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Slope inequalities and a Miyaoka–Yau type inequality

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Abstract. For a minimal smooth projective surface S of general type over a field of characteristic $p > 0$, we prove that

$$K_S^2 \le 32\chi(\mathcal{O}_S).$$

Moreover, if $18\chi(\mathcal{O}_S) < K_S^2 \le 32\chi(\mathcal{O}_S)$, the Albanese morphism of S must induce a genus 2 fibration. A classification of surfaces with $K_S^2 = 32\chi(\mathcal{O}_S)$ is also given. The inequality also implies $\chi(\mathcal{O}_S) > 0$, which answers completely a question of Shepherd-Barron.

Keywords. Slope inequality, fibration of surfaces, Miyaoka–Yau type inequality, Albanese fibration

1. Introduction

Let S be a smooth projective surface of general type over an algebraically closed field $\mathbf{k}$. When $\mathbf{k} = \mathbb{C}$, we have the celebrated Miyaoka–Yau inequality (see [13, 23])

$$c_1^2(S) \le 3c_2(S). \tag{1}$$

By Noether's formula (see [1, Chap. I, (5.5)])

$$12\chi(\mathcal{O}_S) = c_1^2(S) + c_2(S), \tag{2}$$

the Miyaoka–Yau inequality (1) can also be formulated as

$$c_1^2(S) \le 9\chi(\mathcal{O}_S). \tag{3}$$

The Miyaoka–Yau inequality (3) plays an important role in the study of complex algebraic surfaces (see e.g. [3, 16]).

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When char($\mathbf{k}$) $= p > 0$, Noether's formula (2) remains true (see [2, §5]), but the Miyaoka–Yau inequality (1) fails. In fact, Raynaud's examples (see [17] or §4.1 below) show that there exist minimal smooth projective surfaces S of general type with $c_2(S) < 0$, a contradiction to (1) as $c_1^2(S) = K_S^2 > 0$ when S is minimal. As $c_2(S)$ can be negative, a natural question is whether $\chi(\mathcal{O}_S)$ can be negative or not. Shepherd-Barron has shown that $\chi(\mathcal{O}_S) > 0$ unless its Albanese map induces a fibration $f : S \to C$ with singular generic fibre of arithmetic genus $2 \le g \le 4$ and $p \le 7$ (see [18, Theorem 8] or Theorem 3.1 below). However, the question whether there exists such a surface S with $\chi(\mathcal{O}_S) < 0$ remains unsolved (see [18, Remark p. 268]). Shepherd-Barron also suggested that the most obvious place to look for such examples would be where $(p, g) = (2, 2)$. Later, it was proved by the first author [6] that $\chi(\mathcal{O}_S) > 0$ when $p \ge 3$. Our main result in this article is a Miyaoka–Yau type inequality

$$K_S^2 \le 32\chi(\mathcal{O}_S)$$

for all smooth projective surfaces S of general type, which in particularly implies that $\chi(\mathcal{O}_S) > 0$ for any p and answers Shepherd-Barron's question completely.

We observe that the above Miyaoka–Yau type inequality follows in fact from a series of slope inequalities. Let $f : S \to C$ be a relatively minimal surface fibration of genus $g \ge 2$ over $\mathbf{k}$. The slope inequalities are numerical relations between $K_{S/C}^2$ and $\chi_f := \deg(f_*\omega_{S/C})$. When $\mathbf{k} = \mathbb{C}$, we have Xiao's slope inequality

$$K_{S/C}^2 \ge \frac{4g-4}{g}\chi_f. \tag{4}$$

It was proved for any minimal fibration by G. Xiao [22] and for semi-stable fibrations independently by Cornalba and Harris [5]. Some other proofs have also been given [15]. In this paper, we first prove a partial generalization of Xiao's slope inequality in positive characteristic.

Theorem 1.1. *Let $f : S \to C$ be a relatively minimal fibration of genus $g \ge 2$ over an algebraically closed field $\mathbf{k}$ of positive characteristic. Assume any one of the following assumptions is true:*

(a) *the generic fibre of f is hyperelliptic;*

(b) *the generic fibre of f is smooth;*

(c) *the genus $b := g(C)$ satisfies $b \le 1$,*

Then Xiao's slope inequality $K_{S/C}^2 \ge \frac{4g-4}{g}\chi_f$ holds.

Note that in positive characteristic, the generic fibre of f may be non-smooth (i.e. singular). It should also be pointed out that in case the generic fibre of f is singular, we do not have the nefness of $K_{S/C}$ and the semi-positivity of $f_*\omega_{S/C}$. As a result, both $K_{S/C}^2$ and χ_f may be negative (see §4.1 for an example). Under the assumption that f has a smooth generic fibre, Xiao's slope inequality (4) has already been proven [19] by H. Sun and the last two authors.

We then point out that as one of the positive characteristic pathologies, Xiao's slope inequality (4) fails in general.

Proposition 1.2 (see §4.2). *For infinitely many integers $g \geq 3$, there exists a relatively minimal surface fibration $f : S \to C$ of genus g over an algebraically closed field* $\mathbf{k}$ *of positive characteristic such that*

$$K_{S/C}^2 < \frac{4g-4}{g}\chi_f.$$

For general fibrations or fibrations of small genus, we also give some different slope inequalities.

Theorem 1.3. *Let $f : S \to C$ be a relatively minimal fibration of genus $g \geq 3$ over an algebraically closed field* $\mathbf{k}$ *and $b := g(C)$. Then*

(a) *if K_S is nef, then $K_S^2 \geq \frac{2g-2}{g}\deg(f_*\omega_S)$;*

(b) *if $g = 3$ and f is non-hyperelliptic, then $K_{S/C}^2 \geq 3\chi_f$;*

(c) *if $g = 4$ and K_S is nef, then $7K_{S/C}^2 \geq 15\chi_f - 48(b-1)$;*

(d) *if $g \geq 5$ and K_S is nef, then*

$$K_{S/C}^2 \geq \frac{2(g-1)(g-2)}{g^2-3g+1}\chi_f - \frac{4(g-1)(g^2-4g+2)}{g^2-3g+1}(b-1).$$

When $\mathbf{k} = \mathbb{C}$, the slope inequality in (b) for non-hyperelliptic fibrations of genus 3 was proved by Horikawa [7] and Konno (see [8]). For non-hyperelliptic fibrations with $g = 4, 5$, Konno [9] and Chen [4] have also given some other slope inequalities. In positive characteristic, Yuan and Zhang have given a slope inequality for general genus g in [24, Lem. 3.2] in terms of $b = g(C)$. Our slope inequalities in (c) and (d) are different.

Now let us return to the Miyaoka–Yau type inequality. For a minimal surface S of general type, when $c_2(S) \geq 0$, Noether's formula (2) already implies that $K_S^2 \leq 12\chi(\mathcal{O}_S)$. So we assume $c_2(S) < 0$. Shepherd-Barron [18, Thm. 6] has shown that the Albanese map of S induces a fibration $f : S \to C$ of (arithmetic) genus $g \geq 2$ and $b := g(C) \geq 2$. We will call such an $f : S \to C$ the *Albanese fibration* of S. As an application of Theorems 1.1 and 1.3, we have

Theorem 1.4. *If $c_2(S) < 0$, let $f : S \to C$ be the Albanese fibration of genus $g \geq 2$. Then*

$$K_S^2 \leq \begin{cases} \dfrac{(12g+8)(g-1)}{g^2-g-1}\chi(\mathcal{O}_S) & \text{if the generic fibre is hyperelliptic,} \\ 18\chi(\mathcal{O}_S) & \text{if } g = 3, \\ \dfrac{840}{47}\chi(\mathcal{O}_S) & \text{if } g = 4, \\ \dfrac{12(g-1)(3g^2-4g-4)}{g(3g^2-12g+15)}\chi(\mathcal{O}_S) & \text{if } g \geq 5. \end{cases}$$

By the above results and elementary computations, we have

Theorem 1.5. *Let S be a minimal smooth projective surface of general type. Then $K_S^2 \leq 32\chi(\mathcal{O}_S)$. Moreover, when*

$$18\chi(\mathcal{O}_S) < K_S^2 \leq 32\chi(\mathcal{O}_S),$$

the Albanese fibration of S must be a fibration of genus 2.

Examples of surfaces S with $K_S^2 = 32\chi(\mathcal{O}_S)$ are given in §4.3. Moreover, we give in §4.2 an example of S whose Albanese fibration is of genus 3 and $K_S^2 = 18\chi(\mathcal{O}_S)$ to show that the inequality $18\chi(\mathcal{O}_S) < K_S^2 \leq 32\chi(\mathcal{O}_S)$ in the theorem is optimal.

This theorem answers the question of Shepherd-Barron and leads to the following classification of surfaces with $\chi(\mathcal{O}_S) < 0$ after Liedtke [12]. In addition, the above Miyaoka–Yau type inequality can also be used to study the canonical map of surfaces of general type as in [3].

Theorem 1.6 (after [12, Prop. 8.5]). *Let S be a smooth projective surface over an algebraically closed field* $\mathbf{k}$ *of characteristic $p > 0$ with $\chi(\mathcal{O}_S) < 0$. Then either*

(1) *S is birational to $\mathbb{P}^1 \times C$ with $g(C) = 1 - \chi(\mathcal{O}_S)$, or*

(2) *S is quasi-elliptic of Kodaira dimension 1 and $p = 2$ or 3.*

This paper is organized as follows.

In §2, we first recall Xiao's approach to slope inequalities and then prove Theorem 1.1 (cf. Theorem 2.7) and Theorem 1.3 (cf. Propositions 2.9 and 2.10).

In §3, we prove Theorem 1.4 (cf. Theorem 3.2) by applying Theorems 1.1 and 1.3, and then we prove Theorem 1.5 (cf. Corollary 3.4).

Finally, in §4 we recall or give the following examples in positive characteristic:

- Raynaud's examples of minimal surfaces S of general type with $c_2(S) < 0$. In his examples, the Albanese fibration of S is hyperelliptic with $g = (p-1)/2$ and attains equality in Theorem 1.5 for the hyperelliptic case. Moreover, the Albanese fibration $f : S \to C$ satisfies Xiao's equality.
- Examples of fibrations violating Xiao's slope inequality (hence proving Proposition 1.2).
- Examples of general type surfaces with $K_S^2 = 32\chi(\mathcal{O}_S)$.

Conventions

- A *surface fibration* (or simply a *fibration*) is a flat morphism $f : S \to C$ from a projective smooth surface to a smooth curve over an algebraically closed field such that $f_*\mathcal{O}_S = \mathcal{O}_C$. In particular, all geometric fibres of f are connected. In positive characteristic, the generic fibre of f may be singular.
- For a surface fibration $f : S \to C$, we denote
  - K_S (resp. K_C) := the canonical divisor of S (resp. C);
  - $K_{S/C} := K_S - f^*K_C$;
  - $\chi_f := \deg(f_*\omega_{S/C})$;
  - $l(f) := \dim_{\mathbf{k}}(R^1 f_*\mathcal{O}_S)_{\text{tor}}$.

Let g be the genus of fibres of f and $b := g(C)$. Then

$$K_{S/C}^2 = K_S^2 - 8(g-1)(b-1), \tag{5}$$

$$\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1) + l(f) \tag{6}$$

by Riemann–Roch and Leray's spectral sequence.

- An integral curve over $\mathbf{k}$ is called *hyperelliptic* if it admits a flat double cover of $\mathbb{P}^1_{\mathbf{k}}$, and a surface fibration $f : S \to C$ is called hyperelliptic if a general fibre of f is hyperelliptic.

2. Slope inequalities for fibrations in positive characteristic

In this section, we study some slope inequalities in positive characteristic. Our strategy is based on Xiao's approach to slope inequalities. We prove Xiao's slope inequality for some special fibrations (see Theorem 2.7). We also observe that Xiao's slope inequality cannot hold for the general case (see Remark 2.8), and prove some other slope inequalities (see Propositions 2.9 and 2.10).

2.1. *Xiao's slope inequality in positive characteristic*

In [22], Xiao introduces slope inequalities for fibrations $f : S \to C$ by studying the Harder–Narasimhan filtration of $f_*\omega_{S/C}$. For the readers' convenience, we briefly recall the idea.

Let C be a smooth projective curve over an algebraically closed field. For a vector bundle E on C, let

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}$$

where $\mathrm{rk}(E)$ and $\deg(E)$ denote the rank and degree of E respectively. The vector bundle E is called *semi-stable* if $\mu(E') \leq \mu(E)$ for any subbundle $E' \subseteq E$. One has the following well-known theorem.

Theorem 2.1 (Harder–Narasimhan filtration). *For any vector bundle E on C, there exists a unique filtration of subbundles*

$$0 =: E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

called the Harder–Narasimhan filtration*, such that*

(1) *each subquotient bundle E_i/E_{i-1} is semi-stable for $1 \leq i \leq n$,*

(2) *$\mu_1 > \cdots > \mu_n$, where $\mu_i := \mu(E_i/E_{i-1})$ for $1 \leq i \leq n$.*

We denote by $\mu_{\min}(E)$ the last slope μ_n of E.

For a relatively minimal fibration $f : S \to C$ of genus $g \geq 2$ over $\mathbb{C}$, by using the Harder–Narasimhan filtration of $f_*\omega_{S/C}$,

$$0 =: E_0 \subset E_1 \subset \cdots \subset E_n := f_*\omega_{S/C},$$

Xiao constructs a sequence of effective divisors

$$Z_1 \geq \cdots \geq Z_n \geq 0$$

such that $N_i = K_{S/C} - Z_i - \mu_i F$ $(1 \leq i \leq n)$ are nef $\mathbb{Q}$-divisors. Here F is a fibre of f and $\mu_i := \mu(E_i/E_{i-1})$. Then he uses the following elementary lemma, which holds also for characteristic $p > 0$, to get a lower bound of $K^2_{S/C}$.

Lemma 2.2 ([22, Lem. 2]). *Let $f : S \to C$ be a relatively minimal fibration with a general fibre F, and D be a nef (resp. f-nef) divisor on S. Suppose that there are a sequence of effective divisors*

$$Z_1 \geq \cdots \geq Z_n \geq Z_{n+1} = 0$$

(resp. such that Z_n is vertical) and a sequence of rational numbers

$$\mu_1 > \cdots > \mu_n, \quad \mu_{n+1} = 0,$$

such that $N_i := D - Z_i - \mu_i F$ $(1 \leq i \leq n)$ are nef $\mathbb{Q}$-divisors. Then

$$D^2 \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}), \quad \textit{where } d_i = N_i \cdot F.$$

Xiao's approach cannot be applied directly in positive characteristic. A key point is the failure of the following lemma in positive characteristic.

Lemma 2.3 ([14, Thm. 3.1], or [22, Lem. 3]). *Over $\mathbb{C}$, for any vector bundle E on C, the $\mathbb{Q}$-divisor $\mathcal{O}_{\mathbb{P}(E)}(1) - \mu_{\min}(E) \cdot \Gamma$ is nef on $\mathbb{P}(E)$ where Γ is a fibre of $\mathbb{P}(E) \to C$.*

A key observation of [19] is that one can apply Lemma 2.4 below instead of Lemma 2.3 to generalise Xiao's approach to positive characteristic. Until the end of this section we assume C is defined over an algebraically closed field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = p > 0$.

Lemma 2.4. *If the quotient bundles E_i/E_{i-1} of the Harder–Narasimhan filtration of a vector bundle E on C are all strongly semi-stable (definition recalled below), then the $\mathbb{Q}$-divisor $\mathcal{O}_{\mathbb{P}(E)}(1) - \mu_{\min}(E) \cdot \Gamma$ is nef on $\mathbb{P}(E)$, where Γ is a fibre of $\mathbb{P}(E) \to C$.*

Recall that if $F_C : C \to C$ is the (absolute) Frobenius morphism, a bundle E on C is called *strongly semi-stable* (resp., *strongly stable*) if its pull back by the k-th power F_C^k is semi-stable (resp., stable) for any integer $k \geq 0$.

Theorem 2.5 ([10, Thm. 3.1]). *For any vector bundle E on C, there exists an integer k_0 such that all quotients E_i/E_{i-1} $(1 \leq i \leq n)$ of the Harder–Narasimhan filtration*

$$0 =: E_0 \subset E_1 \subset \cdots \subset E_n = F_C^{k*} E$$

are strongly semi-stable whenever $k \geq k_0$.

Remark 2.6. When $g(C) \leq 1$, semi-stable vector bundles are already strongly semi-stable (see [20]). In particular, we can take $k = k_0 = 0$ in Theorem 2.5.

Now suppose $f : S \to C$ is a relatively minimal fibration of genus $g \geq 2$. Take $E = f_*\omega_{S/C}$ and fix a $k \geq k_0$. Denote by

$$0 = E_0 \subset \cdots \subset E_n = F_C^{k*}E = F_C^{k*} f_*\omega_{S/C} \tag{7}$$

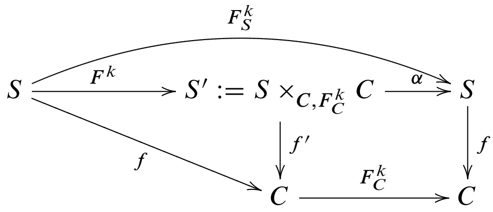
the Harder–Narasimhan filtration, $r_i := \mathrm{rk}(E_i)$, $\mu_i := \mu(E_i/E_{i-1})$ and call $\widetilde{\mu}_i := \mu_i/p^k$ the *normalised slopes*. Then (recall $\mu_{n+1} = 0$)

$$\chi_f = \frac{1}{p^k}\deg(F_C^{k*}E) = \frac{1}{p^k}\sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) = \sum_{i=1}^{n} r_i(\widetilde{\mu}_i - \widetilde{\mu}_{i+1}). \tag{8}$$

Let us now recall the construction given in [19] of effective divisors

$$Z_1 \geq \cdots \geq Z_n \geq 0$$

such that $N_i := p^k K_{S/C} - Z_i - \mu_i F$ is nef. Consider the commutative diagram



and, for each i, the natural homomorphism

$$f^*E_i \hookrightarrow f^*F_C^{k*}E = F_S^{k*}f^*f_*\mathcal{O}_S(K_{S/C}) \to F_S^{k*}\omega_{S/C} = \mathcal{O}_S(p^k K_{S/C}).$$

Denote by $\mathcal{L}_i \subseteq F_S^{k*}\omega_{S/C}$ the image of this homomorphism; then we can write $\mathcal{L}_i = \mathcal{I}_{T_i} \cdot F_S^{k*}\omega_{S/C}(-Z_i)$ for a unique closed subscheme T_i of codimension 2 and a unique effective divisor Z_i. It is clear by construction that $Z_1 \geq \cdots \geq Z_n \geq 0$ and Z_n is vertical. Let $U_i := S \backslash T_i$, so there is a morphism over C

$$\phi_i : U_i \to \mathbb{P}(E_i)$$

such that $\phi_i^*\mathcal{O}_{\mathbb{P}(E_i)}(1) = \mathcal{L}_i|_{U_i}$ by the construction of $\mathcal{L}_i$. Note that the $\mathbb{Q}$-divisor $c_1(\mathcal{L}_i) - \mu_i F$ is nef by Theorem 2.5 and Lemma 2.4 as the complement of U_i consists of finitely many points, here F is a general fibre of f. In other words, we have the nefness of

$$N_i = c_1(\mathcal{L}_i) - \mu_i F$$

by construction. Then Xiao's approach applies in positive characteristic and we have

Theorem 2.7. *Let $f : S \to C$ be a relatively minimal fibration of genus $g \geq 2$, and assume any one of the following conditions holds:*

(a) *the generic fibre of f is hyperelliptic;*

(b) *the generic fibre of f is smooth;*

(c) *the genus $b := g(C)$ satisfies $b \leq 1$.*

Then Xiao's slope inequality $K_{S/C}^2 \geq \frac{4g-4}{g}\chi_f$ holds.

Proof. Let $d_i := N_i \cdot F = \deg(\mathcal{L}_i|_F)$, $d_{n+1} = p^k(2g-2)$ and $\widetilde{d}_i := d_i/p^k$. By Lemma 2.2, we have

$$p^{2k}K_{S/C}^2 \geq \sum_{i=1}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \tag{9}$$

or equivalently

$$K_{S/C}^2 \geq \sum_{i=1}^{n}(\widetilde{d}_i + \widetilde{d}_{i+1})(\widetilde{\mu}_i - \widetilde{\mu}_{i+1}). \tag{10}$$

If the Clifford type inequalities

$$d_i \geq p^k(2r_i - 2) \quad \text{or equivalently} \quad \widetilde{d}_i \geq 2(r_i - 1) \qquad (1 \leq i \leq n) \tag{11}$$

hold, we have by (9) the inequality (see [19, p. 695])

$$K_{S/C}^2 \geq 4\chi_f - \frac{2}{p^k}(\mu_1 + \mu_n) = 4\chi_f - 2(\widetilde{\mu}_1 + \widetilde{\mu}_n), \tag{12}$$

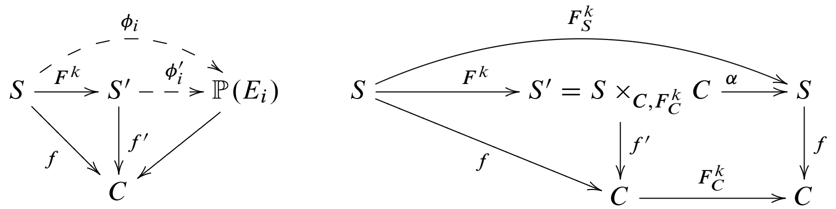
which, together with the inequality

$$p^k K_{S/C}^2 \geq (2g-2)(\mu_1 + \mu_n) \quad \text{or equivalently} \quad K_{S/C}^2 \geq (2g-2)(\widetilde{\mu}_1 + \widetilde{\mu}_2)$$

(obtained by applying Lemma 2.2 to $D = p^k K_{S/C}$, $Z_1 \geq Z_n \geq 0$ and $\mu_1 \geq \mu_n$), implies

$$K_{S/C}^2 \geq \frac{4g-4}{g}\chi_f.$$

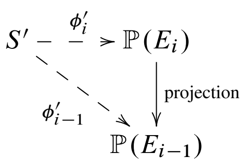
In conclusion, our theorem follows if the Clifford type inequalities (11) hold under any one of the assumptions in the theorem. It remains to estimate d_i in order to prove (11). Consider commutative diagrams



where $\phi_i' : S' \dashrightarrow \mathbb{P}(E_i)$ is defined by the image $\mathcal{L}_i' \subseteq \alpha^*\omega_{S/C}$ of $f'^*(E_i)$ under the canonical homomorphism

$$f'^*(E_i) \subset f'^* F_C^{k*} f_*\omega_{S/C} = \alpha^* f^* f_*\omega_{S/C} \to \alpha^*\omega_{S/C}.$$

Then $\deg(\phi_i') \mid \deg(\phi_{i-1}')$ and $\deg(\phi_n') = \deg(\phi_{|\omega_{S'/C}|})$ since



is commutative. To prove the Clifford type inequalities (11), noting that ϕ_i is well-defined on a general fiber F, we have

$$d_i = N_i \cdot F = p^k \deg(\mathcal{L}'_i|_{F'}), \quad \deg(\mathcal{L}'_i|_{F'}) = \deg(\phi'_i)\deg(\phi_i(F)).$$

Now when $f : S \to C$ is hyperelliptic, we have $2 = \deg(\phi'_n) \mid \deg(\phi'_i)$ for all i and hence $\deg(\phi'_i) \geq 2$ $(1 \leq i \leq n)$. Noting $\deg(\phi_i(F)) \geq r_i - 1$, we have $d_i = p^k \deg(\phi'_i)\deg(\phi_i(F)) \geq p^k(2r_i - 2)$ or equivalently

$$\widetilde{d}_i \geq 2(r_i - 1) \quad (1 \leq i \leq n). \tag{13}$$

Clifford type inequalities (11) hold when $f : S \to C$ is hyperelliptic.

In non-hyperelliptic cases, if $\mathcal{L}'_i$ is locally free along a general fibre F' of $f' : S' \to C$, then ϕ'_i is also defined along F' and we have $d_i = p^k \deg(\mathcal{L}'_i|_{F'})$. Since $\mathcal{L}'_i \subseteq \alpha^*\omega_{S/C} = \omega_{S'/C}$, $\mathcal{L}'_i|_{F'}$ is a special line bundle and we have $\deg(\mathcal{L}'_i|_{F'}) \geq 2r_i - 2$ by Clifford's theorem (see [11]). Thus we have the desired Clifford type inequalities

$$d_i = p^k \deg(\mathcal{L}'_i|_{F'}) \geq p^k(2r_i - 2) \quad (1 \leq i \leq n).$$

Note that one sufficient condition for $\mathcal{L}'_i$ to be locally free on F' is the normality of the generic fibre $S' \times_C \operatorname{Spec}(k(C))$ of f'. In fact, if the generic fibre is normal, then $\mathcal{L}'$ is automatically locally free on it. Hence $\mathcal{L}'$ is locally free on a general fibre F'. The normality of the generic fibre $S' \times_C \operatorname{Spec}(k(C))$ follows if (b) the generic fibre of f is smooth or (c) $g(C) \leq 1$. In case (b) the generic fibre of f' is moreover smooth and in case (c) we can take $k = k_0 = 0$ by Remark 2.6 and hence $f' = f$.

In conclusion, under either assumption of our theorem, we have the Clifford type inequalities (11) and therefore $K_S^2 \geq \frac{4g-4}{g}\deg(f_*\omega_S)$. ∎

Remark 2.8. The Clifford type inequalities (11) can fail in general. As a result, Xiao's slope inequality does not hold in positive characteristic in general. We shall see such counterexamples in §4.2.

2.2. *Other slope inequalities*

In this subsection we give some other slope inequalities.

Proposition 2.9 ([7, 8]). *Let $f : S \to C$ be a non-hyperelliptic, relatively minimal fibration of genus 3. Then $K_{S/C}^2 \geq 3\chi_f$.*

This result is well known over $\mathbb{C}$ and the proof in [8] works in any characteristic.

Proposition 2.10. *Let $f : S \to C$ be a relatively minimal fibration of genus g such that K_S is nef. Let $b := g(C)$. Then*

(1) $K_S^2 \geq \frac{2g-2}{g}\deg(f_*\omega_S)$*;*

(2) *if $g = 4$, then $7K_{S/C}^2 \geq 15\chi_f - 48(b-1)$;*

(3) *if $g \geq 5$, then*

$$K_{S/C}^2 \geq \frac{2(g-1)(g-2)}{g^2 - 3g + 1}\chi_f - \frac{4(g-1)(g^2 - 4g + 2)}{g^2 - 3g + 1}(b-1).$$

Proof. We adopt the notations used in the previous section. Namely, let $E := f_*\omega_{S/C}$ and

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = F_C^{k*}E$$

be the Harder–Narasimhan filtration of $F_C^{k*}E$ for some $k \geq k_0$ (see Theorem 2.5). Take $\widetilde{\mu}_i, r_i, Z_i, N_i$ and $\widetilde{d}_i$ defined as in the previous section, in the paragraphs preceding (7). Then we see that

$$N_1 = p^k K_{S/C} - Z_1 - \mu_1 \cdot F \equiv p^k K_S - (\mu_1 + 2p^k(b-1)) \cdot F$$

is nef. As K_S is nef, we have

$$\begin{aligned} p^{2k}K_S^2 &\geq p^k K_S \cdot N_1 + p^k K_S \cdot (p^k K_S - N_1) \geq p^k K_S \cdot (\mu_1 + 2p^k(b-1))F \\ &\geq p^k(2g-2)(\mu_1 + p^k(2b-2)). \end{aligned}$$

In other words,

$$K_S^2 \geq 4(g-1)(b-1) + (2g-2) \cdot \widetilde{\mu}_1. \tag{14}$$

Note that by definition we have $g \cdot \widetilde{\mu}_1 \geq \chi_f = \deg(f_*\omega_S) - 2g(b-1)$. Combining this inequality with (14), we obtain

$$K_S^2 \geq \frac{2g-2}{g}\deg(f_*\omega_S).$$

Since $K_S^2 = K_{S/C} + 8(g-1)(b-1)$, we can reformulate (14) as

$$K_{S/C}^2 \geq (2g-2) \cdot \widetilde{\mu}_1 - 4(g-1)(b-1). \tag{15}$$

Let $v_i := \widetilde{\mu}_1 - \widetilde{\mu}_i$, so $0 = v_1 < \cdots < v_n$. By (8) and (10), we have

$$K_{S/C}^2 - 4(g-1) \cdot \widetilde{\mu}_1 \geq -\Big(\sum_{i=1}^{n-1}(\widetilde{d}_i + \widetilde{d}_{i+1})(v_i - v_{i+1}) + 4(g-1)v_n\Big),$$

$$\chi_f - g \cdot \widetilde{\mu}_1 = -\Big(\sum_{i=1}^{n-1} r_i(v_{i+1} - v_i) + g v_n\Big).$$

With the help of Lemma 2.11, we immediately have

$$K_{S/C}^2 - 4(g-1) \cdot \widetilde{\mu}_1 \geq \begin{cases} 5(\chi_f - g \cdot \widetilde{\mu}_1), & g = 4, \\ (2g-4)(\chi_f - g \cdot \widetilde{\mu}_1), & g \geq 5. \end{cases}$$

Combining this with (15), after a simple calculation we obtain the desired inequalities by eliminating $\widetilde{\mu}_1$. ∎

Lemma 2.11. *Let*

$$\Phi := \sum_{i=1}^{n-1}(\widetilde{d}_i + \widetilde{d}_{i+1})(v_i - v_{i+1}) + 4(g-1)v_n, \quad \Psi := \sum_{i=1}^{n-1} r_i(v_{i+1} - v_i) + g v_n.$$

Then

$$\Phi \leq \begin{cases} 5\Psi, & g = 4, \\ (2g-4)\Psi, & g \geq 5. \end{cases}$$

Proof. Take $e_1 := \widetilde{d}_1 + \widetilde{d}_2, e_i = \widetilde{d}_{i+1} - \widetilde{d}_{i-1}$ for $i = 2, \ldots, n-1$ and $e_n = 2g - 2 - \widetilde{d}_{n-1}$. Then $\Phi = \sum_{i=1}^{n} e_i v_i$ and $\Psi = \sum_{i=1}^{n} (r_i - r_{i-1}) v_i, r_0 := 0$. Moreover, $\sum_{i=1}^{n} e_i = 4g - 4$.

Case $r_{n-1} < g - 1$ **or** $n = 1$: When $g = 4$, we have

$$\begin{aligned} 5\Psi &\geq 10v_n + 5v_{n-1} \geq e_n \cdot v_n + (15 - e_n) \cdot v_{n-1} \\ &\geq e_n \cdot v_n + (12 - e_n) \cdot v_{n-1} \geq \Phi \end{aligned}$$

since it is clear that $e_n \leq 2g - 2 < 10$. When $g \geq 5$, we have

$$\begin{aligned} (2g-4)\Psi &\geq (4g-8)v_n + (2g-4)v_{n-1} \geq e_n \cdot v_n + (6g - 12 - e_n) \cdot v_{n-1} \\ &\geq e_n \cdot v_n + (4g - 4 - e_n) \cdot v_{n-1} \geq \Phi \end{aligned}$$

since it is clear that $e_n \leq 2g - 2 < 4g - 8$.

Case $r_{n-1} = g - 1, r_{n-2} < g - 2$ **or** $n = 2$: When $g = 4$, we have

$$\begin{aligned} 5\Psi &\geq 5v_n + 10v_{n-1} \geq e_n \cdot v_n + (15 - e_n) \cdot v_{n-1} \\ &\geq e_n \cdot v_n + (12 - e_n) \cdot v_{n-1} \geq \Phi \end{aligned}$$

since in this case $e_n = 6 - \widetilde{d}_{n-1} \leq 6 - r_{n-1} + 1 = 4 < 5$ by (13). When $g \geq 5$, we have

$$\begin{aligned} (2g-4)\Psi &\geq (2g-4)v_n + (4g-8)v_{n-1} \geq e_n \cdot v_n + (6g - 12 - e_n) \cdot v_{n-1} \\ &\geq e_n \cdot v_n + (4g - 4 - e_n) \cdot v_{n-1} \geq \Phi \end{aligned}$$

since in this case $e_n = 2g - 2 - \widetilde{d}_{n-1} \leq g < 2g - 4$.

Case $r_{n-1} = g - 1, r_{n-2} = g - 2$: When $g = 4$, we have

$$\begin{aligned} 5\Psi &\geq 5v_n + 5v_{n-1} + 5v_{n-2} \geq e_n \cdot v_n + e_{n-1} \cdot v_{n-1} + (15 - e_n - e_{n-1}) \cdot v_{n-2} \\ &\geq e_n \cdot v_n + e_{n-1} \cdot v_{n-1} + (12 - e_n - e_{n-1}) \cdot v_{n-2} \geq \Phi \end{aligned}$$

since in this case $e_n \leq 4$ and $e_{n-1} = 6 - \widetilde{d}_{n-2} \leq 6 - r_{n-2} + 1 = 5$. When $g \geq 5$, we have

$$\begin{aligned} (2g-4)\Psi &\geq (2g-4)v_n + (2g-4)v_{n-1} + (2g-4)v_{n-2} \\ &\geq e_n \cdot v_n + e_{n-1} \cdot v_{n-1} + (6g - 12 - e_n - e_{n-1}) \cdot v_{n-2} \\ &\geq e_n \cdot v_n + e_{n-1} \cdot v_{n-1} + (4g - 4 - e_n - e_{n-1}) \cdot v_{n-2} \geq \Phi \end{aligned}$$

since in this case $e_n \leq g$ and $e_{n-1} = (2g - 2) - \widetilde{d}_{n-2} \leq g + 1 \leq 2g - 4$. ∎

3. Miyaoka–Yau type inequality in positive characteristic

We now start to study Miyaoka–Yau type inequalities. Suppose S is a minimal surface of general type over an algebraically closed field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = p > 0$. If $c_2(S) \geq 0$, we have an immediate Miyaoka–Yau type inequality $c_1^2(S) \leq 12\chi(\mathcal{O}_S)$ obtained from (2). Thus, it suffices to discuss S with $c_2(S) < 0$.

We first recall a fundamental theorem on algebraic surfaces of general type with negative c_2 due to Shepherd-Barron.

Theorem 3.1 (Shepherd-Barron [18, Theorem 8]). *If $c_2(S) < 0$, then the Albanese map of S induces a fibration $f : S \to C$ such that*

- *C is a non-singular projective curve of genus $b := g(C) \geq 2$ and $f_*\mathcal{O}_S \cong \mathcal{O}_C$;*
- *the fibre has (arithmetic) genus $g := p_a(F) \geq 2$;*
- *the geometric generic fibre is a singular rational curve with at least one cusp singularity.*

We call $f : S \to C$ the *Albanese fibration* of S. As an application of Theorem 2.7, Proposition 2.9 and Proposition 2.10, we have

Theorem 3.2. *Let $c_2(S) < 0$ and $f : S \to C$ be the Albanese fibration of S. Then*

$$K_S^2 \leq \begin{cases} \dfrac{(12g+8)(g-1)}{g^2-g-1}\chi(\mathcal{O}_S) & \text{if } f \text{ is a hyperelliptic fibration}, \\ 18\chi(\mathcal{O}_S) & \text{if } g = 3, \\ \dfrac{840}{47}\chi(\mathcal{O}_S) & \text{if } g = 4, \\ \dfrac{12(g-1)(3g^2-4g-4)}{g(3g^2-12g+15)}\chi(\mathcal{O}_S) & \text{if } g \geq 5. \end{cases}$$

Proof. We first recall the numerical relations (5) and (6):

$$\begin{aligned} K_{S/C}^2 &= K_S^2 - 8(g-1)(b-1), \\ \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1) + l(f). \end{aligned}$$

If $f : S \to C$ is hyperelliptic, by Theorem 2.7 one has

$$\begin{aligned} K_S^2 - 8(g-1)(b-1) &\geq \frac{4g-4}{g}\big(\chi(\mathcal{O}_S) - (g-1)(b-1) + l(f)\big) \\ &\geq \frac{4g-4}{g}\left(\frac{1}{12}(K_S^2 + c_2(S)) - (g-1)(b-1)\right) \\ &= \frac{g-1}{3g}K_S^2 + \frac{g-1}{3g}c_2(S) - \frac{4(g-1)^2(b-1)}{g}. \end{aligned}$$

By the inequality $c_2(S) \geq -4(b-1)$ (see [6, (3.3)]), one has

$$K_S^2 \geq \frac{(12g+8)(g-1)}{2g+1}(b-1), \quad \text{in other words} \quad \frac{4(b-1)}{K_S^2} \leq \frac{2g+1}{(g-1)(3g+2)},$$

which implies the first inequality in Theorem 3.2:

$$\begin{aligned} \frac{12\chi(\mathcal{O}_S)}{K_S^2} &= 1 + \frac{c_2(S)}{K_S^2} \geq 1 - \frac{4(b-1)}{K_S^2} \\ &\geq 1 - \frac{2g+1}{(g-1)(3g+2)} = \frac{3(g^2-g-1)}{(3g+2)(g-1)}. \end{aligned}$$

Other inequalities follow from the same computations. In fact, if we have a slope inequality $K_{S/C}^2 \geq \psi(g)\chi_f - \phi(g)(b-1)$ with $\psi(g) < 12$, we get

$$\begin{aligned} K_S^2 - 8(g-1)(b-1) & \\ & \geq \psi(g)\left(\frac{K_S^2 + c_2(S)}{12} - (g-1)(b-1) + l(f)\right) - \phi(g)(b-1) \\ & \geq \psi(g)\left(\frac{K_S^2 - 4(b-1)}{12} - (g-1)(b-1)\right) - \phi(g)(b-1). \end{aligned}$$

Thus

$$K_S^2 \geq \frac{12(8-\psi(g))(g-1) - 4\psi(g) - 12\phi(g)}{12-\psi(g)}(b-1).$$

So we have

$$\frac{12\chi(\mathcal{O}_S)}{K_S^2} = 1 + \frac{c_2(S)}{K_S^2} \geq 1 - \frac{4(12-\psi(g))}{12(8-\psi(g))(g-1) - 4\psi(g) - 12\phi(g)}.$$

Now by Propositions 2.9 and 2.10, we can take

- $\psi(3) = 3, \phi(3) = 0$ if $g = 3$ and f is not hyperelliptic;
- $\psi(4) = 15/7, \phi(4) = 48/7$ if $g = 4$;
- $\psi(g) = \dfrac{2(g-1)(g-2)}{g^2-3g+1}, \phi(g) = \dfrac{4(g-1)(g^2-4g+2)}{g^2-3g+1}$ if $g \geq 5$.

Our theorem then follows from a simple calculation. Note that from the computation, when f is a fibration of genus 3, we get

$$K_S^2 \leq \frac{88}{5}\chi(\mathcal{O}_S) < 18\chi(\mathcal{O}_S)$$

if f is hyperelliptic, and $K_S^2 \leq 18\chi(\mathcal{O}_S)$ if f is non-hyperelliptic. ■

Remark 3.3. (1) From the proof, equality holds in the inequalities in the theorem if and only if $c_2(S) = -4(b-1)$, $l(f) = 0$ and equality holds in the associated slope inequality.

(2) In §4.1 below, we will see that Raynaud's examples attain equality for hyperelliptic fibrations in this theorem.

(3) By Tate's genus change formula (cf. [21] or [6, §2.1]), the genus g is such that $(p-1) \mid 2g$.

Corollary 3.4. *Let S be a minimal smooth projective surface of general type. Then*

$$K_S^2 \leq 32\chi(\mathcal{O}_S).$$

Moreover, when

$$18\chi(\mathcal{O}_S) < K_S^2 \leq 32\chi(\mathcal{O}_S),$$

the Albanese fibration of S is a fibration of genus 2.

Proof. If $c_2(S) \geq 0$, Noether's formula implies $K_S^2 \leq 12\chi(\mathcal{O}_S)$, so it is enough to consider the case $c_2(S) < 0$. Then we have the Albanese fibration $f : S \to C$ of genus $g \geq 2$.

If $f : S \to C$ is hyperelliptic, by Theorem 3.2 we have

$$K_S^2 \leq \frac{(12g+8)(g-1)}{g^2-g-1}\chi(\mathcal{O}_S)$$

where $h(g) = \frac{(12g+8)(g-1)}{g^2-g-1}$ is a decreasing function of g with $h(2) = 32$ (note $h'(g) = -\frac{8g^2+8g+4}{(g^2-g-1)^2} < 0$). Thus $K_S^2 \leq 32\chi(\mathcal{O}_S)$.

If $f : S \to C$ is non-hyperelliptic, we have $K_S^2 \leq 18\chi(\mathcal{O}_S)$ for $g = 3, 4$ immediately from Theorem 3.2 and

$$K_S^2 \leq \frac{12(g-1)(3g^2-4g-4)}{g(3g^2-12g+15)}\chi(\mathcal{O}_S)$$

when $g \geq 5$. It is easy to see that $n(g) = \frac{12(g-1)(3g^2-4g-4)}{g(3g^2-12g+15)}$ is also a decreasing function when $g \geq 5$ since

$$n'(g) = -4\frac{5g^2((g-3)^2-2)+4(g-1)(3g-5)}{(g^3-4g^2+5g)^2} < 0.$$

Thus $K_S^2 \leq n(5)\chi(\mathcal{O}_S) = \frac{408}{25}\chi(\mathcal{O}_S) < 18\chi(\mathcal{O}_S)$. Altogether, we have

$$K_S^2 \leq 32\chi(\mathcal{O}_S)$$

for all minimal smooth projective surfaces S of general type and when $K_S^2 > 18\chi(\mathcal{O}_S)$, $f : S \to C$ must be a fibration of genus 2. ■

In the next section, we shall construct the following examples:

(1) Examples of S with $K_S^2 = 32\chi(\mathcal{O}_S)$ (cf. §4.3) ;

(2) An example of S with $K_S^2 = 18\chi(\mathcal{O}_S)$ but with Albanese fibration of genus 3 (cf. Proposition 4.4).

So the bounds in Corollary 3.4 are optimal.

To end this section, it is worth mentioning that Theorem 3.2 implies Gu's conjecture for the "hyperelliptic part" (see [6, Conjecture 1.4]).

Corollary 3.5. *Let S be a minimal algebraic surface of general type in positive characteristic $p \geq 5$. Assume that $c_2(S) < 0$ and the Albanese morphism $f : S \to C$ has generic hyperelliptic fibre. Then*

$$\chi(\mathcal{O}_S) \geq \frac{p^2-4p-1}{4(3p+1)(p-3)}K_S^2 \tag{16}$$

and equality holds exactly for Raynaud's examples (see §4.1).

Proof. Since we always have $g \geq (p-1)/2$ by the genus change formula (see [21] or [6, §2.1]), (16) is a direct consequence of Theorem 3.2 because

$$h(g) = \frac{(12g+8)(g-1)}{g^2-g-1}$$

is a decreasing function of g with

$$h\left(\frac{p-1}{2}\right) = \frac{4(3p+1)(p-3)}{p^2-4p-1}.$$

If S is one of Raynaud's examples, equality holds in (16) by a direct computation (see §4.1). Conversely, equality holds in (16) only if $g = (p-1)/2$ by the above statement, $c_2(S) = -4(b-1), l(f) = 0$ and $K_{S/C}^2 = \frac{4g-4}{g}\chi_f$ by Remark 3.3. The equality $c_2(S) = -4(b-1)$ holds only if all geometric fibres of f are irreducible. Moreover, when $g = (p-1)/2$, there is an integral horizontal divisor Δ contained in the non-smooth locus of f such that $[\Delta : C] = p$. So f has no multiple fibre: each geometric fibre of f is irreducible and reduced. And our result is a direct consequence of Lemma 4.3 below. ∎

4. Examples

4.1. Raynaud's examples

In [17], Raynaud constructed a class of pairs $(S, \mathcal{L})$, where S is a smooth projective algebraic surface in positive characteristic and $\mathcal{L}$ is an ample line bundle on S such that $H^1(S, \mathcal{L}) \neq 0$. These pairs give counterexamples to Kodaira's vanishing theorem in positive characteristic. In fact, Raynaud's examples do not only violate Kodaira's vanishing theorem, but also lead to many other pathologies in positive characteristic.

We now briefly recall their construction; one can also refer to [17] or [6, §4]. Let us start with a smooth projective curve C of genus $b := g(C) \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic $p > 2$ equipped with a rational function $f \in k(C)\backslash k(C)^p$ such that

$$\operatorname{div}(\mathrm{d}f) = pD$$

for some divisor D on C. We have the following examples of C known as a special case of the Artin–Schreier curves.

Example 4.1 (Artin–Schreier curves). Let C be the projective normal curve associated to the following plane equation:

$$y^p - y = \psi(x), \quad \psi(x) \in \mathbf{k}[x].$$

Then $\operatorname{div}(\mathrm{d}x) = (2b-2)\infty$, where $\infty \in C$ is the unique point at infinity. For a suitable choice of $\psi(x)$ (e.g., $\psi(x) = x^{p+1}$), the genus $b = g(C)$ can be such that $p \,|\, 2b-2 > 0$. Therefore $\operatorname{div}(\mathrm{d}x) = p \cdot D$ for

$$D = \frac{2b-2}{p}\infty.$$

Starting from (C, f), Raynaud shows there is a rank 2 vector bundle E on C along with a non-singular effective divisor Σ on $\pi : P = \mathbb{P}(E) \to C$ such that

- $\det(E) \simeq \mathcal{O}_C(D)$;
- $\Sigma \subset P$ is a non-singular divisor consisting of two irreducible components Σ_1, Σ_2 such that
  - Σ_1 is a section of π and $\Sigma_1 \in |\mathcal{O}_P(1)|$;
  - $\pi : \Sigma_2 \to C$ is inseparable and of degree p, so it is the Frobenius;
  - $\Sigma_1 \cap \Sigma_2 = \emptyset$.

Moreover, all such configurations (P, Σ) come from his construction by a suitable choice of (C, f), and we actually have

- $\Sigma_2 \in |\mathcal{O}_P(p) \otimes \pi^*\omega_C^{-1}|$.

In particular, the divisor Σ is an even divisor on P, so we can construct a flat double cover $\sigma : S \to P$ with branch divisor Σ by choosing any line bundle $\mathcal{M}$ on P with $\mathcal{M}^2 \simeq \mathcal{O}_P(\Sigma)$. The resulting surface S is smooth over $\mathbf{k}$ since Σ is (see [6, §2]).

Definition 4.2 (Raynaud's examples). Let S be a smooth projective surface over $\mathbf{k}$. We say that S is one of *Raynaud's examples* if there is a flat double cover $\sigma : S \to P$ with branch divisor Σ,

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & P \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & C & \end{array}$$

Note that by construction, the fibration $f : S \to C$ in Raynaud's construction is hyperelliptic. Let $f : S \to C$ be one of Raynaud's examples associated to the triple (C, P, Σ). Then

- $f : S \to C$ is a fibration of genus $g = (p-1)/2$;
- $K_S^2 = (3p^2 - 8p - 3)(b-1)/p$;
- $\chi(\mathcal{O}_S) = (p^2 - 4p - 1)(b-1)/(8p)$.

Thus

$$K_{S/C}^2 = K_S^2 - 8(g-1)(b-1) = -(p-1)(p-3)(b-1), \tag{17}$$

$$\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1) = -\frac{(p-1)^2(b-1)}{8p}. \tag{18}$$

In particular, we have equality in Xiao's inequality (see Theorem 1.1):

$$K_{S/C}^2 = \frac{4g-4}{g}\chi_f$$

and both sides of the equality are negative. Raynaud's examples are such that

$$K_S^2 = \frac{4(3p+1)(p-3)}{p^2-4p-1}\chi(\mathcal{O}_S),$$

which is the maximal possible slope for the "hyperelliptic part" (see Corollary 3.5).

We end this subsection by a characterization of Raynaud's examples.

Lemma 4.3. *Suppose $f : S \to C$ is a surface fibration. Then S is one of Raynaud's examples if and only if*

(a) *every geometric fibre of f is a singular rational curve of arithmetic genus*

$$g = \frac{p-1}{2};$$

(b) *every geometric fibre is hyperelliptic and integral.*

Proof. The "only if" part can be checked directly. Conversely, let ρ be the hyperelliptic involution, and $\sigma : S \to P' := S/\rho$ be the quotient map. Then condition (b) implies that the canonical homomorphism $\pi : P' \to C$ has integral fibres. Noting that $\pi : P \to C$ is birational to a ruled surface (recall that $k(C)$ is C1 by Tsen's Theorem) and P' is normal with integral π-fibres, we see that P' is exactly a smooth minimal ruled surface over C. Thus the quotient map $\sigma : S \to P'$ is a flat double cover with some branch divisor $\Sigma' \subsetneq P'$, and Σ' itself is smooth over $\mathbf{k}$ (see [6, §2.2]). On the other hand, it can be deduced from [6, §§2.1, 2.2] that $\Sigma_{k(C)} := \Sigma' \times_{\pi,C} k(C)$ is a divisor of $P'_{k(C)} := P' \times_{\pi,C} k(C) \simeq \mathbb{P}^1_{k(C)}$ such that

- $\deg_{k(C)} \Sigma_{k(C)} = p + 1$ (since $p_a = (p-1)/2$);
- $\Sigma_{k(C)}$ contains a point inseparable over $k(C)$ (since all fibres are singular).

By degree counting, one concludes that (P', Σ') falls into the configuration given by Raynaud. ■

4.2. *Counterexamples to Xiao's slope inequality*

Starting from the triple pair $(C, P = \mathbb{P}(E), \Sigma)$ constructed in the previous subsection, we can also take a cyclic cover of P branching at Σ of higher degree, which then gives counterexamples to Xiao's slope inequality.

After an étale base change if necessary, we now assume $p + 1 \mid 2b - 2$ and fix a line bundle $\mathcal{M}$ on C such that $\mathcal{M}^{p+1} \simeq \omega_C$. Now since

$$\Sigma \in |\mathcal{O}_P(p+1) \otimes \pi^*\omega_C^{-1}| = |(\mathcal{O}_P(1) \otimes \pi^*\mathcal{M}^{-1})^{p+1}|,$$

this data gives a cyclic $(p+1)$-cover $\tau : S \to P$ branching at Σ. Since Σ is a smooth divisor, S is smooth over $\mathbf{k}$. Denote by

$$f = \tau \circ \pi : S \to C$$

the associated surface fibration. Then

- S is a minimal surface of general type;
- every closed fibre of f is a singular rational curve of arithmetic genus $(p^2 - p)/2$;
- $K_S^2 = \tau^*(K_P + pc_1(\mathcal{O}_P(1)) + pf^*c_1(\mathcal{M})) = (p+3)(p-2)(2b-2)$;

- $\chi(\mathcal{O}_S) = \sum_{j=0}^{p} \chi(\mathcal{O}(-j) \otimes \pi^*\mathcal{M}^{-j}) = \frac{p^2+p-8}{12}(2b-2)$;
- $K_{S/C}^2 = -(p^2-3p+2)(2b-2) < 0$;
- $\chi_f = -\frac{p^2-2p+1}{6}(2b-2) < 0$.

So

$$\frac{K_{S/C}^2}{\chi_f} = \frac{6(p^2-3p+2)}{p^2-2p+1} > \frac{4(p-2)(p+1)}{p(p-1)} = \frac{4g-4}{g},$$

but since $\chi_f < 0$, this violates Xiao's slope inequality.

When $p = 3$, we have $g = 3$ and $K_S^2 = 18\chi(\mathcal{O}_S)$. Note that f is clearly the Albanese fibration of S, and we have the next proposition.

Proposition 4.4. *There is a surface S of general type in characteristic 3 with $K_S^2 = 18\chi(\mathcal{O}_S)$ and with Albanese fibration of genus 3.*

4.3. Surfaces of general type with maximal slope

Let S be a minimal surface of general type over an algebraically closed field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = p$. We have $K_S^2 \le 32\chi(\mathcal{O}_S)$ by Theorem 3.2. When $K_S^2 = 32\chi(\mathcal{O}_S)$, we say the surface S is *of maximal slope*.

4.3.1. Characterisation of surfaces of maximal slope

Proposition 4.5. *A general type surface S is of maximal slope if and only if there is a fibration $f : S \to C$ of genus 2 such that*

(1) *$b := g(C) \ge 2$;*

(2) *all fibres of f are irreducible, singular and rational.*

Proof. If S is of maximal slope, its Albanese fibration $f : S \to C$ is of genus 2 by Corollary 3.4. Moreover, from Remark 3.3, when S has maximal slope, one must have $c_2(S) = -4(b-1)$, which is equivalent to all fibres of f being irreducible (see [6, (3.3)]).

Conversely, if S admits such a fibration $f : S \to C$, then any fibre of f can have only unibranch singularities and therefore $c_2(S) = -4(b-1)$ by the Grothendieck–Ogg–Shafarevich formula. On the other hand, since all fibres of f are irreducible and reduced (since genus 2 fibres have no multiplicity), we have $l(f) = 0$ and the relative canonical map

$$v : S \to P = \mathbb{P}(f_*\omega_{S/C})$$

is a morphism without base point. In particular,

$$\omega_{S/C} = v^*\mathcal{O}(1).$$

Therefore $K_{S/C}^2 = 2c_1^2(\mathcal{O}(1)) = 2\deg(f_*\omega_{S/C})$. It then follows from Remark 3.3 again that $K_S^2 = 32\chi(\mathcal{O}_S)$. ∎

By Tate's genus change formula (see [21] or [6, §2.1]), the fibration f in Proposition 4.5 may only occur in characteristic $p = 2, 3$ or 5.

4.3.2. Surfaces of maximal slope when $p = 5$

Proposition 4.6. *If $p = 5$, a surface of general type is of maximal slope if and only if it is one of Raynaud's examples.*

Proof. This follows from Lemma 4.3 and Proposition 4.5. ∎

4.3.3. Surfaces of maximal slope when $p = 2$. We give another example of a surface with maximal slope when $p = 2$. Define C to be the quintic plane curve given by the homogeneous equation

$$Y^4 Z + Y Z^4 = X^5 \tag{19}$$

over an algebraically closed field $\mathbf{k}$ of characteristic $p = 2$. One can easily check that C is a smooth curve of genus $b := g(C) = 6$. There are two affine subsets C_i $(i = 0, 1)$ of C:

$$C_0\ (Z=1):\ y^4 + y = x^5, \quad x = \frac{X}{Z}, \quad y = \frac{Y}{Z}, \quad \text{with } C \backslash C_0 = \{(0,1,0)\};$$
$$C_1\ (Y=1):\ z'^4 + z' = x'^5, \quad x' = \frac{X}{Y} = \frac{x}{y}, \quad z' = \frac{Z}{Y} = \frac{1}{y}, \quad \text{with } C \backslash C_1 = \{(0,0,1)\}.$$

For simplicity, we introduce the following notation:

- ∞ is the point $(0, 1, 0)$ which is the complement of C_0 in C;
- $\Lambda := \{(0, 1, \lambda) \mid \lambda \in \mathbb{F}_{16}^*\} = \{(0, \tau, 1) \mid \tau \in \mathbb{F}_{16}^*\} \subsetneq C$.
- $C_1' := C_1 \backslash \Lambda$;
- $C_{10} = C_0 \cap C_1'$.

Over C_0, S is defined as

$$Y_0^2 = S_0 T_0^5 + x S_0^6 \tag{20}$$

in the weighted projective space $\mathrm{Proj}(\mathcal{O}_{C_0}[S_0^1, T_0^1, Y_0^3])$. Here the superscript on each element is its homogeneous degree.

Over C_1', S is defined as

$$Y_1^2 = S_1 T_1^5 + \frac{x'}{1 + z'^6} S_1^6 \tag{21}$$

in the weighted projective space $\mathrm{Proj}(\mathcal{O}_{C_1'}[S_1^1, T_1^1, Y_1^3])$.

The homogeneous translation relation is given by

$$\begin{cases} S_1 = x'^3 S_0, \\ T_1 = x' T_0, \\ Y_1 = x'^4 Y_0 + (1 + z'^3) T_0^3, \end{cases}$$

and this construction makes sense because

$$x'^8(Y_0^2 - S_0T_0^5 + xS_0^6) = Y_1^2 - \left(S_1T_1^5 + \frac{x'}{1+z'^6}S_1^6\right)$$

and x' is invertible on C_{01}.

One can easily check that S is a non-singular surface and the fibration $f : S \to C$ is as in Proposition 4.5. So this gives an example of a surface with maximal slope in characteristic 2. In this example, we actually have $\chi(\mathcal{O}_S) = 1$ and $K_S^2 = 32$. We also mention that S is obtained from $C \times \mathbb{P}^1$ by taking the quotient relative to the foliation $D = s^6\frac{\partial}{\partial s} + \frac{\partial}{\partial x}$, where s is the parameter of $\mathbb{P}^1$.

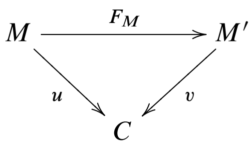
4.3.4. No surface of maximal slope when $p = 3$. Finally, we prove that there is no surface of general type with maximal slope when $p = 3$. Suppose we have such a surface S. Note that the relative canonical map gives a morphism: $\pi : S \to \mathbb{P}(f_*\omega_{S/C})$ since each fibre of f is irreducible and reduced, and π is necessarily a flat double cover (see [6, §2]). Let $M \subsetneq \mathbb{P}(f_*\omega_{S/C})$ be the branch divisor of π, which satisfies

- M is a smooth, horizontal divisor and $[M : C] = 6$;
- each component of M is inseparable over C;
- for each point $c \in C$, its inverse image in M has exactly two points. In fact, if some c has one inverse image, then the fibre of f at c is by construction a flat double cover of $\mathbb{P}^1_{\mathbf{k}}$ branching at a single point of multiplicity 6; such a fibre is clearly not irreducible.

Then there are two possibilities:

(A) $M = M_1 + M_2$ with $M_1 \cdot M_2 = 0$, and the projections $M_i \to C$ $(i = 1, 2)$ are both isomorphic to the Frobenius morphism;

(B) M is irreducible and the projection $u : M \to C$ factors as



where F_M is the Frobenius morphism and v is an étale double cover.

Indeed, we only need to consider case (A), since by replacing C by the base change v above which is an étale double cover, case (B) can be turned into (A).

Finally, we exclude case (A). Let Σ be the divisor class $\mathcal{O}(1)$ of $\mathbb{P}(f_*\omega_{S/C})$, and $M_i \sim_{\mathrm{num}} 3\Sigma + u_iF$ for $i = 1, 2$. Recall that

$$\Sigma^2 = \deg(f_*\omega_{S/C}) = \chi_f,$$

and we have

$$2b - 2 = (3\Sigma + u_iF)^2 + (3\Sigma + u_iF)(-2\Sigma + (\chi_f + 2b - 2)F), \tag{22}$$

$$0 = (3\Sigma + u_1F)(3\Sigma + u_2F). \tag{23}$$

Thus $u_1 = u_2$ and $b = 1$, which is a contradiction.

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