



Itai Benjamini · David Ellis

On the structure of random graphs with constant r -balls

Received September 17, 2019; revised December 1, 2020

Abstract. We continue the study of the properties of graphs in which the ball of radius r around each vertex induces a graph isomorphic to the ball of radius r in some fixed vertex-transitive graph F , for various choices of F and r . This is a natural extension of the study of regular graphs.

More precisely, if F is a vertex-transitive graph and $r \in \mathbb{N}$, we say a graph G is r -locally F if the ball of radius r around each vertex of G induces a graph isomorphic to the graph induced by the ball of radius r around any vertex of F . We consider the following random graph model: for each $n \in \mathbb{N}$, we let $G_n = G_n(F, r)$ be a graph chosen uniformly at random from the set of all unlabelled, n -vertex graphs that are r -locally F . We investigate the properties possessed by the random graph G_n with high probability, i.e. with probability tending to 1 as $n \rightarrow \infty$, for various natural choices of F and r .

We prove that if F is a Cayley graph of a torsion-free group of polynomial growth, and r is sufficiently large depending on F , then the random graph $G_n = G_n(F, r)$ has largest component of order at most $n^{5/6}$ with high probability, and has at least $\exp(n^\delta)$ automorphisms with high probability, where $\delta > 0$ depends upon F alone. Both properties are in stark contrast to random d -regular graphs, which correspond to the case where F is the infinite d -regular tree. We also show that, under the same hypotheses, the number of unlabelled, n -vertex graphs that are r -locally F grows like a stretched exponential in n , again in contrast with d -regular graphs. In the case where F is the standard Cayley graph of \mathbb{Z}^d , we obtain a much more precise enumeration result, and more precise results on the properties of the random graph $G_n(F, r)$. Our proofs use a mixture of results and techniques from geometry, group theory and combinatorics.

We make several conjectures regarding what happens for Cayley graphs of other finitely generated groups.

Keywords. Random graphs, polynomial growth, connectivity, automorphism group

1. Introduction

Many results in combinatorics concern the impact of ‘local’ properties on ‘global’ properties of combinatorial structures (e.g. graphs). Some of these results concern the global

Itai Benjamini: Department of Mathematics, Weizmann Institute of Science, 234 Herzl Street, Rehovot 7610001, Israel; itai.benjamini@weizmann.ac.il

David Ellis: School of Mathematics, University of Bristol, Woodland Road, Bristol, BS8 1UG, United Kingdom; david.ellis@bristol.ac.uk

Mathematics Subject Classification (2020): Primary 05C80; Secondary 05C30, 20H15

properties of *all* structures with a given local property. Others concern the ‘typical’ global properties of a ‘random’ structure with a given local property. As an example of the former, Dirac’s classical theorem [22] states that in any n -vertex graph where each vertex has degree at least $n/2$, there exists a Hamiltonian cycle. On the other hand, the literature on random graphs contains many examples of the latter, among which is the following. Let $G(n, m)$ denote the Erdős–Rényi random graph, chosen uniformly at random from the set of all m -edge graphs with vertex-set $\{1, \dots, n\}$. Erdős and Rényi [24] proved that if $\varepsilon > 0$ is fixed, and $m = \lfloor (1 + \varepsilon)n/2 \rfloor$, then with high probability, the graph $G(n, m)$ has one component of order $\Theta(n)$ (called the ‘giant’ component), and $\Theta(n)$ other connected components, each of order $O(\log n)$. (Here, and henceforth, ‘with high probability’ means ‘with probability tending to 1 as $n \rightarrow \infty$ ’.) It is natural to ask what happens to the global properties of $G \in G(n, m)$ if we impose a local condition at each vertex. For example, if $m = dn/2$, where $d \in \mathbb{N}$ and n is even, and we start with $G(n, m)$ and condition on the event that every vertex has degree d , then what we get is of course the random (labelled) d -regular graph: the graph $G_n(d)$, chosen uniformly at random from the set of all d -regular graphs on $\{1, \dots, n\}$. The following are consequences of theorems of Bollobás, McKay and Wormald.

Theorem 1 (Bollobás [14] / Wormald [61]). *Let $d \geq 3$ be a fixed integer. Then with high probability, $G_n(d)$ is d -connected.*

Theorem 2 (Bollobás [13] / Wormald [61]). *Let $d, g \geq 3$ be fixed integers. Then*

$$\text{Prob} \{ \text{girth}(G_n(d)) \geq g \} = (1 + o(1)) \frac{\exp(-\sum_{l=1}^{g-1} \lambda_l)}{1 - \exp(-(\lambda_1 + \lambda_2))},$$

where

$$\lambda_i = \frac{(d-1)^i}{2^i} \quad (i \in \mathbb{N}).$$

In particular, $G_n(d)$ has girth at least g with positive probability.

Theorem 3 (Bollobás [15] / McKay and Wormald [53]). *Let $d \geq 3$ be a fixed integer. Then*

$$\mathbb{E}[|\text{Aut}(G_n(d))|] \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Hence, $G_n(d)$ is asymmetric with high probability, and so is a uniform random d -regular unlabelled graph on n vertices.

For comparison, note that if $d \in \mathbb{N}$ is fixed, then the Erdős–Rényi random graph $G(n, dn/2)$ (which has average degree d), also has girth at least g with positive probability, for any fixed $g \geq 3$. On the other hand, $G(n, dn/2)$ has $\Theta(n)$ isolated vertices with high probability, and therefore has $\exp(\Theta(n \log n))$ automorphisms with high probability.

(Recall that if G is a graph, the *girth* of G is the length of the shortest cycle in G ; if G is acyclic, we define $\text{girth}(G) = \infty$. We write $\text{Aut}(G)$ for the group of all automorphisms of G . We say that G is *asymmetric* if it has no non-trivial automorphism, i.e. $|\text{Aut}(G)| = 1$.)

It is natural to ask what happens to the global structure of a graph when we impose a ‘local’ condition which is stronger than being d -regular. A natural condition to impose is that the subgraph induced by the ball of radius r in G around any vertex is isomorphic to some fixed graph, for some fixed, small $r \in \mathbb{N}$. We proceed to recall (e.g. from [8]) some definitions that make this precise.

If G is a (simple, undirected) graph, we write $V(G)$ for the vertex-set of G and $E(G)$ for its edge-set. If $S \subset V(G)$, we write $G[S]$ for the subgraph of G induced by S , i.e. the maximal subgraph of G with vertex-set S . If $v, w \in V(G)$, the *distance from v to w in G* is defined to be the minimum number of edges in a path from v to w in G ; it is denoted by $d_G(v, w)$. If $v \in V(G)$, and $r \in \mathbb{N}$, we define $\text{Link}_r(v, G)$ to be the subgraph of G induced by the set of vertices of G with distance at most r from v , and we define

$$\text{Link}_r^-(v, G) := \text{Link}_r(v, G) \setminus \{\{x, y\} \in E(G) : d_G(v, x) = d_G(v, y) = r\}.$$

A *rooted graph* is an ordered pair (G, v) where G is a graph, and $v \in V(G)$.

Our key definition, introduced in [8], is as follows.

Definition 1. If (F, u) is a rooted graph, we say that a graph G is *r -locally (F, u)* if for every vertex $v \in V(G)$, there exists a graph isomorphism $\phi : \text{Link}_r(u, F) \rightarrow \text{Link}_r(v, G)$ such that $\phi(u) = v$.

We remark that if F is vertex-transitive, then Definition 1 is independent of the choice of u . Hence, if F is a vertex-transitive graph, we say that a graph G is *r -locally F* if there exists $u \in V(F)$ such that G is r -locally (F, u) .

A slightly weaker notion, also studied in [8], is as follows.

Definition 2. If (F, u) is a rooted graph, we say that a graph G is *weakly r -locally (F, u)* if for every vertex $v \in V(G)$, there exists a graph isomorphism $\phi : \text{Link}_r^-(u, F) \rightarrow \text{Link}_r^-(v, G)$ such that $\phi(u) = v$.

Note that if a connected graph is viewed as a 1-dimensional simplicial complex, equipped with the usual graph metric that assigns length 1 to every edge, then a graph G is weakly r -locally (F, u) if and only if for every vertex $v \in V(G)$, there exists a (metric) isometry ψ from the (metric) ball of centre u and radius r in F to the (metric) ball of centre v and radius r in G , with $\psi(u) = v$. Definition 2 is therefore perhaps more natural from a geometric point of view, whereas Definition 1 is perhaps more natural from a combinatorial point of view. To avoid confusion, we remark that Georgakopoulos [28] and De La Salle and Tesserà [21] define a graph G to be *r -locally F* (for a connected, vertex-transitive graph F) if it satisfies Definition 2 for some $u \in V(F)$. Note however that we have the implications

$$G \text{ is } r\text{-locally } (F, u) \implies G \text{ is weakly } r\text{-locally } (F, u) \implies G \text{ is } (r - 1)\text{-locally } (F, u),$$

for any $r \in \mathbb{N}$. Hence, if we are interested in these notions for large r (as is the case in [21, 28] and this paper), it does not really matter which notion we work with. In this paper, we work with Definition 1.

As a simple example, if $d \in \mathbb{N}$ with $d \geq 2$, let T_d denote the infinite d -regular tree. A graph G is r -locally T_d if and only if it is a d -regular graph with girth at least $2r + 2$, and G is weakly r -locally T_d if and only if it is d -regular with girth at least $2r + 1$. We remark that regular graphs of high girth have been intensively studied (see e.g. [39, 42, 50]).

Another natural example is as follows. If $d \in \mathbb{N}$, the d -dimensional lattice \mathbb{L}^d is the graph with vertex-set \mathbb{Z}^d , and edge-set

$$\{\{x, x + e_i\} : x \in \mathbb{Z}^d, i \in [d]\},$$

where $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ denotes the i th unit vector in \mathbb{R}^d . It is natural to ask what can be said about the global structure of graphs that are r -locally \mathbb{L}^d , for various integers d and r . We note that \mathbb{L}^d is the standard Cayley graph of the free Abelian group on d generators, whereas T_{2d} is the standard Cayley graph of the free group on d generators, so the case $F = \mathbb{L}^d$ of Definition 1 is perhaps a ‘second’ natural case to study, the ‘first’ natural case perhaps being T_{2d} .

It turns out that for all $d \geq 2$ and all $r \geq 3$, graphs which are r -locally \mathbb{L}^d have a very rigidly prescribed, ‘algebraic’ global structure. Indeed, the authors proved the following in [8].

Theorem 4. *Let $d \in \mathbb{N}$ with $d \geq 2$. Let $r \in \mathbb{N}$ with $r \geq 2$ if $d = 2$ and $r \geq 3$ if $d \geq 3$. Let G be a connected graph. Then G is r -locally \mathbb{L}^d if and only if G is isomorphic to \mathbb{L}^d / Γ , where $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ is such that Γ has minimum displacement at least $2r + 2$, and G is weakly r -locally \mathbb{L}^d if and only if G is isomorphic to \mathbb{L}^d / Γ , where $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ is such that Γ has minimum displacement at least $2r + 2$. In particular, if G is a connected graph that is weakly 3-locally \mathbb{L}^d , then G is covered by \mathbb{L}^d .*

(Here, if F is a graph and $\Gamma \leq \text{Aut}(F)$, the *minimum displacement* of Γ is defined to be $D(\Gamma) := \min \{d_F(v, \gamma(v)) : v \in V(F), \gamma \in \Gamma \setminus \{\text{Id}\}\}$. If F is a graph and $\Gamma \leq \text{Aut}(F)$ with Γ acting freely on F , the *quotient graph* F/Γ is as defined in Definition 11, in Section 3. If F and G are graphs, we say that F *covers* G if there exists a graph homomorphism from F to G that is bijective on neighbourhoods; see Definition 4 in Section 3.)

We remark that similar structure theorems for the case $d = 2$ (albeit with combinatorial rather than algebraic descriptions) were obtained by Thomassen [57] and by Márquez, de Mier, Noy and Revuelta [52].

Viewed as a subgroups of $\text{Isom}(\mathbb{R}^d)$, the groups Γ in Theorem 4 are d -dimensional crystallographic groups, and the orbit spaces \mathbb{R}^d / Γ are compact topological orbifolds. It follows that for any $d \geq 2$, we can view any finite, connected graph which is 3-locally \mathbb{L}^d as a ‘quotient lattice’ of \mathbb{L}^d inside a compact d -dimensional topological orbifold. Bieberbach’s theorems [10, 11] imply that for any $d \in \mathbb{N}$, there are only a finite number ($f(d)$, say) of affine-conjugacy classes of d -dimensional crystallographic groups (where two crystallographic groups are said to be *affine-conjugate* if they are conjugate via an affine transformation of \mathbb{R}^d). It follows that the orbit space \mathbb{R}^d / Γ is homeomorphic to one

of at most $f(d)$ topological spaces. (See Section 3 of this paper, or [8], for the definitions of any unfamiliar terms used above.)

The ‘rigidity’ phenomenon in Theorem 4 can be seen as a discrete analogue (for graphs) of the curvature-zero case of the seminal rigidity results of Ballmann [5] and independently Burns and Spatzier [19] on complete connected Riemannian manifolds of finite volume and bounded nonpositive sectional curvature.

Observe that the ‘algebraic’ structure in Theorem 4 is in stark contrast to the situation for regular graphs of high girth (i.e., graphs which are r -locally T_d for fixed $r, d \in \mathbb{N}$). Indeed, the uniform random d -regular graph $G_n(d)$ can be generated using a simple, purely combinatorial process, namely, the Configuration Model of Bollobás [12], and $G_n(d)$ has girth at least g with positive probability for any fixed $d, g \geq 3$, by Theorem 2.

Georgakopoulos [28] and the first author [7] asked whether the phenomenon in Theorem 4 holds more generally. Following [21], we say that a locally finite, vertex-transitive graph F is *local-to-global rigid* (or *LG-rigid*, for short) if there exists $r \in \mathbb{N}$ such that any connected graph that is r -locally F must be covered by F . (A graph F is said to be *locally finite* if every vertex of F has only finitely many neighbours.) Georgakopoulos and the first author asked the following.

Question 1. Is every locally finite Cayley graph of a finitely presented group LG-rigid?

This question was answered in the negative by De La Salle and Tessera, who constructed in [21] a locally finite Cayley graph of $\text{PGL}(4, \mathbb{Z})$ that is not LG-rigid. (See [20,21] for other counterexamples.) On the other hand, De La Salle and Tessera proved the following beautiful positive result.

Theorem 5 (De La Salle, Tessera, [21]). *If F is a locally finite Cayley graph of a torsion-free group of polynomial growth, then there exists $r \in \mathbb{N}$ such that for any connected graph G that is r -locally F , there exists a group $\Gamma \leq \text{Aut}(F)$ acting freely on F such that the quotient graph F/Γ is isomorphic to G . In particular, F is LG-rigid.*

Recall that a group Γ is said to be *torsion-free* if the only element of Γ with finite order is the identity, and it is said to have *polynomial growth* if there exists $K \in \mathbb{N}$ and a finite generating set S of Γ such that for any $n \in \mathbb{N}$,

$$|\{g \in \Gamma : \exists s_1, \dots, s_n \in S \text{ such that } g = s_1 \dots s_n\}| \leq n^K.$$

Strengthening a result of Georgakopoulos [28], De La Salle and Tessera also proved in [21] that if F is a locally finite Cayley graph of a finitely presented group and F is quasi-isometric to a tree, then F is LG-rigid. We conjecture that the conclusion of LG-rigidity in Theorem 5 holds without the torsion-free hypothesis; the same conjecture has been made independently by De La Salle and Tessera [personal communication].

In this paper, we study the *typical* properties of n -vertex graphs that are r -locally F , and the number of such graphs, for various choices of F and r . We introduce a new random graph model as follows.

Definition 3. Let F be a fixed infinite, locally finite, vertex-transitive graph, and let $r \in \mathbb{N}$ be a fixed integer. For each $n \in \mathbb{N}$, we let $G_n = G_n(F, r)$ be a graph chosen uniformly at random from the set $\mathcal{S}_n = \mathcal{S}_n(F, r)$ of all unlabelled, n -vertex graphs that are r -locally F .

We investigate the properties possessed by the random graph G_n with high probability (meaning, as usual, with probability tending to 1 as $n \rightarrow \infty$). This is a natural extension of the well-studied random regular graph $G_n(d)$, which corresponds roughly to the case $F = T_d$ and $r = 1$. (The random d -regular graph $G_n(d)$ is a labelled graph, but as outlined in [62], many properties possessed by $G_n(d)$ with high probability are also possessed with high probability by the uniform random unlabelled n -vertex d -regular graph.) However, the main combinatorial and probabilistic tools used for studying random regular graphs arise from the aforementioned Configuration Model of Bollobás, whereas in the cases we consider, there is no such simple combinatorial process that generates a uniform (or even approximately uniform) random n -vertex graph that is r -locally F . Instead, we use a mixture of geometric, algebraic and combinatorial arguments, and the results we obtain are somewhat less sharp than their analogues for random regular graphs.

Note that the random graph $G_n = G_n(F, r)$ in the above definition is defined only if $\mathcal{S}_n \neq \emptyset$. In the cases we study, $\mathcal{S}_n \neq \emptyset$ for all n sufficiently large depending on F and r (see Remark 2), so there is no problem with considering properties possessed by G_n with high probability. In the generality of Definition 3, it is possible that $\mathcal{S}_n(F, r) = \emptyset$ for infinitely many n , for example $\mathcal{S}_n(T_3, 1) = \emptyset$ for all odd n , but so long as $\mathcal{S}_n(F, r) \neq \emptyset$ for infinitely many n , we simply interpret ‘with high probability’ to mean with probability tending to 1 as $n \rightarrow \infty$ over integers n such that $\mathcal{S}_n(F, r) \neq \emptyset$, as is usual with random d -regular graphs when d is odd. If F is a locally finite Cayley graph of a residually finite group Γ , then for every $r \in \mathbb{N}$, there exist (arbitrarily large) finite connected graphs that are r -locally F , so in particular, $\mathcal{S}_n(F, r) \neq \emptyset$ for infinitely many n . (This follows easily by considering the quotient graph F/Γ' for appropriate finite-index subgroups Γ' of Γ , as in the proof of Proposition 38.) In all such cases, the random graph model above is non-trivial. However, it is in place to remark that there exists a locally finite Cayley graph F of the Baumslag–Solitar group $BS(2, 3)$, and a positive integer r , such that no non-empty finite graph is r -locally F ; this follows for example from Corollary K in [21]. For all such pairs (F, r) , the random graph model above is of course trivial; this causes a very slight issue in Section 2 on open problems.

Our first aim is to estimate the number of n -vertex graphs that are r -locally \mathbb{L}^d , for various pairs of integers d, r . We prove the following rather precise estimate.

Theorem 6. Define $r^*(2) = 2$, $r^*(d) = 3$ for all $3 \leq d \leq 7$, and $r^*(d) = \lceil (d - 1)/2 \rceil$ for all $d \geq 8$. Let $d \in \mathbb{N}$ with $d \geq 2$, and let $r \in \mathbb{N}$ with $r \geq r^*(d)$. Let $a_{d,r}(n)$ denote the number of unlabelled n -vertex graphs that are r -locally \mathbb{L}^d . Then there exists $\varepsilon_d > 0$ depending upon d alone such that

$$\log a_{d,r}(n) = (1 + O_{d,r}(n^{-\varepsilon_d}))K_d n^{d/(d+1)},$$

where $K_d := \frac{d+1}{d} \left(\frac{1}{2^{d-1}} \prod_{i=2}^d \zeta(i)\right)^{1/(d+1)}$.

This theorem says that, if r grows linearly with d , then the number of unlabelled graphs on n vertices that are r -locally \mathbb{L}^d grows like a stretched exponential in n . (This is in sharp contrast with the number of unlabelled $(2d)$ -regular graphs on n vertices, which grows superexponentially in n .) Note that, in contrast to Theorem 4, Theorem 6 applies only in the case where r grows linearly with d . A less precise enumeration result could be proved under the same hypotheses as in Theorem 4, but for brevity and for clarity of exposition, we do not prove such a result here.

We remark that a much stronger statement holds for $d = 1$ with $r^*(1) := 1$ (see Remark 5). In particular, we have

$$\log a_{1,r}(n) = \pi \sqrt{2n/3} - r \log n + O_r(1) \quad \text{as } n \rightarrow \infty.$$

Our proof of Theorem 6 has several steps (of which the second, in particular, may be of independent interest):

- We first use Theorem 4, together with some standard results and arguments from group theory and topological graph theory, to prove that the *connected* unlabelled n -vertex graphs that are r -locally \mathbb{L}^d are in one-to-one correspondence with conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ such that $|\mathbb{Z}^d/\Gamma| = n$ and $D(\Gamma) \geq 2r + 2$, provided $r \geq r^*(d)$. Our next task is to enumerate the latter.
- We show that for any $x \geq 0$, all but an $o(1)$ -fraction of the conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ such that $|\mathbb{Z}^d/\Gamma| \leq x$ and $D(\Gamma) \geq 2r + 2$, are conjugacy-classes of subgroups consisting only of translations (‘pure-translation subgroups’), provided $r \geq r^*(d)$; we then estimate the number of the latter. We do this by analysing the former subgroups in terms of their lattices of translations and their point groups, using a cohomology argument borrowed from one of the standard proofs of Bieberbach’s Third Theorem, and using a combinatorial argument to show that a uniform random sublattice of \mathbb{Z}^d with index at most x is unlikely to have any non-trivial hyperoctahedral symmetry.

It follows from this part of the proof that for any $x \geq 0$, all but an $o(1)$ -fraction of the connected unlabelled graphs on at most x vertices that are r -locally \mathbb{L}^d (for $r \geq r^*(d)$) are quotient lattices of \mathbb{L}^d inside topological tori of the form \mathbb{R}^d/Γ_0 (where Γ_0 is the group of translations by vectors in a rank- d sublattice of \mathbb{Z}^d). We may call such graphs ‘generalized discrete torus’ graphs, since they generalize the usual discrete torus C_k^d ; the latter is obtained when $\Gamma_0 = \langle \{x \mapsto x + ke_i : i \in [d]\} \rangle$. In general, the ‘fundamental domain’ of a generalized discrete torus graph is a parallelepiped that need not have any of its edges parallel to a coordinate axis.

It is in place to remark that in the case $3 \leq r < \lceil (d - 1)/2 \rceil$, a positive fraction of the connected unlabelled graphs on at most x vertices that are r -locally \mathbb{L}^d are quotient lattices of \mathbb{L}^d inside a topological orbifold obtained by taking the quotient of a topological torus \mathbb{R}^d/Γ_0 by some involution of the form $x \mapsto c - x$, where $c \in (\frac{1}{2} \cdot \mathbb{Z})^d$, so the assumption that r grows linearly with d is necessary for this step of the proof to work. (The rest of the proof works under the weaker assumption $r \geq 2 + 1_{\{d \geq 3\}}$.)

- Combining the previous two steps allows us to estimate, for each $x \geq 0$, the number of *connected* unlabelled graphs with at most x vertices that are r -locally \mathbb{L}^d . We use this, combined with a generating function argument and a variant of a result of Brigham on the asymptotics of partition functions, to obtain the estimate in Theorem 6 on the number of (possibly disconnected) unlabelled graphs with n vertices that are r -locally \mathbb{L}^d .

Using some of the same tools, we also prove the following.

Theorem 7. *For any fixed integers $d, r \in \mathbb{N}$ with $d \geq 2$ and $r \geq 2 + 1_{\{d \geq 3\}}$, the random graph $G_n(\mathbb{L}^d, r)$ can be sampled in time polynomial in n , using a polynomial (in n) number of coins with rational biases.*

We then move on to consider the (much) more general case of connected, locally finite Cayley graphs of torsion-free groups of polynomial growth. (This is a well-studied class of graphs, and a natural extension of \mathbb{L}^d case. We recall that Gromov's celebrated theorem [30] on groups of polynomial growth states that a finitely generated group has polynomial growth if and only if it is virtually nilpotent.) Our first main result in this more general case is as follows.

Theorem 8. *Let F be a connected, locally finite Cayley graph of a torsion-free group of polynomial growth. Then there exists $r_0 = r_0(F) \in \mathbb{N}$ such that for all $r \geq r_0$, the random graph $G_n = G_n(F, r)$ has largest component of order at most $n^{5/6}$ with probability at least $1 - \exp(-n^{1/13})$, provided n is sufficiently large depending on F and r .*

Remark 1. In the case where $F = \mathbb{L}^d$ for $d \geq 2$, we may take $r_0 = 2 + 1_{\{d \geq 3\}}$ in the above theorem.

Remark 2. We also show that if F is a connected, locally finite Cayley graph of a torsion-free group of polynomial growth, and $r \in \mathbb{N}$, then for all n sufficiently large depending on F and r , the set $\mathcal{S}_n = \mathcal{S}_n(F, r)$ in Definition 3 is non-empty (so that the statement of Theorem 8 is meaningful for all sufficiently large n); this is the content of Lemma 39.

Our proof of Theorem 8 proceeds as follows. The first step is to obtain an upper bound on the number of *connected*, unlabelled, n -vertex graphs that are r -locally F , for sufficiently large r and all $n \in \mathbb{N}$; this upper bound is deduced from Gromov's theorem, Theorem 5, and certain other known results from geometric group theory, including another result of De La Salle and Tessera (from [21]), on the structure of automorphism groups of Cayley graphs of torsion-free groups of polynomial growth. The second step is to obtain a lower bound on the above quantity, for an appropriate infinite sequence of positive integers n ; this step requires a rather intricate group-theoretic argument, together with the use of a classical theorem of Gruenberg. In the third and last step of the proof, we use the upper and lower bounds obtained in the first two steps to estimate the probability that a uniform random unlabelled, n -vertex graph that is r -locally F has largest component of order $n^{5/6}$. This part of the proof is combinatorial, and is performed by the construction of explicit maps between certain carefully chosen classes of unlabelled

graphs that are r -locally F , together with the use of classical results from the theory of integer partitions. The upper and lower bounds obtained in the first and second steps in our proof are (in general) far apart and are (probably both) far from the truth; the challenge in the third step is to find arguments robust enough to yield good estimates on the required probability, in spite of this gap. Generating function techniques (often used in similar problems where the gap is narrower) could not be directly used in our case.

Our next main result is that with high probability, the random graph $G_n(F, r)$ has at least $\exp(\text{poly}(n))$ automorphisms:

Theorem 9. *Let F be a connected, locally finite Cayley graph of a torsion-free group of polynomial growth. There exists $r_0 = r_0(F) \in \mathbb{N}$ and $\delta_0 = \delta_0(F) > 0$ such that for all $r \geq r_0$,*

$$\Pr \{ |\text{Aut}(G_n(F, r))| \geq 2^{n^{\delta_0}} \} \geq 1 - \exp(-n^{\delta_0}) \tag{1}$$

provided n is sufficiently large depending on F and r .

In fact, our proof of Theorem 9 yields something somewhat stronger than (1), namely, under the same conditions, with probability at least $1 - \exp(-n^{\delta_0})$, the random graph $G_n(F, r)$ has at least n^{δ_0} vertex-transitive components. The high-level structure of the proof is similar to that of Theorem 8, except that in the second step, it is necessary to obtain a lower bound on the number of connected, *vertex-transitive* unlabelled n -vertex graphs that are r -locally F , for an appropriate infinite sequence of positive integers n .

Turning to the enumeration question, we prove a somewhat less precise result than Theorem 6 in the more general case. Viz., using some of the same tools used in the proofs of Theorems 8 and 9, together with a classical theorem of Brigham on the asymptotics of partition functions, we prove that if F is a connected, locally finite Cayley graph of a torsion-free group of polynomial growth, then there exists $r_0 = r_0(F) \in \mathbb{N}$ such that for each $r \geq r_0$, the number $b_{F,r}(n)$ of unlabelled, n -vertex graphs that are r -locally F grows like a stretched exponential.

Theorem 10. *Let F be a connected, locally finite Cayley graph of a finitely generated, torsion-free group of polynomial growth, and let $r_0 = r_0(F)$ be as in Theorem 5. For each $r \geq r_0$, there exist $n_0 \in \mathbb{N}$, $C > 0$, $c > 0$ and $\alpha \geq 0$ such that*

$$c\sqrt{n} \leq \log b_{F,r}(n) \leq Cn^{(\alpha+1)/(\alpha+2)}$$

for all $n \in \mathbb{N}$ such that $n \geq n_0$.

We do not, however, determine $\lim_{n \rightarrow \infty} (\log \log b_{F,r}(n)) / \log n$ in the general case, as we do in the case of \mathbb{L}^d where $r \geq r^*(d)$.

In the special case where $F = \mathbb{L}^d$, we make the following conjecture, in the spirit of Theorem 8.

Conjecture 1. *Define $r_0(2) = 2$ and $r_0(d) = 3$ for all $d \geq 3$. Let $d \in \mathbb{N}$ with $d \geq 2$, and let $r \in \mathbb{N}$ with $r \geq r_0(d)$. Then with high probability, the largest component of $G_n(\mathbb{L}^d, r)$ has order*

$$\Theta_{d,r}(n^{1/(d+1)} \log n).$$

We remark that the statement of Conjecture 1 holds for $d = 1$ with $r_0(1) = 1$. This is easily deduced from known results from the theory of integer partitions (see Section 3.4).

It is instructive to contrast Theorems 8 and 9 with the situation where $F = T_d$, the infinite d -regular tree, for $d \geq 3$. As remarked above, a graph G is r -locally T_d if and only if it is a d -regular graph of girth at least $2r + 2$. By Theorem 2, for fixed integers $d, g \geq 3$, a uniform random labelled d -regular graph on $\{1, \dots, n\}$ has girth at least g with probability $\Omega_{d,g}(1)$ as $n \rightarrow \infty$. Using Theorems 1 and 3, it follows that if $d, g \geq 3$ are fixed integers and G_n is the random graph chosen uniformly from the set of all *unlabelled*, n -vertex, d -regular graphs of girth at least g , then with high probability, G_n is asymmetric and d -connected. It follows that if $d \geq 3$ and $r \geq 1$, then with high probability, $G_n(T_d, r)$ is asymmetric and d -connected.

We remark that, instead of working with unlabelled graphs, one could consider instead the properties of the random graph $H_n = H_n(F, r)$ chosen uniformly from the set of all *labelled* graphs with vertex-set $\{1, \dots, n\}$ that are r -locally F , for various choices of F and r , and the number of such graphs. The behaviour of H_n does not vary so strikingly with the choice of F and r as in our case: for example, if $d, r \in \mathbb{N}$ are fixed with $d \geq 2$, and F is an infinite, d -regular, connected, vertex-transitive graph, then the number of labelled graphs on $\{1, \dots, n\}$ that are r -locally F is $\exp(\Theta_{d,r}(n \log n))$, as is the number of labelled d -regular graphs on $\{1, \dots, n\}$. Hence, we believe that the random unlabelled graph G_n is a more interesting object of study in this context. While the recent trend in probabilistic combinatorics (perhaps for reasons of tractability) has been to focus more on random *labelled* structures and their enumeration, unlabelled structures have also attracted much attention in combinatorics, probability theory and related fields. We mention for example the work of Tutte [58,59] in the 1960s and 70s on the enumeration of planar maps, and the more recent work of Angel and Schramm [1], Le Gall [43], Gurel-Gurevich and Nachmias [33], and others, on random planar maps. (See also the survey papers [44,45] of Le Gall, and the references contained therein.)

The remainder of this paper is structured as follows. In Section 2, we discuss a broad collection of problems related to our work. In Section 3, we describe the background and tools we will need from topological graph theory, group theory and topology, and we prove some preliminary lemmas using these tools.

In Section 4, we prove Theorem 6, our enumeration result for unlabelled graphs that are r -locally \mathbb{L}^d . In Section 5, we consider the much more general case where \mathbb{L}^d is replaced by a connected, locally finite Cayley graph of a torsion-free group of polynomial growth, and we prove Theorems 8–10.

In Section 6, we prove some more precise results on the typical properties of graphs that are r -locally \mathbb{L}^d , including a more precise version of Theorem 9 (on the order of the automorphism group), for such graphs. (We note that the bulk of the paper is devoted to the proofs of Theorems 6, 8 and 9, and the necessary preliminaries.)

2. Related problems and conjectures

Our results and conjectures above are of course part of a more general class of questions, which we proceed to discuss in this section. Let F be an infinite, locally finite, connected, vertex-transitive graph, and let $r \in \mathbb{N}$. Let us say that the pair (F, r) is *connectivity-forcing* if the random graph $G_n = G_n(F, r)$ in Definition 3 is connected with high probability. (If there exists no non-empty finite graph that is r -locally F , then by convention we say that (F, r) is connectivity-forcing; otherwise, we have $S_n(F, r) \neq \emptyset$ for infinitely many n , so the term ‘with high probability’ is well-defined; see the discussion following Definition 3.) It is natural to pose the following (perhaps rather ambitious) problem.

Problem 1. Characterise the pairs (F, r) that are connectivity-forcing.

Note that it follows immediately from Theorem 8 that if F is a connected, locally finite Cayley graph of a torsion-free group of polynomial growth, then there exists $r_0 \in \mathbb{N}$ depending upon F alone such that for any $r \geq r_0$, the pair (F, r) is not connectivity-forcing. On the other hand, it follows from the discussion at the end of Section 1 that the pair (T_d, r) is connectivity-forcing, for all $r \in \mathbb{N}$.

In the above problem, one may of course replace connectivity with any other graph property (with the same proviso concerning when $S_n(F, r) = \emptyset$ for all $n \in \mathbb{N}$). A closely related question is for which pairs (F, r) it holds that $G_n(F, r)$ has a ‘giant component’ with high probability. (Recall that if (G_n) is a sequence of random graphs, we say that G_n has a giant component with high probability if there exists $\varepsilon > 0$ such that

$$\text{Prob} \{G_n \text{ has a component of order at least } \varepsilon n\} \rightarrow 1$$

as $n \rightarrow \infty$.) In this case, we say that the pair (F, r) forces a giant component. We pose the following.

Problem 2. Characterise the pairs (F, r) that force a giant component.

Theorem 8 implies that if F is a connected, locally finite Cayley graph of a torsion-free group of polynomial growth and $r \geq r_0(F)$, then (F, r) does not force a giant component.

It is also natural to ask for which pairs (F, r) it holds that G_n is an expander with high probability. Let us state this problem in full. Recall that if $G = (V, E)$ is a finite graph, and $S \subset V(G)$, we define the *edge-boundary* $\partial S := E(S, V \setminus S)$, i.e. it is the set of edges of G between S and $V \setminus S$. We define the *vertex-boundary* $b(S) := N(S) \setminus S$, i.e. it is the set of vertices of G which are not in S , but which are adjacent to a point in S . We define the *edge-expansion ratio* of G to be

$$h(G) := \min \{|\partial S|/|S| : 0 < |S| \leq |V|/2\}.$$

Similarly, we define the *vertex-expansion ratio* of G to be

$$h_v(G) := \min \{|b(S)|/|S| : 0 < |S| \leq |V|/2\}.$$

As usual, let $\Delta(G)$ denote the maximum degree of G . Note that $|\partial S| \leq \Delta(G)|b(S)|$ for all $S \subset V$, so $h_v(G) \geq h(G)/\Delta(G)$.

Now let $(G_n)_{n \in \mathbb{N}}$ be any sequence of graphs with $|V(G_n)| \rightarrow \infty$ as $n \rightarrow \infty$, and with uniformly bounded maximum degree (i.e., $\sup_{n \in \mathbb{N}} \Delta(G_n) < \infty$). We say that G_n is an *expander with high probability* if there exists $\varepsilon > 0$ such that

$$\text{Prob} \{h(G_n) \geq \varepsilon\} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Note that this is actually a property of the sequence of random graphs (G_n) , not of a single graph.

Let us say that the pair (F, r) is *expansion-forcing* if $G_n(F, r)$ is an expander with high probability. We pose the following.

Problem 3. Characterise the pairs (F, r) that are expansion-forcing.

Note that if (F, r) is expansion-forcing, then it is clearly connectivity-forcing. Bollobás [16] proved that if $d \geq 3$ is a fixed integer, the random (labelled) d -regular graph $G_n(d)$ has edge-expansion ratio at least $d/18$ with high probability. It follows from this, Theorem 2 and Theorem 3 that for any $d \geq 3$ and any $r \in \mathbb{N}$, the pair (T_d, r) is expansion-forcing. Furthermore, we have $h_v(G_n(T_d, r)) \geq 1/18$ with high probability. On the other hand, if F is a connected, locally finite Cayley graph of a torsion-free group of polynomial growth and $r \geq r_0(F)$, then (F, r) is not connectivity-forcing, so it is certainly not expansion-forcing.

In the case where F is (the graph of) a regular tiling of the hyperbolic plane, we make the following conjecture.

Conjecture 2. *Let F be the graph of a regular tiling of the hyperbolic plane. Then for any $r \in \mathbb{N}$, the pair (F, r) is connectivity-forcing.*

If G is a finite graph, let us write $l(G)$ for the order of the largest component of G . Let \mathcal{R} be the set of all $x \in [0, 1]$ such that there exists a pair (F, r) such that with probability 1, $\log_n(l(G_n(F, r))) \rightarrow x$ as $n \rightarrow \infty$. Conjecture 1 (together with a mild assumption on the rate of convergence to 1 of the probability concerned) would imply that $1/(d + 1) \in \mathcal{R}$ for all $d \in \mathbb{N}$; Corollary 16 implies that $1/2 \in \mathcal{R}$. The fact that $G_n(d)$ is connected with probability at least $1 - O(1/n^2)$ for each fixed integer $d \geq 3$, together with Theorems 2 and 3, implies that $1 \in \mathcal{R}$. It would be of interest to further investigate the set \mathcal{R} . It would also be interesting to determine whether or not there exists a pair (F, r) and an absolute constant $c > 0$ such that with high probability, $cn < l(G_n(F, r)) < n$, i.e. whether it is possible for $G_n = G_n(F, r)$ to have a giant component and yet be disconnected, with high probability.

If Γ is a finitely generated group, let us say that Γ is *strongly-connectivity-forcing* if for any finite generating set S of Γ with $\text{Id} \notin S$ and $S^{-1} = S$, and for any $r \in \mathbb{N}$, the pair $(\text{Cay}(\Gamma, S), r)$ is connectivity-forcing. We pose the following.

Problem 4. Characterise the finitely generated groups that are strongly-connectivity-forcing.

We conjecture that any finitely generated group with superexponential subgroup growth is strongly-connectivity-forcing. (See the book of Lubotzky and Segal [51] for background on such groups.)

We also conjecture that if Γ is a finitely generated, residually finite group, and F is a connected, locally finite Cayley graph of Γ , then the property of (F, r) being connectivity-forcing / expansion-forcing / forcing a giant component, for all sufficiently large r , does not depend upon the generating set chosen for F . More precisely, we make the following.

Conjecture 3. *Let Γ be a finitely generated, residually finite group, and let S_i (for $i = 1, 2$) be two finite generating sets of Γ with $\text{Id} \notin S_i$ and $S_i^{-1} = S_i$. Let $F_i = \text{Cay}(\Gamma, S_i)$. Then (F_1, r) has property P for all sufficiently large r if and only if (F_2, r) has property P for all sufficiently large r , where P is the property of being connectivity-forcing, or the property of being expansion-forcing, or the property of forcing a giant component.*

(The hypothesis of residual finiteness is added to ensure that we never have $S_n(F, r) = \emptyset$ for all $n \in \mathbb{N}$; see the discussion after Definition 3.)

We now turn to questions of symmetry. We say a pair (F, r) is *asymmetry-forcing* if $G_n(F, r)$ has no nontrivial automorphisms, with high probability. We pose the following.

Problem 5. Characterise the pairs (F, r) that are asymmetry-forcing.

It follows from the discussion at the end of Section 1 that for any $d \geq 3$ and any $r \in \mathbb{N}$, the pair (T_d, r) is asymmetry-forcing. On the other hand, it follows immediately from Theorem 9 that if F is a connected, locally finite Cayley graph of a torsion-free group of polynomial growth, then there exists $r_0 \in \mathbb{N}$ depending upon F alone such that for any $r \geq r_0$, the pair (F, r) is not asymmetry-forcing; on the contrary, with high probability, $G_n(F, r)$ has at least $\exp(\text{poly}(n))$ automorphisms. Motivated by the latter, we say a pair (F, r) is *stretched-exponential-symmetry-forcing* if there exists $\delta > 0$ such that with high probability, $G_n(F, r)$ has at least $\exp(n^\delta)$ automorphisms; it would also be of interest to characterise the pairs (F, r) that are stretched-exponential-symmetry-forcing. We thank an anonymous referee for suggesting this problem.

We conclude with a more general suggestion for future research. It is straightforward to generalise Definition 1 and the random graph model in Definition 3 to other well-studied combinatorial structures, in particular to k -uniform hypergraphs and d -dimensional simplicial complexes, for $k \geq 3$ and $d \geq 2$. It would be interesting to investigate the typical properties of random k -uniform hypergraphs, or random d -dimensional simplicial complexes, with given r -local structures. We remark that the topology of random d -dimensional simplicial complexes (for $d > 1$) is a very active area of current research. The most commonly studied model is perhaps the $Y_d(n, p)$ model introduced in 2006 by Linial and Meshulam [46]. (This is the random d -dimensional simplicial complex with n vertices and full $(d - 1)$ -dimensional skeleton, where each d -dimensional simplex is included independently with probability p .)

For d -dimensional simplicial complexes with $d > 1$, there are two natural analogues of (graph-)connectivity, namely, d -collapsibility and the vanishing of the d th homology group over \mathbb{R} . In a series of recent papers [2–4, 47], the thresholds for d -collapsibility

and for the vanishing of the d th real homology group of $Y_d(n, p)$ were established by Aronshtam, Linial, Łuczak, Peled and Meshulam; see also [40] for a survey of related results. In [49], Lubotzky and Meshulam introduced a new model of random 2-dimensional simplicial complexes with bounded upper degrees; these were shown to have positive coboundary expansion with high probability. There is currently much interest in developing a model of random ‘regular’ d -dimensional simplicial complexes (for $d > 1$) with good typical coboundary expansion; see e.g. [48]. We believe our random r -local model is unlikely to solve this particular problem, but may have other interesting combinatorial and topological properties.

3. Definitions, background and tools

This section is structured as follows. In Section 3.1, we list the basic (and standard) graph-theoretic definitions, notations and conventions which we use. In Section 3.2, we list the standard definitions and results we need from topological graph theory. In Section 3.3, we list the definitions and classical results that we need from group theory and topology, and we also obtain a useful group-theoretic lemma (Lemma 14) using the proof of the classical Third Theorem of Bieberbach, and some well-known facts about group cohomology. In Section 3.4, we use classical results from the theory of integer partitions to deduce that the statement of Conjecture 1 holds in the case of \mathbb{L}^1 .

3.1. Basic graph-theoretic definitions, notation and conventions

Unless otherwise stated, all graphs will be undirected and simple (that is, without loops or multiple edges); they need not be finite. An undirected, simple graph is defined to be an ordered pair of sets (V, E) , where $E \subset \binom{V}{2}$; V is called the *vertex-set* and E the *edge-set*. An edge $\{v, w\}$ of a graph will often be written vw , for brevity.

If G is a graph and $S \subset V(G)$, we let $N(S)$ denote the *neighbourhood* of S in G , i.e.

$$N(S) = S \cup \{v \in V(G) : sv \in E(G) \text{ for some } s \in S\}.$$

If $v \in V(G)$, we let $\Gamma(v) = \{u \in V(G) : uv \in E(G)\}$ denote the set of neighbours of v . We say the graph G is *locally finite* if each of its vertices has only finitely many neighbours.

If G is a graph, and $u, v \in V(G)$ are in the same component of G , the *distance from u to v in G* is the minimum length of a path from u to v ; it is denoted by $d_G(u, v)$ (or by $d(u, v)$, when the graph G is understood). If G is a graph, and v is a vertex of G , the *ball of radius r around v* is defined by

$$B_r(v, G) := \{w \in V(G) : d_G(v, w) \leq r\},$$

i.e. it is the set of vertices of G of distance at most r from v .

An *automorphism* of a graph G is a bijection $\phi : V(G) \rightarrow V(G)$ such that $\{\phi(v), \phi(w)\} \in E(G)$ if and only if $\{v, w\} \in E(G)$, for all $v, w \in V(G)$.

If Γ is a group, and $S \subset \Gamma$ is symmetric (meaning that $S^{-1} = S$) and $\text{Id} \notin S$, the *Cayley graph of Γ with respect to S* is the graph with vertex-set Γ and edge-set $\{\{g, gs\} : g \in \Gamma, s \in S\}$; we denote it by $\text{Cay}(\Gamma, S)$.

All logarithms in this paper are to base e .

3.2. Background and tools from topological graph theory

We follow [31, 41].

Definition 4. If F and G are graphs, and $p : V(F) \rightarrow V(G)$ is a graph homomorphism from F to G , we say that p is a *covering map* if p maps $\Gamma(v)$ bijectively onto $\Gamma(p(v))$, for all $v \in V(F)$. In this case, we say that F *covers* G .

It is easy to see that if F and G are graphs with G connected, and $p : V(F) \rightarrow V(G)$ is a covering map, then p is surjective.

Definition 5. Let F and G be graphs, and let $p : V(F) \rightarrow V(G)$ be a covering of G by F . The pre-image of a vertex of G under p is called a *fibre* of p .

Definition 6. Let F and G be graphs, and let $p : V(F) \rightarrow V(G)$ be a covering of G by F . An automorphism $\phi \in \text{Aut}(F)$ is said to be a *covering transformation* of p if $p \circ \phi = p$. The group of covering transformations of p is denoted by $\text{CT}(p)$.

Note that any covering transformation of p acts on each fibre of p , but it need not act transitively on any fibre of p .

Definition 7. Let F and G be graphs, and let $p : V(F) \rightarrow V(G)$ be a covering of G by F . We say that p is a *normal* covering if $\text{CT}(p)$ acts transitively on each fibre of p .

It is well known (and easy to check) that if F is a connected graph, and $p : V(F) \rightarrow V(G)$ is a covering of G by F , then if $\text{CT}(p)$ acts transitively on some fibre of p , it acts transitively on every fibre of p . Hence, in the previous definition, ‘on each fibre’ may be replaced by ‘on some fibre’.

Definition 8. Let Γ be a group, let X be a set, and let $\alpha : \Gamma \times X \rightarrow X$ be an action of Γ on X . For each $x \in X$, we write $\text{Orb}_\Gamma(x) := \{\alpha(\gamma, x) : \gamma \in \Gamma\}$ for the Γ -orbit of x , and $\text{Stab}_\Gamma(x) := \{\gamma \in \Gamma : \alpha(\gamma, x) = x\}$ for the stabiliser of x in Γ . When the group Γ is understood, we suppress the subscript Γ .

Definition 9. If F is a graph and $\Gamma \leq \text{Aut}(F)$, the *minimum displacement* of Γ is defined to be $D(\Gamma) := \min \{d_F(x, \gamma(x)) : x \in V(F), \gamma \in \Gamma \setminus \{\text{Id}\}\}$.

Definition 10. Let Γ be a group, let X be a set, and let $\alpha : \Gamma \times X \rightarrow X$ be an action of Γ on X . We say that α is *free* if $\alpha(\gamma, x) \neq x$ for all $x \in X$ and all $\gamma \in \Gamma \setminus \{\text{Id}\}$.

Definition 11 (Quotient of a graph). Let F be a simple graph, and let $\Gamma \leq \text{Aut}(F)$. Then Γ acts on $V(F)$ via the natural left action $(\gamma, x) \mapsto \gamma(x)$, and on $E(F)$ via the natural (induced) action $(\gamma, \{x, y\}) \mapsto \{\gamma(x), \gamma(y)\}$. We define the *quotient graph* F/Γ to be the

multigraph whose vertices are the Γ -orbits of $V(F)$, and whose edges are the Γ -orbits of $E(F)$, where for any edge $\{x, y\} \in E(F)$, the edge $\text{Orb}(\{x, y\})$ has endpoints $\text{Orb}(x)$ and $\text{Orb}(y)$. Note that F/Γ may have loops (if $\{x, \gamma(x)\} \in E(F)$ for some $\gamma \in \Gamma$ and $x \in V(F)$), and it may also have multiple edges (if there exist $\{u_1, u_2\}, \{v_1, v_2\} \in E(F)$ with $\gamma_1(u_1) = v_1$ and $\gamma_2(u_2) = v_2$ for some $\gamma_1, \gamma_2 \in \Gamma$, but $\{\gamma(u_1), \gamma(u_2)\} \neq \{v_1, v_2\}$ for all $\gamma \in \Gamma$).

Definition 12. Let G be a graph, and let $\Gamma \leq \text{Aut}(G)$. We say that Γ acts freely on G if the natural actions of Γ on $V(G)$ and $E(G)$ are both free actions, or equivalently, if no element of $\Gamma \setminus \{\text{Id}\}$ fixes any vertex or edge of G .

The following two lemmas are well-known, and easy to check.

Lemma 11. Let F be a connected (possibly infinite) graph, let G be a graph, and let $p : V(F) \rightarrow V(G)$ be a covering map from F to G . Then $\text{CT}(p)$ acts freely on F .

Lemma 12. Let F and G be (possibly infinite) graphs with G connected, and suppose that $p : V(F) \rightarrow V(G)$ is a normal covering map from F to G . Then there is a graph isomorphism between G and $F/\text{CT}(p)$. Moreover, if $\Gamma \leq \text{Aut}(F)$, Γ acts freely on F and F/Γ is a simple graph, then the natural quotient map $q : V(F) \mapsto V(F)/\Gamma$ is a normal covering map with $\text{CT}(q) = \Gamma$.

3.3. Some preliminaries from group theory and topology

Fact 1. The group $\text{Isom}(\mathbb{R}^d)$ of isometries of d -dimensional Euclidean space satisfies

$$\begin{aligned} \text{Isom}(\mathbb{R}^d) &= \{t \circ \sigma : t \in T(\mathbb{R}^d), \sigma \in O(d)\} = \{\sigma \circ t : t \in T(\mathbb{R}^d), \sigma \in O(d)\} \\ &= T(\mathbb{R}^d) \rtimes O(d), \end{aligned}$$

where

$$T(\mathbb{R}^d) := \{x \mapsto x + v : v \in \mathbb{R}^d\}$$

denotes the group of all translations in \mathbb{R}^d , and $O(d) \leq \text{GL}(\mathbb{R}^d)$ denotes the group of all real orthogonal $d \times d$ matrices.

Fact 2. For any $d \in \mathbb{N}$, we have

$$\begin{aligned} \text{Aut}(\mathbb{Z}^d) &= \{t \circ \sigma : t \in T(\mathbb{Z}^d), \sigma \in B_d\} = \{\sigma \circ t : t \in T(\mathbb{Z}^d), \sigma \in B_d\} \\ &= T(\mathbb{Z}^d) \rtimes B_d, \end{aligned}$$

where

$$T(\mathbb{Z}^d) := \{x \mapsto x + v : v \in \mathbb{Z}^d\}$$

denotes the group of all translations by elements of \mathbb{Z}^d , and

$$B_d = \{\sigma \in \text{GL}(\mathbb{R}^d) : \sigma(\{\pm e_i : i \in [d]\}) = \{\pm e_i : i \in [d]\}\},$$

denotes the d -dimensional hyperoctahedral group, which is the symmetry group of the d -dimensional (solid) cube with set of vertices $\{-1, 1\}^d$, and can be identified with the permutation group

$$\{\sigma \in \text{Sym}([d] \cup \{-i : i \in [d]\}) : \sigma(-i) = -\sigma(i) \forall i\},$$

in the natural way (identifying e_i with i and $-e_i$ with $-i$ for all $i \in [d]$). We therefore have $|B_d| = 2^d d!$.

Remark 3. It is clear that every element of $\text{Aut}(\mathbb{L}^d)$ can be uniquely extended to an element of $\text{Isom}(\mathbb{R}^d)$. We can therefore view $\text{Aut}(\mathbb{L}^d)$ as a subgroup of $\text{Isom}(\mathbb{R}^d)$.

Definition 13. If X is a topological space, and Γ is a group acting on X , the orbit space X/Γ is the (topological) quotient space X/\sim , where $x \sim y$ iff $y \in \text{Orb}_\Gamma(x)$, i.e. iff x and y are in the same Γ -orbit.

Definition 14. If X is a topological space, a group Γ of homeomorphisms of X is said to be *discrete* if the relative topology on Γ (induced by the compact-open topology on the group of all homeomorphisms of X) is the discrete topology.

Definition 15. If X is a topological space, and Γ is a discrete group of homeomorphisms of X , we say that Γ acts *properly discontinuously* on X if for any $x, y \in X$, there exist open neighbourhoods U of x and V of y such that $|\{\gamma \in \Gamma : \gamma(U) \cap V \neq \emptyset\}| < \infty$.

Fact 3. If $\Gamma \leq \text{Isom}(\mathbb{R}^d)$, then Γ is discrete if and only if for any $x \in \mathbb{R}^d$, the orbit $\{\gamma(x) : \gamma \in \Gamma\}$ is a discrete subset of \mathbb{R}^d . Hence, $\text{Aut}(\mathbb{L}^d)$ is a discrete subgroup of $\text{Isom}(\mathbb{R}^d)$.

Fact 4. If $\Gamma \leq \text{Isom}(\mathbb{R}^d)$ is discrete, then Γ acts properly discontinuously on \mathbb{R}^d . (Note that it is clear directly from the definition that $\text{Aut}(\mathbb{L}^d)$, and any subgroup thereof, acts properly discontinuously on \mathbb{R}^d .)

Definition 16. Let Γ be a discrete subgroup of $\text{Isom}(\mathbb{R}^d)$. The *translation subgroup* T_Γ of Γ is the (normal) subgroup of all translations in Γ . The *lattice of translations* of Γ is the lattice $L_\Gamma := \{\gamma(0) : \gamma \in T_\Gamma\} \subset \mathbb{R}^d$. We have $L_\Gamma \cong \mathbb{Z}^r$ for some $r \in \{0, 1, \dots, d\}$; the integer r is called the *rank* of the lattice L_Γ . We say that Γ is a *pure-translation subgroup* if $\Gamma = T_\Gamma$.

Definition 17. A discrete subgroup $\Gamma \leq \text{Isom}(\mathbb{R}^d)$ is said to be a *d-dimensional crystallographic group* if its lattice of translations has rank d (or, equivalently, if the orbit space \mathbb{R}^d/Γ is compact).

Definition 18. If Γ is a d -dimensional crystallographic group, its *point group* P_Γ is defined by

$$P_\Gamma = \{\sigma \in O(d) : t \circ \sigma \in \Gamma \text{ for some } t \in T(\mathbb{R}^d)\}.$$

Fact 5. If Γ is a d -dimensional crystallographic group, then its point group P_Γ is finite, and $P_\Gamma \cong \Gamma/T_\Gamma$. Hence, Γ is a pure-translation subgroup if and only if $P_\Gamma = \{\text{Id}\}$.

Remark 4. If $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ and $|\mathbb{Z}^d / \Gamma| < \infty$, then Γ is a d -dimensional crystallographic group, viewing $\text{Aut}(\mathbb{L}^d)$ as a subgroup of $\text{Isom}(\mathbb{R}^d)$ in the usual way (see Remark 3). Moreover, in this case, L_Γ is a sublattice of \mathbb{Z}^d (with rank d), and P_Γ is a subgroup of B_d .

We will also need two group-theoretic lemmas. The first is well-known; we provide a proof for completeness.

Lemma 13. *If Γ is a d -dimensional crystallographic group, then $\sigma(L_\Gamma) = L_\Gamma$ for all $\sigma \in P_\Gamma$, i.e. the point group of Γ acts on the lattice of translations of Γ .*

Proof. Let Γ be a d -dimensional crystallographic group and let $\sigma \in P_\Gamma$. Let $a \in L_\Gamma$. Then there exists $\gamma \in T_\Gamma$ such that $a = \gamma(0)$ and there exists $t \in T(\mathbb{R}^d)$ such that $t \circ \sigma \in \Gamma$. Since T_Γ is a normal subgroup of Γ and P_Γ is a group of linear transformations, we have $\sigma \circ \gamma \circ \sigma^{-1} = (t \circ \sigma) \circ \gamma \circ (t \circ \sigma)^{-1} \in T_\Gamma$, and $\sigma(a) = \sigma(\gamma(0)) = (\sigma \circ \gamma \circ \sigma^{-1})(0) \in L_\Gamma$. Hence, P_Γ acts on L_Γ , as required. ■

The next lemma is a straightforward consequence of one of the proofs of Bieberbach’s Third Theorem.

Lemma 14. *For each $d \in \mathbb{N}$, there exists $k = k(d) \in \mathbb{N}$ such that for any rank- d lattice L in \mathbb{R}^d and any group $P \leq O(d)$, there are at most $k(d)$ d -dimensional crystallographic groups $\Gamma \leq \text{Isom}(\mathbb{L}^d)$ with $L_\Gamma = L$ and $P_\Gamma = P$, up to conjugation by translations.*

Proof. We follow the exposition in [37]. Let L be a rank- d lattice in \mathbb{R}^d , and let $P \leq O(d)$. We may assume that there exists at least one d -dimensional crystallographic group with lattice of translations L and point group P , otherwise we are done. By Fact 5, P is a finite group. By Lemma 13, P acts on L , and therefore P acts on the torus \mathbb{R}^d / L , which can be viewed as a P -module.

For each d -dimensional crystallographic group Γ with lattice of translations L and point group P , we define a map $\phi_\Gamma : P \rightarrow \mathbb{R}^d$ as follows. For each $\sigma \in P$, choose a translation $t_c : x \mapsto x + c$ such that $t_c \circ \sigma \in \Gamma$, and define $\phi_\Gamma(\sigma) = c$. Now let $s_\Gamma : P \rightarrow \mathbb{R}^d / L$ be the composition of ϕ_Γ with the natural quotient map $q : \mathbb{R}^d \rightarrow \mathbb{R}^d / L$, $x \mapsto x + L$, i.e., define

$$s_\Gamma(\sigma) = \phi_\Gamma(\sigma) + L \quad \forall \sigma \in P.$$

Observe that the map $s = s_\Gamma$ satisfies the two conditions

$$s(\text{Id}) = 0, \tag{2}$$

$$s(\sigma\tau) = s(\sigma) + \sigma(s(\tau)) \quad \forall \sigma, \tau \in P. \tag{3}$$

In the language of group cohomology, the set of maps $s : P \rightarrow \mathbb{R}^d / L$ satisfying conditions (2) and (3) is termed the *group of 1-cocycles* (of P , with the coefficients in \mathbb{R}^d / L), and is denoted by $Z^1(P, \mathbb{R}^d / L)$. (It is easily checked that the set of all 1-cocycles forms an Abelian group under the operation of pointwise addition.)

Observe also that Γ can be recovered from s_Γ , since

$$\Gamma = \{t_a \circ \sigma : \sigma \in P, a \in s_\Gamma(\sigma)\}.$$

Indeed, clearly $\Gamma \supset \{t_a \circ \sigma : \sigma \in P, a \in s_\Gamma(\sigma)\}$. On the other hand, if $\gamma \in \Gamma$, then $\gamma = t_a \circ \sigma$ for some $\sigma \in P$. Let $c = \phi_\Gamma(\sigma)$. Then we have

$$\Gamma \ni t_a \circ \sigma \circ (t_c \circ \sigma)^{-1} = t_{a-c},$$

so $a - c \in L$ and therefore $a \in c + L = s_\Gamma(\sigma)$. Therefore, $\gamma \in \{t_a \circ \sigma : \sigma \in P, a \in s_\Gamma(\sigma)\}$. It follows that $\Gamma \subset \{t_a \circ \sigma : \sigma \in P, a \in s_\Gamma(\sigma)\}$.

In fact, it is easily checked that for any 1-cocycle s , $\Gamma_s := \{t_a \circ \sigma : \sigma \in P, a \in s(\sigma)\}$ is a d -dimensional crystallographic group with lattice of translations L and point group P . Indeed, the group axioms follow straightforwardly from properties (2) and (3) of s . The group Γ_s is discrete, since for any $x \in \mathbb{R}^d$,

$$\{\gamma(x) : \gamma \in \Gamma_s\} = \bigcup_{\sigma \in P} \{\sigma(x) + \phi(\sigma) + v : v \in L\},$$

which is a discrete subset of \mathbb{R}^d , since $|P| < \infty$ (see Fact 3). Finally, the lattice of translations of Γ_s is L , by property (2) of s . Hence, Γ_s is indeed a d -dimensional crystallographic group with lattice of translations L and point group P .

It follows that the set of d -dimensional crystallographic groups with lattice of translations L and point group P is in one-to-one correspondence with the group of 1-cocycles $Z^1(P, \mathbb{R}^d/L)$.

We now consider the effect on s_Γ of conjugating Γ by a translation $t_a : x \mapsto x + a$, where $a \in \mathbb{R}$. Let $\sigma \in P$, and let $c = \phi_\Gamma(\sigma)$. Then for all $x \in \mathbb{R}^d$, we have

$$(t_a \circ (t_c \circ \sigma) \circ t_a^{-1})(x) = a + c + \sigma(x - a) = a - \sigma(a) + c + \sigma(x) = (t_{a-\sigma(a)+c} \circ \sigma)(x),$$

and therefore

$$t_a \circ (t_c \circ \sigma) \circ t_a^{-1} = t_{c+a-\sigma(a)} \circ \sigma.$$

Passing to the quotient \mathbb{R}^d/L , we see that

$$s_{t_a \Gamma t_a^{-1}}(\sigma) = s_\Gamma(\sigma) + a - \sigma(a) \quad \forall \sigma \in P,$$

i.e. conjugating Γ by t_a changes s_Γ by adding to s_Γ the 1-cocycle s_a , where we define $s_a(\sigma) = a - \sigma(a)$ for all $\sigma \in P$. In the language of group cohomology, the set $\{s_a : a \in \mathbb{R}^d\}$ is termed the *group of 1-coboundaries* (of P , with coefficients in \mathbb{R}^d/L), and is denoted by $B^1(P, \mathbb{R}^d/L)$. (Again, it is clear that the set of 1-coboundaries forms an Abelian group under pointwise addition.) It follows that the set of equivalence classes of d -dimensional crystallographic groups with lattice of translations L and point group P (where two such groups are defined to be equivalent if they are conjugate via a translation by an element of \mathbb{R}^d) is in one-to-one correspondence with the quotient group

$$Z^1(P, \mathbb{R}^d/L)/B^1(P, \mathbb{R}^d/L), \tag{4}$$

which is precisely the 1-dimensional cohomology group of P with coefficients in \mathbb{R}^d/L (denoted $H^1(P, \mathbb{R}^d/L)$). It follows from well-known facts from the theory of group cohomology that the cohomology group (4) is finite. Indeed, for any finite group P , any P -module M and any $n \in \mathbb{N}$, the n th cohomology group of P with coefficients in M , denoted $H^n(P, M)$, is a finitely generated Abelian group with every element of order dividing $|P|$, so is finite (see [18]). Alternatively, see [56, p. 130] for an elementary and direct proof of the finiteness of (4). This completes the proof of the lemma. ■

3.4. The \mathbb{L}^1 case, via the theory of integer partitions

In this section, we use known results from the theory of integer partitions to deduce that the statement of Conjecture 1 holds for $d = 1$, with $r_0(1) = 1$.

Observe that a graph is r -locally \mathbb{L}^1 if and only if it is a vertex-disjoint union of cycles each of length at least $2r + 2$, so choosing an unlabelled n -vertex graph uniformly at random from the set of all unlabelled, n -vertex graphs which are r -locally \mathbb{L}^1 corresponds precisely to choosing a partition of n uniformly at random from the set of all partitions of n with each part of size at least $2r + 2$. (Recall that if $n \in \mathbb{N}$, a partition of n is a non-increasing sequence $(\lambda_1, \dots, \lambda_l)$ of positive integers with $\sum_{i=1}^l \lambda_i = n$. If $\lambda = (\lambda_1, \dots, \lambda_l)$ is a partition of n , the λ_i 's are called the parts of λ .) For $n \in \mathbb{N}$, we let $p(n)$ denote the number of partitions of n . Hardy and Ramanujan [35] proved in 1918 that

$$p(n) = (1 + O(1/\sqrt{n})) \frac{1}{4n\sqrt{3}} \exp(\pi \sqrt{2n/3}). \tag{5}$$

(This was one of the first applications of the celebrated Hardy–Littlewood circle method. It was also proved independently by Uspensky [60] in 1920.) It suffices for us to prove the following.

Proposition 15. *Let $r \in \mathbb{N}$. For each $n \geq 2r + 2$, let λ be a partition chosen uniformly at random from the set of all partitions of n with each part of size at least $2r + 2$. Then with probability at least $1 - O_r(1/n^2)$, the largest part λ_1 of the partition λ satisfies $\lambda_1 = \Theta_r(\sqrt{n} \log n)$.*

Proof. For each $s \in \mathbb{N}$, let $p_{\geq s}(n)$ denote the number of partitions of n with all parts of size at least s . It follows from results of Fristedt [27] that for any $s \in \mathbb{N}$, we have

$$p_{\geq s}(n) = \Theta_s(n^{-(s-1)/2}) p(n). \tag{6}$$

Hence, the probability that a random partition of n has all its parts of size at least $2r + 2$ is

$$\frac{p_{\geq 2r+2}(n)}{p(n)} = \Theta_r(n^{-(2r+1)/2}). \tag{7}$$

On the other hand, for any $s \in \mathbb{N}$, the number of partitions of n with a part of size s is $p(n - s)$. Hence, for any $m \in [n]$, the probability that a uniform random partition of n

has at least one part of size at least m is at most

$$n \frac{p(n-m)}{p(n)}$$

(using the fact that $p(n)$ is a non-decreasing function of n). Choosing $m = C \sqrt{n} \log n$, and using (5) and performing Taylor expansions, we obtain

$$n \frac{p(n-m)}{p(n)} = n \exp\left(-\left(1 + o(1)\right) \frac{\pi}{\sqrt{6}} C \log n\right) = O(1/n^C)$$

if $C \geq 3$. Hence, a uniform random partition of n has largest part of size $O(C \sqrt{n} \log n)$ with probability at least $1 - O(1/n^C)$ provided $C \geq 3$.

Moreover, Erdős and Lehner [23] proved that for any fixed $x \in \mathbb{R}$, if λ is a uniform random partition of n , then its largest part λ_1 satisfies

$$\lambda_1 \leq \frac{\sqrt{n} \log n}{c_0} + x \sqrt{n}$$

with probability

$$(1 + o(1)) \exp\left(-\frac{2}{c_0} \exp\left(-\frac{1}{2} c_0 x\right)\right),$$

where $c_0 := \pi \sqrt{2/3}$. It follows from their proof that the same statement holds when $-\log \log n \leq x \leq 2 \log n$, and therefore

$$\lambda_1 \geq \frac{\sqrt{n} \log n}{c_0} - \sqrt{n} \log \log n$$

with probability at least $1 - O(\exp(-\frac{2}{c_0} (\log n)^{c_0/2}))$. Hence, for all $C \geq 3$, a uniform random partition of n has largest part of size $\Theta_C(\sqrt{n} \log n)$ with probability at least $1 - O(1/n^C)$. By choosing C to be sufficiently large depending on r , it follows from this and (7) that if we choose a partition of n uniformly at random from the set of all partitions with all parts of size at least $2r + 2$, then with probability at least $1 - O_r(1/n^2)$, the largest part has size $\Theta_r(\sqrt{n} \log n)$. ■

The following corollary is immediate.

Corollary 16. *Let $r \in \mathbb{N}$. For each $n \geq 2r + 2$, with probability at least $1 - O_r(1/n^2)$, the largest component of $G_n(\mathbb{L}^1, r)$ has order $\Theta_r(\sqrt{n} \log n)$.*

Remark 5. Observe that using (5), (6) and the correspondence above, we see that for any $r \in \mathbb{N}$, the number $a_{1,r}(n)$ of unlabelled, n -vertex graphs that are r -locally \mathbb{L}^1 satisfies

$$a_{1,r}(n) = p_{\geq 2r+2}(n) = \Theta_r(n^{-(2r+1)/2}) p(n) = \Theta_r(n^{-r}) \exp(\pi \sqrt{2n/3}),$$

so

$$\log a_{1,r}(n) = \pi \sqrt{2n/3} - r \log n + O_r(1).$$

(Note that this is a much more precise estimate than in Theorem 6.)

4. Proof of Theorem 6

In this section, we prove Theorem 6, our enumeration result for graphs that are r -locally \mathbb{L}^d , using the strategy sketched in the Introduction.

We first show that the connected unlabelled n -vertex graphs that are r -locally \mathbb{L}^d are in one-to-one correspondence with the conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ such that $|\mathbb{Z}^d/\Gamma| = n$ and $D(\Gamma) \geq 2r + 2$, provided $r \geq 2 + 1_{\{d \geq 3\}}$.

We recall the following straightforward result, proved in [8].

Proposition 17. *Let $d \in \mathbb{N}$ with $d \geq 2$, and let G be a connected graph that is 2-locally \mathbb{L}^d . Let $v_0 \in V(G)$, and let $\psi : \text{Link}_2(0, \mathbb{L}^d) \rightarrow \text{Link}_2(v_0, G)$ be a graph isomorphism with $\psi(0) = v_0$. Then there is at most one covering map p from \mathbb{L}^d to G such that $p(x) = \psi(x)$ for all $x \in N(0)$.*

(We note that this was stated in [8] under a slightly stronger hypothesis, but it is clear that the proof in [8] relies only upon G being 2-locally \mathbb{L}^d .) We also need the following lemmas.

Lemma 18. *Suppose G_1 and G_2 are connected graphs which are 2-locally \mathbb{L}^2 . Suppose $p_1 : \mathbb{Z}^2 \rightarrow V(G_1)$ is a normal covering map from \mathbb{L}^2 to G_1 , and $p_2 : \mathbb{Z}^2 \rightarrow V(G_2)$ is a normal covering map from \mathbb{L}^2 to G_2 . Then G_1 is isomorphic to G_2 if and only if the subgroups $\text{CT}(p_1)$ and $\text{CT}(p_2)$ are conjugate in $\text{Aut}(\mathbb{L}^2)$.*

Proof. Suppose that $G_1 \cong G_2$. Let $f : V(G_2) \rightarrow V(G_1)$ be a graph isomorphism from G_2 to G_1 . Then $p := f \circ p_2 : \mathbb{Z}^2 \rightarrow V(G_1)$ is a normal covering map from \mathbb{L}^2 to G_1 . Choose $\phi \in \text{Aut}(\mathbb{L}^2)$ such that $p_1 \circ \phi$ and p agree with one another on $N(0)$. Then, by Proposition 17, we have $p_1 \circ \phi = p$. Hence, $\text{CT}(p_2) = \text{CT}(f \circ p_2) = \text{CT}(p) = \text{CT}(p_1 \circ \phi) = \phi^{-1} \text{CT}(p_1)\phi$.

Conversely, suppose $\text{CT}(p_2) = \phi^{-1} \text{CT}(p_1)\phi$ for some $\phi \in \text{Aut}(\mathbb{L}^2)$. Define $p = p_1 \circ \phi$. Then $p : \mathbb{Z}^2 \rightarrow V(G_1)$ is a normal covering map from \mathbb{L}^2 to G_1 , and $\text{CT}(p) = \phi^{-1} \text{CT}(p_1)\phi = \text{CT}(p_2)$. But by Lemma 12, $G_1 \cong \mathbb{L}^2/\text{CT}(p)$ and $G_2 \cong \mathbb{L}^2/\text{CT}(p_2) = \mathbb{L}^2/\text{CT}(p)$, so $G_1 \cong G_2$. ■

In exactly the same way, we may prove the following.

Lemma 19. *Let $d \in \mathbb{N}$ with $d \geq 2$. Suppose G_1 and G_2 are connected graphs which are 3-locally \mathbb{L}^d . Suppose $p_1 : \mathbb{Z}^d \rightarrow V(G_1)$ is a normal covering map from \mathbb{L}^d to G_1 , and $p_2 : \mathbb{Z}^d \rightarrow V(G_2)$ is a normal covering map from \mathbb{L}^d to G_2 . Then G_1 is isomorphic to G_2 if and only if the subgroups $\text{CT}(p_1)$ and $\text{CT}(p_2)$ are conjugate in $\text{Aut}(\mathbb{L}^d)$.*

Clearly, a subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with minimum displacement at least 2 acts freely on \mathbb{L}^d . Hence, from Theorem 4 and Lemmas 12 and 18, we may deduce the following.

Corollary 20. *If $r \in \mathbb{N}$ with $r \geq 2$, then the unlabelled, connected, n -vertex graphs which are r -locally \mathbb{L}^2 are in an explicit one-to-one correspondence with the conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^2)$ with minimum displacement at least $2r + 2$ and with*

$|\mathbb{Z}^2/\Gamma| = n$; this correspondence is given by simply taking the quotient graph of \mathbb{L}^2 by Γ , i.e., $\Gamma \leftrightarrow \mathbb{L}^2/\Gamma$.

Similarly, using Lemma 19 in place of Lemma 18, we may deduce the following.

Corollary 21. *If $r, d \in \mathbb{N}$ with $r, d \geq 3$, then the unlabelled, connected, n -vertex graphs which are r -locally \mathbb{L}^d are in an explicit one-to-one correspondence with the conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with minimum displacement at least $2r + 2$ and with $|\mathbb{Z}^d/\Gamma| = n$; this correspondence is given by taking the quotient graph of \mathbb{L}^d by Γ , i.e., $\Gamma \leftrightarrow \mathbb{L}^d/\Gamma$.*

Our next aim is to deal with the subgroups Γ (as in Corollaries 20 and 21) that are *pure-translation subgroups*. Clearly, a pure-translation subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ is determined by its lattice of translations L_Γ , which must be a sublattice of \mathbb{Z}^d . We now recall some definitions and basic facts concerning sublattices of \mathbb{Z}^d .

A sublattice of \mathbb{Z}^d is a subgroup L of \mathbb{Z}^d ; such a subgroup must be isomorphic (as a group) to \mathbb{Z}^R for some $R \leq d$, and the integer R is called the *rank* of L . If L is a sublattice of \mathbb{Z}^d of rank d , then \mathbb{Z}^d/L is finite, and $|\mathbb{Z}^d/L|$ is called the *index* of L ; if L has rank less than d , then its index is infinite. The *minimum distance* of L is defined to be $\min \{\|x\|_1 : x \in L \setminus \{0\}\}$. If L is a sublattice of \mathbb{Z}^d and $\sigma \in \text{Aut}(\mathbb{L}^d)$, we say that L is σ -invariant if $\sigma(L) = L$.

It is well-known (and easy to see) that there is a one-to-one correspondence between sublattices of \mathbb{Z}^d with index n , and upper-triangular, integer matrices $B = (b_{ij})_{i,j \in [d]}$ with $0 \leq b_{ij} < b_{ii}$ for all $j > i$ and all $i \in [d]$, and with $\prod_{i=1}^d b_{ii} = n$. The matrix B corresponds to the sublattice for which a \mathbb{Z} -basis is the set of columns of B , i.e. the set

$$\left\{ \sum_{i=1}^j b_{ij} e_i : j \in [d] \right\}. \tag{8}$$

It follows that the number of sublattices of \mathbb{Z}^d of index n is

$$\sum_{\substack{c_1, \dots, c_d \in \mathbb{N}: \\ c_1 \dots c_d = n}} c_1^{d-1} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1}.$$

Hence, the number of sublattices of \mathbb{Z}^d of index at most x is

$$\sum_{c_1 \dots c_d \leq x} c_1^{d-1} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1} =: N_d(x).$$

It is well-known that

$$N_d(x) = (1 + O((\log x)/x)) \frac{x^d}{d} \prod_{i=2}^d \zeta(i) \tag{9}$$

(for an elementary proof of this, see for example [29]). We will use this estimate in what follows.

Observe that any pure-translation subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ is invariant under conjugation by any translation. Hence, two pure-translation subgroups $\Gamma, \Gamma' \leq \text{Aut}(\mathbb{L}^d)$ are conjugate in $\text{Aut}(\mathbb{L}^d)$ if and only if they are conjugate via an element of the hyperoctahedral group B_d . Note that if Γ and Γ' are pure-translation subgroups of $\text{Aut}(\mathbb{L}^d)$, and $\sigma \in B_d$, then $\sigma\Gamma\sigma^{-1} = \Gamma'$ if and only if $\sigma(L_\Gamma) = L_{\Gamma'}$. Hence, we obtain the following.

Fact 6. The number of conjugacy-classes of pure-translation subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| = n$ is precisely the number of B_d -orbits of sublattices L of \mathbb{Z}^d with $|\mathbb{Z}^d/L| = n$.

Notice that any sublattice of \mathbb{Z}^d is invariant under both Id and $-\text{Id}$. Our next result says that for any other element $\sigma \in B_d \setminus \{\pm\text{Id}\}$, a uniform random sublattice of \mathbb{Z}^d with index at most x is very unlikely to be invariant under σ .

Lemma 22. Let $\sigma \in B_d \setminus \{\pm\text{Id}\}$. Let $N_{d,\sigma}(n)$ denote the number of index- n sublattices of \mathbb{Z}^d which are invariant under σ . Then for $x > 1$, we have

$$\sum_{n \leq x} N_{d,\sigma}(n) \leq O_d(1)x^{d-1+O(1/\log \log x)}.$$

Proof. For each $i \in [d]$, let $\ell_i = \{e_i, -e_i\}$, and let $P_d = \{\ell_i : i \in [d]\}$. Observe that B_d acts on P_d . Let $\sigma \in B_d \setminus \{\pm\text{Id}\}$. Suppose firstly that σ acts trivially on P_d . Then, since $\sigma \neq \pm\text{Id}$, there exist $i, j \in [d]$ such that $\sigma(e_i) = -e_i$ and $\sigma(e_j) = e_j$. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Let L be a σ -invariant sublattice of \mathbb{Z}^d with index at most x . By the linearity of σ , the matrix B corresponding to L satisfies

$$-b_{12}e_1 + b_{22}e_2 \in \langle B \rangle,$$

where $\langle B \rangle$ denotes the \mathbb{Z} -linear span of the columns of B . This condition holds only if there exist $\lambda_1, \lambda_2 \in \mathbb{Z}$ such that

$$-b_{12}e_1 + b_{22}e_2 = \lambda_1 b_{11}e_1 + \lambda_2 (b_{12}e_1 + b_{22}e_2),$$

which implies $\lambda_2 = 1$ and $\lambda_1 b_{11} = 2b_{12}$. Since $b_{12} < b_{11}$, this condition holds only if $\lambda_1 \in \{0, -1\}$, i.e. only if $b_{12} = 0$ or $b_{12} = b_{11}/2$. Therefore,

$$N_{d,\sigma}(n) \leq 2 \sum_{c_1 \dots c_d = n} c_1^{d-2} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1},$$

so

$$\begin{aligned} \sum_{n \leq x} N_{d,\sigma}(n) &\leq 2 \sum_{c_1 c_2 \dots c_d \leq x} c_1^{d-2} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1} \\ &\leq 2 \sum_{c_2 c_3 \dots c_d \leq x} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1} \sum_{c_1 \leq x/(c_2 c_3 \dots c_d)} c_1^{d-2} \\ &\leq 2 \sum_{c_2 c_3 \dots c_d \leq x} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1} (x/(c_2 c_3 \dots c_d))^{d-1} \end{aligned}$$

$$\begin{aligned}
 &= 2x^{d-1} \sum_{c_2 c_3 \dots c_d \leq x} c_2^{-1} c_3^{-2} \dots c_{d-1}^{-(d-2)} c_d^{-(d-1)} \\
 &\leq 2x^{d-1} \left(\sum_{c_2 \leq x} c_2^{-1} \right) \left(\sum_{c_3 \leq x} c_3^{-2} \right) \dots \left(\sum_{c_d \leq x} c_d^{-(d-1)} \right) \\
 &= O(x^{d-1} \log x).
 \end{aligned}$$

Now suppose instead that σ does not act trivially on P_d . Then there exists $i \in [d]$ such that $\sigma(\ell_i) \neq \ell_i$. Without loss of generality, we may assume that $\sigma(\ell_1) = \ell_2$. Hence, $\sigma(e_1) = \pm e_2$. Since $\sigma(L) = L$ if and only if $(-\sigma)(L) = L$, by considering $-\sigma$ if necessary, we may assume that $\sigma(e_1) = e_2$. Then $\sigma(e_2) \neq \pm e_2$. There are two cases.

Case 1: $\sigma(e_2) = \pm e_1$.

Case 2: $\sigma(e_2) = \pm e_j$ for some $j \geq 3$.

First suppose that Case 1 occurs. Then, if $\sigma(L) = L$, the corresponding matrix B satisfies

$$b_{12}e_2 \pm b_{22}e_1 \in \langle B \rangle.$$

This condition holds if and only if there exist $\lambda_1, \lambda_2 \in \mathbb{Z}$ such that

$$b_{12}e_2 \pm b_{22}e_1 = \lambda_1 b_{11}e_1 + \lambda_2 (b_{12}e_1 + b_{22}e_2),$$

which implies

$$\pm b_{22} = \lambda_1 b_{11} + \lambda_2 b_{12}, \quad b_{12} = \lambda_2 b_{22},$$

which implies $\lambda_1 b_{11} = \pm b_{22} - b_{12}^2/b_{22}$, which implies

$$b_{11} \mid \pm b_{22} - b_{12}^2/b_{22}.$$

Once we have chosen $b_{d,d}, b_{d-1,d-1}, \dots, b_{22}$, there are at most $x/(b_{22}b_{33} \dots b_{dd})$ choices for each of $b_{12}, \dots, b_{1,d}$ (since we must have $b_{1,j} \leq x/(b_{22}b_{33} \dots b_{dd})$ for each $j \geq 2$), and then there are at most $x^{O(1/\log \log x)}$ choices for b_{11} , since

$$b_{11} \mid \pm b_{22} - b_{12}^2/b_{22}, \quad |\pm b_{22} - b_{12}^2/b_{22}| \leq x^2 + 1 \leq 2x^2,$$

and the divisor function $\tau = \tau(m)$ (= number of divisors of m) satisfies

$$\tau(m) \leq m^{O(1/\log \log m)}.$$

Hence,

$$\begin{aligned}
 \sum_{n \leq x} N_{d,\sigma}(n) &\leq \sum_{c_2 \dots c_d \leq x} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1} (x/(c_2 c_3 \dots c_d))^{d-1} x^{O(1/\log \log x)} \\
 &= x^{d-1+O(1/\log \log x)} \sum_{c_2 c_3 \dots c_d \leq x} c_2^{-1} c_3^{-2} \dots c_{d-1}^{-(d-2)} c_d^{-(d-1)} \\
 &\leq O_d(1) x^{d-1+O(1/\log \log x)} \log x \leq O_d(1) x^{d-1+O(1/\log \log x)}.
 \end{aligned}$$

Now assume that Case 2 occurs. Without loss of generality, we may assume that $j = 3$. Then, if $\sigma(L) = L$, the corresponding matrix B satisfies

$$b_{12}e_2 \pm b_{22}e_3 \in \langle B \rangle.$$

This condition holds if and only if there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$ such that

$$b_{12}e_2 \pm b_{22}e_3 = \lambda_1 b_{11}e_1 + \lambda_2(b_{12}e_1 + b_{22}e_2) + \lambda_3(b_{13}e_1 + b_{23}e_2 + b_{33}e_3),$$

which implies

$$0 = \lambda_1 b_{11} + \lambda_2 b_{12} + \lambda_3 b_{13}, \quad b_{12} = \lambda_2 b_{22} + \lambda_3 b_{23}, \quad \pm b_{22} = \lambda_3 b_{33},$$

which implies

$$\lambda_1 b_{11} = -b_{12}(b_{12} - (\pm b_{22}/b_{33})b_{23})/b_{22} - b_{13}(\pm b_{22}/b_{33}),$$

which implies

$$b_{11} \mid -b_{12}(b_{12} - (\pm b_{22}/b_{33})b_{23})/b_{22} - b_{13}(\pm b_{22}/b_{33}).$$

Note that

$$|-b_{12}(b_{12} - (\pm b_{22}/b_{33})b_{23})/b_{22} - b_{13}(\pm b_{22}/b_{33})| \leq 2x^2.$$

Hence, exactly as before, once we have chosen $b_{dd}, b_{d-1,d-1}, \dots, b_{22}$, there are at most $x/(b_{22}b_{33} \dots b_{dd})$ choices for each of $b_{12}, \dots, b_{1,d}$, and then there are at most $x^{O(1/\log \log x)}$ choices for b_{11} . Therefore, by the same calculation as before, we have

$$\sum_{n \leq x} N_{d,\sigma}(n) \leq O_d(1)x^{d-1+O(1/\log \log x)}. \quad \blacksquare$$

Lemma 22 may be restated in the following appealing way.

Corollary 23. *Let $\sigma \in B_d \setminus \{\pm \text{Id}\}$ and let $x > 1$. If L is a lattice chosen uniformly at random from all sublattices of \mathbb{Z}^d of index at most x , then*

$$\text{Prob}\{\sigma(L) = L\} = O_d(x^{-1+O(1/\log \log x)}).$$

We can now estimate the number of B_d -orbits of sublattices of \mathbb{Z}^d with index at most x . Let us write $\mathcal{L}_{\leq x}^d$ for the set of sublattices of \mathbb{Z}^d with index at most x (so that $|\mathcal{L}_{\leq x}^d| = N_d(x)$), and let us write $\tilde{\mathcal{L}}_{\leq x}^d$ for the set of B_d -orbits of sublattices of \mathbb{Z}^d with index at most x .

Proposition 24. *The number $|\tilde{\mathcal{L}}_{\leq x}^d|$ of B_d -orbits of sublattices of \mathbb{Z}^d with index at most x (equivalently, the number of conjugacy-classes of pure-translation subgroups Γ of $\text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$) satisfies*

$$|\tilde{\mathcal{L}}_{\leq x}^d| = (1 + O_d(x^{-1+O(1/\log \log x)}))c_d x^d, \quad \text{where } c_d := \frac{\prod_{i=2}^d \zeta(i)}{2^{d-1}d!d}.$$

Proof. Any sublattice L of \mathbb{Z}^d is invariant under Id and $-\text{Id}$, but for any $\sigma \in B_d \setminus \{\pm\text{Id}\}$, the number of sublattices of \mathbb{Z}^d of index at most x which are invariant under σ is at most $O_d(x^{d-1+O(1/\log \log x)})$, by Lemma 22. Since

$$|B_d| = 2^d d! = O_d(1),$$

it follows that all but at most $O_d(x^{d-1+O(1/\log \log x)})$ of the sublattices of index at most x are not invariant under any element of $B_d \setminus \{\pm\text{Id}\}$. For any such sublattice L ,

$$\sigma(L) = \sigma'(L) \iff \sigma^{-1}\sigma' \in \{\pm\text{Id}\},$$

so there are precisely $2^{d-1}d!$ distinct lattices $L' \in \mathcal{L}_{\leq x}^d$ which are in the same B_d -orbit as L (one for each left coset of $\{\pm\text{Id}\}$ in B_d). For any lattice $L \in \mathcal{L}_{\leq x}^d$, there are at most $2^{d-1}d!$ distinct lattices in the same B_d -orbit as L . Hence, the total number $|\tilde{\mathcal{L}}_{\leq x}^d|$ of B_d -orbits in $\mathcal{L}_{\leq x}^d$ satisfies

$$\begin{aligned} \frac{1}{2^{d-1}d!} |\mathcal{L}_{\leq x}^d| &\leq |\tilde{\mathcal{L}}_{\leq x}^d| \leq \frac{1}{2^{d-1}d!} (1 - O_d(x^{-1+O(1/\log \log x)})) |\mathcal{L}_{\leq x}^d| \\ &\quad + O_d(x^{-1+O(1/\log \log x)}) |\mathcal{L}_{\leq x}^d|, \end{aligned}$$

so

$$\begin{aligned} |\tilde{\mathcal{L}}_{\leq x}^d| &= \frac{1}{2^{d-1}d!} (1 + O_d(x^{-1+O(1/\log \log x)})) |\mathcal{L}_{\leq x}^d| \\ &= (1 + O_d(x^{-1+O(1/\log \log x)})) \frac{\prod_{i=2}^d \zeta(i)}{2^{d-1}d!d} x^d \\ &= (1 + O_d(x^{-1+O(1/\log \log x)})) c_d x^d, \end{aligned}$$

using (9). By Fact 6, $|\tilde{\mathcal{L}}_{\leq x}^d|$ is precisely the number of conjugacy-classes of pure-translation subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$. ■

Our next aim is to show that for fixed $r \in \mathbb{N}$, very few sublattices of \mathbb{Z}^d of index at most x have minimum distance at most r .

Lemma 25. *If $d, r \in \mathbb{N}$ with $d \geq 2$, then the number of sublattices of \mathbb{Z}^d with index at most x and minimum distance at most r is*

$$O_{d,r}(x^{d-1} \log x).$$

Proof. Fix an element $u \in \mathbb{Z}^d$ with $0 < \|u\|_1 \leq r$. We shall bound the number of sublattices of \mathbb{Z}^d with index at most x which contain u . Note that $u = (u^{(1)}, \dots, u^{(d)})$ has at most r non-zero coordinates. By symmetry, we may assume that $u^{(i)} = 0$ for all $i > r$. If a lattice L (with index at most x) contains u , then the corresponding matrix B has $u \in \langle B \rangle$, so there exist $\lambda_1, \dots, \lambda_r \in \mathbb{Z}$ such that

$$u = \sum_{j=1}^r \lambda_j \sum_{i=1}^j b_{ij} e_i.$$

Equating i -coordinates, we obtain

$$u^{(i)} = \sum_{j=i}^r \lambda_j b_{ij} \quad (1 \leq i \leq r).$$

Therefore,

$$\left| \sum_{j=i}^r \lambda_j b_{ij} \right| = |u^{(i)}| \leq r \quad (1 \leq i \leq r).$$

In particular, $|\lambda_r b_{rr}| \leq r$, so $|\lambda_r| \leq r$ (since $b_{rr} \geq 1$). We make the following

Claim 1. We have $|\lambda_{r-k}| \leq 2^{k-1}r$ for all $k \geq 1$.

Proof of claim. We use induction on k . For $k = 1$, we have

$$|\lambda_{r-1} b_{r-1,r-1} + \lambda_r b_{r,r-1}| = |u^{(r-1)}| \leq r,$$

so

$$|\lambda_{r-1}| b_{r-1,r-1} \leq r + |\lambda_r| b_{r,r-1} \leq r + r(b_{r-1,r-1} - 1) = r b_{r-1,r-1},$$

so $|\lambda_{r-1}| \leq r$ (since $b_{r-1,r-1} \geq 1$), as needed. For the induction step, suppose that $k \geq 2$ and $|\lambda_{r-l}| \leq 2^{l-1}r$ for all $l < k$. We have

$$\left| \sum_{j=r-k}^r \lambda_j b_{ij} \right| = |u^{(r-k)}| \leq r,$$

so

$$\begin{aligned} |\lambda_{r-k}| b_{r-k,r-k} &\leq r + \sum_{j=r-k+1}^r |\lambda_j| b_{j,r-k} \leq r + \left(\sum_{j=r-k+1}^r |\lambda_j| \right) (b_{r-k,r-k} - 1) \\ &\leq r + (1 + 1 + 2 + \dots + 2^{k-2})r (b_{r-k,r-k} - 1) \\ &= r + 2^{k-1}r (b_{r-k,r-k} - 1) \leq 2^{k-1}r b_{r-k,r-k}, \end{aligned}$$

so $|\lambda_{r-k}| \leq 2^{k-1}r$, as required. ■

It follows that there are at most

$$(2r + 1) \prod_{k=1}^{r-1} (2^k r + 1) = O_r(1)$$

choices for $(\lambda_1, \dots, \lambda_r)$. Fix one such choice. Since $u \neq 0$, not all of $\lambda_1, \dots, \lambda_r$ are zero.

Let

$$J = \{j : \lambda_j \neq 0\} = \{j_1, \dots, j_s\}.$$

Then

$$\left| \sum_{l=1}^s \lambda_{j_l} b_{1j_l} \right| \leq r. \tag{10}$$

Observe that once we have chosen $b_{dd}, b_{d-1,d-1}, \dots, b_{22}$, there are at most $x/(b_{22}b_{33} \dots b_{dd})$ choices for each b_{1j} with $j \neq j_1$. Once we have chosen each b_{1j} with $j \neq j_1, b_{1j_1}$ is determined by the equation

$$u^{(1)} = \sum_{j=1}^r \lambda_j b_{1j}.$$

Hence, summing over all the $O_r(1)$ possible choices of $(\lambda_1, \dots, \lambda_r)$, the number of sublattices of \mathbb{Z}^d with index at most x and containing u is at most

$$\begin{aligned} O_r(1) \sum_{c_2 \dots c_d \leq x} c_2^{d-2} c_3^{d-3} \dots c_{d-2}^2 c_{d-1} (x/(c_2 c_3 \dots c_d))^{d-1} \\ = O_r(1) x^{d-1} \sum_{c_2 \dots c_d \leq x} c_2^{-1} c_3^{-2} \dots c_{d-1}^{-(d-2)} c_d^{-(d-1)} \leq O_{d,r}(1) x^{d-1} \log x. \end{aligned}$$

Crudely, the number of choices for u is at most

$$\sum_{i=1}^r \binom{d}{i} (2r + 1)^i = O_{d,r}(1),$$

since u has at most r non-zero coordinates, and each of these has modulus at most r . Hence, the total number of sublattices of \mathbb{Z}^d with index at most x and minimum distance at most r is $O_{d,r}(1)x^{d-1} \log x$, as required. ■

Our aim is now to show that there are very few conjugacy-classes of subgroups Γ of $\text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$ and $D(\Gamma) \geq 2r^*(d) + 2$, and which are *not* conjugacy-classes of pure-translation subgroups. (See Theorem 6 for the definition of $r^*(d)$.)

We need the following simple claim.

Claim 2. *If $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $D(\Gamma) \geq d + 1$, then $-\text{Id} \notin P_\Gamma$.*

Proof. Let $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ and suppose that $-\text{Id} \in P_\Gamma$. Then $\gamma : x \mapsto v - x \in \Gamma$ for some $v \in \mathbb{Z}^d$. Define $w_i = \lfloor v_i/2 \rfloor$ for each $i \in [d]$. Then $d_{\mathbb{L}^d}(w, \gamma(w)) \leq d$, so $D(\Gamma) \leq d$. ■

By Corollaries 20 and 21, for each $d \geq 2$, the finite, connected graphs that are r -locally \mathbb{L}^d (where $r \geq 2$ if $d = 2$ and $r \geq 3$ otherwise) correspond to subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $D(\Gamma) \geq 2r + 2$. If $r \geq r^*(d)$, where $r^*(d)$ is defined as in Theorem 6, then we have $D(\Gamma) \geq 2r + 2 \geq d + 1$. Hence, all the subgroups Γ relevant to us have $-\text{Id} \notin P_\Gamma$. We remark that this is the only place where we rely upon $r^*(d)$ growing linearly with d ; the rest of our proof works with $r_0(d) := 2 + 1_{\{d \geq 3\}}$ in place of $r^*(d)$. The subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| < \infty$ and $P_\Gamma = \{\text{Id}, -\text{Id}\}$ are precisely those whose elements are translations and involutions of the form $x \mapsto v - x$; in this case, \mathbb{L}^d/Γ is r -locally \mathbb{L}^d if and only if the lattice of translations of Γ has minimum distance at least $2r + 2$, and whenever $(x \mapsto x - v) \in \Gamma$, v has at least $2r + 2$ odd components. The number of conjugacy-classes of such subgroups with $|\mathbb{Z}^d/\Gamma| \leq x$ is $\Theta(x^d)$, as in the pure-translation case, but the enumeration of conjugacy-classes of these subgroups

(to the required degree of precision) is rather long-winded, so for brevity and clarity of exposition we prefer to rule them out by taking $r^*(d) \geq \lceil (d - 1)/2 \rceil$.

For brevity, let us say that a subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| < \infty$ is *highly symmetric* if $P_\Gamma \setminus \{\text{Id}, -\text{Id}\} \neq \emptyset$. Recall that Γ is a pure-translation subgroup if and only if $P_\Gamma = \{\text{Id}\}$. Hence, all the subgroups Γ relevant to us either are pure-translation subgroups or are highly symmetric.

Our next lemma says that if two subgroups of $\text{Aut}(\mathbb{L}^d)$ are conjugate via a translation by a vector in \mathbb{R}^d , then this vector can be taken to have entries in $(\mathbb{Z} \cup (\mathbb{Z} + 1/2))^d$, implying that when we restrict to subgroups of $\text{Aut}(\mathbb{L}^d)$, the equivalence classes supplied by Lemma 14 (where two crystallographic groups are ‘equivalent’ if they are conjugate via a translation by a vector in \mathbb{R}^d) split into at most 2^d conjugacy-classes in $\text{Aut}(\mathbb{L}^d)$.

Claim 3. *Let $\Gamma_1, \Gamma_2 \leq \text{Aut}(\mathbb{L}^d)$ with $\Gamma_2 = t\Gamma_1t^{-1}$ for some translation t by a vector in \mathbb{R}^d . Then there exists $b \in \mathbb{R}^d$ such that $2b \in \mathbb{Z}^d$ and $\Gamma_2 = t_b\Gamma_1t_{-b}$, where $t_b : x \mapsto x + b$ denotes translation by b .*

Proof. Let $t : x \mapsto x + a$. For any vector $v \in \mathbb{R}^d$, let $t_v : x \mapsto x + v$ denote translation by v . Choose $b \in \mathbb{R}^d$ such that $2b \in \mathbb{Z}^d$, $|b_i - a_i| \leq 1/4$ for all $i \in [d]$, and $b_i \in \mathbb{Z}$ if $a_i \pm 1/4 \in \mathbb{Z}$. Let $\Gamma_3 = t_b\Gamma_1t_{-b} = t_{b-a}\Gamma_2t_{-(b-a)}$. We first observe that $\Gamma_3 \leq \text{Aut}(\mathbb{L}^d)$. Indeed, let $\gamma \in \Gamma_1$; it suffices to prove that $t_b\gamma t_{-b} \in \text{Aut}(\mathbb{L}^d)$. Write $\gamma(x) = Ax + c$, where $A \in B_d$ and $c \in \mathbb{Z}^d$. Since $t_b\gamma t_{-b}$ is an isometry of \mathbb{R}^d , it suffices to prove that $(t_b\gamma t_{-b})(x) \in \mathbb{Z}^d$ for all $x \in \mathbb{Z}^d$, i.e.

$$b + Ax - Ab + c \in \mathbb{Z}^d \quad \forall x \in \mathbb{Z}^d.$$

This is true if and only if $b - Ab \in \mathbb{Z}^d$. Note that $\Gamma_2 = t_a\Gamma_1t_{-a} \leq \text{Aut}(\mathbb{L}^d)$, and therefore $a - Aa \in \mathbb{Z}^d$. Since $A \in B_d$, for each $i \in [d]$ there exist $j \in [d]$ and $\delta \in \{-1, 1\}$ such that $(Ax)_i = \delta x_j$ for all $x \in \mathbb{Z}^d$. We have $(a - Aa)_i = a_i - \delta a_j \in \mathbb{Z}$ (since $a - Aa \in \mathbb{Z}^d$), and therefore, by our choice of b , we have $(b - Ab)_i = b_i - \delta b_j \in \mathbb{Z}$. Hence, $b - Ab \in \mathbb{Z}^d$, as required.

It suffices now to prove that $\Gamma_3 = \Gamma_2$. Let $x \in \mathbb{Z}^d$ and $\gamma \in \Gamma_2$. Write $\gamma(x) = Ax + c$ where $A \in B_d$ and $c \in \mathbb{Z}^d$, and define $\varepsilon = b - a \in \mathbb{R}^d$. Define

$$w = (t_{b-a}\gamma t_{-(b-a)})(x) - \gamma(x) = (t_\varepsilon\gamma t_{-\varepsilon})(x) - \gamma(x).$$

Since $t_{b-a}\Gamma_2t_{-(b-a)} = \Gamma_3 \leq \text{Aut}(\mathbb{L}^d)$, we have $w \in \mathbb{Z}^3$. Moreover,

$$w = (t_\varepsilon\gamma t_{-\varepsilon})(x) - \gamma(x) = \varepsilon + A(x - \varepsilon) + c - Ax - c = \varepsilon - A\varepsilon.$$

For each $i \in [d]$, we have

$$|w_i| = |(\varepsilon - A\varepsilon)_i| = |\varepsilon_i - (A\varepsilon)_i| \leq |\varepsilon_i| + |(A\varepsilon)_i| = |\varepsilon_i| + |\varepsilon_j|$$

for some $j \in [d]$. By our choice of b , we have $|\varepsilon_k| \leq 1/4$ for all $k \in [d]$, and therefore $|w_i| \leq 1/2$ for all $i \in [d]$. Since $w \in \mathbb{Z}^d$, it follows that $w = 0$. Hence, $t_{b-a}\gamma t_{-(b-a)} = \gamma$ for all $\gamma \in \Gamma_2$, and therefore $\Gamma_3 = \Gamma_2$, as required. ■

Our next result says that there are very few conjugacy-classes of highly symmetric subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$ (compared to the number of conjugacy-classes of pure-translation subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$).

Corollary 26. *Let $d \in \mathbb{N}$ with $d \geq 2$. The number of conjugacy-classes of highly symmetric subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$ is at most $O_d(1)x^{d-1+O(1/\log \log x)}$.*

Proof. Let $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$. Then, since $\Gamma/T_\Gamma \cong P_\Gamma$, we have

$$|\mathbb{Z}^d/L_\Gamma| = |\mathbb{Z}^d/T_\Gamma| \leq |P_\Gamma| |\mathbb{Z}^d/\Gamma| \leq 2^d d! x = O_d(x). \tag{11}$$

(We note that this holds whether or not Γ is highly symmetric.) Now assume in addition that Γ is highly symmetric. Then L_Γ is invariant under some element of $B_d \setminus \{\text{Id}, -\text{Id}\}$, so by Lemma 22, L_Γ there are at most $O_d(1)x^{d-1+O(1/\log \log x)}$ possibilities for L_Γ . Trivially, since $P_\Gamma \leq B_d$, there are at most $2^{|B_d|} = 2^{d!2^d}$ possibilities for P_Γ . By Lemma 14, for any fixed sublattice L of \mathbb{Z}^d and any fixed $P \leq B_d$, there are at most $k(d)$ subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $L_\Gamma = L$ and $P_\Gamma = P$, up to conjugation by translations (by vectors in \mathbb{R}^d). If $\Gamma_1, \Gamma_2 \leq \text{Aut}(\mathbb{L}^d)$ and $\Gamma_2 = t\Gamma_1t^{-1}$ for some translation $t : x \mapsto x + c \in T(\mathbb{R}^d)$, we need not have $c \in \mathbb{Z}^d$, but by Claim 3, we may assume that $2c \in \mathbb{Z}^d$. Hence, up to conjugation by translations in $T(\mathbb{Z}^d)$, there are at most $2^d k(d)$ subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $L_\Gamma = L$ and $P_\Gamma = P$. Hence, there are at most

$$O_d(1)x^{d-1+O(1/\log \log x)} \cdot 2^{d!2^d} \cdot 2^d \cdot k(d) = O_d(1)x^{d-1+O(1/\log \log x)}$$

possibilities for the conjugacy-class of Γ in $\text{Aut}(\mathbb{L}^d)$, as required. ■

Putting everything together, we obtain the following two lemmas.

Lemma 27. *Let $r \geq 2$, and let $\gamma_{2,r}(n)$ denote the number of connected, unlabelled, n -vertex graphs which are r -locally \mathbb{L}^2 . Then*

$$\sum_{n \leq x} \gamma_{2,r}(n) = (1 + O_r(x^{-1+O(1/\log \log x)})) \frac{1}{4} \zeta(2) x^2.$$

Proof. The left-hand side is precisely the number of connected, unlabelled graphs on at most x vertices that are r -locally \mathbb{L}^2 . By Corollary 20, this is precisely the number of conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^2)$ which have $|\mathbb{Z}^2/\Gamma| \leq x$ and which have minimum displacement at least $2r + 2$. If Γ has minimum displacement at least 4, then by Claim 2, $-\text{Id} \notin P_\Gamma$, so either Γ is a pure-translation subgroup, or else Γ is highly symmetric. Applying Proposition 24, Lemma 25 and Corollary 26 in the case $d = 2$, we see that the aforesaid number is

$$|\tilde{\mathcal{L}}_{\leq x}^2| - O_r(x \log x) + O(x^{1+O(1/\log \log x)}) = (1 + O_r(x^{-1+O(1/\log \log x)})) \frac{1}{4} \zeta(2) x^2,$$

as required. ■

Similarly, we obtain the following.

Lemma 28. *Let $d, r \in \mathbb{N}$ with $d \geq 3$ and $r \geq r^*(d)$. Let $\gamma_{d,r}(n)$ denote the number of connected, unlabelled, n -vertex graphs which are r -locally \mathbb{L}^d . Then*

$$\sum_{n \leq x} \gamma_{d,r}(n) = (1 + O_{d,r}(x^{-1+O(1/\log \log x)}))c_d x^d, \quad \text{where } c_d = \frac{1}{2^{d-1}d!d} \prod_{i=2}^d \zeta(i).$$

Proof. The left-hand side is the number of connected, unlabelled graphs on at most x vertices that are r -locally \mathbb{L}^d . By Corollary 21, this is precisely the number of conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ which have $|\mathbb{Z}^d/\Gamma| \leq x$ and which have minimum displacement at least $2r + 2$. If Γ has minimum displacement at least $2r^*(d) + 2 \geq d + 1$, then by Claim 2, $-\text{Id} \notin P_\Gamma$, so either Γ is a pure-translation subgroup, or else Γ is highly symmetric. Applying Proposition 24, Lemma 25 and Corollary 26, we see that the aforesaid number is

$$\begin{aligned} |\tilde{\mathcal{L}}_{\leq x}^d| &= O_{d,r}(x^{d-1} \log x) + O_d(x^{d-1+O(1/(\log \log x))}) \\ &= (1 + O_{d,r}(x^{-1+O(1/\log \log x)}))c_d x^d, \end{aligned}$$

as required. ■

We now use a generating function argument to deduce Theorem 6. Recall that $a_{d,r}(n)$ denotes the number of unlabelled (possibly disconnected) graphs on n vertices which are r -locally \mathbb{L}^d . We appeal to the following well-known fact (see for example [26, p. 29]).

Fact 7. Let \mathcal{S} be a set, and let $w : \mathcal{S} \rightarrow \mathbb{N}$ be a function. (If $S \in \mathcal{S}$, we call $w(S)$ the *weight* of S .) Suppose that \mathcal{S} contains exactly $\gamma(n)$ elements of weight n , where $\gamma(n) \in \mathbb{N} \cup \{0\}$ for each $n \in \mathbb{N}$. If \mathcal{T} is a multiset of elements of \mathcal{S} , define the *weight* of \mathcal{T} to be the sum of the weights of the elements of \mathcal{T} (counted with their multiplicities). For each $n \in \mathbb{N}$, let $a(n)$ denote the number of weight- n multisets of elements of \mathcal{S} . Define $a(0) = 1$. Then the (ordinary) generating function of $a(n)$ satisfies

$$\sum_{n \geq 0} a_n z^n = \prod_{j=1}^{\infty} (1 - z^j)^{-\gamma(j)}.$$

Applying this with \mathcal{S} being the set of all finite, connected, unlabelled graphs that are r -locally \mathbb{L}^d , and with $w(G)$ being the number of vertices of a graph G , we obtain

$$\sum_{n \geq 0} a_{d,r}(n) z^n = \prod_{j=1}^{\infty} (1 - z^j)^{-\gamma_{d,r}(j)}.$$

To estimate $a_{d,r}(n)$ from our knowledge of $\gamma_{d,r}(n)$, we use a variant of the following result of Brigham.

Theorem 29 (Brigham, [17]). *Suppose $(b(n))_{n=0}^{\infty}$ is a sequence of non-negative integers with generating function satisfying*

$$\sum_{n=0}^{\infty} b(n) z^n = \prod_{j=1}^{\infty} (1 - z^j)^{-\gamma(j)},$$

where $(\gamma(j))_{j=1}^\infty$ is a sequence of non-negative integers satisfying

$$\sum_{j \leq x} \gamma(j) \sim Kx^u(\log x)^v$$

for some constants $K > 0, u > 0, v \in \mathbb{R}$. Define

$$B(n) = \sum_{k=0}^n b(n).$$

Then

$$\log B(n) \sim \frac{1}{u} (Ku\Gamma(u+2)\zeta(u+1))^{\frac{1}{u+1}} (u+1)^{\frac{u-v}{u+1}} n^{\frac{u}{u+1}} (\log n)^{\frac{v}{u+1}}.$$

If, in addition, every sufficiently large positive integer can be partitioned into integers in the set $\{n : \gamma(n) \geq 1\}$, then

$$\log b(n) \sim \frac{1}{u} (Ku\Gamma(u+2)\zeta(u+1))^{\frac{1}{u+1}} (u+1)^{\frac{u-v}{u+1}} n^{\frac{u}{u+1}} (\log n)^{\frac{v}{u+1}}.$$

(Here, Γ denotes the usual Γ -function, and ζ the Riemann zeta function; all logarithms are to base e .)

Making a slightly different choice of the parameters in Brigham’s proof (and appealing to a theorem of Odlyzko [55] instead of the theorem of Hardy and Ramanujan, Theorem A in [34], which Brigham uses) yields the following theorem in the case $v = 0$. (This is an exercise in well-known techniques in Tauberian theory, and so for brevity, we omit the proof.)

Theorem 30. Suppose $(b(n))_{n=0}^\infty$ is a sequence of non-negative integers with generating function satisfying

$$\sum_{n=0}^\infty b(n)z^n = \prod_{j=1}^\infty (1 - z^j)^{-\gamma(j)},$$

where $(\gamma(j))_{j=1}^\infty$ is a sequence of non-negative integers satisfying

$$\sum_{j \leq x} \gamma(j) = (1 + O(x^{-\varepsilon}))Kx^u$$

for some constants $K > 0, u > 0, \varepsilon \in (0, 1]$. Define

$$B(n) = \sum_{k=0}^n a(n).$$

Then there exists $\delta > 0$ such that

$$\log B(n) = (1 + O(n^{-\delta}))\frac{1}{u} (Ku\Gamma(u+2)\zeta(u+1))^{\frac{1}{u+1}} (u+1)^{\frac{u}{u+1}} n^{\frac{u}{u+1}}.$$

If, in addition, every sufficiently large positive integer can be partitioned into integers in the set $\{n : \gamma(n) \geq 1\}$, then

$$\log b(n) = (1 + O(n^{-\delta})) \frac{1}{u} (Ku\Gamma(u + 2)\zeta(u + 1))^{\frac{1}{u+1}} (u + 1)^{\frac{u}{u+1}} n^{\frac{u}{u+1}}.$$

It is easy to see that $\gamma_{d,r}(n) \geq 1$ for all n sufficiently large depending on d and r ; indeed, the Cayley graph of \mathbb{Z}_n generated by the set

$$\{1, (2r + 1), (2r + 1)^2, \dots, (2r + 1)^{d-1}\} \cup \{-1, -(2r + 1), \dots, -(2r + 1)^{d-1}\} \quad (12)$$

is connected and r -locally \mathbb{L}^d for all $n \geq (2r + 1)^d$. (We thank an anonymous referee for pointing this out, since it simplifies the argument in our original manuscript.) Therefore, certainly, any sufficiently large positive integer can be partitioned into integers in the set $\{n : \gamma_{d,r}(n) \geq 1\}$. Hence, combining Lemmas 27 and 28 and Theorem 30 (with $u = d$, $K = c_d$ and $\varepsilon = 1/2$) yields Theorem 6.

We now use some of the tools above to prove Theorem 7. Much of the proof will be fairly standard to readers familiar with complexity theory, but we give it in full for the convenience of others.

Proof of Theorem 7. Let $d, r \in \mathbb{N}$ with $d \geq 2$ and $r \geq 2 + 1_{\{d \geq 3\}}$, and let $n \in \mathbb{N}$. We write $\text{poly}(n)$ to denote a quantity that is polynomial in n (for fixed $d, r \in \mathbb{N}$). We sample the random graph $G_n(\mathbb{L}^d, r)$ as follows.

We first enumerate, in time $\text{poly}(n)$, the *connected* unlabelled graphs on at most n vertices that are r -locally \mathbb{L}^d . By Corollaries 20 and 21, these are in (explicit) one-to-one correspondence with the conjugacy classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with minimum displacement at least $2r + 2$ and with $|\mathbb{Z}^d / \Gamma| \leq n$, this correspondence being given by taking the quotient graph \mathbb{L}^d / Γ . If $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d / \Gamma| \leq n$, then by (11) we have $|\mathbb{Z}^d / T_\Gamma| \leq 2^d d!n$. Note that T_Γ can be identified with the corresponding sublattice Λ_{T_Γ} of \mathbb{Z}^d (which has index at most $2^d d!n$). Recall from (8) that for any $x > 0$, the sublattices of \mathbb{Z}^d of index at most x are in (explicit) one-to-one correspondence with the upper-triangular, integer matrices $B = (b_{ij})_{i,j \in [d]}$ with $0 \leq b_{ij} < b_{ii}$ for all $j > i$ and $\prod_{i=1}^d b_{ii} \leq x$. This correspondence is given by taking the sublattice of \mathbb{Z}^d spanned (over \mathbb{Z}) by the columns of the corresponding matrix B :

$$B \leftrightarrow \text{Span} \left\{ \sum_{i=1}^j b_{ij} e_i : j \in [d] \right\}. \quad (13)$$

We enumerate all the sublattices of \mathbb{Z}^d with index at most $2^d d!n$ in time $\text{poly}(n)$, by listing the corresponding matrices. Recall from Fact 2 that $\text{Aut}(\mathbb{L}^d) = T(\mathbb{Z}^d) \rtimes B_d$. We can represent the elements of B_d as d by d matrices with entries in $\{0, \pm 1\}$, and therefore we can enumerate them in time $O_d(1)$. For any subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d / \Gamma| < \infty$, we have $\Gamma / T_\Gamma \cong P_\Gamma \leq B_d$. Therefore, the normal subgroup T_Γ has index at most $|B_d| = 2^d d!$ in Γ , which in turn, crudely, has index at most $(2^d d!)^2 n$ in $\text{Aut}(\mathbb{L}^d)$,

as does T_Γ . Hence, for each choice Δ of T_Γ , the corresponding subgroup Γ must be such that

$$\Delta \triangleleft \Gamma, \quad \Gamma \cap T(\mathbb{Z}^d) = \Delta, \quad [\Gamma : \Delta] \leq 2^d d!. \tag{14}$$

Let

$$\mathcal{F} = \{ \Gamma \leq \text{Aut}(\mathbb{L}^d) : (14) \text{ holds for some subgroup } \Delta \text{ of translations} \\ \text{satisfying } |\mathbb{Z}^d / \Delta| \leq 2^d d!n \},$$

i.e. \mathcal{F} is the union (over all possible choices of Δ) of the set of subgroups Γ satisfying (14). Note that some subgroups Γ in \mathcal{F} may have $|\mathbb{Z}^d / \Gamma| > n$, though all necessarily have $|\mathbb{Z}^d / \Gamma| \leq 2^d d!n$. The elements of \mathcal{F} can, however, easily be enumerated in time $\text{poly}(n)$. We may do this, crudely, by finding representatives for the (left) cosets of Δ in Γ which are of the form σt , where $\sigma \in B_d$ and $t = t_v$ is a translation by an element $v \in \mathbb{Z}^d$ satisfying the ‘box’ condition

$$0 \leq v < b_{i,i} \leq n \quad \forall i \in [d]. \tag{15}$$

(Here, B is the matrix corresponding to the lattice $\Lambda = \Lambda_\Delta$ of Δ .) Indeed, for each possible index $M \leq 2^d d!$, and for tuples $(\sigma_1, \dots, \sigma_M) \in (B_d)^M$ and $(t_1, \dots, t_M) \in (T(\mathbb{Z}^d))^M$ satisfying (15) and with $t_1 = \sigma_1 = \text{Id}$, observe that $\{\sigma_i t_i\}_{i \in [M]}$ forms a set of distinct (left) coset representatives of Δ as an index- M normal subgroup of some $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ if and only if the following three conditions are satisfied:

- (i) $\sigma_i(\Lambda) = \Lambda$ for all $i \in [M]$ (guarantees normality of Δ in Γ);
- (ii) for all $i \in [M]$, there exists $j \in [M]$ such that $\sigma_i t_i \sigma_j t_j \in \Delta$ (guarantees closure of Γ under taking inverses);
- (iii) for all $i, j \in [M]$, there exists $k \in [M]$ such that $t_k^{-1} \sigma_k^{-1} \sigma_i t_i \sigma_j t_j \in \Delta$ (guarantees closure of Γ under multiplication).

For brevity, for any $m, d \in \mathbb{N}$ we define

$$\text{Box}_d(m) = \{x \mapsto x + v : 0 \leq v_i < m \forall i \in [d]\}.$$

The condition (15) implies that the translation t satisfies $t \in \text{Box}_d(n)$. Now, condition (i) can be checked by checking that for each $i \in [M]$, we have $\sigma_i(\mathbf{b}) \in \Lambda$ for each column \mathbf{b} of B . This in turn can be done by solving (for $\mathbf{x} \in \mathbb{Z}^d$) the equation $B\mathbf{x} = \sigma_i(\mathbf{b})$, or verifying that no solution exists; solving this has complexity $O_d(\log n \log \log n)$, since integer division (for integers at most n) has complexity $O(\log n \log \log n)$ (see [6, 36]). Hence, the total time required for checking condition (i) is $O_d(M \log n \log \log n) = \text{poly}(n)$, summing over all d columns of B and over all $i \in [M]$. Condition (ii) can be checked in time $\text{poly}(n)$ by checking, for each $i \in [M]$, whether there exists $j \in [M]$ with $\sigma_i \sigma_j = \text{Id}$ (which guarantees that $\sigma_i t_i \sigma_j t_j \in T(\mathbb{Z}_d)$), and then (if this holds) checking whether there exists such a j for which the translation $\sigma_i t_i \sigma_j t_j$ is a translation by an element of Λ (noting that $\sigma_i t_i \sigma_j t_j \in \text{Box}_d(2n)$). Condition (iii) can be checked in time $\text{poly}(n)$, by

checking, for each $i, j \in [M]$, whether there exists $k \in [M]$ with $\sigma_i \sigma_j = \sigma_k$ (which guarantees that $t_k^{-1} \sigma_k^{-1} \sigma_i t_i \sigma_j t_j \in T(\mathbb{Z}_d)$), and then (if this holds), checking whether there exists such a k for which the translation $t_k^{-1} \sigma_k^{-1} \sigma_i t_i \sigma_j t_j$ is a translation by an element of Λ (noting that $t_k^{-1} \sigma_k^{-1} \sigma_i t_i \sigma_j t_j \in \text{Box}_d(3n)$).

The above immediately implies that $|\mathcal{F}| \leq \text{poly}(n)$. We now claim that \mathcal{F} can be partitioned into $\text{Aut}(\mathbb{L}^d)$ -conjugacy-classes in time $\text{poly}(n)$. Indeed, for $\Gamma_1, \Gamma_2 \in \mathcal{F}$, let $\{g_{i,1} T_{\Gamma_i}, g_{i,2} T_{\Gamma_i}, \dots, g_{i,M_i} T_{\Gamma_i}\}$ be the set of (distinct, left) cosets of T_{Γ_i} in Γ_i (for $i = 1, 2$), where each $g_{i,j}$ can be written in the form σt with $\sigma \in B_d$ and $t \in \text{Box}_d(n)$. By the definition of \mathcal{F} , we have $M_1, M_2 \leq 2^d d!$. Now observe that Γ_1 and Γ_2 are conjugate in $\text{Aut}(\mathbb{L}^d)$ if and only if the following three conditions hold:

- (1) $M_1 = M_2$;
- (2) there exists $h \in \text{Aut}(\mathbb{L}^d)$ such that $h^{-1} T_{\Gamma_1} h = T_{\Gamma_2}$;
- (3) for each $j \in [M_1]$ there exists $\ell = \ell(j) \in [M_1]$ such that $h g_{1,j} h^{-1} T_{\Gamma_2} = g_{2,\ell} T_{\Gamma_2}$, or equivalently $g_{2,\ell}^{-1} h g_{1,j} h^{-1} \in T_{\Gamma_2}$.

Since T_{Γ_2} is a normal subgroup of Γ_2 , the conditions $h^{-1} T_{\Gamma_1} h^{-1} = T_{\Gamma_2}$ and $g_{2,\ell}^{-1} h g_{1,j} h^{-1} \in T_{\Gamma_2}$ (in (2) and (3) above) are invariant under multiplying h on the left by an element of T_{Γ_2} . (For $t \in T_{\Gamma_2}$, we have $g_{2,\ell}^{-1} t g_{2,\ell} \in T_{\Gamma_2}$, so $g_{2,\ell}^{-1} h g_{1,j} h^{-1} \in T_{\Gamma_2}$ iff $(g_{2,\ell}^{-1} t g_{2,\ell})(g_{2,\ell}^{-1} h g_{1,j} h^{-1})(t^{-1}) \in T_{\Gamma_2}$ iff $g_{2,\ell}^{-1}(th)g_{1,j}(th)^{-1} \in T_{\Gamma_2}$.) The subgroup T_{Γ_2} has index at most $(2^d d!)^2 n$ in $\text{Aut}(\mathbb{L}^d)$, so to check whether Γ_1 and Γ_2 are conjugate in $\text{Aut}(\mathbb{L}^d)$, we need only check $(2^d d!)^2 n$ possibilities for h and $M_1!$ possibilities for the sequence $(\ell(j) : j \in [M_1])$. For $h = \sigma t$ where $\sigma \in B_d$ and $t \in \text{Box}_d(n)$, we have $h^{-1} T_{\Gamma_1} h = T_{\Gamma_2}$ iff $\sigma(\Lambda_{\Gamma_2}) = \Lambda_{\Gamma_1}$, a condition which can be checked in time $\text{poly}(n)$ in the same way as condition (i) above. Similarly, writing $g_{1,j} = \sigma_1 t_1$ and $g_{2,\ell} = \sigma_2 t_2$, we have $g_{2,\ell}^{-1} h g_{1,j} h^{-1} \in T_{\Gamma_2}$ iff $\sigma^{-1} \sigma_2 \sigma = \sigma_1$ and $t_2^{-1} \sigma_2^{-1} \sigma t \sigma_1 t_1 t^{-1} \sigma^{-1} \in T_{\Gamma_2}$, conditions that can be checked in time $\text{poly}(n)$ similarly to condition (iii) above.

For each of the subgroups $\Gamma \in \mathcal{F}$ found in the previous steps (up to conjugacy), we (simultaneously) check whether Γ has minimum displacement at least $2r + 2$, find the quotient graph \mathbb{L}^d / Γ , and verify that \mathbb{L}^d / Γ has at most n vertices, all in time $\text{poly}(n)$, as follows. We fix a set $\{g_1, \dots, g_M\}$ of (distinct, left) coset representatives of T_Γ in Γ , where each g_i can be written in the form σt with $\sigma \in B_d$ and $t \in \text{Box}_d(n)$; note that $M \leq 2^d d!$. We first construct the quotient graph \mathbb{L}^d / T_Γ (which is the quotient lattice of \mathbb{L}^d inside a d -dimensional torus), using the correspondence (13). We may view \mathbb{L}^d / T_Γ as a (generalised) discrete torus graph. The vertex-set of \mathbb{L}^d / T_Γ is $\mathbb{Z}^d / \Lambda_{T_\Gamma}$, which is naturally identified with the discrete cuboid

$$C := \prod_{i=1}^d \{0, 1, \dots, b_{ii} - 1\}$$

via the quotient map $q : \mathbb{Z}^d \rightarrow \mathbb{Z}^d / \Lambda_{T_\Gamma}$ that reduces modulo Λ_{T_Γ} , i.e., modulo the \mathbb{Z} -span of the columns of the matrix B . The edge-set of \mathbb{L}^d / T_Γ is simply $\{\{w, q(w + e_i)\} :$

$w \in C, i \in [d]$. For any $w \in C$ and any $i \in [d]$, $q(w + e_i)$ can be evaluated in time $O_d(\log n \log \log n)$ by reducing modulo the \mathbb{Z} -span of the columns of the matrix B .

Observe that the action of Γ on \mathbb{L}^d induces an action of Γ/T_Γ on the discrete torus \mathbb{L}^d/T_Γ ; these two actions have the same quotient graph, namely \mathbb{L}^d/Γ . Since \mathbb{L}^d/T_Γ has at most $2^d d!n$ vertices and $|\Gamma/T_\Gamma| = M \leq 2^d d!$, the action of Γ/T_Γ on the discrete torus \mathbb{L}^d/T_Γ can be determined in time $O_d(n)$ (using the left coset representatives g_i), and therefore the quotient graph \mathbb{L}^d/Γ can be determined in time $\text{poly}(n)$. (For each $i \in [M]$ and each $w \in C$, we have $\max_{j \in [d]} |(g_i(w))_j| \leq 2n$ and therefore $q(g_i(w))$ can be evaluated in time $O_d(\log n \log \log n)$.) Checking that Γ has minimum displacement at least $2r + 2$ is equivalent to simply checking directly that the quotient graph \mathbb{L}^d/Γ is r -locally \mathbb{L}^d , which can be done in $O_{d,r}(n)$ steps since \mathbb{L}^d/Γ has at most n vertices, and each ball of radius r in \mathbb{L}^d/Γ contains $O_{d,r}(1)$ vertices.

The rest of the proof follows the method of Nijenhuis and Wilf [54]. Let H_1, \dots, H_N be an enumeration of the connected, unlabelled graphs on at most n vertices that are r -locally \mathbb{L}^d , with $|H_1| \leq \dots \leq |H_N|$; note that $|H_N| \leq n$ and that $N \leq \text{poly}(n)$. For notational convenience, we define an ordering \preceq on the set of such graphs, defined by $H_i \preceq H_j$ iff $i \leq j$ (and $H_i < H_j$ iff $i < j$). For each $i \in [N]$ and each $k \leq n$, we define a_{k,H_i} to be the number of unlabelled graphs on k vertices that are r -locally \mathbb{L}^d , have at least one component equal to H_i , and no components equal to any H_j with $j > i$. We note the recurrence

$$a_{k,H_i} = \sum_{j \leq i} a_{k-|H_j|,H_j} \quad \forall k \leq n,$$

which simply arises from removing a component isomorphic to H_i and considering all possible choices for the ‘next largest’ component (‘next largest’ meaning, of course, with respect to the ordering \preceq). It is clear that this recurrence can be used to calculate (using $\text{poly}(n)$ addition operations), all the numbers a_{k,H_i} (for $k \leq n, i \leq N$).

To sample a graph G , we perform the following process. We first choose the largest component H of G (largest with respect to the ordering \preceq), according to the probability distribution

$$\text{Prob}\{H = H_i\} = \frac{a_{n,H_i}}{\sum_{j=1}^N a_{n,H_j}}.$$

We note that this distribution may be simulated by rolling (once) a fair, $\text{poly}(n)$ -sided die, which can in turn be simulated by flipping $\text{poly}(n)$ biased coins with rational biases (see e.g. [25]). We then simply repeat this process, choosing the largest component H' (largest with respect to \preceq) of $G - V(H)$ according to the probability distribution

$$\text{Prob}\{H' = H_i\} = \frac{a_{n-|H|,H_i}}{\sum_{j=1}^N a_{n-|H|,H_j}},$$

etc. It is easy to see that this process generates the uniform random graph $G_n(\mathbb{L}^d, r)$ in time $\text{poly}(n)$. ■

5. Cayley graphs of torsion-free groups of polynomial growth

In this section, we consider the (much) more general case where \mathbb{L}^d is replaced by a Cayley graph of a torsion-free group of polynomial growth. Our principal objectives here (which we accomplish in order) are to prove Theorems 8, 9 and 10.

Before proving Theorem 8, we first outline some of the notation, definitions and prior results we will use.

Let Γ be a finitely generated group. For each $n \in \mathbb{N}$, we let $a_n(\Gamma) \in \mathbb{N} \cup \{0\}$ denote the number of subgroups of Γ with index n . We say that Γ has *polynomial subgroup growth* if there exists $\alpha \geq 0$ such that $a_n(\Gamma) \leq n^\alpha$ for all $n \in \mathbb{N}$.

We need the following well-known lemma (see e.g. [51]).

Lemma 31. *Let Γ be a finitely generated, virtually nilpotent group. Then Γ has polynomial subgroup growth.*

We recall the celebrated theorem of Gromov [30] characterizing the groups of polynomial growth.

Theorem 32 (Gromov). *Let Γ be a finitely generated group. Then Γ has polynomial growth if and only if it is virtually nilpotent.*

We need the following result of De La Salle and Tessera [21] (relying on a theorem of Trofimov).

Proposition 33. *Let Γ be a finitely generated, torsion-free group of polynomial growth, and let $F = \text{Cay}(\Gamma, S)$ be a connected, locally finite Cayley graph of Γ . Then the vertex-stabilizers of $\text{Aut}(F)$ are finite.*

The following is an immediate consequence of Proposition 33.

Corollary 34. *Let Γ be a finitely generated, torsion-free, virtually nilpotent group, and let $F = \text{Cay}(\Gamma, S)$ be a connected, locally finite Cayley graph of Γ . Then*

$$[\text{Aut}(F) : \Gamma] < \infty,$$

and therefore $\text{Aut}(F)$ is virtually nilpotent.

Lemma 31, Theorem 32 and Corollary 34 together imply the following.

Corollary 35. *Let Γ be a finitely generated group of polynomial growth, and let F be a connected, locally finite Cayley graph of Γ . Then $\text{Aut}(F)$ has polynomial subgroup growth.*

The following is an immediate consequence of Theorem 5 and Corollary 35.

Proposition 36. *Let F be a connected, locally finite Cayley graph of a finitely generated, torsion-free group of polynomial growth. Then there exist $r_0 = r_0(F) \in \mathbb{N}$ and $\alpha = \alpha(F) \geq 0$ such that the number of unlabelled, connected graphs on at most x vertices that are r_0 -locally F is at most x^α .*

For each $n, r \in \mathbb{N}$, let $\gamma_{F,r}(n)$ denote the number of connected, unlabelled graphs on n vertices that are r -locally F ; Proposition 36 says that $\gamma_{F,r}(n) \leq n^\alpha$ if $r \geq r_0$. Similarly, for each $n \in \mathbb{N}$, let $b_{F,r}(n)$ denote the number of (not necessarily connected) unlabelled graphs on n vertices that are r -locally F , and define $b_{F,r}(0) = 1$. (We regard the empty graph to be r -locally F .) When F and r are understood, for brevity, we let $\mathcal{B}(n)$ denote the set of all unlabelled, n -vertex graphs that are r -locally F , so that $b_{F,r}(n) = |\mathcal{B}(n)|$.

We now introduce the following.

Definition 19. Let F be a connected, locally finite Cayley graph of a finitely generated group Γ , and let $r \in \mathbb{N}$. We say that a subgroup $\Lambda \leq \Gamma$ is *good* (for F and r) if $[\Gamma : \Lambda] < \infty$ and $\Lambda \cap B_F(\text{Id}, 2r) = \emptyset$.

Lemma 37. Let F be a connected, locally finite Cayley graph of a finitely generated group Γ , and let $r \in \mathbb{N}$. If Λ is a good subgroup of Γ , then F/Λ (where Λ acts on $V(F) = \Gamma$ by right multiplication) is connected, r -locally F and vertex-transitive. In particular, if Γ has a good subgroup of index n , then $\gamma_{F,r}(n) \geq 1$.

Proof. All properties are clear except (possibly) for the vertex-transitivity. To see the latter, observe that the vertices of F/Λ correspond to the left cosets of Λ in Γ . Now let $F = \text{Cay}(\Gamma, S)$. For any $\gamma, x, y \in \Gamma$, $x\Lambda$ is joined to $y\Lambda$ in F/Λ if and only if there exist $\gamma_1, \gamma_2 \in \Lambda$ such that $(x\gamma_1)^{-1}(y\gamma_2) \in S$, which holds if and only if there exist $\gamma_1, \gamma_2 \in \Lambda$ such that $(\gamma x \gamma_1)^{-1}(\gamma y \gamma_2) \in S$, i.e. if and only if $\gamma x \Lambda$ is joined to $\gamma y \Lambda$ in F/Λ . Hence, for any $\gamma \in \Gamma$, the map $x\Lambda \mapsto \gamma x \Lambda$ is an automorphism of F/Λ , and the left action of Γ is clearly transitive on the left cosets of Λ in Γ . ■

Recall the following.

Definition 20. Let \mathcal{P} be a property of groups, and let Γ be a group. We say that Γ is *residually* \mathcal{P} if for any $\gamma \in \Gamma \setminus \{\text{Id}\}$, there exists a normal subgroup $N \triangleleft \Gamma$ such that $\gamma \notin N$ and Γ/N has the property \mathcal{P} .

It is easy to see that if Γ is a finitely generated, residually finite group and F is a connected, locally finite Cayley graph of Γ , then for every $r \in \mathbb{N}$, there exists a finite graph that is r -locally F (see e.g. [28]). Interestingly, in the other direction, De La Salle and Tessera [21, Corollary K] proved that if Γ is a finitely presented group with an element of infinite order, and for every connected, locally finite Cayley graph F of Γ and every $r \in \mathbb{N}$, there exists a graph that is r -locally F , then Γ must be residually finite. It follows, for example, that there exists a connected, locally finite Cayley graph F of the Baumslag–Solitar group $\text{BS}(2, 3)$, and a positive integer r , such that no finite graph is r -locally F . We need the following slightly stronger variant of the first statement (under stronger hypotheses).

Proposition 38. Let Γ be a finitely generated group of polynomial growth, and let F be a connected, locally finite Cayley graph of Γ . Then there exists $h_1 \in \mathbb{N}$ such that for any multiple n of h_1 , there is a vertex-transitive, connected, n -vertex graph that is r -locally F (so in particular, $\gamma_{F,r}(n) \geq 1$ whenever $h_1 \mid n$).

Proof. Let $B_F(\text{Id}, 2r) = \{\gamma_1, \dots, \gamma_N\}$. By Gromov’s theorem (Theorem 32), Γ is virtually nilpotent. Let $\Gamma' \leq \Gamma$ be such that Γ' is nilpotent and $[\Gamma : \Gamma'] < \infty$. Since Γ' is a finitely generated nilpotent group, it has finite torsion subgroup, which for convenience we denote by $\{\gamma_{N+1}, \dots, \gamma_{N+M}\}$. (Note that we may have $\gamma_i = \gamma_j$ for some $i \leq N$ and $j > N$.) By a theorem of Hirsch [38], Γ' is residually finite, and so for each $i \in [N + M]$ there exists a subgroup $\Lambda_i \leq \Gamma'$ such that $g_i \notin \Lambda_i$ and $[\Gamma' : \Lambda_i] < \infty$. Define $\Lambda = \bigcap_{i=1}^{N+M} \Lambda_i$; then $\Lambda \cap B_F(\text{Id}, 2r) = \emptyset$ and $[\Gamma' : \Lambda] < \infty$ (so $[\Gamma : \Lambda] < \infty$), and therefore Λ is a good subgroup of Γ ; moreover, Γ is torsion-free and nilpotent.

Let $h_1 := [\Gamma : \Lambda]$. Since Λ is a finitely generated, torsion-free nilpotent group, it has a subgroup of every finite index; every such subgroup is a good subgroup of Γ . Hence, Γ has a good subgroup of every index dividing h_1 . The proposition now follows from Lemma 37. ■

We also need the following.

Lemma 39. *Let Γ be a finitely generated, torsion-free group of polynomial growth, let F be a connected, locally finite Cayley graph of Γ and let $r \in \mathbb{N}$. Then there exists $n_0 = n_0(F, r) \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq n_0$, there exists an n -vertex graph that is r -locally F (i.e., $b_{F,r}(n) \geq 1$ for all $n \geq n_0$).*

Proof. It suffices to show that the highest common factor of the set of integers $\{n \in \mathbb{N} : \gamma_{F,r}(n) \geq 1\}$ is equal to 1. By Lemma 37, it suffices to find a finite set of good subgroups of Γ whose indices have highest common factor 1.

As in the proof of Proposition 38, let $B_F(\text{Id}, 2r) = \{\gamma_1, \dots, \gamma_N\}$, and let $\Gamma' \leq \Gamma$ be such that Γ' is nilpotent and $[\Gamma : \Gamma'] < \infty$. Since Γ' is nilpotent, it is residually finite, and so for each $i \in [N]$ there exists a subgroup $\Lambda_i \leq \Gamma'$ such that $g_i \notin \Lambda_i$ and $[\Gamma' : \Lambda_i] < \infty$. Define $\Lambda = \bigcap_{i=1}^N \Lambda_i$; then $\Lambda \cap B_F(\text{Id}, 2r) = \emptyset$ and $[\Gamma' : \Lambda] < \infty$ (so $[\Gamma : \Lambda] < \infty$), and therefore Λ is a good subgroup of Γ .

We now seek a finite-index subgroup $\Lambda' \leq \Lambda$ such that $\Lambda' \triangleleft \Gamma$ and for all $i \in [N]$, $\gamma_i \Lambda'$ does not have prime-power order in Γ/Λ' . If Λ itself satisfies this condition, then we may take $\Lambda = \Lambda'$; otherwise, by relabelling the γ_i if necessary, we may assume that $\gamma_1 \Lambda, \dots, \gamma_k \Lambda$ all have prime-power order in Γ/Λ , where $k \in [N]$, and $\gamma_i \Lambda$ does not have prime-power order in Γ/Λ for any $i > k$. Let $\gamma_i \Lambda$ have order $p_i^{a_i}$ in Γ/Λ for each $i \in [k]$, where p_1, \dots, p_k are primes and $a_1, \dots, a_k \in \mathbb{N}$. Then by definition, we have $\gamma_i^{p_i^{a_i}} \in \Lambda$ for each $i \in [k]$, and since Γ is torsion-free, we have $\gamma_i^{p_i^{a_i}} \neq \text{Id}$ for each $i \in [k]$. Let q be a prime distinct from p_1, \dots, p_k . Since Λ is torsion-free and nilpotent, by a theorem of Gruenberg [32] it is residually a finite p -group for any prime p , and therefore for each $i \in [k]$, it has a normal subgroup Λ'_i such that $[\Lambda : \Lambda'_i]$ is a power of q , and $\gamma_i^{p_i^{a_i}} \notin \Lambda'_i$. Define $\Lambda_0 = \bigcap_{i=1}^k \Lambda'_i$; then $[\Lambda : \Lambda_0]$ is also a power of q , and $\gamma_i^{p_i^{a_i}} \notin \Lambda_0$ for all $i \in [k]$. Finally, let

$$\Lambda' = \bigcap_{g \in \Gamma} g \Lambda_0 g^{-1};$$

then $\Lambda' \triangleleft \Lambda$, $\Lambda' \triangleleft \Gamma$ and $[\Gamma : \Lambda'] < \infty$.

We now claim that for each $i \in [N]$, $\gamma_i \Lambda'$ does not have prime-power order in Γ/Λ' . Indeed, since $\Lambda' \leq \Lambda$, the order of $\gamma_i \Lambda'$ in Γ/Λ' must divide the order of $\gamma_i \Lambda$ in Γ/Λ , so this is trivially true for all $i > k$. Suppose then that $i \leq k$; we may assume without loss of generality that $k \geq 1$ and $i = 1$. Since Λ/Λ_0 has order a power of q , the order of $\gamma_1^{p_1^{a_1}} \Lambda_0$ in Λ/Λ_0 is some power of q , say q^b where $b \in \mathbb{N}$. In other words, $(\gamma_1^{p_1^{a_1}})^{q^b} \in \Lambda_0$, and if $j \in \mathbb{N}$ with $(\gamma_1^{p_1^{a_1}})^j \in \Lambda_0$, then $q^b \mid j$. Let t be the order of $\gamma_1 \Lambda'$ in Γ/Λ' . Then $\gamma_1^t \in \Lambda' \leq \Lambda_0$, so $(\gamma_1^{p_1^{a_1}})^t \in \Lambda_0$, and so $q^b \mid t$. Moreover, since $\gamma_1^t \in \Lambda' \leq \Lambda$, and $\gamma_1 \Lambda$ has order $p_1^{a_1}$ in Γ/Λ , we have $p_1^{a_1} \mid t$. Hence, $q^b p_1^{a_1} \mid t$, so $\gamma_1 \Lambda'$ does not have prime-power order in Γ/Λ' . The same holds for all $i \in [k]$, proving the claim.

Write the order of Γ/Λ' as a product of primes,

$$|\Gamma/\Lambda'| = \prod_{j=1}^M q_j^{b_j}.$$

For each $j \in [M]$, let H_j be a Sylow q_j -subgroup of Γ/Λ' , and let G_j be the corresponding subgroup of Γ containing Λ' , i.e. such that $G_j/\Lambda' = H_j$. Then for each $i \in [N]$ and each $j \in [M]$, we have $\gamma_i \notin G_j$, since for every $g \in G_j$, $g\Lambda' \in H_j$ and therefore $g\Lambda'$ has order a power of q_j in Γ/Λ' , whereas $\gamma_i \Lambda'$ does not have prime-power order in Γ/Λ' , by construction. Hence, all the subgroups G_j are good subgroups of Γ . For each j , we have

$$[\Gamma : G_j] = [\Gamma/\Lambda' : G_j/\Lambda'] = [\Gamma/\Lambda' : H_j] = \prod_{i \neq j} q_i^{b_i},$$

and therefore the indices $\{[\Gamma : G_j] : j \in [M]\}$ have highest common factor 1. This completes the proof of the lemma. ■

We next make a useful ‘monotonicity’ observation.

Claim 4. *Let Γ be a finitely generated, torsion-free group of polynomial growth, let F be a connected, locally finite Cayley graph of Γ and let $r \in \mathbb{N}$. Let n_0 be as in Lemma 39. Then for all $m, n \in \mathbb{N} \cup \{0\}$ with $n \geq m + n_0$, we have $b_{F,r}(n) \geq b_{F,r}(m)$.*

Proof. Let m, n be as in the statement of the claim. Let H_0 be a fixed $(n - m)$ -vertex graph that is r -locally F . Given an m -vertex graph G that is r -locally F , we may produce from G an n -vertex graph $\Phi(G)$ that is r -locally F by adding to G a vertex-disjoint copy of H_0 . The map $\Phi : \mathcal{B}(m) \rightarrow \mathcal{B}(n)$ is clearly injective, proving the claim. ■

We are now in a position to prove Theorem 8.

Proof of Theorem 8. Let r_0 be as in Theorem 5, and let $r \geq r_0$. For brevity, let us write $\gamma(n) := \gamma_{F,r}(n)$ and $b(n) := b_{F,r}(n)$, for each $n \in \mathbb{N}$. For each $k, n \in \mathbb{N}$ with $k \leq n$, let $a(n, k)$ denote the number of unlabelled, n -vertex, r -locally F graphs with largest component of order k .

Let $\delta = \delta(n) > 0$, $\varepsilon = \varepsilon(n) > 0$ to be chosen later. Suppose for a contradiction that for infinitely many $n \in \mathbb{N}$, the probability that a uniform random unlabelled n -vertex r -locally F graph has largest component of order at least εn is greater than δ . Let us call such integers n *bad*. Then for all bad $n \in \mathbb{N}$, we have

$$\sum_{k=\lceil \varepsilon n \rceil}^n a(n, k) > \delta b(n).$$

Observe that for any $k \leq n$, we have $a(n, k) \leq \gamma(k)b(n - k)$, and therefore for all bad $n \in \mathbb{N}$, we have

$$n \max_{\lceil \varepsilon n \rceil \leq k \leq n} \gamma(k)b(n - k) \geq \sum_{k=\lceil \varepsilon n \rceil}^n a(n, k) > \delta b(n).$$

By Proposition 36, we have $\gamma(k) \leq n^\alpha$ for all $k \leq n$, and therefore

$$\max_{\lceil \varepsilon n \rceil \leq k \leq n} b(n - k) > \delta n^{-\alpha-1} b(n) \tag{16}$$

for all bad $n \in \mathbb{N}$. For appropriate choices of δ and ε , this will contradict the following claim.

Claim 5. For all $n, k \in \mathbb{N}$ such that $n \geq n_0$ and $k \geq n^{5/6}$, we have

$$\frac{b(n - k)}{b(n)} \leq C \exp(-cn^{1/12}),$$

where n_0 is as in Lemma 39, and $c, C > 0$ are constants depending upon F and r alone.

Proof of claim. By appropriate choices of c and C , we may assume throughout that n is at least any specified constant depending upon F and r . Let $n^{5/6} \leq k \leq n$. By an appropriate choice of c , and by Claim 4, we may assume that $k \leq n/2$. We now split into two cases.

Case (i). First, suppose that at least half the graphs in $\mathcal{B}(n - k)$ have largest component of order less than $(n - k)^{2/3}$. Let \mathcal{G} denote the set of such graphs; then $|\mathcal{G}| \geq \frac{1}{2} |\mathcal{B}(n - k)|$. Given a graph $G \in \mathcal{G}$, we produce a graph $H \in \mathcal{B}(n)$ as follows. Let $q_0 \in \mathbb{N}$ be minimal such that $q_0 \geq n_0$ and $k - q_0$ is a multiple of h_1 (where h_1 is as in Proposition 38), and define $W = \lceil (n - k)^{2/3} / h_1 \rceil$.

Let $M = \lceil (k - q_0) / ((W + 1)h_1) \rceil$ and let $m = (k - q_0) / h_1 - MW$; then $m \in \mathbb{N}$ and $m \leq M$. Choose any integer partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash m$, and for each $i \in [M]$, add to G a connected, $((W + \lambda_i)h_1)$ -vertex component that is r -locally F . Since $Wh_1 \geq (n - k)^{2/3}$, each of these added components has more vertices than any component of a graph in \mathcal{G} . Now add an additional vertex-disjoint copy of a fixed, q_0 -vertex graph H_0 that is r -locally F , producing an n -vertex graph H that is r -locally F . (Note that the total number of vertices added is $MWh_1 + mh_1 + q_0 = k$.) Given a fixed graph $G \in \mathcal{G}$, each integer partition $\lambda \vdash m$ produces from G in this way a different unlabelled graph $H \in \mathcal{B}(n)$, and for $G_1 \neq G_2 \in \mathcal{G}$, the sets of n -vertex unlabelled graphs $H \in \mathcal{B}(n)$

produced in this way by G_1 and G_2 are disjoint, since the graph in \mathcal{G} that produced a given $H \in \mathcal{B}(n)$ may be recovered from H by deleting a copy of H_0 , and then deleting the M largest remaining components. It follows that

$$b(n) \geq \frac{1}{2}p(m)b(n - k),$$

where $p(m)$ denotes the number of integer partitions of m . It is easy to see that if $n^{5/6} \leq k \leq n/2$, then $m = (1 + o(1))k/(n - k)^{2/3} \geq (1 + o(1))n^{1/6}$ as $n \rightarrow \infty$, and therefore, using the Hardy–Ramanujan asymptotic (5) for $p(m)$, we obtain

$$b(n) \geq (1 + o(1))\frac{1}{8m\sqrt{3}} \exp(\pi\sqrt{2m/3})b(n - k) \geq \frac{1}{C} \exp(cn^{1/12})b(n - k),$$

where the last inequality holds for all n and k as in the statement of the proposition, provided c is sufficiently small and C is sufficiently large depending on h_1 .

Case (ii). Now suppose that more than half the graphs in $\mathcal{B}(n - k)$ have largest component of order at least $(n - k)^{2/3}$. Let \mathcal{H} denote the set of such graphs; then $|\mathcal{H}| > \frac{1}{2}|\mathcal{B}(n - k)|$. Let $G \in \mathcal{H}$. Let G_1, \dots, G_L be the components of G , listed in non-decreasing order of size, i.e. $|V(G_1)| \leq \dots \leq |V(G_L)|$. Let $a, d \in \mathbb{N}$ with $d \mid a, a \leq (n - k)^{2/3}$ and $a^2/(2d) \geq n - k$; we claim that there exists $i \in [L]$ such that $|V(G_{i-1})| \leq a$ and $|V(G_i)| - |V(G_{i-1})| > d$ (where $|V(G_0)| := 0$). Indeed, suppose for a contradiction that for all $i \in [L]$ with $|V(G_{i-1})| \leq a$, we have $|V(G_{i+1})| - |V(G_i)| \leq d$; then

$$n - k = |V(G)| \geq d + 2d + \dots + (a/d)d > a^2/(2d) \geq n - k,$$

a contradiction. From now on, fix $d = \lfloor \frac{1}{2}(n - k)^{1/6} \rfloor$ and $a = 2\lfloor \frac{1}{2}(n - k)^{1/6} \rfloor \lceil (n - k)^{5/12} \rceil$; clearly, $d \mid a$, and provided n is at least an absolute constant, we have $a \leq (n - k)^{2/3}$ and $a^2/(2d) \geq n - k$.

By the pigeonhole principle, there exists $s \leq a$ such that, for at least $|\mathcal{H}|/a$ of the graphs G in \mathcal{H} , there exists i such that $|V(G_{i-1})| = s$ and $|V(G_i)| - |V(G_{i-1})| > d$. Let \mathcal{J}_s be the set of all such graphs; then $|\mathcal{J}_s| \geq |\mathcal{H}|/a \geq |\mathcal{B}(n - k)|/(2a)$. Let $s' > s$ be minimal such that $h_1 \mid s'$, and let $Q = s'/h_1$. Choose d' maximal such that $h_1 \mid d'$ and $d' \leq d + s - s'$. (Note that $d + s - s' > d - h_1 \geq h_1$ provided n is at least some constant depending upon h_1 .) Let $M = \lfloor (k - d')/s' \rfloor$, and let $q_0 = k - d' - Ms'$. Let $m = d'/h_1 \in \mathbb{N}$. (Note that, provided n is at least some constant depending upon F and r , we have $q_0 \geq n_0$ and $m \leq M$.)

Now, given $G \in \mathcal{J}_s$ and a partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash m$, we produce from G a graph $H \in \mathcal{B}(n)$ as follows. For each $i \in [M]$, add to G a connected, $((Q + \lambda_i)h_1)$ -vertex component that is r -locally F . Since $s < s' = Qh_1 \leq (Q + \lambda_i)h_1 = s' + \lambda_i h_1 \leq s' + mh_1 = s' + d' \leq s + d$, all of these added components have orders between $s + 1$ and $s + d$ (inclusive), whereas all orders of components of graphs in \mathcal{J}_s are at most s or greater than $s + d$. Now add an additional vertex-disjoint copy of a fixed, q_0 -vertex graph H_0 that is r -locally F , producing a graph $H \in \mathcal{B}(n)$. (Note that the total number of vertices added is $MQh_1 + mh_1 + q_0 = Ms' + d' + q_0 = k$.) Given a fixed graph $G \in \mathcal{J}_s$, each integer partition $\lambda \vdash m$ produces from G in this way a different unlabelled

graph $H \in \mathcal{B}(n)$, and for $G_1 \neq G_2 \in \mathcal{J}_s$, the sets of n -vertex unlabelled graphs $H \in \mathcal{B}(n)$ produced in this way by G_1 and G_2 are disjoint, since the graph in \mathcal{J}_s that produced a given $H \in \mathcal{B}(n)$ may be recovered from H by deleting first a copy of H_0 , and then deleting the M components of the remaining graph that have orders between $s + 1$ and $s + d$ (inclusive). It follows that

$$b(n) \geq \frac{1}{2a} p(m) b(n - k).$$

Since $m > d/h_1 - 2 = \lfloor \frac{1}{2}(n - k)^{1/6} \rfloor / h_1 - 2 \geq \lfloor \frac{1}{2}(n/2)^{1/6} \rfloor / h_1 - 2$, and since, crudely, $a \leq n$, we obtain (again using the Hardy–Ramanujan asymptotic for the partition function)

$$b(n) \geq (1 + o(1)) \frac{1}{8nm\sqrt{3}} \exp(\pi \sqrt{2m/3}) b(n - k) \geq \frac{1}{C} \exp(cn^{1/12}) b(n - k),$$

provided c is sufficiently small, and n and C are sufficiently large, depending upon h_1 (i.e., upon F and r). By adjusting the values of c and C appropriately, one can ensure that the above holds for all $n \geq n_0$. This completes the proof of the claim. ■

Applying Claim 5 yields a contradiction to (16) for all n sufficiently large depending upon α and h_1 , in the case where $\varepsilon = n^{-1/6}$ and $\delta = \exp(-n^{1/13})$. This proves that there are only finitely many bad integers n (with this choice of δ, ε), yielding Theorem 8. ■

In the special case where $F = \mathbb{L}^d$, by Theorem 4 we may take $r_0 = 2 + 1_{\{d \geq 3\}}$ in the above proof, so we obtain the following.

Theorem 40. *Define $r_0(2) = 2$ and $r_0(d) = 3$ for all $d \geq 3$. Let $d \in \mathbb{N}$ with $d \geq 2$, and let $r \in \mathbb{N}$ with $r \geq r_0(d)$. Then the largest component of $G_n(\mathbb{L}^d, r)$ has order at most $n^{5/6}$ with probability at least $1 - \exp(-n^{1/13})$, provided n is sufficiently large depending upon d and r .*

In the other direction, we prove the following.

Proposition 41. *Let F be a connected, locally finite Cayley graph of a torsion-free group of polynomial growth. Let $r_0 = r_0(F)$ be as in Theorem 5. Then there exists $\varepsilon_0 = \varepsilon_0(F) > 0$ such that for all $r \geq r_0$, the random graph $G_n(F, r)$ has a component of order at least n^{ε_0} with probability at least $1 - \exp(-n^{1/4})$, provided n is sufficiently large depending on F and r .*

Proof. Let $r \geq r_0$. Let $n_0 = n_0(F, r) \in \mathbb{N}$ and $\varepsilon_0 = \varepsilon_0(F) > 0$ to be chosen later. Let $n \geq n_0$, and let \mathcal{G} be the set of all graphs in $\mathcal{B}(n)$ with largest component of order less than n^{ε_0} . Let $G \in \mathcal{G}$. Since G has n vertices and all its components have order less than n^{ε_0} , G has more than $n^{1-\varepsilon_0}$ components, so there exists $k_1 = k_1(G) < n^{\varepsilon_0}$ such that G has at least $n^{1-2\varepsilon_0}$ components of order k_1 . By Proposition 36, there are at most k_1^α unlabelled, connected graphs on k_1 vertices that are r -locally F , so there exists a connected graph $H_1 = H_1(G)$ on $k_1 < n^{\varepsilon_0}$ vertices such that at least

$$n^{1-2\varepsilon_0} / k_1^\alpha \geq n^{1-(\alpha+2)\varepsilon_0}$$

components of G are isomorphic to H_1 .

We now define a bipartite graph \mathbb{B} with vertex-bipartition $(\mathcal{G}, \mathcal{B}(n) \setminus \mathcal{G})$, as follows. For each $G \in \mathcal{G}$, choose some connected graph $H_1 = H_1(G)$ on $k_1 = k_1(G) < n^{\varepsilon_0}$ vertices such that at least $n^{1-(\alpha+2)\varepsilon_0}$ components of G are isomorphic to H_1 (as above). Let $Q = \lfloor n^{1-(\alpha+2)\varepsilon_0} / h_1 \rfloor$ (where h_1 is as in Proposition 38), and delete from G exactly Qh_1 components that are isomorphic to H_1 , producing a graph G' with $n - Qh_1k_1$ vertices. Let $W = \lceil n^{\varepsilon_0} / h_1 \rceil$, let $M = \lceil Qk_1 / (W + 1) \rceil$ and let $m = Qk_1 - MW$; note that $m \leq M$. For any integer partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash m$, we may produce a graph $H_{\lambda,G} \in \mathcal{B}(n) \setminus \mathcal{G}$ by adding to G' a connected, $(W + \lambda_i)h_1$ -vertex component that is r -locally F , for each $i \in [M]$. (Note that the total number of vertices added is $MWh_1 + mh_1 = Qh_1k_1 = |V(G)| - |V(G')|$, and that each added component has order at least $Wh_1 \geq n^{\varepsilon_0}$, i.e. it has order greater than the order of any component of G .) Now we define the edge-set of our bipartite graph \mathbb{B} by joining G to each of the $p(m)$ graphs $H_{\lambda,G}$ obtained in this way (for each $G \in \mathcal{G}$).

Each graph $G \in \mathcal{G}$ has degree exactly $p(m)$ in the bipartite graph \mathbb{B} , whereas each graph $H \in \mathcal{B}(n) \setminus \mathcal{G}$ has degree at most $n^{\alpha\varepsilon_0}$ in \mathbb{B} . Indeed, if in the bipartite graph \mathbb{B} , a graph $H \in \mathcal{B}(n) \setminus \mathcal{G}$ is joined to some $G \in \mathcal{G}$, then the ‘intermediate’ graph G' depends only on H and not on G , since G' can be recovered from H alone by deleting the M components of H that have order at least n^{ε_0} . Given G' , there are at most $n^{\alpha\varepsilon_0}$ possibilities for G , since (by Proposition 36) there are at most $n^{\alpha\varepsilon_0}$ connected graphs on less than n^{ε_0} vertices that are r -locally F , so there are at most $n^{\alpha\varepsilon_0}$ choices for the component $H_1 = H_1(G)$ (and adding Qh_1 copies of H_1 to G' , we recover G).

Counting the edges of \mathbb{B} in two different ways, we obtain

$$|\mathcal{B}(n) \setminus \mathcal{G}| n^{\alpha\varepsilon_0} \geq |\mathcal{G}| p(m),$$

so

$$\frac{|\mathcal{G}|}{|\mathcal{B}(n)|} \leq \frac{n^{\alpha\varepsilon_0}}{p(m)}.$$

It is easy to see that $m \geq (1 + o(1))n^{1-(\alpha+3)\varepsilon_0}$ as $n \rightarrow \infty$, and so using the Hardy–Ramanujan asymptotic (5), we obtain

$$\begin{aligned} \frac{|\mathcal{G}|}{|\mathcal{B}(n)|} &\leq (1 + o(1))4\sqrt{3}mn^{\alpha\varepsilon_0} \exp(-\pi\sqrt{2m/3}) \\ &\leq Cn \exp(-cn^{(1-(\alpha+3)\varepsilon_0)/2}) \leq \exp(-n^{1/4}) \end{aligned}$$

if we choose $\varepsilon_0 = 1/(2\alpha + 7)$ and n_0 sufficiently large depending on F and r . (Here, c and C are positive absolute constants.) This proves the proposition. ■

Remark 6. It is easy to adapt the proof of Proposition 41 to show that with probability at least $1 - \exp(-n^{1/4})$, the random graph $G_n = G_n(F, r)$ has a component of order that is both at least n^{ε_0} and divisible by h_1 , where h_1 is as in Proposition 38, provided n is sufficiently large depending on F and r . (One simply replaces \mathcal{G} in the proof by the set of all graphs in $\mathcal{B}(n)$ with no component of order that is both at least n^{ε_0} and divisible by h_1 , observing that the components added to G' all have orders that are multiples of h_1 .) This fact is useful in what follows.

We now use Proposition 41 and Remark 6 to prove Theorem 9.

Proof of Theorem 9. Let $r_0 = r_0(F)$ be as in Theorem 5, and let $n \in \mathbb{N}$. Let $\delta_0 = \delta_0(F) = \varepsilon_0/3$, where $\varepsilon_0 = \varepsilon_0(F) > 0$ is as in Proposition 41. Let \mathcal{A} denote the set of graphs in $\mathcal{B}(n)$ that have at most n^{δ_0} vertex-transitive components of order divisible by h_1 , and let \mathcal{A}' denote the set of graphs in \mathcal{A} that have a component of order that is at least n^{ε_0} and divisible by h_1 . (Here, h_1 is as in Proposition 38.) We define a bipartite graph \mathbb{B} with vertex-bipartition $(\mathcal{A}', \mathcal{B}(n) \setminus \mathcal{A})$, as follows. For each graph $G \in \mathcal{A}'$, let $H_2 = H_2(G)$ be a component of G of maximal order divisible by h_1 , and let $Q = Q_G = |V(H_2)|/h_1$. Delete H_2 from G , producing a graph G' with $n - Qh_1$ vertices. For any integer partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash Q$, we may produce a graph $H_{\lambda,G} \in \mathcal{B}(n) \setminus \mathcal{A}$ by adding to G' a vertex-transitive component of order $\lambda_i h_1$, for each $i \in [m]$. (Note that the total number of vertices added is $h_1 \sum_{i=1}^m \lambda_i = Qh_1$.) Now we define the edge-set of our bipartite graph \mathbb{B} by joining G to each of the $p(Q_G)$ graphs $H_{\lambda,G}$ obtained in this way (for each $G \in \mathcal{A}'$).

A graph $G \in \mathcal{A}'$ has degree exactly $p(Q_G)$ in the bipartite graph \mathbb{B} , whereas each graph $H \in \mathcal{B}(n) \setminus \mathcal{A}$ has degree at most $n^{\alpha+n\delta_0}$ in \mathbb{B} . Indeed, if in the bipartite graph \mathbb{B} , a graph $H \in \mathcal{B}(n) \setminus \mathcal{A}$ is joined to some $G \in \mathcal{A}'$, then there are at most

$$\sum_{i=0}^{\lfloor n^{\delta_0} \rfloor} \binom{n-1}{i} \leq n^{n^{\delta_0}}$$

ways of choosing the ‘intermediate’ graph G' (since G' may be obtained from H by deleting all but j of the vertex-transitive components of G of order divisible by h_1 , for some $j \leq n^{\delta_0}$, and crudely, H has at most $n/h_1 \leq n - 1$ vertex-transitive components of order divisible by h_1). Given the intermediate graph G' , there are then at most n^α possibilities for G , since (by Proposition 36) there are at most n^α connected graphs on at most n vertices that are r -locally F , so there are at most n^α choices for the component $H_2 = H_2(G)$ (which, when added to G' , produces the graph G).

Counting edges of \mathbb{B} in two different ways, and using the fact that $\delta_0 = \varepsilon_0/3$, we obtain

$$|\mathcal{B}(n)|n^{\alpha+n\varepsilon_0/3} \geq |\mathcal{B}(n) \setminus \mathcal{A}|n^{\alpha+n\delta_0} \geq |\mathcal{A}'| \min_{G \in \mathcal{A}'} p(Q_G) \geq p(\lceil n^{\varepsilon_0}/h_1 \rceil)|\mathcal{A}'|. \tag{17}$$

On the other hand, by Proposition 41 and Remark 6, provided n is sufficiently large depending on F and r , we have

$$|\mathcal{A} \setminus \mathcal{A}'| \leq \exp(-n^{1/4})|\mathcal{B}(n)|. \tag{18}$$

Combining (17) and (18) yields

$$\frac{|\mathcal{A}|}{|\mathcal{B}(n)|} \leq \exp(-n^{-\varepsilon_0/3})$$

for all n sufficiently large depending on F and r , using the Hardy–Ramanujan asymptotic (5). It follows that, with probability at least $1 - \exp(-n^{-\delta_0})$, a uniform random graph

$G_n \in \mathcal{B}(n)$ has more than n^{δ_0} vertex-transitive components of order divisible by h_1 , so in particular $|\text{Aut}(G_n)| \geq (h_1)^{n^{\delta_0}} \geq 2^{n^{\delta_0}}$, proving the theorem. ■

We now use the above machinery, combined with Brigham’s theorem (Theorem 29), to prove Theorem 10, which says that $b_{F,r}(n)$ grows like a stretched exponential for all $r \geq r_0$.

We will need the following easy corollary of Brigham’s theorem.

Corollary 42. *Let $(b(n))_{n=0}^\infty, (\gamma(j))_{j=1}^\infty$ be as in Theorem 29. Let h be the highest common factor of the set $\{n \in \mathbb{N} : \gamma(n) \geq 1\}$. Then*

$$\log b(n) \sim \frac{1}{u} (Ku\Gamma(u + 2)\zeta(u + 1))^{\frac{1}{u+1}} (u + 1)^{\frac{u-v}{u+1}} n^{\frac{u}{u+1}} (\log n)^{\frac{v}{u+1}}$$

for all $n \in \mathbb{N}$ such that $h \mid n$.

(Note that the $h = 1$ case of Corollary 42 appears in [17]; the general case is a straightforward deduction therefrom, using the observation that any sufficiently large multiple of h can be partitioned into integers in the set $\{n : \gamma(n) \geq 1\}$.)

Proof of Theorem 10. Observe that the generating function of $b_{F,r}(n)$ satisfies

$$\sum_{n=0}^\infty b_{F,r}(n)z^n = \prod_{j=1}^\infty (1 - z^j)^{-\gamma_{F,r}(j)}. \tag{19}$$

Define the sequence $(\gamma'(n))_{n=1}^\infty$ by $\gamma'(n) = 1$ if n is a multiple of h_1 , and $\gamma'(n) = 0$ otherwise. Then, by Proposition 38, we have $\gamma_{F,r}(n) \geq \gamma'(n)$ for all $n \in \mathbb{N}$. Note that

$$\sum_{n \leq x} \gamma'(n) = \lfloor x/h_1 \rfloor \quad \forall x > 0. \tag{20}$$

Define the sequence $(b'(n))_{n=0}^\infty$ by

$$\sum_{n=0}^\infty b'(n)z^n = \prod_{j=1}^\infty (1 - z^j)^{-\gamma'(j)};$$

then, since $\gamma_{F,r}(n) \geq \gamma'(n)$ for all $n \in \mathbb{N}$, we have $b_{F,r}(n) \geq b'(n)$ for all $n \in \mathbb{N}$.

By Claim 4, we have $b_{F,r}(n) \geq b_{F,r}(m)$ for all $m, n \in \mathbb{N}$ such that $n \geq m + n_0$, so in particular, whenever $n \geq n_0$, we have

$$b_{F,r}(n) \geq b_{F,r}(\lfloor (n - n_0)/h_1 \rfloor h_1) \geq b'(\lfloor (n - n_0)/h_1 \rfloor h_1). \tag{21}$$

Applying Corollary 42 (with $h = h_1$) to the sequence $(\gamma'(j))_{j=1}^\infty$, and using (20), yields

$$\log b'(n) \geq (1 + o(1))\pi \sqrt{2n/(3h_1)}$$

for $n \rightarrow \infty$ such that $h_1 \mid n$. Using (21) then yields

$$\log b_{F,r}(n) \geq (1 + o(1))\pi \sqrt{2n/(3h_1)} \tag{22}$$

for (all) $n \rightarrow \infty$.

On the other hand, if $r \geq r_0$, then by Proposition 36, we have

$$\gamma_{F,r}(n) \leq n^\alpha \quad \forall n \in \mathbb{N}, \tag{23}$$

where $\alpha = \alpha(F) \geq 0$. Comparing the sequence $(\gamma_{F,r}(j))_{j=1}^\infty$ with the sequence $(\gamma''(j))_{j=1}^\infty$ defined by

$$\gamma''(j) = \begin{cases} \lfloor j^\alpha \rfloor & \text{if } h_0 \mid j, \\ 0 & \text{if } h_0 \nmid j, \end{cases}$$

and appealing to Corollary 42, yields

$$\log b_{F,r}(n) \leq (1 + o(1)) \frac{1}{\alpha + 1} (\Gamma(\alpha + 3)\zeta(\alpha + 2)/h_0)^{\frac{1}{\alpha+2}} (\alpha + 2)^{\frac{\alpha+1}{\alpha+2}} n^{\frac{\alpha+1}{\alpha+2}} \tag{24}$$

for $n \rightarrow \infty$. Combining the upper bound (24) with the lower bound (22) yields the theorem. ■

6. Typical properties of graphs that are r -locally \mathbb{L}^d

In this section, we use some of the results and techniques of Section 4 to obtain some more precise results on the typical properties of unlabelled, n -vertex graphs which are r -locally \mathbb{L}^d , for various integers r and d . We first prove that if $r \geq r^*(d)$ (where $r^*(d)$ is as defined in Theorem 6), then with high probability, all but at most a $1/\text{poly}(n)$ fraction of the vertices of the random graph $G_n = G_n(\mathbb{L}^d, r)$ lie in components that are isomorphic to a quotient of \mathbb{L}^d by a pure-translation subgroup of $\text{Aut}(\mathbb{L}^d)$. (Such quotients can be viewed as quotient lattices of \mathbb{L}^d inside some d -dimensional torus \mathbb{R}^d/Λ , where Λ is a rank- d sublattice of \mathbb{Z}^d , i.e. the lattice of translations of some pure-translation subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| < \infty$; see the Introduction, or alternatively [8].) This is the content of Theorem 44. We then obtain (in Proposition 46) a more precise lower bound on $|\text{Aut}(G_n(\mathbb{L}^d, r))|$ than that which can be deduced from Theorem 9.

We will need the following ratio estimate, an easy consequence of Theorem 6.

Lemma 43. *Let $d \in \mathbb{N}$ with $d \geq 2$, and let $K_d, \varepsilon_d, r^*(d)$ be as in Theorem 6. Let $r \in \mathbb{N}$ with $r \geq r^*(d)$. There exists $\beta_{d,r} > 0$ such for all $m \geq \beta_{d,r}n^{1-\varepsilon_d}$, we have*

$$\frac{a_{d,r}(n-m)}{a_{d,r}(n)} \leq \exp\left(-\frac{1}{5}K_d m n^{-\frac{1}{d+1}}\right).$$

Proof. By taking $\beta_{d,r}$ sufficiently large, we may assume that $n \geq n_0(d, r)$ for any function $n_0 = n_0(d, r)$. We first suppose that $m \leq n/2$. Using the fact that $(1-x)^a \leq 1-ax$ for all $a, x \in (0, 1)$, we have

$$\begin{aligned} & \log(a_{d,r}(n)) - \log(a_{d,r}(n-m)) \\ &= (1 + O(n^{-\varepsilon_d}))K_d n^{\frac{d}{d+1}} - (1 + O((n-m)^{-\varepsilon_d}))K_d (n-m)^{\frac{d}{d+1}} \\ &= K_d n^{\frac{d}{d+1}} (1 + O(n^{-\varepsilon_d}) - (1 - m/n)^{\frac{d}{d+1}}) \end{aligned}$$

$$\begin{aligned} &\geq K_d n^{\frac{d}{d+1}} (1 + O(n^{-\varepsilon d}) - 1 + \frac{d}{d+1} \frac{m}{n}) = K_d n^{\frac{d}{d+1}} (\frac{d}{d+1} \frac{m}{n} + O(n^{-\varepsilon d})) \\ &\geq \frac{1}{2} K_d n^{\frac{d}{d+1}} m/n = \frac{1}{2} K_d m n^{-\frac{1}{d+1}}, \end{aligned}$$

provided $\beta_{d,r}$ is sufficiently large depending on d and r . Now suppose $m \geq n/2$. Given a graph on k vertices which is r -locally \mathbb{L}^d , we may produce a graph on $k + (2r + 2)^d$ vertices which is r -locally \mathbb{L}^d by adding a vertex-disjoint copy of the d -dimensional discrete torus C_{2r+2}^d . Hence, $a_{d,r}(k) \leq a_{d,r}(k + (2r + 2)^d)$ for all $k \geq 0$. It follows that $a_{d,r}(n - m) \leq a_{d,r}(n - m')$ for some $m' \in \mathbb{N}$ with $n/2 - (2r + 2)^d \leq m' \leq n/2$. Hence, we have

$$\begin{aligned} \log(a_{d,r}(n)) - \log(a_d(n - m)) &\geq \log(a_{d,r}(n)) - \log(a_{d,r}(n - m')) \\ &\geq \frac{1}{2} K_d m' n^{-\frac{1}{d+1}} \geq \frac{1}{5} K_d m n^{-\frac{1}{d+1}}, \end{aligned}$$

provided n is sufficiently large depending on d and r . Taking exponents proves the lemma. ■

We now prove the following.

Theorem 44. *Let $d \in \mathbb{N}$ with $d \geq 2$, and let $r, R \in \mathbb{N}$ with $R \geq r \geq r^*(d)$. There exists $\kappa = \kappa_{d,R} > 0$ depending upon d and R alone, such that with high probability, all but at most $n^{1-\kappa}$ of the vertices of $G_n = G_n(\mathbb{L}^d, r)$ are in components that are R -locally \mathbb{L}^d and isomorphic to a quotient of \mathbb{L}^d by a pure-translation subgroup of $\text{Aut}(\mathbb{L}^d)$.*

To prove Theorem 44, we need a straightforward corollary of Theorem 29.

Corollary 45. *Suppose $(b(n))_{n=0}^\infty$ is a sequence of positive integers with generating function satisfying*

$$\sum_{n=0}^\infty b(n)z^n = \prod_{j=1}^\infty (1 - z^j)^{-\gamma(j)}, \tag{25}$$

where $(\gamma(j))_{j=1}^\infty$ is a sequence of non-negative integers satisfying

$$\sum_{j \leq x} \gamma(j) \leq Kx^u (\log x)^v$$

for some constants $K > 0, u > 0, v \in \mathbb{R}$. Then there exists a constant $K' > 0$ (depending only upon u, v and K) such that

$$\log b(n) \leq K' n^{u/(u+1)} (\log n)^{v/(u+1)} \quad \forall n \in \mathbb{N}.$$

Proof. Let $(\eta(j))_{j=1}^\infty$ be the sequence of non-negative integers defined by

$$\sum_{j \leq N} \eta(j) = \lceil K(N + 1)^u (\log(N + 1))^v \rceil + 1 \quad \forall N \in \mathbb{N}. \tag{26}$$

Then we have

$$\sum_{j \leq N} \gamma(j) < \sum_{j \leq N} \eta(j) \quad \forall N \in \mathbb{N}. \tag{27}$$

Define $(a(n))_{n=0}^\infty$ by

$$\sum_{n=0}^\infty a(n)z^n = \prod_{j=1}^\infty (1 - z^j)^{-\eta(j)}. \tag{28}$$

We make the following

Claim 6. *If (27) holds, and $(a(n)), (b(n))$ are defined by (28) and (25) respectively, then $b(n) \leq a(n)$ for all $n \in \mathbb{N}$.*

Proof of claim. Recall from Fact 7 that $b(n)$ is the number of weight- n multisets of elements of \mathcal{T} , where \mathcal{T} contains exactly $\gamma(j)$ elements of weight j for all $j \in \mathbb{N}$, and $a(n)$ is the number of weight- n multisets of elements of \mathcal{S} , where \mathcal{S} contains exactly $\eta(j)$ elements of weight j for all $j \in \mathbb{N}$. Let $\mathcal{A}(n)$ (respectively $\mathcal{B}(n)$) denote the set of weight- n multisets of elements of \mathcal{S} (respectively \mathcal{T}); then $a(n) = |\mathcal{A}(n)|$ and $b(n) = |\mathcal{B}(n)|$. Fix $n \in \mathbb{N}$. It suffices to construct an injection $\Phi : \mathcal{B}(n) \rightarrow \mathcal{A}(n)$. Choose any element $S_1 \in \mathcal{S}$ of weight 1. Let $\mathcal{S}_{\leq n}, \mathcal{T}_{\leq n}$ denote the set of elements of \mathcal{S} (respectively \mathcal{T}) of weight at most n . Let $w : \mathcal{S} \cup \mathcal{T} \rightarrow \mathbb{N}$ denote the weight function. We first construct an injection $f : \mathcal{T}_{\leq n} \rightarrow \mathcal{S}_{\leq n} \setminus \{S_1\}$ such that $w(T) \geq w(f(T))$ for all $T \in \mathcal{T}_{\leq n}$. Define $\eta'(1) = \eta(1) - 1$ and $\eta'(j) = \eta(j)$ for all $j \geq 2$. Then the set $\mathcal{S} \setminus \{S_1\}$ contains exactly $\eta'(j)$ elements of weight j , for each $j \in \mathbb{N}$, and by (27), we have

$$\sum_{j \leq N} \gamma(j) \leq \sum_{j \leq N} \eta'(j) \quad \forall N \in \mathbb{N}. \tag{29}$$

We define f inductively. Order the elements of $\mathcal{T}_{\leq n}$ in non-decreasing order of weight, say as T_1, T_2, \dots, T_M . If we have already defined $f(T_1), \dots, f(T_{i-1})$, define $f(T_i)$ to be an element of $(\mathcal{S} \setminus \{S_1\}) \setminus \{f(T_j) : j < i\}$ of minimal weight. By (29), we have $w(f(T)) \leq w(T)$ for all $T \in \mathcal{T}_{\leq n}$, so $f(\mathcal{T}_{\leq n}) \subset \mathcal{S}_{\leq n} \setminus \{S_1\}$, as needed. Now for any multiset $X = (B_1, \dots, B_k) \in \mathcal{B}(n)$, define

$$\Phi(X) = (f(B_1), \dots, f(B_k), S_1, \dots, S_1),$$

where the number of S_1 's is equal to $\sum_{i=1}^k w(B_i) - \sum_{i=1}^k w(f(B_i))$. Clearly, $\Phi(X)$ is an injection from $\mathcal{B}(n)$ to $\mathcal{A}(n)$. This proves the claim. ■

By (26), we have

$$\sum_{j \leq x} \eta(x) = (1 + o(1))Kx^u(\log x)^v.$$

Hence, by Theorem 29,

$$\log a(n) = (1 + o(1))K_1 n^{u/(u+1)} (\log n)^{v/(u+1)},$$

for some constant $K_1 > 0$ depending only upon u, v and K . Hence,

$$\log b(n) \leq \log a(n) \leq K' n^{u/(u+1)} (\log n)^{v/(u+1)}$$

for some constant $K' > 0$ depending only upon u, v and K , proving the corollary. ■

Proof of Theorem 44. Let $d \geq 2$ and $R \geq r \geq r^*(d)$. Note that, by adjusting the value of $\kappa_{d,R}$ if necessary, we may assume throughout that $n \geq n_0(d, R)$, for any function $n_0 = n_0(d, R)$.

Define $\gamma_{d,r,R}(n)$ (respectively, $b_{d,r,R}(n)$) to be the number of connected (respectively, possibly disconnected), unlabelled, n -vertex graphs which are r -locally \mathbb{L}^d and either not R -locally \mathbb{L}^d or else not isomorphic to a quotient of \mathbb{L}^d by a pure-translation subgroup. Then, using Fact 7, we have

$$\sum_{n=0}^{\infty} b_{d,r,R}(n)z^n = \prod_{j=1}^{\infty} (1 - z^j)^{-\gamma_{d,r,R}(j)}.$$

Observe that

$$\sum_{n \leq x} \gamma_{d,r,R}(n) = O_{d,R}(x^{d-1+O(1/\log \log n)}) \quad \forall x \geq 1. \tag{30}$$

Indeed, the above sum is precisely the number of conjugacy-classes of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ such that $|\mathbb{Z}^d/\Gamma| \leq x$ and $2r + 2 \leq D(\Gamma) \leq 2R + 1$. The total number of pure-translation subgroups with this property is precisely the number of sublattices of \mathbb{Z}^d with index at most x and minimum distance in $\{2r + 2, 2r + 3, \dots, 2R + 1\}$, which, by Lemma 25, is at most

$$O_{d,R}(x^{d-1} \log x).$$

Moreover, the total number of subgroups $\Gamma \leq \text{Aut}(\mathbb{L}^d)$ with $|\mathbb{Z}^d/\Gamma| \leq x$ and $D(\Gamma) \geq 2r^*(d) + 1$ such that Γ is *not* a pure-translation subgroup is at most

$$O_d(x^{d-1+O(1/\log \log x)}),$$

by Corollary 26 and Claim 2. Hence, we have

$$\sum_{n \leq x} \gamma_{d,r,R}(n) = O_{d,R}(x^{d-1+O(1/\log \log x)}),$$

as desired.

Slightly more crudely, it follows that

$$\sum_{n \leq x} \gamma_{d,r,R}(n) = O_{d,R}(x^{d-1/2}).$$

It follows from Corollary 45 that

$$\log b_{d,r,R}(n) \leq K_{d,R} n^{(d-1/2)/(d+1/2)} = K_{d,R} n^{(2d-1)/(2d+1)} \quad \forall n \in \mathbb{N},$$

where $K_{d,R} > 0$ is a constant depending upon d and R alone.

Now let H be a finite, simple, connected graph which is r -locally \mathbb{L}^d . Let us say that H is R -bad if H is not R -locally \mathbb{L}^d or if H is not isomorphic to a quotient of \mathbb{L}^d by a pure-translation subgroup. Let $q_{d,R} = q_{d,R}(n)$ be a function of n to be chosen later. We shall bound the probability that the random graph $G_n(\mathbb{L}^d, r)$ has at least $q_{d,R}(n)$ of its

vertices in R -bad components. Fix an integer $m \geq q_{d,R}(n)$. Using Lemma 43, it follows that the probability that G_n has exactly m vertices in R -bad components is at most

$$\begin{aligned} \frac{b_{d,r,R}(m)a_{d,r}(n-m)}{a_{d,r}(n)} &\leq \exp(K_{d,R}m^{\frac{2d-1}{2d+1}}) \exp(-\frac{1}{5}K_d m n^{-\frac{1}{d+1}}) \\ &\leq \exp(-\frac{1}{6}K_d n^{(2d-1)/(2d+2)}), \end{aligned}$$

provided we choose

$$q_{d,R}(n) = \max \{ \max \{ \beta_{d,r} : r \leq R \} n^{1-\varepsilon_d}, L_{d,R} n^{(2d+1)/(2d+2)} \}$$

for some $L_{d,R} > 0$ sufficiently large. Hence, by the union bound, the probability that G_n has at least $q_{d,R}(n)$ vertices in R -bad components is at most

$$n \exp(-\frac{1}{6}K_d n^{(2d-1)/(2d+2)}) = o(1).$$

Note that for each fixed d, R , we have

$$q_{d,R}(n) \leq n^{1-\kappa_{d,R}}$$

for n sufficiently large depending on d and R , provided we choose $\kappa_{d,R} > 0$ sufficiently small. This proves Theorem 44. ■

From Theorem 44, it follows immediately that the local limit of (G_n) is the rooted lattice $(\mathbb{L}^d, 0)$. For the reader’s convenience, we recall the definition of a local limit. If (F, u) and (G, w) are two rooted graphs, we say that they are *isomorphic as rooted graphs* if there exists a graph isomorphism $\phi : V(F) \rightarrow V(G)$ such that $\phi(u) = w$. Let (G, w) be a random rooted graph, i.e. a probability distribution on the set of rooted graphs. Following [9], we say that a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ has *local limit* (G, w) as $n \rightarrow \infty$ if for every $R \in \mathbb{N}$ and for every rooted graph (H, v) , as $n \rightarrow \infty$, the probability that $\text{Link}_R(G_n, w_n)$ is isomorphic to (H, v) (as a rooted graph) converges to the probability that $\text{Link}_R(G, w)$ is isomorphic to (H, v) (as a rooted graph), where w_n is a vertex chosen uniformly at random from $V(G_n)$. In particular, if (G, w) is constant, the sequence $(G_n)_{n \in \mathbb{N}}$ has local limit (G, w) if the probability that $\text{Link}_R(G_n, w_n)$ is isomorphic to $\text{Link}_R(G, w)$ (as a rooted graph) tends to 1 as $n \rightarrow \infty$, where w_n is a vertex chosen uniformly at random from $V(G_n)$.

Now let $d \geq 2$, let $r \geq r^*(d)$ and let $G_n = G_n(\mathbb{L}^d, r)$. Let $T = \{n : \mathcal{S}_n \neq \emptyset\}$, where \mathcal{S}_n denotes the set of all unlabelled, n -vertex graphs that are r -locally \mathbb{L}^d . (Recall that, by (12), we have $n \in T$ for all n sufficiently large depending on d and r .) Theorem 44 implies that for any $R \in \mathbb{N}$, the probability that $\text{Link}_R(G_n, w_n)$ is isomorphic to $\text{Link}_R(\mathbb{L}^d, 0)$ tends to 1 as $n \rightarrow \infty$, so the local limit of $(G_n)_{n \in T}$ is $(\mathbb{L}^d, 0)$, as claimed.

Theorems 40 and 44 also easily imply the following.

Proposition 46. *Let $d \in \mathbb{N}$ with $d \geq 2$, and let $r \in \mathbb{N}$ with $r \geq r^*(d)$. Then with high probability,*

$$|\text{Aut}(G_n(\mathbb{L}^d, r))| \geq \exp(\Omega(n^{1/6} \log n)).$$

Proof. We may assume that $n \geq n_0(d, r)$, for any function $n_0 = n_0(d, r)$. Observe that if $H = \mathbb{L}^d / \Gamma$ for some pure-translation subgroup $\Gamma \leq \text{Aut}(\mathbb{L}^d)$, then H is vertex-transitive, since the translation map

$$t_a : \mathbb{Z}^d / L_\Gamma \rightarrow \mathbb{Z}^d / L_\Gamma, \quad x + L_\Gamma \mapsto x + a + L_\Gamma,$$

is an automorphism of H , for any $a \in \mathbb{Z}^d$. By Theorem 40, with high probability the largest component of G_n has order at most $n^{5/6}$, and by Theorem 44, with high probability, all but at most $n/2$ vertices of G_n are in components of G_n that are isomorphic to a quotient of \mathbb{L}^d by a pure-translation subgroup of $\text{Aut}(\mathbb{L}^d)$, provided n_0 is sufficiently large depending on d and r . If these two conditions hold, and there are k components $(H_1, \dots, H_k, \text{say})$ that are isomorphic to a quotient of \mathbb{L}^d by a pure-translation subgroup of $\text{Aut}(\mathbb{L}^d)$, then we have

$$\sum_{i=1}^k |H_i| \geq n/2, \quad 3 \leq |H_i| \leq n^{5/6} \quad \forall i \in [k],$$

so

$$|\text{Aut}(G_n)| \geq \prod_{i=1}^k |\text{Aut}(H_i)| \geq \prod_{i=1}^k |H_i| \geq (n^{5/6})^{\lfloor n/(2n^{5/6}) \rfloor} = \exp(\Omega(n^{1/6} \log n)).$$

This proves the proposition. ■

Acknowledgements. We would like to thank Agelos Georgakopoulos, Alex Lubotzky, Mikael De La Salle, Omri Sarig and Romain Tessera for very helpful discussions. We also thank an anonymous referee for their careful reading of the paper, for several suggestions that shortened some of the arguments in our original manuscript, and for recommendations which we have adopted to improve the readability of the paper.

References

- [1] Angel, O., Schramm, O.: Uniform infinite planar triangulations. *Comm. Math. Phys.* **241**, 191–213 (2003) [Zbl 1098.60010](#) [MR 2013797](#)
- [2] Aronshtam, L., Linial, N.: When does the top homology of a random simplicial complex vanish? *Random Structures Algorithms* **46**, 26–35 (2015) [Zbl 1326.55019](#) [MR 3291292](#)
- [3] Aronshtam, L., Linial, N.: The threshold for d -collapsibility in random complexes. *Random Structures Algorithms* **48**, 260–269 (2016) [Zbl 1332.05158](#) [MR 3449598](#)
- [4] Aronshtam, L., Linial, N., Łuczak, T., Meshulam, R.: Collapsibility and vanishing of top homology in random simplicial complexes. *Discrete Comput. Geom.* **49**, 317–334 (2013) [Zbl 1260.05171](#) [MR 3017914](#)
- [5] Ballmann, W.: Nonpositively curved manifolds of higher rank. *Ann. of Math. (2)* **122**, 597–609 (1985) [Zbl 0585.53031](#) [MR 819559](#)
- [6] Barrett, P.: Implementing the Rivest Shamir and Adleman public key encryption algorithm on a standard digital signal processor. In: *Advances in Cryptology—CRYPTO '86* (Santa Barbara, CA, 1986), *Lecture Notes in Computer Sci.* 263, Springer, Berlin, 311–323 (1987) [MR 907099](#)

- [7] Benjamini, I.: *Coarse Geometry and Randomness*. Lecture Notes in Math. 2100, Springer, Cham (2013) Zbl [1282.05001](#) MR [3156647](#)
- [8] Benjamini, I., Ellis, D.: On the structure of graphs which are locally indistinguishable from a lattice. *Forum Math. Sigma* **4**, art. e31, 29 pp. (2016) Zbl [1358.05236](#) MR [3579609](#)
- [9] Benjamini, I., Schramm, O.: Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6**, art. 23, 13 pp. (2001) Zbl [1010.82021](#) MR [1873300](#)
- [10] Bieberbach, L.: Über die Bewegungsgruppen der Euklidischen Räume I. *Math. Ann.* **70**, 297–336 (1911) JFM [42.0144.02](#) MR [1511623](#)
- [11] Bieberbach, L.: Über die Bewegungsgruppen der Euklidischen Räume II. *Math. Ann.* **72**, 400–412 (1912)
- [12] Bollobás, B.: A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. Preprint Series, Matematisk Institut, Aarhus Universitet (1979)
- [13] Bollobás, B.: A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *Eur. J. Combin.* **1**, 311–316 (1980) Zbl [0457.05038](#) MR [595929](#)
- [14] Bollobás, B.: Random graphs. In: *Combinatorics* (Swansea, 1981), London Math. Soc. Lecture Note Ser. 52, Cambridge Univ. Press, Cambridge, 80–102 (1981) Zbl [0489.05050](#) MR [633650](#)
- [15] Bollobás, B.: The asymptotic number of unlabelled regular graphs. *J. London Math. Soc.* (2) **26**, 201–206 (1982) Zbl [0504.05051](#) MR [675164](#)
- [16] Bollobás, B.: The isoperimetric number of random regular graphs. *Eur. J. Combin.* **9**, 241–244 (1988) Zbl [0673.05086](#) MR [947025](#)
- [17] Brigham, N. A.: A general asymptotic formula for partition functions. *Proc. Amer. Math. Soc.* **1**, 182–191 (1950) Zbl [0037.16903](#) MR [34409](#)
- [18] Brown, K. S.: *Cohomology of Groups*. Grad. Texts in Math. 87, Springer, New York (1982) Zbl [0584.20036](#) MR [672956](#)
- [19] Burns, K., Spatzier, R.: Manifolds of nonpositive curvature and their buildings. *Inst. Hautes Études Sci. Publ. Math.* **65** 35–59 (1987) Zbl [0643.53037](#) MR [908215](#)
- [20] De La Salle, M., Tessera, R.: Local-to-global rigidity of Bruhat–Tits buildings. *Illinois J. Math.* **60**, 641–654 (2016) Zbl [1376.20033](#) MR [3707639](#)
- [21] De la Salle, M., Tessera, R.: Characterizing a vertex-transitive graph by a large ball. *J. Topol.* **12**, 705–743 (2019) Zbl [1428.05131](#) MR [4072155](#)
- [22] Dirac, G. A.: Some theorems on abstract graphs. *Proc. London Math. Soc.* (3) **2**, 69–81 (1952) Zbl [0047.17001](#) MR [47308](#)
- [23] Erdős, P., Lehner, J.: The distribution of the number of summands in the partitions of a positive integer. *Duke Math. J.* **8**, 335–345 (1941) JFM [67.0126.02](#) MR [4841](#)
- [24] Erdős, P., Rényi, A.: On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5**, 17–61 (1960) Zbl [0103.16301](#) MR [125031](#)
- [25] Feldman, D., Impagliazzo, R., Naor, M., Nisan, N., Rudich, S., Shamir, A.: On dice and coins: models of computation for random generation. *Inform. and Comput.* **104**, 159–174 (1993) Zbl [0772.68036](#) MR [1221887](#)
- [26] Flajolet, P., Sedgewick, R.: *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge (2009) Zbl [1165.05001](#) MR [2483235](#)
- [27] Fristedt, B.: The structure of random partitions of large integers. *Trans. Amer. Math. Soc.* **337**, 703–735 (1993) Zbl [0795.05009](#) MR [1094553](#)
- [28] Georgakopoulos, A.: On covers of graphs by Cayley graphs. *Eur. J. Combin.* **64**, 57–65 (2017) Zbl [1365.05227](#) MR [3658820](#)
- [29] Gillet, H., Grayson, D. R.: Volumes of symmetric spaces via lattice points. *Doc. Math.* **11**, 425–447 (2006) Zbl [1116.11028](#) MR [2288076](#)
- [30] Gromov, M.: Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.* **53**, 53–73 (1981) Zbl [0474.20018](#) MR [623534](#)

- [31] Gross, J. L., Tucker, T. W.: Topological Graph Theory. Wiley, New York (1987) Zbl 0621.05013 MR 898434
- [32] Gruenberg, K. W.: Residual properties of infinite soluble groups. Proc. London Math. Soc. (3) **7**, 29–62 (1957) Zbl 0077.02901 MR 87652
- [33] Gurel-Gurevich, O., Nachmias, A.: Recurrence of planar graph limits. Ann. of Math. (2) **177**, 761–781 (2013) Zbl 1262.05031 MR 3010812
- [34] Hardy, G. H., Ramanujan, S.: Asymptotic formulae for the distribution of integers of various types. Proc. London Math. Soc. **2**, 112–132 (1917) JFM 46.0198.03
- [35] Hardy, G. H., Ramanujan, S.: Asymptotic formulae in combinatory analysis. Proc. London Math. Soc. (2) **17**, 75–115 (1918) Zbl 46.0198.04 MR 1575586
- [36] Harvey, D., van der Hoeven, J.: Integer multiplication in time $O(n \log n)$. Ann. of Math. (2) **193**, 563–617 (2021) Zbl 07331716 MR 4224716
- [37] Hiller, H.: Crystallography and cohomology of groups. Amer. Math. Monthly **93**, 765–779 (1986) Zbl 0607.20028 MR 867103
- [38] Hirsch, K. A.: On infinite soluble groups. III. Proc. London Math. Soc. (2) **49**, 184–194 (1946) Zbl 0063.02021 MR 17281
- [39] Hoory, S.: The size of bipartite graphs with a given girth, J. Combin. Theory Ser. B **86**, 215–220 (2002) Zbl 1027.05050 MR 1933460
- [40] Kahle, M.: Random simplicial complexes. In: J. E. Goodman et al. (eds.), Handbook of Discrete and Computational Geometry, CRC Press, Boca Raton, FL, 581–603 (2017)
- [41] Kwak, J. H., Nedela, R.: Graphs and Their Coverings. Lecture Notes Ser. 17, Combinatorial and Computational Mathematics Center, POSTECH, Pohang (2005); <http://www.savbb.sk/~nedela/graphcov.pdf>
- [42] Lazebnik, F., Ustimenko, V. A., Woldar, A. J.: A new series of dense graphs of high girth. Bull. Amer. Math. Soc. (N.S.) **32**, 73–79 (1995) Zbl 0822.05039 MR 1284775
- [43] Le Gall, J.-F.: The topological structure of scaling limits of large planar maps. Invent. Math. **169**, 621–670 (2007) Zbl 1132.60013 MR 2336042
- [44] Le Gall, J.-F.: Large random planar maps and their scaling limits. In: A. Ran et al. (eds.), European Congress of Mathematics (Amsterdam, 2008), Eur. Math. Soc., Zürich, 253–276 (2010) Zbl 1195.60017 MR 2648329
- [45] Le Gall, J.-F.: Random geometry on the sphere. In: Proc. Int. Congress of Mathematicians (Seoul, 2014), 421–442 Zbl 1373.60025 MR 3728478
- [46] Linial, N., Meshulam, R.: Homological connectivity of random 2-complexes. Combinatorica **26**, 475–487 (2006) Zbl 1121.55013 MR 2260850
- [47] Linial, N., Peled, Y.: On the phase transition in random simplicial complexes. Ann. of Math. (2) **184**, 745–773 (2016) Zbl 1348.05193 MR 3549622
- [48] Lubotzky, A.: High dimensional expanders. In: Proc. Int. Congress of Mathematicians (Rio de Janeiro, 2018), Vol. I, World Sci., Hackensack, NJ, 705–730 (2018) Zbl 1448.05125 MR 3966743
- [49] Lubotzky, A., Meshulam, R.: Random Latin squares and 2-dimensional expanders. Adv. Math. **272**, 743–760 (2015) Zbl 1306.05020 MR 3303247
- [50] Lubotzky, A., Phillips, R., Sarnak, P.: Ramanujan graphs. Combinatorica **8**, 261–277 (1988) Zbl 0661.05035 MR 963118
- [51] Lubotzky, A., Segal, D.: Subgroup Growth. Progr. Math. 212, Birkhäuser, Basel (2003) Zbl 1071.20033 MR 1978431
- [52] Márquez, A., de Mier, A., Noy, M., Revuelta, M. P.: Locally grid graphs: classification and Tutte uniqueness. Discrete Math. **266**, 327–352 (2003) Zbl 1032.05038 MR 1991727
- [53] McKay, B. D., Wormald, N. C.: Automorphisms of random graphs with specified vertices. Combinatorica **4**, 325–338 (1984) Zbl 0555.60012 MR 779894

-
- [54] Nijenhuis, A., Wilf, H. S.: A method and two algorithms on the theory of partitions. *J. Combin. Theory Ser. A* **18**, 219–222 (1975) Zbl [0298.05015](#) MR [376375](#)
- [55] Odlyzko, A. M.: Explicit Tauberian estimates for functions with positive coefficients. *J. Comput. Appl. Math.* **41**, 187–197 (1992) Zbl [0763.40006](#) MR [1181719](#)
- [56] Schwarzenberger, R. L. E.: *n*-Dimensional Crystallography. Res. Notes in Math. 41, Pitman (Advanced Publishing Program), Boston, MA (1980) Zbl [0419.20001](#) MR [586945](#)
- [57] Thomassen, C.: Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface. *Trans. Amer. Math. Soc.* **323**, 605–635 (1991) Zbl [0722.05031](#) MR [1040045](#)
- [58] Tutte, W. T.: A census of planar triangulations. *Canad. J. Math.* **14**, 21–38 (1962) Zbl [0103.39603](#) MR [130841](#)
- [59] Tutte, W. T.: On the enumeration of planar maps. *Bull. Amer. Math. Soc.* **74**, 64–74 (1968) Zbl [0157.31101](#) MR [218276](#)
- [60] Uspensky, J. V.: Asymptotic formulae for numerical functions which occur in the theory of partitions, *Bull. Acad. Sci. URSS* **14** (1920), 199–218 (in Russian)
- [61] Wormald, N.: Some Problems in the enumeration of labelled graphs. PhD thesis, Univ. of Newcastle (1978)
- [62] Wormald, N. C.: Models of random regular graphs. In: *Surveys in Combinatorics* (Canterbury, 1999), London Math. Soc. Lecture Note Ser. 267, Cambridge Univ. Press, Cambridge, 239–298 (1999) Zbl [0935.05080](#) MR [1725006](#)