)$, given in (4.9), in elements $v \in V$ with $(v, v)_V = \pm 2$, by [54, Sec. 4.3]. The element $-\mathrm{id}_U \in O(U)$ is a product of two reflections. Hence, $-\mathrm{id}_V$ is a product of eight reflections and so $O(V)$ is generated by the set $\{\rho(v) : v \in V,\ (v, v)_V = \pm 2\}$. ∎

The lemma implies that the homomorphism ρ pulls back the character ort to the quotient homomorphism

$$\mathrm{ort} : G(V) \to G(V)/\mathrm{Pin}(V) \cong \{\pm 1\}, \quad x \mapsto x \cdot x^* \in \{\pm 1\}. \tag{4.13}$$

The determinant character of $O(V)$ is pulled back via ρ to the *parity character*

$$p : G(V) \to G(V)/G(V)^{\mathrm{even}} \cong \{\pm 1\}.$$

The product of the parity and orientation characters is the *norm character* denoted[5]

$$N : G(V) \to \{\pm 1\}, \quad x \mapsto x \cdot \tau(x). \tag{4.14}$$

The subgroup $G_0(V)$ of $G(V)$ is the kernel of N.

$H^*(X, \mathbb{Z})$ is a subalgebra of $C(V)$, invariant under both τ and α, and τ acts on $H^i(X, \mathbb{Z})$ by $(-1)^{i(i-1)/2}$. There is a natural symmetric pairing[6] on $H^*(X, \mathbb{Z})$ defined by

$$(s, t)_S := \int_X \tau(s) \cup t. \tag{4.15}$$

[5] In Chevalley's book the norm character is denoted by λ and in Deligne's notes by N [6, 7].

[6] This pairing is denoted by $\beta(s, t)$ in [6, Sec. 3.2]. Another natural symmetric pairing $\widetilde{\beta}(s, t) := \int_X s^* \cup t$ is considered in [6, III.2.5]. The isomorphism $S \to S^*$ induced by $\widetilde{\beta}$ is the one induced by the Fourier–Mukai functor in (7.1) below. We chose to use the former pairing β, since it is the pairing used in the Triality Chapter IV of [6].

We set $Q_S(s) := \frac{1}{2}(s,s)_S$ for $s \in S := H^*(X,\mathbb{Z})$. Given $x \in G(V)$, we have the equality

$$(x \cdot s, x \cdot t)_S = N(x)(s,t)_S,$$

where N is the norm character (4.14). In particular, an element $v \in V$ with $Q(v) = \pm 1$ satisfies

$$(v \cdot s, v \cdot t)_S = Q(v)(s,t)_S. \tag{4.16}$$

The latter equation holds for every $v \in V$ [6, IV.2.3].

Let $(\bullet,\bullet)_{S^+}$ and $(\bullet,\bullet)_{S^-}$ be the restrictions of the pairing (4.15) to S^+ and S^-. Both are even symmetric unimodular bilinear pairings, which are $\mathrm{Spin}(V)$-invariant. Note that $-(\bullet,\bullet)_{S^+}$ is the pairing given in (3.1), which is known as the *Mukai pairing* [38, 56]. The pairing on V is $G(V)$-invariant. The pairings on S^+ and S^- are *not* invariant under $G(V)^{\mathrm{even}}$. Rather, the pairing spans a rank 1 module in $\mathrm{Sym}^2(S^+)$ corresponding to the character (4.13). Let $\widetilde{O}(S^+)$ be the subgroup of $\mathrm{GL}(S^+)$ preserving the bilinear pairing $(\bullet,\bullet)_{S^+}$ only up to sign. Define $\widetilde{O}(S^-)$ similarly. Let $\widetilde{\alpha} \in \mathrm{GL}(S)$ be the element acting as the identity on S^+ and by -1 on S^-. It follows from the Triality Principle recalled below that $\widetilde{\alpha}$ corresponds to the action of $j(-\mathrm{id})$, where j is an order 3 outer automorphism of $\mathrm{Spin}(V)$, and so $\widetilde{\alpha}$ is the image of a unique element of $\mathrm{Spin}(V)$ in $\mathrm{GL}(S)$ (Theorem 4.6). Denote by

$$\widetilde{\alpha} \in \mathrm{Spin}(V) \tag{4.17}$$

the corresponding central element. Set $S\widetilde{O}(S^+) := \widetilde{O}(S^+) \cap \mathrm{SL}(S^+)$ and $S\widetilde{O}(S^-) := \widetilde{O}(S^-) \cap \mathrm{SL}(S^-)$. $G(V)^{\mathrm{even}}$ maps to $S\widetilde{O}(S^+)$ with kernel generated by $\widetilde{\alpha}$, to $S\widetilde{O}(S^-)$ with kernel generated by $-\widetilde{\alpha}$, and onto $SO(V)$ with kernel generated by -1. The image of $G(V)^{\mathrm{even}}$ has index 2 in $S\widetilde{O}(S^+)$.

4.2. *Bilinear operations via the Clifford product*

The Clifford product defines the homomorphisms $V \otimes S^+ \to S^-$, $V \otimes S^- \to S^+$, and

$$S^+ \otimes S^- \to V. \tag{4.18}$$

The latter is the composition of $V^* \to V$, given by the bilinear pairing $(\bullet,\bullet)_V$, and the homomorphism $S^+ \otimes S^- \to V^*$, sending $s \otimes t$ to $((\bullet)\cdot s, t)_{S^-} \in V^*$.

Example 4.2. Let us calculate the homomorphisms

$$m_{s_n} : V \to S^- \tag{4.19}$$

corresponding to Clifford multiplication $(\bullet)\cdot s_n$ with the Chern character $s_n = (1,0,-n) \in S^+$ of the ideal sheaf of a length n subscheme. Let $v = (w,\theta) \in V$, where $w \in H^1(X,\mathbb{Z})$ and $\theta \in H^1(X,\mathbb{Z})^*$. Then

$$\begin{aligned} m_{s_n}(v) := v \cdot (1,0,-n) &= w \wedge (1,0,-n) + D_\theta(1,0,-n) = w - nD_\theta([\mathrm{pt}]) \\ &= w + n\,\mathrm{PD}^{-1}(\theta). \end{aligned}$$

In the last equality we used the isomorphism

$$\mathrm{PD} : H^i(X,\mathbb{Z}) \to H^{4-i}(X,\mathbb{Z})^* \tag{4.20}$$

given by $\beta \mapsto \int_X \beta \wedge (\bullet)$. The equality

$$[\mathrm{PD}(D_\theta([\mathrm{pt}]))](x) := \int_X D_\theta([\mathrm{pt}]) \wedge x = -\theta(x), \quad \forall x \in H^1(X,\mathbb{Z}), \tag{4.21}$$

follows from

$$0 = D_\theta([\mathrm{pt}] \wedge x) = D_\theta([\mathrm{pt}]) \wedge x + \theta(x)[\mathrm{pt}].$$

Note that the homomorphism (4.19) is equivariant with respect to the action of the subgroup $G(V)^{\mathrm{even}}_{s_n}$ of $G(V)^{\mathrm{even}}$ stabilizing s_n. The integral representations V and S^- restrict to $G(V)^{\mathrm{even}}_{s_n}$ as isogenous representations, which are not isomorphic. The cokernel of m_{s_n} is $H^3(X,\mathbb{Z}/n\mathbb{Z})$ and is naturally identified with $H_1(X,\mathbb{Z}/n\mathbb{Z})$ as well as with the group Γ_X of points of order n on X.

The homomorphism (4.18) restricts to $s_n \otimes S^-$ as a homomorphism

$$m^\dagger_{s_n} : S^- \to V, \tag{4.22}$$

which is the adjoint of (4.19) with respect to the pairings $(\bullet,\bullet)_V$ and $(\bullet,\bullet)_{S^-}$. Explicitly, given $(w,\beta) \in S^-$ with $w \in H^1(X,\mathbb{Z})$ and $\beta \in H^3(X,\mathbb{Z})$,

$$m^\dagger_{s_n}(w,\beta) = (-nw, -\mathrm{PD}(\beta)). \tag{4.23}$$

In particular, $m^\dagger_{s_n} \circ m_{s_n} : V \to V$ and $m_{s_n} \circ m^\dagger_{s_n} : S^- \to S^-$ are both multiplication by $Q_{S^+}(s_n) = -n$.

Remark 4.3. Let $m_{s_n} : V \to S^-$ be the homomorphism given in (4.19). Given a primitive element $w \in S^+$ with $(w,w)_{S^+} = -2n$, there exists an element $g \in \mathrm{Spin}(V)$ satisfying $g(s_n) = w$. Given $x \in V$, and regarding x and g as elements of $C(V)$, we have the equalities

$$g(m_{s_n}(x)) = g \cdot x \cdot s_n = (g \cdot x \cdot g^{-1}) \cdot (g \cdot s_n) = m_w(\rho(g)(x)).$$

Let $m^\dagger_{s_n} : S^- \to V$ be the adjoint of m_{s_n} and define $m^\dagger_w$ similarly. We get the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{m_{s_n}} & S^- & \xrightarrow{m^\dagger_{s_n}} & V \\ \downarrow{\scriptstyle \rho(g)} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \rho(g)} \\ V & \xrightarrow{m_w} & S^- & \xrightarrow{m^\dagger_w} & V \end{array}$$

In particular, the cokernels of m_{s_n} and m_w are isomorphic and both $m^\dagger_w \circ m_w$ and $m_w \circ m^\dagger_w$ are multiplication by $-n$.

Let $m_w^{-1} : S^-_{\mathbb{Q}} \to V_{\mathbb{Q}}$ be the inverse of $m_w : V_{\mathbb{Q}} \to S^-_{\mathbb{Q}}$. The quotient $m_w^{-1}(S^-)/V$ is then a subgroup Γ_w of the compact torus $V_{\mathbb{R}}/V$ isomorphic to $(\mathbb{Z}/n\mathbb{Z})^4$ canonically associated to w. The subgroup Γ_w is the kernel of the homomorphism $V_{\mathbb{R}}/V \to S^-_{\mathbb{R}}/S^-$ induced by m_w.

Example 4.4 (An element of $G(S^+)^{\text{even}}_{s_n}$ which does not belong to $\text{Spin}(V)_{s_n}$). Consider the two vectors $s_1 = (1, 0, -1)$ and $s = (1, 0, 1)$ in S^+. Then $Q_S(s_1) = -1$, $Q_S(s) = 1$, and $(s_1, s)_{S^+} = 0$. The composition $m_s^\dagger \circ m_{s_1} : V \to V$ is given by

$$m_s^\dagger \circ m_{s_1}(w, \theta) = m_s^\dagger(w, \text{PD}^{-1}(\theta)) = (w, -\theta).$$

Consequently, $m_{s_1}^\dagger \circ m_s = (m_s^\dagger \circ m_{s_1})^\dagger = -(m_s^\dagger \circ m_{s_1})^{-1} = -(m_s^\dagger \circ m_{s_1})$. This agrees with the relation $s_1 \cdot s + s \cdot s_1 = (s_1, s)_{S^+} = 0$ in the Clifford algebra $C(S^+)$ (see also (4.29) below for a more conceptual interpretation). Similarly, $m_{s_1} \circ m_s^\dagger : S^- \to S^-$ is given by

$$m_{s_1} \circ m_s^\dagger(w, \beta) = (w, -\beta).$$

The reflections $R_{s_1}(r, H, t) = (t, H, r)$ and $R_s(r, H, t) = (-t, H, -r)$ of S^+ commute, and $R_{s_1} \circ R_s(r, H, t) = (-r, H, -t)$. Let $\widetilde{m}_{s_1 \cdot s}$ be the element of $\text{GL}(V) \times \text{GL}(S^+) \times \text{GL}(S^-)$ acting on V via $m_{s_1}^\dagger \circ m_s$, on S^- via $m_{s_1} \circ m_s^\dagger$, and on S^+ via $R_{s_1} \circ R_s$. Let $G(S^+)^{\text{even}}$ be the subgroup of $\text{GL}(V) \times \text{GL}(S^+) \times \text{GL}(S^-)$ generated by $\text{Spin}(V)$ and $\widetilde{m}_{s_1 \cdot s}$. The notation $G(S^+)^{\text{even}}$ is motivated by triality (see (4.30) below). Note that $R_{s_1} \circ R_s(s_n) = -s_n$. Hence, both of the products $-\widetilde{m}_{s_1 \cdot s}$ and $-\widetilde{\alpha}\widetilde{m}_{s_1 \cdot s}$, of $\widetilde{m}_{s_1 \cdot s}$ with -1 or $-\widetilde{\alpha}$ in $\text{Spin}(V)$, belong to the stabilizer $G(S^+)^{\text{even}}_{s_n}$ of $s_n \in S^+$. Set

$$\widetilde{\tau} := -\widetilde{\alpha} \cdot \widetilde{m}_{s_1 \cdot s} \in G(S^+)^{\text{even}}_{s_n}. \tag{4.24}$$

Note that $\widetilde{\tau}$ acts on S^+ and S^- via the restriction of the main anti-involution τ of $C(V)$ to its subalgebra $H^*(X, \mathbb{Z})$. However, the action of $\widetilde{\tau}$ on V is not the identity, and so it does not agree with the restriction of the action of τ to the embedding of V in $C(V)$.

Lemma 4.5. *The image of the homomorphism*

$$G(V)^{\text{even}}_{s_n} \xrightarrow{\rho} SO(V) \to SO(V/nV)$$

is equal to the subgroup of $SO(V/nV)$ leaving invariant the summand $H^1(X, \mathbb{Z}/n\mathbb{Z})^*$ in the direct sum decomposition

$$V/nV := H^1(X, \mathbb{Z}/n\mathbb{Z}) \oplus H^1(X, \mathbb{Z}/n\mathbb{Z})^*.$$

Consequently, we get the homomorphism

$$\bar{\rho} : G(V)^{\text{even}}_{s_n} \to \text{GL}[H^1(X, \mathbb{Z}/n\mathbb{Z})]. \tag{4.25}$$

Proof. Reduce modulo n the homomorphisms m_{s_n} given in (4.19). The kernel in V/nV is $H^1(X, \mathbb{Z}/n\mathbb{Z})^*$ and the kernel is $G(V)^{\text{even}}_{s_n}$-invariant. ∎

4.3. Triality

Set $A_X := V \oplus S^- \oplus S^+$. We endow A_X with the bilinear pairing induced by that of each of the summands, so that A_X is an orthogonal direct sum of the three lattices. Define the commutative, but non-associative, algebra structure on A_X, using the Clifford multiplication and the homomorphism (4.18). The product of two elements in the same summand vanishes [6, IV.2.2]. Any automorphism σ of the algebra A_X which leaves V and the sum $S^- \oplus S^+$ invariant belongs to the image in $\mathrm{GL}(A_X)$ of the kernel $G_0(V) \subset G(V)$ of the norm character (4.14) via the homomorphism

$$\widetilde{\mu} : G(V) \to O(V) \times \mathrm{GL}(S), \tag{4.26}$$

given by $\widetilde{\mu}(g) = (\rho(g), m(g))$, where ρ is given in (4.8) and m in (4.11) (see [6, IV.2.4]). Restriction of $\widetilde{\mu}$ to $G_0(V)$ yields the faithful representation

$$\widetilde{\mu} : G_0(V) \to \mathrm{Aut}(A_X). \tag{4.27}$$

If, in addition, each of the three summands is σ-invariant, then σ belongs to the image of $\mathrm{Spin}(V)$. This leads to a symmetric definition of $\mathrm{Spin}(V)$ in terms of the algebra A_X. Given an element a of A_X, denote by m_a the linear homomorphism of A_X acting via multiplication by a. Note, in particular, that the composition $m_{s_1} \circ m_{s_2}$ of two multiplications by elements s_i of S^+,

$$(m_{s_1} \circ m_{s_2})(a) := s_1 \cdot (s_2 \cdot a), \quad a \in A_X,$$

acts on the subspace $V \oplus S^-$ via an element $m_{s_1 \cdot s_2}$ of $\mathrm{Spin}(V)$, provided $(s_1, s_1)_{S^+} = (s_2, s_2)_{S^+} = 2$ or $(s_1, s_1)_{S^+} = (s_2, s_2)_{S^+} = -2$. The sublattice S^+ belongs to the kernel of $m_{s_1} \circ m_{s_2}$, but $\widetilde{\mu}(m_{s_1 \cdot s_2})$ leaves S^+ invariant and acts on S^+ as an isometry.

Following is the *Triality Principle*, adapted from [6, Th. IV.3.1 and Sec. 4.5].

Theorem 4.6. *There exists an automorphism J of order 3 of the algebra A_X, preserving its bilinear pairing, with the following properties.*

(1) $J(V) = S^+$, $J(S^+) = S^-$, *and* $J(S^-) = V$.

(2) $J^{-1}\widetilde{\mu}(\mathrm{Spin}(V))J = \widetilde{\mu}(\mathrm{Spin}(V))$, *where $\widetilde{\mu}$ is the representation* (4.27). *Consequently, there exists a unique outer automorphism j of $\mathrm{Spin}(V)$ of order 3 satisfying*

$$\widetilde{\mu}(j(g)) = J^{-1}\widetilde{\mu}(g)J.$$

Proof. The proof consists in checking that the explicit construction of the automorphism J in [6, Th. IV.3.1] and of j in [6, Sec. 4.5], which is stated for vector spaces, carries through for our lattices. We outline the construction. Let u_1 be an element of S^+ satisfying $(u_1, u_1)_{S^+} = 2$. We could choose for example $u_1 = (1, 0, 1)$. Let t be the automorphism of the lattice A_X mapping V to S^- via the Clifford action of elements of $V \subset C(V)$ on $u_1 \in S^+ \subset S$, sending S^- to V via the product by u_1 in A_X, so via (4.18), and acting on S^+ by $-R_{u_1}$. Choose an element x_1 of V satisfying $Q(x_1) = 1$, so that $(x_1, x_1)_V = 2$. Regarding x_1 as an element of $C(V)$, via the embedding $V \subset C(V)$, we find that x_1 belongs to the subgroup $G_0(V)$ of $G(V)$. Set $J := \widetilde{\mu}(x_1)t$, the composition of the automorphisms $\widetilde{\mu}(x_1)$ and t of the lattice A_X. It is an isometry and an algebra automorphism of $A_X \otimes_{\mathbb{Z}} \mathbb{Q}$, by [6, Th. IV.3.1] (see also [6, paragraph preceding IV.2.4]), and so of A_X as well. The properties in part (1) of the theorem follow by construction. The identity $J^3 = \mathrm{id}$ holds for the vector space $A_X \otimes_{\mathbb{Z}} \mathbb{Q}$, by [6, proof of Th. IV.3.1], and so must hold also for the J-invariant lattice A_X. The invariance of $\mathrm{Spin}(V)$ in part (2) of the theorem follows from that of $\mathrm{Spin}(V_{\mathbb{Q}})$, by [6, proof in Sec. 4.5], and the existence of j follows, since the representation $\tilde{\mu}$ is faithful. ∎

As a corollary of the Triality Principle we get the following identification of $V \oplus S^-$ with the Clifford module $\bigwedge^* S^+$ of the integral Clifford algebra $C(S^+)$ associated to the decomposition $S^+ = J(H^1(X,\mathbb{Z})) \oplus J(H^1(X,\mathbb{Z})^*)$ as a direct sum of maximal isotropic sublattices. Let $\tilde{J} : C(V) \to C(S^+)$ be the isomorphism extending the isomorphism $J_{|V} : V \to S^+$ induced by the isometry J. Let $\mathrm{Ad}_J : \mathrm{End}(S^+ \oplus S^-) \to \mathrm{End}(S^- \oplus V)$ be the isomorphism sending f to $J \circ f \circ J^{-1}$. Let $m : C(V) \to \mathrm{End}(S^+ \oplus S^-)$ be the algebra homomorphism given in (4.10). Given $x \in C(V)$, let $m_x \in \mathrm{End}(S^+ \oplus S^-)$ be its image under m.

Corollary 4.7. *There exists a unique injective algebra homomorphism*

$$m : C(S^+) \to \mathrm{End}(S^- \oplus V)$$

which restricts to the embedding of S^+ in $C(S^+)$ as multiplication in A_X by elements of S^+. The following diagram is commutative:

$$\begin{array}{ccccc} V & \xrightarrow{\subset} & C(V) & \xrightarrow{m} & \mathrm{End}(S^+ \oplus S^-) \\ \downarrow{\scriptstyle J} & & \downarrow{\scriptstyle \tilde{J}} & & \downarrow{\scriptstyle \mathrm{Ad}_J} \\ S^+ & \xrightarrow{\subset} & C(S^+) & \xrightarrow{m} & \mathrm{End}(S^- \oplus V) \end{array}$$

In particular, for $x \in V$ and $y_1, y_2 \in S^+$, the following equalities hold in $\mathrm{End}(S^- \oplus V)$:

$$m_{J(x)} = J \circ m_x \circ J^{-1}, \tag{4.28}$$

$$m_{y_1} \circ m_{y_2} + m_{y_2} \circ m_{y_1} = (y_1, y_2)_{S^+} \cdot \mathrm{id}_{S^- \oplus V}. \tag{4.29}$$

Proof. The left square is commutative, by definition of $\tilde{J}$. The right lower arrow m is defined by the requirement that the right square is commutative. It is an algebra homomorphism, since the upper arrow m is, and it restricts to S^+ as multiplication in A_X, due to J being an algebra automorphism of A_X. Equality (4.28) follows from the commutativity of the diagram, and (4.29) comes from

$$\begin{aligned} m_{y_1} \circ m_{y_2} &\overset{(4.28)}{=} (J \circ m_{J^{-1}(y_1)} \circ J^{-1}) \circ (J \circ m_{J^{-1}(y_2)} \circ J^{-1}) \\ &= J \circ m_{J^{-1}(y_1)} \circ m_{J^{-1}(y_2)} \circ J^{-1}. \end{aligned}$$

So

$$\begin{aligned} m_{y_1} \circ m_{y_2} + m_{y_2} \circ m_{y_1} &= J \circ (m_{J^{-1}(y_1)} \circ m_{J^{-1}(y_2)} + m_{J^{-1}(y_2)} \circ m_{J^{-1}(y_1)}) \circ J^{-1} \\ &= J \circ (J^{-1}(y_1), J^{-1}(y_2))_V \cdot \mathrm{id}_S \circ J^{-1} \\ &= (y_1, y_2)_{S^+} \cdot \mathrm{id}_{S^- \oplus V}, \end{aligned}$$

where the second equality follows from the $C(V)$-module structure of S and the defining relation (4.2) of $C(V)$, and the last is due to J being an isometry. ∎

Set $G(S^+) := \tilde{J} G(V) \tilde{J}^{-1} \subset C(S^+)$ and define its subgroup $G(S^+)^{\text{even}}$ similarly,

$$G(S^+)^{\text{even}} := \tilde{J} G(V)^{\text{even}} \tilde{J}^{-1}, \quad \text{Spin}(S^+) := \tilde{J}[\text{Spin}(V)]\tilde{J}^{-1}. \tag{4.30}$$

Let $\tilde{m} : G(S^+) \to \text{GL}(A_X)$ be the unique homomorphism making the following diagram commutative:

$$\begin{array}{ccc} G(V) & \xrightarrow{\tilde{\mu}} & O(V) \times \text{GL}(S^+ \oplus S^-) \\ {\scriptstyle \text{Ad}_{\tilde{J}}}\downarrow & & \downarrow{\scriptstyle \text{Ad}_J} \\ G(S^+) & \xrightarrow{\tilde{m}} & O(S^+) \times \text{GL}(S^- \oplus V) \end{array} \tag{4.31}$$

Note the equality

$$\tilde{m}(\text{Spin}(S^+)) = \tilde{\mu}(\text{Spin}(V)) \tag{4.32}$$

as subgroups of $\text{Aut}(A_X)$, by Theorem 4.6 (2). However, $\tilde{m}(-1)$ and $\tilde{\mu}(-1)$ are distinct elements of the center. The element $\tilde{\mu}(-1)$ acts as the identity on V and as $-\text{id}_S$ on $S = S^+ \oplus S^-$. The element $\tilde{m}(-1)$ acts as the identity on S^+ and via scalar multiplication by -1 on $V \oplus S^-$.

Let

$$\rho : G(S^+) \to O(S^+)$$

be the composition of $\tilde{m}$ with the restriction homomorphism to the $G(S^+)$-invariant direct summand S^+ of A_X. The representation ρ of $G(S^+)$ is analogous to that of $G(V)$ given in (4.8). Explicitly, $G(S^+)^{\text{even}}$ is generated by $\tilde{\mu}(\text{Spin}(V))$ and one additional automorphism $\tilde{m}_{s_1 \cdot s_{-1}}$, acting on $V \oplus S^-$ via $m_{s_1} \circ m_{s_{-1}}$, which is the composition of two multiplications

$$(m_{s_1} \circ m_{s_{-1}})(a) := s_1 \cdot (s_{-1} \cdot a), \quad a \in A_X,$$

where s_i, $i = 1, -1$, are two elements of S^+ and $(s_i, s_i)_{S^+} = 2i$. One can take, for instance, $s_1 = (1, 0, 1)$ and $s_{-1} = (1, 0, -1)$ as in Example 4.4. The action of the element $\tilde{m}_{s_1 \cdot s_{-1}}$ on S^+ is the composition $R_{s_1} \circ R_{s_{-1}}$ of the two reflections. Then $G(S^+)^{\text{even}}$ preserves $(\bullet, \bullet)_{S^+}$ and maps into $SO(S^+)$, but its image in $\text{SL}(V)$ is contained in $S\tilde{O}(V)$. In particular, $G(S^+)^{\text{even}}$ is not contained in the image of $G(V)$ in $\text{GL}(A_X)$ via $\rho \times m$.

Note that if $s \in G(S^+)^{\text{even}}$ does not belong to $\text{Spin}(S^+)$, then $\rho(s)$ acts on S^+ as an isometry, V and S^- are s-invariant, but the action of s on $V \oplus S^-$ reverses the sign of the pairing, so its restriction belongs to $\tilde{O}(V \oplus S^-)$ but not to $O(V \oplus S^-)$, in analogy to (4.16).

Consider the composite homomorphism

$$G(S^+)^{\text{even}}_{s_n} \to S\tilde{O}(V) \to S\tilde{O}(V/nV).$$

Its composition with $\text{Ad}_{\tilde{J}} : \text{Spin}(V) \to \text{Spin}(S^+)$ agrees with the restriction of the homomorphism $\bar{\rho}$, given in (4.25), to $\text{Spin}(V)$. The image of $G(S^+)^{\text{even}}_{s_n}$ in $S\tilde{O}(V/nV)$ leaves the subgroup $H^1(X, \mathbb{Z}/n\mathbb{Z})^*$ invariant. The quotient of (V/nV) by $H^1(X, \mathbb{Z}/n\mathbb{Z})^*$ is naturally isomorphic to $H^1(X, \mathbb{Z}/n\mathbb{Z})$. We obtain a homomorphism

$$G(S^+)^{\text{even}}_{s_n} \to \text{GL}[H^1(X, \mathbb{Z}/n\mathbb{Z})]. \tag{4.33}$$

5. Generators for the stabilizer $\mathrm{Spin}(S^+)_{s_n}$

Given a lattice Λ, let $\mathfrak{R}(\Lambda)$ be the group generated by reflections in $+2$ and -2 vectors in Λ. Set $S\mathfrak{R}(\Lambda) := \mathfrak{R}(\Lambda) \cap SO(\Lambda)$ and $S\mathfrak{R}_+(\Lambda) := \mathfrak{R}(\Lambda) \cap SO_+(\Lambda)$. The former is generated by elements which are products of an even number of reflections. $S\mathfrak{R}_+(\Lambda)$ is generated by elements which are either products of an even number of reflections in $+2$ vectors, or products of an even number of reflections in -2 vectors. Let U be the rank 2 even unimodular hyperbolic lattice. The lattice S^+ is isometric to $U^{\oplus 4}$. The orthogonal group is generated by reflections, $O(U^{\oplus k}) = \mathfrak{R}(U^{\oplus k})$, for $k \geq 3$ (see [54, Sec. 4.3]). Consequently, we get the following.

Corollary 5.1. *$SO_+(U^{\oplus k}) = S\mathfrak{R}_+(U^{\oplus k})$ for $k \geq 3$.*

Set $e_1 := (1, 0, 0) \in S^+$ and $e_2 := (0, 0, 1) \in S^+$. Let $\mathrm{Spin}(V)_{e_1,e_2}$ be the subgroup of $\mathrm{Spin}(V)$ stabilizing both e_1 and e_2.

Lemma 5.2. *$\mathrm{Spin}(V)_{e_1,e_2}$ leaves invariant each of the subspaces $H^1(X, \mathbb{Z})$ and $H^1(\widehat{X}, \mathbb{Z})$ of V, its action on $H^1(X, \mathbb{Z})$ factors through an isomorphism*

$$f : \mathrm{Spin}(V)_{e_1,e_2} \to \mathrm{SL}(H^1(X, \mathbb{Z})),$$

and its action on $H^1(\widehat{X}, \mathbb{Z})$ factors through an isomorphism with $\mathrm{SL}(H^1(\widehat{X}, \mathbb{Z}))$.

Proof. Consider V as a subspace of $C(V)$ as in Section 4. Then $H^1(X, \mathbb{Z})$ is the intersection of V with the left ideal of $C(V)$ annihilating e_2 and $H^1(\widehat{X}, \mathbb{Z})$ is the intersection of V with the left ideal annihilating e_1. In the language of [6, Sec. 3.1] the elements e_i, $i = 1, 2$, are *pure spinors* and each of the two maximal isotropic sublattices of V is the one associated to the pure spinor. If g belongs to $\mathrm{Spin}(V)_{e_1,e_2}$ and v to $H^1(X, \mathbb{Z})$, then

$$\rho(g)(v) \cdot e_i = (g \cdot v \cdot g^{-1}) \cdot e_i = (g \cdot v \cdot g^{-1}) \cdot (g \cdot e_i) = g \cdot (v \cdot e_i),$$

where $\cdot$ denotes multiplication in $C(V)$ as well as the module action of $C(V)$ on S, and the second equality follows since g stabilizes e_i. Hence, $\rho(g)(v) \cdot e_i = 0$, if and only if $v \cdot e_i = 0$. Thus, $\mathrm{Spin}(V)_{e_1,e_2}$ leaves each of $H^1(X, \mathbb{Z})$ and $H^1(\widehat{X}, \mathbb{Z})$ invariant. Furthermore, $\widetilde{\mu}(\mathrm{Spin}(V)_{e_1,e_2}) = \widetilde{m}(\mathrm{Spin}(S^+)_{e_1,e_2})$, by (4.32), and the latter is $\mathrm{Spin}(H^2(X, \mathbb{Z}))$, since $H^2(X, \mathbb{Z})$ is the subspace of S^+ orthogonal to $\{e_1, e_2\}$. $\mathrm{Spin}(H^2(X, \mathbb{Q}))$ acts on each of its half-spin representations $H^1(X, \mathbb{Q})$ and $H^1(\widehat{X}, \mathbb{Q})$ via an injective homomorphism into $\mathrm{SL}(H^1(X, \mathbb{Q}))$ and $\mathrm{SL}(H^1(\widehat{X}, \mathbb{Q}))$, by [6, III.7.2]. Hence, the homomorphism f is well defined and injective.

Conversely, $\mathrm{SL}(H^1(X, \mathbb{Z}))$ acts via isometries on $\bigwedge^2 H^1(X, \mathbb{Z})$ and we get the commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}(V)_{e_1,e_2} & \xrightarrow{\quad f \quad} & \mathrm{SL}(H^1(X, \mathbb{Z})) \\ & \searrow \quad \swarrow & \\ & SO_+(\bigwedge^2 H^1(X, \mathbb{Z})) & \end{array}$$

$SO_+(\bigwedge^2 H^1(X,\mathbb{Z}))$ is generated by products of two reflections in classes $c \in H^2(X,\mathbb{Z})$ with $(c,c)_{S^+} = 2$ and products of two reflections in classes $c \in H^2(X,\mathbb{Z})$ with $(c,c)_{S^+} = -2$, by Corollary 5.1. The left slanted arrow is surjective, since these generators are in its image. The kernel of the right slanted arrow is $-\mathrm{id}$, which is in the image of f, since $\widetilde{m}(-1)$, given in (4.31), acts as the identity on S^+ and maps to $-\mathrm{id}_V$. Hence, f is surjective as well. The verification of the statement for $H^1(\widehat{X},\mathbb{Z})$ is identical. ■

Let $s_n := (1,0,-n)$, $n \geq 3$.

Lemma 5.3. (1) *The stabilizer $SO(S^+)_{s_n}$ is equal to the subgroup $S\mathfrak{R}(s_n^\perp)$ of the reflection group of the orthogonal complement $s_n^\perp \subset S^+$ of s_n.*

(2) *$\mathfrak{R}(s_n^\perp)$ is generated by $O(H^2(X,\mathbb{Z}))$ and reflections in vectors of the form $(1,A,n)$, with $A \in H^2(X,\mathbb{Z})$ a primitive class satisfying $\int_X A \cup A = 2n-2$.*

(3) *$S\mathfrak{R}(s_n^\perp)$ is generated by $SO(H^2(X,\mathbb{Z}))$ and products $R_{t_1}R_{t_2}$ of reflections, where $t_i = (1,A_i,n) \in s_n^\perp$ are vectors of the above form.*

Proof. (1, 2) It suffices to prove that $O(S^+)_{s_n}$ is generated by $O(H^2(X,\mathbb{Z})) = \mathfrak{R}(H^2(X,\mathbb{Z}))$ and $+2$ vectors of the form $(1,A,n)$, with A primitive. The argument proving Lemma 7.4 in [26] applies to show that the stabilizer $O(S^+)_{s_n}$ is generated by $O(H^2(X,\mathbb{Z}))$ and reflections in $+2$ elements $(r,A,rn) \in s_n^\perp$, with $A \in H^2(X,\mathbb{Z})$ primitive. If $r=0$, then $R_{(r,A,rn)}$ belongs to $\mathfrak{R}(H^2(X,\mathbb{Z}))$. If $r \neq 0$, then the reflection $R_{(r,A,rn)}$ is a composition of reflections R_{t_i}, with $t_i = (1,A_i,n)$ a $+2$ vector in $s_n^\perp$, and A_i primitive, by the argument proving [26, Lemma 7.7].

(3) Let $t = (0,A,0)$ and $u = (1,B,n)$, with $\int_X A \cup A = \pm 2$ and $\int_X B \cup B = 2n-2$. The equality $R_u R_t = R_t R_t R_u R_t = R_t R_{R_t(u)}$ implies that every element of $S\mathfrak{R}(s_n^\perp)$ can be written as a product

$$R_{t_1} \cdots R_{t_k} R_{u_1} \cdots R_{u_\ell}, \tag{5.1}$$

with $k+\ell$ even, t_i are ± 2 vectors in $H^2(X,\mathbb{Z})$, and $u_j = (1,B_j,n)$ are $+2$ vectors in $s_n^\perp$, with B_j primitive.

It remains to prove that we can choose ℓ to be even as well. The proof is similar to that of [26, Lemma 7.7]. Assume k and ℓ are odd. Choose $A \in H^2(X,\mathbb{Z})$ satisfying $(A,A)_{S^+} = 2$ and $(A,B_1)_{S^+} = -1$. Then $A + B_1 = R_A(B_1)$ is a primitive class in $H^2(X,\mathbb{Z})$. Set $t_{k+1} := (0,A,0)$. Then $(t_{k+1},u_1) = -1$ and $v := R_{u_1}(t_{k+1}) = t_{k+1} + u_1 = (1, A+B_1, n)$. Thus, the subgroup generated by the three reflections $\{R_{t_{k+1}}, R_{u_1}, R_v\}$ is the permutation group Sym_3, and

$$R_{u_1} = R_{t_{k+1}} R_{u_1} R_v.$$

Substitute the right hand side for R_{u_1} in (5.1) to replace ℓ by $\ell+1$. ■

Lemma 5.4. *The stabilizer $\mathrm{Spin}(S^+)_{s_n}$ is generated by $\mathrm{Spin}(S^+)_{e_1,e_2} \cong \mathrm{SL}(H^1(X,\mathbb{Z}))$ and products $t_1t_2 \in \mathrm{Spin}(S^+)_{s_n}$, where each $t_i = (1,A_i,n)$ is a $+2$ vector in $s_n^\perp$ and A_i is a primitive class in $H^2(X,\mathbb{Z})$.*

Proof. Let $SO_+(S^+)_{s_n}$ be the kernel of the restriction of the orientation character (4.12) to $SO(S^+)_{s_n}$. The homomorphism $\mathrm{Spin}(S^+)_{s_n} \to SO_+(S^+)_{s_n}$ is surjective, by Lemma 4.1, and its kernel is equal to the kernel[7] of $\mathrm{Spin}(S^+) \to SO_+(S^+)$ and is contained in $\mathrm{Spin}(S^+)_{e_1,e_2}$. The homomorphism $f : \mathrm{Spin}(S^+)_{e_1,e_2} \to SO_+(H^2(X,\mathbb{Z}))$ is surjective, by Lemma 5.2. Hence, it suffices to prove that $SO_+(S^+)_{s_n}$ is generated by $SO_+(H^2(X,\mathbb{Z}))$ and the products $R_{t_1}R_{t_2}$ of the reflections in the vectors t_i. Let g be an element of $SO_+(S^+)_{s_n}$. Then $g = R_{a_1}\cdots R_{a_k}R_{t_1}\cdots R_{t_\ell}$ with $a_i \in H^2(X,\mathbb{Z})$, $(a_1,a_2)_{S^+} = \pm 2$, t_i of the above form, and ℓ even, by Lemma 5.3 (3). Let $k = k_+ + k_-$, where k_- is the number of a_i with $(a_1,a_1)_{S^+} = -2$. Then k_- is even, since g belongs to $SO_+(S^+)_{s_n}$ and $(t_i,t_i) = 2$. Hence, k_+ is even as well and $R_{a_1}\cdots R_{a_k}$ belongs to $SO_+(H^2(X,\mathbb{Z}))$. ∎

6. Equivariance of the Chern character of a universal sheaf

Fix a non-negative integer n. Let $\mathcal{C}_n$ be the category whose objects are triples (X,H,w), where X is an abelian surface, $w = (r,\beta,s) \in H^{\mathrm{even}}(X,\mathbb{Z})$ is a primitive class with $r \geq 0$, $\beta \in H^{1,1}(X,\mathbb{Z})$, $(w,w)_{S^+} = -2n$, H is a w-generic polarization, and such that there exists a universal sheaf $\mathcal{E}$ over $X \times \mathcal{M}_H(w)$. A morphism in $\mathrm{Hom}_{\mathcal{C}_n}((X_1,w_1,H_1),(X_2,w_2,H_2))$ consists of a pair (g,ϵ), where ϵ is in $\mathbb{Z}/2\mathbb{Z}$ and $g : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ is a group homomorphism mapping $H^{\mathrm{even}}(X_1,\mathbb{Z})$ to $H^{\mathrm{even}}(X_2,\mathbb{Z})$ and $H^{\mathrm{odd}}(X_1,\mathbb{Z})$ to $H^{\mathrm{odd}}(X_2,\mathbb{Z})$ and satisfying the following condition. There exists a ring isomorphism

$$\gamma : H^*(\mathcal{M}_{H_1}(w_1),\mathbb{Z}) \to H^*(\mathcal{M}_{H_2}(w_2),\mathbb{Z})$$

such that for a choice of universal sheaves $\mathcal{E}_i$ over $X_i \times \mathcal{M}_{H_i}(w_i)$ and some class $c \in H^2(\mathcal{M}_{H_2}(w_2),\mathbb{Z})$ we have

$$\begin{aligned} (g\otimes\gamma)(\mathrm{ch}(\mathcal{E}_1)) &= \mathrm{ch}(\mathcal{E}_2)\exp(c) \quad \text{if } \epsilon = 0,\\ ((\tau g\tau)\otimes\gamma)(\mathrm{ch}(\mathcal{E}_1)) &= \mathrm{ch}(\mathcal{E}_2^\vee)\exp(c) \quad \text{if } \epsilon = 1, \end{aligned} \tag{6.1}$$

where $\tau : H^*(X_j,\mathbb{Z}) \to H^*(X_j,\mathbb{Z})$ acts on $H^i(X_j,\mathbb{Z})$ by $(-1)^{i(i-1)/2}$. In the displayed equation above $\mathcal{E}_2^\vee$ is the derived dual, and the product is by the pull-backs of $\exp(c)$ via the projection to $\mathcal{M}_{H_2}(w_2)$.

Composition of morphisms $(h,\epsilon) \in \mathrm{Hom}_{\mathcal{C}_n}((X_1,w_1,H_1),(X_2,w_2,H_2))$ and $(g,\delta) \in \mathrm{Hom}_{\mathcal{C}_n}((X_2,w_2,H_2),(X_3,w_3,H_3))$ is given by

$$(h,\epsilon)\circ(g,\delta) = (\tau^\delta h \tau^\delta g, \epsilon+\delta). \tag{6.2}$$

We prove in this section that $\mathcal{C}_n$ is indeed a category with the above composition rule, namely, the existence of a ring isomorphism $\gamma : H^*(\mathcal{M}_{H_1}(w_1),\mathbb{Z}) \to H^*(\mathcal{M}_{H_3}(w_3),\mathbb{Z})$

[7] It is the order 2 subgroup generated by $s\cdot s$, where $s \in S^+$ is a class with $Q_{S^+}(s) = -1$.

and a class $c \in H^2(\mathcal{M}_{H_3}(w_3), \mathbb{Z})$ needed for the right hand side of (6.2) to satisfy one of the two conditions displayed in (6.1) (Corollary 6.5 and Lemmas 6.7 and 6.8). Furthermore, γ in (6.1) is uniquely determined by (g, ϵ) via an explicit formula (6.3) (Lemma 6.4). Everything follows formally from the expression of the class of the diagonal in $H^*(\mathcal{M}_H(w) \times \mathcal{M}_H(w), \mathbb{Z})$ in terms of a universal sheaf. We do not discuss existence of morphisms in this section. In Section 9, $\mathrm{Hom}_{\mathcal{C}_n}((X_1, w_1, H_1), (X_2, w_2, H_2))$ is shown to be non-empty (Theorem 9.3), and in Corollary 9.4 an injective homomorphism[8] $G(S_X^+)_w^{\mathrm{even}} \to \mathrm{Aut}_{\mathcal{C}_n}((X, w, H))$ is constructed for $n \geq 3$. In the current section we treat the case where X is a $K3$ surface as well.

Given a smooth projective variety M, we denote by

$$\ell : \bigoplus_i H^{2i}(M, \mathbb{Q}) \to \bigoplus_i H^{2i}(M, \mathbb{Q}),$$
$$(r + a_1 + a_2 + \cdots) \mapsto 1 + a_1 + \left(\tfrac{1}{2}a_1^2 - a_2\right) + \cdots,$$

the universal polynomial map, which takes the exponential Chern character of a complex of sheaves to its total Chern class. We let

$$D_M : H^{\mathrm{even}}(M, \mathbb{Z}) \to H^{\mathrm{even}}(M, \mathbb{Z})$$

be the dualization automorphism acting by $(-1)^i$ on $H^{2i}(M, \mathbb{Z})$.

Let X_1 and X_2 be two abelian or $K3$ surfaces and let $w_i \in H^{\mathrm{even}}(X_i, \mathbb{Z})$ be two Mukai vectors. Assume that the moduli space $\mathcal{M}(w_i)$ of Gieseker–Simpson stable sheaves on X_i (with respect to a choice of polarizations when the rank is different from 1) is compact for $i = 1, 2$, and that $\dim(\mathcal{M}(w_1)) = \dim(\mathcal{M}(w_2))$. Set $m := \dim(\mathcal{M}(w_i))$, $i = 1, 2$. Denote by π_i the projection from $X_1 \times \mathcal{M}(w_1) \times X_2 \times \mathcal{M}(w_2)$ onto the i-th factor. Set

$$D := D_{X_1 \times \mathcal{M}(w_1) \times X_2 \times \mathcal{M}(w_2)}.$$

Given a class δ in $H^{\mathrm{even}}(X_1 \times X_2, \mathbb{Z})$, classes α_i in $H^{\mathrm{even}}(X_i \times \mathcal{M}(w_i), \mathbb{Q})$, and an element $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, we define a class $\gamma_{\delta,\epsilon}(\alpha_1, \alpha_2)$ in $H^{2m}(\mathcal{M}(w_1) \times \mathcal{M}(w_2), \mathbb{Q})$ by

$$\gamma_{\delta,\epsilon}(\alpha_1, \alpha_2) = c_m\left(\left\{\ell\left(\pi_{24_*}[D^{1-\epsilon}[\pi_{12}^*(\alpha_1) \cdot \pi_{13}^*(\delta)] \cdot \pi_{34}^*(\alpha_2) \cdot \pi_1^*(\sqrt{\mathrm{td}_{X_1}}) \cdot \pi_3^*(\sqrt{\mathrm{td}_{X_2}})]\right)\right\}^{-1}\right). \quad (6.3)$$

(The Todd classes td_{X_i} are equal to 1 for an abelian surface.) If $\mathcal{E}_i$ is a complex of sheaves on $X_i \times \mathcal{M}(w_i)$, we denote $\gamma_{\delta,\epsilon}(\mathrm{ch}(\mathcal{E}_1) \cdot \sqrt{\mathrm{td}_{X_1}}, \mathrm{ch}(\mathcal{E}_2) \cdot \sqrt{\mathrm{td}_{X_2}})$ also by $\gamma_{\delta,\epsilon}(\mathcal{E}_1, \mathcal{E}_2)$.

Let $\mathcal{M}_H(w)$ be a smooth and projective m-dimensional moduli space of H-stable sheaves on a $K3$ or abelian surface X. Denote by p_i the projection from $\mathcal{M}_H(w) \times X \times \mathcal{M}_H(w)$ onto the i-th factor and by p_{ij} the projection onto the product of the i-th and j-th factors.

[8] The homomorphism sends $f \in G(S_X^+)_w^{\mathrm{even}}$ to $(\tau^{\mathrm{ort}(f)}\widetilde{m}_f, \mathrm{ort}(f))$, where ort is the character (8.3) and $\widetilde{m} : G(S_X^+)^{\mathrm{even}} \to SO(S_X^+) \times \widetilde{SO}(S_X^-)$ is given in (4.31).

Theorem 6.1. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be any two universal families of sheaves over $X \times \mathcal{M}_H(w)$.*

(1) ([24]) *The class of the diagonal in the Chow ring of $\mathcal{M}_H(w) \times \mathcal{M}_H(w)$ is identified by*

$$c_m[-p_{13_!}(p_{12}^*(\mathcal{E}_1)^\vee \overset{L}{\otimes} p_{23}^*(\mathcal{E}_2))],$$

where both the dual $(\mathcal{E}_1)^\vee$ and the tensor product are taken in the derived category.

(2) ([25, Theorem 1]) *The integral cohomology $H^*(\mathcal{M}_H(w), \mathbb{Z})$ is torsion free.*

Remark 6.2. A version of Theorem 6.1 holds for a projective moduli space $\mathcal{M}_H(w)$ of H-stable sheaves on a $K3$ or abelian surface X, even if the universal sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ are twisted with respect to the pull-back to $X \times \mathcal{M}_H(w)$ of a Brauer class $\theta \in H^2_{\mathrm{an}}(\mathcal{M}_H(w), \mathcal{O}^*_{\mathcal{M}_H(w)})$ (a Čech cohomology class for the analytic topology). In that case a universal class e in the topological K-ring $K(X \times \mathcal{M}_H(w))$ of $X \times \mathcal{M}_H(w)$ was constructed in [25, Sec. 3.1], unique up to tensorization by the class of a topological complex line bundle. The statement of Theorem 6.1 then holds with $\mathcal{E}_i$ replaced by any such universal class e_i in $K(X \times \mathcal{M}_H(w))$, by [25, Prop. 24].

We denote by Δ_X the class of the diagonal in $X \times X$. Given a homomorphism $g : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ preserving the parity of the cohomological degree, we get the class $(1 \times g)(\Delta_{X_1})$ in $H^{\mathrm{even}}(X_1 \times X_2, \mathbb{Z})$ inducing g. Set

$$\gamma_{g,\epsilon}(\alpha, \beta) := \gamma_{(1\times g)(\Delta_{X_1}),\epsilon}(\alpha, \beta). \tag{6.4}$$

When the parameter ϵ is omitted, it is understood to be zero. When $X_1 = X_2$ and $g = \mathrm{id}$, Grothendieck–Riemann–Roch yields the equality

$$\gamma_{\mathrm{id}}(\mathcal{E}_1, \mathcal{E}_2) = c_m\{-p_{13_!}(p_{12}^*(\mathcal{E}_1)^\vee \overset{L}{\otimes} p_{23}^*(\mathcal{E}_2))\}. \tag{6.5}$$

In Section 8 we will see examples where $\gamma_g(\mathcal{E}_1, \mathcal{E}_2)$ is the class of the graph of an isomorphism, when g is induced by a stability preserving auto-equivalence of the derived category of the surface.

Identifying $H^*(\mathcal{M}(w_1), \mathbb{Z})$ with its dual, via Poincaré duality $x \mapsto \int_{\mathcal{M}(w_1)} x \cup (\bullet)$, we will view the class $\gamma_{g,\epsilon}(\mathcal{E}_1, \mathcal{E}_2)$ as a homomorphism (preserving the grading) from $H^*(\mathcal{M}(w_1), \mathbb{Z})$ to $H^*(\mathcal{M}(w_2), \mathbb{Z})$. We identify $H^*(X, \mathbb{Z})$ with its dual via Poincaré duality as well and regard classes in $H^*(X_1 \times X_2, \mathbb{Z})$ as homomorphisms from $H^*(X_1, \mathbb{Z})$ to $H^*(X_2, \mathbb{Z})$.

Let $d_X \in \mathrm{Aut}[H^*(X, \mathbb{C})]$ and $d_{\mathcal{M}(w_2)} \in \mathrm{Aut}[H^*(\mathcal{M}(w_2), \mathbb{C})]$ be graded ring automorphisms preserving the intersection pairings and satisfying

$$D_{X\times\mathcal{M}(w_2)} = d_X \otimes d_{\mathcal{M}(w_2)} \tag{6.6}$$

as automorphisms of $H^{\mathrm{even}}(X \times \mathcal{M}(w_2), \mathbb{C})$. Clearly, d_X determines $d_{\mathcal{M}(w_2)}$ uniquely, and each is determined by the above equation, up to a constant factor on each graded summand of the cohomology groups. When X is a $K3$ surface, the odd cohomology groups of both X and $\mathcal{M}(w_2)$ vanish, and we get a natural factorization by setting $d_X = D_X$

and $d_{\mathcal{M}(w_2)} = D_{\mathcal{M}(w_2)}$. When X is an abelian surface, we can let d_X and $d_{\mathcal{M}(w_2)}$ act on the i-th cohomology via multiplication by $(\sqrt{-1})^i$. Note that d_X has order 4. Nevertheless, conjugation by d_X has order 2 and the corresponding inner automorphism of $\mathrm{Aut}[H^*(X,\mathbb{Z})] := GL[H^{\mathrm{even}}(X,\mathbb{Z})] \times GL[H^{\mathrm{odd}}(X,\mathbb{Z})]$ is independent of the choice of d_X in the factorization (6.6).

Let $\tau \in \mathrm{Aut}[H^*(X,\mathbb{Z})]$ be the element acting by $(-1)^{i(i-1)/2}$ on $H^i(X,\mathbb{Z})$. Then d_X commutes with τ and the corresponding inner automorphisms of $\mathrm{Aut}[H^*(X,\mathbb{Z})]$ are equal.[9] Note that d_X is an isometry with respect to the pairing (4.15) on $H^*(X,\mathbb{C})$ (since d_X commutes with τ). Similarly, $d_{\mathcal{M}(w_2)}$ preserves the Poincaré duality pairing.

Let $g : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ be a linear homomorphism preserving the parity of the cohomological degree. Note that

$$d_{X_2} g d_{X_1}^{-1} = \tau g \tau \tag{6.7}$$

and is hence an integral homomorphism. Indeed, $d_X \tau$ acts on $H^{\mathrm{even}}(X)$ as the identity and on $H^{\mathrm{odd}}(X)$ via multiplication by a scalar. Hence, $d_{X_2}\tau f = f d_{X_1}\tau$ for every linear homomorphism $f : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ preserving the parity of the grading. Applying the latter equality with $f = \tau g \tau$ we get $d_{X_2} g d_{X_1}^{-1} = d_{X_2}\tau(\tau g \tau)\tau d_{X_1}^{-1} = \tau g \tau$. Given $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, we set

$$d_{X_i}^{\epsilon} := \begin{cases} d_{X_i} & \text{if } \epsilon = 1, \\ \mathrm{id} & \text{if } \epsilon = 0. \end{cases}$$

We use this notation only in conjugation, where the equality $d_{X_2}^{\epsilon} g (d_{X_1}^{\epsilon})^{-1} = \tau^{\epsilon} g \tau^{\epsilon}$ makes it unambiguous, since τ is an involution.

Let $\gamma : H^*(\mathcal{M}(w_1),\mathbb{Z}) \to H^*(\mathcal{M}(w_2),\mathbb{Z})$ be an isomorphism of graded rings. Assume that universal sheaves $\mathcal{E}_i$ exist over $X_i \times \mathcal{M}(w_i)$ for $i = 1, 2$.

Definition 6.3. (1) Assume that $g(w_1) = w_2$. We say that $g \otimes \gamma$ *maps a universal class of* $\mathcal{M}(w_1)$ *to a universal class of* $\mathcal{M}(w_2)$ if

$$(g \otimes \gamma)(\mathrm{ch}(\mathcal{E}_1)\sqrt{\mathrm{td}_{X_1}}) = [\mathrm{ch}(\mathcal{E}_2)\sqrt{\mathrm{td}_{X_2}}]\pi^*_{\mathcal{M}(w_2)} \exp(c_g),$$

where the class $c_g \in H^2(\mathcal{M}(w_2),\mathbb{Z})$ is characterized by

$$\mathrm{rank}(w_2)\pi^*_{\mathcal{M}(w_2)} c_g = c_1(\mathcal{E}_2) - [(g \otimes \gamma)(\mathrm{ch}(\mathcal{E}_1)\sqrt{\mathrm{td}_{X_1}})]_1. \tag{6.8}$$

(2) Assume that $g(w_1) = (w_2)^{\vee}$. We say that $g \otimes \gamma$ *maps a universal class to the dual of a universal class* if

$$(g \otimes \gamma)(\mathrm{ch}(\mathcal{E}_1)\sqrt{\mathrm{td}_{X_1}}) = [\mathrm{ch}(\mathcal{E}_2^{\vee})\sqrt{\mathrm{td}_{X_2}}]\pi^*_{\mathcal{M}(w_2)} \exp(c_g),$$

where $\mathrm{rank}(w_2)\pi^*_{\mathcal{M}(w_2)} c_g = -c_1(\mathcal{E}_2) - [(g \otimes \gamma)(\mathrm{ch}(\mathcal{E}_1)\sqrt{\mathrm{td}_{X_1}})]_1$.

(3) We say that $\gamma_{g,\epsilon}(\mathcal{E}_1,\mathcal{E}_2)$ *maps a universal class of* $\mathcal{M}(w_1)$ *to a universal class* (*or the dual of a universal class*) *of* $\mathcal{M}(w_2)$ if $d_{X_2}^{\epsilon} g (d_{X_1}^{\epsilon})^{-1} \otimes \gamma_{g,\epsilon}(\mathcal{E}_1,\mathcal{E}_2)$ does.

[9] In Section 4 the automorphism τ is extended to the main anti-automorphism (4.4) of the Clifford algebra $C(V)$; τ is an element of $G(S^+)^{\mathrm{even}}$ (see Example 4.4).

The following is a characterization of the class $\gamma_g(\mathcal{E}_1, \mathcal{E}_2)$. Let $(X_1, \mathcal{L}_1)$ and $(X_2, \mathcal{L}_2)$ be polarized $K3$ or abelian surfaces, $\mathcal{M}_{\mathcal{L}_1}(w_1)$ and $\mathcal{M}_{\mathcal{L}_2}(w_2)$ compact moduli spaces of stable sheaves, and $\mathcal{E}_i$ a universal sheaf over $X_i \times \mathcal{M}_{\mathcal{L}_i}(w_i)$.

Lemma 6.4 ([26, Lemma 5.2]). *Suppose that $f : H^*(\mathcal{M}_{\mathcal{L}_1}(w_1), \mathbb{Z}) \to H^*(\mathcal{M}_{\mathcal{L}_2}(w_2), \mathbb{Z})$ is a* ring *isomorphism, $g : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ a linear homomorphism preserving the parity of the cohomological degree, and $f \otimes g$ maps a universal class of $\mathcal{M}_{\mathcal{L}_1}(w_1)$ to a universal class of $\mathcal{M}_{\mathcal{L}_2}(w_2)$. Then $[f] = \gamma_g(\mathcal{E}_1, \mathcal{E}_2)$. In particular, given g, a ring isomorphism f satisfying the condition above is* unique (*if it exists*).

Corollary 6.5. *Assume that $\gamma_g(\mathcal{E}_1, \mathcal{E}_2)$ maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_2)$, $\gamma_h(\mathcal{E}_2, \mathcal{E}_3)$ maps a universal class of $\mathcal{M}(w_2)$ to a universal class of $\mathcal{M}(w_3)$, and both $\gamma_g(\mathcal{E}_1, \mathcal{E}_2)$ and $\gamma_h(\mathcal{E}_2, \mathcal{E}_3)$ are ring isomorphisms. Then $\gamma_{hg}(\mathcal{E}_1, \mathcal{E}_3) = \gamma_h(\mathcal{E}_2, \mathcal{E}_3) \circ \gamma_g(\mathcal{E}_1, \mathcal{E}_2)$.*

Lemma 6.6. *Let $g : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ be a linear homomorphism.*

(1) $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2) = (d^{-1}_{\mathcal{M}(w_1)} \otimes 1)\gamma_{d^{-1}_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2) = (d_{\mathcal{M}(w_1)} \otimes 1)\gamma_{d_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2)$.

(2) *When regarded as homomorphisms,*

$$\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2) = \gamma_{d^{-1}_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2) \circ d_{\mathcal{M}(w_1)} = \gamma_{d_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2) \circ d^{-1}_{\mathcal{M}(w_1)}. \tag{6.9}$$

Consequently, we have the following equalities:

$$\begin{aligned}(d^{-1}_{X_2} g d_{X_1} \otimes \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)) \circ (d^{-1}_{X_1} \otimes d^{-1}_{\mathcal{M}(w_1)}) &= d^{-1}_{X_2} g \otimes \gamma_{d^{-1}_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2),\\ (d_{X_2} g d^{-1}_{X_1} \otimes \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)) \circ (d_{X_1} \otimes d_{\mathcal{M}(w_1)}) &= d_{X_2} g \otimes \gamma_{d_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2).\end{aligned} \tag{6.10}$$

(3) *If $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$ maps a universal class of $\mathcal{M}(w_1)$ to the* dual *of a universal class of $\mathcal{M}(w_2)$, then $d_{X_2} g \otimes \gamma_{d_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2)$ maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_2)$.*

Proof. (1) The class $\gamma_{g,1}(\alpha_1, \alpha_2)$ involves $\pi_{13*}[\pi^*_{12}(\alpha_1)\pi^*_{24}[(1 \otimes g)(\Delta_{X_1})]\pi^*_{34}(\alpha_2)]$. The corresponding class in the definition of $\gamma_{h,0}(\alpha_1, \alpha_2)$ is

$$\begin{aligned}&\pi_{13*}\big[D\{\pi^*_{12}(\alpha_1)\pi^*_{24}[(1 \otimes h)(\Delta_{X_1})]\}\pi^*_{34}(\alpha_2)\big]\\ &\qquad = \pi_{13*}\big[\pi^*_{12}(D\alpha_1)\pi^*_{24}[(d_{X_1} \otimes d_{X_2})(1 \otimes h)(\Delta_{X_1})]\pi^*_{34}(\alpha_2)\big].\end{aligned}$$

Integrating first along the X_1 factor and using the fact that d_{X_1} is a ring automorphism preserving the intersection pairing we get

$$\pi_{13*}\big[\pi^*_{12}((d_{\mathcal{M}(w_1)} \otimes 1)\alpha_1)\pi^*_{24}[(1 \otimes d_{X_2} h)(\Delta_{X_1})]\pi^*_{34}(\alpha_2)\big].$$

Now we can "pull $d_{\mathcal{M}(w_1)} \otimes 1$ out" as it commutes with the Gysin map π_{13*} (by the projection formula). Being a ring automorphism, $d_{\mathcal{M}(w_1)} \otimes 1$ commutes with ℓ, the inversion, and projection on the class of degree $2m$. Setting $h = d^{-1}_{X_2} g$ we get the first identity. The second identity follows using the same argument, the fact that $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$

and $(1 \otimes h)(\Delta_X)$ are even cohomology classes, and the identities $D^2 = \mathrm{id}$ on the even cohomology of the products $X_i \times \mathcal{M}(w_i)$ and $X_1 \times X_2$, so $d_{\mathcal{M}(w_1)} \otimes d_{X_1}$ and $d^{-1}_{\mathcal{M}(w_1)} \otimes d^{-1}_{X_1}$ restrict to the same automorphism of the even cohomology and so do $d_{X_1} \otimes d_{X_2}$ and $d^{-1}_{X_1} \otimes d^{-1}_{X_2}$.

(2) The equalities in (6.9) follow from part (1). Each equality in (6.10) follows from the corresponding equality in (6.9).

(3) The equality $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2) \circ d_{\mathcal{M}(w_1)} = d_{\mathcal{M}(w_2)} \circ \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$ holds, since $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$ is assumed to be a graded ring isomorphism. The second equality below follows.

$$\begin{aligned} d_{X_2} g \otimes \gamma_{d_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2) &\overset{(6.10)}{=} d_{X_2} g \otimes (\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2) \circ d_{\mathcal{M}(w_1)}) \\ &= d_{X_2} g \otimes (d_{\mathcal{M}(w_2)} \circ \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)) \\ &= (d_{X_2} \otimes d_{\mathcal{M}(w_2)}) \circ (g \otimes \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)). \end{aligned}$$

The latter is assumed to map a universal class to a universal class. Hence, so does $d_{X_2} g \otimes \gamma_{d_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_2)$. ∎

Lemma 6.7. *Assume that $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$ maps a universal class of $\mathcal{M}(w_1)$ to the dual of a universal class of $\mathcal{M}(w_2)$, $\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3)$ maps a universal class of $\mathcal{M}(w_2)$ to the dual of a universal class of $\mathcal{M}(w_3)$, and both $\gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$ and $\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3)$ are ring isomorphisms. Then $\gamma_{\tau h \tau g,0}(\mathcal{E}_1, \mathcal{E}_3) = \gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3) \circ \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)$ and it is a ring isomorphism that maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_3)$.*

Proof. The equality

$$\begin{aligned} &[d_{X_3} h d^{-1}_{X_2} \otimes \gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3)] \circ D^{-1}_{X_2 \times \mathcal{M}(w_2)} \circ [d_{X_2} g d^{-1}_{X_1} \otimes \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)] \circ D_{X_1 \times \mathcal{M}(w_1)} \\ &\qquad = (d_{X_3} h d^{-1}_{X_2} g) \otimes [\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3) \circ \gamma_{g,1}(\mathcal{E}_1, \mathcal{E}_2)] \end{aligned}$$

follows from the definition of d_{X_i} and $D_{\mathcal{M}(w_i)}$, $i = 1, 2, 3$. The left hand side maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_3)$, by Lemma 6.6 (3). The right hand side is the tensor product of the integral homomorphism $d_{X_3} h d^{-1}_{X_2} g$ with a ring isomorphism. Lemma 6.4 implies that the latter must be $\gamma_{d_{X_3} h d^{-1}_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_3)$. Consequently,

$$(d_{X_3} h d^{-1}_{X_2} g) \otimes [\gamma_{d_{X_3} h d^{-1}_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_3)]$$

maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_3)$. Finally, the equality $\gamma_{d_{X_3} h d^{-1}_{X_2} g,0}(\mathcal{E}_1, \mathcal{E}_3) = \gamma_{\tau h \tau g,0}(\mathcal{E}_1, \mathcal{E}_3)$ follows from the equality $d_{X_3} h d^{-1}_{X_2} = \tau h \tau$ (see (6.7)). ∎

Lemma 6.8. *Assume that $\gamma_{g,0}(\mathcal{E}_1, \mathcal{E}_2)$ maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_2)$, $\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3)$ maps a universal class of $\mathcal{M}(w_2)$ to the dual of a universal class of $\mathcal{M}(w_3)$, $\gamma_{f,0}(\mathcal{E}_3, \mathcal{E}_4)$ maps a universal class of $\mathcal{M}(w_3)$ to a universal class of $\mathcal{M}(w_4)$, and $\gamma_{g,0}(\mathcal{E}_1, \mathcal{E}_2)$, $\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3)$, and $\gamma_{f,0}(\mathcal{E}_3, \mathcal{E}_4)$ are ring isomorphisms. Then $\gamma_{hg,1}(\mathcal{E}_1, \mathcal{E}_3) = \gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3) \circ \gamma_{g,0}(\mathcal{E}_1, \mathcal{E}_2)$ and it is a ring isomorphism* *that maps a universal class of $\mathcal{M}(w_1)$ to the dual of a universal class of $\mathcal{M}(w_3)$. Similarly, $\gamma_{f,0}(\mathcal{E}_3, \mathcal{E}_4)\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3) = \gamma_{\tau f \tau h,1}(\mathcal{E}_2, \mathcal{E}_4)$ and it is a ring isomorphism that maps a universal class of $\mathcal{M}(w_2)$ to the dual of a universal class of $\mathcal{M}(w_4)$.*

Proof. The proof is similar to that of Lemma 6.7. We check only the latter equality. We have $\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3) = \gamma_{d_{X_3} h,0}(\mathcal{E}_2, \mathcal{E}_3) \circ d^{-1}_{\mathcal{M}(w_2)}$, by Lemma 6.6 (2). Hence,

$$\gamma_{f,0}(\mathcal{E}_3, \mathcal{E}_4)\gamma_{h,1}(\mathcal{E}_2, \mathcal{E}_3) = \gamma_{f d_{X_3} h,0}(\mathcal{E}_2, \mathcal{E}_4) \circ d^{-1}_{\mathcal{M}(w_2)},$$

by Corollary 6.5. The right hand side is equal to $\gamma_{d^{-1}_{X_4} f d_{X_3} h,1}(\mathcal{E}_2, \mathcal{E}_4)$, by Lemma 6.6 (2). The latter is equal to $\gamma_{\tau f \tau h,1}(\mathcal{E}_2, \mathcal{E}_4)$, by (6.7). ■

Let g and γ be as in Definition 6.3 (1).

Lemma 6.9. *If $g \otimes \gamma$ maps a universal class of $\mathcal{M}(w_1)$ to a universal class of $\mathcal{M}(w_2)$, then $(\tau g \tau) \otimes \gamma$ maps the dual of a universal class of $\mathcal{M}(w_1)$ to the dual of a universal class of $\mathcal{M}(w_2)$.*

Proof. Set $D_i := D_{X_i \times \mathcal{M}(w_i)}$, $i = 1, 2$. We have

$$\begin{aligned}(\tau g \tau \otimes \gamma)(\mathrm{ch}(\mathcal{E}_1)^\vee \sqrt{\mathrm{td}_{X_1}}) &= (D_2(g \otimes \gamma)D_1)\big(D_1(\mathrm{ch}(\mathcal{E}_1)\sqrt{\mathrm{td}_{X_1}})\big) \\ &= (D_2(g \otimes \gamma))(\mathrm{ch}(\mathcal{E}_1)\sqrt{\mathrm{td}_{X_1}}) = D_2\big(\mathrm{ch}(\mathcal{E}_2)\sqrt{\mathrm{td}_{X_2}}\exp(c_g)\big) \\ &= \mathrm{ch}(\mathcal{E}_2)^\vee \sqrt{\mathrm{td}_{X_2}}\exp(-c_g).\end{aligned}$$ ■

7. Equivalences of derived categories

Let X be an abelian surface and let V, S^+, and S^- be the regular and half-spin integral representations of $\mathrm{Spin}(V)$ recalled in Section 4.1. Let $\mathrm{Aut}(D^b(X))$ be the group of auto-equivalences of the bounded derived category of coherent sheaves on X. Mukai, Polishchuk, and Orlov constructed a homomorphism $\mathrm{Aut}(D^b(X)) \to \mathrm{Spin}(V)$ whose image is equal to the subgroup preserving the Hodge structure of V (see [39, Th. 3.5], [9, Prop. 4.3.7], and [17, Prop. 9.48]). Their result holds for abelian varieties of arbitrary dimension. For abelian surfaces we get a homomorphism $\mathrm{Aut}(D^b(X)) \to \mathrm{Spin}(S^+)$ using the equality (4.32). In Corollary 7.8 below we exhibit an explicit lift to $\mathrm{Aut}(D^b(X))$ of products $m_s m_t \in \mathrm{Spin}(S^+)$ for certain pairs $s, t \in S^+$, each of self-intersection 2. These lifts will be shown in Section 8 to induce isomorphisms of certain moduli spaces of stable sheaves.

7.1. Tensorization by a line bundle

Let F be a line bundle on X. Denote by $\phi_F \in \mathrm{Spin}(V)$ the element corresponding to the auto-equivalence of $D^b(X)$ of tensorization by F. Explicitly, ϕ_F is the element of $\mathrm{Spin}(V)$ which maps to the element of $\mathrm{GL}(S) = \mathrm{GL}(H^*(X, \mathbb{Z}))$ acting by multiplication by the Chern character of F. That this element of $\mathrm{GL}(S)$ is the image of a unique element of $\mathrm{Spin}(V)$ is proven directly in [6, III.1.7].

Lemma 7.1. *ϕ_F acts on A_X as follows:*

(1) *On S^+: Given $(r, H, s) \in S^+$,*

$$\phi_F(r, H, s) = (r, H + rc_1(F), s + rc_1(F)^2/2 + H \wedge c_1(F)).$$

(2) *On S^-: Given $(w, w') \in H^1(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z})$,*

$$\phi_F(w, w') = (w, w' + c_1(F) \wedge w).$$

(3) *On V: Given $(w, \theta) \in H^1(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z})^*$,*

$$\rho(\phi_F)(w, \theta) = (w - D_\theta(c_1(F)), \theta).$$

Proof. The action on $H^*(X, \mathbb{Z})$ is multiplication by the Chern character $\mathrm{ch}(F) := 1 + c_1(F) + c_1(F)^2/2$ of F. For the action on V, embed V in $C(V) \subset \mathrm{End}[H^*(X, \mathbb{Z})]$ sending (w, θ) to $L_w + D_\theta$ as in (4.3). Conjugation yields

$$\phi_F \circ (L_w) \circ (\phi_F)^{-1} = L_w, \quad \phi_F \circ (D_\theta) \circ (\phi_F)^{-1} = D_\theta - L_{D_\theta(c_1(F))},$$

where the last equality is verified as follows. Set $f := c_1(F)$. For $a \in S$ we get

$$\mathrm{ch}(F)(D_\theta(\mathrm{ch}(F^{-1})a)) = D_\theta(a) + \mathrm{ch}(F)[D_\theta(\mathrm{ch}(F^{-1}))]a,$$

and $\mathrm{ch}(F)D_\theta(\mathrm{ch}(F^{-1})) = [1 + f + f^2/2][-D_\theta(f) + fD_\theta(f)] = -D_\theta(f)$. (Compare part (3) with [6, proof of III.1.7, p. 74, last displayed formula]). ∎

7.2. *Fourier–Mukai transform with kernel the Poincaré line bundle*

Let π_X and $\pi_{\hat{X}}$ be the projections from $X \times \hat{X}$ onto the corresponding factors. Let $\mathcal{P}$ be the normalized Poincaré line bundle over $X \times \hat{X}$. Then $\mathcal{P}$ restricts to $X \times \{t\}$, $t \in \hat{X}$, as a line bundle in the isomorphism class t and to $\{0\} \times \hat{X}$ as the trivial line bundle. The Fourier–Mukai functor $\Phi_{\mathcal{P}} : D^b(X) \to D^b(\hat{X})$ with kernel $\mathcal{P}$ is given by $R\pi_{\hat{X},*}(L\pi_X^*(\bullet) \otimes \mathcal{P})$. Let $\iota : H^*(X, \mathbb{Z})^* \to H^*(\hat{X}, \mathbb{Z})$ be the natural isomorphism identifying $H^j(X, \mathbb{Z})^*$ and $H^j(\hat{X}, \mathbb{Z})$. Explicitly, $\iota^{-1} : H^1(\hat{X}, \mathbb{Z}) \to H^1(X, \mathbb{Z})^*$ is the dual of the composition of the isomorphisms

$$H^1(X, \mathbb{Z}) \to H_1(\mathrm{Pic}^0(X), \mathbb{Z}) \stackrel{=}{\to} H_1(\hat{X}, \mathbb{Z}) \to H^1(\hat{X}, \mathbb{Z})^*,$$

where the left one is induced by the identification $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ via the exponential sequence. For $k > 1$, ι is given by the composition $[\bigwedge^k H^1(X, \mathbb{Z})]^* \cong \bigwedge^k [H^1(X, \mathbb{Z})^*] \to \bigwedge^k H^1(\hat{X}, \mathbb{Z})$, where the left isomorphism is the natural one and the right is the k-th exterior power of $\iota : H^1(X, \mathbb{Z})^* \to H^1(\hat{X}, \mathbb{Z})$. On the level of cohomology $\Phi_{\mathcal{P}}$ induces $\phi_{\mathcal{P}} := \sum_{i=0}^4 \phi_{\mathcal{P}}^i$, where

$$\phi_{\mathcal{P}}^i := (-1)^{i(i+1)/2} \iota \circ \mathrm{PD}_X : H^i(X, \mathbb{Z}) \to H^{4-i}(\hat{X}, \mathbb{Z}), \tag{7.1}$$

and PD_X is given in (4.20), by [37, Prop. 1.17]. Let $\Psi_{\mathcal{P}} : D^b(\widehat{X}) \to D^b(X)$ be the integral functor with kernel $\mathcal{P}^\vee \otimes \pi^*_{\widehat{X}}\omega_{\widehat{X}}$, where $\mathcal{P}^\vee$ is the line bundle dual to $\mathcal{P}$. Then $\Psi_{\mathcal{P}}[2]$ is the left adjoint of $\Phi_{\mathcal{P}}$, by [17, Prop. 5.9]. We have the natural isomorphism

$$\Psi_{\mathcal{P}}[2] \circ \Phi_{\mathcal{P}} \cong \mathrm{id}_{D^b(X)}, \tag{7.2}$$

since $\Phi_{\mathcal{P}}$ is an equivalence [35, Theorem 2.2].

The following lemma deals with some delicate sign issues.

Lemma 7.2. *The following equalities hold for classes θ in $H^j(X,\mathbb{Z})$ and ω in $H^{4-j}(X,\mathbb{Z})$.*

(1) $\int_X \theta \wedge \omega = \int_{\widehat{X}} \iota(\mathrm{PD}_X(\theta)) \wedge \iota(\mathrm{PD}_X(\omega))$.

(2) $\mathrm{PD}^{-1}_{\widehat{X}}((\iota^*)^{-1}(\theta)) = (-1)^j \iota(\mathrm{PD}_X(\theta))$.

(3) *The isomorphism $\phi_{\mathcal{P}}$ induces an isometry from $S_X := H^*(X,\mathbb{Z})$ to $S_{\widehat{X}} := H^*(\widehat{X},\mathbb{Z})$ with respect to the pairings given in (4.15).*

Proof. (1) Let $\{e_1, e_2, e_3, e_4\}$ be a basis of $H^1(X,\mathbb{Z})$ compatible with the orientation, so satisfying $\int_X e_1 \wedge e_2 \wedge e_3 \wedge e_4 = 1$, and let $\{f_1, f_2, f_3, f_4\}$ be a dual basis of $H^1(\widehat{X},\mathbb{Z})$, so that $(\iota^{-1}(f_i))(e_j) = \delta_{i,j}$. The element $\mathrm{PD}_X(e_1) \in H^3(X,\mathbb{Z})^*$ sends $e_2 \wedge e_3 \wedge e_4$ to 1 and its kernel consists of $\{e_1 \wedge e_i \wedge e_j : 1 < i < j\}$, so $\iota(\mathrm{PD}_X(e_1)) = f_2 \wedge f_3 \wedge f_4$. Similarly, given a permutation σ of $\{1,2,3,4\}$,

$$\begin{aligned}
\iota(\mathrm{PD}_X(e_{\sigma(1)})) &= \mathrm{sgn}(\sigma) f_{\sigma(2)} \wedge f_{\sigma(3)} \wedge f_{\sigma(4)},\\
\iota(\mathrm{PD}_X(e_{\sigma(1)} \wedge e_{\sigma(2)})) &= \mathrm{sgn}(\sigma) f_{\sigma(3)} \wedge f_{\sigma(4)},\\
\iota(\mathrm{PD}_X(e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge e_{\sigma(3)})) &= \mathrm{sgn}(\sigma) f_{\sigma(4)}.
\end{aligned}$$

The sign of the cyclic shift of four elements is -1, and so

$$\begin{aligned}
\iota(\mathrm{PD}_X(e_{\sigma(2)} \wedge e_{\sigma(3)} \wedge e_{\sigma(4)})) &= -\,\mathrm{sgn}(\sigma) f_{\sigma(1)},\\
\iota(\mathrm{PD}_X(e_{\sigma(3)} \wedge e_{\sigma(4)})) &= \mathrm{sgn}(\sigma) f_{\sigma(1)} \wedge f_{\sigma(2)}.
\end{aligned}$$

We get

$$\begin{aligned}
&\int_{\widehat{X}} \iota(\mathrm{PD}_X(e_{\sigma(1)})) \wedge \iota(\mathrm{PD}_X(e_{\sigma(2)} \wedge e_{\sigma(3)} \wedge e_{\sigma(4)}))\\
&\qquad = -\int_{\widehat{X}} f_{\sigma(2)} \wedge f_{\sigma(3)} \wedge f_{\sigma(4)} \wedge f_{\sigma(1)} = \int_{\widehat{X}} f_{\sigma(1)} \wedge f_{\sigma(2)} \wedge f_{\sigma(3)} \wedge f_{\sigma(4)}\\
&\qquad = \mathrm{sgn}(\sigma) = \int_X e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge e_{\sigma(3)} \wedge e_{\sigma(4)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\widehat{X}} \iota(\mathrm{PD}_X(e_{\sigma(1)} \wedge e_{\sigma(2)})) \wedge \iota(\mathrm{PD}_X(e_{\sigma(3)} \wedge e_{\sigma(4)})) &= \int_{\widehat{X}} f_{\sigma(3)} \wedge f_{\sigma(4)} \wedge f_{\sigma(1)} \wedge f_{\sigma(2)}\\
= \int_{\widehat{X}} f_{\sigma(1)} \wedge f_{\sigma(2)} \wedge f_{\sigma(3)} \wedge f_{\sigma(4)} &= \int_X e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge e_{\sigma(3)} \wedge e_{\sigma(4)}.
\end{aligned}$$

Part (3) follows from part (1) and (7.1). Indeed,

$$\int_X \theta \wedge \omega \stackrel{(4.15)}{=} (-1)^{j(j-1)/2}(\theta,\omega)_{S_X},$$

$$\begin{aligned}
&\int_{\widehat{X}} \iota(\mathrm{PD}_X(\theta)) \wedge \iota(\mathrm{PD}_X(\omega)) \\
&\quad \stackrel{(7.1)}{=} (-1)^{j(j+1)/2}(-1)^{(4-j)(4-j+1)/2} \int_{\widehat{X}} \phi_{\mathcal{P}}(\theta) \wedge \phi_{\mathcal{P}}(\omega) \\
&\quad = (-1)^{j(j-1)/2}(-1)^{(4-j)(4-j-1)/2} \int_{\widehat{X}} \phi_{\mathcal{P}}(\theta) \wedge \phi_{\mathcal{P}}(\omega) \\
&\quad \stackrel{(4.15)}{=} (-1)^{j(j-1)/2}(-1)^{(4-j)(4-j-1)/2}(-1)^{(4-j)(4-j-1)/2}(\phi_{\mathcal{P}}(\theta),\phi_{\mathcal{P}}(\omega))_{S_{\widehat{X}}} \\
&\quad = (-1)^{j(j-1)/2}(\phi_{\mathcal{P}}(\theta),\phi_{\mathcal{P}}(\omega))_{S_{\widehat{X}}}.
\end{aligned}$$

Hence, $(\theta,\omega)_{S_X} = (\phi_{\mathcal{P}}(\theta),\phi_{\mathcal{P}}(\omega))_{S_{\widehat{X}}}$, by part (1).

(2) Let γ be a class in $H^j(\widehat{X},\mathbb{Z})$. We have the equalities

$$(\iota^{-1}(\gamma))(\theta) = ((\iota^{-1})^*(\theta))(\gamma) = ((\iota^*)^{-1}(\theta))(\gamma) = \int_{\widehat{X}} \mathrm{PD}_{\widehat{X}}^{-1}((\iota^*)^{-1}(\theta)) \wedge \gamma, \qquad (7.3)$$

where the last equality follows from the definition of $\mathrm{PD}_{\widehat{X}}$. Now $\gamma = \iota(\mathrm{PD}_X(\delta))$ for some $\delta \in H^{4-j}(X,\mathbb{Z})$. Part (1) yields the second equality below:

$$\begin{aligned}
\int_{\widehat{X}} \iota(\mathrm{PD}_X(\theta)) \wedge \gamma &= \int_{\widehat{X}} \iota(\mathrm{PD}_X(\theta)) \wedge \iota(\mathrm{PD}_X(\delta)) \\
&= \int_X \theta \wedge \delta = (-1)^{j(4-j)} \int_X \delta \wedge \theta = (-1)^j \int_X \delta \wedge \theta \\
&= (-1)^j (\mathrm{PD}_X(\delta))(\theta) = (-1)^j (\iota^{-1}(\gamma))(\theta) \\
&\stackrel{(7.3)}{=} (-1)^j \int_{\widehat{X}} \mathrm{PD}_{\widehat{X}}^{-1}((\iota^*)^{-1}(\theta)) \wedge \gamma
\end{aligned}$$

for all γ in $H^j(\widehat{X},\mathbb{Z})$. The equality in part (2) follows. ■

Lemma 7.3. *The equality* $\iota(\mathrm{PD}_X(\beta \wedge w)) = D_{(\iota^*)^{-1}(w)}(\iota(\mathrm{PD}_X(\beta)))$ *holds for all* $w \in H^1(X,\mathbb{Z})$ *and all* $\beta \in H^2(X,\mathbb{Z})$.

Proof. Let γ be a class in $H^1(X,\mathbb{Z})$. Set $\widetilde{w} := (\iota^*)^{-1}(w)$ and $\widetilde{\gamma} := (\iota^*)^{-1}(\gamma)$. Then $(\iota^*)^{-1}(w \wedge \gamma) = \widetilde{w} \wedge \widetilde{\gamma}$. We have

$$\begin{aligned}
D_{\widetilde{\gamma}}(D_{\widetilde{w}}(\iota(\mathrm{PD}_X(\beta)))) &= (\widetilde{w} \wedge \widetilde{\gamma})(\iota(\mathrm{PD}_X(\beta))) = \int_{\widehat{X}} \mathrm{PD}_{\widehat{X}}^{-1}(\widetilde{w} \wedge \widetilde{\gamma}) \wedge \iota(\mathrm{PD}_X(\beta)) \\
&\stackrel{\text{Lem. 7.2 (2)}}{=} \int_{\widehat{X}} \iota(\mathrm{PD}_X(w \wedge \gamma)) \wedge \iota(\mathrm{PD}_X(\beta)) \\
&\stackrel{\text{Lem. 7.2 (1)}}{=} \int_X w \wedge \gamma \wedge \beta = \int_X \beta \wedge w \wedge \gamma = (\mathrm{PD}_X(\beta \wedge w))(\gamma) \\
&= D_{\widetilde{\gamma}}(\iota(\mathrm{PD}_X(\beta \wedge w))).
\end{aligned}$$

■

Let $\varphi_{\mathcal{P}} : V_X \to V_{\hat{X}}$ be the isomorphism given by

$$\varphi_{\mathcal{P}}(w, \theta) = -(\iota(\theta), (\iota^*)^{-1}(w)) \tag{7.4}$$

for all $(w, \theta) \in V_X$, $w \in H^1(X, \mathbb{Z})$ and $\theta \in H^1(X, \mathbb{Z})^*$.

Lemma 7.4. *The following diagram is commutative:*

$$\begin{array}{ccccccc}
V_X & \xrightarrow{\subset} & C(V_X) & \longrightarrow & \mathrm{End}(S_X) \ni & & f \\
\downarrow{\scriptstyle \varphi_{\mathcal{P}}} & & & & \downarrow{\scriptstyle \mathrm{Ad}_{\phi_{\mathcal{P}}}} & & \downarrow \\
V_{\hat{X}} & \xrightarrow{\subset} & C(V_{\hat{X}}) & \longrightarrow & \mathrm{End}(S_{\hat{X}}) \ni & & \phi_{\mathcal{P}} f \phi_{\mathcal{P}}^{-1}
\end{array}$$

Proof. For $w \in H^1(X, \mathbb{Z})$ we need to show that $D_{\varphi_{\mathcal{P}}(w)} = \mathrm{Ad}_{\phi_{\mathcal{P}}}(L_w)$. Evaluating both sides on $\gamma \in H^*(\hat{X}, \mathbb{Z})$ the equality becomes

$$(\phi_{\mathcal{P}} \circ L_w \circ \phi_{\mathcal{P}}^{-1})(\gamma) = -D_{(\iota^*)^{-1}(w)}(\gamma). \tag{7.5}$$

The above equality holds for all $\gamma \in H^2(\hat{X}, \mathbb{Z})$, by Lemma 7.3 applied with $\beta := \phi_{\mathcal{P}}^{-1}(\gamma)$. Both sides of (7.5) vanish for $\gamma \in H^0(\hat{X}, \mathbb{Z})$. For $\gamma := [\mathrm{pt}_{\hat{X}}] \in H^4(\hat{X}, \mathbb{Z})$, the left hand side of (7.5) is $\phi_{\mathcal{P}}(w)$. The right hand side is

$$-D_{(\iota^*)^{-1}(w)}[\mathrm{pt}_{\hat{X}}] \overset{(4.21)}{=} \mathrm{PD}_{\hat{X}}^{-1}((\iota^*)^{-1}(w)) \overset{\text{Lem. 7.2 (2)}}{=} -\iota(\mathrm{PD}_X(w)) = \phi_{\mathcal{P}}(w).$$

Equation (7.5) thus holds for all $\gamma \in H^{\mathrm{even}}(\hat{X}, \mathbb{Z})$. We have $\phi_{\mathcal{P}}(L_{e_{\sigma(1)}}(\phi_{\mathcal{P}}^{-1}(f_{\sigma(1)}))) = \phi_{\mathcal{P}}(L_{e_{\sigma(1)}}(-\operatorname{sgn}(\sigma) e_{\sigma(2)} \wedge e_{\sigma(3)} \wedge e_{\sigma(4)})) = \phi_{\mathcal{P}}(-[\mathrm{pt}_X]) = -1 \in H^0(\hat{X}, \mathbb{Z})$. Similarly, $-D_{(\iota^*)^{-1}(e_{\sigma(1)})}(f_{\sigma(1)}) = -1$. Both sides of (7.5) vanish for $\gamma = f_i$ and $w = e_j$ if $i \neq j$. The case $\gamma \in H^3(\hat{X}, \mathbb{Z})$ checks as well.

We denote by $\iota_{\hat{X}} : H^j(\hat{X}, \mathbb{Z})^* \to H^j(\hat{\hat{X}}, \mathbb{Z})$ the homomorphism analogous to ι. Let $\hat{\mathcal{P}}$ be the Poincaré line bundle over $\hat{X} \times \hat{\hat{X}}$. The composition $\phi_{\hat{\mathcal{P}}} \circ \phi_{\mathcal{P}} : S_X \to S_{\hat{\hat{X}}}$ is equal to $\iota_{\hat{X}} \circ (\iota^*)^{-1}$. Indeed,

$$\begin{aligned}
\phi_{\hat{\mathcal{P}}} \circ \phi_{\mathcal{P}} &= (-1)^{j(j+1)/2} (-1)^{(4-j)(4-j+1)/2} \iota_{\hat{X}} \, \mathrm{PD}_{\hat{X}} \, \iota \, \mathrm{PD}_X = (-1)^j \iota_{\hat{X}} \, \mathrm{PD}_{\hat{X}} \, \iota \, \mathrm{PD}_X \\
&= \iota_{\hat{X}} \circ (\iota^*)^{-1},
\end{aligned}$$

where the last equality follows from Lemma 7.2 (2). Hence, given $\theta \in H^1(\hat{X}, \mathbb{Z})^*$, we have

$$\mathrm{Ad}_{\phi_{\hat{\mathcal{P}}}}(-D_\theta) \overset{(7.5)}{=} \mathrm{Ad}_{\phi_{\hat{\mathcal{P}}}} \circ \mathrm{Ad}_{\phi_{\mathcal{P}}}(L_{\iota^*\theta}) = \mathrm{Ad}_{[\iota_{\hat{X}}(\iota^*)^{-1}]}(L_{\iota^*\theta}).$$

Now, $\iota_{\hat{X}} \circ (\iota^*)^{-1}$ is a cohomology ring isomorphism. Hence the right hand side above is equal to $L_{(\iota_{\hat{X}} \circ (\iota^*)^{-1})(\iota^*\theta)} = L_{\iota_{\hat{X}}(\theta)} = -L_{\varphi_{\hat{\mathcal{P}}}(\theta)}$. We deduce that $\mathrm{Ad}_{\phi_{\hat{\mathcal{P}}}}(D_\theta) = L_{\varphi_{\hat{\mathcal{P}}}(\theta)}$ for the dual of every abelian surface. Hence

$$\mathrm{Ad}_{\phi_{\mathcal{P}}}(D_\theta) = L_{\varphi_{\mathcal{P}}(\theta)} \quad \text{for every } \theta \in H^1(X, \mathbb{Z})^*. \qquad \blacksquare$$

Remark 7.5. The isomorphism $\iota_{\hat{X}} \circ (\iota^*)^{-1} : H^1(X, \mathbb{Z}) \to H^1(\hat{\hat{X}}, \mathbb{Z})$ corresponds to the standard isomorphism st $: X \to \hat{\hat{X}}$. The isomorphism $\varphi_{\mathcal{P}}$ corresponds to an isomorphism $\tilde{\varphi}_{\mathcal{P}} : X \times \hat{X} \to \hat{X} \times \hat{\hat{X}}$, which pulls back the line bundle $\hat{\mathcal{P}}$ to $\mathcal{P}^{-1}$. The isomorphism $-\tilde{\varphi}_{\mathcal{P}}$ pulls back the line bundle $\hat{\mathcal{P}}$ to $\mathcal{P}^{-1}$ as well and it restricts to the second factor $\hat{X}$ as the identity onto the first factor $\hat{X}$ of the image. Similarly, $-\tilde{\varphi}_{\mathcal{P}}$ restricts to the first factor X as the standard isomorphism st onto the second factor $\hat{\hat{X}}$ of the image. Mukai and Orlov composed $\tilde{\varphi}_{\mathcal{P}}$ with the isomorphism $\mathrm{id}_{\hat{X}} \times -(\mathrm{st})^{-1} : \hat{X} \times \hat{\hat{X}} \to \hat{X} \times X$ obtaining an isomorphism $\tilde{\phi}_{\mathcal{P}} : X \times \hat{X} \to \hat{X} \times X$ which pulls back $\hat{\mathcal{P}}$ to $\mathcal{P}$, and whose square is minus the identity (see [17, Rem. 9.12 and Ex. 9.38 (v), p. 213] or [45, Rem. 2.2]).

The natural isomorphisms $\varphi_{\mathcal{P}} : V_X \to V_{\hat{X}}$ and $\phi_{\mathcal{P}} : S_X \to S_{\hat{X}}$ combine to yield the linear isomorphism $\phi_{\mathcal{P}} : A_X \to A_{\hat{X}}$, which is an isometry. The isometry $\varphi_{\mathcal{P}}$ has a unique extension to an algebra isomorphism $\tilde{\varphi}_{\mathcal{P}} : C(V_X) \to C(V_{\hat{X}})$, by the definition of the Clifford algebras, and $\tilde{\varphi}_{\mathcal{P}}(\mathrm{Spin}(V_X)) = \mathrm{Spin}(V_{\hat{X}})$, by definition of the spin groups. Lemma 7.4 implies

$$\tilde{\mu}(\tilde{\varphi}_{\mathcal{P}}(g)) = \mathrm{Ad}_{\phi_{\mathcal{P}}}(\tilde{\mu}(g))$$

for all $g \in \mathrm{Spin}(V_X)$, where $\tilde{\mu} : \mathrm{Spin}(V_X) \to \mathrm{Aut}(A_X)$ is the restriction of the homomorphism given in (4.27) and we denote by $\tilde{\mu} : \mathrm{Spin}(V_{\hat{X}}) \to \mathrm{Aut}(A_{\hat{X}})$ its analogue.

7.3. *Lifting products of two* $(+2)$*-reflections to auto-equivalences*

Consider the natural homomorphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(\hat{X}), \quad F \mapsto \hat{F} := \det(\Phi_{\mathcal{P}}(F))^{-1}. \tag{7.6}$$

Then $c_1(\hat{F}) = \iota(\mathrm{PD}(c_1(F)))$, by (7.1).

Let F be a line bundle on X, and $\phi_F \in \mathrm{Spin}(V_X)$ the isometry of $H^*(X, \mathbb{Z})$ induced by tensorization with F. Set $s := (1,0,1) \in S_X^+$ and $\hat{s} := (1,0,1) \in S_{\hat{X}}^+$. Then $(s,s)_{S_X^+} = 2$. Denote by R_s the reflection (4.9) in s.

Lemma 7.6. *The auto-equivalence* $\Phi_{\mathcal{P}}^{-1} \circ (\otimes \hat{F}) \circ \Phi_{\mathcal{P}} \circ (\otimes F^{-1})$ *of* $D^b(X)$ *maps to the element* $\tilde{m}_s \cdot \tilde{m}_{\phi_F(s)}$ *of* $\mathrm{Aut}(A_X)$*. In other words,*

$$\tilde{\mu}(\phi_{\mathcal{P}}^{-1} \circ \phi_{\hat{F}} \circ \phi_{\mathcal{P}} \circ \phi_F^{-1}) = \tilde{m}_s \cdot \tilde{m}_{\phi_F(s)}, \tag{7.7}$$

where $\tilde{\mu}$ *is the homomorphism in* (4.26) *and* $\tilde{m}$ *is defined in* (4.31)*. The displayed element acts on* S_X^+ *via the composition* $R_s \circ R_{\phi_F(s)}$ *of the two reflections.*

Proof. Consider the composition $\eta := \tilde{m}_{\hat{s}} \circ \phi_{\mathcal{P}} : A_X \to A_{\hat{X}}$. Then η maps $H^{\mathrm{even}}(X, \mathbb{Z})$ to $H^{\mathrm{even}}(\hat{X}, \mathbb{Z})$ and preserves the grading and $\eta : S_X^+ \to S_{\hat{X}}^+$ is given by

$$\eta(r, H, t) = -(r, \iota(\mathrm{PD}(H)), t).$$

Set $\tilde{\phi}_F := \tilde{\mu}(\phi_F) \in \mathrm{Aut}(A_X)$ and define $\tilde{\phi}_{\hat{F}}$ similarly. We claim that conjugation yields the equality

$$\eta \circ \tilde{\phi}_F \circ \eta^{-1} = \tilde{\phi}_{\hat{F}} \tag{7.8}$$

in $\tilde{\mu}(\mathrm{Spin}(V_{\hat{X}}))$. It suffices to verify that both sides act the same way on $S^+_{\hat{X}}$ and on $V_{\hat{X}}$, since $\mathrm{Spin}(V_{\hat{X}})$ acts faithfully on the direct sum $S^+_{\hat{X}} \oplus V_{\hat{X}}$. Both sides of (7.8) map to the same element of $SO(S^+_{\hat{X}})$, by the above computation of η. Let θ be a class in $H^1(X,\mathbb{Z})$, ω a class in $H^3(X,\mathbb{Z})$, and set $\tilde{\theta} := (\iota^*)^{-1}(\theta) \in H^1(\hat{X},\mathbb{Z})^*$ and $\hat{\omega} := \iota(\mathrm{PD}_X(\omega)) \in H^1(\hat{X},\mathbb{Z})$. Consider the element $(\hat{\omega},\tilde{\theta}) \in V_{\hat{X}}$. We have

$$\tilde{\phi}_{\hat{F}}(\hat{\omega},\tilde{\theta}) = (\hat{\omega} - D_{\tilde{\theta}}(c_1(\hat{F})),\tilde{\theta}) = (\hat{\omega} - D_{\tilde{\theta}}(\iota(\mathrm{PD}_X(c_1(F)))),\tilde{\theta}), \tag{7.9}$$

where the first equality follows from Lemma 7.1 (3) and the second from the equality $c_1(\hat{F}) = \iota(\mathrm{PD}_X(c_1(F)))$ observed above. Let us evaluate $\eta\tilde{\phi}_F\eta^{-1}(\hat{\omega},\tilde{\theta})$. Note that $\tilde{m}_{\hat{s}}^{-1} = \tilde{m}_{\hat{s}}$. Example 4.2 with $n = -1$ yields the equality

$$\tilde{m}_{\hat{s}}^{-1}(\hat{\omega},\tilde{\theta}) = (\hat{\omega}, -\mathrm{PD}_{\hat{X}}^{-1}(\tilde{\theta}))$$

in $S^-_{\hat{X}}$. Now $\mathrm{PD}_{\hat{X}}^{-1}(\tilde{\theta}) = \phi_{\mathcal{P}}(\theta)$, by Lemma 7.2 (2), and $\hat{\omega} = \phi_{\mathcal{P}}(\omega)$. Hence,

$$\phi_{\mathcal{P}}^{-1}(\hat{\omega}, -\mathrm{PD}_{\hat{X}}^{-1}(\tilde{\theta})) = (-\theta,\omega) \tag{7.10}$$

in S^-_X. The equality $\tilde{\phi}_F(-\theta,\omega) = (-\theta,\omega - c_1(F)\wedge\theta)$ holds by Lemma 7.1 (2). We have the following two equalities, the first by (7.10):

$$\phi_{\mathcal{P}}(-\theta,\omega - c_1(F)\wedge\theta) = (\hat{\omega} - \iota(\mathrm{PD}_X(c_1(F)\wedge\theta)), -\mathrm{PD}_{\hat{X}}^{-1}(\tilde{\theta})),$$
$$\tilde{m}_{\hat{s}}(\hat{\omega} - \iota(\mathrm{PD}_X(c_1(F)\wedge\theta)), -\mathrm{PD}_{\hat{X}}^{-1}(\tilde{\theta})) = (\hat{\omega} - \iota(\mathrm{PD}_X(c_1(F)\wedge\theta)),\tilde{\theta}).$$

The right hand side above is equal to the right hand side of (7.9), by Lemma 7.3. Hence, both sides of (7.8) map to the same isometry of $V_{\hat{X}}$ as well, and (7.8) is verified.

Substitute the equality (7.8) into (7.7) to get the following equalities in $\mathrm{Aut}(A_{\hat{X}})$:

$$\begin{aligned}\phi_{\mathcal{P}}^{-1}\circ(\eta\circ\tilde{\phi}_F\circ\eta^{-1})\circ\phi_{\mathcal{P}}\circ\tilde{\phi}_F^{-1} &= (\phi_{\mathcal{P}}^{-1}\circ\tilde{m}_{\hat{s}}\circ\phi_{\mathcal{P}})\circ\tilde{\phi}_F\circ(\phi_{\mathcal{P}}^{-1}\circ(\tilde{m}_{\hat{s}})^{-1}\circ\phi_{\mathcal{P}})\circ\tilde{\phi}_F^{-1}\\ &\overset{(7.11)}{=} \tilde{m}_s\circ\tilde{\phi}_F\circ(\tilde{m}_s)^{-1}\tilde{\phi}_F^{-1} \overset{(7.12)}{=} \tilde{m}_s\cdot\tilde{m}_{\phi_F(s)}.\end{aligned}$$ ■

Lemma 7.7. *The following equalities hold in* $\mathrm{GL}(A_X)$*:*

$$\phi_{\mathcal{P}}^{-1}\tilde{m}_{\hat{s}}\phi_{\mathcal{P}} = \tilde{m}_s, \tag{7.11}$$
$$\tilde{\phi}_F\tilde{m}_s\tilde{\phi}_F^{-1} = \tilde{m}_{\phi_F(s)}. \tag{7.12}$$

Proof. Equation (7.11): $\phi_{\mathcal{P}}$ restricts to an isometry from S^+_X to $S^+_{\hat{X}}$, by Lemma 7.2. The right hand side acts on S^+ by $-R_s$, where R_s is the reflection in s. The left hand side acts by $-R_{\phi_{\mathcal{P}}^{-1}(\hat{s})} = -R_s$ as well.

The element $\widetilde{m}_s$ maps S_X^- to V_X and V_X to S_X^-, and $\widetilde{m}_s^2 = 1 \in \mathrm{Aut}(A_X)$. Hence, it suffices to check that both sides restrict to the same homomorphism from S_X^- to V_X. Let $(\alpha, \beta) \in S_X^-$, $\alpha \in H^1(X, \mathbb{Z})$ and $\beta \in H^3(X, \mathbb{Z})$. This checks as follows:

$$\phi_{\mathcal{P}}(\widetilde{m}_s(\alpha,\beta)) \stackrel{(4.23)}{=} \varphi_{\mathcal{P}}(\alpha, -\mathrm{PD}_X(\beta)) \stackrel{\text{Lem. 7.4}}{=} (\iota(\mathrm{PD}_X(\beta)), -(\iota^*)^{-1}(\alpha)).$$

$$\widetilde{m}_{\hat{s}}(\phi_{\mathcal{P}}(\alpha,\beta)) = \widetilde{m}_{\hat{s}}(\iota\,\mathrm{PD}_X(\beta), -\iota\,\mathrm{PD}_X(\alpha)) \stackrel{(4.23)}{=} (\iota\,\mathrm{PD}_X(\beta), -\mathrm{PD}_{\hat{X}}(-\iota\,\mathrm{PD}_X(\alpha)))$$
$$\stackrel{\text{Lem. 7.2 (2)}}{=} (\iota(\mathrm{PD}_X(\beta)), -(\iota^*)^{-1}(\alpha)).$$

Equation (7.12): The restrictions of both sides to S_X^+ are equal, since $\widetilde{\phi}_F$ restricts to S_X^+ as an isometry. Both sides map S_X^- to V_X and V_X to S_X^-. The square of both sides is the identity, so it suffices to check that both sides restrict to the same homomorphism from S_X^- to V_X. This checks as follows. Let $(\alpha, \beta) \in S_X^-$. We have

$$\widetilde{\phi}_F(\widetilde{m}_s(\alpha,\beta)) \stackrel{(4.23)}{=} \widetilde{\phi}_F(\alpha, -\mathrm{PD}_X(\beta)) \stackrel{\text{Lem. 7.1}}{=} (\alpha + D_{\mathrm{PD}_X(\beta)}(c_1(F)), -\mathrm{PD}_X(\beta)),$$

$$\widetilde{m}_{\phi_F(s)}(\widetilde{\phi}_F(\widetilde{m}_s(\alpha,\beta))) = \widetilde{m}_{\phi_F(s)}(\alpha + D_{\mathrm{PD}_X(\beta)}(c_1(F)), -\mathrm{PD}_X(\beta))$$
$$= (\alpha + D_{\mathrm{PD}_X(\beta)}(c_1(F))) \wedge \phi_F(s) - D_{\mathrm{PD}_X(\beta)}(\phi_F(s)),$$

$$\widetilde{\phi}_F(\alpha,\beta) \stackrel{\text{Lem. 7.1}}{=} (\alpha, \beta + c_1(F) \wedge \alpha).$$

Substituting $\phi_F(s) = 1 + c_1(F) + c_1(F)^2/2 + [\mathrm{pt}_X]$ we see that the right hand sides of the two lines above are equal, using also the equality $D_{\mathrm{PD}_X(\beta)}([\mathrm{pt}]) = -\beta$, which follows from (4.21). ∎

Assume now that F_1 and F_2 are two line bundles on X. Let $\Phi_{F_i} : D^b(X) \to D^b(X)$ be tensorization by F_i.

Corollary 7.8. *The auto-equivalence* $(\Phi_{F_2^{-1}} \circ [1] \circ \Psi_{\mathcal{P}} \circ \Phi_{\hat{F}_2}) \circ (\Phi_{\hat{F}_1^{-1}} \circ [1] \circ \Phi_{\mathcal{P}} \circ \Phi_{F_1})$ *of* $D^b(X)$ *is mapped to the element* $\widetilde{m}_{\phi_{F_2}^{-1}(s)} \circ \widetilde{m}_{\phi_{F_1}^{-1}(s)}$ *of* $\mathrm{Aut}(A_X)$*, which acts on* S_X^+ *as the composition* $R_{\phi_{F_2}^{-1}(s)} \circ R_{\phi_{F_1}^{-1}(s)}$ *of two reflections in the* $+2$ *vectors* $\phi_{F_i}^{-1}(s)$*,* $i = 1, 2$*:*

$$(\Phi_{F_2^{-1}} \circ [1] \circ \Psi_{\mathcal{P}} \circ \Phi_{\hat{F}_2}) \circ (\Phi_{\hat{F}_1^{-1}} \circ [1] \circ \Phi_{\mathcal{P}} \circ \Phi_{F_1}) \mapsto \widetilde{m}_{\phi_{F_2}^{-1}(s)} \circ \widetilde{m}_{\phi_{F_1}^{-1}(s)}. \quad (7.13)$$

Proof. The left hand side translates via (7.2) to the left hand side below:

$$\phi_{F_2^{-1}}(\phi_{\mathcal{P}}^{-1}\phi_{\hat{F}_2 \otimes \hat{F}_1^{-1}}\phi_{\mathcal{P}}\phi_{F_1 \otimes F_2^{-1}})\phi_{F_2} \stackrel{\text{Lem. 7.6}}{=} \phi_{F_2^{-1}}(\widetilde{m}_s \circ \widetilde{m}_{\phi_{F_1^{-1} \otimes F_2}(s)})\phi_{F_2}$$
$$\stackrel{(7.12)}{=} \widetilde{m}_{\phi_{F_2}^{-1}(s)} \circ \widetilde{m}_{\phi_{F_1}^{-1}(s)}.$$ ∎

8. Monodromy of moduli spaces via Fourier–Mukai functors

Let $\Phi : D^b(X_1) \to D^b(X_2)$ be an equivalence of derived categories of abelian surfaces which maps H_1-stable sheaves with a primitive Chern character w_1 to H_2-stable sheaves with Chern character w_2. In Section 8.1 we note that Φ induces an isometry $\phi : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ and an isomorphism $f : \mathcal{M}_{H_1}(w_1) \to \mathcal{M}_{H_2}(w_2)$ such that

$\phi \otimes f_* : H^*(X_1 \times \mathcal{M}_{H_1}(w_1), \mathbb{Z}) \to H^*(X_2 \times \mathcal{M}_{H_2}(w_2), \mathbb{Z})$ maps a universal class to a universal class. In Section 8.2 we prove Theorem 1.2 constructing the homomorphism $\mathrm{mon} : G(S^+)^{\mathrm{even}}_{s_n} \to \mathrm{Mon}(\mathcal{M}(s_n))$. In Section 8.3 we construct monodromy equivariant homomorphisms from the half-spin representations S^+_X and S^-_X to canonical quotients of $H^d(\mathcal{M}(s_n), \mathbb{Z})$.

8.1. *Stability preserving Fourier–Mukai transformations*

Let X_i, $i = 1, 2$, be abelian surfaces, $v_i \in S^+_{X_i}$ a primitive Mukai vector, and H_i a v_i-generic polarization on X_i. Assume that the integral transform $\Phi_F : D^b(X_1) \to D^b(X_2)$ with kernel an object F in $D^b(X_1 \times X_2)$ is an equivalence, and $\Phi_F(E)$ is an H_2-stable sheaf with Mukai vector v_2, for every H_1-stable coherent sheaf E on X_1 with Mukai vector v_1. Denote by $\Phi_F \boxtimes \mathrm{id}_{\mathcal{M}(v_1)} : D^b(X_1 \times \mathcal{M}_{H_1}(v_1)) \to D^b(X_2 \times \mathcal{M}_{H_1}(v_1))$ the integral transform with kernel the object in $D^b(X_1 \times \mathcal{M}_{H_1}(v_1) \times X_2 \times \mathcal{M}_{H_1}(v_1))$, which is the derived tensor product of the pull-backs of the object F and the structure sheaf of the diagonal in the cartesian square of $\mathcal{M}_{H_1}(v_1)$. Then $\Phi_F \boxtimes \mathrm{id}_{\mathcal{M}(v_1)}$ is an equivalence [45, Assertion 1.7]. Let $\mathcal{E}_{v_1}$ be a (possibly twisted) universal sheaf over $X_1 \times \mathcal{M}_{H_1}(v_1)$. The object $\Phi_F \boxtimes \mathrm{id}_{\mathcal{M}(v_1)}(\mathcal{E}_{v_1})$ is represented by a flat (possibly twisted by the pull-back of a Brauer class from $\mathcal{M}_{H_1}(v_1)$) family of H_2-stable coherent sheaves with Mukai vector v_2 on X_2, by [37, Theorem 1.6]. Let $f : \mathcal{M}_{H_1}(v_1) \to \mathcal{M}_{H_2}(v_2)$ be the classifying morphism associated to this family. Then f is easily seen to be an open immersion, which must be surjective, by compactness of $\mathcal{M}_{H_1}(v_1)$ and irreducibility of $\mathcal{M}_{H_2}(v_2)$ (Theorem 3.1). Hence, f is an isomorphism. Let $\phi_F : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ be the parity preserving isomorphism induced by Φ_F [17, Cor. 9.43]. Then $\phi_F(v_1) = v_2$. Let $e_{v_i} \in K(X_i \times \mathcal{M}_{H_i}(v_i))$ be a universal class (see Remark 6.2).

Lemma 8.1. *$\phi_F \otimes f_* : H^*(X_1 \times \mathcal{M}_{H_1}(v_1), \mathbb{Q}) \to H^*(X_2 \times \mathcal{M}_{H_2}(v_2), \mathbb{Q})$ maps a universal class to a universal class, in the sense of Definition 6.3. In particular, $f_* = \gamma_{\phi_F}(e_{v_1}, e_{v_2})$.*

Proof. The statement was proven in detail in [26, Lemma 5.6] in the case of moduli spaces of sheaves on $K3$ surfaces. The proof for abelian surfaces is identical. We briefly outline the argument in the case of fine moduli spaces admitting untwisted universal sheaves $\mathcal{E}_{v_i}$, $i = 1, 2$. In that case $(\mathrm{id}_{X_1} \times f)^* \mathcal{E}_{v_2}$ represents the object $\Phi_F \boxtimes \mathrm{id}_{\mathcal{M}(v_1)}(\mathcal{E}_{v_1})$, possibly after replacing $\mathcal{E}_{v_2}$ by its tensor product with the pull-back of a line bundle on $\mathcal{M}_{H_2}(v_2)$. Now $\mathrm{ch}[\Phi_F \boxtimes \mathrm{id}_{\mathcal{M}(v_1)}(\mathcal{E}_{v_1})] = (\phi_F \otimes \mathrm{id})(\mathrm{ch}(\mathcal{E}_{v_1}))$. Hence, $(\phi_F \otimes f_*)$ maps $\mathrm{ch}(\mathcal{E}_{v_1})$ to $\mathrm{ch}(\mathcal{E}_{v_2})$. The equality $f_* = \gamma_{\phi_F}(\mathcal{E}_{v_1}, \mathcal{E}_{v_2})$ now follows from Lemma 6.4. ∎

8.2. *The monodromy representation of* $G(S^+)^{\mathrm{even}}_{s_n}$

Let X be an abelian surface and let H be an ample line bundle on X with $\chi(H) = n \geq 2$ (and genus $n + 1$ and degree $2n$). Given a length $n + 1$ subscheme $Z \subset X$, we get the equality

$$\mathrm{ch}(I_Z \otimes H) = (1, H, -1).$$

In particular, $z_{n+1} := \mathrm{ch}(I_Z \otimes H)$ is orthogonal to $s := (1,0,1)$, $(z_{n+1}, s)_{S^+} = 0$. Let $\widehat{X}$ be the dual surface and $\widehat{H}$ the line bundle associated to H via (7.6). Then $\widehat{H}$ is ample as well.

Proposition 8.2 ([56, Proposition 3.5]). *Let E be a μ-stable sheaf with* $\mathrm{ch}(E) = (r, H, a)$, *$a < 0$. Then the sheaf cohomology $\mathcal{H}^i(\Phi_{\mathcal{P}}(E))$ vanishes for $i \neq 1$, and $\mathcal{H}^1(\Phi_{\mathcal{P}}(E))$ is a μ-stable sheaf with Chern character $(-a, \widehat{H}, -r)$. In particular, if H generates* $\mathrm{Pic}(X)$, *then the composition $[1] \circ \Phi_{\mathcal{P}}$ induces an isomorphism of moduli spaces*

$$\mathcal{M}_H(r, H, a) \stackrel{\cong}{\rightarrow} \mathcal{M}_{\widehat{H}}(-a, \widehat{H}, -r).$$

We conclude that the composition $\Phi_{\widehat{H}^{-1}} \circ [1] \circ \Phi_{\mathcal{P}} \circ \Phi_H$ induces an isomorphism of the moduli spaces

$$X^{[n+1]} \times \widehat{X} \cong \mathcal{M}_H(1,0,-n-1) \stackrel{\cong}{\rightarrow} \mathcal{M}_{\widehat{H}}(1,0,-n-1) \cong \widehat{X}^{[n+1]} \times X \tag{8.1}$$

as well as of the Albanese fibers (the generalized Kummer varieties). An isomorphism of cohomology rings induced via push-forward by an isomorphism is a parallel transport operator, by definition (see footnote 2 in Sec. 1.6). Let $\mathcal{E}_X$ and $\mathcal{E}_{\widehat{X}}$ be the universal ideal sheaves. Setting $g := -\phi_{\widehat{H}}^{-1} \circ \phi_{\mathcal{P}} \circ \phi_H$, we see that (8.1) induces the parallel transport operator $\gamma_g(\mathcal{E}_X, \mathcal{E}_{\widehat{X}})$, and the isomorphism $g \otimes \gamma_g(\mathcal{E}_X, \mathcal{E}_{\widehat{X}})$ maps a universal class of $\mathcal{M}_H(1,0,-n-1)$ to a universal class of $\mathcal{M}_{\widehat{H}}(1,0,-n-1)$, by Lemma 8.1. The latter property is stable under deformation of the pair (X, H), dropping the condition that H is ample. Consequently, we get the following corollary.

Corollary 8.3. *Let F_1 and F_2 be two line bundles with $c_1(F_i)$ primitive and $\chi(F_i) = n$. The left hand side of equality (7.13), and hence its right hand side*

$$g' := \widetilde{m}_{(1,F_2^{-1},n+1)} \circ \widetilde{m}_{(1,F_1^{-1},n+1)},$$

has the property that $\gamma_{g'}(\mathcal{E}_X, \mathcal{E}_X)$ is a monodromy operator which maps a universal class of $\mathcal{M}_H(1,0,-n-1)$ to another universal class of $\mathcal{M}_H(1,0,-n-1)$.

Note that $(1, F_i^{-1}, n+1) = \phi_{F_i}^{-1}(s)$, $i = 1,2$, are two vectors in S_X^+ orthogonal to the Chern character s_{n+1} and of square 2 with respect to $(\bullet,\bullet)_{S_X^+}$, so that g' belongs to $\mathrm{Spin}(S_X^+)_{s_{n+1}}$.

Let the ample line bundle H on X have Euler characteristic $n+2$, $n \geq 2$. Given a length $n+1$ subscheme $Z \subset X$, we get

$$z_{n+1} := \mathrm{ch}(I_Z \otimes H) = (1, H, 1).$$

In particular, z_{n+1} is orthogonal to the -2 vector $s_1 := (1,0,-1)$. Let $D_{\widehat{X}} : D^b(\widehat{X}) \to D^b(\widehat{X})^{\mathrm{op}}$ be the functor taking $E \in D^b(\widehat{X})$ to $E^\vee := R\mathcal{H}om(E, \mathcal{O}_{\widehat{X}})$. Set

$$\mathcal{G}_{\mathcal{P}} := D_{\widehat{X}} \circ \Phi_{\mathcal{P}}[2].$$

Given $E \in D^b(X)$, set

$$\mathcal{G}^i_{\mathcal{P}}(E) := \mathcal{H}^i(\mathcal{G}_{\mathcal{P}}(E)),$$

the i-th sheaf cohomology. If E is a coherent sheaf, then $\mathcal{G}^i_{\mathcal{P}}(E)$ is isomorphic to the relative extension sheaf $\mathcal{E}xt^i_{\pi_{\hat{X}}}(\mathcal{P}\otimes\pi_X^*(E),\mathcal{O}_{X\times\hat{X}})$, by Grothendieck–Verdier duality and the triviality of the relative canonical line bundle $\omega_{\pi_{\hat{X}}}$.

Proposition 8.4 ([56, Proposition 3.2]). *Let E be a μ-stable sheaf with* $\mathrm{ch}(E)=(r,H,a)$*, $a>0$. Then the sheaf cohomology $\mathcal{G}^i_{\mathcal{P}}(E)$ vanishes for $i\neq 2$, and $\mathcal{G}^2_{\mathcal{P}}$ is a μ-stable sheaf with Chern character $(a,c_1(\widehat{H}),r)$. In particular, if H generates* $\mathrm{Pic}(X)$*, then the composition $\mathcal{G}_{\mathcal{P}}[2]$ induces an isomorphism of moduli spaces*

$$\mathcal{M}_H(r,H,a)\xrightarrow{\cong}\mathcal{M}_{\widehat{H}}(a,\widehat{H},r).$$

We conclude that the composition $\Phi_{\widehat{H}^{-1}}\circ\mathcal{G}_{\mathcal{P}}[2]\circ\Phi_H$ induces another isomorphism between the moduli spaces in (8.1).

Denote by

$$\tau_X : A_X\to A_X \tag{8.2}$$

the element of $G(S^+)^{\mathrm{even}}$ in (4.24). Then τ_X acts on S^+ via $D_X:=-R_s\circ R_{s_1}$, where $s=(1,0,1)$ and $s_1=(1,0,-1)$.

Corollary 8.5. (1) *The composition*

$$\gamma_{\phi_{\widehat{H}}^{-1}}(\mathcal{E}_{(1,\widehat{H},1)},\mathcal{E}_{\hat{X}})\circ\gamma_{\phi_{\mathcal{P}},1}(\mathcal{E}_{(1,H,1)},\mathcal{E}_{(1,\widehat{H},1)})\circ\gamma_{\phi_H}(\mathcal{E}_X,\mathcal{E}_{(1,H,1)}) :$$
$$H^*(\mathcal{M}_H(1,0,-1-n),\mathbb{Z})\to H^*(\mathcal{M}_{\widehat{H}}(1,0,-1-n),\mathbb{Z})$$

is equal to $\gamma_{\tau_{\hat{X}}\phi_{\widehat{H}}^{-1}\tau_{\hat{X}}\phi_{\mathcal{P}}\phi_H,1}(\mathcal{E}_X,\mathcal{E}_{\hat{X}})$, is induced by an isomorphism of the moduli spaces, and maps a universal class to the dual of a universal class.

(2) *Let H and F be line bundles on X with $\chi(F)=n$, $\chi(H)=n+2$. Assume that the classes $c_1(F)$ and $c_1(H)$ in $H^2(X,\mathbb{Z})$ are primitive. Set*

$$\psi:=-\tau_X\phi_F^{-1}\phi_{\mathcal{P}}^{-1}\phi_{\hat{F}}\phi_{\widehat{H}}^{-1}\tau_{\hat{X}}\phi_{\mathcal{P}}\phi_H.$$

Then the automorphism $\gamma_{\psi,1}(\mathcal{E}_X,\mathcal{E}_X)$ of $H^*(\mathcal{M}(1,0,-1-n),\mathbb{Z})$ is induced by a monodromy operator.

Proof. (1) Let $f:\mathcal{M}_H(1,0,-1-n)\to\mathcal{M}_{\widehat{H}}(1,0,-1-n)$ be the isomorphism in Proposition 8.4. Let $\eta:\mathcal{M}_H(1,H,1)\to\mathcal{M}_{\widehat{H}}(1,\widehat{H},1)$ be the associated isomorphism. By construction, we have an isomorphism, in the derived category of $\widehat{X}\times M_H(1,H,1)$,

$$(1_{\hat{X}}\times\eta)^*(\mathcal{E}^\vee_{(1,\widehat{H},1)})\cong\Phi_{\mathcal{P}\boxtimes\mathcal{O}_{\Delta_{\mathcal{M}(1,H,1)}}}(\mathcal{E}_{(1,H,1)})$$

for suitably chosen universal sheaves. Consequently, we have the equality

$$\mathrm{ch}(\mathcal{E}^\vee_{(1,\widehat{H},1)})=(\phi_{\mathcal{P}}\otimes\eta_*)(\mathrm{ch}(\mathcal{E}_{(1,H,1)})),$$

and so $\phi_{\mathcal{P}}\otimes\eta_*$ maps a universal class of $\mathcal{M}_H(1,H,1)$ to the dual of a universal class of $\mathcal{M}_{\widehat{H}}(1,\widehat{H},1)$. Hence, $(d_{\hat{X}}^{-1}\phi_{\mathcal{P}})\otimes(d^{-1}_{\mathcal{M}(1,\widehat{H},1)}\eta_*)$ maps a universal class of $\mathcal{M}_H(1,H,1)$

to a universal class of $\mathcal{M}_{\widehat{H}}(1, \widehat{H}, 1)$. Lemma 6.4 implies the equality

$$d^{-1}_{\mathcal{M}(1,\widehat{H},1)}\eta_* = \gamma_{d^{-1}_{\widehat{X}}\phi_{\mathcal{P}},0}(\mathcal{E}_{(1,H,1)}, \mathcal{E}_{(1,\widehat{H},1)}).$$

We get

$$\eta_* = d_{\mathcal{M}(1,\widehat{H},1)}\gamma_{d^{-1}_{\widehat{X}}\phi_{\mathcal{P}},0}(\mathcal{E}_{(1,H,1)}, \mathcal{E}_{(1,\widehat{H},1)}) = \gamma_{d^{-1}_{\widehat{X}}\phi_{\mathcal{P}},0}(\mathcal{E}_{(1,H,1)}, \mathcal{E}_{(1,\widehat{H},1)})d_{\mathcal{M}(1,H,1)}.$$

The right hand side is equal to $\gamma_{\phi_{\mathcal{P}},1}(\mathcal{E}_{(1,H,1)}, \mathcal{E}_{(1,\widehat{H},1)})$, by (6.9). Hence, the graph of η is Poincaré dual to the class $\gamma_{\phi_{\mathcal{P}},1}(\mathcal{E}_{(1,H,1)}, \mathcal{E}_{(1,\widehat{H},1)})$. The composition in the statement of part (1) is equal to $\gamma_{\tau_{\widehat{X}}\phi^{-1}_{\widehat{H}}\tau_{\widehat{X}}\phi_{\mathcal{P}}\phi_H,1}(\mathcal{E}_X, \mathcal{E}_X)$, by Lemma 6.8.

(2) The homomorphism $\gamma_{-\phi^{-1}_F\phi^{-1}_{\mathcal{P}}\phi_{\widehat{F}},0}(\mathcal{E}_{\widehat{X}}, \mathcal{E}_X)$ is a deformation of the isomorphism in Proposition 8.2 and is hence a parallel transport operator. The equality

$$\gamma_{\psi,1}(\mathcal{E}_X, \mathcal{E}_X) = \gamma_{-\phi^{-1}_F\phi^{-1}_{\mathcal{P}}\phi_{\widehat{F}},0}(\mathcal{E}_{\widehat{X}}, \mathcal{E}_X)\gamma_{\tau_{\widehat{X}}\phi^{-1}_{\widehat{H}}\tau_{\widehat{X}}\phi_{\mathcal{P}}\phi_H,1}(\mathcal{E}_X, \mathcal{E}_X)$$

follows from Lemma 6.8. The latter is a composition of a parallel transport operator and an isomorphism, and is hence a parallel transport operator (so a monodromy operator). ∎

Let

$$\mathrm{ort} : G(S^+)^{\mathrm{even}} \to \mathbb{Z}/2\mathbb{Z} \tag{8.3}$$

be the pull-back of the character of $O(S^+)$ corresponding to the orientation of the positive cone in $S^+_{\mathbb{R}}$ and analogous to the character (4.12). The kernel of ort is $\mathrm{Spin}(S^+)$. Let $\mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{Spin}(S^+)$ be the semidirect product with multiplication given by $(\epsilon_1, g_1)(\epsilon_2, g_2) := (\epsilon_1 + \epsilon_2, \tau_X^{\epsilon_2} g_1 \tau_X^{\epsilon_2} g_2)$. We have the isomorphism

$$\lfloor : G(S^+)^{\mathrm{even}} \to \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{Spin}(S^+),$$

given by $\lfloor(g) = (\mathrm{ort}(g), \tau_X^{\mathrm{ort}(g)} g)$. Set $s_n := (1, 0, -n)$, $n \geq 3$. Note that τ_X belongs to $G(S^+)^{\mathrm{even}}_{s_n}$, by Example 4.4. Let $\gamma : \mathbb{Z}/2\mathbb{Z} \rtimes \mathrm{Spin}(S^+) \to \mathrm{End}[H^*(\mathcal{M}(s_n), \mathbb{Z})]$ send (ϵ, g) to the correspondence homomorphism induced by the class $\gamma_{g,\epsilon}(\mathcal{E}_{s_n}, \mathcal{E}_{s_n})$. Set

$$\mathrm{mon} := \gamma \circ \lfloor : G(S^+)^{\mathrm{even}} \to \mathrm{End}[H^*(\mathcal{M}(s_n), \mathbb{Z})]. \tag{8.4}$$

Let $D_{X\times\mathcal{M}(s_n)}$ be the automorphism of $H^{\mathrm{even}}(X \times \mathcal{M}(s_n), \mathbb{Z})$ acting on the group $H^{2i}(X \times \mathcal{M}(s_n), \mathbb{Z})$ by $(-1)^i$. Let $\tau_{\mathcal{M}}$ be the unique solution of the equation

$$1 \otimes \tau_{\mathcal{M}} = (\tau_X \otimes 1) \circ D_{X\times\mathcal{M}(s_n)}. \tag{8.5}$$

Explicitly, $\tau_{\mathcal{M}}$ acts on $H^j(\mathcal{M}(s_n), \mathbb{Z})$ via $(-1)^{j(j+1)/2}$. We get the factorization $D_{X\times\mathcal{M}} = \tau_X \otimes \tau_{\mathcal{M}}$. We get the group homomorphism

$$G(S^+)^{\mathrm{even}}_{s_n} \to G(S^+)^{\mathrm{even}}_{s_n} \times \mathrm{Aut}[H^*(\mathcal{M}(s_n), \mathbb{Z})],$$
$$g \mapsto (\tau_X^{\mathrm{ort}(g)} g \tau_X^{\mathrm{ort}(g)}, \tau_{\mathcal{M}}^{\mathrm{ort}(g)} \mathrm{mon}(g)),$$

whose image consists of pairs mapping a universal class to a universal class, by the following theorem. Set $\tau := \tau_X$.

Theorem 8.6. (1) *The class*

$$\gamma_{\tau^{\mathrm{ort}(g)} g, \mathrm{ort}(g)}(\mathcal{E}_{s_n}, \mathcal{E}_{s_n}) \in H^{4n+4}(\mathcal{M}(s_n) \times \mathcal{M}(s_n), \mathbb{Z})$$

induces a graded ring automorphism for every $g \in G(S^+)^{\mathrm{even}}_{s_n}$. *The resulting map*

$$\mathrm{mon} : G(S^+)^{\mathrm{even}}_{s_n} \to \mathrm{Aut}[H^*(\mathcal{M}(s_n), \mathbb{Z})] \tag{8.6}$$

is a group homomorphism.

(2) *If* $\mathrm{ort}(g) = 0$, *then* $g \otimes \gamma_{g,0}(\mathcal{E}_{s_n}, \mathcal{E}_{s_n})$ *maps a universal class to a universal class.*

(3) *If* $\mathrm{ort}(g) = 1$, *then* $g\tau \otimes \gamma_{\tau g,1}(\mathcal{E}_{s_n}, \mathcal{E}_{s_n})$ *maps a universal class to the dual of a universal class.*

(4) *The image of the homomorphism* mon *in* (8.6) *is contained in the monodromy group* $\mathrm{Mon}(\mathcal{M}(s_n))$.

Proof. (1) The fact that the map mon is a group homomorphism would follow, once the rest of the statements in parts (1)–(3) of the theorem are proven, by Corollary 6.5, and Lemmas 6.7 and 6.8.

We first prove part (1) for the subgroup of $\mathrm{Spin}(S^+)_{s_n}$ stabilizing both $(1,0,0)$ and $(0,0,1)$. This subgroup is the image of $SL(H^1(X,\mathbb{Z}))$ via the inverse of the isomorphism f in Lemma 5.2. A *marked compact complex torus of dimension* 2 is a pair (X', η) consisting of a compact complex torus X' of dimension 2 and an isomorphism $\eta : H^1(X',\mathbb{Z}) \to H^1(X,\mathbb{Z})$, where X is our fixed abelian surface. Two pairs (X_1, η_1) and (X_2, η_2) are *isomorphic* if there exists an isomorphism $f : X_1 \to X_2$ such that $\eta_2 = \eta_1 f^*$. Let $\mathfrak{M}$ be the moduli space of isomorphism classes of marked compact complex 2-dimensional tori. Let $\mathfrak{M}^0$ be the connected component containing (X, id). The group $GL(H^1(X,\mathbb{Z}))$ acts on $\mathfrak{M}$, via $g(X', \eta) = (X', g\eta)$, and the subgroup $SL(H^1(X,\mathbb{Z}))$ leaves $\mathfrak{M}^0$ invariant. We have a universal torus $\pi : \mathcal{X} \to \mathfrak{M}^0$, a relative Douady space $\mathcal{X}^{[n]} \to \mathfrak{M}^0$ of length n zero-dimensional subschemes of fibers of π, a relative dual torus $\hat{\pi} : \hat{\mathcal{X}} \to \mathfrak{M}^0$, and so a relative moduli space $\mathcal{M}_{\mathcal{X}}(s_n) := \hat{\mathcal{X}} \times_{\mathfrak{M}^0} \mathcal{X}^{[n]} \to \mathfrak{M}^0$ of rank 1 torsion free sheaves on fibers of π with Chern character s_n. Furthermore, we have a universal sheaf $\mathcal{E}$ over $\mathcal{X} \times_{\mathfrak{M}^0} \mathcal{M}_{\mathcal{X}}(s_n)$.

Let $\mathcal{M}_X(s_n)$ be the fiber of $\mathcal{M}_{\mathcal{X}}(s_n)$ over (X, id). The group $SL(H^1(X,\mathbb{Z}))$ acts via monodromy operators on $H^*(X,\mathbb{Z})$ and on $H^*(\mathcal{M}_X(s_n),\mathbb{Z})$, by [26, Lemma 6.6]. The Chern character $\mathrm{ch}(\mathcal{E})$ maps to a global flat section of the local system $R\Pi_*\mathbb{Q}$, where $\Pi : \mathcal{X} \times_{\mathfrak{M}^0} \mathcal{M}_{\mathcal{X}}(s_n) \to \mathfrak{M}^0$ is the natural morphism. Hence, $\mathrm{ch}(\mathcal{E})$ restricts to an $SL(H^1(X,\mathbb{Z}))$-invariant class in $H^*(X \times \mathcal{M}_X(s_n))$, under the diagonal monodromy action. The statement of part (1) follows for the image of $SL(H^1(X,\mathbb{Z}))$ in $\mathrm{Spin}(S^+)_{s_n}$, by Lemma 6.4. The statement of part (1) follows for the elements of $\mathrm{Spin}(S^+)_{s_n}$ which are the compositions $\widetilde{m}_{t_1}\widetilde{m}_{t_2}$, where $t_i = (1, A_i, n) \in s_n^{\perp}$, satisfying $(t_i, t_i)_{S^+} = 2$, and A_i is a primitive class in $H^2(X,\mathbb{Z})$, by Corollary 8.3. The statement of part (1) follows for the whole of $\mathrm{Spin}(S^+)_{s_n}$, since the latter is generated by the image of $SL(H^1(X,\mathbb{Z}))$ and compositions $\widetilde{m}_{t_1}\widetilde{m}_{t_2}$ as above, by Lemma 5.4.

Part (2) is evident for elements of $\mathrm{SL}(H^1(X,\mathbb{Z}))$ and it was verified in Corollary 8.3 for the above mentioned compositions $\widetilde{m}_{t_1}\widetilde{m}_{t_2}$, so it follows for all elements of $\mathrm{Spin}(S^+)_{s_n}$.

In the verification of part (3) we use the evident identity $d_X(\tau g)d_X^{-1} = \tau(\tau g)\tau^{-1} = g\tau$ and Definition 6.3 (3). Then Corollary 8.5 verifies part (3) for $g = \tau\psi$, which belongs to $G(S^+)^{\mathrm{even}}_{s_n}$ but not to $\mathrm{Spin}(S^+)_{s_n}$. The result of part (1) thus extends to $G(S^+)^{\mathrm{even}}_{s_n}$.

(4) The statement was verified for elements of $\mathrm{SL}(H^1(X,\mathbb{Z}))$ above, and is established in Corollaries 8.3 and 8.5, hence it is verified for a set of generators of $G(S^+)^{\mathrm{even}}_{s_n}$. ∎

8.3. The half-spin representations as subquotients of the cohomology ring

We identify next $H^2(K_X(n-1),\mathbb{Z})$ as a $G(S_X^+)^{\mathrm{even}}_{s_n}$-module by determining the equivariance property of the isomorphism θ in (3.2). We do this more generally for all degrees as follows. Let $\widetilde{I}_d$ be the ideal in $H^*(\mathcal{M}(s_n),\mathbb{Z})$ generated by $H^i(\mathcal{M}(s_n),\mathbb{Z})$, $1 \le i \le d$, denote by $\widetilde{I}_d^j$ its graded summand of degree j, and set $I := \bigoplus_{d\ge 2}\widetilde{I}^d_{d-1}$. Set $Q(\mathcal{M}(s_n)) := H^*(\mathcal{M}(s_n),\mathbb{Z})/I$. In particular, $Q^1(\mathcal{M}(s_n)) = H^1(\mathcal{M}(s_n),\mathbb{Z})$, and $Q^d(\mathcal{M}(s_n)) = H^d(\mathcal{M}(s_n),\mathbb{Z})/I^d$, where $I^d := \widetilde{I}^d_{d-1}$. For example,

$$
\begin{aligned}
I^2 &= H^1(\mathcal{M}(s_n),\mathbb{Z}) \cup H^1(\mathcal{M}(s_n),\mathbb{Z}),\\
I^3 &= H^1(\mathcal{M}(s_n),\mathbb{Z}) \cup H^2(\mathcal{M}(s_n),\mathbb{Z}),\\
I^4 &= H^1(\mathcal{M}(s_n),\mathbb{Z}) \cup H^3(\mathcal{M}(s_n),\mathbb{Z}) + H^2(\mathcal{M}(s_n),\mathbb{Z}) \cup H^2(\mathcal{M}(s_n),\mathbb{Z}).
\end{aligned}
$$

We have the homomorphism $\widetilde{\theta} : S_X \to H^*(\mathcal{M}(s_n),\mathbb{Q})$, given by

$$
\widetilde{\theta}(\lambda) := \pi_{\mathcal{M},*}[\pi_X^*(\tau_X(\lambda)) \cup \mathrm{ch}(\mathcal{E})]. \tag{8.7}
$$

Set $S_X^1 := S_X^-$, $S_X^2 := S_X^+ \cap s_n^\perp$, and for $j > 2$ let S_X^j be S_X^+ if j is even, and S_X^- if j is odd. Let

$$
\widetilde{\theta}_j : S_X^j \to Q^j(\mathcal{M}(s_n)) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{8.8}
$$

be the composition of $\widetilde{\theta}$ with the inclusion $S_X^j \subset S_X$ and the projection $H^*(\mathcal{M}(s_n),\mathbb{Q}) \to Q^j(\mathcal{M}(s_n)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that $\widetilde{\theta}_j$ is independent of the choice of the universal sheaf, by our definition of S_X^j and $Q^j(\mathcal{M}(s_n))$. Note also that $Q^2(\mathcal{M}(s_n))$ is isomorphic to the second integral cohomology $H^2(K_a(s_n),\mathbb{Z})$ of the generalized Kummer, $\widetilde{\theta}_2$ above is injective and has integral values, and the integral isomorphism θ in (3.2) factors through $\widetilde{\theta}_2$, since $H^1(K_a(s_n),\mathbb{Z})$ vanishes.

The action of any graded ring automorphism of $H^*(\mathcal{M}(s_n),\mathbb{Z})$ descends to an action on $Q^j(\mathcal{M}(s_n))$ for all $j \ge 1$. Similarly, the action of $\tau_{\mathcal{M}}$ given in (8.5) descends to one on $Q^j(\mathcal{M}(s_n))$ for all $j \ge 1$.

Corollary 8.7. *The image of $\widetilde{\theta}_j$, $j \ge 1$, is invariant under the monodromy action of $G(S_X^+)^{\mathrm{even}}_{s_n}$ via the homomorphism* mon *given in (8.6) and the image spans $Q^j(\mathcal{M}(s_n)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Furthermore, for all $\lambda \in S_X^j$ and all $g \in G(S_X^+)^{\mathrm{even}}_{s_n}$ we have*

$$
\mathrm{mon}_g(\widetilde{\theta}_j(\lambda)) = \tau_{\mathcal{M}}^{\mathrm{ort}(g)}[\widetilde{\theta}_j(\tau_X^{\mathrm{ort}(g)} g \tau_X^{\mathrm{ort}(g)}(\lambda))]. \tag{8.9}
$$

In particular, $\widetilde{\theta}_j$ *is* mon*-equivariant with respect to the subgroup* $\mathrm{Spin}(S_X^+)_{s_n}$ *of* $G(S_X^+)_{s_n}^{\mathrm{even}}$.

Proof. The image of $\widetilde{\theta}_j$ spans $Q^j(\mathcal{M}(s_n)) \otimes_{\mathbb{Z}} \mathbb{Q}$, since the Künneth factors of $\mathrm{ch}(\mathcal{E})$ generate $H^*(\mathcal{M}(s_n), \mathbb{Q})$, by [24, Cor. 2]. Let $\{x_1, \ldots, x_{16}\}$ be a basis of S_X with each class either even or odd. Let $\mathrm{ch}(\mathcal{E}) := \sum_i x_i \otimes e_i$ be its Künneth decomposition. Then $\widetilde{\theta}(\lambda) = \sum_i (\lambda, x_i)_{S_X} e_i$. If g belongs to $\mathrm{Spin}(V)_{s_n}$, so that $\mathrm{ort}(g) = 0$, then

$$\begin{aligned}\mathrm{mon}_g(\widetilde{\theta}(g^{-1}(\lambda))) &= \mathrm{mon}_g\Big(\sum_i (g^{-1}(\lambda), x_i)_{S_X} e_i\Big) = \sum_i (\lambda, g(x_i))_{S_X}\, \mathrm{mon}_g(e_i)\\ &= \pi_{\mathcal{M},*}[\pi_X^*(\tau_X(\lambda)) \cup (g \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E}))].\end{aligned}$$

Consequently, we get

$$\mathrm{mon}_g(\widetilde{\theta}(g^{-1}(\lambda))) = \widetilde{\theta}(\lambda)\exp(c_g), \tag{8.10}$$

where the class c_g is given in (6.8), since $(g \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E}))$ is a universal class, by Theorem 8.6. The projection of the right hand side to $Q(\mathcal{M}(s_n)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to that of $\widetilde{\theta}(\lambda)$. The identity (8.9) follows.

Assume $\mathrm{ort}(g) = 1$. Then

$$\begin{aligned}\mathrm{mon}_g(\widetilde{\theta}(\tau_X g^{-1} \tau_X(\lambda))) &= \sum_i (\lambda, \tau_X g \tau_X(x_i))_{S_X}\, \mathrm{mon}_g(e_i)\\ &= \pi_{\mathcal{M},*}[\pi_X^*(\tau_X(\lambda)) \cup \{((\tau_X g \tau_X) \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E}))\}]. \end{aligned}\tag{8.11}$$

Now, $((g\tau_X) \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E}))$ is the dual of a universal class, by Theorem 8.6. So

$$((\tau_X g \tau_X) \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E}))$$

projects to the image of $(1 \otimes \tau_{\mathcal{M}})(\mathrm{ch}(\mathcal{E}))$ in $H^*(X, \mathbb{Z}) \otimes Q(\mathcal{M}(s_n))$. We conclude that the right hand side of (8.11) and $\tau_{\mathcal{M}}(\widetilde{\theta}(\lambda))$ project to the same class in $Q(\mathcal{M}(s_n))$. Hence, so does the left hand side of (8.11). We have thus verified the equation obtained from (8.9) by substituting $(\tau_X g \tau_X)(\lambda)$ for λ. We conclude that (8.9) follows in this case as well. ■

We construct next an integral analogue of the homomorphism (8.8) into $Q^j(\mathcal{M}(s_n))$. Given a topological space M, let $K(M) := K^0(M) \oplus K^1(M)$ be the topological K-ring of M. The Chern character induces a ring isomorphism $\mathrm{ch} : K(M) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^*(M, \mathbb{Q})$ sending $K^0(M)$ into $H^{\mathrm{even}}(M, \mathbb{Q})$ and $K^1(M)$ into $H^{\mathrm{odd}}(M, \mathbb{Q})$ [21, V.3.26]. The Chern classes $c_{j/2}(x) \in H^j(M, \mathbb{Z})$ are defined for an odd integer $j \geq 1$ in [25, Def. 19]. They satisfy the equality

$$\mathrm{ch}_{k-1/2}(x) = (-1)^{k-1}/(k-1)!\,c_{k-1/2}(x) \tag{8.12}$$

for $x \in K^1(\mathcal{M}(s_n))$ and a positive integer k, by [25, Lemma 22 (2)].

For the abelian surface X the Chern character is integral and we get an isomorphism $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Z})$. Integrality follows for $K^0(X)$, since the intersection pairing on $H^2(X, \mathbb{Z})$ is even, and for $K^1(X)$ since the coefficient of the right hand side of (8.12) is ± 1 for $k = 1, 2$. Surjectivity of $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Z})$ follows, since $H^*(X, \mathbb{Z})$ is generated by $H^1(X, \mathbb{Z})$, the latter is spanned by pull-back of classes via maps to a circle S^1, and surjectivity of $\mathrm{ch} : K(S^1) \to H^*(S^1, \mathbb{Z})$ is clear. The Künneth theorem in K-theory yields an isomorphism

$$[K^0(X) \otimes K^0(\mathcal{M}(s_n))] \oplus [K^1(X) \otimes K^1(\mathcal{M}(s_n))] \to K^0(X \times \mathcal{M}(s_n)),$$

by [1, Cor. 2.7.15]. Choose a basis $\{x_1, \dots, x_{16}\}$ of $K(X)$ which is a union of a basis of $K^0(X)$ and a basis of $K^1(X)$. Let $[\mathcal{E}] \in K(X \times \mathcal{M}(s_n))$ be the class of a universal sheaf. We get the Künneth decomposition

$$[\mathcal{E}] = \sum_{i=1}^{16} x_i \otimes e_i$$

with e_i either in $K^0(\mathcal{M}(s_n))$ or $K^1(\mathcal{M}(s_n))$.

Theorem 8.8 ([25, Theorem 1]). *The Chern classes of* $\{e_i : 1 \leq i \leq 16\}$ *generate the integral cohomology ring* $H^*(\mathcal{M}(s_n), \mathbb{Z})$.

Let $\pi_X^! : K(X) \to K(X \times \mathcal{M}(s_n))$ be the pull-back homomorphism and denote by $\pi_{\mathcal{M},!} : K(X \times \mathcal{M}(s_n)) \to K(\mathcal{M}(s_n))$ the Gysin homomorphism. The involution τ_X acts on $H^i(X, \mathbb{Z})$ via $(-1)^{i(i-1)/2}$, as in (4.15), and we denote the involution $\mathrm{ch}^{-1} \circ \tau_X \circ \mathrm{ch} : K(X) \to K(X)$ by τ_X as well. We get the homomorphism $e : K(X) \to K(\mathcal{M}(s_n))$ given by

$$e(\lambda) := \pi_{\mathcal{M},!}(\pi_X^!(\tau_X(\lambda)) \cup [\mathcal{E}]).$$

The Chern classes of $e(x_i)$, $1 \leq i \leq 16$, generate $H^*(\mathcal{M}(s_n), \mathbb{Z})$, by Theorem 8.8.

Let $\bar{c}_j : K(\mathcal{M}(s_n)) \to Q^{2j}(\mathcal{M}(s_n))$ be the composition of the Chern class map $c_j : K(\mathcal{M}(s_n)) \to H^{2j}(\mathcal{M}(s_n), \mathbb{Z})$ with the natural projection $H^{2j}(\mathcal{M}(s_n), \mathbb{Z}) \to Q^{2j}(\mathcal{M}(s_n))$. Then $\bar{c}_j$ is a group homomorphism. This is proven in [27, Prop. 2.6] for an integer $j \geq 0$ and $K^0(\mathcal{M}(s_n))$, and for a half-integer j and $K^1(\mathcal{M}(s_n))$ it follows from (8.12) and the linearity of the Chern character homomorphism, hence of the integral homomorphism $(j - 1/2)!\,\mathrm{ch}_j$.

Let

$$\bar{\theta}_j : S_X^j \to Q^j(\mathcal{M}(s_n)), \quad j \geq 1, \tag{8.13}$$

be the composition of

$$S_X := H^*(X, \mathbb{Z}) \xrightarrow{\mathrm{ch}^{-1}} K(X) \xrightarrow{e} K(\mathcal{M}(s_n)) \xrightarrow{\bar{c}_j} Q^j(\mathcal{M}(s_n)) \tag{8.14}$$

with the inclusion $S_X^j \to S_X$.

Lemma 8.9. *If the ranks of* S_X^j *and* $Q^j(\mathcal{M}(s_n))$ *are equal, then* $\bar{\theta}_j$ *is an isomorphism and* $Q^j(\mathcal{M}(s_n))$ *is torsion free. This is the case for* $1 \leq j \leq 3$.

Proof. The first statement is clear, since the homomorphism $\bar{\theta}_j$ is surjective for all $j \geq 0$, by Theorem 8.8. $Q^1(\mathcal{M}(s_n)) = H^1(\mathcal{M}(s_n), \mathbb{Z})$ and its rank is equal to the first Betti number 8 of the Albanese $X \times \hat{X}$ of $\mathcal{M}(s_n)$. The rank of $Q^2(\mathcal{M}(s_n))$ is equal to that of $H^2(K_X(n-1), \mathbb{Z})$, which is 7 by Theorem 3.1, since $H^1(K_X(n-1), \mathbb{Z})$ vanishes. $Q^3(\mathcal{M}(s_n))$ is isomorphic to $Q^3(X^{[n]})$, since $\mathcal{M}(s_n)$ is isomorphic to $\hat{X} \times X^{[n]}$ and $H^*(\hat{X}, \mathbb{Z})$ is generated by $H^1(\hat{X}, \mathbb{Z})$. Now $b_3(X^{[n]}) = 40$ by Göttsche's formula [10, Cor. 2.3.13], and

$$\begin{aligned} 8 = \operatorname{rank}(S^-) &\geq \operatorname{rank}(Q^3(X^{[n]})) \geq 40 - b_3(X) - \operatorname{rank}(Q^2(X^{[n]}))b_1(X) \\ &= 40 - 4 - 7 \times 4 = 8. \end{aligned}$$

Hence, the ranks of S^- and $Q^3(\mathcal{M}(s_n))$ are equal. ∎

Lemma 8.10. *The composition of $\bar{\theta}_{2j}$ with the natural homomorphism $Q^{2j}(\mathcal{M}(s_n)) \to Q^{2j}(\mathcal{M}(s_n)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to $(-1)^{j-1}(j-1)!\tilde{\theta}_{2j}$ if j is an integer, and to*

$$(-1)^{j-1/2}(j-1/2)!\tilde{\theta}_{2j}$$

if j is half an odd integer. Consequently, the integral version of (8.9), with $\tilde{\theta}_k$ replaced by $\bar{\theta}_k$ holds whenever $Q^k(\mathcal{M}(s_n))$ is torsion free, and in particular for $k \in \{1, 2, 3\}$.

Proof. We have the commutative diagram

$$\begin{array}{ccccccccc} K(X) & \xrightarrow{\tau_X} & K(X) & \xrightarrow{\pi_X^!} & K(X \times \mathcal{M}) & \xrightarrow{\cup[\mathcal{E}]} & K(X \times \mathcal{M}) & \xrightarrow{\pi_{\mathcal{M},!}} & K(\mathcal{M}) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(X, \mathbb{Z}) & \xrightarrow{\tau_X} & H^*(X, \mathbb{Z}) & \xrightarrow{\pi_X^*} & H^*(X \times \mathcal{M}, \mathbb{Q}) & \xrightarrow{\cup \mathrm{ch}(\mathcal{E})} & H^*(X \times \mathcal{M}, \mathbb{Q}) & \xrightarrow{\pi_{\mathcal{M},*}} & H^*(\mathcal{M}, \mathbb{Q}) \end{array}$$

The first (left) square commutes by definition of the top τ_X, the second and third by well known properties of the Chern character, and the fourth by the topological version of Grothendieck–Riemann–Roch and the triviality of the Todd class of X. Finally, let $\bar{\mathrm{ch}}_j : K(\mathcal{M}) \to Q^{2j}(\mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the composition of ch_j with the quotient homomorphism $H^{2j}(\mathcal{M}, \mathbb{Q}) \to Q^{2j}(\mathcal{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\bar{c}_j = (-1)^{j-\epsilon}(j-\epsilon)!\,\bar{\mathrm{ch}}_j$, where $\epsilon = 1$ if j is an integer, and $\epsilon = 1/2$ if j is half an odd integer (see for example [25, Lemma 22]). ∎

9. Four groupoids

We extend in Corollary 9.4 the representation of $G(S^+)^{\text{even}}_{s_n}$ in the monodromy group of the moduli space $\mathcal{M}(s_n)$ of rank 1 sheaves, given in Theorem 8.6, to more general moduli spaces. This is achieved by extending the monodromy group symmetry of a single moduli space to a symmetry of the collection of all smooth and compact moduli spaces $\mathcal{M}_H(w)$ of stable sheaves on abelian surfaces with respect to a groupoid, whose morphisms are parallel transport operators. In Corollary 9.6 we construct a $\mathrm{Spin}(S^+)_w$-equivariant homomorphism from the cohomology $H^*(\mathcal{M}_H(w), \mathbb{Z})$ of a moduli space of sheaves on an abelian surface X to $H^*(X \times \hat{X}, \mathbb{Z})$.

A *groupoid* is a category whose morphisms are all isomorphisms. Let $\mathcal{G}_1$ be the groupoid whose objects are abelian varieties and whose morphisms $\mathrm{Hom}_{\mathcal{G}_1}(X,Y)$ are objects $\mathcal{E}$ in $D^b(X\times Y)$ such that the integral transform $\Phi_{\mathcal{E}}: D^b(X)\rightarrow D^b(Y)$ with kernel $\mathcal{E}$ is an equivalence of triangulated categories. Composition of morphisms corresponds to convolution of the objects. Given abelian varieties X and Y of the same dimension, let $U(X\times\widehat{X},Y\times\widehat{Y})$ be the set of isomorphisms $f: X\times\widehat{X}\rightarrow Y\times\widehat{Y}$ such that $f^*: H^1(Y\times\widehat{Y},\mathbb{Z})\rightarrow H^1(X\times\widehat{X},\mathbb{Z})$ induces an isometry with respect to the symmetric bilinear pairing (4.1).[10] Let $\mathcal{G}_2$ be the groupoid whose objects are abelian varieties and whose morphisms $\mathrm{Hom}_{\mathcal{G}_2}(X,Y)$ are isomorphisms $f: X\times\widehat{X}\rightarrow Y\times\widehat{Y}$ in the subset $U(X\times\widehat{X},Y\times\widehat{Y})$. Composition of morphisms is the usual composition of isomorphisms. The following combines results of Orlov and Polishchuk.

Theorem 9.1 ([17, Prop. 9.39, Exercise 9.41, and Prop. 9.48]). *There exists an explicit full functor $f: \mathcal{G}_1\rightarrow\mathcal{G}_2$ sending the object X to itself and associating to an equivalence $\Phi_{\mathcal{E}}: D^b(X)\rightarrow D^b(Y)$ with kernel $\mathcal{E}\in D^b(X\times Y)$ an isomorphism $f_{\mathcal{E}}: X\times\widehat{X}\rightarrow Y\times\widehat{Y}$ in $U(X\times\widehat{X},Y\times\widehat{Y})$.*

Let $\mathcal{G}_3$ be the groupoid whose objects are triples (X,w,H), where X is an abelian surface, $w\in S_X^+$ is a primitive Mukai vector, and H is a w-generic polarization, such that the moduli space $\mathcal{M}_H(w)$ has dimension ≥ 4. Morphisms in $\mathrm{Hom}_{\mathcal{G}_3}[(X_1,w_1,H_1),(X_2,w_2,H_2)]$ are pairs (g,γ), where $g: H^*(X_1,\mathbb{Z})\rightarrow H^*(X_2,\mathbb{Z})$ is an isometry, with respect to the pairings (4.15), preserving the parity of the grading and satisfying $g(w_1)=w_2$, and γ is a graded ring isomorphism

$$\gamma: H^*(\mathcal{M}_{H_1}(w_1),\mathbb{Z})\rightarrow H^*(\mathcal{M}_{H_2}(w_2),\mathbb{Z}).$$

Each pair (g,γ) is assumed to be the composition $(g_k,\gamma_k)\circ(g_{k-1},\gamma_{k-1})\circ\cdots\circ(g_1,\gamma_1)$, where (g_i,γ_i) is of one of three types. Type 1: $g_i: H^*(X_i,\mathbb{Z})\rightarrow H^*(X_{i+1},\mathbb{Z})$ is induced by an equivalence $\Phi_{\mathcal{E}_i}: D^b(X_i)\rightarrow D^b(X_{i+1})$ of triangulated categories which maps H_i-stable sheaves with Mukai vector w_i to H_{i+1}-stable sheaves with Mukai vector w_{i+1}, and γ_i is induced by an isomorphism $\widetilde{\gamma}_i: \mathcal{M}_{H_i}(w_i)\rightarrow\mathcal{M}_{H_{i+1}}(w_{i+1})$ of moduli spaces, which is in turn induced by $\Phi_{\mathcal{E}_i}$ (see Section 8.1). Type 2: $g_i: H^*(X_i,\mathbb{Z})\rightarrow H^*(X_{i+1},\mathbb{Z})$ and $\gamma_i: H^*(\mathcal{M}_{H_i}(w_i),\mathbb{Z})\rightarrow H^*(\mathcal{M}_{H_{i+1}}(w_{i+1}),\mathbb{Z})$ are parallel transport operators associated to a continuous path from a point b_0 to a point b_1 in the complex analytic base B of a smooth family $\Pi: \mathcal{M}\rightarrow B$ of moduli spaces of stable sheaves corresponding to a family $\pi: \mathcal{X}\rightarrow B$ of abelian surfaces, and a section w of $R^{\mathrm{even}}\pi_*\mathbb{Z}$ of Hodge type, as well as a section h of $R^2\pi_*\mathbb{Z}$ of Hodge type (not necessarily continuous) such that $h(b)$ is a $w(b)$-generic polarization on the fiber X_b of π, and the fiber $\mathcal{M}_b$ of Π is a smooth and projective moduli space $\mathcal{M}_{h(b)}(w(b))$ of $h(b)$-stable sheaves over X_b for all $b\in B$. An isomorphism is chosen between X_i and X_{b_0}, mapping w_1 and H_1 to $w(b_0)$ and $h(b_0)$. An isomorphism is chosen between X_{i+1} and X_{b_1} with the analogous properties. The chosen isomorphisms yield isomorphisms between $\mathcal{M}_{b_0}$ and $\mathcal{M}_{H_i}(w_i)$ and between $\mathcal{M}_{b_1}$ and

[10] See [17, Def. 9.46] for a matrix form characterization of elements of $U(X\times\widehat{X},Y\times\widehat{Y})$.

$\mathcal{M}_{H_{i+1}}(w_{i+1})$. Type 3: Analogous to type 2 but the Mukai vectors are all $(1,0,-n)$, the data h is dropped, the family $\pi : \mathcal{X} \to B$ is a smooth and proper family of 2-dimensional complex tori, and each fiber $\mathcal{M}_b$ is the product $X_b^{[n]} \times \hat{X}_b$, where the first factor is the Douady space of length n subschemes.

Remark 9.2. Note that in each of the three types of morphisms (g_i, γ_i) above, $g_i \otimes \gamma_i$ maps a universal class to a universal class, in the sense of Definition 6.3. This is obvious for types 2 and 3, and for type 1 it follows from Lemma 8.1. Consequently, the same holds for their composition (g,γ), by Corollary 6.5. In particular, a morphism (g,γ) is determined already by g, since $\gamma = \gamma_g(\mathcal{E}_{w_1}, \mathcal{E}_{w_2})$, by Lemma 6.4. Whenever non-empty, $\mathrm{Hom}_{\mathcal{G}_3}[(X, s_n, H), (X, w, H)]$ is a right $\mathrm{Aut}_{\mathcal{G}_3}(X, s_n, H)$-torsor and a left $\mathrm{Aut}_{\mathcal{G}_3}(X, w, H)$-torsor. $\mathrm{Aut}_{\mathcal{G}_3}(X, s_n, H)$ contains $\mathrm{Spin}(S_X^+)_{s_n}$ for $s_n := (1,0,-n)$ and $n \geq 3$, by Theorem 8.6.

The main result of [56] may be stated as follows.

Theorem 9.3. *The set* $\mathrm{Hom}_{\mathcal{G}_3}[(X_1, w_1, H_1), (X_2, w_2, H_2)]$ *is non-empty for any two objects* (X_i, w_i, H_i), $i = 1, 2$, *of* $\mathcal{G}_3$ *with* $(w_1, w_1)_{S^+_{X_1}} = (w_2, w_2)_{S^+_{X_2}}$.

Let (X, w, H) be an object of $\mathcal{G}_3$ with $(w,w)_{S_X^+} = -2n$, $n \geq 3$, and denote by $e_w \in K(X \times \mathcal{M}_H(w))$ the class of a possibly twisted universal sheaf.

Corollary 9.4. *The class* $\gamma_{\tau_X^{\mathrm{ort}(g)} g, \mathrm{ort}(g)}(e_w, e_w) \in H^{4n+4}(\mathcal{M}_H(w) \times \mathcal{M}_H(w), \mathbb{Z})$ *induces a monodromy operator* $\mathrm{mon}(g) \in \mathrm{Aut}(H^*(\mathcal{M}_H(w), \mathbb{Z}))$ *for every* $g \in G(S_X^+)_w^{\mathrm{even}}$. *The resulting map*

$$\mathrm{mon} : G(S_X^+)_w^{\mathrm{even}} \to \mathrm{Aut}(H^*(\mathcal{M}_H(w), \mathbb{Z}))$$

is a group homomorphism. The analogues of statements (2) *and* (3) *of Theorem* 8.6 *hold as well.*

Proof. Choose a morphism $(g,\gamma) \in \mathrm{Hom}_{\mathcal{G}_3}[(X_1, s_n, H_1), (X, w, H)]$, where $n = -(w,w)_{S_X^+}/2$. Such a morphism exists by Theorem 9.3. Let e_{s_n} be the class of a universal sheaf over $X_1 \times \mathcal{M}_{H_1}(s_n)$. Then $\gamma = \gamma_g(e_{s_n}, e_w)$, by Remark 9.2. Now $g : S_{X_1} \to S_X$ is the composition of parallel transport operators and isomorphisms induced by equivalences of derived categories of abelian surfaces and $g(s_n) = w$. Thus, g conjugates $G(S^+_{X_1})^{\mathrm{even}}_{s_n}$ to $G(S^+_X)^{\mathrm{even}}_w$. Given $f \in G(S^+_{X_1})^{\mathrm{even}}_{s_n}$, let $\mathrm{mon}(f) \in \mathrm{Mon}(\mathcal{M}_{H_1}(s_n))$ be the monodromy operator of Theorem 8.6. Then $\mathrm{mon}(f) = \gamma_{\tau_{X_1}^{\mathrm{ort}(f)} f, \mathrm{ort}(f)}(e_{s_n}, e_{s_n})$. The conjugate $\gamma \circ \mathrm{mon}(f) \circ \gamma^{-1}$ is equal to $\gamma_{\tau_X^{\mathrm{ort}(f)} gfg^{-1}, \mathrm{ort}(f)}(e_w, e_w)$, by Lemma 6.8. The latter is just $\mathrm{mon}(gfg^{-1})$, since $\mathrm{ort}(gfg^{-1}) = \mathrm{ort}(f)$. Hence, the map mon of the current corollary is the conjugate via γ and g of the homomorphism mon of Theorem 8.6:

$$\mathrm{mon}(h) = \gamma \circ \mathrm{mon}(g^{-1}hg) \circ \gamma^{-1}$$

for every $h \in G(S_X^+)_w^{\mathrm{even}}$. It is thus a group homomorphism into the monodromy group. If $\mathrm{ort}(h) = 0$, then $h \otimes \mathrm{mon}(h)$ maps a universal class to a universal class, since $g^{-1}hg \otimes \mathrm{mon}(g^{-1}hg)$ and $g \otimes \gamma$ do. The case $\mathrm{ort}(h) = 1$ is similar. ∎

Let $\mathcal{G}_4$ be the groupoid whose objects are 2-dimensional compact complex tori X, and let $\mathrm{Hom}_{\mathcal{G}_4}(X, Y)$ consist of ring isomorphisms $f : H^*(X \times \widehat{X}, \mathbb{Z}) \to H^*(Y \times \widehat{Y}, \mathbb{Z})$, each of which is the composition $f_k \circ f_{k-1} \circ \cdots \circ f_1$ of a sequence of isomorphisms $f_i : H^*(X_i \times \widehat{X}_i, \mathbb{Z}) \to H^*(X_{i+1} \times \widehat{X}_{i+1}, \mathbb{Z})$ of one of two types. Type 1: X_i and X_{i+1} are projective and f_i is induced by an isomorphism $\widetilde{f} : X_i \times \widehat{X}_i \to X_{i+1} \times \widehat{X}_{i+1}$ in $U(X_i \times \widehat{X}_i, X_{i+1} \times \widehat{X}_{i+1})$. Type 2: f_i is the parallel transport operator associated to a continuous path from a point b_0 to a point b_1 in the base B of a smooth and proper family $\pi : \mathcal{X} \to B$ of 2-dimensional compact complex tori. Isomorphisms are chosen between $\mathcal{X}_{b_0}$ is and X_i and between $\mathcal{X}_{b_1}$ and X_{i+1}.

We define next a functor $F : \mathcal{G}_3 \to \mathcal{G}_4$ as follows. F sends the object (X, w, H) to X. F sends a morphism $(g, \gamma) : (X_1, w_1, H_1) \to (X_2, w_2, H_2)$ of type 1, corresponding to a Fourier–Mukai transformation $\Phi_{\mathcal{E}} : D^b(X_1) \to D^b(X_2)$ with kernel $\mathcal{E} \in D^b(X_1 \times X_2)$, to the isomorphism $f_{\mathcal{E},*} : H^*(X_1 \times \widehat{X}_1, \mathbb{Z}) \to H^*(X_2 \times \widehat{X}_2, \mathbb{Z})$ induced by the isomorphism $f_{\mathcal{E}} : X_1 \times \widehat{X}_1 \to X_2 \times \widehat{X}_2$ of Theorem 9.1. Morphisms of types 2 and 3 in $\mathcal{G}_3$ are associated to continuous paths in the bases of families of 2-dimensional complex tori X_b and F sends these to the associated parallel transport operators of the fourfolds $X_b \times \widehat{X}_b$.

Let Alg be the category of commutative algebras with a unit. Let $\Psi : \mathcal{G}_3 \to \mathrm{Alg}$ be the functor which sends an object (X, w, H) to $H^*(\mathcal{M}_H(w), \mathbb{Z})$. The functor Ψ sends a morphism (g, γ) in $\mathcal{G}_3$ to γ. Let $\Sigma : \mathcal{G}_4 \to \mathrm{Alg}$ be the functor which sends X to $H^*(X \times \widehat{X}, \mathbb{Z})$. The functor Σ sends a morphism in $\mathcal{G}_4$ to itself. We get a second functor $\Sigma \circ F$ from $\mathcal{G}_3$ to Alg.

Given an object (X, w, H) of $\mathcal{G}_3$ and a generic H-stable coherent sheaf F on X of Mukai vector w, we get the embedding

$$\iota_F : X \times \widehat{X} \to \mathcal{M}_H(w) \tag{9.1}$$

given by $\iota_F(x, L) = \tau_{x,*}(F) \otimes L$, where $L \in \widehat{X}$, $x \in X$, and $\tau_x : X \to X$ sends x' to $x + x'$. We postpone the proof that ι_F is an embedding to Lemma 10.1 (1). The homomorphism $\iota_F^* : H^*(\mathcal{M}_H(w), \mathbb{Z}) \to H^*(X \times \widehat{X}, \mathbb{Z})$ is independent of the choice of such a generic F, as the data ι_F^* is discrete and depends continuously on F. We thus denote ι_F^* also by

$$q_w : H^*(\mathcal{M}_H(w), \mathbb{Z}) \to H^*(X \times \widehat{X}, \mathbb{Z}). \tag{9.2}$$

Proposition 9.5. *The assignment $(X, w, H) \mapsto q_w$ defines a natural transformation q from Ψ to $\Sigma \circ F$.*

Proof. Given an isomorphism $\gamma : H^*(\mathcal{M}_{H_1}(w_1), \mathbb{Z}) \to H^*(\mathcal{M}_{H_2}(w_2), \mathbb{Z})$ corresponding to a morphism (g, γ) in $\mathrm{Hom}_{\mathcal{G}_3}[(X_1, w_1, H_1), (X_2, w_2, H_2)]$ and H_i-stable sheaves F_i on X_i with Mukai vectors w_i, $i = 1, 2$, we need to prove that the following diagram is commutative:

$$\begin{array}{ccc} H^*(\mathcal{M}_{H_1}(w_1), \mathbb{Z}) & \xrightarrow{\gamma} & H^*(\mathcal{M}_{H_2}(w_2), \mathbb{Z}) \\ \iota_{F_1}^* \downarrow & & \downarrow \iota_{F_2}^* \\ H^*(X_1 \times \widehat{X}_1, \mathbb{Z}) & \xrightarrow{\Sigma(F(g,\gamma))} & H^*(X_2 \times \widehat{X}_2, \mathbb{Z}) \end{array}$$

The commutativity for morphisms (g, γ) of type 2 and 3 is obvious. Consider next the case where (g, γ) is of type 1, associated to a stability preserving Fourier–Mukai transformation $\Phi_{\mathcal{E}} : D^b(X_1) \to D^b(X_2)$, which induces an isomorphism

$$\widetilde{\gamma} : \mathcal{M}_{H_1}(w_1) \to \mathcal{M}_{H_2}(w_2)$$

of the two moduli spaces, and we choose F_2 to be a sheaf representing the object $\Phi_{\mathcal{E}}(F_1)$. The commutativity of the above diagram follows from the commutativity of the diagram

$$\begin{array}{ccccc} X_1 \times \widehat{X}_1 & \longrightarrow & \mathrm{Aut}(\mathcal{M}_{H_1}(v_1)) & & g \\ {\scriptstyle f_{\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_{\widetilde{\gamma}}} & & \downarrow \\ X_2 \times \widehat{X}_2 & \longrightarrow & \mathrm{Aut}(\mathcal{M}_{H_2}(v_2)) & & \widetilde{\gamma} g \widetilde{\gamma}^{-1} \end{array}$$

The commutativity of the latter diagram follows from the analogous commutativity when we regard $X_i \times \widehat{X}_i$ as a subgroup of the group of auto-equivalences of the derived categories $D^b(X_i)$ of X_i and regard $\mathcal{M}_{H_i}(v_i)$ as a subset of objects in $D^b(X_i)$ for $i = 1, 2$ (see [17, Cor. 9.58]). ∎

The group $\mathrm{Spin}(S_X^+)_{s_n}$ acts on $H^*(\mathcal{M}_H(s_n), \mathbb{Z})$ via the monodromy representation mon in (8.6). Corollary 9.4 implies that $\mathrm{Spin}(S_X^+)_w$ similarly acts on $H^*(\mathcal{M}_H(w), \mathbb{Z})$. The group $H^1(X \times \widehat{X}, \mathbb{Z})$ is the representation V_X of $\mathrm{Spin}(S_X^+)$ and so $\mathrm{Spin}(S_X^+)_w$ acts on $H^*(X \times \widehat{X}, \mathbb{Z}) \cong \bigwedge^* V_X$.

Corollary 9.6. *The homomorphism $q_w : H^*(\mathcal{M}_H(w), \mathbb{Z}) \to H^*(X \times \widehat{X}, \mathbb{Z})$, given in (9.2), is $\mathrm{Spin}(S_X^+)_w$ equivariant.*

Proof. The proof of Theorem 8.6 exhibits the image of $\mathrm{Spin}(S_X^+)_{s_n}$ via mon as a subgroup of the automorphism group $\mathrm{Aut}_{\mathcal{G}_3}(X, s_n, H)$. Conjugating by a morphism in $\mathrm{Hom}_{\mathcal{G}_3}[(X, s_n, H), (X, w, H)]$ we find that $\mathrm{Spin}(S_X^+)_w$ is a subgroup of $\mathrm{Aut}_{\mathcal{G}_3}(X, w, H)$, for every object (X, w, H) of $\mathcal{G}_3$, by Corollary 9.4. Now q_w is $\mathrm{Aut}_{\mathcal{G}_3}(X, w, H)$-equivariant, by Proposition 9.5. ∎

Let $\bar{\theta}_1 : S_X^- \to H^1(\mathcal{M}(w), \mathbb{Z})$ be the isomorphism given in (8.13). Let $m : S_X^+ \to \mathrm{Hom}(S_X^-, V_X)$ be the homomorphism given in Corollary 4.7.

Lemma 9.7. *The composition $q_w \circ \bar{\theta}_1 : S_X^- \to H^1(X \times \widehat{X}, \mathbb{Z}) = V_X$ is either m_w or $-m_w$.*

Proof. Both q_w and $\bar{\theta}_1$ are $\mathrm{Spin}(S_X^+)_w$-equivariant, and thus so is their composition. The homomorphism q_w is equivariant, by Corollary 9.6. Equivariance of $\bar{\theta}_1$ is proven in Lemma 8.10 when $w = s_n$ and the proof goes through in the general case, once we replace Theorem 8.6 by Corollary 9.4. Hence, $q_w \circ \bar{\theta}_1$ is a multiple km_w for some integer k. It remains to prove that $|k| = 1$. It suffices to prove it for $w = s_n$, by Theorem 9.3. It suffices to prove that the cardinality of $\mathrm{coker}(q_{s_n})$ is equal to that of the group Γ_{s_n} of n-torsion points of X, by Remark 4.3 and the fact that $\widetilde{\theta}_1$ is an isomorphism. The cokernel of q_{s_n} is equal to the cokernel of the composition

$$H^1(\mathrm{Alb}(\mathcal{M}(s_n)),\mathbb{Z}) \to H^1(\mathcal{M}(s_n),\mathbb{Z}) \stackrel{\iota_F^*}{\longrightarrow} H^1(X\times\widehat{X},\mathbb{Z}),$$

since the left arrow is an isomorphism and the right is q_{s_n}. Now, $\mathrm{Alb}(\mathcal{M}(s_n)) = X\times\widehat{X}$ and the above displayed composition is the pull-back by the homomorphism $X\times\widehat{X}\to X\times\widehat{X}$ corresponding to multiplication by n on the first factor and the identity on the second, whose kernel is Γ_{s_n} (see [11, proof of Th. 7]). ∎

10. The monodromy of a generalized Kummer

We prove Theorem 1.4, about $\mathrm{Mon}^2(Y)$ for an irreducible holomorphic symplectic manifold Y deformation equivalent to a generalized Kummer, in Section 10.1. In Section 10.2 we relate the Lie algebra of the Zariski closure of the monodromy integral Spin(7)-representation we constructed on the cohomology of Y to an action of a Lie algebra constructed by Verbitsky. We use it to show that Spin(7)-invariant classes are Hodge classes (Lemma 10.8).

10.1. The monodromy action on the translation-invariant subring

Let $\mathcal{M}(v) := \mathcal{M}_H(v)$ be a smooth and compact moduli space of H-stable sheaves of primitive Mukai vector v of dimension $m\geq 8$ over an abelian surface X. The Albanese variety $\mathrm{Alb}^0(\mathcal{M}(v))$ is the connected component of the identity in the larger group $D(\mathcal{M}(v))$, the Deligne cohomology group of $\mathcal{M}(v)$ (see [8]). They fit in the exact sequence

$$0\to \mathrm{Alb}^0(\mathcal{M}(v))\to D(\mathcal{M}(v))\to H^{m,m}(\mathcal{M}(v),\mathbb{Z})\to 0.$$

Denote by $\mathrm{Alb}^d(\mathcal{M}(v))$ the connected component of Deligne cohomology mapping to d times the class Poincaré dual to the class of a point in $H^{m,m}(\mathcal{M}(v),\mathbb{Z})$. Let

$$\mathrm{alb}:\mathcal{M}(v)\to \mathrm{Alb}^1(\mathcal{M}(v))$$

be the Albanese morphism. The abelian fourfold $A := X\times \mathrm{Pic}^0(X)$ acts on $\mathcal{M}(v)$. Given a point $F\in\mathcal{M}(v)$ the action yields the morphisms $\iota_F : X\times\mathrm{Pic}^0(X)\to\mathcal{M}(v)$ and $\mathrm{alb}\circ\iota_F : A\to \mathrm{Alb}^1(\mathcal{M}(v))$. Define the morphism $\bar{q}: A\to\mathrm{Alb}^0(\mathcal{M}(v))$ by

$$\bar{q}(g) := (\mathrm{alb}\circ\iota_F)(g) - \mathrm{alb}(F),$$

where the difference is defined, since $\mathrm{Alb}^1(\mathcal{M}(v))$ is an $\mathrm{Alb}^0(\mathcal{M}(v))$-torsor. Then $\bar{q}$ is a group homomorphism, since every morphism of abelian varieties mapping the identity to the identity is a group homomorphism. The morphism $\bar{q}$ is independent of the point F of $\mathcal{M}(v)$, since it depends on F continuously and varies in a discrete group. We have

$$\begin{aligned}(\mathrm{alb}\circ\iota_F)(g_1+g_2) &= \bar{q}(g_1+g_2)+\mathrm{alb}(F) = \bar{q}(g_1)+[\bar{q}(g_2)+\mathrm{alb}(F)]\\ &= \bar{q}(g_1)+(\mathrm{alb}\circ\iota_F)(g_2).\end{aligned}$$

Thus, the action fits in the commutative diagram

$$\begin{array}{ccc} A\times \mathcal{M}(v) & \xrightarrow{\quad\lambda\quad} & \mathcal{M}(v) \\ \downarrow{\scriptstyle \bar{q}\times \mathrm{alb}} & & \downarrow{\scriptstyle \mathrm{alb}} \\ \mathrm{Alb}^0(\mathcal{M}(v))\times \mathrm{Alb}^1(\mathcal{M}(v)) & \xrightarrow{\bar{\lambda}} & \mathrm{Alb}^1(\mathcal{M}(v)) \end{array} \tag{10.1}$$

where λ and $\bar{\lambda}$ are the action morphisms. The *anti-diagonal action* of $a\in A$ on $A\times\mathcal{M}(v)$ sends (b,F) to $(b-a,\lambda(a,F))$.

Choose a point $a\in \mathrm{Alb}^1(\mathcal{M}(v))$ and denote by $K_a(v)$ the fiber of alb over a. Let $\iota_a : K_a(v)\to \mathcal{M}(v)$ be the inclusion. The pull-back homomorphism $\iota_a^* : H^i(\mathcal{M}(v),\mathbb{Z})\to H^i(K_a(v),\mathbb{Z})$ factors through a homomorphism

$$h_i : Q^i(\mathcal{M}(v))\to H^i(K_a(v),\mathbb{Z}) \tag{10.2}$$

for $i=2,3$, since $H^1(K_a(v),\mathbb{Z})$ vanishes. Furthermore, h_2 is an isomorphism, by Theorem 3.1, and h_3 is injective with finite cokernel, since $Q^3(\mathcal{M}(v))$ is torsion free, by Lemma 8.9, and pull-back by the covering map $q : A\times K_a(v)\to\mathcal{M}(v)$ induces an isomorphism $q^* : H^3(\mathcal{M}(v),\mathbb{Q})\to H^3(A\times K_a(v),\mathbb{Q})$, by Göttsche's formula for the Betti numbers of $K_a(v)$ [10, Prop. 2.4.12].

Lemma 10.1. (1) *The morphism ι_F is an embedding for generic F. In particular, the abelian fourfold $A := X\times \mathrm{Pic}^0(X)$ acts faithfully on $\mathcal{M}(v)$.*

(2) *The composition*

$$S_X^- \xrightarrow[\cong]{\tilde{\theta}_1} H^1(\mathcal{M}(v),\mathbb{Z}) \xrightarrow[\cong]{(\mathrm{alb}^*)^{-1}} H^1(\mathrm{Alb}^0(\mathcal{M}(v)),\mathbb{Z}) \xrightarrow{\bar{q}^*} H^1(A,\mathbb{Z})\cong V$$

is m_v or $-m_v$. In particular, the kernel of the homomorphism $\bar{q} : A\to \mathrm{Alb}^0(\mathcal{M}(v))$ is the subgroup Γ_v of Remark 4.3. Consequently, Γ_v acts on each fiber $K_a(v)$ of the Albanese morphism.

(3) *$\mathcal{M}(v)$ is isomorphic to the quotient of $A\times K_a(v)$ by the anti-diagonal action of Γ_v.*

(4) *Γ_v acts trivially on $H^i(K_a(v),\mathbb{Z})$, $i=2,3$, but embeds in* $\mathrm{Mon}(K_a(v))$. *The image of Γ_v in* $\mathrm{Mon}(K_a(v))$ *is characterized as the subgroup of* $\mathrm{Mon}(K_a(v))$ *acting trivially on $H^i(K_a(v),\mathbb{Z})$, $i=2,3$.*

Proof. Part (2) was established in Lemma 9.7. Let s_n be the Mukai vector $(1,0,-n)$ of the ideal sheaf of a length n subscheme of X. Parts (1) and (3) are known when $v=s_n$: see for example [11, proof of Th. 7]. These statements follow from the case of s_n whenever there exists a Fourier–Mukai equivalence $\Phi : D^b(X)\to D^b(X')$ of the derived categories mapping the Mukai vector v to s_n and inducing an isomorphism between the moduli spaces $\mathcal{M}(v)$ and $\mathcal{M}(s_n)$. The fact that the action of $X\times\mathrm{Pic}^0(X)$ on $\mathcal{M}(v)$ conjugates to that of $X'\times\mathrm{Pic}^0(X')$ on $\mathcal{M}(s_n)$ follows from Orlov's characterization of $X\times\mathrm{Pic}^0(X)$ as the connected component of the identity of the subgroup of the group of auto-equivalences of $D^b(X)$ which act trivially on the cohomology of X [17, Cor. 9.57]. The statements follow for a general moduli space $\mathcal{M}(v)$ as above by Yoshioka's proof that $\mathcal{M}(v)$ is connected to $\mathcal{M}(s_n)$ via a sequence of stability preserving Fourier–Mukai transformations (inducing isomorphisms of moduli spaces) and deformations of the abelian surface (Theorem 9.3).

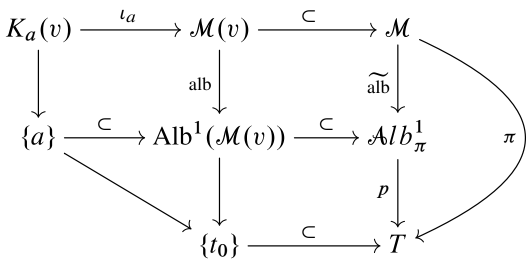
Part (4) again reduces to the case $v = s_n$. When $v = s_n$, the abelian fourfold $X \times \mathrm{Pic}^0(X)$ naturally acts on $\mathcal{M}(v)$, and the subgroup Γ_X, of torsion points of X of order n, is a subgroup of the first factor, which coincides with the subgroup Γ_{s_n} of Remark 4.3. Write $a = (x_0, L)$, $x_0 \in X$ and $L \in \mathrm{Pic}^0(X)$. Let ν_a be the automorphism of $K_a(s_n)$ induced by pull-back of sheaves on X by the automorphism $x \mapsto 2x_0 - x$ of X, followed by tensorization by L^2. The group Γ_X acts on $K_a(s_n)$ by translation and is equal to the subgroup of its automorphism group which acts trivially on $H^i(K_a(s_n), \mathbb{Z})$, $i = 2, 3$, while the subgroup of the automorphism group of $K_a(s_n)$ which acts trivially on $H^2(K_a(s_n), \mathbb{Z})$ is generated by Γ_X and ν_a, by [3, Th. 3 and Cor. 5]. The automorphism ν_a acts on $H^3(K_a(s_n), \mathbb{Q})$ via multiplication by -1 and Γ_X embeds in $\mathrm{Mon}(K_a(s_n))$, by [44, Th. 1.3]. The subgroup of $\mathrm{Mon}(K_a(s_n))$ acting trivially on $H^2(K_a(s_n), \mathbb{Z})$ is known to be induced by automorphisms, by [13, Th. 2.1], and thus contains the image of Γ_X as an index 2 subgroup. ∎

Proposition 10.2. *There exists a unique injective homomorphism*[11]

$$\overline{\mathrm{mon}} : G(S_X^+)_v^{\mathrm{even}} \to \mathrm{Mon}(K_a(v))/\Gamma_v$$

such that both h_2 and h_3, given in (10.2), are $G(S_X^+)_v^{\mathrm{even}}$-equivariant with respect to the homomorphisms mon*, given in Corollary 9.4, and* $\overline{\mathrm{mon}}$.

Proof. Let $\pi : \mathcal{M} \to T$ be a smooth and proper family of Kähler manifolds with fiber $\mathcal{M}(v)$ over a point t_0 of an analytic space T. We get the commutative diagram of the relative Albanese variety $\mathcal{A}lb^1_\pi$ of degree 1:



The morphism p is a fibration with connected fibers. Hence, the homomorphism $p_* : \pi_1(\mathcal{A}lb^1_\pi, a) \to \pi_1(T, t_0)$ is surjective. Let $g \in \mathrm{Mon}(\mathcal{M}(v))$ be a monodromy operator corresponding to a class $\gamma \in \pi_1(T, t_0)$. Choose a class $\tilde{\gamma}$ in $\pi_1(\mathcal{A}lb^1_\pi, a)$ such that

[11] Once Theorem 1.4 is proven it would follow that this homomorphism is in fact an isomorphism.

$p_*(\tilde{\gamma}) = \gamma$. Let $\tilde{g}$ be the monodromy operator of $K_a(v)$ corresponding to $\tilde{\gamma}$. Then the pull-back homomorphism $\iota_a^* : H^*(\mathcal{M}(v), \mathbb{Z}) \to H^*(K_a(v), \mathbb{Z})$ is $(g, \tilde{g})$-equivariant,

$$\iota_a^*(g(x)) = \tilde{g}(\iota_a^*(x))$$

for all $x \in H^*(\mathcal{M}(v), \mathbb{Z})$, since the evaluation homomorphism

$$p^* R\pi_*\mathbb{Z} \to R\widetilde{\mathrm{alb}}_*\mathbb{Z}$$

is a global section, hence monodromy invariant. It follows that the homomorphisms h_2 and h_3 are $(g, \tilde{g})$-equivariant as well. Now, the image of $\tilde{g}$ in $\mathrm{Mon}(K_a(v))/\Gamma_v$ is determined by its action on $H^i(K_a(v), \mathbb{Q})$ for $i = 2, 3$. Indeed, if $\tilde{g}_1, \tilde{g}_2 \in \mathrm{Mon}(K_a(v))$ and $\tilde{g}_1\tilde{g}_2^{-1}$ acts trivially on $H^i(K_a(v), \mathbb{Q})$ for $i = 2, 3$, then $\tilde{g}_1\tilde{g}_2^{-1}$ belongs to Γ_v, since Γ_v is equal to the subgroup of $\mathrm{Mon}(K_a(v))$ acting trivially on both $H^2(K_a(v), \mathbb{Z})$ and $H^3(K_a(v), \mathbb{Z})$, by Lemma 10.1. The homomorphisms

$$h_i : Q^i(\mathcal{M}(v)) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^i(K_a(v), \mathbb{Q})$$

are isomorphisms for $i = 2, 3$, as noted in the paragraph below (10.2). Hence, the image of $\tilde{g}$ in $\mathrm{Mon}(K_a(v))/\Gamma_v$ is uniquely determined by g. We get a canonical homomorphism

$$\mathrm{Mon}(\mathcal{M}(v)) \to \mathrm{Mon}(K_a(v))/\Gamma_v.$$

Define $\overline{\mathrm{mon}}$ as the composition of the above homomorphism with the homomorphism mon given in Corollary 9.4. The homomorphism $\overline{\mathrm{mon}}$ is injective, since h_i is injective and $G(S_X^+)_v^{\mathrm{even}}$-equivariant for $i = 2, 3$, and $G(S_X^+)_v^{\mathrm{even}}$ acts faithfully on the direct sum of $Q^i(\mathcal{M}(v)) \otimes \mathbb{Q}$ for $i = 2, 3$. ■

Pulling back the extension

$$0 \to \Gamma_v \to \mathrm{Mon}(K_a(v)) \to \mathrm{Mon}(K_a(v))/\Gamma_v \to 0$$

via $\overline{\mathrm{mon}}$ we get the extension

$$0 \to \Gamma_v \to \widetilde{G}(S^+)_v^{\mathrm{even}} \to G(S_X^+)_v^{\mathrm{even}} \to 0, \tag{10.3}$$

where $\widetilde{G}(S^+)_v^{\mathrm{even}}$ is a subgroup of $\mathrm{Mon}(K_a(v))$, by the injectivity of $\overline{\mathrm{mon}}$.

Proof of Theorem 1.4. We prove the inclusion $\mathcal{W}^{\det\cdot\chi} \subset \overline{\mathrm{mon}}^2(G(S_X^+)_{s_n}^{\mathrm{even}})$, as the reverse inclusion was proven by Mongardi [33]. The restriction homomorphism from $H^2(\mathcal{M}(s_n), \mathbb{Z})$ to $H^2(K_X(n-1), \mathbb{Z})$ factors through the isomorphism $h_2 : Q^2(\mathcal{M}(s_n)) \to H^2(K_X(n-1), \mathbb{Z})$ given in (10.2), by Theorem 3.1. Denote by $\mathcal{W}^{\det\cdot\chi}$ the corresponding subgroup of $O(s_n^\perp)$ as well. Proposition 10.2 reduces the proof to checking that the isomorphism $\bar{\theta}_2 : s_n^\perp \to Q^2(\mathcal{M}(s_n))$ given in (8.13) conjugates the image of $\mathrm{mon}(G(S_X^+)_{s_n}^{\mathrm{even}})$ in $GL[Q^2(\mathcal{M}(s_n), \mathbb{Z})]$ onto $\mathcal{W}^{\det\cdot\chi}$. The subgroup $\mathrm{Spin}(S^+)_{s_n}$ of $G(S_X^+)_{s_n}^{\mathrm{even}}$ maps onto $SO_+(S^+)_{s_n}$, by Corollary 5.1. The image of $\mathrm{mon}(\mathrm{Spin}(S^+)_{s_n})$ in $GL[Q^2(\mathcal{M}(s_n), \mathbb{Z})]$ is conjugated onto $SO_+(S^+)_{s_n}$ via $\bar{\theta}_2$, by the $\mathrm{Spin}(S^+)_{s_n}$-equivariance of the latter established in Lemma 8.10. $\mathcal{W}^{\det\cdot\chi}$ is generated by $SO_+(S^+)_{s_n}$ and the involution of $s_n^\perp$ sending $(1,0,n)$ to $-(1,0,n)$ and acting as the identity on $H^2(X,\mathbb{Z})$. The involution τ_X given in (8.2) is an element of $G(S_X^+)_{s_n}^{\text{even}}$. The former involution of $s_n^\perp$ is precisely $\bar{\theta}_2^{-1} \circ \text{mon}_{\tau_X} \circ \bar{\theta}_2$, by the equality

$$\text{mon}_{\tau_X}(\widetilde{\theta}_2(\lambda)) = -\bar{\theta}_2(\tau_X(\lambda))$$

for all $\lambda \in s_n^\perp$, which is a special case of Corollary 8.7 and Lemma 8.10.

It remains to prove the surjectivity of the homomorphism (1.2). Let $\text{Aut}_0(K_a(s_n))$ be the subgroup of $\text{Aut}(K_a(s_n))$ consisting of automorphisms acting trivially on $H^2(K_a(s_n),\mathbb{Z})$. The subgroup of $\text{Mon}(K_a(s_n))$ acting trivially on $H^2(K_a(s_n),\mathbb{Z})$ is known to be the image of $\text{Aut}_0(K_a(s_n))$, by [13, Th. 2.1]. It suffices to prove that the image of $\overline{\text{mon}}$ contains the image of $\text{Aut}_0(K_a(s_n))$, by the surjectivity of $\overline{\text{mon}}^2$. Now $\text{Aut}_0(K_a(s_n))$ is generated by Γ_{s_n} and the automorphism ν_a of order 2 in the proof of Lemma 10.1 (4), and action of the coset $\nu_a\Gamma_{s_n}$ on $H^3(K_a(s_n),\mathbb{Q})$ is equal to the action of $\overline{\text{mon}}(-1)$, as both act as minus the identity. Furthermore, $\overline{\text{mon}}(-1)$ acts trivially on $H^2(K_a(s_n),\mathbb{Z})$. It follows that the subgroup $\{1,-1\} \subset G(S_X^+)_{s_n}^{\text{even}}$ is mapped via $\overline{\text{mon}}$ onto the image of $\text{Aut}_0(K_a(s_n))$ in $\text{Mon}(K_a(s_n))/\Gamma_{s_n}$. ∎

10.2. Comparison with Verbitsky's Lie algebra representation

Lemma 10.3. *The homomorphism $\overline{\text{mon}}$ of Proposition 10.2 embeds $\text{Spin}(S_X^+)_v$ as a normal subgroup of $\text{Mon}(K_a(v))/\Gamma_v$.*

Proof. The image of $\text{Spin}(S_X^+)_v$ has index 2 in $\text{Mon}(K_a(v))/\Gamma_v$, by Proposition 10.2 and the surjectivity statement of Theorem 1.4, and is thus a normal subgroup. We provide next a proof independent of the surjectivity result, so independent of the result in [33]. We may assume that $v=(1,0,-1-n)$. The homomorphism $\overline{\text{mon}}$ is injective, by Proposition 10.2. The group $O_+(S^+)_v$ naturally embeds in $O_+(v^\perp)$ and the image is a normal subgroup, by [26, Lemma 4.10] (that lemma is stated for the Mukai lattice of a $K3$ surface, but the same proof applies to the Mukai lattice of an abelian surface). Hence, $SO_+(S^+)_v$ embeds as a normal subgroup, the image being the intersection of two normal subgroups of $O_+(v^\perp)$. The restriction homomorphism $r:\text{Mon}(K_a(v))\to\text{Mon}^2(K_a(v))$ factors through a homomorphism $\bar{r}:\text{Mon}(K_a(v))/\Gamma_v\to\text{Mon}^2(K_a(v))$. The group $\text{Mon}^2(K_a(v))$ is naturally identified with a subgroup of $O_+(v^\perp)$ and the inverse image via $\bar{r}$ of $SO_+(S^+)_v$ is thus a normal subgroup of $\text{Mon}(K_a(v))/\Gamma_v$. It remains to show that this inverse image is $\overline{\text{mon}}(\text{Spin}(S_X^+)_v)$. Note that $\text{Spin}(S_X^+)_v$ surjects onto $SO_+(S^+)_v$, by Lemma 4.1, and its kernel has order 2 and is generated by an element acting via scalar multiplication by -1 on V and S^-. The group Γ_v has index 2 in the kernel of r, by the proof of Lemma 10.1 (4). Hence, the kernel of $\bar{r}$ has order 2 and is thus contained in $\overline{\text{mon}}(\text{Spin}(S_X^+)_v)$. ∎

Let Y be an irreducible holomorphic symplectic manifold of complex dimension $2n$. We recall next Verbitsky's construction of a Lie algebra representation on the cohomology of Y. Set $b_2 := \dim H^2(Y,\mathbb{R})$. Let $h \in \text{End}[H^*(Y,\mathbb{R})]$ be the endomorphism acting via scalar multiplication by $i-2n$ on $H^i(Y,\mathbb{R})$. Given a class $a \in H^2(Y,\mathbb{R})$ let

$e_a \in \mathrm{End}[H^*(Y,\mathbb{R})]$ be given by cup product with a. The class a is called of *Lefschetz type* if there exists an endomorphism $f_a \in \mathrm{End}[H^*(Y,\mathbb{R})]$ satisfying the $\mathfrak{sl}_2$ commutation relations

$$[e_a, f_a] = h, \quad [h, e_a] = 2e_a, \quad [h, f_a] = -2f_a.$$

Such an f_a is unique if it exists. The triple $\{e_a, h, f_a\}$ is called a *Lefschetz triple.*

Let $\mathfrak{g}(Y)$ be the Lie subalgebra of $\mathrm{End}[H^*(Y,\mathbb{R})]$ generated by all Lefschetz triples. Denote by $\mathfrak{g}_k(Y)$ its graded summand of degree k. Let $\mathrm{Prim}^k(Y) \subset H^k(Y,\mathbb{R})$ be the subspace annihilated by $\mathfrak{g}_{-2}(Y)$ and set $\mathrm{Prim}(Y) := \bigoplus_k \mathrm{Prim}^k(Y)$. Let $A_2 \subset H^*(Y,\mathbb{R})$ be the subring generated by $H^2(Y,\mathbb{R})$. The following theorem was proven by Verbitsky [50] and in a detailed form by Looijenga and Lunts.

Theorem 10.4. (1) ([23, Prop. 4.5]) $\mathfrak{g}(Y)$ *is isomorphic to* $\mathfrak{so}(4, b_2 - 2, \mathbb{R})$ *and* $\mathfrak{g}_0(Y) \cong \mathfrak{so}(H^2(Y,\mathbb{R})) \oplus \mathbb{R}h$. *The homomorphism* $e : H^2(Y,\mathbb{R}) \to \mathfrak{g}_2(Y)$*, sending* a *to* e_a*, is an isomorphism.* $\mathfrak{g}_k(Y)$ *vanishes if* k *does not belong to* $\{-2, 0, 2\}$*.*

(2) ([23, Prop. 1.6]) $\mathfrak{g}(Y)$ *preserves, infinitesimally, the Poincaré pairing on* $H^*(Y,\mathbb{R})$*.*

(3) ([23, Prop. 1.6 and Cor. 2.3]) $H^*(Y,\mathbb{R})$ *is the orthogonal direct sum, with respect to the Poincaré pairing, of the* A_2*-submodules generated by* $\mathrm{Prim}^k(Y)$*,* $0 \le k \le 2n$*,*

$$H^*(Y,\mathbb{R}) = \bigoplus_{k=0}^{2n} A_2 \cdot \mathrm{Prim}^k(Y).$$

(4) ([23, Cor. 1.13]) *Let* W *be an irreducible* $\mathfrak{g}_0(Y)$*-submodule of* $\mathrm{Prim}^k(Y)$*. Then the* A_2*-submodule generated by* W *is an irreducible* $\mathfrak{g}(Y)$*-submodule. Conversely, all irreducible* $\mathfrak{g}(Y)$*-submodules are of this type.*

(5) ([51, Th. 7.1]) *The Hodge endomorphism of* $H^*(Y,\mathbb{C})$*, which acts on* $H^{p,q}(Y)$ *via scalar multiplication by* $\sqrt{-1}\,(p-q)$*, is an element of the semisimple summand* $\mathfrak{so}(H^2(Y,\mathbb{C}))$ *of* $\mathfrak{g}_0(Y) \otimes_{\mathbb{R}} \mathbb{C}$*.*

Let Y be an irreducible holomorphic symplectic manifold of generalized Kummer type. Let Γ be the subgroup of $\mathrm{Aut}(Y)$ acting trivially on $H^i(Y,\mathbb{Q})$, $i = 2, 3$. The Γ-action commutes with the $\mathfrak{g}(Y)$-action, since Γ acts trivially on $H^2(Y,\mathbb{R})$ and f_a is uniquely determined by e_a, for each Lefschetz triple $\{e_a, h, f_a\}$. Hence, the Γ-invariant subring $H^*(Y,\mathbb{R})^\Gamma$ is a $\mathfrak{g}(Y)$-submodule of $H^*(Y,\mathbb{R})$. Let $A_k \subset H^*(Y,\mathbb{R})^\Gamma$, $k \ge 0$, be the subalgebra generated by $\bigoplus_{i=0}^k H^i(Y,\mathbb{R})^\Gamma$. This definition of A_2 agrees with the one above, since Γ acts trivially on $H^2(Y,\mathbb{R})$, by Lemma 10.1 (4). Set $(A_k)^j := A_k \cap H^j(Y,\mathbb{R})$. Let A'_k be the A_2-submodule of $H^*(Y,\mathbb{R})^\Gamma$ generated by $\mathrm{Prim}(Y) \cap A_k$. Then A'_k is the maximal $\mathfrak{g}(Y)$-submodule of $H^*(Y,\mathbb{R})^\Gamma$ which is contained in A_k, by parts (3) and (4) of Theorem 10.4. The Poincaré pairing restricts to a non-degenerate pairing on each irreducible $\mathfrak{g}(Y)$-submodule, by the second paragraph in the proof of [23, Prop. 1.6]. Hence, the Poincaré pairing restricts to A'_k as a non-degenerate pairing. Set $C_k := H^k(Y,\mathbb{R})$ for $0 \le k \le 3$. For $k \ge 4$, set

$$C_k := (A'_{k-2})^\perp \cap H^k(Y,\mathbb{R})^\Gamma,$$

where the orthogonal complement is taken with respect to the Poincaré pairing.

Lemma 10.5. (1) $H^k(Y,\mathbb{R})^\Gamma$ *admits a monodromy invariant and* $\mathfrak{g}_0(Y)$*-invariant decomposition*

$$H^k(Y,\mathbb{R})^\Gamma = (A_{k-2})^k \oplus C_k. \tag{10.4}$$

(2) *The subspaces* C_k, $k \geq 2$, *generate* $H^*(Y,\mathbb{R})^\Gamma$ *as a ring.*

Proof. (1) Γ is a normal subgroup of $\mathrm{Mon}(Y)$ and so $H^*(Y,\mathbb{R})^\Gamma$ is $\mathrm{Mon}(Y)$-invariant. Indeed, if $\alpha \in H^*(Y,\mathbb{R})^\Gamma$, $g \in \mathrm{Mon}(Y)$, and $\gamma \in \Gamma$, then $g^{-1}\gamma g$ belongs to Γ and so $\gamma(g(\alpha)) = g(g^{-1}\gamma g)(\alpha) = g(\alpha)$. Hence, $g(\alpha)$ belongs to $H^*(Y,\mathbb{R})^\Gamma$. The $\mathrm{Mon}(Y)$-action on $H^*(Y,\mathbb{R})^\Gamma$ factors through $\mathrm{Mon}(Y)/\Gamma$. The proof of [26, Cor. 4.6 (1)] now applies to the $\mathrm{Mon}(Y)/\Gamma$-action on $H^*(Y,\mathbb{R})^\Gamma$ (instead of the $\mathrm{Mon}(X)$-action on $H^*(X,\mathbb{R})$ for X of $K3^{[n]}$-deformation type).

(2) follows easily by induction from part (1). ∎

Lemma 10.6. (1) C_{2i}, $i \geq 2$, *admits a* $\mathrm{Mon}(Y)/\Gamma$*-invariant decomposition*

$$C_{2i} = C'_{2i} \oplus C''_{2i}.$$

Here C'_{2i} *either vanishes, or is a* 1*-dimensional representation of* $\mathrm{Mon}(Y)/\Gamma$*; and* C''_{2i} *either vanishes, or is isomorphic to the tensor product of* $H^2(Y,\mathbb{R})$ *with a* 1*-dimensional representation of* $\mathrm{Mon}(Y)/\Gamma$.

(2) C_{2i+1}, $i \geq 1$, *either vanishes, or is an irreducible* 8*-dimensional representation of* $\mathrm{Mon}(Y)/\Gamma$. *If* C_{2i+1} *does not vanish and* $Y = K_a(v)$ *for* $v = (1,0,-1-n) \in S^+_X$, *then* C_{2i+1} *is the spin representation for the monodromy representation of* $\mathrm{Spin}(S^+_X)_v$ *given in Proposition* 10.2.

Proof. As the decomposition (10.4) is $\mathrm{Mon}(Y)$-invariant, we may prove the statements for $Y = K_a(v)$ and $\Gamma = \Gamma_v$, where $v = (1,0,-1-n) \in S^+_X$ for an abelian surface X.

The moduli space $\mathcal{M}(v)$ is the quotient of $K_a(v) \times A$ by the anti-diagonal action of Γ_v, by Lemma 10.1 (3). Denote the quotient morphism by $j : K_a(v) \times A \to \mathcal{M}(v)$. Then the pull-back homomorphism $j^* : H^*(\mathcal{M}(v),\mathbb{R}) \to H^*(K_a(v) \times A,\mathbb{R})^{\Gamma_v}$ is surjective. Choose a point $\tilde{a}$ of A over a and let $\iota_{\tilde{a}} : K_a(v) \to K_a(V) \times A$ be the natural inclusion onto $K_a(v) \times \{\tilde{a}\}$. Then $\iota_a = j \circ \iota_{\tilde{a}}$. The homomorphism $\iota^*_{\tilde{a}} : H^*(K_a(v) \times A,\mathbb{R}) \to H^*(K_a(v),\mathbb{R})$ is surjective and Γ_v-equivariant, since the action of Γ_v on $H^*(A,\mathbb{R})$ is trivial. Hence, $\iota^*_{\tilde{a}} : H^*(K_a(v) \times A,\mathbb{R})^{\Gamma_v} \to H^*(K_a(v),\mathbb{R})^{\Gamma_v}$ is surjective as well. It follows that

$$\iota^*_a : H^*(\mathcal{M}(v),\mathbb{R}) \to H^*(K_a(v),\mathbb{R})^{\Gamma_v} \tag{10.5}$$

is the composition $\iota^*_{\tilde{a}} \circ j^*$ of two surjective homomorphisms, hence itself surjective.

Let B_k be the projection to $H^k(\mathcal{M}(v),\mathbb{R})$ of the image of the homomorphism $\tilde{\theta} : S^k_X \otimes_{\mathbb{Z}} \mathbb{R} \to H^*(\mathcal{M}(v),\mathbb{R})$, given in (8.7). The subspaces B_k generate the cohomology ring $H^*(\mathcal{M}(v),\mathbb{R})$, by [24, Cor. 2]. Let $\overline{B}_k$ be the image of B_k in $H^*(K_a(v),\mathbb{R})^{\Gamma_v}$ via the restriction homomorphism ι^*_a. The subspaces $\overline{B}_k$ generate the cohomology ring $H^*(K_a(v),\mathbb{R})^{\Gamma_v}$, by the surjectivity of (10.5). Hence, $B_k + (A_{k-2})^k = H^k(K_a(v),\mathbb{R})^{\Gamma_v}$. Consequently, $\overline{B}_k$ surjects onto the direct summand C_k for all k. We get a surjective

$\mathrm{Spin}(S_X^+)_v$-equivariant homomorphism $S_X^k \otimes_{\mathbb{Z}} \mathbb{R} \to C_k$, with respect to the spin representation on S_X and the monodromy representation on C_k. The rest of the proof of (1) and (2) is identical to that of [26, Lemma 4.8], where the decomposition for even k follows from the decomposition $S_X^+ \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}v \oplus (v^\perp \otimes_{\mathbb{Z}} \mathbb{R})$. ∎

The quotient $\mathrm{Mon}(Y)/\Gamma$ has a canonical normal subgroup N obtained by conjugating the subgroup $\overline{\mathrm{mon}}(\mathrm{Spin}(S_X^+)_v)$ of Lemma 10.3 via a parallel transport operator from $H^*(K_a(v), \mathbb{Z})$ to $H^*(Y, \mathbb{Z})$.

Lemma 10.7. *The Lie algebra of the Zariski closure $N_{\mathbb{C}}$ of N in $\mathrm{GL}[H^*(Y, \mathbb{C})^\Gamma]$ is equal to the semisimple direct summand $\mathfrak{so}(H^2(Y, \mathbb{C}))$ of the complexification $\mathfrak{g}_0(Y) \otimes_{\mathbb{R}} \mathbb{C}$ of Verbitsky's Lie algebra $\mathfrak{g}_0(Y)$ introduced in Theorem 10.4 (1).*

Proof. Verbitsky's Lie algebra $\mathfrak{g}_0(Y)$ is a monodromy invariant subalgebra of $\mathfrak{gl}[H^*(Y, \mathbb{C})^\Gamma]$, and so it suffices to prove the statement for $Y = K_a(v)$, $v = (1, 0, -1-n)$. Let

$$\nu : \mathrm{Spin}(H^2(K_a(v), \mathbb{C})) \to \mathrm{GL}[H^*(K_a(v), \mathbb{C})^{\Gamma_v}]$$

be the integration of the infinitesimal action of the semisimple part of Verbitsky's Lie algebra $\mathfrak{g}_0(K_a(v))$ in Theorem 10.4 (1) to the group action of the corresponding simply connected group. Under the identification of $v^\perp \otimes_{\mathbb{Z}} \mathbb{C}$ with $H^2(K_a(v), \mathbb{C})$ we may view $\mathrm{Spin}(S_X^+)_v$ as a Zariski dense arithmetic subgroup of $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$. We claim that the $\overline{\mathrm{mon}}$ representation of $\mathrm{Spin}(S_X^+)_v$, given in Proposition 10.2, extends to a representation of $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$, which we again denote by $\overline{\mathrm{mon}}$. The proof is identical to that of [26, Lemma 4.11 (3)], and uses the fact that the subspaces C_k generate $H^*(K_a(v), \mathbb{C})^{\Gamma_v}$, by Lemma 10.5 (2), and each of the representations $C_k \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathrm{Spin}(S_X^+)_v$ is induced from a representation of $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$, by Lemma 10.6. (Contrast this with the congruence representation in Lemma 4.5.)

We adapt next the proof of [26, Lemma 4.13] to our set-up. The monodromy equivariance of Verbisky's representation ν yields the equality

$$\overline{\mathrm{mon}}(g)\nu(f)\,\overline{\mathrm{mon}}(g)^{-1} = \nu(gfg^{-1}) \tag{10.6}$$

for all $f \in \mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$ and all $g \in \mathrm{Spin}(S_X^+)_v$. The equality holds also for all g in $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$, by the density of $\mathrm{Spin}(S_X^+)_v$. Let

$$\eta : \mathrm{Spin}(H^2(K_a(v), \mathbb{C})) \to \mathrm{GL}[H^*(K_a(v), \mathbb{C})^{\Gamma_v}]$$

be given by $\eta(g) := \nu(g)^{-1}\,\overline{\mathrm{mon}}(g)$. We have

$$\eta(g)\nu(f)\eta(g)^{-1} = \nu(g)^{-1}\overline{\mathrm{mon}}(g)\nu(f)\overline{\mathrm{mon}}(g)^{-1}\nu(g) \overset{(10.6)}{=} \nu(g)^{-1}\nu(gfg^{-1})\nu(g) = \nu(f).$$

Hence, $\nu(f)$ commutes with $\eta(g)$ for all $f, g \in \mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$. We get

$$\eta(fg) = \nu(fg)^{-1}\,\overline{\mathrm{mon}}(fg) = \nu(g)^{-1}\eta(f)\,\overline{\mathrm{mon}}(g) = \eta(f)\eta(g).$$

Hence, η is a representation of $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$.

Taking $g = f$ the commutation of $\nu(g)$ and $\eta(g)$ yields

$$\nu(g)^{-1}\,\overline{\mathrm{mon}}(g)\nu(g) = \eta(g)\nu(g) = \nu(g)\eta(g) = \overline{\mathrm{mon}}(g).$$

Hence, $\nu(g)$ commutes with $\overline{\mathrm{mon}}(g)$ for all $g \in \mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$. The subspaces C_k are $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$-invariant with respect to both ν and $\overline{\mathrm{mon}}$, by Lemma 10.5. The irreducible monodromy subrepresentations C'_{2i} and C''_{2i} are non-isomorphic if they do not vanish, by Lemma 10.6. Hence, each is also ν-invariant, by the latter commutativity.

We claim that each of C_{2i+1}, C'_{2i}, and C''_{2i} is an irreducible ν-representation as well, if it does not vanish. The statement is clear for the 1-dimensional C'_{2i}. Note that each non-trivial representation of $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$ of dimension 7 is necessarily irreducible. Similarly, each 8-dimensional representation which does not contain a trivial subrepresentation is necessarily irreducible. The Hodge structure of each $\mathfrak{g}_0(K_a(v))$-submodule in $H^*(K_a(v), \mathbb{C})$ is determined by the $\mathfrak{g}_0(K_a(v))$-action, by Theorem 10.4 (4). Each ν-subrepresentation of $H^{2p}(K_a(v), \mathbb{C})$ which is not of Hodge type (p, p) is thus a non-trivial ν representation and each odd-degree subrepresentation is non-trivial. Irreducibility of C_{2i+1} follows, if it does not vanish, as it is 8-dimensional in that case. If non-zero, C''_{2i} is 7-dimensional with Hodge numbers $(h^{i+1,i-1}, h^{i,i}, h^{i-1,i+1}) = (1, 5, 1)$, since the surjective homomorphism $S^k_X \otimes_{\mathbb{Z}} \mathbb{R} \to C^k$ constructed in the proof of Lemma 10.6 was a Hodge homomorphism. Hence, if non-zero, C''_{2i} is an irreducible ν-representation.

We have seen that $\nu(f)$ commutes with $\eta(g)$ for all $f, g \in \mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$. Hence, η must act on C'_{2i} and C''_{2i} via scalar multiplication, as they are irreducible subrepresentations of ν, which appear with multiplicity 1 in the η-invariant ν-representation C_{2i}. Similarly, η acts on C_k for odd k, via scalar multiplication. But $\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))$ does not have any non-trivial 1-dimensional representations. Hence, each C_k is a trivial η-representation, and hence so is the subring $H^*(K_a(v), \mathbb{C})^{\Gamma_v}$ they generate (Lemma 10.5 (2)). We conclude that $\nu = \overline{\mathrm{mon}}$ and so

$$N_{\mathbb{C}} := \overline{\mathrm{mon}}(\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))) = \nu(\mathrm{Spin}(H^2(K_a(v), \mathbb{C}))). \qquad \blacksquare$$

Lemma 10.8. *Every class in $H^{2p}(Y, \mathbb{Q})^{\Gamma}$ which is N-invariant is of Hodge type (p, p).*

Proof. An N-invariant class $\alpha \in H^{2p}(Y, \mathbb{R})^{\Gamma}$ is annihilated by the Lie algebra of the identity component of the Zariski closure $N_{\mathbb{C}}$ of N in $\mathrm{GL}[H^*(Y, \mathbb{C})^{\Gamma}]$. The latter Lie algebra is equal to the semisimple summand of the complexification $\mathfrak{g}_0(Y) \otimes_{\mathbb{R}} \mathbb{C}$, by Lemma 10.7. Hence, α is annihilated by the Hodge endomorphism, by Theorem 10.4 (5). ■

11. The Cayley class as a characteristic class

We prove Theorem 1.8 in this section exhibiting the $\mathrm{Spin}(V)_w$-invariant Cayley class $c_w \in H^4(X \times \hat{X}, \mathbb{Z})$ as the second Chern class of the pull-back of a sheaf $\mathcal{E}nd(E_F)$ on $\mathcal{M}_H(w)$ (Proposition 11.2) such that $c_2(\mathcal{E}nd(E_F))$ is $\mathrm{Spin}(V)_w$-invariant (Theorem 11.1).

Let $\mathcal{M}(w) := \mathcal{M}_H(w)$ be a smooth and compact moduli space of H-stable sheaves with a primitive Mukai vector w over an abelian surface X. Assume[12] that the dimension m of $\mathcal{M}(w)$ is greater than or equal 8. Set $\widehat{X} := \mathrm{Pic}^0(X)$. Given a point of $\mathcal{M}(w)$ representing the isomorphism class of a sheaf F, let $\iota_F : X \times \widehat{X} \to \mathcal{M}(w)$ be the morphism given in (9.1). Assume F is generic, so that the morphism ι_F is an embedding, by Lemma 10.1. Let $\mathcal{U}$ be a universal sheaf over $X \times \mathcal{M}(w)$, possibly twisted by a Brauer class $\theta \in H^2_{\mathrm{an}}(\mathcal{M}(w), \mathcal{O}^*_{\mathcal{M}(w)})$. Let pr_i be the projection from $X \times \mathcal{M}(w)$ onto the i-th factor, $i = 1, 2$, and let

$$\Phi_{\mathcal{U}} : D^b(X) \to D^b(\mathcal{M}(w), \theta)$$

be the integral functor $R\,\mathrm{pr}_{2,*}(L\,\mathrm{pr}_1^*(\bullet) \overset{L}{\otimes} \mathcal{U})$ with kernel $\mathcal{U}$. Let E_F be the first sheaf cohomology $\mathcal{H}^1(\Phi_{\mathcal{U}}(F^\vee))$, where $F^\vee := R\mathcal{H}om(F, \mathcal{O}_X)$ is the object derived dual to F. Then E_F is the relative extension sheaf $\mathcal{E}xt^1_{\mathrm{pr}_2}(L\,\mathrm{pr}_1^*\,F, \mathcal{U})$. Under the identification $V \cong H^1(X \times \widehat{X}, \mathbb{Z})$ we see that $\bigwedge^4 V \cong H^4(X \times \widehat{X}, \mathbb{Z})$ is a $\mathrm{Spin}(S^+_X)$-representation and so restricts to a $\mathrm{Spin}(S^+_X)_w$-representation. The $\mathrm{Spin}(S^+_X)_w$-invariant subgroup of $\bigwedge^4 V$ has rank 1 [40, Sec. 2.1]. We will refer to either one of its integral generators as "the" *Cayley class* of $\mathrm{Spin}(S^+_X)_w$ (which belongs to $H^4(X \times \widehat{X}, \mathbb{Z})$ and depends on w). The Cayley class is algebraic, by the following result.

Theorem 11.1. (1) *The sheaf E, given in (1.6), is a reflexive sheaf of rank $m-2$, which is locally free away from the diagonal in $\mathcal{M}(w) \times \mathcal{M}(w)$. The class $c_2(\mathcal{E}nd(E)) \in H^4(\mathcal{M}(w) \times \mathcal{M}(w), \mathbb{Z})$ is $\mathrm{Spin}(S^+_X)_w$-invariant with respect to the diagonal monodromy representation of Corollary 9.4.*

(2) *E_F is a reflexive sheaf of rank $m-2$, which is locally free over $\mathcal{M}(w) \setminus \{F\}$.*

(3) *The class $c_2(\mathcal{E}nd(E_F))$ is $\mathrm{Spin}(S^+_X)_w$-invariant with respect to the monodromy representation of Corollary 9.4.*

(4) *The class $\iota_F^*(c_2(\mathcal{E}nd(E_F)))$ in $H^4(X \times \widehat{X}, \mathbb{Z})$ is non-zero and $\mathrm{Spin}(S^+_X)_w$-invariant.*

The theorem is proved at the end of this section. We will need to treat first the case $w = s_n := (1, 0, -n)$. Let $\mathcal{P}$ be the Poincaré line bundle over $X \times \widehat{X}$, normalized so that its restriction to $\{0\} \times \widehat{X}$ is trivial. Denote by $[\mathrm{pt}_X] \in H^4(X, \mathbb{Z})$ the class Poincaré dual to a point and define $[\mathrm{pt}_{\widehat{X}}] \in H^4(\widehat{X}, \mathbb{Z})$ similarly. Let π_i be the projection from $X \times \widehat{X}$ onto the i-th factor, $i = 1, 2$.

Proposition 11.2. *If $w = s_n$, then*

$$\iota_F^*(c_2(\mathcal{E}nd(E_F))) = -n^2 c_1(\mathcal{P})^2 + 4n^3 \pi_1^*[\mathrm{pt}_X] + 4n \pi_2^*[\mathrm{pt}_{\widehat{X}}], \tag{11.1}$$

and the above class is $\mathrm{Spin}(S^+_X)_{s_n}$-invariant.

[12] Theorem 8.6 should hold, more generally, for moduli spaces of dimension ≥ 4. Once verified, Corollary 9.4 would then follow for moduli spaces of dimension ≥ 4 as well. The results of this section would then extend to moduli spaces of dimension ≥ 4.

Proof. $\mathcal{M}(s_n)$ is isomorphic to $X^{[n]} \times \hat{X}$. Let π_{ij} be the projection from $X \times X^{[n]} \times \hat{X}$ onto the product of the i-th and j-th factors. Let $\mathcal{Z} \subset X \times X^{[n]}$ be the universal subscheme and $I_{\mathcal{Z}}$ its ideal sheaf. Then $\mathcal{U} := \pi_{12}^* I_{\mathcal{Z}} \otimes \pi_{13}^* \mathcal{P}$ is a universal sheaf. Set $F := I_Z$, where Z is a reduced length n subscheme supported on the set $\{z_i : 1 \leq i \leq n\}$, consisting of n distinct points. Let $\tau_x : X \to X$ be the translation $\tau_x(x') = x + x'$ by $x \in X$. The image of $\iota_F : X \times \hat{X} \to \mathcal{M}(s_n)$ is the subset

$$\{I_{\tau_x(Z)} \otimes L : x \in X,\ L \in \hat{X}\}.$$

If $\tau_{x_1}(Z) = \tau_{x_2}(Z)$ then $n(x_2 - x_1) = 0$ and translation by $x_2 - x_1$ permutes the support of Z. Assume that $n(z_i - z_j) \neq 0$ for some pair i, j, so that ι_F is an embedding. Let $\Delta_i \subset X \times X$ be the translate of the diagonal Δ by $(z_i, 0)$, $1 \leq i \leq n$. Let p_{ij} be the projection from $X \times X \times \hat{X}$ onto the product of the i-th and j-th factors. The pull-back $\iota_F^* \mathcal{U}$ of $\mathcal{U}$ to $X \times (X \times \hat{X})$ via $\mathrm{id}_X \times \iota_F$ is thus $p_{12}^* I_{\bigcup_{i=1}^n \Delta_i} \otimes p_{13}^* \mathcal{P}$. Furthermore, $\iota_F^* \mathcal{U}$ is isomorphic to the derived pull-back $L\iota_F^* \mathcal{U}$, as $\mathcal{U}$ is flat over $\mathcal{M}(s_n)$. Let $\delta : X \times \hat{X} \to X \times X \times \hat{X}$ be the diagonal embedding $(x, L) \mapsto (x, x, L)$. Then the class $[\iota_F^* \mathcal{U}]$ of $\iota_F^* \mathcal{U}$ in the topological K-group of $X \times X \times \hat{X}$ is equal to $[p_{13}^* \mathcal{P}] - n[\delta_* \mathcal{P}]$. The class $[F^\vee]$ of $F^\vee$ in the topological K-group of X is equal to $[F]$, hence to $[\mathcal{O}_X] - n[\mathbb{C}_0]$, where $\mathbb{C}_0$ is the sky-scraper sheaf at the origin. Consider the cartesian diagram

$$\begin{array}{ccccc} X \times X \times \hat{X} & \xrightarrow{p_{13}} & X \times \hat{X} & \xrightarrow{\pi_1} & X \\ \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle \pi_2} & & \\ X \times \hat{X} & \xrightarrow{\pi_2} & \hat{X} & & \end{array}$$

We have an isomorphism of functors $Rp_{23,*} \circ Lp_{13}^* \cong L\pi_2^* \circ R\pi_{2,*}$, by cohomology and base change. Hence,

$$\begin{aligned} \Phi_{p_{13}^* \mathcal{P}}(\bullet) &:= Rp_{23,*} \circ Lp_{13}^*(L\pi_1^*(\bullet) \otimes \mathcal{P}) \cong L\pi_2^* \circ R\pi_{2,*}(L\pi_1^*(\bullet) \otimes \mathcal{P}) \\ &\cong L\pi_2^* \circ \Phi_{\mathcal{P}}(\bullet). \end{aligned}$$

We get an isomorphism of integral functors $\Phi_{p_{13}^* \mathcal{P}} \cong L\pi_2^* \circ \Phi_{\mathcal{P}}$. Now, $\Phi_{\mathcal{P}}(\mathbb{C}_0) \cong \mathcal{O}_{\hat{X}}$ and $\Phi_{\mathcal{P}}(\mathcal{O}_X) \cong \mathbb{C}_{\hat{0}}[-2]$, where $\hat{0}$ is the origin of $\hat{X}$. Hence, $[\Phi_{p_{13}^* \mathcal{P}}(F^\vee)] = -n[\mathcal{O}_{X \times \hat{X}}] + \pi_2^![\mathbb{C}_{\hat{0}}]$.

The integral functor $\Phi_{\delta_* \mathcal{P}} : D^b(X) \to D^b(X \times \hat{X})$ is just the composition

$$D^b(X) \xrightarrow{L\pi_1^*} D^b(X \times \hat{X}) \xrightarrow{\otimes \mathcal{P}} D^b(X \times \hat{X})$$

of derived pull-back and tensorization by $\mathcal{P}$, since $p_{23} \circ \delta : X \times \hat{X} \to X \times \hat{X}$ is the identity morphism. Hence, $\Phi_{\delta_* \mathcal{P}}(\mathbb{C}_0) \cong L\pi_1^* \mathbb{C}_0$, $\Phi_{\delta_* \mathcal{P}}(\mathcal{O}_X) \cong \mathcal{P}$, and

$$[\Phi_{\delta_* \mathcal{P}}(F^\vee)] = [\mathcal{P}] - n\pi_1^![\mathbb{C}_0].$$

We conclude that

$$[\Phi_{\iota_F^* \mathcal{U}}(F^\vee)] = -n[\mathcal{O}_{X \times \hat{X}}] + \pi_2^![\mathbb{C}_{\hat{0}}] - n[\mathcal{P}] + n^2 \pi_1^![\mathbb{C}_0].$$

The first three terms of its Chern character are thus

$$-\mathrm{ch}[\Phi_{\iota_F^*\mathcal{U}}(F^\vee)] = 2n + nc_1(\mathcal{P}) + (n/2)c_1(\mathcal{P})^2 - n^2\pi_1^*[\mathrm{pt}_X] - \pi_2^*[\mathrm{pt}_{\hat{X}}] + \cdots.$$

Given a class α in the topological K-group, of non-zero rank r, set

$$\kappa(\alpha) := \mathrm{ch}(\alpha)\exp(-c_1(\alpha)/r).$$

So the graded summand of degree 4 of the class

$$\kappa(-[\Phi_{\iota_F^*\mathcal{U}}(F^\vee)]) := \mathrm{ch}[-\Phi_{\iota_F^*\mathcal{U}}(F^\vee)]\big(1 - (1/2)c_1(\mathcal{P}) + (1/8)c_1(\mathcal{P})^2 + \cdots\big)$$

is

$$\kappa_2(-[\Phi_{\iota_F^*\mathcal{U}}(F^\vee)]) = (n/4)c_1(\mathcal{P})^2 - n^2\pi_1^*[\mathrm{pt}_X] - \pi_2^*[\mathrm{pt}_{\hat{X}}].$$

If we set $b := [-\Phi_{\iota_F^*\mathcal{U}}(F^\vee)]$, then $c_2(b \otimes b^*) = -\mathrm{ch}_2(b \otimes b^*) = -4n\kappa_2(b)$, which is equal to the right hand side of (11.1). Finally, $\mathcal{H}^i(\Phi_{\mathcal{U}}(F^\vee))$ vanishes if $i \notin \{1, 2\}$, and $\mathcal{H}^2(\Phi_{\mathcal{U}}(F^\vee))$ is supported on the point of $\mathcal{M}(w)$ representing F, and so on a subvariety of codimension ≥ 4. Hence, $c_i(E_F) = c_i(\mathcal{H}^1(\Phi_{\mathcal{U}}(F^\vee)) = c_i(-\Phi_{\mathcal{U}}(F^\vee))$ for $i = 1, 2$, and so

$$\begin{aligned}\iota_F^* c_2(\mathcal{E}nd(E_F)) &= \iota_F^* c_2([\Phi_{\mathcal{U}}(F^\vee)] \otimes [\Phi_{\mathcal{U}}(F^\vee)]^*)\\ &= c_2(L\iota_F^*([\Phi_{\mathcal{U}}(F^\vee)] \otimes [\Phi_{\mathcal{U}}(F^\vee)]^*)) = c_2(L\iota_F^*[\Phi_{\mathcal{U}}(F^\vee)] \otimes L\iota_F^*[\Phi_{\mathcal{U}}(F^\vee)]^*)\\ &= c_2([\Phi_{\iota_F^*\mathcal{U}}(F^\vee)] \otimes [\Phi_{\iota_F^*\mathcal{U}}(F^\vee)]^*),\end{aligned}$$

where the last equality follows from cohomology and base change and the isomorphism $L\iota_F^*\mathcal{U} \cong \iota_F^*\mathcal{U}$ observed above. Equality (11.1) thus follows.

The invariance of the class (11.1) would follow once we show that

$$\mathrm{ch}([\Phi_{\mathcal{U}}(F^\vee)] \otimes [\Phi_{\mathcal{U}}(F^\vee)]^*) \tag{11.2}$$

is invariant with respect to the monodromy action of $\mathrm{Spin}(S_X^+)_{s_n}$ via mon in (8.6), since the homomorphism $\iota_F^* : H^4(\mathcal{M}(s_n), \mathbb{Z}) \to H^4(X \times \hat{X}, \mathbb{Z})$ is $\mathrm{Spin}(S_X^+)_{s_n}$-equivariant, by Corollary 9.6. The Chern character $\mathrm{ch}(\Phi_{\mathcal{U}}(F^\vee))$ is equal to $\tilde{\theta}(\mathrm{ch}(F))$, where $\tilde{\theta}$ is given in (8.7). Now, $\mathrm{mon}_g(\tilde{\theta}(g^{-1}(\mathrm{ch}(F)))) = \tilde{\theta}(\mathrm{ch}(F))\exp(c_g)$ for $g \in \mathrm{Spin}(S_X^+)_{s_n}$, by (8.10). The invariance of the class (11.2) follows, since $\mathrm{ch}(F) = s_n$, and so $g^{-1}(\mathrm{ch}(F)) = \mathrm{ch}(F)$. ∎

Proof of Theorem 11.1. (1) The reflexivity and rank statements are proven in [29, Prop. 4.1 and Rem. 4.6]. We prove the $\mathrm{Spin}(S_X^+)_w$-invariance of $c_2(\mathcal{E}nd(E))$. Given $g \in \mathrm{Spin}(S_X^+)_w$, we see that $(g \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E})) = \mathrm{ch}(\mathcal{E})\exp(c_g)$, by Corollary 9.4, where c_g is given in (6.8). Hence, $(\tau g \tau \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{E})^\vee) = \mathrm{ch}(\mathcal{E})^\vee \exp(-c_g)$, by Lemma 6.9. The Chern character of the object $\mathcal{F} := R\pi_{13,*}R\mathcal{H}om(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$ in $D^b(\mathcal{M}(w) \times \mathcal{M}(w))$ is obtained by contracting the tensor product $\mathrm{ch}(\mathcal{E})^\vee \otimes \mathrm{ch}(\mathcal{E})$ in $H^*(X \times \mathcal{M}(w) \times X \times \mathcal{M}(w))$ with the class in $H^*(X) \otimes H^*(X)$ corresponding to minus the Mukai pairing (4.15). The latter class is invariant under $(\tau g \tau) \otimes g$. We get the equality

$$(\mathrm{mon}_g \otimes \mathrm{mon}_g)(\mathrm{ch}(\mathcal{F})) = \mathrm{ch}(\mathcal{F})\pi_1^* \exp(-c_g)\pi_2^* \exp(c_g).$$

The $\mathrm{Spin}(S_X^+)_w$-invariance of $\mathrm{ch}(R\mathcal{H}om(\mathcal{F}, \mathcal{F}))$ follows. Now, $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{O}_{\Delta_{\mathcal{M}(w)}}) - \mathrm{ch}(E)$, since the first sheaf cohomology of $\mathcal{F}$ is E and the second is $\mathcal{O}_{\Delta_{\mathcal{M}(w)}}$ and all other sheaf cohomologies vanish, by [29, Prop. 4.1 and Rem. 4.6]. The statement follows, since the classes $\mathrm{ch}_i(\mathcal{O}_{\Delta_{\mathcal{M}(w)}})$ vanish for $i < m$.

(2) is proven in [29, Prop. 4.1 and Rem. 4.6].

(3) The $\mathrm{Spin}(S_X^+)_w$-invariance of $c_2(\mathcal{E}nd(E_F))$ with respect to the monodromy representation of Corollary 9.4 follows from (8.10) by the same argument used in Proposition 11.2 to prove the invariance when $w = s_n$.

(4) The $\mathrm{Spin}(S_X^+)_w$-equivariance of ι_F^* is established in Corollary 9.6. The non-vanishing of the pulled-back class $\iota_F^* c_2(\mathcal{E}nd(E_F))$ is checked for $w = s_n$ in Proposition 11.2. It follows for all w by Remark 9.2 and Theorem 9.3. ■

12. Period domains

In Section 12.1 we construct the universal torus $\mathcal{T}$, given in (1.5), over the period domain $\Omega_{w^\perp}$ of irreducible holomorphic symplectic manifolds of generalized Kummer deformation type. In Section 12.2 we prove Proposition 1.7; we construct the polarization Θ_h and the complex multiplication for the complex tori with periods in the 4-dimensional subloci $\Omega_{\{w,h\}^\perp}$ in the 5-dimensional period domain $\Omega_{w^\perp}$. In Section 12.3 we construct a hyperkähler structure on the complex torus T_ℓ associated with a Kähler class on an irreducible holomorphic symplectic manifold with period ℓ (Proposition 12.6). In Section 12.4 we prove that the subloci $\Omega_{\{w,h\}^\perp}$ parametrize abelian fourfolds of Weil type of discriminant 1. In Section 12.5 we construct the universal deformation $\pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp}$ of the moduli space of sheaves over the moduli space of marked irreducible holomorphic symplectic manifolds of generalized Kummer deformation type. In Section 12.6 we prove that the torus T_ℓ is isogenous to the third intermediate Jacobian of the irreducible holomorphic symplectic manifold of generalized Kummer deformation type with period ℓ.

12.1. Two isomorphic period domains

Keep the notation of Section 4.1. Set $S_{\mathbb{C}}^+ := S^+ \otimes_{\mathbb{Z}} \mathbb{C}$ and define $S_{\mathbb{C}}^-$ and $V_{\mathbb{C}}$ similarly. Let ℓ be an isotropic line in $S_{\mathbb{C}}^+$. Clifford multiplication $S_{\mathbb{C}}^+ \otimes V_{\mathbb{C}} \to S_{\mathbb{C}}^-$ restricts to $\ell \otimes V_{\mathbb{C}}$ as a homomorphism of rank 4, whose kernel is $\ell \otimes Z_\ell$ for a maximal isotropic subspace Z_ℓ of $V_{\mathbb{C}}$, by [6, III.1.4 and IV.1.1]. The image of $\ell \otimes V_{\mathbb{C}}$ is a maximal isotropic subspace of $S_{\mathbb{C}}^-$. Conversely, ℓ is the kernel of the homomorphism $S_{\mathbb{C}}^+ \to \mathrm{Hom}(Z_\ell, S_{\mathbb{C}}^-)$, induced by Clifford multiplication. We get an isomorphism between the quadric in $\mathbb{P}(S^+_{\mathbb{C}})$ of isotropic lines and a connected component $IG^+(4, V_{\mathbb{C}})$ of the Grassmannian $IG(4, V_{\mathbb{C}})$ of maximal isotropic subspaces of $V_{\mathbb{C}}$. Similarly, interchanging the roles of $S^+_{\mathbb{C}}$ and $S^-_{\mathbb{C}}$ we find that the quadric of isotropic lines in $S^-_{\mathbb{C}}$ is isomorphic to the other connected component $IG^-(4, V_{\mathbb{C}})$ of the Grassmannian of maximal isotropic subspaces of $V_{\mathbb{C}}$ [6, III.1.6]. A maximal isotropic subspace of $V_{\mathbb{C}}$ associated to an isotropic line of $S^+_{\mathbb{C}}$ is called *even*. It is called *odd* if it is associated to an isotropic line in $S^-_{\mathbb{C}}$.

Set

$$\Omega_{S^+} := \{\ell \in \mathbb{P}(S^+_{\mathbb{C}}) : (\ell, \ell) = 0,\ (\ell, \bar{\ell}) < 0\},$$

where the pairing is associated to the pairing $(\bullet, \bullet)_S$ given in (4.15). Then Ω_{S^+} is a connected open analytic subset of the quadric hyperplane of isotropic lines. The discussion above yields a $\mathrm{Spin}(S^+_{\mathbb{R}})$-equivariant embedding

$$\zeta : \Omega_{S^+} \to IG^+(4, V_{\mathbb{C}}) \tag{12.1}$$

of Ω_{S^+} as an open subset of $IG^+(4, V_{\mathbb{C}})$ in the analytic topology. $\mathrm{Spin}(S^+_{\mathbb{R}})$ acts transitively on Ω_{S^+} (see [16, Sec. 4.1]), and so the image of ζ is an open $\mathrm{Spin}(S^+_{\mathbb{R}})$-orbit. The two maximal isotropic subspaces Z_ℓ and $Z_{\bar{\ell}}$ of $V_{\mathbb{C}}$ are transversal. Indeed, their intersection is even-dimensional by [6, III.1.10], it is not 4-dimensional since $\ell \neq \bar{\ell}$, and it is not 2-dimensional by [6, III.1.12], since the 2-dimensional subspace $\ell + \bar{\ell}$ is not isotropic. $Z_{\bar{\ell}}$ is the complex conjugate of Z_ℓ as the map ζ is defined over $\mathbb{R}$, since Clifford multiplication was defined over $\mathbb{Z}$. Let $J_\ell : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be the endomorphism acting on Z_ℓ by i and on $Z_{\bar{\ell}}$ by $-i$. Then $V_{\mathbb{R}}$ is J_ℓ-invariant and J_ℓ induces a complex structure on $V_{\mathbb{R}}$. So the choice of $\ell \in \Omega_{S^+}$ endows S^+ with an integral weight 2 Hodge structure such that $(S^+_{\mathbb{C}})^{2,0} = \ell$, and it endows V with an integral weight 1 Hodge structure such that $V^{1,0} = Z_\ell$.

Consider the real plane

$$P_\ell := [\ell + \bar{\ell}] \cap S^+_{\mathbb{R}}. \tag{12.2}$$

Let $\{e_1, e_2\}$ be an orthogonal basis of P_ℓ satisfying $(e_1, e_1)_{S^+} = (e_2, e_2)_{S^+} = -2$ such that ℓ is spanned by the isotropic vector $e_1 - ie_2$. Let

$$m : C(S^+) \to \mathrm{End}(S^- \oplus V)$$

be the homomorphism of Corollary 4.7, and denote also by m its extension to the corresponding complex vector spaces. Recall the equality $\tilde{m}(\mathrm{Spin}(S^+)) = \tilde{\mu}(\mathrm{Spin}(V))$, established in (4.32). It identifies $\mathrm{Spin}(S^+)$ with $\mathrm{Spin}(V)$.

Lemma 12.1. *The complex structure J_ℓ is the element of $SO(V_{\mathbb{R}})$ which is the image of the element $m_{e_1} \circ m_{e_2}$ of* $\mathrm{Spin}(S^+_{\mathbb{R}})$.

Proof. $m_{e_1 - ie_2}$ is a nilpotent element of square zero, and $Z_\ell = m_{e_1 - ie_2}(S^-_{\mathbb{C}}) \subset V_{\mathbb{C}}$. Now,

$$(m_{e_1} \circ m_{e_2}) \circ m_{e_1 - ie_2} = i m_{e_1 - ie_2}.$$

Hence, $m_{e_1} \circ m_{e_2}$ acts on Z_ℓ via multiplication by i. Similarly, $Z_{\bar{\ell}} = m_{e_1 + ie_2}(S^-_{\mathbb{C}})$ and $m_{e_1} \circ m_{e_2}$ acts on $Z_{\bar{\ell}}$ by $-i$. Hence, $m_{e_1} \circ m_{e_2}$ is a lift of J_ℓ to an element of $\mathrm{Spin}(S^+_{\mathbb{R}})$. ■

Note that $m_{e_1} \circ m_{e_2}$ acts on $S^+_{\mathbb{R}}$ with P_ℓ as the -1 eigenspace and $P_\ell^\perp$ as the 1 eigenspace. Let $\mathrm{Spin}(V_{\mathbb{R}})_\ell$ be the subgroup of elements acting as the identity on the plane P_ℓ and let $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$ be the subgroup of elements acting as the identity on the orthogonal complement $P_\ell^\perp$. Then $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$ is isomorphic to $U(1)$ and contains $m_{e_1} \circ m_{e_2}$. It consists of elements of the form $a + bm_{e_1} \circ m_{e_2}$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$. Clearly, J_ℓ commutes with the action of the subgroup $\mathrm{Spin}(V_{\mathbb{R}})_\ell \times \mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$ of $\mathrm{Spin}(V_{\mathbb{R}})$. The group $\mathrm{Spin}(V_{\mathbb{R}})_\ell$ acts on Z_ℓ and $Z_{\bar{\ell}}$ and the two representations are dual with respect to the bilinear pairing of $V_{\mathbb{C}}$. Elements of $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$ act on Z_ℓ via scalar product by $e^{i\theta}$, and on $Z_{\bar{\ell}}$ by $e^{-i\theta}$, for some $\theta \in \mathbb{R}$.

Given a class $w \in S^+$ with $(w, w) < 0$, let

$$\Omega_{w^\perp} := \{\ell \in \Omega_{S^+} : (\ell, w) = 0\} \tag{12.3}$$

be the corresponding hyperplane section of Ω_{S^+}. The space $\Omega_{w^\perp}$ is connected as well, and it is the period domain of irreducible holomorphic symplectic manifolds deformation equivalent to generalized Kummers of dimension $2n$ if $(w, w) = -2n - 2$ and $n \geq 2$, by Theorem 3.1. Given $\ell \in \Omega_{w^\perp}$, the integral weight 1 Hodge structure (V, J_ℓ) has the additional property that $\bigwedge^4 V$ admits the integral $\mathrm{Spin}(S^+)_w$-invariant Cayley class, recalled in Section 11, which is of Hodge type $(2, 2)$, by the lemma below. Given a class $h \in w^\perp$, let $\mathrm{Spin}(S^+)_{w,h}$ be the subgroup of $\mathrm{Spin}(S^+)$ stabilizing both w and h.

Lemma 12.2. (1) *Any* $\mathrm{Spin}(S^+)_w$ *invariant class in* $\bigwedge^{2p} V_{\mathbb{C}}$ *is of Hodge type* (p, p) *with respect to* J_ℓ *for all* $\ell \in \Omega_{w^\perp}$.

(2) *Any* $\mathrm{Spin}(S^+)_{w,h}$*-invariant class in* $\bigwedge^{2p} V_{\mathbb{C}}$ *is of Hodge type* (p, p) *with respect to* J_ℓ *for all* $\ell \in \Omega_{w^\perp}$ *such that* $(h, \ell) = 0$.

Proof. (1) The Zariski closure of the image of $\mathrm{Spin}(S^+)_w$ in $\mathrm{GL}(\bigwedge^{2p} V_{\mathbb{C}})$ contains the image of $\mathrm{Spin}(S^+_{\mathbb{R}})_w$, and hence also that of $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$ for all $\ell \in \Omega_{w^\perp}$. A class in $\bigwedge^{2p} V_{\mathbb{C}}$ is of Hodge type (p, p) with respect to J_ℓ if and only if it is $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$-invariant.

(2) $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$ is contained in $\mathrm{Spin}(V_{\mathbb{R}})_{w,h}$. Hence, the Zariski closure of the image of $\mathrm{Spin}(S^+)_{w,h}$ contains that of $\mathrm{Spin}(V_{\mathbb{R}})_{\ell^\perp}$. ∎

Given a negative definite 3-dimensional subspace W of $w^\perp_{\mathbb{R}}$ we get a subgroup $\mathrm{Spin}(S^+_{\mathbb{R}})_{W^\perp}$ of $\mathrm{Spin}(S^+_{\mathbb{R}})_w$, isomorphic[13] to $SU(2)$, consisting of elements of $\mathrm{Spin}(S^+_{\mathbb{R}})$ acting as the identity on $W^\perp$. It fits in the cartesian diagram

$$\begin{array}{ccc}
\mathrm{Spin}(S^+_{\mathbb{R}})_{W^\perp} & \xrightarrow{\subset} & \mathrm{Spin}(S^+_{\mathbb{R}})_w \\
{\scriptstyle 2:1}\downarrow & & \downarrow{\scriptstyle 2:1} \\
SO(W) & \xrightarrow{\subset} & SO(S^+_{\mathbb{R}})_w
\end{array}$$

[13] $\mathrm{Spin}(W)$ is isomorphic to $SU(2)$ and the even Clifford algebra $C(W)^{\mathrm{even}}$ is the quaternion algebra $\mathbb{H}$, by [22, Ch. V, Ex. 1.5 (3) and Cor. 2.10].

where the bottom horizontal homomorphism extends an isometry of W to an isometry of $S^+_{\mathbb{R}}$ acting as the identity on $W^\perp$. If W is spanned by the real and imaginary parts of an element of ℓ and a Kähler class on a marked irreducible holomorphic symplectic manifold Y with period ℓ, then W corresponds to a hyperkähler structure on Y [15, Sec. 1.17]. We show in Sections 12.2 and 12.3 that the subgroup $\mathrm{Spin}(S^+_{\mathbb{R}})_{W^\perp}$ associated to W determines a hyperkähler structure on the complex torus $T_\ell := V_{\mathbb{C}}/[Z_\ell + V]$ and $\mathrm{Spin}(S^+_{\mathbb{R}})_{W^\perp}$ acts on $V_{\mathbb{C}}$ as the group of unit quaternions (Proposition 12.6 below).

12.2. The polarization map $w^\perp \to \bigwedge^2 V^*$

The construction in this section is inspired by O'Grady's recent observation that the third intermediate Jacobians of projective irreducible holomorphic symplectic varieties of generalized Kummer type are abelian fourfolds of Weil type [43]. Let $w \in S^+$ be a primitive class satisfying $(w, w)_{S^+} = -2n$, where n is a positive integer. Let $w^\perp$ be the sublattice of S^+ orthogonal to w. Given a class $h \in S^+$ we get the endomorphism m_h of $S^- \oplus V$, given in Corollary 4.7. It maps V to S^- and S^- to V. Multiplication by h in A_X leaves the direct summand $S^- \oplus V$ invariant and restricts to m_h, by definition.

Let

$$\Theta' : w^\perp \to \mathrm{Hom}(V, V) \tag{12.4}$$

send h to the restriction Θ'_h of $m_w \circ m_h$ to V. Note that $m_w \circ m_h + m_h \circ m_w$ restricts to V as $(w, h)_{S^+} \cdot \mathrm{id}_V$, by (4.29). The latter scalar endomorphism vanishes due to the fact that h is in $w^\perp$. Furthermore, $m_h \circ m_w$ restricts to V as the adjoint of the restriction of $m_w \circ m_h$ with respect to the pairing $(\bullet, \bullet)_V$, by definition of multiplication in A_X. Hence, Θ'_h is anti-self-dual with respect to $(\bullet, \bullet)_V$. The isomorphism $V \to V^*$ given by $x \mapsto (x, \bullet)_V$ induces an isomorphism $a : \mathrm{Hom}(V, V) \to V^* \otimes V^*$ given by

$$a(f)(x, y) := (f(x), y)_V$$

for all $x, y \in V$. An anti-self-dual homomorphism is sent by a to $\bigwedge^2 V^*$. Hence, we get the composite homomorphism

$$\Theta := a \circ \Theta' : w^\perp \to \bigwedge^2 V^*, \tag{12.5}$$

sending h to Θ_h, where $\Theta_h(x, y) := (\Theta'_h(x), y)_V$. More equivariantly, Θ is the composition of the embedding $w^\perp \to \bigwedge^2 S^+$, sending h to $w \wedge h$, with a $\mathrm{Spin}(V)$-module isomorphism $\bigwedge^2 S^+ \cong \bigwedge^2 V^*$ (see [6, Sec. II.4, p. 96, 5-th displayed formula] for the latter).

Let $\mathrm{ort}_{S^+} : G(S^+)^{\mathrm{even}} \to \{\pm 1\}$ be the character (8.3), except that here it will be convenient to have it take values in the multiplicative group $\{\pm 1\}$ rather than in $\mathbb{Z}/2\mathbb{Z}$. An element g of $G(S^+)^{\mathrm{even}}$ acts on V as an isometry if $\mathrm{ort}_{S^+}(g) = 1$, and it reverses the sign of $(\bullet, \bullet)_V$ if $\mathrm{ort}_{S^+}(g) = -1$.

Lemma 12.3. Θ *spans a* $G(S^+)^{\mathrm{even}}_w$*-invariant saturated rank* 1 *subgroup in* $\mathrm{Hom}(w^\perp, \bigwedge^2 V^*)$ *and* $G(S^+)^{\mathrm{even}}_w$ *acts on it via the character* ort_{S^+}*. Moreover,* Θ *spans the* $\mathrm{Spin}(V)_w$*-invariant subgroup of* $\mathrm{Hom}(w^\perp, \bigwedge^2 V^*)$.

Proof. The homomorphism $\Theta'': S^+ \to \mathrm{Hom}(V, S^-)$, $h \mapsto m_h$, is $\mathrm{Spin}(V)$-equivariant, since $\mathrm{Spin}(V)$ acts by algebra automorphisms of A_X. The homomorphism $m_w : S^- \to V$ is $\mathrm{Spin}(V)_w$-equivariant, since $\mathrm{Spin}(V)_w$ acts on A_X by algebra automorphisms fixing w. Invariance of Θ' follows. Invariance of Θ with respect to $\mathrm{Spin}(V)_w$ follows from that of Θ' and the invariance of the bilinear pairing $(\bullet, \bullet)_V$. If $h \in w^\perp$ satisfies $(h.h)_{S^+} = \pm 2$, then m_h restricts to an isomorphism from V to S^-, so Θ is indivisible and spans a saturated subgroup of $\mathrm{Hom}(w^\perp, \bigwedge^2 V^*)$.

We claim that the homomorphism Θ'' spans a rank 1 $G(S^+)^{\mathrm{even}}_w$-invariant sublattice of $\mathrm{Hom}(S^+, \mathrm{Hom}(V, S^-))$ and $G(S^+)^{\mathrm{even}}_w$ acts on it via the character ort_{S^+}. The group $\mathrm{Spin}(V)_w$ is a normal subgroup of $G(S^+)^{\mathrm{even}}_w$, hence the latter maps any $\mathrm{Spin}(V)_w$-invariant submodule to a $\mathrm{Spin}(V)_w$-invariant submodule. Once we prove that there exists a unique rank 1 $\mathrm{Spin}(V)_w$-invariant submodule in $(S^+)^* \otimes V^* \otimes S^-$, it would follow that it is necessarily $G(S^+)^{\mathrm{even}}_w$-invariant and equal to $\mathrm{span}_{\mathbb{Z}}\{\Theta''\}$. We then need to show that $G(S^+)^{\mathrm{even}}_w$ acts on it via the character ort_{S^+}. The bilinear pairing of S^+ is $G(S^+)$-invariant, so that S^+ is a self-dual $G(S^+)^{\mathrm{even}}$-module. V^* is isomorphic to $V \otimes \mathrm{ort}_{S^+}$ as a $G(S^+)^{\mathrm{even}}$-module. Hence, it suffices to prove that $S^+ \otimes V \otimes S^-$ contains a unique rank 1 $\mathrm{Spin}(S^+)_w$-invariant submodule, which is the trivial character of $G(S^+)^{\mathrm{even}}_w$. Now $G(S^+)^{\mathrm{even}} = JG(V)^{\mathrm{even}}J^{-1}$ and $J^{-1} \otimes J^{-1} \otimes J^{-1}$ maps a $G(S^+)^{\mathrm{even}}_w$-invariant element of $S^+ \otimes V \otimes S^-$ to a $G(V)^{\mathrm{even}}_{J^{-1}(w)}$-invariant element of $V \otimes S^- \otimes S^+$. The Triality Principle thus reduces the verification of the above claim to the statement that $\mathrm{Spin}(V)_{J^{-1}(w)}$ has a unique invariant submodule in $V \otimes S^- \otimes S^+$, which is the trivial character of $G(V)^{\mathrm{even}}_{J^{-1}(w)}$. Now V is a self-dual $G(V)^{\mathrm{even}}$-module and $S^-_{\mathbb{Q}} \otimes S^+_{\mathbb{Q}}$ decomposes as a direct sum of the representations $V_{\mathbb{Q}}$ and $\bigwedge^3 V_{\mathbb{Q}}$ as a $G(V_{\mathbb{Q}})^{\mathrm{even}}$-representation, by [6, Sec. 3.4, p. 96, third displayed formula]. $V_{\mathbb{Q}}$ contains a 1-dimensional trivial $\mathrm{Spin}(V)_{J^{-1}(w)}$-submodule, the one spanned by $J^{-1}(w)$, which is also a trivial $G(V)^{\mathrm{even}}_{J^{-1}(w)}$-module. The $\mathrm{Spin}(V)_{J^{-1}(w)}$-invariant submodule of $\bigwedge^3 V_{\mathbb{Q}}$ vanishes.[14] Hence, the statement about Θ'' is proven.

The image m_w of w in $\mathrm{Hom}(V, S^-)$ via Θ'' spans a 1-dimensional $G(S^+)^{\mathrm{even}}_w$-module isomorphic to the restriction of ort_{S^+}, since w spans an invariant $G(S^+)^{\mathrm{even}}_w$-module in S^+, and Θ'' spans a character isomorphic to ort_{S^+}, by the previous paragraph. A similar argument shows that $m_w \in \mathrm{Hom}(S^-, V)$ spans a 1-dimensional $G(S^+)^{\mathrm{even}}_w$-module isomorphic to the restriction of ort_{S^+}. The inverse of the bilinear pairing $(\bullet, \bullet)_V$ spans a 1-dimensional $G(S^+)^{\mathrm{even}}_w$-submodule of $\mathrm{Hom}(V^*, V)$ isomorphic to the restriction of ort_{S^+}. The homomorphism

$$(S^+)^* \otimes V^* \otimes S^- \to (w^\perp)^* \otimes V \otimes V$$

[14] Set $u := J^{-1}(w)$. The 28-dimensional representation $\bigwedge^2 V_{\mathbb{Q}}$ of $\mathrm{Spin}(V_{\mathbb{Q}})_u \cong \mathrm{Spin}((u^\perp)_{\mathbb{Q}})$ decomposes as the direct sum of the 21-dimensional adjoint representation $\mathfrak{so}((u^\perp)_{\mathbb{Q}}) \cong \mathfrak{so}(7)$ and the 7-dimensional fundamental representation $(u^\perp)_{\mathbb{Q}}$, both irreducible. Hence, the $\mathrm{Spin}(V)_u$-invariant submodule of $\mathrm{Hom}((u^\perp)^*, \bigwedge^2 V)$ has rank 1, it is contained in the image of $\mathrm{Sym}^2(u^\perp)_{\mathbb{Q}} \otimes u$, hence its image in $\bigwedge^3 V$ vanishes.

induced by the restriction on the first factor, by the inverse of the bilinear pairing $(\bullet,\bullet)_V$ on the second factor, and by m_w on the third factor, is thus $G(S^+)^{\text{even}}_w$-equivariant. The above displayed homomorphism maps Θ'' to Θ. Hence, $G(S^+)^{\text{even}}_w$ acts via the character ort_{S^+} on the rank 1 submodule spanned by Θ. ∎

Remark 12.4. The $\text{Spin}(V)_w$-equivariant composition

$$\text{Sym}^2(w^\perp) \xrightarrow{\text{Sym}^2\Theta} \text{Sym}^2(\textstyle\bigwedge^2(V^*)) \to \textstyle\bigwedge^4(V^*)$$

is injective and maps the rank 1 trivial submodule of $\text{Sym}^2(w^\perp)$ to the submodule spanned by the Cayley class (see [40, Sec. 2.1]).

Lemma 12.5. *The endomorphism $\Theta'_h := m_w \circ m_h : V \to V$ satisfies*

$$(\Theta'_h)^2 = \frac{-(w,w)(h,h)}{4}\,\text{id}_V = \frac{n(h,h)}{2}\,\text{id}_V\,.$$

Given a period $\ell \in \Omega_{w^\perp}$, the endomorphism $\Theta'_h \in \text{End}(V)$ is a Hodge endomorphism of the integral Hodge structure determined by ℓ whenever h belongs to $\{\ell, w\}^\perp \cap S^+$.

Proof. We have $(m_w \circ m_h)^2 = -(m_w \circ m_w) \circ (m_h \circ m_h) = \frac{n(h,h)}{2}\,\text{id}_V$, where the first equality is due to the identity $m_w \circ m_h = -m_h \circ m_w$ observed above. It remains to prove that the endomorphism $\Theta'_h \in \text{End}(V)$ is a Hodge endomorphism of the integral Hodge structure determined by ℓ whenever h belongs to $\{\ell, w\}^\perp \cap S^+$. Indeed, such an h is of Hodge type, the 2-form $\Theta_h \in \bigwedge^2 V^*$ is of Hodge type, since Θ is an integral homomorphism of Hodge structures, and Θ'_h is obtained from Θ_h via pairing with the class $(\bullet,\bullet)_V$, which is of Hodge type as Z_ℓ is isotropic with respect to $(\bullet,\bullet)_V$. ∎

12.3. Diagonal twistor lines

We have the $\text{Spin}(V)_w$-equivariant injective homomorphism $\Theta : w^\perp \to \bigwedge^2 V^*$, given in (12.5). The explicit construction in terms of w and the symmetric pairing of V exhibits Θ as an integral homomorphism of Hodge structures, since w and the pairing are both of Hodge type. Given a class $h \in w^\perp \cap S^+$ satisfying $(h,h)_{S^+} < 0$, let

$$\Omega_{\{w,h\}^\perp} \tag{12.6}$$

be the hyperplane section of $\Omega_{w^\perp}$ consisting of periods orthogonal to both w and h. Given a period $\ell \in \Omega_{\{w,h\}^\perp}$ we get the $(1,1)$-form Θ_h in $\bigwedge^2 V^*_{\mathbb{R}}$ given in (12.5). We fix an orientation of the negative cone of $w^\perp_{\mathbb{R}}$, which determines an orientation for every negative definite 3-dimensional subspace of $w^\perp_{\mathbb{R}}$ (see, for example, [28, Lemma 4.1]). The real plane P_ℓ, given in (12.2), is naturally oriented by its isomorphism with the complex line ℓ. We will always choose the sign of h so that given a basis $\{e_1, e_2\}$ of P_ℓ compatible with its orientation, the basis $\{e_1, e_2, h\}$ is compatible with the orientation of the negative definite 3-dimensional subspace $P_\ell + \mathbb{R}h \subset w^\perp_{\mathbb{R}}$. The complex torus T_ℓ is an abelian variety for every $\ell \in \Omega_{\{w,h\}^\perp}$, by the following result.

Proposition 12.6. (1) *Θ_h is a $(1,1)$-form on the complex torus*

$$T_\ell := V_{\mathbb{C}}/[Z_\ell + V],$$

whose associated symmetric pairing $g(x,y) := \Theta_h(J_\ell(x), y)$ is definite (and so Θ_h or Θ_{-h} is a Kähler form). We may choose the orientation of the negative cone of $w_{\mathbb{R}}^{\perp}$ so that Θ_h is a Kähler form.

(2) *Let W be a negative definite 3-dimensional subspace of $w_{\mathbb{R}}^{\perp}$ and ℓ, ℓ' two points of $\Omega_{w^\perp}$ such that the planes P_ℓ and $P_{\ell'}$ are both contained in W. Let $h \in P_\ell^{\perp} \cap W$ and $h' \in P_{\ell'}^{\perp} \cap W$ be classes satisfying $(h', h') = (h, h)$ and such that the pairs (h, ℓ) and (h', ℓ') are both compatible with the orientation of W. Then $g'(x,y) := \Theta_{h'}(J_{\ell'}(x), y)$ is the same Kähler metric on $V_{\mathbb{R}}$ as the metric $g(x,y) := \Theta_h(J_\ell(x), y)$. If $(h,h') = 0$, then J_ℓ and $J_{\ell'}$ satisfy $J_\ell J_{\ell'} = -J_{\ell'} J_\ell$.*

Proof. (1) We first express the bilinear form g in terms of the Clifford action using the description of J_ℓ provided by Lemma 12.1:

$$g(x,y) = \Theta_h(J_\ell(x), y) = (\Theta'_h(J_\ell(x)), y)_V = (m_w \circ m_h \circ m_{e_1} \circ m_{e_2}(x), y)_V, \qquad (12.7)$$

where Θ'_h is given in (12.4). The ordered set $\{w, h, e_1, e_2\}$ is an orthogonal basis of a negative definite subspace of $S^+_{\mathbb{R}}$. All negative definite subspaces of $S^+_{\mathbb{R}}$ belong to a single $\mathrm{Spin}(S^+_{\mathbb{R}})$-orbit. It suffices to prove the analogous statement for some orthogonal basis of a negative definite subspace of $S^+_{\mathbb{R}}$. The latter statement translates via the commutative diagram in Corollary 4.7 to the statement that the bilinear pairing

$$(m_{f_1} \circ m_{f_2} \circ m_{f_3} \circ m_{f_4}(x), y)_{S^-}$$

on $S^-_{\mathbb{R}}$ is definite for some orthogonal basis $\{f_1, f_2, f_3, f_4\}$ of a negative definite subspace of $V_{\mathbb{R}}$. Let $\{v_1, v_2, v_3, v_4\}$ be a basis of $H^1(X, \mathbb{Z})$ satisfying $\int_X v_1 \wedge v_2 \wedge v_3 \wedge v_4 = 1$, $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ the dual basis of $H^1(X, \mathbb{Z})^*$ and set $f_i := v_i - \theta_i$, $1 \leq i \leq 4$. Then $(f_i, f_i) = -2$ and so $m^2_{f_i} = -1$. Set $\gamma := m_{f_1} \circ m_{f_2} \circ m_{f_3} \circ m_{f_4}$. Then $\gamma^2 = 1$. Being an isometry, the adjoint of γ is γ^{-1}, and so γ is self-adjoint and the pairing $(\gamma(\bullet), \bullet)_{S^-}$ is symmetric. Regarding $H^1(X, \mathbb{Z})$ as a subgroup of S^- and considering the action of γ on S^-, we have

$$\gamma(v_4) = m_{f_1} \circ m_{f_2} \circ m_{f_3}(m_{f_4}(v_4)) = m_{f_1} \circ m_{f_2} \circ m_{f_3}(-1) = -v_1 \wedge v_2 \wedge v_3,$$

so that $\tau(\gamma(v_4)) = -\gamma(v_4)$ and $(\gamma(v_4), v_4)_{S^-} = -\int_X \gamma(v_4) \wedge v_4 = 1$. Given a permutation σ of $\{1,2,3,4\}$ we similarly have

$$\mathrm{sgn}(\sigma)\gamma(v_{\sigma(4)}) = m_{f_{\sigma(1)}} \circ m_{f_{\sigma(2)}} \circ m_{f_{\sigma(3)}} \circ m_{f_{\sigma(4)}}(v_{\sigma(4)}) = -v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)},$$

so that $\mathrm{sgn}(\sigma)\gamma(v_{\sigma(4)}) \wedge v_{\sigma(4)} = -v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge v_{\sigma(3)} \wedge v_{\sigma(4)} = -\mathrm{sgn}(\sigma) v_1 \wedge v_2 \wedge v_3 \wedge v_4$. We conclude that

$$(\gamma(v_i), v_i)_{S^-} = \int_X v_1 \wedge v_2 \wedge v_3 \wedge v_4 = 1$$

for $1 \leq i \leq 4$. Hence, $(\gamma(\gamma(v_i)), \gamma(v_i)) = 1$, since γ acts as an isometry. But $\{v_i\}_{i=1}^4 \cup \{\gamma(v_i)\}_{i=1}^4$ is a basis of S^-. Hence, the bilinear form $(\gamma(\bullet), \bullet)_{S^-}$ is positive definite.

(2) Given a negative definite 4-dimensional subspace Σ of $S^+_{\mathbb{R}}$ and an orthogonal basis $\{f_1, f_2, f_3, f_4\}$ of Σ, the element $m_{f_1} \circ m_{f_2} \circ m_{f_3} \circ m_{f_4}$ of $C(S^+_{\mathbb{R}})$ depends only on the element $f_1 \wedge f_2 \wedge f_3 \wedge f_4$ of the line $\bigwedge^4 \Sigma$. Indeed, set $\tilde{f_i} := f_i/\sqrt{-Q(f_i)}$, so that $m_{f_i} = \sqrt{-Q(f_i)}\, m_{\tilde{f_i}}$. Then $m_{f_1} \circ m_{f_2} \circ m_{f_3} \circ m_{f_4} = \sqrt{\prod_{i=1}^4 Q(f_i)}\, m_{\tilde{f}_1} \circ m_{\tilde{f}_2} \circ m_{\tilde{f}_3} \circ m_{\tilde{f}_4}$, where $m_{\tilde{f}_1} \circ m_{\tilde{f}_2} \circ m_{\tilde{f}_3} \circ m_{\tilde{f}_4}$ is an element of $\mathrm{Spin}(S^+_{\mathbb{R}})$ which acts on Σ by -1 and it acts on the orthogonal complement $\Sigma^\perp$ in $S^+_{\mathbb{R}}$ by 1. This determines $m_{\tilde{f}_1} \circ m_{\tilde{f}_2} \circ m_{\tilde{f}_3} \circ m_{\tilde{f}_4}$ up to sign, and the sign depends on the orientation of the basis $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\}$. Let $\{e'_1, e'_2\}$ be an orthogonal basis of $P_{\ell'}$ satisfying $(e'_i, e'_i) = -2$, $i = 1, 2$, and such that ℓ' is spanned by $e'_1 - ie'_2$. We get a second orthogonal basis $\{w, h', e'_1, e'_2\}$ of the negative definite subspace $\Sigma := W + \mathbb{R}w$. The two elements $h' \wedge e'_1 \wedge e'_2$ and $h \wedge e_1 \wedge e_2$ of $\bigwedge^3 W$ are equal. Consequently, $w \wedge h \wedge e_1 \wedge e_2 = w \wedge h' \wedge e'_1 \wedge e'_2$ and so

$$m_w \circ m_{h'} \circ m_{e'_1} \circ m_{e'_2} = m_w \circ m_h \circ m_{e_1} \circ m_{e_2}.$$

The equality of the metrics g and g' follows from (12.7).

It remains to prove that $J_\ell J_{\ell'} = -J_{\ell'} J_\ell$ when $(h, h') = 0$. Assume, possibly after rescaling by a positive real factor, that $(h, h)_{S^+} = (h', h')_{S^+} = -2$. Let f be an element of $\{h, h'\}^\perp \cap W$ such that $(f, f) = -2$ and the ordered basis $\{h, h', f\}$ corresponds to the orientation of W. Then $\{h', f\}$ is a basis of P_ℓ, $\{h, f\}$ is a basis of $P_{\ell'}$, J_ℓ or $-J_\ell$ lifts to the element $m_{h'} \circ m_f$ of $\mathrm{Spin}(S^+_{\mathbb{R}})$, and $J_{\ell'}$ or $-J_{\ell'}$ lifts to $m_f \circ m_h$, by Lemma 12.1. We have

$$(m_{h'} \circ m_f) \circ (m_f \circ m_h) = -m_{h'} \circ m_h = -(m_f \circ m_h) \circ (m_{h'} \circ m_f).$$

Now, $-1 \in \mathrm{Spin}(S^+)$ acts on V via multiplication by -1. ∎

Let $\pi : \mathcal{T} \to \Omega_{w^\perp}$ be the pull-back of the universal torus over $IG^+(4, V_{\mathbb{C}})$ via the restriction to $\Omega_{w^\perp}$ of the embedding ζ given in (12.1). Given a 3-dimensional negative definite subspace W of $w^\perp_{\mathbb{R}}$, let $\mathbb{P}_W$ be the smooth conic of isotropic lines in $W_{\mathbb{C}}$. We will refer to $\mathbb{P}_W$ as a *twistor line*, denote by $\pi_W : \mathcal{T}_W \to \mathbb{P}_W$ the pulled-back family, and refer to it as the *twistor family* associated to W. Proposition 12.6 (2) shows that the metric $g_W(x, y) := \Theta_h(J_\ell(x), y)$, $\ell \in \mathbb{P}_W$, $h \in W \cap P_\ell^\perp$, (ℓ, h) compatible with the orientation of W, and $(h, h) = -2$, is independent of ℓ and is indeed a hyperkähler metric, and the twistor family π_W is the one associated to this metric. Given a point $\ell \in \mathbb{P}_W$ we get the commutative diagram

$$\begin{array}{ccccc} T_\ell & \xrightarrow{\subset} & \mathcal{T}_W & \xrightarrow{\subset} & \mathcal{T} \\ \downarrow & & \downarrow{\scriptstyle \pi_W} & & \downarrow{\scriptstyle \pi} \\ \{\ell\} & \xrightarrow{\subset} & \mathbb{P}_W & \xrightarrow{\subset} & \Omega_{w^\perp} \end{array} \qquad (12.8)$$

Remark 12.7. Any two points ℓ, ℓ' in the period domain $\Omega_{w^\perp}$ are connected by a *twistor path*, a sequence $\ell_0, W_1, \ell_1, W_2, \dots, \ell_{k-1}, W_k, \ell_k$ such that $\ell = \ell_0$, $\ell' = \ell_k$, W_i is a negative definite 3-dimensional subspace, and both ℓ_i and ℓ_{i+1} belong to $\mathbb{P}_{W_i}$ for $0 \leq i \leq k$ (see [15, Lemma 8.4]).

Remark 12.8. If $(w, w) = -2n$, $n \geq 3$, then $\Omega_{w^\perp}$ is the period domain of generalized Kummers of dimension $2n - 2$. For all $n \geq 2$, the family $\pi : \mathcal{T} \to \Omega_{w^\perp}$ should be related to generalized (not necessarily commutative) deformations of the derived categories of coherent sheaves over abelian surfaces, in a sense similar to [32]. The 4-dimensional compact complex torus should be thought of as the identity component of the subgroup of the group of auto-equivalences of the deformed triangulated category, which acts trivially on its numerical K-group.

12.4. Abelian fourfolds of Weil type

The following corollary asserts that (T_ℓ, Θ_h) is a polarized abelian variety of Weil type according to [49, Def. 4.9]. This was first observed by O'Grady for the isogenous intermediate Jacobians of projective irreducible holomorphic symplectic manifolds of generalized Kummer deformation type [43]. Given a positive integer d, let the *norm* map $\mathrm{Nm} : \mathbb{Q}[\sqrt{-d}] \to \mathbb{Q}$ be given by $\mathrm{Nm}(a + b\sqrt{-d}) := (a + b\sqrt{-d})(a - b\sqrt{-d}) = a^2 + b^2 d$. Let $n \geq 1$ be an integer and w a primitive element of S^+ satisfying $(w, w) = -2n$.

Corollary 12.9. *Let $h \in w^\perp$ be an integral class and $\ell \in \Omega_{w^\perp}$ be such that the pair (h, ℓ) satisfies the assumptions of Proposition 12.6. Then $d := -n(h, h)/2$ is a positive integer, T_ℓ is an abelian variety, and the ring $\mathbb{Z}[\sqrt{-d}]$ acts on T_ℓ via integral Hodge endomorphisms such that $\lambda^*(\Theta_h) = \mathrm{Nm}(\lambda)\Theta_h$ for all $\lambda \in \mathbb{Z}[\sqrt{-d}]$.*

Proof. If h is integral, then Θ_h is an ample class, by Proposition 12.6 (1), and so T_ℓ is an abelian variety. Integrality of d is due to the fact that the lattice V is even. $\mathbb{Z}[\sqrt{-d}]$ acts by sending $\sqrt{-d}$ to the endomorphism Θ'_h which satisfies $(\Theta'_h)^2 = (-d)\,\mathrm{id}_V$, by Lemma 12.5. Finally, we compute

$$\Theta_h(\Theta'_h(x), \Theta'_h(y)) = ((\Theta'_h)^2(x), \Theta'_h(y)) = -d(x, \Theta'_h(y)) = d(\Theta'_h(x), y) = d\Theta_h(x, y),$$

where the third equality follows from the anti-self-duality of Θ'_h. Set $\lambda := a + b\sqrt{-d}$. We get

$$\begin{aligned}(\lambda^*\Theta_h)(x, y) &= \Theta_h(ax + b\Theta'_h(x), ay + b\Theta'_h(y)) \\ &= (a^2 + b^2 d)\Theta_h(x, y) + ab[\Theta_h(x, \Theta'_h(y)) + \Theta_h(\Theta'_h(x), y)] \\ &= (a^2 + b^2 d)\Theta_h(x, y) = \mathrm{Nm}(\lambda)\Theta_h(x, y).\end{aligned}$$ ■

We recall next a discrete isogeny invariant of abelian varieties of Weil type. Set $K := \mathbb{Q}[\sqrt{-d}]$. Consider the map $H : V_{\mathbb{Q}} \otimes V_{\mathbb{Q}} \to K$ given by

$$H(x, y) := \Theta_h(x, \Theta'_h(y)) + \sqrt{-d}\,\Theta_h(x, y) = d(x, y) + \sqrt{-d}\,(\Theta'_h(x), y). \quad (12.9)$$

Then H is a non-degenerate Hermitian form on the 4-dimensional K-vector space $V_{\mathbb{Q}}$, by [49, Lemma 5.2]. Choose a K-basis $\beta := \{x_1, x_2, x_3, x_4\}$ of $V_{\mathbb{Q}}$ and denote by $\Psi := (H(x_i, x_j))$ the Hermitian matrix of H with respect to β.

Definition 12.10. The *discriminant* $\det H$ of H is the image of $\det(\Psi)$ in $\mathbb{Q}^*/\mathrm{Nm}(K^*)$.

The discriminant $\det H$ is independent of the choice of β, by [49, Lemma 5.2 (3)].

Lemma 12.11. *The Hermitian forms of the abelian fourfolds of Weil type in Corollary 12.9 all have trivial discriminants.*

Proof. Let U be the rank 2 even unimodular lattice with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The isometry group of the orthogonal direct sum of three or more copies of U acts transitively on the set of primitive elements with a fixed self-intersection, by [42, Th. 1.14.4]. The lattice S^+ is isometric to $U^{\oplus 4}$. Hence, the sublattice spanned by $\{w, h\}$ is contained in a sublattice of S^+ isometric to $U \oplus U$. Consequently, the orthogonal sublattice $\{w, h\}^\perp$ contains a sublattice $U_1 \oplus U_2$ of S^+ isometric to $U \oplus U$. Here and below, the notation $(\bullet)^\perp$ is with respect to the bilinear parings $(\bullet, \bullet)_{S^+}$ or $(\bullet, \bullet)_V$, but not with respect to H. In this proof $(\bullet, \bullet)_V$ will be denoted by $(\bullet, \bullet)$. Let $e_i, f_i \in U_i$ satisfy

$$(e_i, e_i) = 2, \quad (f_i, f_i) = -2, \quad (e_i, f_i) = 0, \quad i = 1, 2.$$

We get the four isotropic classes $z_1 := e_1 - f_1$, $z_2 := e_1 + f_1$, $y_1 := e_2 - f_2$, and $y_2 := e_2 + f_2$. The elements $\eta_i := m_{e_i} \circ m_{f_i} \in G(S^+)^{\text{even}}$ commute with Θ'_h (and so are $\mathbb{Z}[\sqrt{-d}]$-module automorphisms) and satisfy $\eta_i^2 = 1 \in C(S^+)$ and

$$(\eta_i(x), \eta_i(x)) = -(x, x), \quad \forall x \in V. \tag{12.10}$$

Let L_{z_i} and L_{y_i} be the 4-dimensional isotropic subspaces of $V_{\mathbb{Q}}$ associated to the isotropic vectors z_i and y_i, $i = 1, 2$, by [6, IV.1.1]. Then L_{z_1} and L_{z_2} are transversal, by [6, III.1.10 and III.1.12]. The automorphism η_i acts on U_i via multiplication by -1 and on $U_i^\perp$ as the identity. The four isotropic lines, and hence also the four isotropic subspaces L_{z_i} and L_{y_i}, $i = 1, 2$, are each invariant with respect to both η_1 and η_2. The action of η_1 on L_{z_i} commutes with that of the subgroup $\mathrm{Spin}(S^+)_{e_1, f_1}$ of $\mathrm{Spin}(S^+)$ stabilizing both e_1 and f_1. Further, L_{z_i} is an irreducible representation of $\mathrm{Spin}(S^+)_{e_1, f_1}$. Hence, η_1 acts on each L_{z_i} via multiplication by a scalar, which is 1 or -1, since $\eta_1^2 = 1$. The automorphism η_1 acts on one of L_{z_1} or L_{z_2} via -1 and on the other as the identity, by (12.10) and the transversality of L_{z_1} and L_{z_2}. Similarly, η_2 acts on one of L_{y_1} or L_{y_2} via -1 and on the other as the identity. The subspaces $L_{z_i, y_j} := L_{z_i} \cap L_{y_j}$, $i, j \in \{1, 2\}$, are 2-dimensional, by [6, III.1.12], since the subspace spanned by $\{z_i, y_j\}$ is isotropic. We conclude that each of $L_{z_i, y_j} := L_{z_i} \cap L_{y_j}$, $i, j \in \{1, 2\}$, is the direct sum of two copies of the same character of the group G generated by η_1 and η_2 and the four characters are distinct. It follows that Θ'_h leaves each L_{z_i, y_j} invariant, since it commutes with G. Hence, each of L_{z_i, y_j} is a 1-dimensional K-subspace of $V_{\mathbb{Q}}$. Let $\hat{}$ be the transposition permutation of $\{1, 2\}$. Then

$$L_{z_i, y_j}^\perp = L_{z_i, y_j} + L_{z_{\hat{i}}, y_j} + L_{z_i, y_{\hat{j}}}.$$

Being Θ'_h-invariant, the right hand side is also the H-orthogonal K-subspace to L_{z_i, y_j}. Furthermore, both $(\bullet, \bullet)_V$ and H induce a non-degenerate bilinear pairing between L_{z_i, y_j} and $L_{z_{\hat{i}}, y_{\hat{j}}}$.

Let $a \in L_{z_1,y_1}$ and $b \in L_{z_2,y_2}$ be elements satisfying

$$(a,b) \neq 0 \quad \text{and} \quad (a, \Theta'_h(b)) = 0.$$

Note that $(a,a) = 0 = (b,b)$. Set $x_1 := a + b$ and $x_2 = a - b$. Then $(x_1, x_2) = 0$, $(x_1, x_1) = 2(a,b) = -(x_2, x_2)$, and

$$H(x_1, x_2) = d(x_1, x_2) + \sqrt{-d}\,(\Theta'_h(x_1), x_2) = -2\sqrt{-d}\,(a, \Theta'_h(b)) = 0.$$

Choose $a' \in L_{z_1,y_2}$ and $b' \in L_{z_2,y_1}$ satisfying

$$(a', b') \neq 0 \quad \text{and} \quad (a', \Theta'_h(b')) = 0.$$

Then $(x_3, x_3) = 2(a', b') = -(x_4, x_4)$ and $H(x_3, x_4) = 0$. We conclude that $\beta := \{x_1, x_2, x_3, x_4\}$ is an H-orthogonal K-basis for $V_{\mathbb{Q}}$ and the β-matrix Ψ of H satisfies

$$\det(\Psi) = \prod_{i=1}^{4} H(x_i, x_i) = d^4 \prod_{i=1}^{4} (x_i, x_i) = d^4 (x_1, x_1)^2 (x_3, x_3)^2 \in (\mathbb{Q}^*)^2.$$

The discriminant is trivial, since $(\mathbb{Q}^*)^2$ is contained in $\mathrm{Nm}(K^*)$. ■

Lemma 12.12. *The subgroup* $\mathrm{Spin}(S^+)_{w,H}$ *of* $\mathrm{Spin}(S^+)_w$ *leaving invariant the Hermitian form H given in* (12.9) *is equal to the subgroup* $\mathrm{Spin}(S^+)_{w,h}$ *stabilizing both w and h.*

Proof. $\mathrm{Spin}(S^+)_{w,h}$ preserves the bilinear pairing $(\bullet, \bullet)_V$, acting as a subgroup of $\mathrm{Spin}(V)$ via the identification (4.32), and $\mathrm{Spin}(S^+)_{w,h}$ commutes with the endomorphism $\Theta'_h := m_w \circ m_h$ of V. Hence, $\mathrm{Spin}(S^+)_{w,h}$ leaves the Hermitian form H invariant. Conversely, the subgroup $\mathrm{Spin}(S^+)_{w,H}$ consists of elements of $\mathrm{Spin}(S^+)_w$ which commute with Θ'_h, by definition of the Hermitian form H. But Θ'_h is the element corresponding to h in an irreducible $\mathrm{Spin}(S^+)_w$-subrepresentation of $\mathrm{Hom}(V, V)$ isomorphic to $w^\perp$. Hence, $\mathrm{Spin}(S^+)_{w,H}$ is contained in $\mathrm{Spin}(S^+)_{w,h}$. ■

12.5. A universal deformation of a moduli space of sheaves

Let $\mathcal{M}(w) := \mathcal{M}_H(w)$ be a smooth and compact moduli space of H-stable sheaves of primitive Mukai vector w of dimension ≥ 8 over an abelian surface X. Let alb : $\mathcal{M}(w) \to \mathrm{Alb}^1(\mathcal{M}(w))$ be the Albanese morphism to the Albanese variety of degree 1. Choose a point $a \in \mathrm{Alb}^1(\mathcal{M}(w))$ and denote by $K_a(w)$ the fiber of alb over a. Let $\iota_a : K_a(w) \to \mathcal{M}(w)$ be the inclusion. A Λ*-marking* for an irreducible holomorphic symplectic manifold M is an isometry $\eta : H^2(M, \mathbb{Z}) \to \Lambda$ with a fixed lattice Λ. There exists a moduli space $\mathfrak{M}_\Lambda$ of Λ-marked irreducible holomorphic symplectic manifolds, which is a non-Hausdorff complex manifold [15]. Let $\eta_0 : H^2(K_a(w), \mathbb{Z}) \to w^\perp$ be the isometry of Theorem 3.1 with respect to the Beauville–Bogomolov–Fujiki pairing on $H^2(K_a(w), \mathbb{Z})$ and the Mukai pairing $-(\bullet, \bullet)_{S^+}$ on S^+. Below we will continue to work with the pairing $(\bullet, \bullet)_{S^+}$ rather than the Mukai pairing. Let $t_0 \in \mathfrak{M}_{w^\perp}$ be the point representing the isomorphism class of $(K_a(w), \eta_0)$ in the moduli space $\mathfrak{M}_{w^\perp}$ of $w^\perp$-marked irreducible holomorphic symplectic manifolds and let $\mathfrak{M}^0_{w^\perp}$ be the connected component of $\mathfrak{M}_{w^\perp}$ containing t_0. Denote by $\mathrm{Per} : \mathfrak{M}^0_{w^\perp} \to \Omega_{w^\perp}$ the period map, sending a marked pair (Y, η) to $\eta(H^{2,0}(Y))$.

Let $\underline{w}^\perp$ be the trivial local system over $\mathfrak{M}^0_{w^\perp}$ with fiber $w^\perp$. There exists a universal family

$$p : \mathcal{Y} \to \mathfrak{M}^0_{w^\perp} \tag{12.11}$$

and a trivialization $\eta : R^2 p_* \mathbb{Z} \to \underline{w}^\perp$ with value η_0 at t_0, by [30, Th. 1.1]. The groups of automorphisms of the fibers of p which act trivially on the second cohomology form a trivial local system $\mathrm{Aut}_0(p)$ over $\mathfrak{M}^0_{w^\perp}$, by [30, Th. 1.1]. The local subsystem $\mathcal{Z}$ of $\mathrm{Aut}_0(p)$ of subgroups which act trivially on the third cohomology as well is thus a trivial local system. We may thus extend the isomorphism of the fiber of $\mathcal{Z}$ over t_0 with the group Γ_w, given in Lemma 10.1, to a trivialization $\psi : \mathcal{Z} \to \underline{\Gamma_w}$, where $\underline{\Gamma_w}$ is the trivial local system with fiber Γ_w.

Let $\mathrm{Per}^*(\pi) : \mathrm{Per}^*\mathcal{T} \to \mathfrak{M}^0_{w^\perp}$ be the pull-back via the period map of the universal torus $\pi : \mathcal{T} \to \Omega_{w^\perp}$ given in diagram (12.8). Ignoring the complex structure, π is a differentiably trivial fibration with fiber the compact torus $V_{\mathbb{R}}/V$. Hence, the local system $\underline{\Gamma_w}$ embeds naturally as a subsystem of torsion subgroups of $\mathrm{Per}^*\mathcal{T}$. Let

$$\mathcal{M} := \mathrm{Per}^*\mathcal{T} \times_{\underline{\Gamma_w}} \mathcal{Y}$$

be the quotient of the fiber product of $\mathrm{Per}^*\mathcal{T}$ and $\mathcal{Y}$ over $\mathfrak{M}^0_{w^\perp}$ by the anti-diagonal action of $\underline{\Gamma_w}$ (this action is defined below diagram (10.1)). Denote by

$$\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp} \tag{12.12}$$

the natural projection and let $\mathcal{M}_t$ be the fiber of Π over $t \in \mathfrak{M}^0_{w^\perp}$. The fiber $\mathcal{M}_{t_0}$ is naturally isomorphic to $\mathcal{M}(w)$, by Lemma 10.1. The relative Albanese map is then

$$\mathrm{alb} : \mathcal{M} \to (\mathrm{Per}^*\mathcal{T})/\underline{\Gamma_w}.$$

Let (Y_t, η_t) be a fiber of $\mathcal{Y}$ over $t \in \mathfrak{M}^0_{w^\perp}$ endowed with the marking determined by η. Set $\ell := \mathrm{Per}(Y_t, \eta_t)$. Let κ be a Kähler class on Y_t and set $h := \eta_t(\kappa)$. Let W be the subspace of $(w^\perp)_{\mathbb{R}}$ spanned by the negative definite plane P_ℓ and h. Then W is negative definite with respect to $(\bullet, \bullet)_{S^+}$. Again denote by $\mathbb{P}_W$ the conic of isotropic lines in $W_{\mathbb{C}}$. We get a twistor family $p_W : \mathcal{Y}_W \to \mathbb{P}_W$ of deformations of Y_t [15, Sec. 1.17]. The marking η_t extends to a trivialization η_W of $R^2 p_{W,*}\mathbb{Z}$, since $\mathbb{P}_W$ is simply connected. The pair $(\mathcal{Y}_W, \eta_W)$ determines an embedding $\iota_W : \mathbb{P}_W \to \mathfrak{M}^0_{w^\perp}$ such that $\mathrm{Per} \circ \iota_W$ is the inclusion of $\mathbb{P}_W$ in $\Omega_{w^\perp}$. The image $\widetilde{\mathbb{P}}_W := \iota_W(\mathbb{P}_W)$ is called the *twistor line through the point* (Y_t, η_t) *associated to the Kähler class* κ.

The twistor family $p_W : \mathcal{Y}_W \to \mathbb{P}_W$ admits a differential geometric construction, which we now recall following [2]. Let M be an irreducible holomorphic symplectic manifold, I its complex structure, and κ a Kähler class. There exists a unique Ricci-flat Kähler metric g such that $\omega_I(\bullet, \bullet) := g(I(\bullet), \bullet)$ is a Kähler form with class κ. Furthermore, there exists an action of the quaternion algebra $\mathbb{H} := \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ on the real tangent bundle of M via parallel endomorphisms such that $\omega_J + i\omega_K$ is a non-degenerate holomorphic 2-form and I is the original complex structure. We will refer to g as the *hyperkähler metric associated to the Kähler form* κ. Every purely imaginary unit quaternion $\lambda := aI + bJ + cK, a^2 + b^2 + c^2 = 1$, yields a complex structure on M and a Kähler form $\omega_\lambda := g(\lambda(\bullet), \bullet)$ with class in the positive definite 3-dimensional subspace $W := \mathbb{R}\kappa + [H^{2,0}(M) + H^{0,2}(M)] \cap H^2(M, \mathbb{R})$ such that the line $\ell_\lambda := H^{2,0}(M, \lambda)$ is one of the two isotropic lines in $W_{\mathbb{C}}$ orthogonal to ω_λ. The sphere of unit purely imaginary quaternions gets identified with the complex plane conic $\mathbb{P}_W \subset \mathbb{P}(W_{\mathbb{C}})$ of isotropic lines with respect to the Beauville–Bogomolov–Fujiki pairing, by sending λ to ℓ_λ. One gets a complex structure on the differentiable manifold $\mathbb{P}_W \times M$ such that the first projection p_W is holomorphic and the fiber over $\ell_\lambda \in \mathbb{P}_W$ is endowed with the complex structure λ. The resulting family $p_W : \mathbb{P}_W \times M \to \mathbb{P}_W$ is the twistor family $p_W : \mathcal{Y}_W \to \mathbb{P}_W$, if we let $M = Y_t$.

Given a marked pair $(Y, \eta) \in \mathfrak{M}^0_{w^\perp}$ and a Kähler class κ on Y, we get the negative definite 3-dimensional subspace W of $w^\perp_{\mathbb{R}}$ containing $h := \eta(\kappa)$ and such that $W_{\mathbb{C}}$ contains $\ell := \mathrm{Per}(Y, \eta)$. We get the hyperkähler metric g_W on Y, associated to the Kähler form κ, and the metric g on T_ℓ associated to the Kähler form Θ_h in Proposition 12.6. Hence, we get a hyperkähler metric on the product $T_\ell \times Y$, which we call the *product hyperkähler metric associated to* (Y, η, κ). The assignment $\lambda \mapsto [\omega_\lambda] \in W$, associating to a purely imaginary unit quaternion λ the class in W of the Kähler form, identifies the sphere in W of self-intersection $-(\kappa, \kappa)$ with the sphere of complex structures on Y associated to κ. Similarly, the sphere in W of self-intersection -2 was identified with the sphere of complex structures on T_ℓ associated to Θ_h. Rescaling κ so that $(\kappa, \kappa) = 2$, we get an identification of the sphere of complex structures on Y and T_ℓ and so an action of $\mathbb{H}$ on the real tangent bundle of $T_\ell \times Y$, so that the purely imaginary unit quaternions act via parallel complex structures.

Definition 12.13. (1) The *product hyperkähler structure on $T_\ell \times Y$, associated to a marked pair $(Y, \eta) \in \mathfrak{M}^0_{w^\perp}$ with period ℓ and a Kähler class κ on Y,* is the data consisting of the product hyperkähler metric associated to (Y, η, κ) and the above action of the quaternion algebra $\mathbb{H}$.

(2) The above product hyperkähler structure on $T_\ell \times Y$ is equivariant with respect to the anti-diagonal action of Γ_w and it thus descends to a hyperkähler structure on the quotient $[T_\ell \times Y]/\Gamma_w$, which we call the *natural hyperkähler structure on $[T_\ell \times Y]/\Gamma_w$ associated to a marked pair $(Y, \eta) \in \mathfrak{M}^0_{w^\perp}$ with period ℓ and a Kähler class κ on Y.*

Denote by

$$\Pi_W : \mathcal{M}_W \to \widetilde{\mathbb{P}}_W \tag{12.13}$$

the restriction of $\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp}$ to the twistor line $\widetilde{\mathbb{P}}_W$ through (Y_t, η_t) associated to the Kähler class κ. Note that $\mathcal{M}_W$ is the quotient by the anti-diagonal action of Γ_w on the fiber product of the two twistor families $\pi_W : \mathcal{T}_W \to \mathbb{P}_W$, given in (12.8), and $p_W : \mathcal{Y}_W \to \mathbb{P}_W$ over the same twistor line $\mathbb{P}_W$ in $\Omega_{w^\perp}$. The fiber product is itself a twistor deformation with respect to the product hyperkähler structure on $T_\ell \times Y_t$. Similarly, the twistor family

Π_W displayed above is the one associated to the natural hyperkähler structure on $\mathcal{M}_t := [T_\ell \times Y_t]/\Gamma_w$.

Let (Y, η) be a marked pair in $\mathfrak{M}^0_\Lambda$ and set $\ell := \mathrm{Per}(Y, \eta)$. Let $H^{2p}(Y \times T_\ell, \mathbb{Q})^{\Gamma_w}$ be the subspace invariant under the anti-diagonal action of Γ_w. Note the isomorphism $H^*(Y \times T_\ell, \mathbb{Q})^{\Gamma_w} \cong H^*(Y, \mathbb{Q})^{\Gamma_w} \otimes H^*(T_\ell, \mathbb{Q})$. Recall that the quotient $\mathrm{Mon}(Y)/\Gamma_w$ has a canonical normal subgroup N, obtained by conjugating the image of $\mathrm{Spin}(S^+)_w$ in $\mathrm{Mon}(K_a(w))/\Gamma_w$ via a parallel transport operator (Lemma 10.7). Every local system over $\mathfrak{M}^0_{w^\perp}$ is trivial, by [30, Lemma 2.1]. Hence, we get an identification of N with $\mathrm{Spin}(S^+)_w$.

Lemma 12.14. *Any class in $H^{2p}(Y \times T_\ell, \mathbb{Q})^{\Gamma_w}$ which is invariant under the diagonal $\mathrm{Spin}(S^+)_w$ monodromy action is of Hodge type (p, p).*

Proof. If the Hodge operator belongs to the Lie algebra of the identity component of the Zariski closure of a group acting on the cohomology rings of two compact Kähler manifolds M_1 and M_2, then it belongs to the Lie algebra of the identity component of the Zariski closure of its diagonal action on $M_1 \times M_2$ (see for example [29, proof of Lemma 3.2]). Hence, the statement follows from Lemmas 10.8 and 12.2. ∎

12.6. Third intermediate Jacobians of generalized Kummers

The third Betti number of a generalized Kummer is 8 [10, p. 50]. The Hodge group $H^{0,3}(\mathcal{Y}_t)$ vanishes for the fiber $\mathcal{Y}_t$ of p over $t \in \mathfrak{M}^0_{w^\perp}$, and so the third intermediate Jacobian $H^{1,2}(\mathcal{Y}_t)/H^3(\mathcal{Y}_t, \mathbb{Z})$ is an abelian fourfold whenever $\mathcal{Y}_t$ is projective, by the Hodge–Riemann bilinear relations.

Lemma 12.15. *There exists a global isogeny between the family of third intermediate Jacobians of $p : \mathcal{Y} \to \mathfrak{M}^0_{w^\perp}$ and the family* $\mathrm{Per}^*\mathcal{T}$ *over $\mathfrak{M}^0_{w^\perp}$.*

Proof. Let $\underline{V}$ be the trivial local system with fiber V over $\mathfrak{M}^0_{w^\perp}$. It is isomorphic to the weight 1 variation of integral Hodge structures of the family $\mathrm{Per}^*\mathcal{T}$. It suffices to construct a global $\mathrm{Spin}(V)_w$-equivariant isogeny between the integral local systems $\underline{V}$ and $R^3p_*\mathbb{Z}$, by Lemma 12.14.

Let $\underline{S^-}$ be the trivial local system with fiber S^- over $\mathfrak{M}^0_{w^\perp}$. Clifford multiplication $m_w : V \to S^-$ by w induces a $\mathrm{Spin}(V)_w$-equivariant isogeny $m_w : \underline{V} \to \underline{S^-}$. It remains to construct a $\mathrm{Spin}(V)_w$-equivariant isogeny from $\underline{S^-}$ to $R^3p_*\mathbb{Z}$. Let $\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp}$ be the universal deformation of $\mathcal{M}(w)$ given in (12.12). Let $\underline{Q}^3$ be the quotient of $R^3\Pi_*\mathbb{Z}$ by the cup product image of $R^1\Pi_*\mathbb{Z} \otimes R^2\Pi_*\mathbb{Z}$. Recall that the fiber $\mathcal{Y}_{t_0}$ was the Albanese fiber of the moduli space $\mathcal{M}(w)$, by construction. $\underline{Q}^3$ has rank 8, since $\mathcal{M}(w) = [\mathcal{Y}_{t_0} \times X \times \hat{X}]/\Gamma_w$ and Γ_w acts trivially on $H^i(\mathcal{Y}_{t_0} \times X \times \hat{X}, \mathbb{Q})$ for $i \leq 3$. The homomorphism $\bar{\theta}_3 : S^- \to Q^3(\mathcal{M}(w))$, given in (8.13), is a $\mathrm{Spin}(V)_w$-equivariant isomorphism, by Lemmas 8.9 and 8.10 (extended to general Mukai vector w via Theorem 9.3). The restriction homomorphism $H^3(\mathcal{M}(w), \mathbb{Z}) \to H^3(\mathcal{Y}_{t_0}, \mathbb{Z})$ factors through an injective[15] $\mathrm{Spin}(V)_w$-equivariant homomorphism $Q^3(\mathcal{M}(w)) \to H^3(\mathcal{Y}_{t_0}, \mathbb{Z})$, since $H^1(\mathcal{Y}_{t_0}, \mathbb{Z}) = 0$. Composing the latter with $\bar{\theta}_3$ we get a $\mathrm{Spin}(V)_w$-equivariant isogeny $S^- \to H^3(\mathcal{Y}_{t_0}, \mathbb{Z})$. Every local system over $\mathfrak{M}^0_{w^\perp}$ is trivial, by [30, Lemma 2.1]. We get a $\mathrm{Spin}(V)_w$-equivariant isogeny from $\underline{S}^-$ to $R^3 p_* \mathbb{Z}$. ■

Remark 12.16. Note that for fourfolds Y of generalized Kummer type the polarization map $\Theta : w^\perp \to \bigwedge^2 V^*$ given in (12.5) is conjugate via Mukai's isometry $H^2(Y, \mathbb{Z}) \cong w^\perp$ and the isogeny between $H^3(Y, \mathbb{Z})$ and V of the above lemma to a map proportional to

$$H^2(Y, \mathbb{Z}) \to \bigwedge^2 H^3(Y, \mathbb{Z})^*,$$

given by $h \mapsto \int_Y h \cup x \cup y$ for $x, y \in H^3(Y, \mathbb{Z})$, as both belong to the rank 1 invariant subgroup under the $\mathrm{Spin}(S^+)_w$ monodromy action on $\mathrm{Hom}(H^2(Y, \mathbb{Z}), \bigwedge^2 H^3(Y, \mathbb{Z})^*)$ (see Lemma 12.3). O'Grady used the latter map to construct the polarization on the intermediate Jacobians, and the positivity of the metric in Proposition 12.6 (1) follows in this case by the Hodge–Riemann bilinear relations. When Y is of generalized Kummer type of dimension $2n \geq 6$, O'Grady integrates the product $h \cup x \cup y \cup \beta^{n-2}$, where $\beta \in H^{2,2}(Y, \mathbb{Q})$ is the Beauville–Bogomolov–Fujiki class [43].

13. Hyperholomorphic sheaves

We prove in this section Theorem 1.5 about the algebraicity of the Hodge–Weil classes on abelian fourfolds of Weil type of discriminant 1. Let $\mathcal{M}(w) := \mathcal{M}_H(w)$ be a smooth and compact moduli space of H-stable sheaves of primitive Mukai vector w of dimension ≥ 8 over an abelian surface X. Given a class $\lambda \in S^+_X := H^{\mathrm{even}}(X, \mathbb{Z})$, denote by λ_i its projection to $H^i(X, \mathbb{Z})$.

Lemma 13.1. *The Brauer class $\alpha \in H^2_{\mathrm{an}}(\mathcal{M}(w), \mathcal{O}^*_{\mathcal{M}(w)})$ of the universal sheaf has order divisible by $g_w := \gcd\{(w, \lambda) : \lambda \in S^+_X,\ \lambda_2 \in H^{1,1}(X, \mathbb{Z})\}$.*

Proof. The fiber $K_a(w)$ of $\mathcal{M}(w)$ over $a \in \mathrm{Alb}^1(\mathcal{M}(w))$ has dimension ≥ 4, by assumption, and so $H^2(K_a(w), \mathbb{Z})$ is Hodge isometric to $w^\perp$, by Yoshioka's Theorem 3.1. It suffices to prove that g_w divides the order of the restriction of α to $K_a(w)$. The proof of the latter fact is identical to that of [29, Lemma 7.5 (2)]. ■

Let r be an even integer satisfying $r \geq 6$. Let X be an abelian surface with a cyclic Picard group generated by an ample class H with $h := c_1(H)$ satisfying

$$(h, h)_{S^+_X} = -(2r^2 + r)$$

(so $\int_X h^2 = 2r^2 + r$). Set $w := (r, h, r)$. Then $(w, w)_{S^+_X} = -r$ and $g_w = r$.

[15] That restriction homomorphism is known to be surjective for generalized Kummer fourfolds, by [20, Th. 6.33].

Lemma 13.2. (1) *The sheaf E_F over $\mathcal{M}(w)$ in Theorem 11.1 is α-twisted by a Brauer class α of order equal to the rank r of E_F. Consequently, the sheaf E_F does not have any non-trivial subsheaf of lower rank and $\mathcal{E}nd(E_F)$ is κ-slope-polystable with respect to every Kähler class κ on $\mathcal{M}(w)$. Furthermore, the first Chern class of every direct summand of $\mathcal{E}nd(E_F)$ vanishes.*

(2) *The sheaf E of Theorem 11.1 (1) is $\pi_1^*(\alpha^{-1})\pi_2^*\alpha$-twisted, where α is a Brauer class of order equal to the rank r of E. Consequently, E is $(\pi_1^*\kappa + \pi_2^*\kappa)$-slope-polystable with respect to every Kähler class κ on $\mathcal{M}(w)$. Furthermore, the first Chern class of every direct summand of $\mathcal{E}nd(E)$ vanishes.*

Proof. The proofs of the two parts are identical. We prove (1). The order of the Brauer class necessarily divides the rank of the sheaf. In our case the rank r divides the order of α by Lemma 13.1. Hence, they are equal. The polystability of $\mathcal{E}nd(E)$ is proven for any torsion free reflexive sheaf E twisted by a Brauer class of order equal to its rank in [29, Prop. 6.6]. The vanishing of the first Chern classes of the direct summands is proven in [29, Lemma 7.2]. ∎

Theorem 13.3. *The sheaf E_F deforms with $\mathcal{M}(w)$ to a reflexive sheaf, locally free on the complement of a point, over every fiber of the universal family* (12.12)*. The sheaf E deforms with $\mathcal{M}(w) \times \mathcal{M}(w)$ to a reflexive sheaf, locally free away from the diagonal, over the cartesian square of every fiber of the universal family* (12.12).

Proof. The sheaf E_F is κ-slope-stable with respect to every Kähler class κ on $\mathcal{M}(w)$, by Lemma 13.2. The class $c_2(\mathcal{E}nd(E_F))$ is $\mathrm{Spin}(S_X^+)_w$-invariant with respect to the monodromy representation of Theorem 8.6, by Theorem 11.1. Hence, $c_2(\mathcal{E}nd(E_F))$ remains of Hodge type $(2,2)$ along any flat deformation to every fiber of the family Π in (12.12), by Lemma 12.14. Let $\eta : H^2(K_a(w), \mathbb{Z}) \to w^\perp$ be the inverse of Mukai's Hodge isometry. It follows that the sheaf E_F deforms as a twisted sheaf along the twistor family (12.13) of the natural hyper-Kähler structure on $\mathcal{M}(w)$ (Definition 12.13 (2)) associated to any Kähler class κ on the generalized Kummer $K_a(w)$ and the marking η, by [53, Th. 3.19], which is generalized to the case of twisted sheaves in [29, Cor. 6.12]. The sheaf E_F deforms, furthermore, along every generic twistor path in $\mathfrak{M}^0_{w^\perp}$, by [29, Prop. 6.17]. The statement follows from the fact that every point in $\mathfrak{M}^0_{w^\perp}$ is connected to $(\mathcal{M}(w), \eta)$ via a generic twistor path, by [50, Ths. 3.2 and 5.2.e]. The proof of the statement for the sheaf E is identical. ∎

Let T_ℓ, $\ell \in \Omega_{w^\perp}$, be an abelian fourfold of Weil type with ample class Θ_h, $h \in w^\perp$, as in Corollary 12.9. It admits complex multiplication by $K := \mathbb{Q}[\sqrt{-d}]$, where $d = (w,w)(h,h)/4$. Let $\mathrm{Spin}(S^+)_{w,h}$ be the subgroup of $\mathrm{Spin}(S^+)$ stabilizing both w and h. The group $\mathrm{Spin}(S^+)_{w,h}$ is an arithmetic subgroup of $\mathrm{Spin}(S^+_{\mathbb{R}})_{w,h} \cong \mathrm{Spin}(4,2,\mathbb{R})$ and $\mathrm{Spin}(4,2,\mathbb{R})$ is isomorphic to $SU(2,2)$ [14, IX.4.3 B (vi)]. $SU(2,2)$ is the special Mumford–Tate group of polarized abelian fourfolds of Weil type [55], [49, Theorem 6.11].

Theorem 13.4. *The subspace* $H^4(T_\ell,\mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}}$*, consisting of classes invariant under* $\mathrm{Spin}(S^+)_{w,h}$*, is* 3*-dimensional and consists of algebraic classes.*

Proof. The complexification $\mathrm{Spin}(S^+_{\mathbb{C}})_{w,h}$ of $\mathrm{Spin}(S^+)_{w,h}$ is isomorphic to $SL(4,\mathbb{C})$. The invariant subspace $H^4(T_\ell,\mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}}$ is 3-dimensional, by [40, Prop. 2], and it consists of Hodge type $(2,2)$ classes, by Lemma 12.2 (2). This agrees with Weil's observation that $H^{2,2}(A,\mathbb{Q})$ is 3-dimensional for the general polarized abelian fourfold A of Weil type with complex multiplication by the field $K=\mathbb{Q}[\sqrt{-d}]$ in each complete family [55], [49, Ths. 4.11 and 6.12]. Regarding $H^1(A,\mathbb{Q})$ as a 4-dimensional K vector space, we get that $\bigwedge^4_K H^1(A,\mathbb{Q})$ is a 1-dimensional K vector space, which is a 2-dimensional $\mathbb{Q}$-subspace of $\bigwedge^4_{\mathbb{Q}} H^1(A,\mathbb{Q})$. Weil proved that $H^{2,2}(A,\mathbb{Q})$ contains $\bigwedge^4_K H^1(A,\mathbb{Q})$ and for a generic A of Weil type the equality

$$H^{2,2}(A,\mathbb{Q}) = \mathrm{span}_{\mathbb{Q}}\{\Theta_h^2\} + \bigwedge\nolimits^4_K H^1(A,\mathbb{Q})$$

holds [55], [49, Ths. 4.11 and 6.12]. It follows that $\bigwedge^4_K H^1(T_\ell,\mathbb{Q})$ is contained in $H^4(T_\ell,\mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}}$ for a generic $\ell\in\Omega_{w^\perp}$ such that $(\ell,h)=0$. The inclusion

$$\bigwedge\nolimits^4_K H^1(T_\ell,\mathbb{Q}) \subset H^4(T_\ell,\mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}} \tag{13.1}$$

must thus hold for all $\ell\in\Omega_{w^\perp}$ such that $(\ell,h)=0$, as it is a closed condition.

A class $\alpha\in\bigwedge^4_{\mathbb{Q}} H^1(T_\ell,\mathbb{Q})$ belongs to $\bigwedge^4_K H^1(T_\ell,\mathbb{Q})$ if and only if

$$(\alpha,\lambda(v_1)\wedge v_2\wedge v_3\wedge v_4) = (\alpha, v_1\wedge\cdots\wedge\lambda(v_i)\wedge\cdots\wedge v_4) \tag{13.2}$$

for $2\leq i\leq 4$, for all $v_i\in H_1(T_\ell,\mathbb{Q})$ and all $\lambda\in K$. The structure of a 1-dimensional K-vector space on $\bigwedge^4_K H^1(T_\ell,\mathbb{Q})$ is given by

$$(\lambda\alpha, v_1\wedge v_2\wedge v_3\wedge v_4) := (\alpha,\lambda(v_1)\wedge v_2\wedge v_3\wedge v_4).$$

The inclusion (13.1) can be seen more directly using the following description of the subspace $\bigwedge^4_K H^1(T_\ell,\mathbb{Q})$ of $H^4(T_\ell,\mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}}$. Let Z_1 and Z_2 be the two maximal isotropic subspaces of $H^1(T_\ell,\mathbb{C})\cong V_{\mathbb{C}}$ corresponding to the two isotropic lines in the plane $\mathrm{span}_{\mathbb{C}}\{w,h\}$ in $S^+_{\mathbb{C}}$. Explicitly, Z_i is the kernel of $m_{\lambda_i}: V_{\mathbb{C}}\to S^-_{\mathbb{C}}$, where $\lambda_i = w\pm\frac{2\sqrt{-d}}{(h,h)}h$. Note that Z_i is defined over K. The $\mathrm{Spin}(S^+_{\mathbb{C}})_{w,h}$-action on Z_i factors through $SL(Z_i)$ and so it acts trivially on $\bigwedge^4 Z_i$. Each $\bigwedge^4 Z_i$, $i=1,2$, is defined over K and the non-trivial element in $\mathrm{Gal}(K/\mathbb{Q})$ interchanges the two, so their direct sum is defined over $\mathbb{Q}$. Now, Z_1 and Z_2 are the two eigenspaces of the endomorphism Θ'_h of $V_{\mathbb{C}}$ in Lemma 12.5. If $\{z_1,z_2,z_3,z_4\}$ is a basis for Z_i, then

$$z_1\wedge\cdots\wedge(a+b\Theta'_h)(z_i)\wedge\cdots\wedge z_4 = (a+bc_i)(z_1\wedge z_2\wedge z_3\wedge z_4)$$

for all $a,b\in\mathbb{Q}$, where the eigenvalue c_i of Θ'_h is $\pm\sqrt{-d}$. Hence, (13.2) holds for $\alpha := z_1\wedge z_2\wedge z_3\wedge z_4$ and for $v_i\in V^*\otimes_{\mathbb{Q}} K$. The subspace $\bigwedge^4_K H^1(T_\ell,\mathbb{Q})$ thus contains the intersection of $\bigwedge^4_{\mathbb{Q}} H^1(T_\ell,\mathbb{Q})$ with the 2-dimensional complex subspace $\bigwedge^4 Z_1 + \bigwedge^4 Z_2$ of $\bigwedge^4_{\mathbb{C}} H^1(T_\ell,\mathbb{Q})$. Both subspaces are 2-dimensional over $\mathbb{Q}$, hence equal.

Let $\varphi : K \to \mathrm{End}_{\mathbb{Q}}(H^1(T_\ell, \mathbb{Q}))$ be the homomorphism sending $\sqrt{-d}$ to the endomorphism Θ'_h given in (12.4). We get the degree 4 polynomial map $\varphi_4 : K \to \mathrm{End}_{\mathbb{Q}}(\bigwedge^4_{\mathbb{Q}} H^1(T_\ell, \mathbb{Q}))$ sending λ to $\bigwedge^4 \varphi(\lambda)$ and the latter restricts to the 1-dimensional K-vector space $\bigwedge^4_K H^1(T_\ell, \mathbb{Q})$ as scalar multiplication by $\lambda^4 \in K$. The image of the map $\lambda \mapsto \lambda^4$, from K to K, spans K as a $\mathbb{Q}$-vector space. Hence, given any non-zero $\alpha \in \bigwedge^4_K H^1(T_\ell, \mathbb{Q})$, the set $\varphi_4(\lambda)(\alpha)$, $\lambda \in K$, spans $\bigwedge^4_K H^1(T_\ell, \mathbb{Q})$. In contrast, $\varphi_4(\lambda)(\Theta_h^2) = \mathrm{Nm}(\lambda)^2\Theta_h^2$, by Corollary 12.9, and so the 1-dimensional subspace $\mathrm{span}_{\mathbb{Q}}\{\Theta_h^2\}$ is invariant under $\varphi_4(K)$.

The Cayley class C, associated to the $\mathrm{Spin}(S^+)_w$-action on $H^4(T_\ell, \mathbb{Z})$, and Θ_h^2 are linearly independent, by [40, Prop. 2]. Hence, the 2-dimensional $\mathbb{Q}$-subspaces $\mathrm{span}_{\mathbb{Q}}\{C, \Theta_h^2\}$ and $\bigwedge^4_K H^1(T_\ell, \mathbb{Q})$ of the 3-dimensional $H^4(T_\ell, \mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}}$ intersect non-trivially along a 1-dimensional $\mathbb{Q}$-subspace. Choose a non-zero class α in their intersection. It follows that $H^4(T_\ell, \mathbb{Q})^{\mathrm{Spin}(S^+)_{w,h}}$ is spanned by Θ_h^2 and the 2-dimensional $\mathbb{Q}$-subspace spanned by the $\varphi_4(K)$-translates of α.

Let $\mathcal{M}_t$, $t \in \mathfrak{M}^0_{w^\perp}$, be any fiber of the universal family $\Pi : \mathcal{M} \to \mathfrak{M}^0_{w^\perp}$ given in (12.12) and let $(\mathcal{Y}_t, \eta_t)$ be the marked fiber of the universal family $p : \mathcal{Y} \to \mathfrak{M}^0_{w^\perp}$ of generalized Kummer type given in (12.11). Let ℓ be the period of $(\mathcal{Y}_t, \eta_t)$. Let $\iota : T_\ell \to \mathcal{M}_t$ be the inclusion of a general fiber of $\mathcal{M}_t \to \mathcal{Y}_t/\Gamma_w$. Let E_t be a deformation of E_F to the fiber $\mathcal{M}_t$ as in Theorem 13.3. Then $c_2(\mathcal{E}nd(E_t))$ restricts to T_ℓ as a non-zero $\mathrm{Spin}(S^+)_w$-invariant class $\iota^* c_2(\mathcal{E}nd(E_t))$, by Theorem 11.1, hence the restriction is a non-zero integral multiple of the Cayley class. The Cayley class is thus algebraic. Hence, so is the class α above. The ring $\mathbb{Z}[\sqrt{-d}]$ acts on T_ℓ via holomorphic group endomorphisms, which are necessarily algebraic, by Corollary 12.9. This algebraic action induces the cohomological action on $H^4(T_\ell, \mathbb{Q})$ by $\varphi_4(\mathbb{Z}[\sqrt{-d}])$. Hence, the 2-dimensional subspace spanned by the $\varphi_4(K)$-translates of α consists of algebraic classes as well. ∎

Proof of Theorem 1.5. The two discrete invariants, K and the discriminant, of a polarized abelian fourfold of Weil type (A, K, h) determine a 4-dimensional connected period domain of all polarized abelian fourfolds of Weil type with these two invariants, up to an isogeny compatible with the subspaces of Hodge–Weil classes, by [48, Lemma 4, Sec. 6 and Sec. 7]. Every polarized abelian fourfold of Weil type (A, K, h) with discriminant 1 and imaginary quadratic field $K := \mathbb{Q}[\sqrt{-d}]$ is thus isogenous to T_ℓ for some period ℓ in the period domain $\Omega_{\{w,h'\}^\perp}$ given in (12.6), for some integral classes $w \in S^+$ and $h' \in w^\perp$ of negative self-intersection such that $(h', h')(w, w)/4 = d$, by Corollary 12.9 and Lemma 12.11. The push-forward of an algebraic class via an isogeny of abelian varieties is algebraic. Theorem 1.5 thus follows from Theorem 13.4. ∎

14. The generalized Hodge conjecture for codimension 2 cycles on IHSM's of Kummer type

We prove Theorem 1.10 in this section verifying the generalized Hodge conjecture for codimension 2 algebraic cycles homologous to 0 on every projective irreducible holomorphic symplectic manifold of generalized Kummer deformation type. We will need a few preparatory results. Let X be an abelian surface, H a polarization on X, $w \in H^{\text{even}}(X, \mathbb{Z})$ a primitive Mukai vector, and assume that the moduli space $\mathcal{M}$ of H-stable sheaves on X with Chern character w is smooth and projective of dimension ≥ 8. Assume that there exists a universal sheaf $\mathcal{U}$ over $X \times \mathcal{M}$ (untwisted). The latter assumption is equivalent to the equality $\gcd\{(w, \lambda) : \lambda \in S_X^+,\ \lambda_2 \in H^{1,1}(X, \mathbb{Z})\} = 1$, by [38, Appendix 2]. Let π_{ij} be the projection from $\mathcal{M} \times X \times \mathcal{M}$ onto the product of the i-th and j-th factors. Let $E := \mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^*\mathcal{U}, \pi_{23}^*\mathcal{U})$ be the relative first extension sheaf over $\mathcal{M} \times \mathcal{M}$. Let $\text{alb} : \mathcal{M} \to \text{Alb}(\mathcal{M})$ be the Albanese map. Denote by $K_t(w)$, $t \in \text{Alb}(\mathcal{M})$, the fiber $\text{alb}^{-1}(t)$. Let $e_t : K_t(w) \hookrightarrow \mathcal{M}$ be the inclusion. Set $k := \frac{1}{2}\dim_{\mathbb{C}}(K_t(w))$.

Given $F \in \mathcal{M}$, let E_F be the restriction of E to $\{F\} \times \mathcal{M}$. Fix $F_0 \in \mathcal{M}$ and consider the map

$$\text{AJ}_E : \mathcal{M} \to J^2(K_0(w)) \tag{14.1}$$

sending F to the Abel–Jacobi image of an algebraic cycle representing the Chow class $e_t^*[c_2(E_F^\vee \overset{L}{\otimes} E_F) - c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0})]$. The latter class is the same as the one given in (1.8). In the introduction the sheaf E depended on a parameter b, and AJ_{E_b} was denoted by AJ_b for short. The proof of the surjectivity of AJ_E requires a few lemmas.

Set $\hat{X} := \text{Pic}^0(X)$. The group $X \times \hat{X}$ acts on $\mathcal{M}$ via $(x, L)F = \tau_{x,*}(F) \otimes L$, where $\tau_x : X \to X$ is the translation by the point $x \in X$. Given $g := (x, L) \in X \times \hat{X}$, denote by $\tilde{g} : \mathcal{M} \to \mathcal{M}$ the automorphism of $\mathcal{M}$.

Lemma 14.1. *The isomorphism $E_{\tilde{g}(F)} \cong \tilde{g}_*(E_F)$ holds for all $F \in \mathcal{M}$ and $g \in X \times \hat{X}$.*

Proof. The sheaf E is homogeneous with respect to the diagonal action of $X \times \hat{X}$ on $\mathcal{M} \times \mathcal{M}$, i.e., we have an isomorphism $(\tilde{g} \times \tilde{g})^*(E) \cong E$ for every $g \in X \times \hat{X}$. Indeed, given $x \in X$ set $\tilde{\tau}_x := \tau_x \times \text{id}_{\mathcal{M}} : X \times \mathcal{M} \to X \times \mathcal{M}$ and observe that for every $L \in \text{Pic}^0(X)$ we have the natural isomorphism

$$\begin{aligned}\mathcal{E}xt^1_{\pi_{13}}(\pi_2^*L \otimes \pi_{12}^*\tilde{\tau}_{x,*}\mathcal{U}, \pi_2^*L \otimes \pi_{23}^*\tilde{\tau}_{x,*}\mathcal{U}) &\cong \mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^*\tilde{\tau}_{x,*}\mathcal{U}, \pi_{23}^*\tilde{\tau}_{x,*}\mathcal{U})\\ &\xrightarrow{\tau_x^*} \mathcal{E}xt^1_{\pi_{13}}(\pi_{12}^*\mathcal{U}, \pi_{23}^*\mathcal{U}).\end{aligned}$$

The isomorphism $(\tilde{g} \times \tilde{g})^*(E) \cong E$ yields $(\tilde{g} \times \text{id}_{\mathcal{M}})^*(E) \cong (\text{id}_{\mathcal{M}} \times \tilde{g})_*(E)$, explaining the second isomorphism below:

$$E_{\tilde{g}(F)} \cong ((\tilde{g} \times \text{id}_{\mathcal{M}})^*(E))_F \cong ((\text{id}_{\mathcal{M}} \times \tilde{g})_*(E))_F \cong \tilde{g}_*(E_F). \tag{14.2}$$

■

Set $\widetilde{\mathcal{M}} := X \times \hat{X} \times K_0(w)$. Let

$$a : \widetilde{\mathcal{M}} \to \mathcal{M} \tag{14.3}$$

be the restriction of the action morphism $X \times \hat{X} \times \mathcal{M} \to \mathcal{M}$, given by $a(x, L, F) = \tau_{x,*}(F) \otimes L$. Then a is a surjective étale morphism. Denote by $\mathcal{U}_0$ the pull-back of

$\mathcal{U}$ to $X \times K_0(w)$ via $\mathrm{id}_X \times e_0$. Let π_{ij} be the projection from $X \times X \times \widehat{X} \times K_0(w)$ onto the product of the i-th and j-th factors. Set $\widetilde{\mathcal{U}} := (\mathrm{id}_X \times a)^*\mathcal{U}$. The restriction of $\widetilde{\mathcal{U}}$ to $X \times \{(x, L, F)\}$ is isomorphic to $\tau_{x,*}(F) \otimes L$. The restriction of $\pi_{14}^*\mathcal{U}_0$ to $X \times \{(x, L, F)\}$ is isomorphic to F. Define $\eta : X \times X \to X \times X$ by $\eta(x_1, x_2) = (x_1 + x_2, x_2)$. Let $\widetilde{\eta}$ be the automorphism of $X \times X \times \widehat{X} \times K_0(w)$ given by $\eta \times \mathrm{id}_X \times \mathrm{id}_{K_0(w)}$. We conclude that there is a line bundle N over $X \times \widehat{X} \times K_0(w)$ and an isomorphism

$$\widetilde{\mathcal{U}} \cong \pi_{234}^* N \otimes \widetilde{\eta}^* \pi_{14}^* \mathcal{U}_0 \otimes \pi_{13}^* \mathcal{P}. \tag{14.4}$$

Let p_{ij} be the projection from $\widetilde{\mathcal{M}} = X \times \widehat{X} \times K_0(w)$ onto the product of the i-th and j-th factors. Let $\mathbb{C}_0$ be the sky-scraper sheaf supported on the origin in X. Let $\Phi_{\widetilde{\mathcal{U}}} : D^b(X) \to D^b(\widetilde{\mathcal{M}})$ be the integral functor with kernel $\widetilde{\mathcal{U}}$.

Lemma 14.2. $\Phi_{\widetilde{\mathcal{U}}}(\mathbb{C}_0) \cong N \otimes p_{13}^* \mathcal{U}_0$.

Proof. Note that η restricts to $\{0\} \times X$ as the diagonal embedding of X in $X \times X$. Hence, the restriction of $\pi_{14} \circ \widetilde{\eta}$ to $\{0\} \times X \times \widehat{X} \times K_0(w)$ is equal to that of π_{24}. The restriction of $\pi_{13}^* \mathcal{P}$ to $\{0\} \times \widetilde{\mathcal{M}}$ is the trivial line bundle. The statement follows from (14.4). ∎

Let $\gamma : X \times K_0(w) \to \widetilde{\mathcal{M}}$ be given by $\gamma(x, F) = (x, \widehat{0}, F)$, where $\widehat{0} \in \widehat{X}$ represents $\mathcal{O}_X$. Let $\pi_{K_0(w)}$ be the projection from $X \times K_0(w)$ to $K_0(w)$.

Lemma 14.3. $L\gamma^* \Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X) \cong (\gamma^* N) \otimes L\pi_{K_0(w)}^* R\pi_{K_0(w),*} \mathcal{U}_0$.

Proof. Let q_{ij} be the projection from $X \times X \times K_0(w)$ to the product of the i-th and j-th factors. Set $\widetilde{\gamma} := (\mathrm{id}_X \times \gamma) : X \times X \times K_0(w) \to X \times \widetilde{\mathcal{M}}$. We have $\Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X) \cong R\pi_{234,*}[\pi_{234}^* N \otimes \widetilde{\eta}^* \pi_{14}^* \mathcal{U}_0 \otimes \pi_{13}^* \mathcal{P}] \cong N \otimes R\pi_{234,*}[\widetilde{\eta}^* \pi_{14}^* \mathcal{U}_0 \otimes \pi_{13}^* \mathcal{P}]$. Hence, $L\gamma^* \Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X)$ is isomorphic to $\gamma^* N \otimes Rq_{23,*}[L\widetilde{\gamma}^* \{\widetilde{\eta}^* \pi_{14}^* \mathcal{U}_0\}]$, by the triviality of $(\pi_{13} \circ \widetilde{\gamma})^* \mathcal{P}$ and cohomology and base change for the right square in the cartesian diagram

$$\begin{array}{ccccc}
X \times K_0(w) & \xleftarrow{q_{13}} & X \times X \times K_0(w) & \xrightarrow{\widetilde{\gamma}} & X \times X \times \widehat{X} \times K_0(w) \\
\downarrow{\scriptstyle \pi_{K_0(w)}} & & \downarrow{\scriptstyle q_{23}} & & \downarrow{\scriptstyle \pi_{234}} \\
K_0(w) & \xleftarrow{\pi_{K_0(w)}} & X \times K_0(w) & \xrightarrow{\gamma} & X \times \widehat{X} \times K_0(w)
\end{array}$$

Set $\widehat{\eta} := \eta \times \mathrm{id}_{K_0(w)}$. The isomorphism $L\gamma^* \Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X) \cong (\gamma^* N) \otimes Rq_{23,*}[\widehat{\eta}^* q_{13}^* \mathcal{U}_0]$ thus follows from the equality $\pi_{14} \circ \widetilde{\eta} \circ \widetilde{\gamma} = q_{13} \circ \widehat{\eta}$. The isomorphism

$$L\gamma^* \Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X) \cong (\gamma^* N) \otimes Rq_{23,*}[q_{13}^* \mathcal{U}_0]$$

follows from the equality $q_{23} = q_{23} \circ \widehat{\eta}$ and the fact that $R\widehat{\eta}_* L\widehat{\eta}^*$ is the identity. The statement follows by cohomology and base change with respect to the left square in the above diagram. ∎

Proof of Theorem 1.10. Up to translation, the morphism $\overline{\mathrm{AJ}}_b$, given in (1.9), is determined by the homomorphism $\overline{\mathrm{AJ}}_{b,*} : H_1(\mathrm{Alb}(\mathcal{M}_b), \mathbb{Z}) \to H_1(J^2(Y_b), \mathbb{Z})$. The latter depends continuously on b and E_b. Any two points b_1, b_2 of $\mathfrak{M}^0_{\omega^\perp}$ with projective Y_{b_1} and Y_{b_2} can be connected by a subfamily of Π with projective fibers. It thus suffices to prove Theorem 1.10 for one point in $\mathfrak{M}^0_{\omega^\perp}$. We will prove it for a moduli space of sheaves $\mathcal{M}$ as in (14.1).

Let $\mathrm{CH}^i(\mathcal{M})$ be the group of codimension i algebraic cycles in $\mathcal{M}$ and $\mathrm{CH}^i(\mathcal{M})_0$ its subgroup of cycles homologous to zero. Given a point $[F]$ in $\mathcal{M}$ representing the isomorphism class of a sheaf F, let $\iota_F : X \times \widehat{X} \to \mathcal{M}$ be the map onto the orbit of $[F]$ under the $X \times \widehat{X}$-action. Set

$$J^2(\mathcal{M}) := H^3(\mathcal{M}, \mathbb{C})/[F^2H^3(\mathcal{M}, \mathbb{C}) + H^3(\mathcal{M}, \mathbb{Z})],$$

where $F^2H^3(\mathcal{M}, \mathbb{C}) := H^{3,0}(\mathcal{M}) \oplus H^{2,1}(\mathcal{M})$ is the second subspace in the Hodge filtration. We have the commutative diagram of Abel–Jacobi maps

$$\begin{array}{ccccccc} X\times\widehat{X} & \xrightarrow{\iota_{F_0}} & \mathcal{M} & \xrightarrow{\psi} & \mathrm{CH}^2(\mathcal{M})_0 & \xrightarrow{\mathrm{AJ}_{\mathcal{M}}} & J^2(\mathcal{M}) \\ & & & \searrow & \downarrow{\scriptstyle e_0^*} & & \downarrow{\scriptstyle r} \\ & & & & \mathrm{CH}^2(K_0(w))_0 & \xrightarrow{\mathrm{AJ}_K} & J^2(K_0(w)) \end{array}$$

where the horizontal map ψ sends F to $c_2(E_F^\vee \overset{L}{\otimes} E_F) - c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0})$, with both the dual $E_F^\vee$ and the tensor product taken in the derived category, and the right vertical homomorphism r is induced by the restriction homomorphism $e_0^* : H^3(\mathcal{M}, \mathbb{C}) \to H^3(K_0(w), \mathbb{C})$.

It suffices to prove the surjectivity of $r \circ \mathrm{AJ}_{\mathcal{M}} \circ \psi \circ \iota_{F_0}$, as $r \circ \mathrm{AJ}_{\mathcal{M}} \circ \psi$ is equal to AJ_E given in (14.1). Being a morphism of complex tori, the composition is induced by a linear homomorphism (its differential) from $H_1(X \times \widehat{X}, \mathbb{R}) \cong V_{\mathbb{R}}^*$ to $H^3(K_0(w), \mathbb{R})$. Both are irreducible $\mathrm{Spin}(V)_w$-representations. Hence, it suffices to prove that the differential of $r \circ \mathrm{AJ}_{\mathcal{M}} \circ \psi \circ \iota_{F_0}$ is $\mathrm{Spin}(V)_w$-equivariant and it does not vanish.

Let Z_0 be an algebraic cycle representing the Chow class $c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0})$. Then $\widetilde{g}_*(Z_0)$ represents $c_2(E_{\widetilde{g}(F_0)}^\vee \overset{L}{\otimes} E_{\widetilde{g}(F_0)})$, by (14.2). Given a smooth path γ from 0 to $g_1 \in X \times \widehat{X}$ we get the cochain $\Gamma := \bigcup_{g \in \gamma} \widetilde{g}(Z_0)$ with boundary $\widetilde{g}_1(Z_0) - Z_0$. The point $(\mathrm{AJ}_{\mathcal{M}} \circ \psi)(\widetilde{g}_1(F_0))$ is the projection to $J^2(\mathcal{M})$ of the class in $H^{1,2}(\mathcal{M}) \oplus H^{0,3}(\mathcal{M})$ which corresponds to the linear functional sending a class ϕ in $H^{2k+3,2k+2}(\mathcal{M}) \oplus H^{2k+4,2k+1}(\mathcal{M})$ to $\int_\Gamma \phi$. Let ξ be a tangent vector to γ at 0. Let

$$da : T_0[X \times \widehat{X}] \to H^0(T\mathcal{M})$$

be the homomorphism induced by the action of $X \times \widehat{X}$ on $\mathcal{M}$. Then the restriction of $da(\xi)$ to Z_0 maps to a global section of the real normal bundle of Z_0 in Γ. The differential of $(\mathrm{AJ}_{\mathcal{M}} \circ \psi)$ maps $da(\xi)$ to $H^{1,2}(\mathcal{M}) \oplus H^{0,3}(\mathcal{M})$, hence to a linear functional on $H^{2k+3,2k+2}(\mathcal{M}) \oplus H^{2k+4,2k+1}(\mathcal{M})$, whose value at a cohomology class ϕ is equal to $\int_{Z_0}(\phi, da(\xi))$, where $(\bullet, da(\xi)) : H^q(\mathcal{M}, \Omega^p_{\mathcal{M}}) \to H^q(\mathcal{M}, \Omega^{p-1}_{\mathcal{M}})$ is induced by contraction with $da(\xi)$ (see [12, Lecture 6]). The differential of $\mathrm{AJ}_{\mathcal{M}} \circ \psi \circ \iota_{F_0}$ at $0 \in X \times \widehat{X}$

thus maps ξ to the linear functional

$$\phi \mapsto \int_{\mathcal{M}} (\phi, da(\xi)) \cup c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0}),$$

since the homology class of Z_0 is Poincaré dual to the cohomology class of $c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0})$. The value at $\xi \otimes \phi$ of the differential of $\mathrm{AJ}_E \circ \iota_{F_0}$ at 0 is thus given by

$$d_0(\mathrm{AJ}_E \circ \iota_{F_0})(\xi \otimes \phi) = \int_{\mathcal{M}} (e_{0,*}(\phi), da(\xi)) \cup c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0}), \tag{14.5}$$

where $e_{0,*} : H^{4k-3}(K_0(w), \mathbb{C}) \to H^{4k+5}(\mathcal{M}, \mathbb{C})$ is the Gysin homomorphism, since dr^* is induced by $(e_0^*)^* = e_{0,*}$.

Set $\widetilde{\mathcal{M}} := X \times \widehat{X} \times K_0(w)$. Let $[\mathrm{pt}] \in H^8(X \times \widehat{X}, \mathbb{Z})$ be the class Poincaré dual to a point in $X \times \widehat{X}$. We prove next that

$$(e_{0,*}(\phi), da(\xi)) = a_*(\phi \boxtimes ([\mathrm{pt}], \xi)) \tag{14.6}$$

for all $\phi \in H^{4k-3}(K_0(w), \mathbb{C})$, where $\boxtimes$ denotes the outer product and we consider $H^{4k-3}(K_0(w), \mathbb{C}) \otimes H^7(X \times \widehat{X}, \mathbb{C})$ as a subspace of $H^{4k+4}(\widetilde{\mathcal{M}}, \mathbb{C})$ via the Künneth decomposition. Let $\tilde{e}_0 : K_0(w) \to \widetilde{\mathcal{M}}$ be given by $t \mapsto (0, 0, t)$. We have $e_0 = a \circ \tilde{e}_0$, where a is given in (14.3), and so the Gysin map $e_{0,*}$ is the composition $a_* \circ \tilde{e}_{0,*}$. Let Γ_w be the Galois group of $a : \widetilde{\mathcal{M}} \to \mathcal{M}$. A class β in $H^*(\widetilde{\mathcal{M}}, \mathbb{C})$ decomposes as $\beta = a^*(\beta') + \beta''$, where β'' belongs to the direct sum of non-trivial Γ_w-representations. Then

$$\int_{\mathcal{M}} \alpha \cup a_*(\beta) = \int_{\widetilde{\mathcal{M}}} a^*(\alpha) \cup \beta = \int_{\widetilde{\mathcal{M}}} a^*(\alpha \cup \beta') = \deg(a) \int_{\mathcal{M}} \alpha \cup \beta'$$

for all $\alpha \in H^*(\mathcal{M})$. Hence, $a_*(\beta) = \deg(a)\beta'$. Now, $\tilde{e}_{0,*}(\phi) = \phi \boxtimes [\mathrm{pt}]$, the outer product of ϕ with $[\mathrm{pt}]$. The group Γ_w acts on $X \times \widehat{X} \times K_0(w)$ via its translation action on $X \times \widehat{X}$ and its action on $K_0(w)$ as automorphisms acting trivially on $H^i(K_0(w), \mathbb{C})$ for $i \leq 3$ and so also for $i \geq 4k - 3$, by Lemma 10.1 (4). Hence, given $\phi \in H^{2k-1,2k-2}(K_0(w))$, the class $\phi \boxtimes [\mathrm{pt}]$ is Γ_w-invariant and equal to $a^*\phi'$ for some $\phi' \in H^{2k+3,2k+2}(\mathcal{M})$, and $e_{0,*}(\phi) = \deg(a)\phi' = (k + 1)^4\phi'$. Let $\tilde{\xi}$ be the global tangent vector of $\widetilde{\mathcal{M}} = X \times \widehat{X} \times K_0(w)$ corresponding to ξ via the natural isomorphism $H^0(T[X \times \widehat{X}]) = H^0(T\widetilde{\mathcal{M}})$. Then $a^*(da(\xi)) = \tilde{\xi}$ via the isomorphism $a^*T\mathcal{M} \cong T\widetilde{\mathcal{M}}$. We have

$$\frac{1}{\deg(a)} a^*(e_{0,*}(\phi), da(\xi)) = a^*(\phi', da(\xi)) = (a^*\phi', \tilde{\xi}) = (\phi \boxtimes [\mathrm{pt}], \tilde{\xi}) = \phi \boxtimes ([\mathrm{pt}], \xi).$$

Applying a_* to both sides we get (14.6).

Combining (14.5) and (14.6) we get

$$d_0(\mathrm{AJ}_E \circ \iota_{F_0})(\xi \otimes \phi) = \int_{\widetilde{\mathcal{M}}} (\phi \boxtimes ([\mathrm{pt}], \xi)) \cup a^* c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0}).$$

The differential $d_0(\mathrm{AJ}_E \circ \iota_{F_0})$ is $\mathrm{Spin}(V)_w$-equivariant, by the $\mathrm{Spin}(V)_w$-invariance of $c_2(E_{F_0}^\vee \overset{L}{\otimes} E_{F_0})$ established in Theorem 11.1. The right hand side in the above displayed equation does not vanish for some $\xi \otimes \phi \in H^0(T[X \times \widehat{X}]) \otimes H^{2k-1,2k-2}(K_0(w))$

if and only if the Künneth direct summand of $a^*c_2(E^\vee_{F_0} \overset{L}{\otimes} E_{F_0})$ in $H^1(X \times \widehat{X}) \otimes H^3(K_0(w))$ does not vanish. This is the case if and only if the Künneth direct summand of $a^*c_2(E_{F_0})$ in $H^1(X \times \widehat{X}) \otimes H^3(K_0(w))$ does not vanish, since the corresponding direct summand of $a^*c_1(E_{F_0})^2$ vanishes, as $H^2(X \times \widehat{X} \times K_0(w))$ decomposes as $H^2(X \times \widehat{X}) \oplus H^2(K_0(w))$. For the same reason, the direct summand of $a^*c_2(E_{F_0})$ in $H^1(X \times \widehat{X}) \otimes H^3(K_0(w))$ does not vanish if and only if that of $a^* \mathrm{ch}_2(E_{F_0})$ does not vanish.

Assume next that w is the Mukai vector $(1, 0, -1-k)$ of the ideal sheaf of a length $k+1$ subscheme. The class $a^* \mathrm{ch}_2(E_{F_0})$ is equal to $-\mathrm{ch}_2(\Phi_{\widetilde{\mathcal{U}}}(F_0^\vee))$, as $a^*E_{F_0}$ is the first sheaf cohomology of $\Phi_{\widetilde{\mathcal{U}}}(F_0^\vee)$, the second sheaf cohomology is the direct sum of the sky-scraper sheaves of the points of $\widetilde{\mathcal{M}}$ over $[F_0]$, and all other sheaf cohomologies vanish. We have

$$\mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(F_0^\vee)] = \mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X)] - (k+1)\,\mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(\mathbb{C}_0)].$$

Let $\widehat{0}$ be the point of $\widehat{X}$ representing the isomorphism class of the trivial line bundle. It suffices to prove that the Künneth direct summand of $\mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(\mathbb{C}_0)]$ in $H^1(X \times \widehat{X}) \otimes H^3(K_0(w))$ restricts non-trivially to $X \times \{\widehat{0}\} \times K_0(w)$, while the Künneth direct summand of $\mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X)]$ in $H^1(X \times \widehat{X}) \otimes H^3(K_0(w))$ restricts to zero in $X \times \{\widehat{0}\} \times K_0(w)$.

The object $(\mathrm{id}_X \times \widetilde{e}_0)^*\Phi_{\widetilde{\mathcal{U}}}(\mathbb{C}_0)$ is isomorphic to the tensor product of a line bundle with $\mathcal{U}_0$, by Lemma 14.2. Hence, the Künneth component of $(\mathrm{id}_X \times \widetilde{e}_0)^* \mathrm{ch}_2(\Phi_{\widetilde{\mathcal{U}}}(\mathbb{C}_0))$ in $H^1(X) \otimes H^3(K_0(w))$ is equal to that of $\mathrm{ch}_2(\mathcal{U}_0)$. We have the equality

$$e_0^*\pi_{\mathcal{M},*}[\pi_X^*(\lambda) \cup \mathrm{ch}(\mathcal{U})] = \pi_{K_o(w),*}[\pi_X^*(\lambda) \cup \mathrm{ch}(\mathcal{U}_0)], \tag{14.7}$$

by the projection formula applied to the cartesian diagram

$$\begin{array}{ccc} X \times K_0(w) & \xrightarrow{\mathrm{id}_X \times e_0} & X \times \mathcal{M} \\ {\scriptstyle \pi_{K_0(w)}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{M}}} \\ K_0(w) & \xrightarrow{\quad e_0 \quad} & \mathcal{M} \end{array}$$

The only graded summand of $\mathrm{ch}(\mathcal{U}_0)$ (resp. $\mathrm{ch}(\mathcal{U})$) which contributes to the homomorphism (14.7) from $H^3(X)$ to $H^3(K_0(w))$ (resp. to $H^3(\mathcal{M})$) is $\mathrm{ch}_2(\mathcal{U}_0)$ (resp. $\mathrm{ch}_2(\mathcal{U})$). The left hand side of (14.7) induces an isomorphism from $H^{\mathrm{odd}}(X, \mathbb{Q})$ onto $H^3(K_0(w), \mathbb{Q})$, by Lemma 8.9 and the surjectivity of h_3, given in (10.2), established in the paragraph preceding Lemma 10.1. Hence, so does the right hand side, and the Künneth direct summand of $\mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(\mathbb{C}_0)]$ in $H^1(X \times \widehat{X}) \otimes H^3(K_0(w))$ restricts non-trivially to $X \times \{\widehat{0}\} \times K_0(w)$.

The Künneth component in $H^1(X) \otimes H^3(K_0(w))$ of $\gamma^* \mathrm{ch}_2[\Phi_{\widetilde{\mathcal{U}}}(\mathcal{O}_X)]$ is equal to that of $(\gamma^*N) \otimes L\pi^*_{K_0(w)}R\pi_{K_0(w),*}\mathcal{U}_0$, by Lemma 14.3, and hence also to that of $L\pi^*_{K_0(w)}R\pi_{K_0(w),*}\mathcal{U}_0$, as observed above. The Künneth component in $H^1(X) \otimes H^3(K_0(w))$ of the latter clearly vanishes. This completes the proof of Theorem 1.10. ∎

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15. Glossary of notation

V	the lattice $H^1(X,\mathbb{Z}) \oplus H^1(\hat{X},\mathbb{Z})$	Eq. (4.1)
$\mathrm{Spin}(V)$	the spin group of a lattice or a vector space V	Eq. (4.6)
$\mathrm{Spin}(V)_w$	the stabilizer of w in $\mathrm{Spin}(V)$	Sec. 1.1
$\mathrm{Pin}(V)$	the pin group of a lattice or a vector space V	Eq. (4.6)
$G(V)$	one of the Clifford groups	Eq. (4.6)
$G_0(V)$	one of the Clifford groups	Eq. (4.6)
$G(V)^{\mathrm{even}}$	the even Clifford group	Eq. (4.6)
$G(V)^{\mathrm{even}}_w$	the stabilizer of w in $G(V)^{\mathrm{even}}$	Sec. 1.1
$C(V)$	the Clifford algebra of a lattice or a vector space V	Sec. 4.1
$C(V)^{\mathrm{even}}$	the even direct summand of $C(V)$	Sec. 4.1
$C(V)^{\mathrm{odd}}$	the odd direct summand of $C(V)$	Sec. 4.1
S	the cohomology $H^*(X,\mathbb{Z})$ as the spin representation	Sec. 4.1
S^+	$H^{\mathrm{even}}(X,\mathbb{Z})$ as the half-spin representation	Sec. 4.1
S^-	$H^{\mathrm{odd}}(X,\mathbb{Z})$ as the half-spin representation	Sec. 4.1
A_X	the algebra $V \oplus S^+ \oplus S^-$	Sec. 4.3
m	spin representation of the Clifford algebra $C(V)$ or $C(S^+)$	Eq. (4.10), Cor. 4.7
m_w	the value of m on $w \in S^+$	Eq. (4.29)
m	spin representation of the Clifford group	Eq. (4.11)
$\tilde{m}$	embedding of $G(S^+)$ in $\mathrm{GL}(A_X)$	Eq. (4.31)
$\tilde{O}(S^+)$	subgroup of $\mathrm{GL}(S^+)$ preserving the pairing up to sign	Sec. 4.1
$S\tilde{O}(S^+)$	subgroup of $\mathrm{SL}(S^+)$ preserving the pairing up to sign	Sec. 4.1
$(\bullet,\bullet)_V$	the pairing on V	Eq. (1.1)
$\langle\bullet,\bullet\rangle$	the Mukai pairing on $H^{\mathrm{even}}(X,\mathbb{Z})$	Eq. (3.1)
$(\bullet,\bullet)_S$	the pairing on S	Eq. (4.15)
τ	main anti-automorphism of $C(V)$	Eq. (4.4)
$\tilde{\tau}$	lift of τ to $G(S^+)^{\mathrm{even}}$	Eq. (4.24)
τ_X	the automorphism of A_X induced by $\tilde{\tau}$	Eq. (8.2)
α	main involution of $C(V)$	Eq. (4.5)
$\tilde{\alpha}$	element in the center of $\mathrm{Spin}(V)$	Eq. (4.17)
ρ	the homomorphism $G(V) \to O(V)$	Eq. (4.8)
$\tilde{\mu}$	the homomorphism $G(V) \to \mathrm{GL}(A_X)$	Eq. (4.26)
ort	the orientation character	Eq. (4.12), (8.3)
$SO_+(V)$	the kernel of the orientation character in $SO(V)$	Sec. 4.1
$O_+(V)$	the kernel of the orientation character in $O(V)$	Sec. 4.1
L_w	the endomorphism $w \wedge \bullet : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$	Eq. (4.3)
D_θ	endomorphism of $H^*(X,\mathbb{Z})$ of contraction with θ	Sec. 4.1
PD	Poincaré duality homomorphism	Eq. (4.20)
$\hat{X}$	$\mathrm{Pic}^0(X)$ of an abelian surface X	Sec. 2

$X^{[n]}$	Hilbert scheme of length n subschemes of a surface X	Sec. 1.1
$X^{(n)}$	the n-th symmetric product of an abelian surface X	Sec. 1.1
$K_X(m)$	generalized Kummer variety of an abelian surface X	Sec. 1.1
$\mathcal{M}_H(v)$	moduli space of stable sheaves with Mukai vector v	Sec. 3
$K_a(v)$	fiber of the Albanese map $\mathcal{M}_H(v) \to X \times \hat{X}$ over a	Sec. 3
$\mathrm{Mon}(Y)$	monodromy group of a compact Kähler manifold Y	Def. 1.1
mon	the monodromy representation on $H^*(\mathcal{M}(w), \mathbb{Z})$	Eq. (8.6)
$\overline{\mathrm{mon}}$	the monodromy representation on $H^*(K_a(v), \mathbb{Z})$	Prop. 10.2
Γ_X	group of points of order n on the abelian surface X	Sec. 1.1
Γ_w	subgroup of torsion points in the torus $V_{\mathbb{R}}/V$	Rem. 4.3
$\gamma_{g,\epsilon}(\mathcal{E}_1, \mathcal{E}_2)$	correspondence in $H^{2m}(\mathcal{M}(w_1) \times \mathcal{M}(w_2), \mathbb{Q})$	Eq. (6.3)
D_M	homomorphism acting by $(-1)^i$ on $H^{2i}(M)$	Sec. 6
$d_X, d_{\mathcal{M}(w)}$	a choice of factorization $D_{X\times\mathcal{M}(w)} = d_X \otimes d_{\mathcal{M}(w)}$	Eq. (6.6)
$\tilde{\theta}$	homomorphism $S \to H^*(\mathcal{M}(s_n), \mathbb{Q})$	Eq. (8.7)
$Q^d(\mathcal{M}(w))$	quotient of $H^d(\mathcal{M}(w), \mathbb{Z})$	Sec. 8.3
S_X^j	$\begin{cases} S_X^+ & \text{if } j \text{ is even, } j \neq 2, \\ S_X^+ \cap s_n^\perp & \text{if } j = 2, \\ S_X^- & \text{if } j \text{ is odd} \end{cases}$	Eq. (8.8)
$\tilde{\theta}_j$	homomorphism $S_X^j \to Q^j(\mathcal{M}(w)) \otimes_{\mathbb{Z}} \mathbb{Q}$	Eq. (8.8)
h_i	homomorphism $Q^i(\mathcal{M}(w)) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^i(K_a(v), \mathbb{Q})$	Eq. (10.2)
ι_F	embedding of $X \times \hat{X}$ in the orbit of F in $\mathcal{M}_H(w)$	Eq. (9.1)
q_w	$\iota_F^* : H^*(\mathcal{M}_H(w), \mathbb{Z}) \to H^*(X \times \hat{X}, \mathbb{Z})$	Eq. (9.2)
$\Omega_{w^\perp}$	period domain	Eq. (12.3)
$\Omega_{\{w,h\}^\perp}$	period domain	Eq. (12.6)
$\mathfrak{M}_{w^\perp}$	moduli space of marked hyperkähler manifolds	Sec. 12.5
$\mathfrak{M}^0_{w^\perp}$	a connected component of $\mathfrak{M}_{w^\perp}$	Sec. 12.5
Θ'	homomorphism $w^\perp \to \mathrm{Hom}(V, V)$	Eq. (12.4)
Θ	homomorphism $w^\perp \to \bigwedge^2 V^*$	Eq. (12.5)
J_ℓ	complex structure on $V_{\mathbb{C}}$ associated to $\ell \in \Omega_{w^\perp}$	Sec. 12.1
T_ℓ	a complex torus associated to a period $\ell \in \Omega_{w^\perp}$	Sec. 12.1
Per	the period map $\mathfrak{M}_{w^\perp} \to \Omega_{w^\perp}$	Sec. 12.5

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