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Rectifiability of the reduced boundary for sets of finite perimeter over RCD(K, N) spaces

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Abstract. This paper is devoted to the study of sets of finite perimeter in RCD(K, N) metric measure spaces. Its aim is to complete the picture of the generalization of De Giorgi's theorem within this framework. Starting from the results of Ambrosio et al. (2019) we obtain uniqueness of tangents and rectifiability for the reduced boundary of sets of finite perimeter. As an intermediate tool, of independent interest, we develop a Gauss–Green integration-by-parts formula tailored to this setting. These results are new and non-trivial even in the setting of Ricci limits.

Keywords. Set of finite perimeter, rectifiability, reduced boundary, RCD space, tangent cone, Gauss–Green formula

Contents

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Introduction

In the last years the theory of $\mathrm{RCD}(K, N)$ metric measure spaces has undergone a fast and remarkable development. After the introduction of the so-called *curvature-dimension* condition $\mathrm{CD}(K, N)$ in the seminal and independent works [51, 52] and [44], the notion of $\mathrm{RCD}(K, N)$ space was proposed in [28] after the study of its infinite-dimensional counterpart $\mathrm{RCD}(K, \infty)$ in [5] (see also [4] for the case of σ-finite reference measure). In the infinite-dimensional case the equivalence with the Bochner inequality was studied in [6], and then [26] established equivalence with the dimensional Bochner inequality for the so-called class $\mathrm{RCD}^*(K, N)$ (see also [10]). Equivalence between $\mathrm{RCD}^*(K, N)$ and $\mathrm{RCD}(K, N)$ has been established in [14].

We know nowadays that, apart from smooth weighted Riemannian manifolds (with generalized Ricci tensor bounded from below), this class includes Ricci limits (see [16–18]) and Alexandrov spaces [48].

One of the main research lines within this theory in recent times has been aimed at understanding the structure of $\mathrm{RCD}(K, N)$ spaces. After [23,33,40,47] we know that they are rectifiable as metric measure spaces. Moreover, in [12] the first and the third named authors proved that these spaces have constant dimension, in the almost everywhere sense.

This being the state of the art, we have reached a good understanding of the structure of $\mathrm{RCD}(K, N)$ spaces *up to measure zero*. It sounds therefore quite natural to try to push the study further, investigating their structure, both from the analytic and from the geometric points of view, up to sets of positive codimension. In this perspective in the last two years there have been some independent and remarkable developments. We wish to mention a few of them below, without the aim of being complete in this list.

- In the setting of non-collapsed Ricci limit spaces, Cheeger–Jiang–Naber have obtained in [19] rectifiability for singular sets of any codimension. Let us also mention [20], where the codimension-4 conjecture for non-collapsed limits of Einstein manifolds was solved, and [11], where some estimates (actually much weaker than those in [19]) are proved for singular strata of non-collapsed RCD spaces.
- There have been some efforts aimed at defining a notion of boundary for metric measure spaces and relating it to singular sets of codimension 1. See [38] and the very recent [39].
- One of the main contributions of [30] was the development of the language of tensor fields defined almost everywhere (with respect to the reference measure) on RCD spaces. In [21] the notion of tensor field defined "2-capacity-almost everywhere" is introduced and it is proved that Sobolev vector fields on RCD spaces have a representative in this class.
- In [2], the first and third named authors together with Ambrosio initiated a fine study of sets of finite perimeter over $\mathrm{RCD}(K, N)$ spaces, with a view to generalizing the Euclidean De Giorgi theorem to this framework.

One of the main results in [2] was the existence of a Euclidean half-space as tangent space to a set of finite perimeter at almost every point (with respect to the perimeter measure). This conclusion could be improved to a uniqueness statement (up to negligible sets) only in the case of a non-collapsed ambient space. The state of the theory of sets of finite perimeter was at that stage comparable to that of the structure theory after [32], where existence of Euclidean tangent spaces almost everywhere with respect to the reference measure was proved. Uniqueness of tangents in the possibly collapsed case and rectifiability for the boundary were conjectured by analogy with the Euclidean theory, but left as open questions in [2]. Let us point out that, up to our knowledge, no general rectifiability criterion is known at this stage for (subsets of) metric measure spaces.

The aim of this note is to provide a positive answer to these questions, providing a counterpart in codimension 1 of [47] and of De Giorgi's theorem in this setting.

Together with uniqueness of tangents (Theorem 3.2) and rectifiability (Theorem 4.1) we also establish a representation formula for the perimeter measure in terms of the codimension 1 Hausdorff measure (Corollary 3.15). As an intermediate tool, which, however, we find to have independent interest, we prove in Theorem 2.4 a Gauss–Green integration-by-parts formula for Sobolev vector fields.

The proof of uniqueness for blow-ups of sets of finite perimeter follows a strategy quite similar to that of the uniqueness theorem for tangents to $\mathrm{RCD}(K, N)$ spaces adopted in [47]. As in that case, closeness to a rigid configuration (half-space in Euclidean space) at a certain location and along a certain scale, which we learn from [2], can be turned into closeness to the same configuration at almost any location and at any scale, yielding uniqueness.

To encode the "closeness information" in analytic terms we rely on the use of harmonic δ-splitting maps, which were introduced in [15] and turned out to be an extremely powerful tool in the study of Ricci limits (see [16–18] and the more recent [19, 20]). To the best of our knowledge this is the first time they are explicitly used in RCD-theory, even though their use is implicit in [9], and we establish some of their properties within this framework.

Propagation of regularity at almost every location and at any scale, which was a consequence of a maximal function argument in [47], this time follows from a weighted maximal function argument suitably adapted to the codimension 1 framework. This argument heavily relies on the interplay between the fact that the perimeter measure is a codimension 1 measure (which was proved in a fairly more general context in [1]) and the fact that harmonic functions satisfy L^2 Hessian bounds on $\mathrm{RCD}(K, N)$ spaces.

In order to explain the strategy and the difficulties in the proof of rectifiability for the reduced boundary, let us recall how things work on $\mathbb{R}^n$. Therein a crucial role is played by the exterior normal to the set of finite perimeter, which is an almost everywhere unit valued vector field providing the representation $D\chi_E = \nu_E |D\chi_E|$ for the distributional derivative of the set E of finite perimeter. Relying on the properties of the exterior normal one can obtain a characterization of blow-ups in a much simpler way than in [2] and even get rectifiability of the boundary, proving that sets where the unit normal is not oscillating too much are bi-Lipschitz to subsets of $\mathbb{R}^{n-1}$.

When trying to reproduce the Euclidean approach in the *non-smooth* and *non-flat* realm of RCD spaces, one faces two main difficulties. The first one is due to the fact that the theory of tangent modules, as developed in [30], allows one to talk about vector fields only up to negligible sets with respect to the reference measure; observe that reduced boundaries of sets with finite perimeter are negligible with respect to the reference measure. The second one is that controlling the behaviour of the normal vector cannot be enough to control the behaviour of the set in this framework, since the space itself might "oscillate". This is a common feature of geometry on metric measure spaces (see also the introduction of [19]), which can be understood by looking at the following example: Let $(X, \mathsf{d}, \mathfrak{m})$ be any $\mathrm{RCD}(K, N)$ m.m.s. and take its product with the Euclidean line. Then consider the "generalized half-space" $\{t < 0\}$, where t denotes the coordinate along the line; it is easily seen that it is a set of locally finite perimeter and one can identify its reduced boundary with X. Moreover, whatever notion of unit normal we have in mind, this will be non-oscillating in this case. Still, rectifiability of $(X, \mathsf{d}, \mathfrak{m})$ is highly non-trivial and requires using [47] to be achieved.

To handle the first difficulty we mentioned above, we rely on the very recent [21], where a notion of cotangent module with respect to 2-capacity is introduced and studied. Building upon the fact that 2-capacity controls the perimeter measure in great generality, we introduce the notion of tangent module over the boundary of a set of finite perimeter (see Theorem 2.2).

Furthermore, we prove that there is a well-defined unit normal to a set of finite perimeter as an element of this module, that it satisfies the Gauss–Green integration-by-parts formula and, relying on functional analysis tools, that it can be approximated by regular vector fields (see Theorem 2.4 for a rigorous statement).

The results obtained in the study of the unit normal are then combined in a new way with the theory of δ-splitting maps to prove rectifiability of the reduced boundary for sets of finite perimeter.

We introduce a notion of δ-orthogonality to the unit normal for δ-splitting maps. Then we prove on the one hand that δ-splitting maps δ-orthogonal to the unit normal control both the geometry of the space and that of the boundary of the set of finite perimeter (and vice versa). On the other hand, the combination of δ-orthogonality and δ-splitting is seen to be suitable for propagation at many locations and any scale with maximal function arguments (Propositions 4.5 and 4.7).

We wish to emphasize that, on the one hand, the coarea formula (which holds in great generality in metric measure spaces) provides plenty of sets of finite perimeter even in the non-smooth context; on the other hand, there is no hope to have a notion of smooth hypersurface within this setting. Therefore we expect the range of applications to be large in the development of the theory of spaces satisfying lower curvature bounds, both for the techniques we develop in the paper and for our main results that, to the best of our knowledge, are new also for Ricci limits.

A number of questions remain open and suitable for future investigation. In particular, we point out that neither the constancy of the dimension result of [12], nor the absolute continuity of the reference measure with respect to the Hausdorff measure [23, 33, 40], play a role in the proofs of our results. It might be interesting to investigate whether one can prove constancy of the dimension for tangents also in the case of sets of finite perimeter and sharpen the representation formula for the perimeter measure (maybe relying on the good understanding we have of the top-dimensional case). In this regard let us point out that, in none of these cases, the techniques adopted to solve the analogous problems in codimension 0 seem to work when dealing with sets of finite perimeter.

This paper is organized as follows. In Section 1 we collect all the preliminary material to be used in the paper. We dedicate Section 2 to the construction of the tangent module over the boundary of a set of finite perimeter and to establishing a Gauss–Green integration-by-parts formula. Uniqueness of blow-ups is the main outcome of Section 3, while rectifiability for the reduced boundary is obtained in Section 4.

1. Preliminaries and notations

1.1. Calculus tools

Throughout this paper a *metric measure space* is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative Borel measure on X finite on bounded sets. From now on when we write *m.m.s.* we mean *metric measure space(s)*.

In order to simplify the notation, numerical constants depending only on the parameters entering into play will be denoted by the same letter C even if they vary. Often we will make explicit their dependence on the parameters, writing for instance C_N, $C_{N,K}$.

We will denote by $B_r(x) = \{\mathsf{d}(\cdot, x) < r\}$ and $\bar{B}_r(x) = \{\mathsf{d}(\cdot, x) \leq r\}$ the open and closed balls respectively, by $\mathrm{Lip}(X, \mathsf{d})$ (resp. $\mathrm{Lip}_{\mathrm{b}}(X, \mathsf{d})$, $\mathrm{Lip}_{\mathrm{c}}(X, \mathsf{d})$, $\mathrm{Lip}_{\mathrm{bs}}(X, \mathsf{d})$, $\mathrm{Lip}_{\mathrm{loc}}(X, \mathsf{d})$) the space of Lipschitz (resp. bounded Lipschitz, compactly supported Lipschitz, Lipschitz with bounded support, Lipschitz on bounded sets) functions and for any $f \in \mathrm{Lip}(X, \mathsf{d})$ we shall denote its slope by

$$\mathrm{lip}\, f(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}.$$

We shall use the standard notation $L^p(X, \mathfrak{m}) = L^p(\mathfrak{m})$ for L^p spaces, and $\mathscr{L}^n$ for the n-dimensional Lebesgue measure on $\mathbb{R}^n$. We shall denote by ω_n the Lebesgue measure of the unit ball in $\mathbb{R}^n$. If $f \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$ and $U \subset X$ is such that $0 < \mathfrak{m}(U) < \infty$, then $⨍_U f \,\mathrm{d}\mathfrak{m}$ denotes the average of f over U.

The *Cheeger energy* $\mathrm{Ch} : L^2(X, \mathfrak{m}) \to [0, \infty]$ is the convex and lower semicontinuous functional defined by

$$\mathrm{Ch}(f) := \inf \left\{ \liminf_{n \to \infty} \int_X (\mathrm{lip}\, f_n)^2 \,\mathrm{d}\mathfrak{m} \;\middle|\; f_n \in \mathrm{Lip}_{\mathrm{b}}(X) \cap L^2(X, \mathfrak{m}),\ \|f_n - f\|_2 \to 0 \right\}, \tag{1.1}$$

and its finiteness domain will be denoted by $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$; sometimes we write $H^{1,2}(X)$ when d and $\mathfrak{m}$ are clear from the context. Looking at the optimal approximating sequence in (1.1), it is possible to identify a canonical object $|\nabla f|$, called the *minimal relaxed slope*, providing the integral representation

$$\mathrm{Ch}(f) = \int_X |\nabla f|^2 \,\mathrm{d}\mathfrak{m} \quad \forall f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}).$$

Any metric measure space such that Ch is a quadratic form is said to be *infinitesimally Hilbertian*. Let us recall from [5, 28] that, under this assumption, the function

$$\nabla f_1 \cdot \nabla f_2 := \lim_{\varepsilon \to 0} \frac{|\nabla(f_1 + \varepsilon f_2)|^2 - |\nabla f_1|^2}{2\varepsilon}$$

defines a symmetric bilinear form on $H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \times H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with values in $L^1(X, \mathfrak{m})$.

It is possible to define a Laplacian operator $\Delta : L^2(X, \mathfrak{m}) \supset \mathcal{D}(\Delta) \to L^2(X, \mathfrak{m})$ in the following way. We let $\mathcal{D}(\Delta)$ be the set of those $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ such that, for some $h \in L^2(X, \mathfrak{m})$, one has

$$\int_X \nabla f \cdot \nabla g \, \mathrm{d}\mathfrak{m} = -\int_X hg \, \mathrm{d}\mathfrak{m} \quad \forall g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}), \tag{1.2}$$

and in that case we put $\Delta f = h$. It is easy to check that the definition is well-posed and that the Laplacian is linear (because Ch is a quadratic form).

The heat flow P_t is defined as the $L^2(X, \mathfrak{m})$-gradient flow of $\frac{1}{2}$Ch. Its existence and uniqueness follow from the Komura–Brezis theory. The heat flow can be equivalently characterized by saying that for any $u \in L^2(X, \mathfrak{m})$ the curve $t \mapsto P_t u \in L^2(X, \mathfrak{m})$ is locally absolutely continuous in $(0, \infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t u = \Delta P_t u \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0, \infty), \qquad \lim_{t \downarrow 0} P_t u = u \quad \text{in } L^2(X, \mathfrak{m}).$$

Under the infinitesimal Hilbertianity assumption the heat flow provides a linear, continuous and self-adjoint contraction semigroup in $L^2(X, \mathfrak{m})$. Moreover, P_t extends to a linear, continuous and mass preserving operator, still denoted by P_t, in all the L^p spaces for $1 \le p < \infty$.

Definition 1.1 (Function of bounded variation). We say that $f \in L^1(X, \mathfrak{m})$ belongs to the space $\mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ of functions *of bounded variation* if there exist locally Lipschitz functions f_i converging to f in $L^1(X, \mathfrak{m})$ such that

$$\limsup_{i \to \infty} \int_X |\nabla f_i| \, \mathrm{d}\mathfrak{m} < \infty.$$

If $f \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ one can define

$$|Df|(A) := \inf \left\{ \liminf_{i \to \infty} \int_A |\nabla f_i| \, \mathrm{d}\mathfrak{m} \;\middle|\; f_i \in \mathrm{Lip}_{\mathrm{loc}}(A), \; f_i \to f \text{ in } L^1(A, \mathfrak{m}) \right\}$$

for any open $A \subset X$. In [3] (see also [46] for the case of locally compact spaces) it is proven that this set function is the restriction to open sets of a finite Borel measure that we call the *total variation* of f and still denote by $|Df|$.

Dropping the global integrability condition on $f = \chi_E$, let us now recall the analogous definition of a set of finite perimeter in a metric measure space (see again [1, 3, 46]).

Definition 1.2 (Perimeter and sets of finite perimeter). Given a Borel set $E \subset X$ and an open set A, the *perimeter* $\mathrm{Per}(E, A)$ is defined by

$$\mathrm{Per}(E, A) := \inf\left\{\liminf_{n\to\infty} \int_A |\nabla u_n|\, \mathrm{d}\mathfrak{m} \;\middle|\; u_n \in \mathrm{Lip}_{\mathrm{loc}}(A),\ u_n \to \chi_E \text{ in } L^1_{\mathrm{loc}}(A, \mathfrak{m})\right\}.$$

We say that E has *finite perimeter* if $\mathrm{Per}(E, X) < \infty$. In that case it can be proved that the set function $A \mapsto \mathrm{Per}(E, A)$ is the restriction to open sets of a finite Borel measure $\mathrm{Per}(E, \cdot)$ defined by

$$\mathrm{Per}(E, B) := \inf\{\mathrm{Per}(E, A) \mid B \subset A,\ A \text{ open}\}.$$

Let us remark for the sake of clarity that $E \subset X$ with finite $\mathfrak{m}$-measure is a set of finite perimeter if and only if $\chi_E \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ and that $\mathrm{Per}(E, \cdot) = |D\chi_E|(\cdot)$. In the following we will say that $E \subset X$ is a set *of locally finite perimeter* if χ_E is a function of locally bounded variation, that is, $\eta\chi_E \in \mathrm{BV}(X, \mathsf{d}, \mathfrak{m})$ for any $\eta \in \mathrm{Lip}_{\mathrm{bs}}(X, \mathsf{d})$.

1.1.1. Total variation of BV *functions via integration by parts.* Let us present an equivalent approach to the study of BV functions in m.m.s. introduced by Di Marino [24]. Before stating Theorem 1.7 we need to recall the notion of derivation.

Definition 1.3 (Derivation). Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then a *derivation* on X is a linear map $\boldsymbol{b} : \mathrm{Lip}_{\mathrm{bs}}(X) \to L^0(\mathfrak{m})$ such that the following properties are satisfied:

(i) (Leibniz rule) $\boldsymbol{b}(fg) = \boldsymbol{b}(f)g + f\boldsymbol{b}(g)$ for all $f, g \in \mathrm{Lip}_{\mathrm{bs}}(X)$.

(ii) (Weak locality) There exists $G \in L^0(\mathfrak{m})$ such that[1]

$$|\boldsymbol{b}(f)| \le G\, \mathrm{lip}_{\mathrm{a}}(f) \quad \mathfrak{m}\text{-a.e. for every } f \in \mathrm{Lip}_{\mathrm{bs}}(X).$$

The least function G (in the $\mathfrak{m}$-a.e. sense) with this property is denoted by $|\boldsymbol{b}|$.

The space of all derivations on X is denoted by $\mathrm{Der}(X)$. Given any derivation $\boldsymbol{b} \in \mathrm{Der}(X)$, we define its *support* $\mathrm{supp}(\boldsymbol{b}) \subset X$ as the essential closure of $\{|\boldsymbol{b}| \neq 0\}$. For any open set $U \subset X$, we write $\mathrm{supp}(\boldsymbol{b}) \Subset U$ if $\mathrm{supp}(\boldsymbol{b})$ is bounded and $\mathrm{dist}(\mathrm{supp}(\boldsymbol{b}), X \setminus U) > 0$. Given any $\boldsymbol{b} \in \mathrm{Der}(X)$ with $|\boldsymbol{b}| \in L^1_{\mathrm{loc}}(X)$, we say that $\mathrm{div}(\boldsymbol{b}) \in L^p(\mathfrak{m})$ (for some exponent $p \in [1, \infty]$) if there exists $h \in L^p(\mathfrak{m})$ such that

$$-\int \boldsymbol{b}(f)\, \mathrm{d}\mathfrak{m} = \int f h\, \mathrm{d}\mathfrak{m} \quad \text{for every } f \in \mathrm{Lip}_{\mathrm{bs}}(X). \tag{1.3}$$

The function h is uniquely determined, thus it can be unambiguously denoted by $\mathrm{div}(\boldsymbol{b})$. We set

$$\mathrm{Der}^p(X) := \{\boldsymbol{b} \in \mathrm{Der}(X) \mid |\boldsymbol{b}| \in L^p(\mathfrak{m})\},$$
$$\mathrm{Der}^{p,p}(X) := \{\boldsymbol{b} \in \mathrm{Der}^p(X) \mid \mathrm{div}(\boldsymbol{b}) \in L^p(\mathfrak{m})\}$$

[1] Here $\mathrm{lip}_{\mathrm{a}}(f)(x) := \lim_{r\to 0} \sup_{\mathsf{d}(x,y)<r} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)}$ is the so-called asymptotic Lipschitz constant.

for any $p \in [1,\infty]$. The set $\mathrm{Der}^p(X)$ is a Banach space if endowed with the norm $\|\boldsymbol{b}\|_p := \| |\boldsymbol{b}| \|_{L^p(\mathfrak{m})}$.

Remark 1.4. We claim that for every $\boldsymbol{b} \in \mathrm{Der}^{p,p}(X)$ (where $p \in [1,\infty]$),

$$\mathrm{supp}(\mathrm{div}(\boldsymbol{b})) \subset \mathrm{supp}(\boldsymbol{b}). \tag{1.4}$$

To prove it, fix any open bounded subset U of $X \setminus \mathrm{supp}(\boldsymbol{b})$. Then formula (1.3) guarantees that $\int f \,\mathrm{div}(\boldsymbol{b})\,\mathrm{d}\mathfrak{m} = -\int \boldsymbol{b}(f)\,\mathrm{d}\mathfrak{m} = 0$ for every $f \in \mathrm{Lip_{bs}}(U)$, whence $\mathrm{div}(\boldsymbol{b}) = 0$ $\mathfrak{m}$-a.e. on U. By arbitrariness of U, (1.4) follows.

In the next proposition the notions of tangent module $L^2(TX)$ and, more generally, of Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module, play a role. We will denote by $\nabla\colon H^{1,2}(X) \to L^2(TX)$ the gradient map. We refer to Section 1.3 below for the definition of these objects.

Proposition 1.5. *Let $(X,\mathsf{d},\mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Let $\overline{\mathbb{D}}$ be the closure in $\mathrm{Der}^2(X)$ of the pre-Hilbert space $\mathbb{D} := (\mathrm{Der}^{2,2}(X), \|\cdot\|_2)$. Then $\overline{\mathbb{D}}$ has a natural structure of Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module and the map $A\colon L^2(TX) \to \overline{\mathbb{D}}$, defined as*

$$A(v)(f) := v \cdot \nabla f \quad \textit{for all } v \in L^2(TX) \textit{ and } f \in \mathrm{Lip}_{bs}(X),$$

is a normed module isomorphism between $L^2(TX)$ and $\overline{\mathbb{D}}$. Moreover, $A(D(\mathrm{div})) = \mathbb{D}$ and

$$\mathrm{div}(A(v)) = \mathrm{div}(v) \quad \textit{for every } v \in D(\mathrm{div}).$$

Proof. See [25, proof of Proposition 6.5]. ∎

Remark 1.6. Given an infinitesimally Hilbertian space $(X,\mathsf{d},\mathfrak{m})$ and any f in $\mathrm{BV}(X,\mathsf{d},\mathfrak{m})$,

$$\int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} \le |Df|(X) \quad \text{for all } v \in D(\mathrm{div}) \text{ with } |v| \le 1 \ \mathfrak{m}\text{-a.e. and } \mathrm{div}(v) \in L^\infty(\mathfrak{m}).$$

This readily follows from [24, Theorem 3.3] and Proposition 1.5.

Theorem 1.7 (Representation formula for $|Df|$). *Let $(X,\mathsf{d},\mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Let $f \in \mathrm{BV}(X,\mathsf{d},\mathfrak{m})$. Then for every open set $U \subset X$,*

$$|Df|(U) = \sup\left\{ \int_U f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} \;\middle|\; v \in D(\mathrm{div}),\ |v| \le 1 \ \mathfrak{m}\text{-}a.e.,\ \mathrm{div}(v) \in L^\infty(\mathfrak{m}),\ \mathrm{supp}(v) \Subset U \right\}.$$

Proof. Combine [24, Theorem 3.4] with Proposition 1.5 (recall that $\boldsymbol{b} \in \mathrm{Der}^{2,2}(X)$ for every $\boldsymbol{b} \in \mathrm{Der}^{\infty,\infty}(X)$ such that $\mathrm{supp}(\boldsymbol{b})$ is bounded, thanks to Remark 1.4). ∎

1.1.2. PI spaces. Let us recall that $(X, \mathsf{d}, \mathfrak{m})$ satisfies a *weak local* $(1,2)$-*Poincaré inequality* with constants $C_P > 0$ and $\lambda \geq 1$ if

$$\begin{aligned}&\fint_{B_r(x)} |f - (f)_{x,r}| \, \mathrm{d}\mathfrak{m} \\ &\qquad \leq C_P r \bigg(\fint_{B_{\lambda r}(x)} |Df|^2 \, \mathrm{d}\mathfrak{m} \bigg)^{1/2} \quad \text{for all } f \in H^{1,2}(X),\ x \in X,\ r > 0, \end{aligned} \tag{1.5}$$

where

$$(f)_{x,r} := \fint_{B_r(x)} f \, \mathrm{d}\mathfrak{m}. \tag{1.6}$$

Before giving the definition of PI space we need to recall the notion of locally doubling m.m.s. We say that $(X, \mathsf{d}, \mathfrak{m})$ is *locally doubling* if for any $R > 0$ there exists $C_D > 0$ depending only on R such that

$$\mathfrak{m}(B_{2r}(x)) \leq C_D \mathfrak{m}(B_r(x)) \quad \text{for all } 0 < r < R,\ x \in X. \tag{1.7}$$

Definition 1.8. A *PI space* is a locally doubling metric measure space supporting a weak local $(1,2)$-Poincaré inequality.

1.1.3. Capacity and Hausdorff measures. We briefly recall the notion of capacity and its main properties in this setting, referring to [21] for a detailed discussion. The *capacity* of a set $E \subset X$ is defined as

$$\begin{aligned}&\mathrm{Cap}(E) \\ &:= \inf\{\|f\|^2_{H^{1,2}(X)} \mid f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}),\ f \geq 1\ \mathfrak{m}\text{-a.e. on some neighbourhood of } E\}.\end{aligned}$$

It turns out that Cap is a submodular outer measure on X, finite on all bounded sets, such that $\mathfrak{m}(E) \leq \mathrm{Cap}(E)$ for any Borel set $E \subset X$. Any function $f : X \to [0, \infty]$ can be integrated with respect to capacity via Cavalieri's formula:

$$\int f \, \mathrm{d}\,\mathrm{Cap} := \int_0^\infty \mathrm{Cap}(\{f > t\}) \, \mathrm{d}t.$$

(The function $t \mapsto \mathrm{Cap}(\{f > t\})$ is non-increasing, thus in particular it is Lebesgue measurable.) The integral operator $f \mapsto \int f \, \mathrm{d}\,\mathrm{Cap}$ is subadditive as a consequence of the submodularity of Cap. Given any set $E \subset X$, we shall use the shorthand notation $\int_E f \, \mathrm{d}\,\mathrm{Cap} := \int \chi_E f \, \mathrm{d}\,\mathrm{Cap}$.

Let us now introduce the codimension-α Hausdorff measure. We refer to [1] for a more detailed discussion.

Definition 1.9. Given a locally doubling metric measure space $(X, \mathsf{d}, \mathfrak{m})$, for any $\alpha > 0$ we set

$$h_\alpha(B_r(x)) := \mathfrak{m}(B_r(x))/r^\alpha \quad \text{for any } x \in X,\ r \in (0, \infty).$$

The *codimension-α Hausdorff measure* $\mathscr{H}^{h_\alpha}$ is the Borel regular outer measure built from h_α with the Carathéodory construction. We will denote by $\mathscr{H}^{h_\alpha}_\delta$ the pre-measure with parameter δ.

The codimension-1 measure plays a crucial role in the theory of sets of finite perimeter over PI spaces, since $\mathrm{Per}(E,\cdot) \ll \mathscr{H}^{h_1}$ for any set of finite perimeter E. This result has been proved by Ambrosio [1, Lemma 5.2].

Lemma 1.10. *Let $(X,\mathsf{d},\mathfrak{m})$ be a PI space. For any set $E \subset X$ of locally finite perimeter and any $\delta > 0$,*

$$\mathscr{H}^{h_1}_\delta(B) = 0 \implies \mathrm{Per}(E,B) = 0 \quad \textit{for any Borel set } B \subset X.$$

To be precise, in [1, Lemma 5.2] Ambrosio proved the result above only for $\delta = 0$. However, it easily follows from the definitions that

$$\mathscr{H}^{h}_\delta(B) = 0 \implies \mathscr{H}^{h}(B) = 0.$$

Let us now prove two results connecting the codimension-α Hausdorff measure and the capacity. Their proofs are inspired by those given for the analogous results in the Euclidean setting in [27]. We also refer to [41, 42] for other such results in the framework of metric measure spaces.

Lemma 1.11. *Let $(X,\mathsf{d},\mathfrak{m})$ be a locally doubling m.m.s. Let $f \in L^1(X,\mathfrak{m})$, $f \geq 0$. Then for any exponent $\alpha > 0$,*

$$\mathscr{H}^{h_\alpha}(\Lambda_\alpha) = 0, \quad \textit{where} \quad \Lambda_\alpha := \left\{ x \in X \;\middle|\; \limsup_{r \searrow 0} r^\alpha (f)_{x,r} > 0 \right\}.$$

Proof. By Lebesgue's differentiation theorem, $\lim_{r\searrow 0} (f)_{x,r}$ exists and is finite for $\mathfrak{m}$-a.e. $x \in X$, thus for any $\alpha > 0$ we have $\limsup_{r\searrow 0} r^\alpha (f)_{x,r} = 0$ for $\mathfrak{m}$-a.e. $x \in X$. This means that $\mathfrak{m}(\Lambda_\alpha) = 0$. Setting

$$\Lambda^k_\alpha := \left\{ x \in X \;\middle|\; \limsup_{r \searrow 0} r^\alpha (f)_{x,r} \geq 1/k \right\} \quad \text{for every } k \in \mathbb{N},$$

we see that $\Lambda_\alpha = \bigcup_k \Lambda^k_\alpha$, thus in particular $\mathfrak{m}(\Lambda^k_\alpha) = 0$ for every $k \in \mathbb{N}$. Since $f \in L^1(X,\mathfrak{m})$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A f \, \mathrm{d}\mathfrak{m} \leq \varepsilon$ for any Borel set $A \subset X$ satisfying $\mathfrak{m}(A) < \delta$. Fix $k \in \mathbb{N}$ and pick an open set $U \subset X$ such that $\Lambda^k_\alpha \subset U$ and $\mathfrak{m}(U) < \delta$. Define

$$\mathcal{F} := \left\{ B_r(x) \;\middle|\; x \in \Lambda^k_\alpha,\ r \in (0,\varepsilon),\ B_r(x) \subset U,\ \int_{B_r(x)} f \, \mathrm{d}\mathfrak{m} \geq \mathfrak{m}(B_r(x))/(r^\alpha k) \right\}.$$

By the Vitali covering theorem we can find a sequence $(B_i)_{i\in\mathbb{N}} \subset \mathcal{F}$ of pairwise disjoint balls $B_i = B_{r_i}(x_i)$ such that $\Lambda^k_\alpha \subset \bigcup_i B_{5r_i}(x_i)$. Since $\mathfrak{m}$ is locally doubling, there exists a constant $C_D \geq 1$ such that $\mathfrak{m}(B_{5r}(x)) \leq C_D\, \mathfrak{m}(B_r(x))$ for all $x \in X$ and $r < \varepsilon$.

Consequently,

$$\begin{aligned}\mathscr{H}^{h_\alpha}_{10\varepsilon}(\Lambda^k_\alpha) &\le \frac{1}{5^\alpha}\sum_{i=1}^\infty \frac{\mathfrak{m}(B_{5r_i}(x_i))}{r_i^\alpha} \le \frac{C_D}{5^\alpha}\sum_{i=1}^\infty \frac{\mathfrak{m}(B_i)}{r_i^\alpha} \le \frac{C_D k}{5^\alpha}\sum_{i=1}^\infty \int_{B_i} f\,\mathrm{d}\mathfrak{m}\\ &\le \frac{C_D k}{5^\alpha}\int_U f\,\mathrm{d}\mathfrak{m} \le \frac{C_D k}{5^\alpha}\varepsilon.\end{aligned}$$

Letting $\varepsilon \searrow 0$ yields $\mathscr{H}^{h_\alpha}(\Lambda^k_\alpha) = 0$, whence $\mathscr{H}^{h_\alpha}(\Lambda_\alpha) = \lim_k \mathscr{H}^{h_\alpha}(\Lambda^k_\alpha) = 0$. ■

Theorem 1.12. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a PI space. Then $\mathscr{H}^{h_\alpha} \ll \mathrm{Cap}$ for every $\alpha \in (0, 2)$.*

Proof. Fix $\alpha \in (0, 2)$ and a set $A \subset X$ with $\mathrm{Cap}(A) = 0$. We aim to prove that $\mathscr{H}^{h_\alpha}(A) = 0$. By definition of capacity, we can find a sequence $(f_i)_i \subset H^{1,2}(X)$ such that $f_i \ge 1$ on some neighbourhood of A and $\|f_i\|_{H^{1,2}(X)} \le 1/2^i$ for every $i \in \mathbb{N}$. Since $\sum_{i=1}^\infty \|f_i\|_{H^{1,2}(X)} < \infty$, one sees that $g := \sum_{i=1}^\infty f_i$ is a well-defined element of the Banach space $H^{1,2}(X)$. For any $k \in \mathbb{N}$, clearly $g \ge k$ on some neighbourhood of A, whence for any $x \in A$ we have $(g)_{x,r} \ge k$ for every $r < \mathrm{dist}(x, \{g < k\})$ and accordingly

$$\lim_{r\searrow 0} (g)_{x,r} = \infty \quad \text{for every } x \in A. \tag{1.8}$$

Furthermore, we claim that

$$\limsup_{r\searrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2\,\mathrm{d}\mathfrak{m} = \infty \quad \text{for every } x \in A. \tag{1.9}$$

Indeed, suppose that this lim sup is finite for some $x \in A$, so that there exists a constant $M > 0$ such that

$$r^\alpha \fint_{B_r(x)} |Dg|^2\,\mathrm{d}\mathfrak{m} \le M \quad \text{for every } r \in (0, 1). \tag{1.10}$$

Let C_D be the doubling constant of $\mathfrak{m}$ (for $r < 1/2$). Then, for every $r < 1/(2\lambda)$,

$$\begin{aligned}|(g)_{x,r} - (g)_{x,2r}| &= \frac{1}{\mathfrak{m}(B_r(x))}\left|\int_{B_r(x)} (g - (g)_{x,2r})\,\mathrm{d}\mathfrak{m}\right|\\ &\le C_D \fint_{B_{2r}(x)} |g - (g)_{x,2r}|\,\mathrm{d}\mathfrak{m}\\ &\overset{(1.5)}{\le} 2C_D C_P r\left(\fint_{B_{2\lambda r}(x)} |Dg|^2\,\mathrm{d}\mathfrak{m}\right)^{1/2}\\ &\overset{(1.10)}{\le} (2^{1-\alpha/2} C_D C_P \lambda^{-\alpha/2} M^{1/2}) r^{1-\alpha/2}.\end{aligned}$$

Set $C := 2^{1-\alpha/2} C_D C_P \lambda^{-\alpha/2} M^{1/2}$ and $\theta := 1 - \alpha/2 \in (0, 1)$. Then the previous computation gives $\sum_{i=2}^\infty |(g)_{x,2^{-i}} - (g)_{x,2^{-i+1}}| \le C \sum_{i=2}^\infty (2^\theta)^{-i} < \infty$, contradicting (1.8). This proves (1.9).

Finally, it immediately follows from (1.9) that A is contained in the set of all $x \in X$ that satisfy $\limsup_{r\searrow 0} r^\alpha \fint_{B_r(x)} |Dg|^2\,\mathrm{d}\mathfrak{m} > 0$, which is $\mathscr{H}^{h_\alpha}$-negligible by Lemma 1.11. Therefore, $\mathscr{H}^{h_\alpha}(A) = 0$, completing the proof of the statement. ■

1.2. RCD *metric measure spaces*

The main object of our investigation is $\mathrm{RCD}(K,N)$ metric measure spaces, that is, infinitesimally Hilbertian spaces satisfying a lower Ricci curvature bound and an upper dimension bound in synthetic sense according to [44, 51, 52]. Before passing to the description of the main properties of $\mathrm{RCD}(K,N)$ spaces that will be relevant for this article, let us briefly focus on the adimensional case.

The class of $\mathrm{RCD}(K,\infty)$ spaces was introduced in [6] (see also [4] for the extension to the case of σ-finite reference measures) adding to the $\mathrm{CD}(K,\infty)$ condition, formulated in terms of K-convexity properties of the logarithmic entropy over the Wasserstein space (P_2, W_2), the infinitesimal Hilbertianity assumption.

Under the $\mathrm{RCD}(K,\infty)$ condition it was proved that the dual heat semigroup $P_t^* : \mathrm{P}_2(X) \to \mathrm{P}_2(X)$, defined by

$$\int_X f \, \mathrm{d}P_t^*\mu = \int_X P_t f \, \mathrm{d}\mu \quad \forall \mu \in \mathrm{P}_2(X), \ \forall f \in \mathrm{Lip}_{\mathrm{bs}}(X,\mathsf{d}),$$

is K-contractive with respect to the W_2-distance and, for $t > 0$, maps probability measures to probability measures absolutely continuous with respect to $\mathfrak{m}$. With a slight abuse of notation we shall denote by $P_t^*\mu$ also the density of the measure $P_t^*\mu$ with respect to $\mathfrak{m}$. Then, for any $t > 0$, one can define the *heat kernel* $p_t : X \times X \to [0,\infty)$ by

$$p_t(x,\cdot)\mathfrak{m} = P_t^*\delta_x. \tag{1.11}$$

We now state a few regularization properties of $\mathrm{RCD}(K,\infty)$ spaces, referring again to [4, 6] for a more detailed discussion and for the proofs.

First we have the *Bakry–Émery contraction estimate*

$$|\nabla P_t f|^2 \le e^{-2Kt} P_t |\nabla f|^2 \quad \mathfrak{m}\text{-a.e.} \tag{1.12}$$

for any $t > 0$ and any $f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$. This estimate can be generalized to the whole range of exponents $1 < p < \infty$. Furthermore in [31] it has been proved that on any proper $\mathrm{RCD}(K,\infty)$ m.m.s.,

$$|DP_t f| \le e^{-Kt} P_t^* |Df| \tag{1.13}$$

for any $t > 0$ and any $f \in \mathrm{BV}(X,\mathsf{d},\mathfrak{m})$.

Next we have the so-called *Sobolev-to-Lipschitz property*: any $f \in H^{1,2}(X,\mathsf{d},\mathfrak{m})$ such that $|\nabla f| \in L^\infty(X,\mathfrak{m})$ admits a representative $\tilde{f} \in \mathrm{Lip}(X,\mathsf{d})$ with Lipschitz constant bounded from above by $\| |\nabla f| \|_{L^\infty}$.

Let us introduce the space $\mathrm{Test}(X,\mathsf{d},\mathfrak{m})$ of test functions following [30]:

$$\mathrm{Test}(X,\mathsf{d},\mathfrak{m}) := \{ f \in D(\Delta) \cap L^\infty(X,\mathfrak{m}) \,|\, |\nabla f| \in L^\infty(X,\mathfrak{m}) \text{ and } \Delta f \in H^{1,2}(X,\mathsf{d},\mathfrak{m}) \}. \tag{1.14}$$

The notion of $\mathrm{RCD}(K,N)$ m.m.s. was proposed and extensively studied in [10, 26, 28] (see also [14] for the equivalence between the RCD and the RCD^* condition when the reference measure is finite), as a finite-dimensional counterpart to $\mathrm{RCD}(K,\infty)$ m.m.s.

which were introduced and first studied in [6]. Here we just recall that they can be characterized by requiring the quadraticity of Ch, the volume growth condition $\mathfrak{m}(B_r(x)) \le c_1 \exp(c_2 r^2)$ for some (and thus for all) $x \in X$, the validity of the Sobolev-to-Lipschitz property and of a weak form of Bochner's inequality,

$$\frac{1}{2}\Delta|\nabla f|^2 - \nabla f \cdot \nabla \Delta f \ge \frac{(\Delta f)^2}{N} + K|\nabla f|^2 \quad \text{for any } f \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m}).$$

We refer to [10, 26] for a more detailed discussion and equivalent characterizations of the $\mathrm{RCD}(K, N)$ condition.

Note that if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ m.m.s., then so is $(\operatorname{supp} \mathfrak{m}, \mathsf{d}, \mathfrak{m})$, hence in the following we will always tacitly assume $\operatorname{supp} \mathfrak{m} = X$.

We recall that any $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfies the *Bishop–Gromov inequality*

$$\frac{\mathfrak{m}(B_R(x))}{v_{K,N}(R)} \le \frac{\mathfrak{m}(B_r(x))}{v_{K,N}(r)} \quad \text{for any } 0 < r < R \text{ and } x \in X, \tag{1.15}$$

where $v_{K,N}(r)$ is the volume of the ball with radius r in the model space with dimension N and Ricci curvature K. We refer to [54, Theorem 30.11] for the proof of (1.15). In particular, $(X, \mathsf{d}, \mathfrak{m})$ is locally uniformly doubling. Furthermore, it was proved in [49] that it satisfies a local Poincaré inequality. Therefore $\mathrm{RCD}(K, N)$ spaces fit in the framework of PI spaces that we introduced above.

We assume the reader to be familiar with the notion of pointed measured Gromov–Hausdorff convergence (pmGH-convergence for short), referring to [54, Chapter 27] for an overview.

Remark 1.13. A fundamental property of $\mathrm{RCD}(K, N)$ spaces, to be used several times in this paper, is the stability with respect to pmGH-convergence, meaning that a pmGH-limit of a sequence of (pointed) $\mathrm{RCD}(K_n, N_n)$ spaces for some $K_n \to K$ and $N_n \to N$ is an $\mathrm{RCD}(K, N)$ m.m.s.

Let us finally recall the construction of good cut-off functions over $\mathrm{RCD}(K, N)$ metric measure spaces; see [47, Lemma 3.1] for a proof.

Lemma 1.14. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. For any* $0 < 2r < R$ *and* $x \in X$ *there exists a test function* $\eta : X \to \mathbb{R}$ *satisfying*

(i) $0 \le \eta \le 1$ *on* X, $\eta = 1$ *on* $B_r(x)$ *and* $\eta = 0$ *on* $X \setminus B_{2r}(x)$*;*

(ii) $r^2|\Delta\eta| + r|\nabla\eta| \le C_{N,K,R}$.

1.2.1. Structure theory. Let us briefly review the main results concerning the state of the art about the so-called structure theory of $\mathrm{RCD}(K, N)$ spaces.

Given a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, $x \in X$ and $r \in (0, 1)$, we consider the rescaled and normalized pointed m.m.s. $(X, r^{-1}\mathsf{d}, \mathfrak{m}_r^x, x)$, where

$$\mathfrak{m}_r^x := \left(\int_{B_r(x)} \left(1 - \frac{\mathsf{d}(x, y)}{r}\right) \mathrm{d}\mathfrak{m}(y)\right)^{-1} \mathfrak{m} = C(x, r)^{-1}\mathfrak{m}.$$

Definition 1.15. We say that a pointed m.m.s. $(Y, \mathsf{d}_Y, \eta, y)$ is *tangent* to $(X, \mathsf{d}, \mathfrak{m})$ at x if there exists a sequence $r_i \downarrow 0$ such that $(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_{r_i}^x, x) \to (Y, \mathsf{d}_Y, \eta, y)$ in the pmGH-topology. The collection of all the tangent spaces of $(X, \mathsf{d}, \mathfrak{m})$ at x is denoted by $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$.

A compactness argument, due to Gromov, together with the rescaling and stability properties of the $\mathrm{RCD}(K, N)$ condition (see Remark 1.13), shows that $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ is non-empty for every $x \in X$ and its elements are all $\mathrm{RCD}(0, N)$ pointed m.m.s.

Let us recall the notion of k-regular point and k-regular set.

Definition 1.16. Given any natural $1 \le k \le N$, we say that $x \in X$ is a *k-regular point* if

$$\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}) = \{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathscr{L}^k, 0)\}.$$

We shall denote by $\mathcal{R}_k$ the set of k-regular points in X.

Observe that, by explicit computation, the constant c_k in Definition 1.16 equals $\frac{\omega_k}{k+1}$.

Remark 1.17. Observe that if $x \in \mathcal{R}_k$, then

$$\lim_{r\to 0} \frac{\int_{B_r(x)} (1 - \frac{\mathsf{d}(x,y)}{r}) \,\mathrm{d}\mathfrak{m}(y)}{\mathfrak{m}(B_r(x))} = \frac{1}{k+1}. \tag{1.16}$$

Moreover, it can be easily checked that $x \in \mathcal{R}_k$ if and only if

$$\lim_{r\to 0} \mathsf{d}_{\mathrm{pmGH}}\left(\left(X, r^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_r(x))}, x \right), \left(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \frac{1}{\omega_k} \mathscr{L}^k, 0^k \right) \right) = 0.$$

From the works [23, 32, 33, 40, 47] and [12] we have the following structure theorem for $\mathrm{RCD}(K, N)$ spaces.

Theorem 1.18. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. with $K \in \mathbb{R}$ and $N \ge 1$. Then there exists a natural number $1 \le n \le N$, called the* essential dimension *of X, such that $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$. Moreover, $\mathcal{R}_n$ is $(\mathfrak{m}, n)$-rectifiable and $\mathfrak{m}$ is representable as $\theta \mathscr{H}^n \llcorner \mathcal{R}_n$ for some non-negative density $\theta \in L^1_{\mathrm{loc}}(X, \mathscr{H}^n \llcorner \mathcal{R}_n)$.*

Recall that X is said to be $(\mathfrak{m}, n)$-*rectifiable* if there exists a family $\{A_i\}_{i\in\mathbb{N}}$ of Borel subsets of X such that each A_i is bi-Lipschitz to a Borel subset of $\mathbb{R}^n$ and $\mathfrak{m}(X \setminus \bigcup_{i\in\mathbb{N}} A_i) = 0$.

1.2.2. Sobolev functions and Laplacian on balls. Following a standard approach let us give a notion of Sobolev functions and Laplacian on balls; we refer to [8] for a more detailed presentation.

We define $H^{1,2}_0(B_r(x), \mathsf{d}, \mathfrak{m})$ to be the closure of $\mathrm{Lip_c}(B_r(x), \mathsf{d})$ in $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$. Let us also define $H^{1,2}_{\mathrm{loc}}(B_r(x), \mathsf{d}, \mathfrak{m})$ as the space of those $f : B_r(x) \to \mathbb{R}$ such that $\eta f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ for any $\eta \in \mathrm{Lip_c}(B_r(x), \mathsf{d})$. Exploiting the locality of the minimal relaxed slope one can easily define $|\nabla f|$ for any $f \in H^{1,2}_{\mathrm{loc}}(B_r(x), \mathsf{d}, \mathfrak{m})$. This allows us to introduce $H^{1,2}(B_r(x), \mathsf{d}, \mathfrak{m})$ as the space of $f \in H^{1,2}_{\mathrm{loc}}(B_r(x), \mathsf{d}, \mathfrak{m})$ such that $f, |\nabla f|$ are in $L^2(X, \mathfrak{m})$.

Definition 1.19. A function $f \in H^{1,2}(B_r(x), \mathsf{d}, \mathfrak{m})$ belongs to $D(\Delta, B_r(x))$ if there exists $g \in L^2(B_r(x), \mathfrak{m})$ satisfying

$$\int_{B_r(x)} \nabla f \cdot \nabla h \, \mathrm{d}\mathfrak{m} = -\int_{B_r(x)} fg \, \mathrm{d}\mathfrak{m} \quad \text{for all } h \in H_0^{1,2}(B_r(x), \mathsf{d}, \mathfrak{m}).$$

With a slight abuse of notation we write $\Delta f = g$ in $B_r(x)$.

It is easily seen that if $f \in D(\Delta, B_r(x))$ and if $\eta \in \mathrm{Lip}_c(B_r(x), \mathsf{d}) \cap D(\Delta)$ satisfies $\Delta\eta \in L^\infty(X, \mathfrak{m})$ then $\eta f \in D(\Delta)$.

1.2.3. Stability and convergence results. Let us fix a pointed measured Gromov–Hausdorff convergent sequence

$$(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \to (Y, \varrho, \mu, y)$$

of $\mathrm{RCD}(K, N)$ m.m.s. Recall that, in the setting of uniformly locally doubling spaces, the pointed measured Gromov–Hausdorff convergence can be equivalently characterized by requiring the existence of a proper metric space (Z, d_Z) in which (X_i, d_i) and (Y, ϱ) are isometrically embedded, and $x_i \to y$ and $\mathfrak{m}_i \rightharpoonup \mu$ in duality with $\mathrm{C_{bs}}(Z)$ (the space of continuous functions with bounded supports in Z). This is the so-called extrinsic approach and it is convenient when formulating various notions of convergence.

Definition 1.20. Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$, (Y, ϱ, μ, y), (Z, d_Z) be as above and $f_i : X_i \to \mathbb{R}$, $f : Y \to \mathbb{R}$. We say that $f_i \to f$ *pointwise* if $f_i(z_i) \to f(z)$ for every sequence of points $z_i \in X_i$ such that $z_i \to z$ in Z. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_i(z_i) - f(z)| \le \varepsilon$ for all $i \ge \delta^{-1}$ and $z_i \in X_i$, $z \in Y$ with $\mathsf{d}_Z(z_i, z) \le \delta$, then we say that $f_i \to f$ *uniformly*.

The next proposition is a version of the Ascoli–Arzelà compactness theorem for sequences of functions defined on varying spaces. We omit the proof, which can be obtained by arguing as in the case of a fixed space.

Proposition 1.21. *Let $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ and (Y, ρ, μ, y) be as above and let $R, L > 0$. Then for any sequence of L-Lipschitz functions $f_i : B_R(x_i) \to \mathbb{R}$ such that $\sup_i |f_i(x_i)| < \infty$ there exists a subsequence that converges uniformly to some L-Lipschitz function $f : B_R(y) \to \mathbb{R}$.*

We recall below the notions of convergence in L^p and Sobolev spaces for functions defined over converging sequences of metric measure spaces. We will be concerned only with the cases $p = 2$ and $p = 1$ in the rest of the article. We refer again to [7, 8] for a more general treatment and the proofs of the results we state below.

Definition 1.22. We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ *converges L^2-weakly* to $f \in L^2(Y, \mu)$ if $f_i \mathfrak{m}_i \rightharpoonup f\mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\sup_i \|f_i\|_{L^2(X_i, \mathfrak{m}_i)} < \infty$.

We say that $f_i \in L^2(X_i, \mathfrak{m}_i)$ *converges L^2-strongly* to $f \in L^2(Y, \mu)$ if $f_i \mathfrak{m}_i \rightharpoonup f\mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\lim_i \|f_i\|_{L^2(X_i, \mathfrak{m}_i)} = \|f\|_{L^2(Y, \mu)}$.

Definition 1.23. We say that a sequence $(f_i) \subset L^1(X_i, \mathfrak{m}_i)$ *converges* L^1*-strongly* to $f \in L^1(Y, \mu)$ if

$$\sigma \circ f_i \mathfrak{m}_i \rightharpoonup \sigma \circ f \mu \quad \text{and} \quad \int_{X_i} |f_i| \, \mathrm{d}\mathfrak{m}_i \to \int_Y |f| \, \mathrm{d}\mu,$$

where $\sigma(z) := \mathrm{sign}(z)\sqrt{|z|}$ and the weak convergence is in duality with $\mathrm{C_{bs}}(Z)$, or equivalently, if $\sigma \circ f_i$ converges L^2-strongly to $\sigma \circ f$.

Dealing with characteristic functions one has the following equivalent notion of L^1-convergence.

Definition 1.24. We say that a sequence of Borel sets $E_i \subset X_i$ such that $\mathfrak{m}_i(E_i) < \infty$ for any $i \in \mathbb{N}$ *converges* L^1*-strongly* to a Borel set $F \subset Y$ with $\mu(F) < \infty$ if $\chi_{E_i} \mathfrak{m}_i \rightharpoonup \chi_F \mu$ in duality with $\mathrm{C_{bs}}(Z)$ and $\mathfrak{m}_i(E_i) \to \mu(F)$.

We also say that a sequence of Borel sets $E_i \subset X_i$ *converges in* L^1_{loc} to a Borel set $F \subset Y$ if $E_i \cap B_R(x_i) \to F \cap B_R(y)$ L^1-strongly for any $R > 0$.

Remark 1.25. It follows from the very definition of L^1-strong convergence that if a sequence of sets E_i converges to F in L^1, then $\chi_{E_i} \to \chi_F$ L^2-strongly.

Definition 1.26. We say that a sequence of sets $E_i \subset X_i$ with locally finite perimeter *converges locally strongly in* BV to a set $F \subset Y$ of locally finite perimeter if $E_i \to F$ strongly in L^1_{loc} and $|D\chi_{E_i}| \rightharpoonup |D\chi_F|$ in duality with $\mathrm{C_{bs}}(Z)$.

A proof of the technical result below can be found in [7].

Proposition 1.27. *Let us fix* $p = 1, 2$.

(i) *For any* $f_i, g_i \in L^p(X_i, \mathfrak{m}_i)$ *such that* $f_i \to f \in L^p(Y, \mu)$ *and* $g_i \to g \in L^p(Y, \mu)$ L^p*-strongly one has* $f_i + g_i \to f + g$ L^p*-strongly.*

(ii) *If* $f_i \to f$ *and* $g_i \to g$ L^2*-strongly then* $f_i g_i \to fg$ L^1*-strongly.*

(iii) *If* $f_i \to f$ L^1*-strongly and* $\sup_{i \in \mathbb{N}} \|f_i\|_{L^\infty(X_i, \mathfrak{m}_i)} < \infty$ *then* $\|f_i\|_{L^2(X_i, \mathfrak{m}_i)} \to \|f\|_{L^2(Y, \mu)}$. *In particular* $f_i \to f$ L^2*-strongly.*

Let us present a compactness result for sets with finite perimeter that is partially taken from [2].

Proposition 1.28. *Let* $E_i \subset X_i$ *be sets of finite perimeter satisfying*

$$\sup_{i \in \mathbb{N}} \mathrm{Per}(E_i, B_1(x_i)) < \infty.$$

Then there exists $F \subset Y$ *of finite perimeter such that, up to a subsequence,* $E_i \cap B_1(x_i) \to F \cap B_1(y)$ L^1*-strongly and*

$$\liminf_{i \to \infty} \int g \, \mathrm{d}|D\chi_{E_i}| \geq \int g \, \mathrm{d}|D\chi_F| \quad \textit{for any } 0 \leq g \in \mathrm{C}(Z) \textit{ with } \mathrm{supp}(g) \subset \bar{B}_{1/2}(y). \tag{1.17}$$

If we further assume that

$$\lim_{i\to\infty}|D\chi_{E_i}|(B_{1/2}(x_i)) = |D\chi_F|(B_{1/2}(y)), \tag{1.18}$$

then (1.17) *improves to*

$$\lim_{i\to\infty}\int g\,\mathrm{d}|D\chi_{E_i}| = \int g\,\mathrm{d}|D\chi_F| \quad \textit{for any } g\in\mathrm{C}(Z) \textit{ with } \mathrm{supp}(g)\subset B_{1/2}(y). \tag{1.19}$$

Proof. The L^1-strong convergence $E_i\cap B_1(x_i)\to F\cap B_1(y)$ up to subsequence can be obtained by arguing as in [2, proof of Corollary 3.4].

Inequality (1.17) follows from [2, Proposition 3.6] along with a localization argument that we sketch briefly. For any $i\in\mathbb{N}$, using Lemma 1.14 we build a good cut-off function $\eta_i\in\mathrm{Lip}(X_i,\mathsf{d}_i)$ satisfying $\eta_i=1$ in $B_{1/2}(x_i)$ and $\eta_i=0$ in $X_i\setminus B_{3/4}(x_i)$. By Proposition 1.21, up to a subsequence, we can assume that $\eta_i\to\eta_\infty\in\mathrm{Lip}(Y,\rho)$ uniformly and L^2-strongly. It is easily seen that $\eta_\infty=1$ in $B_{1/2}(y)$ and $\eta_\infty=0$ in $Y\setminus B_1(y)$. The sequence $(\eta_i\chi_{E_i})_i$ satisfies

$$\eta_i\chi_{E_i}\to\eta_\infty\chi_F\ L^1\text{-strongly} \quad\text{and}\quad \sup_{i\in\mathbb{N}}|D(\eta_i\chi_{E_i})|(X_i)<\infty,$$

thanks to Proposition 1.27(ii) and standard calculus rules. Applying [2, Proposition 3.6] to the sequence $(\eta_i\chi_{E_i})_i$ we get (1.17).

Inequality (1.19) is a weak convergence result in the ball $B_{1/2}(y)\subset Z$, which can be proved as in [2, proof of Corollary 3.7] taking into account (1.17) and (1.18). ∎

Let us now introduce a notion of $H^{1,2}$-convergence along with its local counterpart.

Definition 1.29. We say that $f_i\in H^{1,2}(X_i,\mathsf{d}_i,\mathfrak{m}_i)$ *weakly converges* to $f\in H^{1,2}(Y,\varrho,\mu)$ if it converges L^2-weakly and $\sup_i\mathrm{Ch}^i(f_i)<\infty$. *Strong $H^{1,2}$-convergence* is defined by asking that f_i converges to f L^2-strongly and $\lim_i\mathrm{Ch}^i(f_i)=\mathrm{Ch}(f)$.

Definition 1.30. We say that $f_i\in H^{1,2}(B_R(x_i),\mathsf{d}_i,\mathfrak{m}_i)$ *converges in $H^{1,2}$* to $f\in H^{1,2}(B_R(y),\varrho,\mu)$ on $B_R(y)$ if f_i converges L^2-weakly (or L^2-strongly, equivalently) to f on $B_R(y)$ with $\sup_{i\in\mathbb{N}}\|f_i\|_{H^{1,2}}<\infty$. *Strong convergence in $H^{1,2}$* on $B_R(y)$ is defined by requiring

$$\lim_{i\to\infty}\int_{B_R(x_i)}|\nabla f_i|^2\,\mathrm{d}\mathfrak{m}_i=\int_{B_R(y)}|\nabla f|^2\,\mathrm{d}\mu.$$

Let us now collect those results from [8] that will play a role in this paper.

Lemma 1.31 ([8, Lemma 2.10]). *For any $f\in\mathrm{Lip_c}(B_R(y),\varrho)$ there exist functions $f_i\in\mathrm{Lip_c}(B_R(x_i),\mathsf{d}_i)$ satisfying*

$$\sup_{i\in\mathbb{N}}\|\,|\nabla f_i|\,\|_{L^\infty(X_i,\mathfrak{m}_i)}<\infty$$

and strongly converging to f in $H^{1,2}$.

Theorem 1.32 ([8, Theorem 4.4]). *Let $f_i \in D(\Delta, B_R(x_i))$ with*

$$\sup_{i\in\mathbb{N}} \int_{B_R(x_i)} (|f_i|^2 + |\nabla f_i|^2 + (\Delta f_i)^2)\,\mathrm{d}\mathfrak{m}_i < \infty,$$

and let f be an L^2-strong limit of f_i on $B_R(y)$. Then:

(i) $f \in D(\Delta, B_R(y))$*;*

(ii) $\Delta f_i \to \Delta f$ *on* $B_R(y)$ *weakly in* L^2*;*

(iii) $|\nabla f_i|^2 \to |\nabla f|^2$ *strongly in* L^1 *on* $B_R(y)$*.*

Proposition 1.33 ([8, Corollary 4.12]). *Let $f \in H^{1,2}(B_R(y), \varrho, \mu)$ be a harmonic function (i.e., $f \in D(\Delta, B_R(y))$ with $\Delta f = 0$). Then, for any $0 < r < R$ there exist $f_i \in H^{1,2}(B_r(x_i), \mathsf{d}_i, \mathfrak{m}_i)$ harmonic such that $f_i \to f$ strongly in $H^{1,2}$ on $B_r(y)$.*

1.3. Normed modules

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We begin by briefly recalling the definitions of normed module over $(X, \mathsf{d}, \mathfrak{m})$, which have been introduced in [30] and are in turn inspired by the theory developed in [55].

Let R be either $L^\infty(\mathfrak{m})$ or $L^0(\mathfrak{m})$. Let $\mathscr{M}$ be a module over the commutative ring R. Then an L^p*-pointwise norm* on $\mathscr{M}$, for some $p \in \{0\} \cup [1, \infty)$, is any mapping $|\cdot| : \mathscr{M} \to L^p(\mathfrak{m})$ such that

$$\begin{aligned} |v| &\geq 0 && \text{for every } v \in \mathscr{M}, \text{ with equality if and only if } v = 0,\\ |v + w| &\leq |v| + |w| && \text{for all } v, w \in \mathscr{M},\\ |fv| &= |f|\,|v| && \text{for all } f \in R \text{ and } v \in \mathscr{M}, \end{aligned} \tag{1.20}$$

where all (in)equalities are in the $\mathfrak{m}$-a.e. sense. We shall consider two classes of normed modules:

- $L^p(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-modules, with* $p \in [1, \infty)$*:* A module $\mathscr{M}^p$ over $L^\infty(\mathfrak{m})$ endowed with an L^p-pointwise norm $|\cdot|$ such that $\|v\|_{\mathscr{M}^p} := \||v|\|_{L^p(\mathfrak{m})}$ is a complete norm on $\mathscr{M}^p$.
- $L^0(\mathfrak{m})$*-normed* $L^0(\mathfrak{m})$*-modules:* A module $\mathscr{M}^0$ over $L^0(\mathfrak{m})$ endowed with an L^0-pointwise norm $|\cdot|$ such that $\mathsf{d}_{\mathscr{M}^0}(v, w) := \int \min\{|v - w|, 1\}\,\mathrm{d}\mathfrak{m}'$ (where $\mathfrak{m}'$ is any probability measure that is mutually absolutely continuous with $\mathfrak{m}$) is a complete distance on $\mathscr{M}^0$.

We refer to [29] for an account of the abstract normed modules theory on metric measure spaces.

Assume $(X, \mathsf{d}, \mathfrak{m})$ is *infinitesimally Hilbertian*, i.e., its Sobolev space $H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is Hilbert. Then a key example of normed module on X is the *tangent module* $L^0(TX)$, which is characterized as follows: there is a unique couple $(L^0(TX), \nabla)$, where $L^0(TX)$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and $\nabla : H^{1,2}(X) \to L^0(TX)$ is a linear *gradient* map, such that the following hold:

$|\nabla f|$ coincides with the minimal relaxed slope of f for every $f \in H^{1,2}(X)$,

$$\Big\{\sum_{i=1}^{n} \chi_{E_i} \nabla f_i \;\Big|\; (E_i)_{i=1}^n \text{ Borel partition of } X,\ (f_i)_{i=1}^n \subset H^{1,2}(X)\Big\} \text{ is dense in } L^0(TX).$$

For any $p \in [1,\infty]$, we set $L^p(TX) := \{v \in L^0(TX) \mid |v| \in L^p(\mathfrak{m})\}$. It can be readily checked that the space $L^p(TX)$ has a natural $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module structure (for $p < \infty$).

1.3.1. Second order calculus over RCD *spaces.* Gigli [30] has developed a second order calculus for $\mathrm{RCD}(K,\infty)$ metric measure spaces. The notions of Hessian and covariant derivative have been introduced as bilinear forms on $L^2(TX)$, along with the spaces $H^{2,2}(X,\mathsf{d},\mathfrak{m}) \subset H^{1,2}(X,\mathsf{d},\mathfrak{m})$ and $H^{1,2}_{\mathrm{C}}(TX) \subset L^2(TX)$ [30, Definitions 3.3.1, 3.4.1, 3.3.17, 3.4.3].

Let us recall that, as proved in [30, Proposition 3.3.18], we have the inclusion

$$D(\Delta) \subset H^{2,2}(X,\mathsf{d},\mathfrak{m}). \tag{1.21}$$

Moreover, assuming $(X,\mathsf{d},\mathfrak{m})$ to be $\mathrm{RCD}(K,N)$ m.m.s. one has the local estimate

$$\begin{aligned}\int_{B_1(x)} |\mathrm{Hess}\, f|^2 \,\mathrm{d}\mathfrak{m} \le C_{N,K}\Big(\int_{B_2(x)} |\Delta f|^2 \,\mathrm{d}\mathfrak{m} + \inf_{m\in\mathbb{R}} \int_{B_2(x)} \big||\nabla f|^2 - m\big| \,\mathrm{d}\mathfrak{m}\Big)\\ - K\int_{B_2(x)} |\nabla f|^2 \,\mathrm{d}\mathfrak{m},\end{aligned} \tag{1.22}$$

which can be checked by integrating the improved Bochner inequality proved in [35] against a good cut-off function (see Lemma 1.14 above).

Let us recall that the Hessian enjoys the following locality property that has been proved in [30, Proposition 3.3.24].

Proposition 1.34. *If* $f_1, f_2 \in H^{2,2}(X,\mathsf{d},\mathfrak{m})$ *then*

$$|\mathrm{Hess}\, f_1| = |\mathrm{Hess}\, f_2| \quad \mathfrak{m}\text{-a.e. in } \{f_1 = f_2\}.$$

In addition we shall use the following inequality that has been proved in [30, Proposition 3.3.22]:

$$|\nabla(\nabla f \cdot \nabla g)| \le |\mathrm{Hess}\, f|\,|\nabla g| + |\mathrm{Hess}\, g|\,|\nabla f| \quad \text{for all } f, g \in H^{2,2}(X,\mathsf{d},\mathfrak{m}). \tag{1.23}$$

1.3.2. Module with respect to the capacity measure. We recall a variant of the notion of L^0-normed L^0-module, where the Borel measure $\mathfrak{m}$ is replaced by capacity, which has been proposed in [21]. Fix a metric measure space $(X,\mathsf{d},\mathfrak{m})$. The space of all Borel functions on X, considered up to Cap-a.e. equality, is denoted by $L^0(\mathrm{Cap})$. If continuous functions are strongly dense in $H^{1,2}(X)$ (this holds, for instance, if the space is infinitesimally Hilbertian), then there exists a unique "quasi-continuous representative" map $\mathsf{QCR} : H^{1,2}(X) \to L^0(\mathrm{Cap})$ that is characterized as follows: QCR is a continuous map, and for any $f \in H^{1,2}(X)$, $\mathsf{QCR}(f)$ is (the equivalence class of) a quasi-continuous function that $\mathfrak{m}$-a.e. coincides with f itself. Recall that a function $f : X \to \mathbb{R}$ is said to be *quasi-continuous* if for any $\varepsilon > 0$ there exists a set $E \subset X$ with $\mathrm{Cap}(E) < \varepsilon$ such that $f : X \setminus E \to \mathbb{R}$ is continuous. We refer to [21, Theorem 1.20] for a proof of this result.

Given a module $\mathscr{M}_{\mathrm{Cap}}$ over the ring $L^0(\mathrm{Cap})$, we say that a mapping $|\cdot| : \mathscr{M}_{\mathrm{Cap}} \to L^0(\mathrm{Cap})$ is a *pointwise norm* if it satisfies the (in)equalities in (1.20) in the Cap-a.e. sense for any $v, w \in \mathscr{M}_{\mathrm{Cap}}$ and $f \in L^0(\mathrm{Cap})$. Then the space $\mathscr{M}_{\mathrm{Cap}}$ is said to be an $L^0(\mathrm{Cap})$-*normed* $L^0(\mathrm{Cap})$-*module* if it is complete when endowed with the distance

$$\mathsf{d}_{\mathscr{M}_{\mathrm{Cap}}}(v, w) := \sum_{k \in \mathbb{N}} \frac{1}{2^k \max\{\mathrm{Cap}(A_k), 1\}} \int_{A_k} \min\{|v - w|, 1\} \, \mathrm{d}\,\mathrm{Cap},$$

where $(A_k)_k$ is any increasing sequence of open subsets of X with finite capacity such that any bounded set $B \subset X$ is contained in A_k for some $k \in \mathbb{N}$ sufficiently large.

Since this fact plays a crucial role below, we recall that $|\nabla f|^2 \in H^{1,2}(X)$ for any $f \in \mathrm{Test}(X)$ (see [50]), and thus $|\nabla f| \in H^{1,2}(X)$ as well (see [21]). In particular, for any $f \in \mathrm{Test}(X)$, $|\nabla f|$ admits a quasi-continuous representative.

Theorem 1.35 (Tangent $L^0(\mathrm{Cap})$-module [21]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space. Then there exists a unique couple* $(L^0_{\mathrm{Cap}}(TX), \tilde{\nabla})$*, where* $L^0_{\mathrm{Cap}}(TX)$ *is an* $L^0(\mathrm{Cap})$-*normed* $L^0(\mathrm{Cap})$-*module and* $\tilde{\nabla} : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ *is a linear operator, such that*

$$|\tilde{\nabla} f| = \mathsf{QCR}(|\nabla f|) \ \mathrm{Cap}\text{-}a.e.\ for\ every\ f \in \mathrm{Test}(X),$$

$$\Big\{\sum_{n \in \mathbb{N}} \chi_{E_n} \tilde{\nabla} f_n \ \Big|\ (E_n)_n \ Borel\ partition\ of\ X,\ (f_n)_n \subset \mathrm{Test}(X)\Big\} \ is\ dense\ in\ L^0_{\mathrm{Cap}}(TX).$$

The space $L^0_{\mathrm{Cap}}(TX)$ *is called the* capacitary tangent module *on* X*, while* $\tilde{\nabla}$ *is the* capacitary gradient.

Fix any Radon measure μ on a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and suppose that $\mu \ll \mathrm{Cap}$. Then there is a natural projection $\pi_\mu : L^0(\mathrm{Cap}) \to L^0(\mu)$. Given an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module $\mathscr{M}_{\mathrm{Cap}}$, we define an equivalence relation $\sim_\mu$ on $\mathscr{M}_{\mathrm{Cap}}$ as follows: for $v, w \in \mathscr{M}_{\mathrm{Cap}}$, we declare that

$$v \sim_\mu w \iff |v - w| = 0 \ \mu\text{-a.e. on } X.$$

Then the quotient $\mathscr{M}^0_\mu := \mathscr{M}_{\mathrm{Cap}}/\sim_\mu$ inherits a natural structure of $L^0(\mu)$-normed $L^0(\mu)$-module. Let $\bar{\pi}_\mu : \mathscr{M}_{\mathrm{Cap}} \to \mathscr{M}^0_\mu$ be the canonical projection. Moreover, for any $p \in [1, \infty)$ define

$$\mathscr{M}^p_\mu := \{v \in \mathscr{M}^0_\mu \mid |v| \in L^p(\mu)\}. \tag{1.24}$$

It turns out that $\mathscr{M}^p_\mu$ is an $L^p(\mu)$-normed $L^\infty(\mu)$-module. Notice that $|\bar{\pi}_\mu(v)| = \pi_\mu(|v|)$ μ-a.e. for every $v \in \mathscr{M}_{\mathrm{Cap}}$.

Lemma 1.36. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s., and $\mathscr{M}_{\mathrm{Cap}}$ an $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module. Fix a finite Borel measure $\mu \geq 0$ on X such that $\mu \ll \mathrm{Cap}$. Let V be a linear subspace of $\mathscr{M}_{\mathrm{Cap}}$ such that $|v|$ admits a bounded* Cap*-a.e. representative for every $v \in V$ and*

$$\mathcal{V} := \Big\{ \sum_{n\in\mathbb{N}} \chi_{E_n} v_n \;\Big|\; (E_n)_{n\in\mathbb{N}} \text{ Borel partition of } X,\ (v_n)_{n\in\mathbb{N}} \subset V \Big\} \text{ is dense in } \mathscr{M}_{\mathrm{Cap}}.$$

Then for any $p \in [1, \infty)$,

$$\mathcal{W} := \Big\{ \sum_{i=1}^{n} \chi_{E_i} \bar{\pi}_\mu(v_i) \;\Big|\; n \in \mathbb{N},\ (E_i)_{i=1}^n \text{ Borel partition of } X,\ (v_i)_{i=1}^n \subset V \Big\} \text{ is dense in } \mathscr{M}^p_\mu.$$

Proof. Fix $v \in \mathscr{M}^p_\mu$ and $\varepsilon > 0$. Since $|v|^p \in L^1(\mu)$, there is $\delta > 0$ such that $(\int_E |v|^p \,\mathrm{d}\mu)^{1/p} \leq \varepsilon/3$ for any Borel set $E \subset X$ with $\mu(E) < \delta$. Choose any $\bar{v} \in \mathscr{M}_{\mathrm{Cap}}$ such that $\bar{\pi}_\mu(\bar{v}) = v$. We can find $(\bar{v}_k)_k \subset \mathcal{V}$ with $|\bar{v}_k - \bar{v}| \to 0$ in $L^0(\mathrm{Cap})$. Hence $|\bar{\pi}_\mu(\bar{v}_k) - \bar{\pi}_\mu(\bar{v})| = \pi_\mu(|\bar{v}_k - \bar{v}|) \to 0$ in $L^0(\mu)$. Thanks to the Egorov theorem, there exists a compact set $K \subset X$ with $\mu(X \setminus K) < \delta$ such that (possibly taking a non-relabelled subsequence) $|\bar{\pi}_\mu(\bar{v}_k) - v| \to 0$ uniformly on K. Consequently, by the dominated convergence theorem, $\chi_K \bar{\pi}_\mu(\bar{v}_k) \to \chi_K v$ in $\mathscr{M}^p_\mu$. Thus we can pick $k \in \mathbb{N}$ so that the element $\bar{w} := \bar{v}_k$ satisfies $\|\chi_K \bar{\pi}_\mu(\bar{w}) - \chi_K v\|_{\mathscr{M}^p_\mu} \leq \varepsilon/3$. If $\bar{w}$ is written as $\sum_{n\in\mathbb{N}} \chi_{E_n} \bar{w}_n$, then $\chi_K \bar{\pi}_\mu(\bar{w}) = \sum_{n\in\mathbb{N}} \chi_{K\cap E_n} \bar{\pi}_\mu(\bar{w}_n)$. By the dominated convergence theorem, for $N \in \mathbb{N}$ sufficiently large the element $z := \sum_{n=1}^N \chi_{K\cap E_n} \bar{\pi}_\mu(\bar{w}_n) \in \mathcal{W}$ satisfies $\|z - \chi_K \bar{\pi}_\mu(\bar{w})\|_{\mathscr{M}^p_\mu} \leq \varepsilon/3$. Therefore,

$$\|z - v\|_{\mathscr{M}^p_\mu} \leq \|z - \chi_K \bar{\pi}_\mu(\bar{w})\|_{\mathscr{M}^p_\mu} + \|\chi_K \bar{\pi}_\mu(\bar{w}) - \chi_K v\|_{\mathscr{M}^p_\mu} + \|\chi_{X\setminus K} v\|_{\mathscr{M}^p_\mu} \leq \varepsilon,$$

proving the statement. ■

1.4. Hodge Laplacian of vector fields on RCD *spaces*

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Consider the space $H^{1,2}_{\mathrm{H}}(TX)$ and the Hodge Laplacian $\Delta_{\mathrm{H}} : H^{1,2}_{\mathrm{H}}(TX) \supset D(\Delta_{\mathrm{H}}) \to L^2(TX)$, which have been defined in [30, Definition 3.5.13] and [30, Definition 3.5.15], respectively (see [29, first paragraph of Section 2.6] for the identification between vector and covector fields).

It follows from its definition that the Hodge Laplacian is self-adjoint,

$$\int \langle \Delta_{\mathrm{H}} v, w\rangle \,\mathrm{d}\mathfrak{m} = \int \langle v, \Delta_{\mathrm{H}} w\rangle \,\mathrm{d}\mathfrak{m} \quad \text{for all } v, w \in D(\Delta_{\mathrm{H}}). \tag{1.25}$$

Let us consider the *augmented Hodge energy functional* $\tilde{\mathcal{E}}_{\mathrm{H}} : L^2(TX) \to [0, \infty]$ which is defined in [30, (3.5.16)] (up to identifying $L^2(T^*X)$ with $L^2(TX)$ via the musical isomorphism). Then we denote by $(\mathsf{h}_{\mathrm{H},t})_{t\geq 0}$ the gradient flow in $L^2(TX)$ of the functional $\tilde{\mathcal{E}}_{\mathrm{H}}$. This means that for any vector field $v \in L^2(TX)$, $t \mapsto \mathsf{h}_{\mathrm{H},t}(v) \in L^2(TX)$ is the unique continuous curve on $[0, \infty)$ with $\mathsf{h}_{\mathrm{H},0}(v) = v$ which is locally absolutely continuous on $(0,\infty)$ and satisfies

$$\mathsf{h}_{\mathrm{H},t}(v) \in D(\Delta_{\mathrm{H}}) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_{\mathrm{H},t}(v) = -\Delta_{\mathrm{H}}\mathsf{h}_{\mathrm{H},t}(v) \quad \text{for every } t>0$$

(see [30, discussion that precedes Proposition 3.6.10]). Furthermore,

$$\mathsf{h}_{\mathrm{H},t}(\nabla f) = \nabla P_t f \text{ for all } f \in H^{1,2}(X) \text{ and } t \geq 0 \tag{1.26}$$

Finally, we recall that vector fields satisfy the following Bakry–Émery contraction estimate (see [30, Proposition 3.6.10]):

$$|\mathsf{h}_{\mathrm{H},t}(v)|^2 \leq e^{-2Kt} P_t(|v|^2) \quad \mathfrak{m}\text{-a.e. for all } v \in L^2(TX) \text{ and } t \geq 0. \tag{1.27}$$

Lemma 1.37 ($\mathsf{h}_{\mathrm{H},t}$ is self-adjoint). *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. Then*

$$\int \langle \mathsf{h}_{\mathrm{H},t}(v), w\rangle \,\mathrm{d}\mathfrak{m} = \int \langle v, \mathsf{h}_{\mathrm{H},t}(w)\rangle \,\mathrm{d}\mathfrak{m} \quad \textit{for all } v,w \in L^2(TX) \textit{ and } t \geq 0. \tag{1.28}$$

Proof. Fix $v,w \in L^2(TX)$ and $t>0$. We define $\varphi:[0,t]\to\mathbb{R}$ by

$$\varphi(s) := \int \langle \mathsf{h}_{\mathrm{H},s}(v), \mathsf{h}_{\mathrm{H},t-s}(w)\rangle \,\mathrm{d}\mathfrak{m} \quad \text{for every } s\in[0,t].$$

Then φ is absolutely continuous and

$$\varphi'(s) = -\int \langle \Delta_{\mathrm{H}}\mathsf{h}_{\mathrm{H},s}(v), \mathsf{h}_{\mathrm{H},t-s}(w)\rangle \,\mathrm{d}\mathfrak{m} + \int \langle \mathsf{h}_{\mathrm{H},s}(v), \Delta_{\mathrm{H}}\mathsf{h}_{\mathrm{H},t-s}(w)\rangle \,\mathrm{d}\mathfrak{m} \overset{(1.25)}{=} 0$$

for a.e. $t>0$. Thus φ is constant, thus in particular $\int \langle \mathsf{h}_{\mathrm{H},t}(v), w\rangle \,\mathrm{d}\mathfrak{m} = \varphi(t) = \varphi(0) = \int \langle v, \mathsf{h}_{\mathrm{H},t}(w)\rangle \,\mathrm{d}\mathfrak{m}$. ∎

Proposition 1.38. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. Then for any* $v \in D(\mathrm{div})$*,*

$$\mathsf{h}_{\mathrm{H},t}(v) \in H^{1,2}_{\mathrm{C}}(TX) \cap D(\mathrm{div}) \quad \textit{and} \quad \mathrm{div}(\mathsf{h}_{\mathrm{H},t}(v)) = P_t(\mathrm{div}(v)) \quad \textit{for every } t>0.$$

Proof. First of all, observe that $\mathsf{h}_{\mathrm{H},t}(v) \in H^{1,2}_{\mathrm{H}}(TX) \subset H^{1,2}_{\mathrm{C}}(TX)$ by [30, Corollary 3.6.4]. Moreover, let $f \in H^{1,2}(X)$. Then

$$\begin{aligned}\int \langle \nabla f, \mathsf{h}_{\mathrm{H},t}(v)\rangle \,\mathrm{d}\mathfrak{m} &\overset{(1.28)}{=} \int \langle \mathsf{h}_{\mathrm{H},t}(\nabla f), v\rangle \,\mathrm{d}\mathfrak{m} \overset{(1.26)}{=} \int \langle \nabla P_t f, v\rangle \,\mathrm{d}\mathfrak{m} \\ &= -\int P_t f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} = -\int f P_t(\mathrm{div}(v))\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By arbitrariness of f, we conclude that $\mathsf{h}_{\mathrm{H},t}(v) \in D(\mathrm{div})$ and $\mathrm{div}(\mathsf{h}_{\mathrm{H},t}(v)) = P_t(\mathrm{div}(v))$. ∎

2. A Gauss–Green formula on RCD spaces

Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ m.m.s. and $E\subset X$ a set of finite perimeter. We recall that, by Lemma 1.10, $|D\chi_E| \ll \mathscr{H}^{h_1}$, so $|D\chi_E| \ll \mathrm{Cap}$ by Theorem 1.12. It thus makes sense to consider the projection $\pi_{|D\chi_E|} : L^0(\mathrm{Cap}) \to L^0(|D\chi_E|)$. Recall also that $\mathsf{QCR} : H^{1,2}(X) \to L^0(\mathrm{Cap})$ stands for the "quasi-continuous representative" operator. Then define

$$\mathrm{tr}_E : H^{1,2}(X) \to L^0(|D\chi_E|), \quad \mathrm{tr}_E := \pi_{|D\chi_E|} \circ \mathsf{QCR},$$

the trace operator over the boundary of E. Observe that $\mathrm{tr}_E(f) \in L^\infty(|D\chi_E|)$ for every $f \in \mathrm{Test}(X)$.

Remark 2.1. When $(X, \mathsf{d}, \mathfrak{m})$ is the Euclidean space of dimension n and $E \subset \mathbb{R}^n$ is open and smooth, $\mathrm{tr}_E : H^1(\mathbb{R}^n) \to L^0(|D\chi_E|)$ coincides with the canonical trace operator. Indeed, the two operators coincide on smooth functions and they are continuous. In the case of the canonical trace this is a standard result, while for tr_E it is a consequence of [21, Proposition 1.19] and the continuity of $\pi_{|D\chi_E|} : L^0(\mathrm{Cap}) \to L^0(|D\chi_E|)$.

This being said, let us state the two main results of this section. The first one gives existence and uniqueness of the tangent module over the boundary of a set of finite perimeter. The second theorem provides a Gauss–Green formula tailored to finite-dimensional RCD spaces along with a strong approximation result for the exterior normal of sets with finite perimeter. This approximation result, whose proof heavily relies on the abstract machinery of normed modules and on functional-analytic tools, plays a key role in the study of rectifiability properties for boundaries of sets with finite perimeter that we are going to perform in the last section of this paper.

Let us point out that in the very recent [13] the problem of obtaining a Gauss–Green formula on $\mathrm{RCD}(K, \infty)$ spaces has been treated and in [53] an integration-by-parts formula is considered under stronger assumptions. A comparison between our stronger result, heavily relying on finite-dimensionality, and those in [13, 53] is outside the scope of this paper.

Theorem 2.2 (Tangent module over ∂E). *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Let $E \subset X$ be a set of finite perimeter. Then there exists a unique couple $(L^2_E(TX), \bar\nabla)$, where $L^2_E(TX)$ is an $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module and $\bar\nabla : \mathrm{Test}(X) \to L^2_E(TX)$ is linear, such that:*

(i) $|\bar\nabla f| = \mathrm{tr}_E(|\nabla f|)$ *$|D\chi_E|$-a.e. for every $f \in \mathrm{Test}(X)$.*

(ii) $\{\sum_{i=1}^n \chi_{E_i} \bar\nabla f_i \mid (E_i)_{i=1}^n$ *Borel partition of X,* $(f_i)_{i=1}^n \subset \mathrm{Test}(X)\}$ *is dense in $L^2_E(TX)$.*

Uniqueness is understood up to unique isomorphism: given another couple $(\mathscr{M}, \bar\nabla')$ satisfying (i)–(ii) *above, there exists a unique normed module isomorphism $\Phi : L^2_E(TX) \to \mathscr{M}$ such that $\Phi \circ \bar\nabla = \bar\nabla'$. The space $L^2_E(TX)$ is called the* tangent module over the boundary of E *and $\bar\nabla$ is the* gradient.

We denote by $\bar{\mathsf{QCR}} : H^{1,2}_C(TX) \to L^0_{\mathrm{Cap}}(TX)$ the “quasi-continuous representative” map for Sobolev vector fields, whose existence has been proven in [21, Theorem 2.14] (see [21, Definition 2.12] for a notion of “quasi-continuous vector field” suitable for this context). Moreover, with a slight abuse of notation we define

$$\mathrm{tr}_E : H^{1,2}_C(TX) \cap L^\infty(TX) \to L^2_E(TX), \quad \mathrm{tr}_E := \bar\pi_{|D\chi_E|} \circ \bar{\mathsf{QCR}}.$$

Notice that $|\mathrm{tr}_E(v)| = \mathrm{tr}_E(|v|)$ $|D\chi_E|$-a.e. for every $v \in H^{1,2}_C(TX) \cap L^\infty(TX)$.

Remark 2.3. Arguing as in Remark 2.1 one can prove that tr_E coincides with the canonical trace in the case of smooth domains in $\mathbb{R}^n$.

Theorem 2.4 (Gauss–Green formula on RCD spaces). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and* $E \subset X$ *be a set of finite perimeter such that* $\mathfrak{m}(E) < \infty$. *Then there exists a unique vector field* $\nu_E \in L^2_E(TX)$ *such that* $|\nu_E| = 1$ $|D\chi_E|$*-a.e. and*

$$\int_E \mathrm{div}(v)\,\mathrm{d}\mathfrak{m} = -\int \langle \mathrm{tr}_E(v), \nu_E\rangle\,\mathrm{d}|D\chi_E| \quad \text{for all } v \in H^{1,2}_C(TX) \cap D(\mathrm{div}) \text{ with } |v| \in L^\infty(\mathfrak{m}). \tag{2.1}$$

Moreover, there exists a sequence $(v_n)_n \subset \mathrm{TestV}_E(X)$ *of test vector fields over the boundary of* E (*see Lemma* 2.9 *below for the precise definition of this class*) *such that* $v_n \to \nu_E$ *in the strong topology of* $L^2_E(TX)$.

Remark 2.5. If X is a Riemannian manifold and $E \subset X$ is a domain with smooth boundary, then $L^2_E(TX)$ is the space of all Borel vector fields over X which are concentrated on the boundary of E and 2-integrable with respect to the surface measure, and in this case $\bar\nabla$ is the classical gradient for smooth functions.

Remark 2.6. The tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ is a Hilbert module [21, Proposition 2.8]. Therefore, it is immediate to see by passing to the quotient that $L^2_E(TX)$ is a Hilbert module as well.

The remaining part of this section is dedicated to the proofs of Theorems 2.2 and 2.4.

Proof of Theorem 2.2. *Uniqueness.* Denote by $\mathcal{W}$ the family of elements of $L^2_E(TX)$ considered in (ii). Given any $\omega = \sum_{i=1}^n \chi_{E_i}\bar\nabla f_i \in \mathcal{W}$, we are forced to set $\Phi(\omega) := \sum_{i=1}^n \chi_{E_i}\bar\nabla' f_i$. Well-posedness of this definition stems from the $|D\chi_E|$-a.e. identity

$$\Big|\sum_{i=1}^n \chi_{E_i}\bar\nabla' f_i\Big| = \sum_{i=1}^n \chi_{E_i}|\bar\nabla' f_i| = \sum_{i=1}^n \chi_{E_i}\,\mathrm{tr}_E(|\nabla f_i|) = \sum_{i=1}^n \chi_{E_i}|\bar\nabla f_i| = |\omega|,$$

which also shows that Φ preserves the pointwise norm. Then Φ is linear continuous, thus it can be uniquely extended to a continuous linear map $\Phi : L^2_E(TX) \to \mathscr{M}$ by density of $\mathcal{W}$ in $L^2_E(TX)$. By an approximation argument, it is easy to see that the extended Φ preserves the pointwise norm and is an $L^\infty(|D\chi_E|)$-module morphism. Finally, the map Φ is surjective, because its image is dense (as $\mathscr{M}$ satisfies (ii)) and closed (as Φ is an isometry). Consequently, we have proved that there exists a unique normed module isomorphism $\Phi : L^2_E(TX) \to \mathscr{M}$ such that $\Phi \circ \bar\nabla = \bar\nabla'$.

Existence. Let us consider the tangent $L^0(\mathrm{Cap})$-module $L^0_{\mathrm{Cap}}(TX)$ and the relative capacitary gradient operator $\tilde\nabla : \mathrm{Test}(X) \to L^0_{\mathrm{Cap}}(TX)$ associated to the space $(X, \mathsf{d}, \mathfrak{m})$ (see Theorem 1.35). We define $L^0_E(TX)$ as $L^0_{\mathrm{Cap}}(TX)/\sim_{|D\chi_E|}$ and the $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module $L^2_E(TX)$ as in (1.24). Moreover, we define the differential

$\bar{\nabla}:\mathrm{Test}(X)\to L^2_E(TX)$ as $\bar{\nabla}:=\bar{\pi}_{|D\chi_E|}\circ\tilde{\nabla}$. Clearly, the map $\bar{\nabla}$ is linear by construction. Given any function $f\in\mathrm{Test}(X)$, it follows that $|D\chi_E|$-a.e. we have

$$|\bar{\nabla}f|=|\bar{\pi}_{|D\chi_E|}(\tilde{\nabla}f)|=\pi_{|D\chi_E|}(|\tilde{\nabla}f|)=\pi_{|D\chi_E|}(\mathsf{QCR}(|\nabla f|))=\mathrm{tr}_E(|\nabla f|),$$

which shows that (i) is satisfied. We also set $V:=\mathrm{Test}(X)$ and consider the associated space $\mathcal{V}\subset L^0_{\mathrm{Cap}}(TX)$ as in the statement of Lemma 1.36. By the defining property of the cotangent Cap-module we know that $\mathcal{V}$ is dense in $L^0_{\mathrm{Cap}}(TX)$, whence Lemma 1.36 ensures that $\mathcal{W}$ is dense in $L^2_E(TX)$. This means that property (ii) holds. Therefore, the existence part of the statement is proven. ∎

To prove Theorem 2.4 we need some auxiliary results. Let us begin with the following one, which was obtained as an intermediate step in [2, proof of Theorem 4.2].

Lemma 2.7. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Let $E\subset X$ be a set of finite perimeter. Then*

$$\lim_{t\searrow 0}\int\left|1-e^{Kt}\frac{|\nabla P_t\chi_E|}{P_t^*|D\chi_E|}\right|P_t^*|D\chi_E|\,\mathrm{d}\mathfrak{m}=0. \tag{2.2}$$

Lemma 2.8. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Let $E\subset X$ be a set of finite perimeter. Then*

$$\int fP_t^*|D\chi_E|\,\mathrm{d}\mathfrak{m}=\int\mathrm{tr}_E(P_tf)\,\mathrm{d}|D\chi_E|\quad\textit{for all } f\in H^{1,2}(X)\cap L^\infty(\mathfrak{m})\textit{ and } t>0. \tag{2.3}$$

Moreover,

$$\lim_{t\searrow 0}\int\mathrm{tr}_E(P_tf)\,\mathrm{d}|D\chi_E|=\int\mathrm{tr}_E(f)\,\mathrm{d}|D\chi_E|\quad\textit{for all } f\in H^{1,2}(X)\cap L^\infty(\mathfrak{m}). \tag{2.4}$$

Proof. To prove (2.3), fix any $f\in H^{1,2}(X)\cap L^\infty(\mathfrak{m})$ and $t>0$. We claim that

$$\begin{aligned}&\exists(f_n)_n\subset\mathrm{Lip}_{\mathrm{bs}}(X,\mathsf{d})\text{ bounded in }L^\infty(\mathfrak{m}):\\&\qquad\qquad f_n\to f\text{ strongly in }H^{1,2}(X),\ \text{weakly}^*\text{ in }L^\infty(\mathfrak{m}).\end{aligned} \tag{2.5}$$

We argue as follows. Given any $s>0$, the function P_sf has a Lipschitz representative (still denoted by P_sf) thanks to the L^∞-Lip regularization of the heat flow. Since $\{P_sf\}_{s>0}$ is bounded in $L^\infty(\mathfrak{m})$ by the weak maximum principle and $P_s|\nabla f|^2\to|\nabla f|^2$ strongly in $L^1(\mathfrak{m})$, we can find $G\in L^1(\mathfrak{m})$ and a sequence $s_n\searrow 0$ such that $P_{s_n}|\nabla f|^2\leq G$ $\mathfrak{m}$-a.e. for all n and $P_{s_n}f\to f$ weakly* in $L^\infty(\mathfrak{m})$. Fix $\bar{x}\in X$ and for any $n\in\mathbb{N}$ choose a compactly supported 1-Lipschitz function $\eta_n:X\to[0,1]$ such that $\eta_n=1$ on $B_n(\bar{x})$. Standard computations (based on the Leibniz rule $\nabla(\eta_nP_{s_n}f)=\eta_n\nabla P_{s_n}f+P_{s_n}f\nabla\eta_n$, the dominated convergence theorem, and the Bakry–Émery contraction estimate) show that $f_n:=\eta_nP_{s_n}f\in\mathrm{Lip}_{\mathrm{bs}}(X,\mathsf{d})$ satisfy (2.5). Now observe that $P_t:H^{1,2}(X)\to H^{1,2}(X)$ is continuous, as a consequence of the Bakry–Émery contraction estimate and the continuity of $P_t:L^2(\mathfrak{m})\to L^2(\mathfrak{m})$. This ensures that

$P_t f_n \to P_t f$ strongly in $H^{1,2}(X)$ as $n \to \infty$, whence we know from [21, Propositions 1.12, 1.17 and 1.19] that (possibly passing to a non-relabelled subsequence) $\mathsf{QCR}(P_t f_n) \to \mathsf{QCR}(P_t f)$ Cap-a.e., and accordingly $\mathrm{tr}_E(P_t f_n) \to \mathrm{tr}_E(P_t f)$ $|D\chi_E|$-a.e. Moreover, since $|P_t f_n| \le \sup_k \|f_k\|_{L^\infty(\mathfrak{m})} =: C$ in the $\mathfrak{m}$-a.e. sense for all $n \in \mathbb{N}$, we deduce that $|\mathsf{QCR}(P_t f_n)| \le C$ Cap-a.e. for all $n \in \mathbb{N}$, and thus $\mathrm{tr}_E(P_t f_n) \le C$ $|D\chi_E|$-a.e. for all $n \in \mathbb{N}$. All in all, we obtain (2.3) by letting $n \to \infty$ in $\int f_n P_t^*|D\chi_E| \,\mathrm{d}\mathfrak{m} = \int \mathrm{tr}_E(P_t f_n) \,\mathrm{d}|D\chi_E|$, which is satisfied thanks to the defining property of $P_t^*|D\chi_E|$; here we use the dominated convergence theorem and the L^∞-weak* convergence $f_n \to f$.

Let us now pass to the proof of (2.4). Fix $f \in H^{1,2}(X) \cap L^\infty(\mathfrak{m})$. By arguing as above, we see that $|\mathrm{tr}_E(P_t f)| \le \|f\|_{L^\infty(\mathfrak{m})}$ $|D\chi_E|$-a.e. for all $t > 0$, and any given sequence $t_n \searrow 0$ admits a subsequence $t_{n_i} \searrow 0$ such that $\mathrm{tr}_E(P_{t_{n_i}} f) \to \mathrm{tr}_E(f)$ $|D\chi_E|$-a.e. Therefore, by the dominated convergence theorem we conclude that $\lim_i \int \mathrm{tr}_E(P_{t_{n_i}} f) \,\mathrm{d}|D\chi_E| = \int \mathrm{tr}_E(f) \,\mathrm{d}|D\chi_E|$, which yields (2.4). ∎

Lemma 2.9 (Test vector fields over ∂E). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Let* $E \subset X$ *be a set of finite perimeter and finite mass. Define the class* $\mathrm{TestV}_E(X) \subset L^2_E(TX)$ *of* test vector fields over the boundary of E *as*

$$\mathrm{TestV}_E(X) := \mathrm{tr}_E(\mathrm{TestV}(X)) = \left\{ \sum_{i=1}^n \mathrm{tr}_E(g_i)\bar{\nabla} f_i \,\middle|\, n \in \mathbb{N},\, (f_i)_{i=1}^n, (g_i)_{i=1}^n \subset \mathrm{Test}(X) \right\}.$$

Then $\mathrm{TestV}_E(X)$ *is dense in* $L^2_E(TX)$.

Proof. By Theorem 2.2 (ii), it suffices to show that each $v \in L^2_E(TX)$ of the form $v = \chi_E \bar{\nabla} f$, where $E \subset X$ is a Borel set and $f \in \mathrm{Test}(X)$, can be approximated by elements of $\mathrm{TestV}_E(X)$ in the strong topology of $L^2_E(TX)$. Fix $\varepsilon > 0$ and choose $h \in \mathrm{Lip}_c(X)$ such that $\|h - \chi_E\|_{L^2(|D\chi_E|)} \le \varepsilon/(2\,\mathrm{Lip}(f))$. By [30, (3.2.3)] we can find a sequence $(g_n)_n \subset \mathrm{Test}(X)$ such that $\sup_n \|g_n\|_{L^\infty(\mathfrak{m})} < \infty$ and $g_n \to h$ in $H^{1,2}(X)$. Hence, by using [22, Propositions 2.13 and 2.20] we see that (up to a non-relabelled subsequence) we have $\mathrm{tr}_E(g_n)(x) \to h(x)$ for $|D\chi_E|$-a.e. $x \in X$. Accordingly, by the dominated convergence theorem, $|(\mathrm{tr}_E(g_n) - h)\bar{\nabla} f| \to 0$ in $L^2(|D\chi_E|)$. Now choose $n \in \mathbb{N}$ so large that $g := g_n$ satisfies $\|(\mathrm{tr}_E(g) - h)\bar{\nabla} f\|_{L^2_E(TX)} < \varepsilon/2$. Hence,

$$\begin{aligned} \|\mathrm{tr}_E(g)\bar{\nabla} f - v\|_{L^2_E(TX)} &\le \|(\mathrm{tr}_E(g) - h)\bar{\nabla} f\|_{L^2_E(TX)} + \|(h - \chi_E)\bar{\nabla} f\|_{L^2_E(TX)} \\ &\le \varepsilon/2 + \|h - \chi_E\|_{L^2(|D\chi_E|)} \,\mathrm{Lip}(f) < \varepsilon. \end{aligned}$$

Since $\mathrm{tr}_E(g)\bar{\nabla} f \in \mathrm{TestV}_E(X)$, the statement is proved. ∎

The last ingredient we need is an improvement of Theorem 1.7 in the special case of $\mathrm{RCD}(K, \infty)$ spaces. As we are going to see in the ensuing result, to obtain the total variation of a BV function it is sufficient to restrict attention to those competitors that are Sobolev regular. The proof is based on a parabolic approximation argument that builds upon the technical results developed in Section 1.4.

Theorem 2.10 (Representation formula for $|Df|$ on RCD spaces). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space and* $f \in \mathrm{BV}(X)$. *Then*

$$|Df|(X) = \sup\left\{ \int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} \;\middle|\; v \in H^{1,2}_{\mathrm{C}}(TX) \cap D(\mathrm{div}),\ |v| \leq 1\ \mathfrak{m}\text{-a.e.}, \mathrm{div}(v) \in L^\infty(\mathfrak{m}) \right\}.$$

Proof. Denote by S the right hand side of the above formula. We know by Remark 1.6 that $|Df|(X) \geq S$. To prove the converse inequality, fix any $\varepsilon > 0$. Theorem 1.7 guarantees the existence of a vector field $v \in D(\mathrm{div})$, with $|v| \leq 1$ $\mathfrak{m}$-a.e. and $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$, such that $\int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} > |Df|(X) - \varepsilon/2$. Now define $v_t := e^{Kt}\mathsf{h}_{\mathrm{H},t}(v)$ for every $t > 0$. Notice that $v_t \in H^{1,2}_{\mathrm{C}}(TX) \cap D(\mathrm{div})$ by Proposition 1.38. Since $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$ and $\mathrm{div}(v_t) = e^{Kt}P_t(\mathrm{div}(v))$, we deduce from the weak maximum principle that $\mathrm{div}(v_t) \in L^\infty(\mathfrak{m})$ as well. More precisely, $\|\mathrm{div}(v_t)\|_{L^\infty(\mathfrak{m})} \leq e^{Kt}\|\mathrm{div}(v)\|_{L^\infty(\mathfrak{m})}$ for all $t > 0$. Moreover, the weak maximum principle also guarantees that

$$|v_t| = e^{Kt}|\mathsf{h}_{\mathrm{H},t}(v)| \overset{(1.27)}{\leq} \sqrt{P_t(|v|^2)} \leq 1 \quad \text{in the } \mathfrak{m}\text{-a.e. sense.}$$

Since $\lim_{t\searrow 0}\mathrm{div}(v_t) = \mathrm{div}(v)$ in $L^2(\mathfrak{m})$, we can find $t_n \searrow 0$ such that $\mathrm{div}(v_{t_n})(x) \to \mathrm{div}(v)(x)$ for $\mathfrak{m}$-a.e. $x \in X$. As $(\mathrm{div}(v_{t_n}))_n$ is a bounded sequence in $L^\infty(\mathfrak{m})$, we can finally conclude that $\lim_n \int f \,\mathrm{div}(v_{t_n})\,\mathrm{d}\mathfrak{m} = \int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m}$ by the dominated convergence theorem. Therefore, there exists $n \in \mathbb{N}$ such that $w := v_{t_n}$ satisfies

$$\int f \,\mathrm{div}(w)\,\mathrm{d}\mathfrak{m} > \int f \,\mathrm{div}(v)\,\mathrm{d}\mathfrak{m} - \varepsilon/2 > |Df|(X) - \varepsilon.$$

This shows that $|Df|(X) < S + \varepsilon$, whence $|Df|(X) \leq S$ by arbitrariness of ε, as desired. ∎

Proof of Theorem 2.4. First of all, define $\mu_t := P_t^*|D\chi_E|\mathfrak{m}$ for every $t > 0$. Recall that $\mu_t \rightharpoonup |D\chi_E|$ in duality with $\mathrm{C_b}(X)$ as $t \searrow 0$. Set

$$\nu_t := \chi_{\{P_t^*|D\chi_E|>0\}}\frac{\nabla P_t\chi_E}{P_t^*|D\chi_E|} \in L^0(TX) \quad \text{for every } t > 0.$$

It follows from the 1-Bakry–Émery estimate (1.13) that $|DP_t\chi_E| \leq e^{-Kt}P_t^*|D\chi_E|$ $\mathfrak{m}$-a.e., thus $\nu_t \in L^\infty(TX)$ and $|\nu_t| \leq e^{-Kt}$ $\mathfrak{m}$-a.e. Set

$$\mathcal{V} := \{v \in H^{1,2}_{\mathrm{C}}(TX) \cap D(\mathrm{div}) \mid |v| \in L^\infty(\mathfrak{m})\}$$

and fix $v \in \mathcal{V}$. The Leibniz rule for divergence ensures that $\varphi v \in D(\mathrm{div})$ for any $\varphi \in \mathrm{Lip_b}(X)$, so the usual integration-by-parts formula yields

$$\int P_t\chi_E \,\mathrm{div}(\varphi v)\,\mathrm{d}\mathfrak{m} = -\int \varphi\langle\nabla P_t\chi_E, v\rangle\,\mathrm{d}\mathfrak{m} = -\int \varphi\langle v, \nu_t\rangle\,\mathrm{d}\mu_t \quad \text{for all } \varphi \in \mathrm{Lip_b}(X). \tag{2.6}$$

Moreover, $\langle v, \nu_t \rangle \in L^\infty(\mu_t)$ and $\|\langle v, \nu_t \rangle\|_{L^\infty(\mu_t)} \le e^{-Kt} \| |v| \|_{L^\infty(\mathfrak{m})}$ for every $t > 0$. Let $\sigma_t := \langle v, \nu_t \rangle \mu_t$ for all $t > 0$. Fix any sequence $t_n \searrow 0$. Since $\mu_{t_n} \rightharpoonup |D\chi_E|$ in duality with $\mathrm{C_b}(X)$, we know that $(\mu_{t_n})_n$ is tight by the Prokhorov theorem. As $\sup_n \|\langle v, \nu_{t_n} \rangle\|_{L^\infty(\mu_{t_n})}$ is finite, we deduce that $(\sigma_{t_n})_n$ is tight as well. By using the Prokhorov theorem again, we can thus take a subsequence $(t_{n_i})_i$ such that $\sigma_{t_{n_i}} \rightharpoonup \sigma$ in duality with $\mathrm{C_b}(X)$ for some finite (signed) Borel measure σ on X. Since $\mathrm{Lip_b}(X)$ is dense in $\mathrm{C_b}(X)$ and the identity in (2.6) gives

$$\int \varphi \,\mathrm{d}\sigma = \lim_{i \to \infty} \int \varphi \,\mathrm{d}\sigma_{t_{n_i}} = -\int_E \operatorname{div}(\varphi v) \,\mathrm{d}\mathfrak{m} \quad \text{for every } \varphi \in \mathrm{Lip_b}(X),$$

we see that σ is independent of the chosen sequence $(t_{n_i})_i$. Hence, $\sigma_t \rightharpoonup \sigma$ in duality with $\mathrm{C_b}(X)$ as $t \searrow 0$. Thus, for any non-negative function $\varphi \in \mathrm{C_b}(X)$,

$$\begin{aligned}\left|\int \varphi \,\mathrm{d}\sigma\right| &\le \lim_{t \searrow 0} \int \varphi |\langle v, \nu_t \rangle| \,\mathrm{d}\mu_t \le e^{|K|} \| |v| \|_{L^\infty(\mathfrak{m})} \lim_{t \searrow 0} \int \varphi \,\mathrm{d}\mu_t \\ &= e^{|K|} \| |v| \|_{L^\infty(\mathfrak{m})} \int \varphi \,\mathrm{d}|D\chi_E|,\end{aligned}$$

whence $\sigma \ll |D\chi_E|$ and the Radon–Nikodým derivative $L(v) := \frac{\mathrm{d}\sigma}{\mathrm{d}|D\chi_E|}$ belongs to $L^\infty(|D\chi_E|)$. Consequently, taking into account (2.6) we deduce that

$$\int_E \operatorname{div}(\varphi v) \,\mathrm{d}\mathfrak{m} = -\int \varphi L(v) \,\mathrm{d}|D\chi_E| \quad \text{for all } v \in \mathcal{V} \text{ and } \varphi \in \mathrm{Lip_b}(X). \tag{2.7}$$

Furthermore,

$$\lim_{t \searrow 0} \int \varphi \langle v, \nu_t \rangle \,\mathrm{d}\mu_t = \int \varphi L(v) \,\mathrm{d}|D\chi_E| \quad \text{for all } v \in \mathcal{V} \text{ and } \varphi \in \mathrm{Lip_b}(X). \tag{2.8}$$

Observe that for any $v \in \mathcal{V}$ and $\varphi \in \mathrm{Lip_b}(X)$, $\varphi \ge 0$,

$$\begin{aligned}\left|\int \varphi L(v) \,\mathrm{d}|D\chi_E|\right| &\overset{(2.8)}{=} \lim_{t \searrow 0} \left| e^{Kt} \int \varphi \langle v, \nu_t \rangle \,\mathrm{d}\mu_t \right| \\ &\le \lim_{t \searrow 0} \left(\|\varphi\|_{L^\infty(\mathfrak{m})} \| |v| \|_{L^\infty(\mathfrak{m})} \int \left|1 - e^{Kt} |\nu_t|\right| \mathrm{d}\mu_t + \int \varphi \left\langle v, \frac{\nu_t}{|\nu_t|} \right\rangle \mathrm{d}\mu_t \right) \\ &\overset{(2.2)}{\le} \lim_{t \searrow 0} \int \varphi |v| \,\mathrm{d}\mu_t \overset{(2.3)}{=} \lim_{t \searrow 0} \int \mathrm{tr}_E(P_t(\varphi |v|)) \,\mathrm{d}|D\chi_E| \\ &\overset{(2.4)}{=} \int \varphi \,\mathrm{tr}_E(|v|) \,\mathrm{d}|D\chi_E|.\end{aligned}$$

In the last two equalities we have used the fact that $|v| \in H^{1,2}(X)$. By arbitrariness of φ, we find that $|L(v)| \le \mathrm{tr}_E(|v|)$ $|D\chi_E|$-a.e. for all $v \in \mathcal{V}$. Now define $\omega : \mathrm{tr}_E(\mathcal{V}) \to L^1(|D\chi_E|)$ by

$$\omega(\mathrm{tr}_E(v)) := L(v) \quad \text{for every } v \in \mathcal{V}. \tag{2.9}$$

The operator $L : \mathcal{V} \to L^\infty(|D\chi_E|)$ is linear by its very construction, and the inequality $|L(v)| \le \mathrm{tr}_E(|v|)$ shows that ω is well-posed, linear and satisfies

$$|\omega(v)| \le |v| \quad |D\chi_E|\text{-a.e. for every } v \in \mathrm{tr}_E(\mathcal{V}).$$

Since $\mathrm{TestV}_E(X) \subset \mathcal{V}$ and $\mathrm{TestV}_E(X)$ is dense in $L^2_E(TX)$, we infer from Lemma 2.9 that $\mathrm{tr}_E(\mathcal{V})$ is a dense linear subspace of $L^2_E(TX)$. Therefore, by [30, Proposition 1.4.8], ω can be uniquely extended to an element $\omega \in L^2_E(T^*X) := L^2_E(TX)^*$ satisfying $|\omega| \le 1$ $|D\chi_E|$-a.e. We denote by $\nu_E \in L^2_E(TX)$ the vector field corresponding to ω via the Riesz isomorphism. By combining (2.7) (with $\varphi \equiv 1$) and (2.9), we conclude that (2.1) is satisfied. It only remains to show that $|\nu_E| \ge 1$ $|D\chi_E|$-a.e. To do so, just observe that Theorem 2.10 yields

$$\begin{aligned}|D\chi_E|(X) &\le \sup_{\substack{v\in\mathcal{V}\\ |v|\le 1\ \mathfrak{m}\text{-a.e.}}} \int_E \mathrm{div}(v)\,\mathrm{d}\mathfrak{m} \overset{(2.1)}{=} \sup_{\substack{v\in\mathcal{V}\\ |v|\le 1\ \mathfrak{m}\text{-a.e.}}} -\int \langle \mathrm{tr}_E(v), \nu_E\rangle\,\mathrm{d}|D\chi_E| \\ &\le \int |\nu_E|\,\mathrm{d}|D\chi_E| \le |D\chi_E|(X),\end{aligned}$$

whence each inequality must be an equality. This clearly forces the $|D\chi_E|$-a.e. equality $|\nu_E| = 1$. The element ν_E is uniquely determined by (2.1) as the space $\mathrm{tr}_E(\mathcal{V})$ is dense in $L^2_E(TX)$. Finally, the last part of the statement is an immediate consequence of Lemma 2.9. ■

3. Uniqueness of tangents for sets of finite perimeter

In this section we prove a uniqueness theorem (up to negligible sets) for blow-ups of sets with finite perimeter over $\mathrm{RCD}(K, N)$ metric measure spaces. This is a further step towards generalizing De Giorgi's theorem to the framework of RCD spaces.

We recall the notion of tangent to a set of finite perimeter, introduced in [2].

Definition 3.1 (Tangents to a set of finite perimeter). Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s., $x \in X$ and let $E \subset X$ be a set of locally finite perimeter. We denote by $\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ the collection of quintuples (Y, ϱ, μ, y, F) satisfying the following two properties:

(a) $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ and there are $r_i \downarrow 0$ are such that the rescaled spaces $(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x)$ converge to (Y, ϱ, μ, y) in the pointed measured Gromov–Hausdorff topology;

(b) F is a set of locally finite perimeter in Y with $\mu(F) > 0$, and if r_i are as in (a), then the sequence $f_i = \chi_E$ converges in L^1_{loc} to χ_F according to Definition 1.24.

We identify $(Y_1, \rho_1, \mu_1, y), (Y_2, \rho_2, \mu_2, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ if there exists an isometry $\Psi : (Y_1, \rho_1) \to (Y_2, \rho_2)$ such that $\Psi(y_1) = y_2$, $\Psi_*\mathfrak{m}_1 = \mathfrak{m}_2$ and $\int |\chi_{F_2} \circ \Psi - \chi_{F_1}|\,\mathrm{d}\mathfrak{m}_1 = 0$.

Let us point out that, up to a $|D\chi_E|$-negligible set, the perimeter measures $|D^i\chi_E|$ on the rescaled spaces weakly converge to $|D\chi_F|$ in duality with C_{bs}. This statement, which is part of [2, Corollary 4.10], plays a role in the rest of this paper.

We are ready to state the main theorem of this section.

Theorem 3.2. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. with essential dimension* $1 \le n \le N$ *and* $E \subset X$ *be a set of finite perimeter. Then, for* $|D\chi_E|$*-a.e.* $x \in X$*, there exists* $k = 1, \dots, n$ *such that*

$$\mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \left\{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\})\right\}.$$

Let us explain the strategy of the proof. The starting point is [2, Theorem 4.3], which we recall below.

Theorem 3.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and* $E \subset X$ *be a set of locally finite perimeter. Then* E *admits a Euclidean half-space as tangent at* x *for* $|D\chi_E|$*-a.e.* $x \in X$*, that is,*

$$(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \quad \textit{for some } k \in [1, N].$$

Let us point out that if n denotes again the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$, then we can sharpen the conclusion above to $1 \le k \le n$. Indeed, by the lower semicontinuity of the essential dimension with respect to pmGH convergence (see [9, 43]) one can prove that all the tangent spaces to an $\mathrm{RCD}(K, N)$ space of essential dimension n have essential dimension no greater than n. In particular, no Euclidean space of dimension $k > n$ can be a tangent space.

After establishing Theorem 3.3 the state of the art in the theory of sets of finite perimeter was similar to that of the structure theory of RCD spaces after [32], where the authors proved the existence of a Euclidean tangent space up to negligible sets. The content of this and of the next section instead can be seen as a counterpart in codimension 1 of the main results obtained by Mondino–Naber [47].

Also the main ideas underlying the proofs of the uniqueness of tangents and the rectifiability result are quite similar to those in [47]. As in that case, the existence of a Euclidean tangent along a fixed scale is a regularity information which can be propagated at any location and scale up to a set which is small with respect to the relevant measure, yielding uniqueness of tangents.

From a technical point of view, our construction heavily relies on the use of the so-called *harmonic* δ*-splitting maps*, a kind of good replacement for coordinate functions within the theory of lower Ricci bounds, which played a crucial role in the development of the theory of Ricci limits (see [16–18] and the more recent [19, 20]). Since, up to our knowledge, this is the first time they are explicitly used in the RCD framework, we dedicate Section 3.1 below to establishing some of their properties. With this tool at our disposal, the *propagation of regularity step* is a consequence of a weighted maximal argument which was suggested in [20]. Let us point out that, in order for the whole procedure to work, the fact that perimeter measures have codimension 1 (see Lemma 1.10) and the fact that harmonic functions satisfy L^2 Hessian bounds play a key role. The strategy would completely fail if perimeter measures had codimension greater than or equal to 2.

3.1. *Splitting maps and propagation of regularity*

This subsection is devoted to the study of δ-splitting maps. Their introduction in the study of spaces with lower Ricci curvature bounds dates back to [15].

Definition 3.4. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(-1, N)$ metric measure space, and let $x \in X$ and $\delta > 0$. We say that $u := (u_1, \ldots, u_k) : B_r(x) \to \mathbb{R}^k$ is a δ-*splitting map* if it is harmonic (meaning that $u_a \in D(\Delta, B_r(x))$ with $\Delta u_a = 0$ for any $a = 1, \ldots, k$) and satisfies:

(i) u_a is C_N-Lipschitz for any $a = 1, \ldots, k$;

(ii) $r^2 \fint_{B_r(x)} |\mathrm{Hess}\, u_a|^2 \, \mathrm{d}\mathfrak{m} < \delta$ for any $a = 1, \ldots, k$;

(iii) $\fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{a,b}| \, \mathrm{d}\mathfrak{m} < \delta$ for any $a, b = 1, \ldots, k$.

Remark 3.5. Let us clarify the meaning of $|\mathrm{Hess}\, u|$ when $u : B_r(x) \to \mathbb{R}$ is harmonic and not necessarily globally defined. For any ball $B_{2s}(y) \subset B_r(x)$ we take a good cut-off function η according to Lemma 1.14 that satisfies $\eta = 1$ in $B_s(y)$ and $\eta = 0$ in $X \setminus B_{2s}(y)$. As we already remarked in Section 1.2.2, one has $\eta u \in D(\Delta)$, therefore $\eta u \in H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ as a consequence of (1.21). We can now set $|\mathrm{Hess}\, u| := |\mathrm{Hess}(\eta u)|$ in $B_s(y)$. Observe that this is a good definition thanks to the locality of the Hessian (see Proposition 1.34).

Remark 3.6. Compared to the definition of δ-splitting map which is nowadays adopted within the theory of Ricci limits (see for instance [20, Definition 1.20]) the main difference is condition (i). In [20] the sharper bound $|\nabla u| \le 1 + \delta$ is imposed, though, as the authors observe, it can be obtained as a consequence of the bound $|\nabla u| \le C_N$ and of the other defining properties (when working in the smooth framework).

3.1.1. δ-splitting maps and ε-closeness. The power of δ-splitting maps in the theory of lower Ricci bounds is that, roughly speaking, they allow one to pass from analysis to geometry and vice versa. Namely, the existence of a δ-splitting map with k components on a Riemannian manifold with Ricci curvature bounded below by $-\delta$ can be turned into ε-GH closeness (in the scale invariant sense) to a space which splits a factor $\mathbb{R}^k$ and vice versa (see [15] and [20, Lemma 1.21]).

Below we provide rigorous statements of the above-mentioned results in the framework of RCD spaces. The convergence and stability results of [7, 8] allow us to argue by compactness avoiding the explicit constructions of [15]. The price we have to pay is that the results become less local in nature compared to [20, Lemma 1.21]. Still, they are sufficient for our purposes.

The first result presented below, Proposition 3.7, corresponds to the rough statement "the existence of a δ-splitting map with k components implies that the m.m.s. is ε-close to a product $\mathbb{R}^k \times Z$". The second one, Proposition 3.9, ensures that, over an $\mathrm{RCD}(-\varepsilon, N)$ space ε-close to a product $\mathbb{R}^k \times Z$, one can build a δ-splitting map with k components.

In order to shorten the notation for the rest of the paper we write $(\mathbb{R}^k \times Z, (0^k, z))$ to denote the p.m.m.s. (pointed m.m.s.) $(\mathbb{R}^k \times Z, \mathsf{d}_{\mathrm{eucl}} \times \mathsf{d}_Z, \mathscr{L}^k \times \mathfrak{m}_Z, (0^k, z))$.

Proposition 3.7. *Fix $N > 1$. Then, for any $\varepsilon > 0$, there exists $\delta = \delta_{N,\varepsilon} > 0$ such that, for any $\mathrm{RCD}(-\delta, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and for any $x \in X$, if there exists a map $u : B_{\delta^{-1}}(x) \to \mathbb{R}^k$ such that u is a δ-splitting map over $B_s(x)$ for any $0 < s < \delta^{-1}$, then*

$$\mathsf{d}_{\mathrm{pmGH}}\big((X, \mathsf{d}, \mathfrak{m}, x), (\mathbb{R}^k \times Z, (0^k, z))\big) < \varepsilon$$

for some pointed $\mathrm{RCD}(0, N - k)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$.

Proof. We argue by contradiction. Suppose that, for any $n \geq 1$, there exist an $\mathrm{RCD}(-1/n, N)$ m.m.s. $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$, a point $x_n \in X_n$ and a map $u_n : B_n(x_n) \to \mathbb{R}^k$ which is a $1/n$-splitting map when restricted to $B_s(x_n)$ for any $0 < s < n$. We assume without loss of generality that $\mathfrak{m}_n(B_1(x)) = 1$. Up to extracting a subsequence, which we do not relabel, we can assume that $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converges in the pmGH-topology to an $\mathrm{RCD}(0, N)$ p.m.m.s. $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$. Here we have used the stability and compactness property of $\mathrm{RCD}(K, N)$ spaces (see Remark 1.13). We claim that X_∞ splits off a factor $\mathbb{R}^k$. Observe that if this is the case, then we reach the sought contradiction. The rest of this proof is dedicated to establishing the claim.

We wish to prove that there exists a function $v : X_\infty \to \mathbb{R}^k$ such that, letting $v := (v^1, \dots, v^k)$, v^i is Lipschitz, harmonic and with vanishing Hessian for any $i = 1, \dots, k$ and $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ $\mathfrak{m}_\infty$-a.e. for any $i, j = 1, \dots, k$. The function v will be obtained as a limit of $1/n$-splitting maps $u_n : B_n(x_n) \to \mathbb{R}^k$. Indeed, since by the definition of a δ-splitting map the u_n are C_N-Lipschitz for any $n \in \mathbb{N}$ and we can assume without loss of generality that $u_n(x_n) = 0^k$ for any $n \in \mathbb{N}$, by a generalized version of the Ascoli–Arzelà theorem (Proposition 1.21) we can infer the existence of $v : X_\infty \to \mathbb{R}^k$ such that u_n converges to v locally uniformly on $B_R(x_n)$ for any $R > 0$. As a consequence, it is easy to check that u_n converges strongly in L^2 (see Definition 1.22) to v on $B_R(x_n)$ for any $R > 0$. Since the functions u_n are harmonic on $B_{2R}(x_n)$, at least for n sufficiently large, by Theorem 1.32 and Proposition 1.27 it follows that v is harmonic and that, for any $R > 0$ and $i, j = 1, \dots, k$,

$$\fint_{B_R(x_\infty)} |\nabla v^i \cdot \nabla v^j - \delta_{ij}| \,\mathrm{d}\mathfrak{m}_\infty = \lim_{n \to \infty} \fint_{B_R(x_n)} |\nabla u_n^i \cdot \nabla u_n^j - \delta_{ij}| \,\mathrm{d}\mathfrak{m}_n = 0.$$

Hence $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ $\mathfrak{m}_\infty$-a.e. on X_∞.

Since $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ is an $\mathrm{RCD}(0, N)$ m.m.s., from $\Delta v^i = 0$ and $|\nabla v^i|^2 = 1$ we infer by (1.22) that $\mathrm{Hess}\, v^i = 0$ for any $i = 1, \dots, k$. All in all we find by a standard argument (see [11, proof of Lemma 1.21]) that X_∞ splits a factor $\mathbb{R}^k$, as claimed. ■

Corollary 3.8. *Let $N > 1$ and $K \in \mathbb{R}$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $r > 0$, any $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and any $x \in X$, if there exists $u : B_r(x) \to \mathbb{R}^k$ such that $u : B_s(x) \to \mathbb{R}^k$ is a δ-splitting map for any $0 < s < r$, then for any $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ there exists an $\mathrm{RCD}(0, N - k)$ p.m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that*

$$\mathsf{d}_{\mathrm{pmGH}}\big((Y, \varrho, \mu, y), (Z \times \mathbb{R}^k, (z, 0^k))\big) < \varepsilon.$$

Proof. Choose $\delta = \delta(K, N, \varepsilon/2)$ given by Proposition 3.7. If $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$ then there exists $t > 0$ such that $t^{-1}r > \delta^{-1}$, $t^2|K| \le \delta$ and

$$\mathsf{d}_{\mathrm{pmGH}}\big((X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x), (Y, \varrho, \mu, y)\big) < \varepsilon/2. \tag{3.1}$$

Thanks to Proposition 3.7 applied to $(X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x)$, there exists an $\mathrm{RCD}(0, N-k)$ p.m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ such that

$$\mathsf{d}_{\mathrm{pmGH}}\big((X, t^{-1}\mathsf{d}, \mathfrak{m}_x^t, x), (Z \times \mathbb{R}^k, (z, 0^k))\big) < \varepsilon/2. \tag{3.2}$$

The conclusion follows from (3.1) and (3.2) by the triangle inequality. ■

Proposition 3.9. *Let $N > 1$. For any $\delta > 0$ there exists $\varepsilon = \varepsilon_{N,\delta} > 0$ such that if $(X, \mathsf{d}, \mathfrak{m})$ is an* $\mathrm{RCD}(-\varepsilon, N)$ *m.m.s., $x \in X$ and*

$$\mathsf{d}_{\mathrm{pmGH}}\big((X, \mathsf{d}, \mathfrak{m}, x), (\mathbb{R}^k \times Z, (0^k, z))\big) < \varepsilon$$

for some pointed $\mathrm{RCD}(0, N-k)$ *metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$, then there exists a δ-splitting map $u : B_5(x) \to \mathbb{R}^k$.*

Proof. We are going to build upon the local convergence and stability results that we recalled in Section 1.2.3, arguing by contradiction.

Suppose the conclusion is false. Then we can find a sequence of pointed $\mathrm{RCD}(-1/n, N)$ m.m.s. $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ such that, for some $\mathrm{RCD}(0, N-k)$ p.m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$,

$$\mathsf{d}_{\mathrm{pmGH}}\big((X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n), (\mathbb{R}^k \times Z, (0^k, z))\big) < 1/n$$

for any $n \ge 1$. Furthermore, there should be $\delta_0 > 0$ such that there is no δ_0-splitting map over $B_5(x_n)$ for any $n \ge 1$.

Let $v : Z \times \mathbb{R}^k \to \mathbb{R}^k$ be defined by $v(p, x) = x$ and denote by $v^1, \dots, v^k$ its components (they are the coordinate functions of the split factor). Observe that $\Delta v^i = 0$ for any $i = 1, \dots, k$ and $\nabla v^i \cdot \nabla v^j = \delta_{ij}$ for any $i, j = 1, \dots, k$. In particular, v^i is harmonic on $B_{10}((z, 0^k))$. Hence we can apply Proposition 1.33 to get harmonic functions $v_n^i : B_9(x_n) \to \mathbb{R}$ that converge strongly in $H^{1,2}$ to v^i on $B_9((z, 0^k))$.

Observe that, thanks to [36, Theorem 1.1], we can assume that v_n^i is C_N-Lipschitz for any $n \in \mathbb{N}$ and any $i = 1, \dots, k$. We wish to prove that $v_n = (v_n^1, \dots, v_n^k)$ is a δ_0-splitting map on $B_5(x_n)$ for n sufficiently large.

To this end, recall that Theorem 1.32 yields strong L^1-convergence of $\nabla v_n^i \cdot \nabla v_n^j$ to δ_{ij} on $B_9((z, 0^k))$ and on $B_5((z, 0^k))$ for any $i, j = 1, \dots, k$ (as a consequence of the L^1-convergence of $\nabla v_n^i \cdot \nabla v_n^i$ and of $\nabla(v_n^i + v_n^j) \cdot \nabla(v_n^i + v_n^j)$). In particular, due to the uniform boundedness of the gradients we obtained above, we get

$$\lim_{n\to\infty} \fint_{B_R(x_n)} |\nabla v_n^i \cdot \nabla v_n^j - \delta_{ij}| \,\mathrm{d}\mathfrak{m}_n = 0$$

for any $i, j = 1, \dots, k$ and $R = 5, 9$. The choice $R = 5$ gives that the third defining condition of δ-splitting map is satisfied for n sufficiently large and it remains to verify the second one. We wish to prove that

$$\lim_{n\to\infty} \int_{B_5(x_n)} |\mathrm{Hess}\, v_n^i|^2 \,\mathrm{d}\mathfrak{m}_n = 0$$

for any $i = 1, \dots, k$. To this end we choose cut-off functions η_n for the pairs $B_5(x_n) \subset B_9(x_n)$ as in Lemma 1.14 and, taking into account (1.22), we get

$$\int_{B_9(x_n)} \Delta\eta_n (|\nabla v_n^i|^2 - 1) \,\mathrm{d}\mathfrak{m}_n + C_N \frac{\mathfrak{m}_n(B_9(x_n))}{n} \ge \int_{B_5(x_n)} |\mathrm{Hess}\, v_n^i|^2 \,\mathrm{d}\mathfrak{m}_n \tag{3.3}$$

for any $i = 1, \dots, k$ and any $n \ge 1$. Since $|\Delta\eta_n| \le C_N$ by construction, and as we already observed, $|\nabla v_n^i|^2 - 1$ converges to 0 in $L^1(B_9)$ and is uniformly bounded, we find that the left-hand side in (3.3) converges to 0 as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int_{B_5(x_n)} |\mathrm{Hess}\, v_n^i|^2 \,\mathrm{d}\mathfrak{m}_n = 0,$$

as claimed. ■

Arguing by scaling starting from Proposition 3.9, one can obtain the following statement.

Corollary 3.10. *If* $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *m.m.s.,* $r^2|K| \le \varepsilon$ *and*

$$\mathsf{d}_{\mathrm{pmGH}}\big((X, r^{-1}\mathsf{d}, \mathfrak{m}_x^r, x), (\mathbb{R}^k \times Z, (0^k, z))\big) < \varepsilon$$

for some pointed $\mathrm{RCD}(0, N-k)$ *metric measure space* $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$*, then there exists a* δ*-splitting map* $u : B_{5r}(x) \to \mathbb{R}^k$.

3.1.2. Propagation of δ*-splitting.* In the next result we are concerned with the propagation of the property of being a δ-splitting map. We are going to prove that if $\alpha \in (0, 2)$, outside a set of small codimension-α content any δ-splitting map at a given scale is a $C_{N,\alpha}\delta^{1/4}$-splitting map at any scale. The proof is based on a weighted maximal function argument; see [37] for a similar argument.

Proposition 3.11. *Let* $\alpha \in (0, 2)$ *and* $N > 1$. *There exist constants* $C_N, C_{N,\alpha} > 0$ *such that, for any* $0 < \delta < 1$*, any* $\mathrm{RCD}(-1, N)$ *m.m.s.* $(X, \mathsf{d}, \mathfrak{m})$*, any* $p \in X$ *and any* δ*-splitting map* $u := (u_1, \dots, u_k) : B_2(p) \to \mathbb{R}^k$*, there exists a Borel set* $G \subset B_1(p)$ *with* $\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) < C_N\sqrt{\delta}\,\mathfrak{m}(B_2(p))$ *such that for any* $x \in G$,

$$\sup_{0<r<1} r^\alpha \fint_{B_r(x)} |\mathrm{Hess}\, u_a|^2 \,\mathrm{d}\mathfrak{m} \le \sqrt{\delta} \quad \textit{for any} \quad a = 1, \dots, k, \tag{3.4}$$

and

$$u : B_r(x) \to \mathbb{R}^k \textit{ is } C_{N,\alpha}\delta^{1/4}\textit{-splitting for any } 0 < r < 1/2. \tag{3.5}$$

Proof. To prove (3.4), fix any $a = 1, \dots, k$ and denote by C_P and C_D the Poincaré and the doubling constants over balls of radius 10 of $(X, \mathsf{d}, \mathfrak{m})$. More precisely, C_P is a constant in the $(1, 2)$-Poincaré inequality with $\lambda = 2$ as in (1.5). This inequality is available on $\mathrm{RCD}(K, N)$ m.m.s. (see for instance [54, Theorem 30.26]) with constant depending only on K and N. In particular, since $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(-1, N)$, C_P depends only on N. The same conclusion holds for C_D thanks to the Bishop–Gromov inequality (1.15).

Set

$$G := \left\{ x \in B_1(p) \;\middle|\; \sup_{0<r<1} r^\alpha \fint_{B_r(x)} |\mathrm{Hess}\, u_a|^2 \,\mathrm{d}\mathfrak{m} \le \sqrt{\delta} \right\}.$$

We claim that $\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) < C_N \sqrt{\delta}\, \mathfrak{m}(B_2(p))$. For any $x \in B_1(p) \setminus G$ we choose $\rho_x \in (0, 1)$ satisfying

$$\rho_x^\alpha \fint_{B_{\rho_x}(x)} |\mathrm{Hess}\, u_a|^2 \,\mathrm{d}\mathfrak{m} > \sqrt{\delta}. \tag{3.6}$$

Then the family $\{B_{\rho_x}(x)\}_{x \in B_1(p) \setminus G}$ covers $B_1(p) \setminus G$. Using Vitali's covering lemma we can find a subfamily $\{B_{\rho_i}(x_i)\}_{i \in \mathbb{N}}$ of disjoint balls such that $B_1(p) \setminus G \subset \bigcup_{i \in \mathbb{N}} B_{5\rho_i}(x_i)$. This gives the sought conclusion:

$$\begin{aligned}
\mathscr{H}_5^{h_\alpha}(B_1(p) \setminus G) &\le \sum_{i \in \mathbb{N}} h_\alpha(B_{5\rho_i}(x_i)) = \sum_{i \in \mathbb{N}} \frac{\mathfrak{m}(B_{5\rho_i}(x_i))}{(5\rho_i)^\alpha} \\
&\le C_N \sum_{i \in \mathbb{N}} \frac{\mathfrak{m}(B_{\rho_i}(x_i))}{\rho_i^\alpha} \le C_N \sum_{i \in \mathbb{N}} \frac{1}{\sqrt{\delta}} \int_{B_{\rho_i}(x_i)} |\mathrm{Hess}\, u_a|^2 \,\mathrm{d}\mathfrak{m} \\
&\le C_N \frac{1}{\sqrt{\delta}} \int_{B_2(p)} |\mathrm{Hess}\, u_a|^2 \,\mathrm{d}\mathfrak{m} \le C_N \sqrt{\delta}\, \mathfrak{m}(B_2(p)),
\end{aligned}$$

where we have used the definition of $\mathscr{H}_5^{h_\alpha}$, the Bishop–Gromov inequality, (3.6) and the fact that u is a δ-splitting map.

In order to verify (3.5) we just need to check that, for $a, b = 1, \dots, k$,

$$\fint_{B_r(x)} |\nabla u_a \cdot \nabla u_b - \delta_{a,b}| \,\mathrm{d}\mathfrak{m} < C_{N,\alpha} \delta^{1/4} \quad \text{for any } x \in G,\ 0 < r < 1.$$

To see this set $f_{a,b} := |\nabla u_a \cdot \nabla u_b - \delta_{a,b}|$ and note that $|\nabla f_{a,b}| \le C_N(|\mathrm{Hess}\, u_a| + |\mathrm{Hess}\, u_b|)$ as a consequence of Definition 3.4 (i) and (1.23). Hence, the Poincaré inequality and (3.4) yield

$$\begin{aligned}
&\left| \fint_{B_r(x)} f_{a,b} \,\mathrm{d}\mathfrak{m} - \fint_{B_{r/2}(x)} f_{a,b} \,\mathrm{d}\mathfrak{m} \right| \le C_P\, r \left(\fint_{B_{2r}(x)} |\nabla f_{a,b}|^2 \,\mathrm{d}\mathfrak{m} \right)^{1/2} \\
&\le C_N C_P \left(r^2 \fint_{B_{2r}(x)} |\mathrm{Hess}\, u_a|^2 \,\mathrm{d}\mathfrak{m} + r^2 \fint_{B_{2r}(x)} |\mathrm{Hess}\, u_b|^2 \,\mathrm{d}\mathfrak{m} \right)^{1/2} \le C_N C_P \delta^{1/4} r^{1-\alpha/2}
\end{aligned}$$

for any $0 < r < 1/2$. Applying a telescopic argument it is easy to see that

$$\left| \fint_{B_{2^{-1}}(x)} f_{a,b} \,\mathrm{d}\mathfrak{m} - \fint_{B_{2^{-k}}(x)} f_{a,b} \,\mathrm{d}\mathfrak{m} \right| \le C_\alpha C_N C_P \delta^{1/4} \quad \text{for any } k > 1.$$

Therefore, for any $0 < r < 1/2$ we take $k \in \mathbb{N}$ such that $2^{-k-1} < r \le 2^{-k}$ and using the fact that $u : B_2(p) \to \mathbb{R}^k$ is a δ-splitting map we get

$$\begin{aligned}\fint_{B_r(x)} f_{a,b}\,\mathrm{d}\mathfrak{m} &\le C_D 2^N \fint_{B_{2^{-k}}(x)} f_{a,b}\,\mathrm{d}\mathfrak{m}\\ &\le C_D 2^N \left| \fint_{B_{1/2}(x)} f_{a,b}\,\mathrm{d}\mathfrak{m} - \fint_{B_r(x)} f_{a,b}\,\mathrm{d}\mathfrak{m} \right| + C_D 2^N \fint_{B_{1/2}(x)} f_{a,b}\,\mathrm{d}\mathfrak{m}\\ &\le 2^N C_D C_\alpha C_N C_P \delta^{1/4} + 8^N C_D^2 \fint_{B_2(p)} f_{a,b}\,\mathrm{d}\mathfrak{m} \le C_{N,\alpha}\delta^{1/4}. \qquad \blacksquare\end{aligned}$$

For the purposes of this paper we just need to consider the case $\alpha = 1$ in Proposition 3.11. This is related to the fact that boundaries of sets with finite perimeter are codimension-1 objects. For simplicity we will write h for h_1 below.

We are going to use the following scale invariant version of Proposition 3.11 several times.

Corollary 3.12. *Let $(X, \mathsf{d}, \mathfrak{m}, p)$ be an $\mathrm{RCD}(K, N)$ p.m.m.s. and $u : B_{4r}(p) \to \mathbb{R}^k$ a δ-splitting map for some $r > 0$ such that $K^- r^2 \le 4$, where $K^- := -\min\{K, 0\}$ and $r < 1/2$. Then there exists $G \subset B_{2r}(p)$ with*

$$\mathscr{H}^h_5(B_{2r}(p) \setminus G) \le \mathscr{H}^h_{10r}(B_{2r}(p) \setminus G) \le C_N \sqrt{\delta}\, \frac{\mathfrak{m}(B_{2r}(p))}{2r}$$

such that $u : B_s(x) \to \mathbb{R}^k$ is a $C_N \delta^{1/4}$-splitting map for any $x \in G$ and any $0 < s < r$.

Proof. Apply Proposition 3.11 to the rescaled space $(X, (2r)^{-1}\mathsf{d}, \mathfrak{m}(B_{2r}(p))^{-1}\mathfrak{m}, p)$. $\blacksquare$

3.2. Uniqueness of tangents and consequences

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space with essential dimension $n \le N$ (see Theorem 1.18) and let $E \subset X$ be a set of locally finite perimeter. For any $k = 1, \dots, n$ we set

$$\begin{aligned}A_k := \{x \in X \mid\ &(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathscr{L}^k, 0^k, \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E), \text{ but}\\ &(Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\mathrm{eucl}}, \mu \times \mathscr{L}^k, (y, 0^k), \{x_k > 0\}) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)\\ &\text{for any } (Y, \varrho, \mu, y) \text{ with } \mathrm{diam}(Y) > 0\}.\end{aligned}$$

Let us point out that with arguments analogous to those in [47, Lemma 6.1] one can show that A_k is a $|D\chi_E|$-measurable set for any $k = 1, \dots, n$.

Aiming at proving that the family $\{A_k\}_{k=1}^n$ covers X up to a $|D\chi_E|$-negligible set we need to use the following result that has been proven in [2, appendix].

Theorem 3.13. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and let $E \subset X$ be a set of locally finite perimeter. Then for $|D\chi_E|$-a.e. $x \in X$ and all $(Y, \varrho, \mu, y, F) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E)$ one has*

$$\mathrm{Tan}_{y'}(Y, \varrho, \mu, F) \subset \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \quad \textit{for every } y' \in \mathrm{supp}\, |D\chi_F|.$$

Lemma 3.14. *Under the assumptions above,*

$$|D\chi_E|\Big(X \setminus \bigcup_{k=1}^{n} A_k\Big) = 0.$$

Proof. As a consequence of Theorem 3.3, together with the lower semicontinuity of the essential dimension with respect to pmGH convergence, we have

$$|D\chi_E|\Big(X \setminus \bigcup_{k=1}^{n} A'_k\Big) = 0,$$

where

$$A'_k := \Big\{x \in X \mid (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E),\ \text{but} \\ (\mathbb{R}^m, \mathsf{d}_{\mathrm{eucl}}, c_m\mathscr{L}^m, 0^m, \{x_m > 0\}) \notin \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) \text{ for any } m > k\Big\}.$$

The measurability of the A'_k's can be verified as in the case of the A_k's.

It is clear that $A_k \subset A'_k$; let us prove $|D\chi_E|(A'_k \setminus A_k) = 0$. We argue by contradiction. If the claim is false we can find $x \in A'_k \setminus A_k$ such that the iterated tangent property of Theorem 3.13 holds true. Since $x \in A'_k \setminus A_k$ we can find $(Y, \varrho, \mu, y) \in \mathrm{RCD}(0, N-k)$ with $\mathrm{diam}(Y) > 0$ such that

$$(Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\mathrm{eucl}}, \mu \times \mathscr{L}^k, (y, 0^k), \{x_k > 0\}) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E).$$

Moreover, $\mathrm{Tan}_{(y',x,0)}(Y \times \mathbb{R}^k, \varrho \times \mathsf{d}_{\mathrm{eucl}}, \mu \times \mathscr{L}^k, \{x_k > 0\}) \subset \mathrm{Tan}(E, x)$ for any (y', x) in $Y \times \mathbb{R}^{k-1}$, thanks to Theorem 3.13. Thus, choosing $(y', x, 0) \in Y \times \mathbb{R}^k$ such that Theorem 3.3 holds and y' is regular in Y we get the sought contradiction, since the essential dimension of Y is at least 1 (otherwise $\mathrm{diam}(Y) = 0$). ■

We are now in a position to conclude the proof of Theorem 3.2.

Proof of Theorem 3.2. In light of Lemma 3.14 it is enough to prove that A_k coincides up to a $|D\chi_E|$-negligible set with

$$\Big\{x \in X \;\Big|\; \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathscr{L}^k, 0^k, \{x_k > 0\})\}\Big\}.$$

Assume without loss of generality that $A_k \subset B_2(p)$ for some $p \in X$. We claim that, for any $\eta > 0$, there exists $G^\eta \subset A_k$ with

$$\mathscr{H}^h_5(A_k \setminus G^\eta) \le C_N \eta \,\mathrm{Per}(E, B_2(p)) \tag{3.7}$$

such that, for any $x \in G^\eta$ and any $(Y, \varrho, \mu, y) \in \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m})$, there exists a pointed $\mathrm{RCD}(0, N-k)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z, z)$ satisfying

$$\mathsf{d}_{\mathrm{pmGH}}\big((Y, \varrho, \mu, y), (\mathbb{R}^k \times Z, (0, z))\big) \le \eta. \tag{3.8}$$

Observe that the claim implies our conclusion. Indeed, if we fix $\eta > 0$ and set $\eta_i := \eta 2^{-i}$ then $G_\eta := \bigcup_{i \in \mathbb{N}} G^{\eta_i}$ satisfies $\mathscr{H}^h_5(A_k \setminus G_\eta) = 0$ and thus $\mathrm{Per}(E, A_k \setminus G_\eta) = 0$ thanks to Lemma 1.10. Moreover, for any $x \in G_\eta$, (3.8) holds. We conclude by observing that $G := \bigcap_{k\in\mathbb{N}} G_{2^{-k}}$ still satisfies $\mathrm{Per}(E, A_k \setminus G) = 0$ and any tangent cone at $x \in G$ splits off a factor $\mathbb{R}^k$. By definition of A_k we deduce that the only tangent at $x \in G$ is the Euclidean space of dimension k.

Let us pass to the verification of the claim. Fix $\delta \in (0, 1/2)$ and take $\varepsilon > 0$ as in Proposition 3.9. Of course we can assume $\varepsilon \le \delta$. We wish to prove that there exists a disjoint family $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ of balls such that $r_i^2|K| \le \varepsilon$ for any $i \in \mathbb{N}$ and

(i) $A_k \cap B_1(p) \subset \bigcup_{i\in\mathbb{N}} B_{5r_i}(x_i)$;

(ii) $\mathsf{d}_{\mathrm{pmGH}}\big((X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x_i), (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathscr{L}^k, 0^k)\big) \le \varepsilon$;

(iii) $\dfrac{\omega_{k-1}}{\omega_k}(1-\varepsilon)\dfrac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \le \mathrm{Per}(E, B_{r_i}(x_i)) \le \dfrac{\omega_{k-1}}{\omega_k}(1+\varepsilon)\dfrac{\mathfrak{m}(B_{r_i}(x_i))}{r_i}$.

Indeed, for any $x \in A_k$ there exists a sequence of radii $r_i \to 0$ such that

$$\lim_{i\to\infty} \mathsf{d}_{\mathrm{pmGH}}\big((X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x), (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \mathscr{L}^k, 0^k)\big) = 0,$$
$$\lim_{i\to\infty} \frac{r_i\,\mathrm{Per}(E, B_{r_i}(x))}{\mathfrak{m}(B_{r_i}(x))} = \frac{\omega_{k-1}}{\omega_k},$$

as a consequence of Theorem 3.3; see also (1.16). Therefore, for any $x \in A_k$ we can choose $r_x^2|K| \le \varepsilon$ such that the pair (x, r_x) satisfies (ii) and (iii). In order to get a disjoint family of balls satisfying (i) we have just to apply Vitali's Lemma to $\{B_{r_x}(x)\}_{x\in A_k\cap B_1(p)}$.

Let us now focus on a single ball $B_{20r_i}(x_i) \subset X$. Corollary 3.10 yields the existence of a δ-splitting map

$$u^i : B_{5r_i}(x_i) \to \mathbb{R}^k.$$

Thanks to Corollary 3.12 we can find $G_i \subset B_{5r_i}(x_i)$ with

$$\mathscr{H}_5^h(B_{5r_i}(x_i) \setminus G_i) \le C_N\sqrt{\delta}\,\frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \tag{3.9}$$

and such that $u^i : B_s(x) \to \mathbb{R}^k$ is $C_N\delta^{1/4}$-splitting for any $x \in G_i$ and any $0 < s < 5r_i$. Applying Corollary 3.8, up to assuming δ small enough, we deduce that at any $x \in G_i$, (3.8) holds true.

To conclude let us verify that $G := \bigcup_{i\in\mathbb{N}} G_i$ satisfies (3.7). Using (iii), (3.9) and the Bishop–Gromov inequality (1.15) we get

$$\begin{aligned}
\mathscr{H}_5^h(A_k \setminus G) &\le \sum_{i\in\mathbb{N}} \mathscr{H}_5^{h_1}(B_{5r_i}(x_i) \setminus G_i) \le \sum_{i\in\mathbb{N}} C_N\sqrt{\delta}\,\frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \\
&\le C_N\sqrt{\delta}\sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \le C_N\sqrt{\delta}\sum_{i\in\mathbb{N}} \mathrm{Per}(E, B_{r_i}(x_i)) \\
&\le C_N\sqrt{\delta}\,\mathrm{Per}(E, B_2(p)).
\end{aligned}$$

Since we can assume $\delta < \eta^2$ we get the sought estimate. ∎

Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ metric measure space and $E \subset X$ a set of locally finite perimeter. For any $k = 1, \dots, n$, where n is the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$, we set

$$\mathcal{F}_k E := \left\{ x \in X \;\middle|\; \mathrm{Tan}_x(X, \mathsf{d}, \mathfrak{m}, E) = \{(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathcal{L}^k, 0^k, \{x_k > 0\})\} \right\}.$$

We know thanks to Theorem 3.2 that $\mathrm{Per}(E, \cdot)$ is concentrated on $\mathcal{F} E := \bigcup_{k=1}^n \mathcal{F}_k E$ and, from now on, we shall call $\mathcal{F} E$ the *reduced boundary* of E.

The result about uniqueness of tangents that we have just proved allows us to obtain a representation formula for the perimeter measure in terms of the codimension-1 Hausdorff measure.

Corollary 3.15. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. with essential dimension n. Let $E \subset X$ be a set of locally finite perimeter. Then*

$$|D\chi_E| = \sum_{k=1}^n \frac{\omega_{k-1}}{\omega_k} \mathcal{H}^h \llcorner \mathcal{F}_k E. \tag{3.10}$$

Proof. The proof can be obtained as in the case of the representation formula for the perimeter on non-collapsed spaces obtained in [2, Corollary 4.7], relying on [45, Theorem 3] in place of [45, Theorem 5]. We just report here the key computation.

If $x \in \mathcal{F}_k E$, then we can compute

$$\lim_{r\to 0} \frac{r|D\chi_E|(B_r(x))}{\mathfrak{m}(B_r(x))} = \lim_{r\to 0} \frac{r|D\chi_E|(B_r(x))}{C(x,r)} \cdot \frac{C(x,r)}{\mathfrak{m}(B_r(x))} = \lim_{r\to 0} \frac{|D^r\chi_E|(B_1(x))}{\mathfrak{m}_x^r(B_1(x))}$$
$$= \frac{\mathcal{H}^{k-1}(B_1(0))}{\mathcal{H}^k(B_1(0))} = \frac{\omega_{k-1}}{\omega_k},$$

where the regularity of the point and the weak convergence of the rescaled perimeter measures to the perimeter measure of a half-space play a role.

This computation, together with the rigid structure of the tangent, allows us then to infer, arguing as in the non-collapsed case, that

$$\lim_{r\to 0} \sup_{x\in B_s(y),\, s\le r} \frac{s|D\chi_E|(B_s(y))}{\mathfrak{m}(B_s(y))} = \frac{\omega_{k-1}}{\omega_k},$$

which is the density estimate needed to obtain the representation formula (3.10). ■

4. Rectifiability of the reduced boundary

The main achievement of this section is a rectifiability result for the reduced boundary of sets with finite perimeter. With this theorem we complete the picture of the generalization of De Giorgi's theorem to the framework of $\mathrm{RCD}(K, N)$ spaces.

Theorem 4.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ be a set of locally finite perimeter. Then, for any $k = 1, \dots, n$, $\mathcal{F}_k E$ is $(|D\chi_E|, (k-1))$-rectifiable.*

Recall that a set is $(|D\chi_E|, \ell)$-rectifiable if up to a $|D\chi_E|$-negligible set it can be covered by $\bigcup_{i\in\mathbb{N}} A_i$ where any A_i is bi-Lipschitz equivalent to a Borel subset of $\mathbb{R}^\ell$.

When specialized to the non-collapsed case (see [22]), where the only non-empty regular set is the top-dimensional one, Theorem 4.1 turns into

Corollary 4.2. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a non-collapsed* $\mathrm{RCD}(K, N)$ *m.m.s. and* $E \subset X$ *a set of locally finite perimeter. Then* $\mathcal{F}E = \mathcal{F}_N E$ *is* $(|D\chi_E|, N-1)$*-rectifiable (equivalently,* $(\mathcal{H}^{N-1}, N-1)$*-rectifiable, where* $\mathcal{H}^{N-1}$ *denotes the* $(N-1)$*-dimensional Hausdorff measure). Furthermore,*[2]

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \mathcal{F}E. \tag{4.1}$$

Remark 4.3. We point out that, given any $\varepsilon > 0$, the maps providing rectifiability of the reduced boundary in Theorem 4.1 and Corollary 4.2 can be taken $(1+\varepsilon)$-bi-Lipschitz (compare with the analogous statement in [47]).

In particular, if $(X, \mathsf{d}, \mathfrak{m})$ is non-collapsed, then $(X, \mathsf{d}, |D\chi_E|)$ is a strongly $|D\chi_E|$-rectifiable m.m.s. according to [34].

Remark 4.4. It is worth mentioning that Theorem 4.1 is stronger than [47, Theorem 1.1]. Indeed, given an $\mathrm{RCD}(K, N)$ m.m.s. $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ we can consider $X := Z \times \mathbb{R}$ endowed with the product structure, and the set $E := \{(z, t) \in Z \times \mathbb{R} \mid t > 0\}$ of finite perimeter. Applying Theorem 4.1 to $E \subset X$ we get the rectifiability result for Z.

Let us outline the strategy of the proof of Theorem 4.1. First of all, up to intersecting with a ball and thanks to the locality of perimeter and tangents, we can assume that E has finite measure and perimeter. The bi-Lipschitz maps from subsets of $\mathcal{F}_k E$ to $\mathbb{R}^{k-1}$ providing rectifiability are going to be *suitable approximations* of the $k-1$ coordinate maps over the hyperplane where the perimeter concentrates after the blow-up; or, in better terms, they will be the first $k-1$ components of a (k, δ)-splitting map "δ-orthogonal to the exterior normal ν_E to the boundary of E".

Proving existence of such maps requires some technical work which builds upon the Gauss–Green formula of Theorem 2.4. The rigorous statement is as follows.

Proposition 4.5. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *m.m.s. and* $E \subset X$ *a set of finite perimeter and measure. For any* $\delta > 0$, $r_0 > 0$ *and* $|D\chi_E|$*-a.e.* $x \in \mathcal{F}_k E$ *there exist* $r = r_{x,\delta} < r_0$ *and a* δ*-splitting map* $u = (u_1, \dots, u_{k-1}) : B_r(x) \to \mathbb{R}^{k-1}$ *such that*

$$\frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu \cdot \nabla u_\alpha| \, \mathrm{d}|D\chi_E| < \delta \quad \textit{for } \alpha = 1, \dots, k-1.$$

The second step in the proof of Theorem 4.1 is to show that the map built in Proposition 4.5 is indeed bi-Lipschitz onto its image if restricted to suitable subsets of $\mathcal{F}_k E$ (see Proposition 4.7 below for the rigorous statement). These subsets are obtained by collecting points $x \in \mathcal{F}_k E$ such that $B_s(x) \cap E$ is ε-close, in a suitable sense, to $B_s(0^k) \cap \{x_k > 0\}$ for any $s \le r_0$, where $r_0 > 0$ is a fixed radius.

[2]In [2] it was proved that $|D\chi_E| = \mathcal{S}^{N-1} \llcorner \mathcal{F}E$, where $\mathcal{S}$ denotes the spherical Hausdorff measure. Coincidence with the Hausdorff measure $\mathcal{H}$ is a consequence of rectifiability.

Definition 4.6. Given $\varepsilon > 0$ and $r_0 > 0$, we define $(\mathcal{F}_k E)_{r_0,\varepsilon}$ as the set of points $x \in \mathcal{F}_k E$ satisfying

(i) $\mathsf{d}_{\text{pmGH}}((X, s^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_s(x))}, x), (\mathbb{R}^k, \mathsf{d}_{\text{eucl}}, \frac{1}{\omega_k}\mathscr{L}^k, 0^k)) < \varepsilon$ for any $s \leq r_0$;

(ii)

$$\left|\frac{\mathfrak{m}(B_s(x) \cap E)}{\mathfrak{m}(B_s(x))} - \frac{1}{2}\right| + \left|\frac{s|D\chi_E|(B_s(x))}{\mathfrak{m}(B_s(x))} - \frac{\omega_{k-1}}{\omega_k}\right| < \varepsilon \quad \text{for any } s \leq r_0. \tag{4.2}$$

Observe that, as a consequence of Theorem 3.2 and Remark 1.17, for any $\varepsilon > 0$ we have

$$\mathcal{F}_k E = \bigcup_{0<r<1} (\mathcal{F}_k E)_{r,\varepsilon} \quad \text{and} \quad (\mathcal{F}_k E)_{r,\varepsilon} \subset (\mathcal{F}_k E)_{r',\varepsilon} \quad \text{for } r' < r.$$

Hence for any $\eta > 0$ there exists $r = r(\eta) > 0$ such that

$$|D\chi_E|(\mathcal{F}_k E \setminus (\mathcal{F}_k E)_{s,\varepsilon}) < \eta \quad \text{for any } \ 0 < s < r. \tag{4.3}$$

Proposition 4.7. *Let $N > 1$, $K \in \mathbb{R}$ and $k \in [1, N]$. For any $\eta > 0$ there exists $\varepsilon = \varepsilon(\eta, N) < \eta$ such that if $(X, \mathsf{d}, \mathfrak{m})$ is an* RCD(K, N) *m.m.s., $E \subset X$ is a set of finite perimeter and finite measure, $p \in (\mathcal{F}_k E)_{2s,\varepsilon}$ for some $s \in (0, |K|^{-1/2})$ and there exists an ε-splitting map $u : B_{2s}(p) \to \mathbb{R}^{k-1}$ such that*

$$\frac{s}{\mathfrak{m}(B_{2s}(x))} \int_{B_{2s}(p)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E| < \varepsilon \quad \textit{for any } a = 1, \ldots, k-1, \tag{4.4}$$

then there exists $G \subset B_s(p)$ such that

(i) *$G \cap (\mathcal{F}_k E)_{2s,\varepsilon}$ is bi-Lipschitz to a Borel subset of $\mathbb{R}^{k-1}$, more precisely,*

$$\big||u(x) - u(y)| - \mathsf{d}(x, y)\big| \leq C_N \eta \mathsf{d}(x, y), \quad \forall x, y \in (\mathcal{F}_k E)_{2s,\varepsilon} \cap G; \tag{4.5}$$

(ii) *$\mathscr{H}^h_5(B_s(p) \setminus G) < C_N \eta \mathfrak{m}(B_s(p))/s$.*

Let us now prove Theorem 4.1 assuming Propositions 4.5 and 4.7.

Proof of Theorem 4.1. Assume without loss of generality that E has finite perimeter and measure, and that $\mathcal{F}_k E \subset B_2(p)$ for some $p \in X$. We claim that, for any $\eta > 0$, we can decompose $\mathcal{F}_k E = G^\eta \cup B^\eta \cup R^\eta$, where G^η is $(k-1)$-rectifiable and

$$\mathscr{H}^h_5(B^\eta) + |D\chi_E|(R^\eta) \leq C_{N,K}|D\chi_E|(B_2(p))\eta + \eta. \tag{4.6}$$

Observe that the claim easily gives the sought conclusion. Indeed, setting $\eta_i := \eta 2^{-i}$, $G_\eta := \bigcup_i G^{\eta_i}$ and $R_\eta := \bigcup_{i\in\mathbb{N}} R^{\eta_i}$, G_η is still $(k-1)$-rectifiable and

$$\mathscr{H}^h_5((\mathcal{F}_k E \setminus G_\eta) \setminus R_\eta) = 0;$$

hence, as a consequence of Lemma 1.10, $|D\chi_E|(\mathcal{F}_k E \setminus G_\eta) \setminus R_\eta) = 0$. Therefore

$$|D\chi_E|(\mathcal{F}_k E \setminus G_\eta) \leq |D\chi_E|(R_\eta) \leq C_N|D\chi_E|(B_2(p))\eta + \eta.$$

Setting $G := \bigcup_{i\in\mathbb{N}} G_{2^{-i}}$, we find that G is still $(k-1)$-rectifiable and coincides with $\mathcal{F}_k E$ up to a $|D\chi_E|$-negligible set.

Let us now prove the claim. To this end, fix $r > 0$ and $\varepsilon > 0$. We cover $(\mathcal{F}_k E)_{r,\varepsilon}$ with balls of radius smaller than $r/5$ with centre in $(\mathcal{F}_k E)_{r,\varepsilon}$ such that the assumptions of Proposition 4.7 are satisfied. The possibility of building such a covering is a consequence of Theorem 3.2 and of Proposition 4.5. By Vitali's lemma, we can extract a disjoint family $\{B_{r_i/5}(x_i)\}_{i\in\mathbb{N}}$ such that $(\mathcal{F}_k E)_{r,\varepsilon} \subset \bigcup_i B_{r_i}(x_i)$. Applying Proposition 4.7 above, for any $i \in \mathbb{N}$ we can find $G_i \subset B_{r_i}(x_i)$ such that $G_i \cap (\mathcal{F}_k E)_{r,\varepsilon}$ is $(k-1)$-rectifiable and $\mathscr{H}^h_5(B_{r_i}(x_i) \setminus G_i) < C_N \eta \mathfrak{m}(B_{r_i}(x_i))/r_i$. Set $G^\eta_r := (\mathcal{F}_k E)_{r,\varepsilon} \cap \bigcup_i G_i$ and observe that

$$
\begin{aligned}
\mathscr{H}^h_5((\mathcal{F}_k E)_{r,\varepsilon} \setminus G^\eta_r) &\le \sum_{i\in\mathbb{N}} \mathscr{H}^h_5(B_{r_i}(x_i) \setminus G_i) \le \sum_{i\in\mathbb{N}} C_N \eta \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \\
&\le C_N \eta \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i/5}(x_i))}{r_i/5} \le C_{N,K} \eta \sum_{i\in\mathbb{N}} |D\chi_E|(B_{r_i/5}(x_i)) \\
&\le C_{N,K} \eta |D\chi_E|(B_2(p)),
\end{aligned}
$$

where we have used the Bishop–Gromov inequality (1.15) and

$$
\frac{\mathfrak{m}(B_{r_i/5}(x_i))}{r_i/5} \le \mathrm{C}(k)|D\chi_E|(B_{r_i/5}(x_i)),
$$

which holds true provided ε is small enough.

Setting $B^\eta_r := (\mathcal{F}_k E)_{r,\varepsilon} \setminus G^\eta_r$, the argument above gives the decomposition

$$
(\mathcal{F}_k E)_{r,\varepsilon} = G^\eta_r \cup B^\eta_r,
$$

where G^η_r is $(k-1)$-rectifiable and $\mathscr{H}^h_5(B^\eta_r) \le C_{N,K}\eta|D\chi_E|(B_2(p))$. Let us now choose $r > 0$ small enough to have (4.3). This allows us to write

$$
\mathcal{F}_k E = G^\eta_r \cup B^\eta_r \cup (\mathcal{F}_k E \setminus (\mathcal{F}_k E)_{r,\varepsilon}) =: G^\eta \cup B^\eta \cup R^\eta
$$

and to conclude the proof. ∎

4.1. *Proof of Proposition 4.5*

Let us start by recalling that one of the main results of the previous part of the paper was that the exterior normal is indeed an element of $L^2_E(TX)$ (see Theorem 2.4). In the following, to simplify the notation, we shall write v in place of $\mathrm{tr}_E(v)$ for any v in $H^{1,2}_C(TX) \cap D(\mathrm{div})$.

Definition 4.8. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and let $E \subset X$ be a set of finite perimeter. Given $x \in X$ and a sequence $r_i \downarrow 0$ we say that $\{u^{r_i} := (u^{r_i}_1, \dots, u^{r_i}_{k-1}) : B_{r_i}(x) \to \mathbb{R}^{k-1}\}_{i\in\mathbb{N}}$ is a *good approximation of the boundary* of E at x if the following conditions hold true:

(i) there exists a sequence $\delta_i \to 0$ such that $u^{r_i} : B_{r_i}(x) \to \mathbb{R}^{k-1}$ is a δ_i-splitting map with $u^{r_i}(x) = 0$;

(ii) there exists (Z, d_Z) that realizes the convergences

$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x) \to (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathscr{L}^k, 0^k)$ and $E_{r_i} \to \{x_k > 0\}$ locally strongly in BV,

$r_i^{-1} u_\alpha^{r_i} \to x_\alpha$ $H^{1,2}$-strongly on $B_1(0^k)$ along the sequence

$$(X, r_i^{-1}\mathsf{d}, \mathfrak{m}_x^{r_i}, x) \to (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathscr{L}^k, 0^k)$$

for any $\alpha = 1, \dots, k-1$.

Lemma 4.9. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ m.m.s. and $E \subset X$ a set of finite perimeter and finite measure. Then for any $p \in X$ and any $\varepsilon > 0$ there exists $V \in \mathrm{TestV}_E(X)$ such that*

$$\int_{B_2(p)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \le \varepsilon,$$

where ν is the exterior normal of E. Moreover, there exists a set $G \subset B_1(p)$ with $\mathscr{H}^h(B_1(p) \setminus G) \le C_{K,N}\sqrt{\varepsilon}$ such that, for any $x \in G$,

$$\limsup_{r \to 0} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \le \sqrt{\varepsilon}.$$

Proof. The first conclusion follows from Theorem 2.4, where we proved that the normal is an element of $L^2_E(TX)$, and Lemma 2.9, yielding density of $\mathrm{tr}_E(\mathrm{TestV}_E(X))$ in $L^2_E(TX)$.

To prove the second part of the statement we set

$$G := \left\{ x \in B_1(p) \;\middle|\; \limsup_{r \downarrow 0} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \le \sqrt{\varepsilon} \right\}.$$

Then, for any $r_0 > 0$ and any $x \in B_1(p) \setminus G$, there exists $r_x < r_0$ such that

$$\frac{r_x}{\mathfrak{m}(B_{r_x}(x))} \int_{B_{r_x}(x)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| > \sqrt{\varepsilon}.$$

Hence, applying Vitali's covering theorem we can find a disjoint set $\{B_{r_i}(x_i)\}$ of balls such that $\{B_{5r_i}(x_i)\}$ is a covering of $B_1(p) \setminus G$. Now we can estimate, for any $r_0 > 0$,

$$\begin{aligned}
\mathscr{H}^h_{5r_0}(B_1(p) \setminus G) &\le \sum_{i=0}^{\infty} \frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \le C_{K,N} \sum_{i=0}^{\infty} \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \\
&\le \frac{C_{K,N}}{\sqrt{\varepsilon}} \sum_{i=0}^{\infty} \int_{B_{r_i}(x_i)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \le \frac{C_{K,N}}{\sqrt{\varepsilon}} \int_{B_2(p)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \le C_{K,N}\sqrt{\varepsilon}.
\end{aligned}$$

The conclusion follows by letting $r_0 \downarrow 0$. ■

Proof of Proposition 4.5. The proof is divided into three steps. The aim of the first one is to prove that good approximations of the boundary are regular enough to guarantee that the scalar product between their gradient and the gradient of any given test function leaves a well-defined trace over the reduced boundary of E. In the second step we combine the outcome of the first one, the approximation result of Lemma 4.9 and *orthogonality in the weak sense* between the normal vector and the coordinates of its orthogonal hyperplane guaranteed by the Gauss–Green formula, to deduce that the gradients of good approximations of the boundary leave a trace even when coupled with the normal to the boundary, and that this trace is 0. In the last step we prove existence of good approximations of the boundary and combine it with Steps 1 and 2 to get the sought conclusion.

Step 1. Observe that it suffices to restrict attention to the ball $B_1(p) \subset X$ for any $p \in X$.

We claim that for any function $\phi \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ there exists a $|D\chi_E|$-negligible set $\mathrm{N} \subset \mathcal{F}_k E \cap B_1(p)$ such that, for any $x \in \mathcal{F}_k E \cap B_1(p) \setminus \mathrm{N}$ and any good approximation of the boundary of E at x with radii $r_i \downarrow 0$ and maps $\{u^{r_i} := (u_1^{r_i}, \dots, u_{k-1}^{r_i}) \mid B_{r_i}(x) \to \mathbb{R}^{k-1}\}_{i \in \mathbb{N}}$, there exist a subsequence r_{i_j} and $c(x) = (c_1(x), \dots, c_{k-1}(x)) \in \mathbb{R}^{k-1}$ such that

$$\lim_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot \nabla\phi - c_\alpha(x)|^2 \,\mathrm{d}|D\chi_E| = 0 \qquad \text{for any } \alpha = 1, \dots, k-1. \tag{4.7}$$

Assume without loss of generality that $|\nabla\phi| \le 1$. Let us also fix $\alpha \in \{1, \dots, k-1\}$ and set $g_i := \nabla u_\alpha^{r_i} \cdot \nabla\phi$. We have

(i) $\|g_i\|_{L^\infty(B_{r_i}(x))} \le C_N$;

(ii) $r_i^2 ⨍_{B_{r_i}(x)} |\nabla g_i|^2 \,\mathrm{d}\mathfrak{m} \le 2\delta_i + C_N r_i^2 ⨍_{B_{r_i}(x)} |\mathrm{Hess}\,\phi|^2 \,\mathrm{d}\mathfrak{m}$, where δ_i is as in Definition 4.8.

Since $|\mathrm{Hess}\,\phi| \in L^2(B_2(p), \mathfrak{m})$, by Lemmas 1.10 and 1.11 we deduce that

$$\lim_{r\to 0} r^2 ⨍_{B_r(x)} |\mathrm{Hess}\,\phi|^2 \,\mathrm{d}\mathfrak{m} = 0$$

for any $x \in X$ outside a $|D\chi_E|$-negligible set depending only on ϕ. Therefore we can assume that x does not belong to this set, obtaining

$$\lim_{i\to\infty} r_i^2 ⨍_{B_{r_i}(x)} |\nabla g_i|^2 \,\mathrm{d}\mathfrak{m} = 0. \tag{4.8}$$

This gives that, up to a subsequence, $g_i \to c_\alpha(x)$ $H^{1,2}$-strongly on $B_1(0^k)$ along the sequence in Definition 4.8 (ii). Here we have used (1.16). Taking into account Proposition 1.27, it follows that $g_i - c_\alpha(x) \to 0$ $H^{1,2}$-strongly on $B_1(0^k)$ and thus, reading the convergence in the starting space,

$$⨍_{B_{r_{i_j}}(x)} |g_{i_j} - c_\alpha(x)|^2 \,\mathrm{d}\mathfrak{m} + r_{i_j}^2 ⨍_{B_{r_{i_j}}(x)} |\nabla g_{i_j}|^2 \,\mathrm{d}\mathfrak{m} =: \varepsilon_j \to 0 \quad \text{as } j \to \infty. \tag{4.9}$$

We wish to prove that, up to excluding another $|D\chi_E|$-negligible set depending only on E, (4.9) gives (4.7). More precisely, we are going to prove that (4.9) implies (4.7) at any $x \in X$ such that $x \in E_{r_0,C}$ for some $r_0 > 0$ and $C > 1$, where

$$E_{r_0,C} := \left\{ y \in X \;\middle|\; C^{-1} \le \frac{r|D\chi_E|(B_r(y))}{\mathfrak{m}(B_r(y))} \le C \;\forall r < r_0 \right\}, \tag{4.10}$$

and

$$\lim_{r\to 0} \frac{|D\chi_E|(E_{r_0,C} \cap B_r(x))}{|D\chi_E|(B_r(x))} = 1. \tag{4.11}$$

Observe that (4.10) and (4.11) are satisfied at $|D\chi_E|$-a.e. point in $\mathcal{F}E$ thanks to Theorem 3.2, the asymptotic doubling property of $|D\chi_E|$ and elementary considerations. In order to keep notations short, from now on we set $r_j := r_{i_j}$ and $g_j := g_{i_j}$.

We claim that, for any j such that $r_j \le r_0/5$,

$$|D\chi_E|\big(E_{r_0,C} \cap B_{r_j}(x) \cap \{|g_j - c_\alpha(x)|^2 \ge \sqrt{\varepsilon_j}\}\big) \le C C_{N,K} \sqrt{\varepsilon_j}\, \frac{\mathfrak{m}(B_{r_j}(x))}{r_j}, \tag{4.12}$$

where ε_j is as in (4.9) and r_0 and C are as in (4.10).

Notice that (4.12), together with the Chebyshev inequality, (i) and (4.11), gives (4.7). To establish (4.12), fix any j such that $r_j \le r_0/5$ and set

$$(X_j, \mathsf{d}_j, \mathfrak{m}_j, x) := \left(X, r_j^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{r_j}(x))}, x \right).$$

With a slight abuse of notation we use the notations $E_{r_0,C}$ and g_j also in X_j. Observe that, when read in X_j, (4.9) turns into

$$\fint_{B_1^j(x)} |g_j - c_\alpha(x)|^2 \,\mathrm{d}\mathfrak{m}_j + \fint_{B_1^j(x)} |\nabla g_j|^2 \,\mathrm{d}\mathfrak{m}_j \le \varepsilon_j.$$

Moreover, a telescopic argument as in the proof of Proposition 3.11 gives

$$\begin{aligned} &B_1^j(x) \cap E_{r_0,C} \cap \{|g_j - c_\alpha(x)|^2 \ge C_{N,K}\sqrt{\varepsilon_j}\} \\ &\qquad \subset B_1^j(x) \cap E_{r_0,C} \cap \left\{ z \;\middle|\; \sup_{0<s<1} s \fint_{B_s^j(z)} |\nabla g_j|^2 \,\mathrm{d}\mathfrak{m}_j > \sqrt{\varepsilon_j} \right\}. \end{aligned}$$

Using Vitali's lemma we can find a disjoint family $\{B_{s_i}^j(z_i)\}_{i\in\mathbb{N}}$ with $s_i \le 1$ and $z_i \in B_1^j(x) \cap E_{r_0,C} \cap \{z \mid \sup_{0<s<1} s \fint_{B_s^j(z)} |\nabla g_j|^2 \,\mathrm{d}\mathfrak{m}_j > \sqrt{\varepsilon_j}\}$ for any $i \in \mathbb{N}$ such that

$$B_1^j(x) \cap E_{r_0,C} \cap \left\{ z \;\middle|\; \sup_{0<s<1} s \fint_{B_s^j(z)} |\nabla g_j|^2 \,\mathrm{d}\mathfrak{m}_j > \sqrt{\varepsilon_j} \right\} \subset \bigcup_{i\in\mathbb{N}} B_{5s_i}^j(z_i).$$

Taking into account (4.10) and the defining identities

$$B_{s_i}^j(z_i) = B_{r_j s_i}(z_i), \quad \mathfrak{m}_j = \frac{\mathfrak{m}}{\mathfrak{m}(B_{r_j}(x))},$$

we get

$$\begin{aligned}
&\frac{r_j}{\mathfrak{m}(B_{r_j}(x))}|D\chi_E|(E_{r_0,C} \cap B_{r_j}(x) \cap \{|g_j - c_\alpha(x)|^2 \geq \sqrt{\varepsilon_j}\}) \\
&\quad \leq \frac{r_j}{\mathfrak{m}(B_{r_j}(x))} \sum_{i\in\mathbb{N}} |D\chi_E|(B_{5r_js_i}(z_i)) \leq \frac{CC_{N,K}r_j}{\mathfrak{m}(B_{r_j}(x))} \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_js_i}(z_i))}{r_js_i} \\
&\quad = CC_{N,K} \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}_j(B^j_{s_i}(z_i))}{s_i} \leq \frac{CC_{N,K}}{\sqrt{\varepsilon_j}} \int_{B^j_1(x)} |\nabla g_j|^2 \,\mathrm{d}\mathfrak{m}_j \leq CC_{N,K}\sqrt{\varepsilon_j}.
\end{aligned}$$

Step 2. We wish to prove that, for $|D\chi_E|$-a.e. $x \in \mathcal{F}_k E$ and any good approximation of the boundary of E at x with radii $r_i \downarrow 0$ and maps $\{u^{r_i} := (u_1^{r_i}, \ldots, u_{k-1}^{r_i}) \mid B_{r_i}(x) \to \mathbb{R}^{k-1}\}_{i\in\mathbb{N}}$, there exists a subsequence $r_{i_j} \to 0$ such that

$$\lim_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu \cdot \nabla u_\alpha^{r_{i_j}}| \,\mathrm{d}|D\chi_E| = 0 \quad \text{for any } \alpha = 1, \ldots, k-1. \tag{4.13}$$

Let us restrict attention as above to $\mathcal{F}_k E \cap B_1(p)$.

We claim that for any $\varepsilon > 0$ there is $G_\varepsilon \subset B_1(p) \cap \mathcal{F}_k E$ with $\mathcal{H}^h(B_1(p) \cap \mathcal{F}_k E \setminus G_\varepsilon) \leq C_{N,K}\sqrt{\varepsilon}$ and such that, for any $x \in G_\varepsilon$ and any good approximation $\{u^{r_i} := (u_1^{r_i}, \ldots, u_{k-1}^{r_i}) \mid B_{r_i}(x) \to \mathbb{R}^{k-1}\}_{i\in\mathbb{N}}$ of the boundary of E at x, there exists a subsequence $r_{i_j} \to 0$ satisfying

$$\limsup_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu \cdot \nabla u_\alpha^{r_{i_j}}| \,\mathrm{d}|D\chi_E| \leq C_{N,K}\varepsilon^{1/4} \quad \text{for any } \alpha = 1, \ldots, k-1. \tag{4.14}$$

Before proving the claim let us see how it implies (4.13). Fix $\varepsilon > 0$, set $\varepsilon_i := \varepsilon 2^{-i}$ and take $G^\varepsilon := \bigcup_{i\in\mathbb{N}} G_{\varepsilon_i}$. Then we have $|D\chi_E|(B_1(p) \cap \mathcal{F}_k E \setminus G^\varepsilon) = 0$, thanks to Lemma 1.10, and (4.14) holds for any $x \in G^\varepsilon$. Therefore $\bigcap_{i\in\mathbb{N}} G^{\varepsilon_i}$ has full $|D\chi_E|$-measure in $B_1(p) \cap \mathcal{F}_k E$ and has the sought property.

The remaining part of this step is the proof of (4.14). Fix $\varepsilon > 0$, and take G and V as in Lemma 4.9. Recalling that any test vector field can be represented as $\sum_{i=1}^m \eta_i \nabla\phi_i$ with $\eta_i, \phi_i \in \mathrm{Test}(X, \mathsf{d}, \mathfrak{m})$ for some $m \in \mathbb{N}$ and using Step 1, we conclude that there exists $G_\varepsilon \subset G \cap \mathcal{F}_k E$ with $|D\chi_E|(G \cap \mathcal{F}_k E \setminus G_\varepsilon) = 0$ and such that, for any $x \in G_\varepsilon$ and any good approximation $\{u^{r_i} := (u_1^{r_i}, \ldots, u_{k-1}^{r_i}) : B_{r_i}(x) \to \mathbb{R}^{k-1}\}_{i\in\mathbb{N}}$ of the boundary of E at x, there exists $c(x) := (c_1(x), \ldots, c_{k-1}(x)) \in \mathbb{R}^{k-1}$ and a subsequence $r_{i_j} \to 0$ such that

$$\lim_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot V - c_\alpha(x)|^2 \,\mathrm{d}|D\chi_E| = 0 \quad \text{for } \alpha = 1, \ldots, k-1. \tag{4.15}$$

In order to conclude the proof it suffices to show that

$$|c(x)| \leq C_{K,N}\varepsilon^{1/4}. \tag{4.16}$$

Indeed, in that case,

$$\begin{aligned}
&\limsup_{j\to\infty} \frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu \cdot \nabla u_\alpha^{r_{i_j}}| \,\mathrm{d}|D\chi_E| \\
&\quad \le C_N \limsup_{j\to\infty} \left(\frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nu - V|^2 \,\mathrm{d}|D\chi_E| \right)^{1/2} \\
&\qquad + \limsup_{j\to\infty} \frac{C_N r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot V| \,\mathrm{d}|D\chi_E| \\
&\quad \le C_N \varepsilon^{1/4} + \lim_{j\to\infty} C_N \left(\frac{r_{i_j}}{\mathfrak{m}(B_{r_{i_j}}(x))} \int_{B_{r_{i_j}}(x)} |\nabla u_\alpha^{r_{i_j}} \cdot V - c_\alpha(x)|^2 \,\mathrm{d}|D\chi_E| \right)^{1/2} \\
&\qquad + |c_\alpha(x)| \frac{r_{i_j} |D\chi_E|(B_{r_{i_j}}(x))}{\mathfrak{m}(B_{r_{i_j}}(x))} \\
&\quad \le C_{K,N} \varepsilon^{1/4},
\end{aligned}$$

where we have used (4.15), (4.16) and the fact that $x \in \mathcal{F}_k E$.

To prove (4.16) we simplify the notation setting $r_{i_j} =: r_j$. Choose a smooth function $\psi_\infty : \mathbb{R}^k \to \mathbb{R}$ with compact support in $B_1(0^k)$ and such that $\int_{\{x_k=0\}} \psi_\infty \,\mathrm{d}\mathcal{L}^{k-1} =: C_k > 0$. Then we consider a sequence $\psi_j \in \mathrm{Lip}(X, \mathsf{d})$ with $\mathrm{supp}(\psi_j) \subset B_{r_j}(x)$, $\|\psi_j\|_{L^\infty} \le 2$ and $\psi_j \to \psi_\infty$ strongly in $H^{1,2}$ along the sequence in Definition 4.8 (ii), whose existence is proved in Lemma 1.31. Observe now that for $\alpha = 1, \dots, k-1$,

$$\lim_{j\to\infty} \frac{r_j}{\mathfrak{m}(B_{r_j}(x))} \int_E \nabla\psi_j \cdot \nabla u_\alpha^{r_j} \,\mathrm{d}\mathfrak{m} = c_k \int_{\{x_k>0\}} \nabla\psi_\infty \cdot e_\alpha \,\mathrm{d}\mathcal{L}^k = 0, \tag{4.17}$$

$$\lim_{j\to\infty} \frac{r_j}{\mathfrak{m}(B_{r_j}(x))} \int \psi_j V \cdot \nabla u_\alpha^{r_j} \,\mathrm{d}|D\chi_E| = C_k c_\alpha(x), \tag{4.18}$$

where the last equality in (4.17) is obtained by integrating by parts, and to prove (4.18) we use (4.15). Building upon (4.17), (4.18), Theorem 2.4 and Lemma 4.9, we get (4.15):

$$\begin{aligned}
C_k |c_\alpha(x)| &= \left| \lim_{j\to\infty} \frac{r_j}{\mathfrak{m}(B_{r_j}(x))} \left(\int_E \nabla\psi_j \cdot \nabla u_\alpha^{r_j} \,\mathrm{d}\mathfrak{m} + \int \psi_j V \cdot \nabla u_\alpha^{r_j} \,\mathrm{d}|D\chi_E| \right) \right| \\
&= \left| \lim_{j\to\infty} \frac{r_j}{\mathfrak{m}(B_{r_j}(x))} \left(-\int \psi_j \nu \cdot \nabla u_\alpha^{r_j} \,\mathrm{d}|D\chi_E| + \int \psi_j V \cdot \nabla u_\alpha^{r_j} \,\mathrm{d}|D\chi_E| \right) \right| \\
&\le \limsup_{j\to\infty} \frac{C_N r_j}{\mathfrak{m}(B_{r_j}(x))} \int_{B_{r_j}(x)} |\nu - V| \,\mathrm{d}|D\chi_E| \le C_{N,K} \varepsilon^{1/4}.
\end{aligned}$$

Note that in order to apply the Gauss–Green formula in the previous estimate, the fact that $u_\alpha^{r_j}$ is locally the restriction of an $H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ function (see Remark 3.5) plays a role.

Step 3. In order to conclude the proof we just observe that, since

$$\mathsf{d}_{\mathrm{pmGH}}\big((X, r^{-1}\mathsf{d}, \mathfrak{m}_x^r, x), (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k \mathcal{L}^k, 0^k)\big) \to 0$$

as $r \downarrow 0$ and the blow-up of the set of finite perimeter is a half-space (in the sense of $\mathrm{BV}_{\mathrm{loc}}$ convergence, as pointed out after Definition 3.1), a slight modification of Proposition 3.9[3] provides, for any sequence $r_i \downarrow 0$, a good approximation of the boundary of E at x with maps $\{u^{r_i} := (u_1^{r_i}, \ldots, u_{k-1}^{r_i}) : B_{r_i}(x) \to \mathbb{R}^{k-1}\}_{i\in\mathbb{N}}$ (observe that Proposition 3.9 gives δ_i-splitting maps defined on balls of radius 1 of the rescaled spaces for a sequence $\delta_i \downarrow 0$, and then rescale these functions). The sought conclusion follows now from what we obtained in the previous step. ■

4.2. *Proof of Proposition 4.7*

The proof is divided into three steps.

The aim of the first one is to provide a bridge between analysis and geometry suitable for this context. We prove that whenever at a certain location and scale the set of finite perimeter is quantitatively close to a half-space in a Euclidean space and there exists a $(k-1,\delta)$-splitting map which is also δ-orthogonal to the normal vector in the sense of (4.4), then the $(k-1,\delta)$-splitting map is an η-isometry (in the scale invariant sense) when restricted to the support of the perimeter.

The second step is analytic and dedicated to the propagation of the δ-orthogonality condition.

In the last one we get the bi-Lipschitz property, since a map which is an η-isometry (in the scale invariant sense) at any location and scale is bi-Lipschitz.

Step 1. Let $N > 0$, $K \in \mathbb{R}$ and $k \in [1, N]$, $k \in \mathbb{N}$. We claim that, for any $\eta > 0$, there exists $\delta = \delta_{\eta,N} \le \eta$ such that the following holds. For any pointed $\mathrm{RCD}(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m}, x)$ and for any set of finite perimeter and finite measure $E \subset X$ such that, for some $0 < r < |K|^{-1/2}$,

(i) $\mathsf{d}_{\mathrm{pmGH}}((X, (2r)^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{2r}(x))}, x), (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \frac{1}{\omega_k}\mathcal{L}^k, 0^k)) < \delta$;

(ii)

$$\left| \frac{\mathfrak{m}(B_t(x) \cap E)}{\mathfrak{m}(B_t(x))} - \frac{1}{2} \right| + \left| \frac{t|D\chi_E|(B_t(x))}{\mathfrak{m}(B_t(x))} - \frac{\omega_{k-1}}{\omega_k} \right| < \delta \quad \text{for any } t \le 2r; \tag{4.19}$$

(iii) there exists a δ-splitting map $u := (u_1, \ldots, u_{k-1}) : B_{2r}(x) \to \mathbb{R}^{k-1}$ satisfying

$$\frac{r}{\mathfrak{m}(B_{2r}(x))} \int_{B_{2r}(x)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E| < \delta \quad \text{for any } a = 1, \ldots, k-1, \tag{4.20}$$

the map $u : \operatorname{supp}|D\chi_E| \cap B_r(x) \to B_r^{\mathbb{R}^{k-1}}(u(x))$ is an ηr-GH isometry.

By scaling it is enough to prove the claim when $r = 1/2$ and $|K| \le 4$. Let us argue by contradiction. Then we could find $\eta > 0$, a sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, E_n, x_n)$, points $z_1^n, z_2^n \in \operatorname{supp}|D\chi_{E_n}| \cap B_{1/2}(x_n)$, and $1/n$-splitting maps $u^n : B_1(x_n) \to \mathbb{R}^{k-1}$ satisfying

[3] With the splitting functions defined on balls of radius 1 in place of 5.

(i)–(iii) with $\delta = 1/n$, $u^n(x_n) = 0$ and

$$\big||u^n(z_1^n) - u^n(z_2^n)| - \mathsf{d}_n(z_1^n, z_2^n)\big| \geq \eta, \quad \forall n \in \mathbb{N}. \tag{4.21}$$

Notice that $\mathsf{d}_n(z_1^n, z_2^n) \geq \min\{\eta/(C_N - 1), \eta\}$ since u^n is C_N-Lipschitz.

Observe that, by (i), $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$ converges in the pmGH topology to $(\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, \frac{1}{\omega_k}\mathscr{L}^k, 0^k)$. We can assume the existence of a metric space (Z, d_Z) realizing this convergence (see Section 1.2.3). Since E_n satisfies the bound

$$\left|\frac{\mathfrak{m}_n(E_n \cap B_t(x_n))}{\mathfrak{m}_n(B_t(x_n))} - \frac{1}{2}\right| + \left|\frac{t|D\chi_{E_n}|(B_t(x_n))}{\mathfrak{m}_n(B_t(x_n))} - \frac{\omega_{k-1}}{\omega_k}\right| < 1/n \quad \text{for any } t \leq 1, \tag{4.22}$$

up to extracting a subsequence, $E_n \cap B_1(x_n) \to F \cap B_1(0^k)$ L^1-strongly, where F is of locally finite perimeter in $B_1(0^k)$ thanks to Proposition 1.28.

Up to extracting again a subsequence we can assume $u^n \to u^\infty$ strongly in $H^{1,2}$ on $B_1(0^k)$, where $u^\infty : B_1^{\mathbb{R}^k}(0) \to \mathbb{R}^{k-1}$ is the restriction of an orthogonal projection, as a consequence of Proposition 1.21 and Theorem 1.32. Without loss of generality we assume that $u^\infty(x) = (x_1, \ldots, x_{k-1})$ for any $x \in B_1(0^k)$.

We claim that $\mathscr{L}^k((F \cap B_1(0^k)) \,\triangle\, (\{x_k > 0\} \cap B_1(0^k))) = 0$ and

$$\int g \,\mathrm{d}|D\chi_{E_n}| \to \int g \,\mathrm{d}|D\chi_{\{x_k>0\}}| \quad \text{for any } g \in \mathrm{C}(Z) \text{ with } \mathrm{supp}(g) \subset B_{1/2}(0^k). \tag{4.23}$$

This would imply that $z_1^\infty, z_2^\infty \in \{x_k = 0\}$, and therefore $|u^\infty(z_1^\infty) - u^\infty(z_2^\infty)| = \mathsf{d}_{\mathrm{eucl}}(z_1^\infty, z_2^\infty)$, which contradicts (4.21).

In order to verify the claim we argue as in the proof of the second step of Proposition 4.5. We choose a smooth function $\psi_\infty : \mathbb{R}^k \to \mathbb{R}$ with compact support in $B_1(0^k)$. Then we consider a sequence $\psi_n \in \mathrm{Lip}(X_n, \mathsf{d}_n)$ with $\mathrm{supp}(\psi_n) \subset B_1(x_n)$, $\|\psi_n\|_{L^\infty} + \||\nabla\psi_n|\|_{L^\infty} \leq 4$ and $\psi_n \to \psi_\infty$ strongly in $H^{1,2}$ along the sequence $(X_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)$, whose existence is proved in Lemma 1.31. Observe now that

$$\nabla\psi_n \cdot \nabla u_a^n \to \nabla\psi_\infty \cdot e_a = \frac{\partial\psi_\infty}{\partial x_a} \quad L^2\text{-strongly, for any } a = 1, \ldots, k-1,$$

by Proposition 1.27 (i, iii). This observation, along with Proposition 1.27 (ii) and Remark 1.25, gives

$$\int_F \frac{\partial\psi_\infty}{\partial x_a} \,\mathrm{d}\frac{\mathscr{L}^k}{\omega_k} = \lim_{n\to\infty} \int_{E_n} \nabla\psi_n \cdot \nabla u_a^n \,\mathrm{d}\mathfrak{m}_n. \tag{4.24}$$

We can now use (4.24), Theorem 2.4 and (iii) to conclude that

$$\begin{aligned}\left|\int_F \frac{\partial\psi_\infty}{\partial x_a} \,\mathrm{d}\frac{\mathscr{L}^k}{\omega_k}\right| &= \lim_{n\to\infty}\left|\int_{E_n} \nabla\psi_n \cdot \nabla u_a^n \,\mathrm{d}\mathfrak{m}_n\right| = \lim_{n\to\infty}\left|\int \psi_n \nabla u_a^n \cdot \nu_{E_n} \,\mathrm{d}|D\chi_{E_n}|\right| \\ &\leq \lim_{n\to\infty} \int |\psi_n| \, |\nabla u_a^n \cdot \nu_{E_n}| \,\mathrm{d}|D\chi_{E_n}| = 0\end{aligned}$$

for $a = 1, \dots, k-1$. Since $\psi_\infty \in C_c^\infty(B_1(0^k))$ is arbitrary we obtain

$$\mathcal{L}^k\big((F \cap B_1(0^k)) \triangle (\{x_k > \lambda\} \cap B_1(0^k))\big) = 0 \quad \text{for some } \lambda \in \mathbb{R}.$$

Using again (4.22) we get $\mathcal{L}^k(F \cap B_1(0^k)) = \omega_k/2$ which forces $\lambda = 0$.

Let us finally prove (4.23). To this end we use again (4.22) with $t = 1/2$, obtaining

$$\lim_{n\to\infty} |D\chi_{E_n}|(B_{1/2}(x_n)) = \frac{\omega_{k-1}}{2^{k-1}} = |D\chi_{\{x_k>0\}}|(B_{1/2}(0^k)).$$

We can now apply the third conclusion of Proposition 1.28 to conclude.

Step 2. By assumption there exists an ε-splitting map $u : B_{2s}(p) \to \mathbb{R}^{k-1}$ such that (4.4) holds true. We wish to propagate now both the ε-splitting condition and the orthogonality condition (4.4) at any scale and point outside a set of small $\mathscr{H}_5^h$-measure. More precisely, we are going to prove that there exists a set $G \subset B_s(p)$ with $\mathscr{H}_5^h(B_s(p) \setminus G) \le C_N \sqrt{\varepsilon}\, \mathfrak{m}(B_s(p))/s$ such that

(i) for any $x \in G$ and $0 < r < s$, $u : B_r(x) \to \mathbb{R}^{k-1}$ is $C_N \varepsilon^{1/4}$-splitting;

(ii) for any $x \in G$ and $0 < r < s$,

$$\frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E| < \sqrt{\varepsilon} \quad \text{for } a = 1, \dots, k-1. \tag{4.25}$$

We can find a set G' satisfying the measure estimate and (i) by applying Corollary 3.12. Hence it is enough to find a set G'' satisfying the measure estimate and (ii) and to take $G := G' \cap G''$.

To do so we apply a standard maximal argument. Fix $a = 1, \dots, k-1$ and set

$$M(x) := \sup_{0<r<s} \frac{r}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E|.$$

We claim that $G'' := \{x \in B_s(p) \mid M(x) < \sqrt{\varepsilon}\}$ has the sought properties.

Indeed, for any $x \in B_s(p) \setminus G''$, there exists $\rho_x \in (0, s)$ such that

$$\frac{\rho_x}{\mathfrak{m}(B_{\rho_x}(x))} \int_{B_{\rho_x}(x)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E| \ge \sqrt{\varepsilon}. \tag{4.26}$$

Applying the Vitali lemma to the family $\{B_{\rho_x}(x)\}_{x\in B_s(p)\setminus G''}$ we find a disjoint subfamily $\{B_{r_i}(x_i)\}_{i\in\mathbb{N}}$ such that $B_s(p) \setminus G'' \subset \bigcup_i B_{5r_i}(x_i)$. Taking into account the disjointedness of the covering, (4.26), (4.4) and the Bishop–Gromov inequality, we can compute

$$\begin{aligned}
\mathscr{H}_5^h(B_s(p) \setminus G'') &\le \sum_{i\in\mathbb{N}} h(B_{5r_i}(x_i)) = \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{5r_i}(x_i))}{5r_i} \\
&\le C_N \sum_{i\in\mathbb{N}} \frac{\mathfrak{m}(B_{r_i}(x_i))}{r_i} \le C_N \sum_{i\in\mathbb{N}} \varepsilon^{-1/2} \int_{B_{r_i}(x_i)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E| \\
&\le C_N \varepsilon^{-1/2} \int_{B_{2s}(p)} |\nu \cdot \nabla u_a| \,\mathrm{d}|D\chi_E| \le C_N \sqrt{\varepsilon}\, \frac{\mathfrak{m}(B_{2s}(p))}{s}.
\end{aligned}$$

Step 3. We claim now that for any $\eta > 0$ there exists $\varepsilon = \varepsilon_{\eta,N} > 0$ small enough such that for any $0 < r < s$ and $x \in G \cap (\mathscr{F}_k E)_{2s,\varepsilon}$ the map

$$u = (u_1, \ldots, u_{k-1}) : \operatorname{supp} |D\chi_E| \cap B_r(x) \to \mathbb{R}^{k-1} \text{ is an } r\eta\text{-GH isometry.} \tag{4.27}$$

The claim is a consequence of Step 1. Indeed, for any $x \in G \cap (\mathscr{F}_k E)_{2s,\varepsilon}$ and any $r \in (0, s)$, conditions (i) and (ii) of Step 1 are satisfied by definition of $(\mathscr{F}_k E)_{2s,\varepsilon}$. Moreover, u is a $C_N \varepsilon^{1/4}$-splitting map on $B_r(x)$ satisfying (4.25), hence also assumption (iii) of Step 1 is satisfied for ε small enough.

To conclude the proof we have just to check conclusion (i) of Proposition 4.7, since conclusion (ii) follows from Step 2 by choosing ε so small that $\sqrt{\varepsilon} < \eta$. To this end, take $x, y \in G \cap (\mathscr{F}_k E)_{2s,\varepsilon}$ and choose $r := \mathsf{d}(x, y)$. Our claim (4.27) ensures that

$$\big| |u(x) - u(z)| - \mathsf{d}(x, z) \big| \le r\eta \quad \text{for any } z \in \operatorname{supp} |D\chi_E| \cap B_r(x),$$

therefore we can take $z = y$ and conclude.

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