J. Eur. Math. Soc. 25, 467–513 (2023) DOI 10.4171/JEMS/1187

Will Johnson

The classification of dp-minimal and dp-small fields

Received September 12, 2019

Abstract. Continuing the work of Johnson (2018), we classify the pure fields and pure valued fields of dp-rank 1, up to elementary equivalence. In the process, we give a few new examples of dp-minimal fields. Specializing to the case of dp-small fields, we prove that dp-small fields and VC-minimal fields are all algebraically closed or real closed, as had been conjectured by Guingona (2014). This generalizes earlier work of Haskell and Macpherson (1994) and Macpherson et al. (2000), showing that C-minimal fields are algebraically closed, and weakly o-minimal fields are real closed. In fact, we obtain slight strengthenings of these earlier results, since we assume no compatibility between the field structure on the one hand and the C-relation or order on the other hand. We also give a new proof of Jahnke, Simon, and Walsberg's (2017) theorem that dp-minimal valued fields are henselian. Lastly, we apply the Kaplan–Scanlon–Wagner (2011) theorem (NIP fields are Artin–Schreier closed) to prove some general properties of strongly dependent valued fields. For example, strongly dependent henselian valued fields are defectless.

Keywords. Dp-minimality, strong dependence, valued fields

1. Introduction

The class of *NIP structures* plays a central role in contemporary model theory. It contains the model-theoretically important class of *stable structures*, as well as many important structures from algebra and number theory, such as the power series ring $\mathbb{C}[\![X]\!]$, the local fields $\mathbb{R}, \mathbb{Q}_p$ (obtained by completing $\mathbb{Q}$ along its absolute values), and *all* abelian ordered groups.

The class of *dp-minimal* structures can be regarded as the simplest type of NIP structure: dp-rank measures the complexity of an NIP structure, and dp-minimal structures are the structures of dp-rank 1. But at the same time, dp-minimality generalizes many of the important "minimality" notions in model theory, such as strong minimality, o-minimality, VC-minimality, and p-minimality. Many of the NIP structures arising "naturally" in algebra and number theory are already dp-minimal.

Will Johnson: School of Philosophy, Fudan University, Room 2104, Guanghua East Main Building, 220 Handan Road, Shanghai, China 200433; willij6@gmail.com

Mathematics Subject Classification (2020): Primary 03C45; Secondary 12J10

This paper continues [18], and culminates in a classification of dp-minimal (pure) fields and dp-minimal valued fields, up to elementary equivalence. The classification is a bit complicated. The bulk of the examples come from various theories of henselian valued fields. In addition to the usual suspects like $\mathbb{R}, \mathbb{Q}_p, \mathbb{C}((X))$, there are some positive characteristic and mixed characteristic cases. See Theorem 1.3 for details.

The classification results rely on three key ingredients. First, we use the "canonical topology" on dp-minimal fields, developed in [18]. This is a canonically determined field topology on any dp-minimal field K that fails to be algebraically closed. The canonical topology has several nice properties:

- There is a definable basis of opens.
- The topology is a V-topology, essentially meaning that the infinitesimals are a valuation ideal.

Second, we use the henselianity of ($\vee$-)definable valuation rings on dp-minimal fields. This was first proven in [17], but we give an alternative proof using the canonical topology. This proof uses a new method for proving henselianity that may be of independent interest.[1] Lastly, we use Jahnke and Koenigsmann's work on the definability of canonical (p-)henselian valuations [16]. Morally, this allows us to pin down the finest definable valuation ring and gain control of the residue field.

Along the way, we produce some (apparently) new examples of dp-minimal fields and valued fields. We also prove a few elementary facts about strongly dependent valued fields.

1.1. Strong dependence, dp-minimality, and dp-smallness

Let $\mathbb{M}$ be a model and $\{X_{\alpha,i}\}_{\alpha\in\kappa, i\in\omega}$ be a rectangular array of definable sets. The $X_{\alpha,i}$ form an *ict-pattern* or *randomness pattern* if the following two conditions hold:

- For each α, the sets $X_{\alpha,0}, X_{\alpha,1}, X_{\alpha,2}, \ldots$ in row α are uniformly definable.
- For each function $\eta : \kappa \to \omega$, there is an $a \in \mathbb{M}^* \succeq \mathbb{M}$ such that

$$a \in X_{\alpha,i} \iff i = \eta(\alpha), \quad \text{for all } \alpha \in \kappa, i \in \omega.$$

The number κ is called the *depth* of the ict-pattern. If all the $X_{\alpha,i}$ are subsets of a definable set Y, we say that the $X_{\alpha,i}$ form an ict-pattern *in the set Y*.

The *dp-rank* of a definable set Y is the supremum of cardinals κ such that there is an ict-pattern of depth κ in the set Y, possibly in an elementary extension $\mathbb{M}^* \succeq \mathbb{M}$.

Using this, one defines:

(1) $\mathbb{M}$ is *NIP* (or *dependent*) if dp-rk$(\mathbb{M}) < \infty$, i.e., if there is an absolute bound on the depth of ict-patterns in $\mathbb{M} = \mathbb{M}^1$.

[1] For example, it has recently been used to prove the henselianity of definable valuation rings in positive characteristic NIP fields. See [19, Theorem 2.8].

(2) $\mathbb{M}$ is *strongly dependent* if there is no ict-pattern of depth $\aleph_0$ in $\mathbb{M} = \mathbb{M}^1$.

(3) $\mathbb{M}$ is *dp-minimal* if dp-rk$(\mathbb{M}) = 1$, i.e., there is no ict-pattern of depth 2 in $\mathbb{M} = \mathbb{M}^1$.

One can show that (1) is equivalent to the usual definition of NIP (no formula has the independence property). Note that

$$\text{dp-minimal} \implies \text{strongly dependent} \implies \text{NIP}.$$

Concretely, a theory T is dp-minimal if there does *not* exist a model $M \models T$, formulas $\phi(x; y), \psi(x; z)$ with $|x| = 1$, and elements b_i, c_j such that for every i_0, j_0, the type

$$\phi(x; b_{i_0}) \wedge \psi(x; c_{j_0}) \wedge \bigwedge_{i \neq i_0} \neg\phi(x; b_i) \wedge \bigwedge_{j \neq j_0} \neg\psi(x; c_j)$$

is consistent. Here, the $\phi(M; b_i)$ are the sets of the first row, and the $\psi(M; c_j)$ are the sets of the second row; (i_0, j_0) is the function $2 \to \omega$.

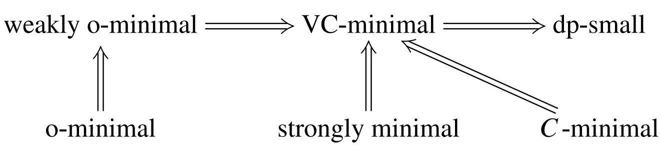
In [10, Definition 1.4], Guingona defines a related notion: a theory T is *dp-small* if there does not exist a model $M \models T$, formulas $\phi_i(x; y), \psi(x; z)$ with $|x| = 1$, and elements b_i, c_j such that for every i_0, j_0, the type

$$\phi_{i_0}(x; b_{i_0}) \wedge \psi(x; c_{j_0}) \wedge \bigwedge_{i \neq i_0} \neg\phi_i(x; b_i) \wedge \bigwedge_{j \neq j_0} \neg\psi(x; c_j)$$

is consistent. The subtle difference is that the definable sets of the first row are no longer uniformly definable. Note that

$$\text{dp-small} \implies \text{dp-minimal}.$$

It turns out that many theories are dp-small. One has the following implications (for any sense of C-minimality):



On the other hand, we will see that most dp-minimal fields fail to be dp-small. For example, the field $\mathbb{Q}_p$ of p-adic numbers and the field $\mathbb{C}((X))$ of Laurent series are dp-minimal but not dp-small.

1.2. Main results

If (K, v) is a valued field, we let Kv denote the residue field and vK denote the value group. See [7] for the basics of valuation theory.

Theorem 1.1. *Let (K, v) be a strongly dependent non-trivially valued field.*

(1) *If (K, v) is henselian, then (K, v) is defectless.*

(2) *If Kv is finite of characteristic p, then K has characteristic* 0 *and the absolute ramification is bounded: the interval $[-v(p), v(p)] \subseteq vK$ is finite.*

(3) *If Kv is infinite of characteristic p, then the interval $[-v(p), v(p)]$ is p-divisible. Here, if K has characteristic p, then $[-v(p), v(p)]$ denotes all of vK.*

This is proven in §2.4. Parts (2)–(3) determine a dichotomy for strongly dependent valued fields of positive residue characteristic. Theorem 1.1 was mostly already known. The equicharacteristic 0 statements are trivial, and the equicharacteristic p statements were proven by Kaplan, Scanlon, and Wagner [22]. The new statements are in mixed characteristic, and are easily obtained by combining [22] with the technique of Shelah expansions.

Theorem 1.2. *Let K be a dp-minimal field, possibly with extra structure. Then at least one of the following holds:*

- *K is finite.*
- *K is real closed.*
- *K is algebraically closed.*
- *K has a non-trivial definable henselian valuation.*

In fact, if $\mathcal{O}$ is the intersection of the definable henselian valuation rings, then $\mathcal{O}$ is a type-definable henselian valuation ring whose residue field is finite, real closed, or algebraically closed.

We prove this in §4.4. Theorem 1.2 is an instance of the more general "Shelah conjecture," which changes "dp-minimal" to "NIP" and "algebraically closed" to "separably closed" [12].

Using Theorem 1.2, we prove the following classification theorem in §7.

Theorem 1.3 (Classification of dp-minimal fields).

(1) *Let Γ be an ordered abelian group such that $\Gamma/n\Gamma$ is finite for all $n > 1$.*
   (a) *Let k be a local field of characteristic* 0. *The theory of henselian valued fields (K, v) with $vK \equiv \Gamma$ and $Kv \equiv k$ is complete and dp-minimal.*
   (b) *If p is a prime and Γ is p-divisible, then the theory of henselian defectless characteristic p valued fields (K, v) with $vK \equiv \Gamma$ and $Kv \models \mathrm{ACF}_p$ is complete and dp-minimal.*
   (c) *If p is a prime, $a \in \Gamma_{>0}$, and $[-a, a] \subseteq \Gamma$ is p-divisible, then the theory of henselian defectless characteristic* 0 *valued fields (K, v) with $Kv \models \mathrm{ACF}_p$ and $(vK, v(p)) \equiv (\Gamma, a)$ is complete and dp-minimal.*

(2) *If F is a pure field which is infinite and dp-minimal, then $\mathrm{Th}(F)$ is the reduct to the language of rings of one of the above theories, for some Γ, k, p, a.*

The important part is (2), though (1b)–(1c) are apparently new as well.

Corollary 1.4. *An infinite field K of characteristic $p > 0$ is dp-minimal if and only if it is elementarily equivalent to a Hahn series field $\mathbb{F}_p^{\mathrm{alg}}((t^\Gamma))$ where Γ is a p-divisible ordered abelian group with $\Gamma/n\Gamma$ finite for all $n > 0$.*

This is analogous to [17]'s classification of dp-minimal ordered fields—up to elementary equivalence, they are the Hahn fields $\mathbb{R}((t^\Gamma))$ where Γ is an ordered abelian group with $\Gamma/n\Gamma$ finite for all $n > 0$.

In characteristic 0, case (1a) consists of exactly the fields elementarily equivalent to $K((t^\Gamma))$ where K is a characteristic 0 local field and $\Gamma/n\Gamma$ is finite for all n. However, there seems to be no clean way to describe case (1c), which includes annoyances like the spherical completions of

$$\mathbb{Q}_p^{\mathrm{un}}(p^{1/p}, p^{1/p^2}, p^{1/p^3}, \dots),$$

where $\mathbb{Q}_p^{\mathrm{un}}$ is the maximal unramified extension of $\mathbb{Q}_p$.

We also get a classification of dp-minimal valued fields in §8.

Theorem 1.5. *Let (K, v) be a valued field with* infinite *residue field. Then (K, v) is dp-minimal (as a pure valued field) if and only if the following conditions all hold:*

(1) *The residue field Kv and value group vK are dp-minimal.*

(2) *The valuation v is henselian and defectless.*

(3) *In mixed characteristic, every element of $[-v(p), v(p)]$ is divisible by p.*

(4) *In pure characteristic p, the value group vK is p-divisible.*

Theorem 1.6. *Let (K, v) be a valued field with* finite *residue field. Then (K, v) is dp-minimal (as a valued field) if and only if the following conditions all hold:*

(1) *The value group vK is dp-minimal.*

(2) *The valuation v is henselian.*

(3) *The valuation is finitely ramified, in the sense that $[-v(p), v(p)]$ is finite. (In particular, K has characteristic 0 if v is non-trivial.)*

Meanwhile, for dp-small fields, we prove the following results in §5.

Theorem 1.7. *Let K be an infinite field, possibly with extra structure.*

(1) *If K is VC-minimal (or more generally, dp-small), then K is real closed or algebraically closed.*

(2) *If K is (densely) C-minimal, then K is algebraically closed.*

(3) *If K is weakly o-minimal, then K is real closed.*

In case (2), we mean C-minimal in the sense of Haskell and Macpherson [14] rather than Delon [5]. In particular, we assume the C-relation is dense; see §5.1 for details.

Cases (2) and (3) are exactly the main results of [14] and [24], *except* that we have generalized slightly: we do not assume any compatibility between the field operations and the C-predicate or ordering.

Case (1) is the "VC-minimal fields conjecture" of [10].

1.3. Related work

Dp-small ordered fields were classified by Guingona [10], and dp-minimal ordered fields were classified by Jahnke, Simon, and Walsberg [17], who also classified the dp-minimal ordered abelian groups and proved henselianity for dp-minimal valued fields, both of which we shall use here.

In the four years since the classification of dp-minimal fields was announced, considerable progress has been made on NIP fields. Here is a brief summary of the current state of affairs. By work of Anscombe, Dolich, Farré, Goodrick, Halevi, Hasson, Jahnke, and Sinclair [6,8,11–13,26], the classification of strongly dependent fields—and possibly even NIP fields—has been reduced to the following two conjectures:

Conjecture 1.8 (Henselianity conjecture). *If (K, v) is an NIP valued field, then v is henselian.*

Conjecture 1.9 ("Shelah" conjecture). *If K is an NIP field, then at least one of the following holds:*

- *K is separably closed.*
- *K is real closed.*
- *K is finite.*
- *K admits a definable henselian valuation.*

Unfortunately, these two conjectures have proven difficult to attack. The main known results are

(1) The henselianity conjecture for dp-minimal (K, v), proven in [17].

(2) The Shelah conjecture for dp-minimal K, proven in the present paper (Theorem 1.2).

(3) The implication

$$\text{Shelah conjecture} \implies \text{henselianity conjecture},$$

proven in [13].

In [19,20], I extend the techniques of the present paper to prove

(4) The henselianity conjecture for (K, v) of positive characteristic.

(5) The Shelah conjecture for K of positive characteristic and finite dp-rank.

2. Strongly dependent valued fields

In [22], it is shown that all NIP fields are Artin–Schreier closed. We will use this fact to prove several nice properties of strongly dependent valued fields.

2.1. Some valuation theory

The *characteristic exponent* of a valued field K is p if K has residue characteristic p, and 1 if K has residue characteristic 0.

Fact 2.1. *If K is henselian and L/K is a finite extension, then*

$$[L:K] = |vL/vK| \cdot [Lv : Kv] \cdot p^d,$$

where p is the characteristic exponent, and $d \in \mathbb{N}$.

One says that the extension L/K is *defectless* if $p^d = 1$. A henselian field K is *defectless* if every finite extension is defectless.

Fact 2.2. *The henselian field K is defectless if any of the following three conditions hold:*

- *K has residue characteristic* 0.
- *K has residue characteristic p, and p does not divide the degree of any finite extension of K.*
- *K is spherically complete.*

2.2. *Perfection*

Recall that a set X has *rudimentarily finite* dp-rank if there is no ict-pattern of depth $\aleph_0$ in X. Here, X can be type-definable, or even $*$-definable (i.e., pro-definable). In Lemmas 2.3 and 2.6, we shall use the fact that an infinite product $D_1 \times D_2 \times \cdots$ of infinite definable sets has an ict-pattern of depth $\aleph_0$, and therefore fails to have rudimentarily finite dp-rank.

Lemma 2.3. *Let K be a strongly dependent field. Then K is perfect.*

Proof. If K is imperfect, then there is a definable injection $f : K \times K \hookrightarrow K$, namely $f(x, y) = x^p + b \cdot y^p$ for any $b \notin K^p$.

Let $X_0 = K$ and let $X_{i+1} = f(X_0, X_i)$. Note $X_0 \supseteq X_1 \supseteq \cdots$. Let X_∞ be the type definable set $\bigcap_i X_i$. In the category of $*$-definable sets, there is a surjection $X_\infty \to \prod_{i=0}^\infty K$, roughly sending

$$f(x_0, f(x_1, f(x_2, \ldots))) \mapsto (x_0, x_1, x_2, \ldots).$$

More precisely, note that $X_\infty = f(K, X_\infty)$, and $f : K \times X_\infty \xrightarrow{\sim} X_\infty$ is a bijection. Let π_1 and π_2 be the two projections

$$X_\infty \xrightarrow{\sim} K \times X_\infty \twoheadrightarrow K, \qquad X_\infty \xrightarrow{\sim} K \times X_\infty \twoheadrightarrow X_\infty.$$

Then the surjection $X_\infty \to \prod_{i=0}^\infty K$ is the map

$$x \mapsto (\pi_1(x), \pi_1(\pi_2(x)), \pi_1(\pi_2(\pi_2(x))), \ldots).$$

Since $\prod_{i=0}^\infty K$ does not have rudimentarily finite weight, neither does X_∞, nor does its superset K. ∎

Remark 2.4. Let K be a strongly dependent field of characteristic p, and L/K be a finite extension. Then p does not divide $[L:K]$.

Proof. By perfection, L/K is a separable extension, so this follows by [22, Corollary 4.5]. ■

Lemma 2.5. *Let K be an infinite strongly dependent field of positive characteristic p. Then* any *valuation on K has p-divisible value group, and* any *henselian valuation on K is defectless.*

Proof. The first claim follows because K is Artin–Schreier closed, or because it is perfect. For the second claim, if v is a henselian valuation, if L/K is a finite extension, and if w is a prolongation of v to L, then

$$[L : K] = [Lw : Kv] \cdot |wL/vK| \cdot p^d,$$

for some $d \in \mathbb{N}$. By Remark 2.4, p does not divide $[L : K]$, so $d = 0$, i.e., the valuation is defectless. ■

2.3. Finite ramification

Lemma 2.6. *Let (K, v) be a strongly dependent mixed characteristic valued field. Suppose the interval $[-v(p), v(p)]$ in the value group is finite. Then the residue field Kv is finite.*

Proof. We may replace K with a sufficiently saturated elementary extension. Note that Kv is itself strongly dependent, hence perfect.

Let $\mathcal{O}$ be the valuation ring. By finite ramification, the maximal ideal of $\mathcal{O}$ is a principal ideal (π) for some generator π. Let $\hat{\mathcal{O}}$ denote the $*$-definable set

$$\hat{\mathcal{O}} = \varprojlim \mathcal{O}/(\pi^n).$$

Then $\mathcal{O}$ surjects onto $\hat{\mathcal{O}}$ via the obvious map.

Suppose the map $\hat{\mathcal{O}} \to Kv$ had a $*$-definable section

$$s : Kv \to \hat{\mathcal{O}}.$$

We would then obtain a $*$-definable bijection

$$Kv \times \hat{\mathcal{O}} \xrightarrow{\sim} \hat{\mathcal{O}}, \quad (\alpha, x) \mapsto s(\alpha) + \pi \cdot x.$$

This would then yield $*$-definable surjections

$$\mathcal{O} \to \hat{\mathcal{O}} \to Kv \times \hat{\mathcal{O}} \to Kv \times Kv \times \hat{\mathcal{O}} \to \cdots \to Kv \times Kv \times \cdots,$$

showing that $\mathcal{O}$ is not strongly dependent unless Kv is finite.

So it suffices to produce a $*$-definable section of the projection $\hat{\mathcal{O}} \to Kv$. We will use the Teichmüller character.

Claim 2.7. *For each n, if $m > n$ and* $\mathrm{res}(y_1) = \mathrm{res}(y_2) \neq 0$, *then* $y_1^{p^m} - y_2^{p^m} \in (\pi^n)$.

Proof. Note first that if I is any principal proper ideal of $\mathcal{O}$, then $(1+I)^p \subseteq 1+J$ for a strictly smaller principal ideal, namely $J = I^2 + p \cdot I$. It follows that

$$(1+(\pi))^{p^m} \subseteq 1+(\pi^m).$$

Then for $y_1, y_2 \in \mathcal{O}^\times$,

$$y_1 - y_2 \in (\pi) \implies \frac{y_1}{y_2} \in 1+(\pi) \implies \frac{y_1^{p^m}}{y_2^{p^m}} \in 1+(\pi^m) \implies y_1^{p^m} - y_2^{p^m} \in (\pi^m) \subseteq (\pi^n).$$

∎

Now define a section $s : (Kv)^\times \to \hat{\mathcal{O}}^\times$ as follows: given non-zero $\alpha \in Kv$, choose a sequence $y_1, y_2, \dots$ in $\mathcal{O}$ such that $(\operatorname{res}(y_n))^{p^n} = \alpha$, using perfection of Kv. Then let

$$s(\alpha) = \lim_{n \to \infty} y_n^{p^n}.$$

To see that this is well-defined and $*$-definable, note that for $m > n$, the class of $y_m^{p^m}$ modulo (π^n) does not depend on y_m, by the claim, nor on m, because for $m' > m$,

$$y_{m'}^{p^{m'-m}} - y_m \in (\pi) \quad \text{and so} \quad y_{m'}^{p^{m'}} - y_m^{p^m} \in (\pi^n).$$

And $\operatorname{res}(s(\alpha)) = \alpha$, by choice of the y_i's.

Therefore there is a $*$-definable section of $\hat{\mathcal{O}}^\times \to (Kv)^\times$. We can extend this to a section of $\hat{\mathcal{O}} \to Kv$ by sending 0 to 0. ∎

2.4. *Defectlessness and a dichotomy*

Definition 2.8. If Γ is an ordered abelian group and p is prime, let $\operatorname{Int}_p \Gamma$ denote the maximal convex p-divisible subgroup of Γ.

The subgroup $\operatorname{Int}_p \Gamma$ is first-order definable in Γ, because

$$\gamma \in \operatorname{Int}_p \Gamma \iff [-|\gamma|, |\gamma|] \subseteq p \cdot \Gamma.$$

(Note that $[-|\gamma|, |\gamma|]$ generates a convex subgroup.)

Definition 2.9. A valuation $v : K \to \Gamma$ is *roughly p-divisible* if $[-v(p), v(p)] \subseteq p \cdot \Gamma$, where $[-v(p), v(p)]$ denotes $\{0\}$ in pure characteristic 0, denotes Γ in pure characteristic p, and denotes the usual interval $[-v(p), v(p)]$ in mixed characteristic.

In mixed characteristic, (K, v) is roughly p-divisible if and only if $v(p) \in \operatorname{Int}_p$. In pure characteristic p, (K, v) is roughly p-divisible if and only if the value group is p-divisible.

Remark 2.10. Let P be one of the following properties of valuation data:

- Roughly p-divisible.
- Henselian.
- Henselian and defectless.
- Every countable chain of balls has non-empty intersection.

If $K_1 \to K_2$ and $K_2 \to K_3$ are places, the composition $K_1 \to K_3$ has property P if and only if each of $K_1 \to K_2$ and $K_2 \to K_3$ has property P.

For each property, this is straightforward to check.

In what follows, we will use the *Shelah expansion*. If M is an NIP structure, M^{sh} denotes the expansion of M by all externally definable sets. By [25, Proposition 3.23], M^{sh} eliminates quantifiers. Using this, one sees that if M is dp-minimal or strongly dependent, then so is M^{sh}. Of course, properties like saturation will often be lost.

Theorem 2.11. *Let (K, v) be a strongly dependent valued field. If Kv is infinite, then v is roughly p-divisible. If Kv is finite, then K has characteristic 0 and the interval $[-v(p), v(p)]$ is finite. If v is henselian, then v is defectless.*

Proof. All the properties described here are elementary properties, so we may replace K with a sufficiently saturated elementary extension. We break into cases by the characteristic and residue characteristic of v.

In equicharacteristic 0, Kv is infinite, rough p-divisibility is vacuous, and henselian implies defectless (Fact 2.2).

In equicharacteristic p, Kv is infinite by [22, Proposition 5.3]. The value group is p-divisible and the valuation is defectless if henselian, by Lemma 2.5.

This leaves the case of mixed characteristic. Let Δ_0 be the biggest convex subgroup not containing $v(p)$, and Δ be the smallest convex subgroup containing $v(p)$. These convex subgroups decompose the place $K \to Kv$ as a composition of three places:

$$K \xrightarrow{vK/\Delta} K_1 \xrightarrow{\Delta/\Delta_0} K_2 \xrightarrow{\Delta_0} Kv, \tag{2.1}$$

where each arrow is labeled by its value group. The fields K and K_1 have characteristic 0, while K_2 and Kv have characteristic p.

Note that Δ/Δ_0 embeds into $\mathbb{R}$, so is small. Because K is sufficiently saturated, we get the following chain of implications:

$$\begin{aligned}\Delta_0 = 0 &\implies \Delta \text{ small} \implies [-v(p), v(p)] \text{ small} \implies [-v(p), v(p)] \text{ finite} \\ &\implies \Delta_0 \text{ finite} \implies \Delta_0 = 0,\end{aligned}$$

so Δ_0 vanishes if and only if $[-v(p), v(p)]$ is finite.

Both Δ_0 and Δ are externally definable, hence definable in K^{sh}. So the sequence of places in (2.1) is interpretable in the strongly dependent structure K^{sh}.

In particular, Δ_0 is p-divisible, by [22, Proposition 5.4]. So $\mathrm{Int}_p\, vK$ is non-trivial or $[-v(p), v(p)]$ is finite.

Note that we have just proven the following general fact:

> *If (K, v) is a strongly dependent mixed characteristic valued field, then* $\mathrm{Int}_p\, vK$ *is non-trivial or* $[-v(p), v(p)]$ *is finite,*

because this depends only on the elementary equivalence class of (K, v).

Combining with Lemma 2.6, we have actually shown

If (K, v) is a strongly dependent mixed characteristic valued field, then $\mathrm{Int}_p\, vK$ *is non-trivial or Kv is finite.*

In particular, we can apply this fact to the strongly dependent place $K_1 \to K_2$ in (2.1). We see that

$$\mathrm{Int}_p(\Delta/\Delta_0) \text{ is non-trivial, or } K_2 \text{ is finite.} \tag{2.2}$$

Now we prove the three claims of the theorem.

First suppose that Kv is infinite. Then K_2 is infinite, so Δ/Δ_0 has a non-trivial p-divisible convex subgroup by (2.2). Being archimedean, Δ/Δ_0 has very few convex subgroups and must be p-divisible itself. As Δ_0 is p-divisible, it follows that Δ is p-divisible, so v is roughly p-divisible.

Next suppose that Kv is finite. If $[-v(p), v(p)]$ is infinite, then Δ_0 is non-trivial, so $K_2 \to Kv$ is an infinite strongly dependent valued field of characteristic p with a finite residue field. This contradicts [22, Proposition 5.3].

Next suppose that v is henselian. Then all three of the places in (2.1) are henselian. By the equicharacteristic cases, $K \to K_1$ and $K_2 \to Kv$ are defectless, so it remains to show that $K_1 \to K_2$ is defectless. Because (K, v) is saturated, any countable chain of balls in (K, v) has non-empty intersection. So the place $K \to Kv$ satisfies the countable intersection property of Remark 2.10. Therefore, so does $K_1 \to K_2$. However, the value group of $K_1 \to K_2$ is Δ/Δ_0. This group has countable cofinality, because it embeds into $\mathbb{R}$. Consequently, $K_1 \to K_2$ is spherically complete, hence defectless (Fact 2.2). ∎

3. Henselianity

Until Theorem 3.15, we assume that $\mathbb{M}$ is a sufficiently saturated dp-minimal field that is *not* strongly minimal. For any small model $K \preceq \mathbb{M}$, recall the type-definable set

$$I_K = \bigcap \{X - X : X \text{ infinite and } K\text{-definable}\}$$

of K-infinitesimals from [18, Definition 4.3, Corollary 5.7]. Recall that I_K is the maximal ideal of some $\vee$-definable valuation ring $\mathcal{O}_K$ [18, Proposition 6.2].

In this section, we prove that $\mathcal{O}_K$ is henselian. More generally, so is any $\vee$-definable valuation ring on $\mathbb{M}$. The henselianity of $\vee$-definable valuation rings on dp-minimal fields was independently obtained by [17]. We give an alternative proof here using the canonical topology constructed in [18]. This technique for proving henselianity has potential applications to fields of higher rank.[2]

First we prove a general fact about $\vee$-definable valuation rings: their prolongations to finite extension fields are still $\vee$-definable.

[2]For example, it is used in [19] to prove that positive characteristic NIP valuation rings are henselian.

Lemma 3.1. *Let F be a field with some structure, and L/F be a finite extension. Suppose $\mathcal{O}$ is a $\vee$-definable valuation ring on F. Then each extension of $\mathcal{O}$ to L is $\vee$-definable.*

Proof. Replacing L with the normal closure of L over F, we may assume L/F is a normal extension of some degree n. Let $\mathcal{O}'$ be some extension of $\mathcal{O}$ to L. We can find some finite set $S \subseteq \mathcal{O}'$ such that $\mathcal{O}'$ is the unique extension of $\mathcal{O}$ containing S, because there are only finitely many extensions of $\mathcal{O}$ to L and the extensions are pairwise incomparable. Then

$$\mathcal{O}' = \left\{x \in L : \text{some extension of } \mathcal{O} \text{ to } L \text{ contains } S \cup \{x\}\right\}.$$

Write $S = \{a_1, \ldots, a_{k-1}\}$. By the equivalence (1)⇔(2) in Lemma 3.2 below, there is $d \in \mathbb{N}$ such that

$$L \setminus \mathcal{O}' = \{x \in L : 1 = P(a_1, \ldots, a_{k-1}, x) \text{ for some polynomial } P(X_1, \ldots, X_k) \\ \text{with coefficients from } \mathfrak{m} \text{ and degree less than } d\}.$$

Because $\mathcal{O}$ is $\vee$-definable, $\mathfrak{m}$ is type-definable, so the right hand side is type-definable. Therefore $\mathcal{O}'$ is $\vee$-definable. ∎

Lemma 3.2. *Let L/F be a normal extension of degree $n < \infty$. Let $\{a_1, \ldots, a_k\} \subseteq L$ be a finite subset. Let $\mathcal{O}$ be a valuation ring on F, and let $\mathcal{O}_L$ be some extension of $\mathcal{O}$ to L. Then for some $d = d(n, k)$ depending only on n and k, the following are equivalent:*

(1) *No extension of $\mathcal{O}$ to L contains $\{a_1, \ldots, a_k\}$.*

(2) $1 = P(a_1, \ldots, a_k)$ *for some polynomial* $P(X_1, \ldots, X_k) \in \mathfrak{m}[X_1, \ldots, X_k]$ *of degree less than* $d(k, n)$.

(3) $1 = P(a_1, \ldots, a_k)$ *for some* $P(X_1, \ldots, X_k) \in \mathfrak{m}[X_1, \ldots, X_k]$.

(4) $\{a_1, \ldots, a_k\} \not\subseteq \sigma(\mathcal{O}_L)$ *for any* $\sigma \in \mathrm{Aut}(L/F)$.

Proof. Valued fields can be amalgamated, so $\mathrm{Aut}(L/F)$ acts transitively on the set of extensions of $\mathcal{O}$ to L, by normality of L/F. Therefore (4)⇔(1). The equivalence (3)⇔(1) holds on general valuation-theoretic grounds:

- If some extension of $\mathcal{O}$ contains $\{a_1, \ldots, a_k\}$, then every element of $\mathfrak{m}[a_1, \ldots, a_k]$ has positive valuation, so $1 \notin \mathfrak{m}[a_1, \ldots, a_k]$.
- Conversely, if $1 \notin \mathfrak{m}[a_1, \ldots, a_k]$, then the ideal $\mathfrak{m}[a_1, \ldots, a_k] \lhd \mathcal{O}[a_1, \ldots, a_k]$ is non-trivial, so we can find a prime ideal $\mathfrak{n} \lhd \mathcal{O}[a_1, \ldots, a_k]$ containing $\mathfrak{m}[a_1, \ldots, a_k]$. By Chevalley's theorem, there is a valuation w on L taking non-negative values on $\mathcal{O}[a_1, \ldots, a_k]$ and positive values on $\mathfrak{n} \supseteq \mathfrak{m}[a_1, \ldots, a_k] \supseteq \mathfrak{m}$. Then w extends $\mathcal{O}$ and $a_1, \ldots, a_k$ are in the valuation ring of w.

Thus (1)⇔(3)⇔(4). Now consider the theory T_n whose models are pairs of valued fields $(L, \mathcal{O}_L, K, \mathcal{O})$ with L/K a normal extension of degree n. In this language, condition (4) is first-order, so compactness yields a bound on the degree in (3). This yields the stronger statement (2), for some d depending on fixed n and k. ∎

Remark 3.3. If $\mathcal{O}$ is a $\vee$-definable valuation ring on a stable field K, then $\mathcal{O}$ is trivial. More generally, if $A \subseteq K$ and K is $|A|^+$-saturated, and if $\mathcal{O}$ is an $\mathrm{Aut}(K/A)$-invariant valuation ring, then $\mathcal{O}$ is trivial.

Proof. Let p be the generic type of the field K. If a realizes p over A, then $a \equiv_A a^{-1}$ by the uniqueness of the generic type. Thus $v(a) \geq 0 \Leftrightarrow v(a^{-1}) \geq 0 \Leftrightarrow v(a) \leq 0$, and so $v(a) = 0$.

Now suppose d is any element of $K^\times$. Take $a \in K$ generic over Ad. Then $d \cdot a$ is also generic over Ad, so $v(d \cdot a) = 0 = v(a)$, implying $v(d) = 0$. Therefore the valuation is trivial. ∎

We now return to the specific setting of a sufficiently saturated dp-minimal field $\mathbb{M}$, not strongly minimal, with a small submodel $K \preceq \mathbb{M}$. Recall that a *basic neighborhood* is a set of the form $X - X$ with X infinite and $\mathbb{M}$-definable. These form a neighborhood basis of 0 for the canonical topology on $\mathbb{M}$, and in particular they form a downward-directed family [18, Corollary 5.7]. By definition, I_K is the intersection of the K-definable basic neighborhoods. As a consequence, we have

Remark 3.4 ([18, Remark 7.9]). For K-definable $X \subseteq \mathbb{M}$,

$$0 \in \mathrm{Int}(X) \iff I_K \subseteq X,$$

with $\mathrm{Int}(X)$ denoting the interior of X in the canonical topology on $\mathbb{M}$.

Lemma 3.5. *The group I_K is open in the canonical topology on $\mathbb{M}$.*

Proof. Because I_K is a subgroup of $(\mathbb{M}, +)$, it suffices to show that I_K is a neighborhood of 0. Note that I_K is type-definable and $\mathcal{O}_K$ is $\vee$-definable, both over K. Therefore we can find a K-definable set B lying between them:

$$I_K \subseteq B \subseteq \mathcal{O}_K.$$

By directedness of the family of K-definable basic neighborhoods, there is a K-definable basic neighborhood $X - X$ such that

$$I_K \subseteq X - X \subseteq B \subseteq \mathcal{O}_K.$$

Now choose some non-zero $\epsilon \in I_K$. Then

$$(\epsilon \cdot X) - (\epsilon \cdot X) = \epsilon \cdot (X - X) \subseteq \epsilon \cdot \mathcal{O}_K \subseteq I_K,$$

so I_K contains the basic neighborhood $(\epsilon \cdot X) - (\epsilon \cdot X)$. ∎

Lemma 3.6. *Let $f : \mathbb{M}^n \to \mathbb{M}^n$ be a finite-to-one definable map. Let $X \subseteq \mathbb{M}^n$ be a set with non-empty interior with respect to the product topology on $\mathbb{M}^n$. Then $f(X)$ also has non-empty interior.*

Proof. We may assume X is definable, by shrinking it (recall the basis of definable opens). By [18, Proposition 8.2], X has interior if and only if $\text{dp-rk}(X) = n$, and $f(X)$ has interior if and only if $\text{dp-rk}(f(X)) = n$. By basic properties of dp-rank, $\text{dp-rk}(X) = \text{dp-rk}(f(X))$. ∎

Proposition 3.7. *Let K be a small submodel of $\mathbb{M}$. Let L/K be a finite algebraic extension, and $\mathbb{L} = L \otimes_K \mathbb{M}$. (So $\mathbb{L}$ is a saturated elementary extension of L.) Then $\mathcal{O}_K$ has a unique extension to $\mathbb{L}$.*

(This does not immediately give henselianity of $\mathcal{O}_K$, because we are only considering finite extensions $\mathbb{L}/\mathbb{M}$ defined over the small field K.)

Proof. We first give the proof in characteristic $\neq 2$.

Replacing L with its normal closure over K, we may assume L/K is normal.

Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ denote the extensions of $\mathcal{O}_K$ to $\mathbb{L}$. By Lemma 3.1, these are all $\vee$-definable. Let $\mathfrak{m}_i$ be the maximal ideal of $\mathcal{O}_i$; this is type-definable. Let v_i be the valuation on $\mathbb{L}$ from $\mathcal{O}_i$.

Write $L = K(\alpha)$ (possible because K is perfect by Lemma 2.3). So $\mathbb{L} = \mathbb{M}(\alpha)$ and $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a basis for $\mathbb{L}$ over $\mathbb{M}$.

Claim 3.8. $\bigcap_i \mathfrak{m}_i = \sum_{i=0}^{n-1} I_K \cdot \alpha^i$. *Consequently, $\bigcap_i \mathfrak{m}_i$ is type-definable over K.*

Proof. Let $(F, \mathcal{O})$ be some algebraically closed valued field extending $(\mathbb{M}, \mathcal{O}_K)$, and let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. All the extensions of $\mathcal{O}_K$ to $\mathbb{L}$ come from embeddings of $\mathbb{L}$ into F, so if $\iota_1, \ldots, \iota_n$ denote the embeddings of $\mathbb{L}$ into F, then

$$\{\mathfrak{m}_1, \ldots, \mathfrak{m}_m\} = \{\iota_i^{-1}(\mathfrak{m}) : 1 \leq i \leq n\}.$$

Thus

$$\bigcap_i \mathfrak{m}_i = \bigcap_{i=1}^{n} \iota_i^{-1}(\mathfrak{m}).$$

Because $K \subseteq \mathcal{O}_K$ [18, Proposition 4.4.6]), it follows that $K^{\text{alg}} \subseteq \mathcal{O}$, where K^{alg} is the algebraic closure of K inside F. Let $\alpha_1, \ldots, \alpha_n$ be the images of α under $\iota_1, \ldots, \iota_n$. These are pairwise distinct because $\mathbb{L}/\mathbb{M}$ is separable (by Lemma 2.3 again). Let M be the Vandermonde matrix whose (i, j) entry is α_i^{j-1}. Then $M \in \text{GL}_n(K^{\text{alg}}) \subseteq \text{GL}_n(\mathcal{O})$.

It follows that multiplication by M and M^{-1} preserves $\mathfrak{m}^n \subseteq F^n$. Concretely, this means that if $(x_0, x_1, \ldots, x_{n-1}) \in F^n$, then the following are equivalent:

- Each x_i is in $\mathfrak{m}$.
- $\sum_{i=0}^{n-1} x_i \alpha_j^i \in \mathfrak{m}$ for each j.

Specializing to the case where $x_0, \ldots, x_{n-1} \in \mathbb{M}$, and writing $x = \sum_{i=0}^{n-1} x_i \alpha^i$, the following are equivalent:

- Each x_i is in I_K.
- $\iota_j(x) \in \mathfrak{m}$ for each $j \leq n$, or equivalently $x \in \mathfrak{m}_i$ for each $i \leq m$. ∎

Our goal is to show $m = 1$. Suppose for the sake of contradiction that $m > 1$. Because the finite group $\mathrm{Aut}(L/K)$ acts transitively on the $\mathcal{O}_i$'s, they are pairwise incomparable. By the approximation theorem for valuations [3, VI.7.1, Corollaire 1], we can find an element $x \in \mathbb{L}$ such that $x \in 1 + \mathfrak{m}_1$ and $x \in -1 + \mathfrak{m}_i$ for $i > 1$.

Let $I = \bigcap_i \mathfrak{m}_i$. This is type-definable over K. Then

$$x \notin 1 + I,$$
$$-x \notin 1 + I,$$
$$x^2 \in 1 + I.$$

By basic valuation theory, each $1 + \mathfrak{m}_i$ is a subgroup of $\mathbb{M}^\times$. The intersection $1 + I$ is therefore also a subgroup of $\mathbb{M}^\times$. The intersection $1 + I$ is also topologically open: by Claim 3.8,

$$1 + I = (1 + I_K) + I_K \cdot \alpha + I_K \cdot \alpha^2 + \cdots + I_K \cdot \alpha^{n-1},$$

and I_K is open by Lemma 3.5. Similarly, any K-definable neighborhood of 1 in $\mathbb{L}$ contains $1 + I$, because of Remark 3.4.

The squaring map on $\mathbb{L}^\times$ is finite-to-one, so by Lemma 3.6, $(1 + I)^2$ has interior. Since $(1 + I)^2$ is a group, it is actually open, hence contains a neighborhood of 1:

$$(1 + I)^2 \text{ is a neighborhood of } 1. \tag{3.1}$$

Now $x \notin 1 + I$ and $-x \notin 1 + I$, and I is type-definable over K. So there is some K-definable set U containing I, such that $x \notin 1 + U$ and $-x \notin 1 + U$. By (3.1), $(1 + U)^2$ is a neighborhood of 1. It is K-definable, so it contains $1 + I$, hence x^2. Then there is $y \in 1 + U$ such that $y^2 = x^2$. Either $x \in 1 + U$ or $-x \in 1 + U$, contradicting the choice of U.

If K has characteristic 2, replace -1 and 1 with 0 and 1, replace the squaring map with the Artin–Schreier map, and replace $1 + I < \mathbb{L}^\times$ with $I < \mathbb{L}$. ■

There are several variants of the argument in Proposition 3.7:

- If $\mathrm{char}(K) \neq p$ and K has the primitive pth roots of unity, one can use the pth power map instead of the squaring map.
- If $\mathrm{char}(K) = p$, one can use the pth Artin–Schreier map (as in the case $p = 2$ in the proof).

Surprisingly, the Artin–Schreier technique can be generalized from dp-minimal fields to the much broader setting of NIP fields. This yields a proof that NIP valued fields in positive characteristic are henselian [19, Theorem 2.8].

Lemma 3.9. *If $\mathcal{O}$ is a non-trivial valuation ring on $\mathbb{M}$, $\vee$-definable over K, then $\mathcal{O}_K$ is a coarsening of $\mathcal{O}$. If $\mathcal{O}$ is definable, then the canonical topology on K is induced by the valuation ring $\mathcal{O}(K) = \mathcal{O} \cap K$ on K.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. Then $\mathfrak{m}$ is infinite because $\mathcal{O}$ is non-trivial, and $\mathfrak{m}$ is type-definable over K because $\mathcal{O}$ is $\vee$-definable over K. By [18, Remark 5.8],

$$I_K \subseteq \mathfrak{m} - \mathfrak{m} = \mathfrak{m},$$

implying that $\mathcal{O} \subseteq \mathcal{O}_K$.

Now suppose $\mathcal{O}$ is definable. By [18, proof of Claim 6.6], the inclusions $I_K \subseteq \mathcal{O} \subseteq \mathcal{O}_K$ imply that $\{a \cdot \mathcal{O}(K) : a \in K^\times\}$ is a neighborhood basis of 0 in the canonical topology on K. ■

Remark 3.10. Suppose F is a field with some structure, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable $\vee$-definable valuation rings on F. Then the join $\mathcal{O}_1\mathcal{O}_2$ is *definable*.

Proof. The join can be written as either $\{x \cdot y : x \in \mathcal{O}_1,\ y \in \mathcal{O}_2\}$ (which is $\vee$-definable) or as $\{x \cdot y : x \in \mathfrak{m}_1, y \in \mathfrak{m}_2\}$, which is type-definable.[3] ■

Lemma 3.11. *Let $\mathbb{L}/\mathbb{M}$ be a finite algebraic extension. Any two non-trivial $\vee$-definable valuation rings on $\mathbb{L}$ are* not *independent, i.e., they induce the same topology.*

Proof. Let w_1, w_2 be two $\vee$-definable valuations on $\mathbb{L}$, and let v_1 and v_2 be their restrictions to $\mathbb{M}$. Let Γ_i be the value group of w_i. Let K be a small model over which everything is defined (including the extension $\mathbb{L}/\mathbb{M}$). Let v_K be the valuation on $\mathbb{M}$ coming from $\mathcal{O}_K$ and I_K. By Lemma 3.9, v_K is a coarsening of v_1 and v_2. So there are convex subgroups $\Delta_i < \Gamma_i$ such that v_K is the coarsening of v_i by Δ_i. Let w_i' be the coarsening of w_i by Δ_i. Then w_1' and w_2' are valuations on $\mathbb{L}$ extending v_K. By Proposition 3.7, $w_1' = w_2'$. The induced topology is invariant under coarsening, so w_1, w_1', w_2', w_2 all induce the same topology. ■

Proposition 3.12. *Let $\mathbb{L}$ be a finite extension of $\mathbb{M}$. Any two $\vee$-definable valuation rings on $\mathbb{L}$ are comparable.*

Proof. Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are incomparable. Let $\mathcal{O} = \mathcal{O}_1 \cdot \mathcal{O}_2$ be their join, which is definable by Remark 3.10. Let w be the valuation corresponding to $\mathcal{O}$, and let v be its restriction to $\mathbb{M}$.

The residue field $\mathbb{L}' := \mathbb{L}w$ is a finite extension of $\mathbb{M}' := \mathbb{M}v$. Moreover, $\mathbb{L}'$ has two independent $\vee$-definable valuations, induced by $\mathcal{O}_1$ and $\mathcal{O}_2$. By Remark 3.3, this ensures that $\mathbb{L}'$ is infinite and unstable, so $\mathbb{M}'$ is also infinite and unstable. But $\mathbb{M}'$ has dp-rank at most 1, so $\mathbb{M}'$ is a dp-minimal unstable field. It is also as saturated as $\mathbb{M}$, so all our results so far apply to $\mathbb{M}'$. By Lemma 3.11, $\mathbb{L}'$ cannot have two independent $\vee$-definable valuation rings, and we have a contradiction. ■

Corollary 3.13. *Any $\vee$-definable valuation ring $\mathcal{O}$ on $\mathbb{M}$ is henselian.*

[3] Here, we are using the fact that if $\mathcal{O}$ is a valuation ring with maximal ideal $\mathfrak{m}$, and S is any set, then $S \cdot \mathcal{O}$ and $S \cdot \mathfrak{m}$ are closed under addition, and are equal to each other unless S has an element of minimum valuation. Incomparability of $\mathcal{O}_1$ and $\mathcal{O}_2$ ensures that e.g. $v_1(\mathcal{O}_2)$ has no minimum.

Proof. Otherwise, $\mathcal{O}$ would have two incomparable extensions to some finite Galois extension of $\mathbb{M}$. ■

Corollary 3.13 was obtained independently by Jahnke, Simon, and Walsberg [17, Proposition 4.5]).

Theorem 3.14. *The valuation ring $\mathcal{O}_K$ (whose maximal ideal is the set of K-infinitesimals) is henselian.*

We summarize the state of affairs for a general dp-minimal field as follows:

Theorem 3.15. *Let K be a dp-minimal field.*

(1) *If K is infinite, sufficiently saturated, and not algebraically closed, then K admits a non-trivial Henselian valuation (not necessarily definable).*

(2) *Any definable valuation on K is henselian. Any two definable valuations on K are comparable.*

Proof. (1) If K is strongly minimal, then K is algebraically closed by a well-known theorem of Macintyre. Otherwise, this is Theorem 3.14.

(2) We may assume K is sufficiently saturated. If K is not strongly minimal, this is Proposition 3.12 and Corollary 3.13. Otherwise, K is NSOP, so has only the trivial valuation. ■

4. The canonical valuation

We now turn to proving Theorem 4.8. This relies crucially on Jahnke and Koenigsmann's work on the definability of canonical valuations [16].

4.1. *Review of canonical p-henselian valuations*

Fix a prime p. Following [16], if K is any field, let $K(p)$ denote the *p-closure* of K, the compositum of all finite Galois extensions L/K with $\mathrm{Gal}(L/K)$ a p-nilpotent group.

A valuation v on K is *p-henselian* if it has a unique extension to $K(p)$. This is a weakening of henselianity. On any field K there is a *canonical p-henselian* valuation v_K^p, which might be trivial. It has the following properties:

Fact 4.1. (1) *If the residue field Kv_K^p is not p-closed, then v_K^p is the finest p-henselian valuation on K.*

(2) *Every p-henselian valuation strictly finer than v_K^p has p-closed residue field.*

(3) *If K admits no orderings and contains the pth roots of unity, then the valuation ring of v_K^p is 0-definable in K from the field language.*

The first two points appear in the discussion following [16, Theorem 2.2], and the non-trivial third point is [16, Main Theorem 3.1].

Say that a field K is "p-corrupted" if no finite extension is p-closed.

Lemma 4.2. *Let K be a perfect field which is neither algebraically nor real closed. Then some finite extension of K is p-corrupted for some p.*

Proof. Replace K with $K(\sqrt{-1})$ in characteristic 0. Take some non-trivial finite Galois extension L/K. Take p dividing $|\mathrm{Gal}(L/K)|$. By Sylow theory there is some intermediate field $K < F < L$ such that L/F is a p-nilpotent Galois extension. Then $F(p) \neq F$. A theorem of Becker [2] says that if F is not p-closed and admits no orderings, then $[F(p) : F] = \infty$. We forced F to contain $\sqrt{-1}$ in characteristic 0, and F is not p-closed, so $[F(p) : F] = \infty$. Then no finite extension F' of F will contain $F(p)$, nor $F'(p) \supseteq F(p)$, so F is p-corrupted. ∎

4.2. *Applying canonical p-henselian valuations*

Say that a fairly saturated field L, perhaps with extra structure, is *special* if it is a finite extension of an infinite dp-minimal field.[4]

If L is a special field, then L is a finite extension of a perfect field K, so L is perfect. (Or use Lemma 2.3 directly.) Moreover, the analogue of Theorem 3.15 holds for special fields:

Remark 4.3. Let L be a special field.

(1) L admits a non-trivial Henselian valuation if L is sufficiently saturated, and not algebraically closed.

(2) Any definable valuation on L is henselian. Any two definable valuations on L are comparable.

All of these facts follow easily from Theorem 3.15.

Also, special fields are closed under the following operations:

- Any finite extension of a special field is special.
- If L is special and w is a definable valuation on L, then Lw is finite or special.

To see the second point, let $v = w|K$, and note that Lw is a finite extension of Kv, which is dp-minimal.

We get a handle on special fields via the following trick:

Proposition 4.4. *Let L be a sufficiently saturated special field. Suppose L is not orderable, and contains all the pth roots of unity. Then the canonical p-henselian valuation v_L^p on L is definable, and its residue field is finite or p-closed.*

Proof. Definability of v_L^p follows by the work of Jahnke and Koenigsmann [16, Main Theorem 3.1]. Suppose the residue field Lv_L^p is infinite and not p-closed. Then Lv_L^p is special

[4] More precisely, L is special if there is an L-definable infinite subfield K with $[L : K]$ finite, and the induced structure on K is dp-minimal.

and not algebraically closed, so it admits a non-trivial henselian place $Lv_L^p \to L'$. The place $L \to Lv_L^p$ is henselian because it is definable, so the composition

$$L \to Lv_L^p \to L'$$

is itself a henselian place, which corresponds to a *finer* p-henselian valuation than v_L^p. But the canonical p-henselian valuation is the finest p-henselian valuation, unless its residue field is p-closed, so we have a contradiction. ■

Mostly we will use the following consequence:

Corollary 4.5. *Let L be a sufficiently saturated special field containing all the $4p$th roots of unity, and let v be a henselian valuation on L which is as fine as every definable valuation on L. Then Lv is finite or p-closed.*

Proof. Because v_L^p is definable, v is as fine as v_L^p. If $v = v_L^p$, then Lv is Lv_L^p, which is finite or p-closed. If v is strictly finer than v_L^p, then Lv is p-closed by Fact 4.1.2. ■

4.3. *The saturated case*

Remark 4.6. Let K be a field, $(I, <)$ a totally ordered set, and $\langle \mathcal{O}_x \rangle_{x \in I}$ be a totally ordered chain of valuation rings on K. Then the intersection

$$\mathcal{O} = \bigcap_{x \in I} \mathcal{O}_x$$

is itself a valuation ring on K. If the intersection has residue characteristic p, then some $\mathcal{O}_x$ does: either K itself has characteristic p, or $1/p \notin \mathcal{O}$, hence $1/p \notin \mathcal{O}_x$ for some $x \in I$.

Theorem 4.7. *Let K be a sufficiently saturated dp-minimal field. Let $\mathcal{O}_\infty$ be the intersection of all the definable valuation rings on K. (So $\mathcal{O}_\infty = K$ if K admits no definable non-trivial valuations.)*

(1) *$\mathcal{O}_\infty$ is a henselian valuation ring on K.*

(2) *$\mathcal{O}_\infty$ is type-definable, without parameters. In fact, it is the intersection of all 0-definable valuation rings on K.*

(3) *The residue field of $\mathcal{O}_\infty$ is finite, real-closed, or algebraically closed. If it is finite, then $\mathcal{O}_\infty$ is definable.*

Proof. (1) By Theorem 3.15 (2), the class of definable valuation rings on K is totally ordered, and they are all henselian. The intersection of a chain of valuation rings is a valuation ring. The intersection of a chain of henselian valuation rings is henselian.

(2) We need to show that $\mathcal{O}_\infty$ is a small intersection. Suppose $\mathcal{O}$ is a definable valuation ring on K, defined by a formula $\phi(K; b)$. Let $\psi(x)$ be the formula asserting that $\phi(K; x)$ is a valuation ring. Then $\bigcap_{b \in \psi(K)} \phi(K; b)$ is a 0-definable valuation ring contained in $\mathcal{O}$. Thus every definable valuation ring on K contains a 0-definable valuation

ring. Therefore $\mathcal{O}_\infty$ is the intersection of the 0-definable valuation rings on K. It is therefore type-definable over $\emptyset$.

(2) First suppose that the residue field of $\mathcal{O}_\infty$ is finite. Let $\mathfrak{m}_\infty$ denote the maximal ideal of $\mathcal{O}_\infty$. Then $\mathfrak{m}_\infty = K \setminus \mathcal{O}_\infty^{-1}$, so $\mathfrak{m}_\infty$ is $\vee$-definable. On the other hand, $\mathcal{O}_\infty$ is a finite union of translates of $\mathfrak{m}_\infty$, so $\mathcal{O}_\infty$ is also $\vee$-definable, hence definable.

Now suppose that the residue field is infinite. Let v_∞ denote the valuation associated with $\mathcal{O}_\infty$. Note that v_∞ is as fine as any definable valuation on K, by choice of $\mathcal{O}_\infty$.

In particular, for every definable valuation v on K, the place $K \to Kv_\infty$ factors as a composition of two places

$$K \to Kv \to Kv_\infty.$$

We first show that Kv_∞ is perfect. If Kv_∞ has characteristic p, then Kv has characteristic p for some definable valuation v, by Remark 4.6. The field Kv is perfect by Lemma 2.3, so the place $Kv \to Kv_\infty$ ensures that Kv_∞ is perfect as well (perfect equicharacteristic valued fields have perfect residue fields).

Suppose for the sake of contradiction that Kv_∞ is not algebraically closed or real closed. As Kv_∞ is perfect, Lemma 4.2 applies, and some finite extension F of Kv_∞ is p-corrupted for some prime p (not necessarily the characteristic).

Choose a finite extension L of K such that

- L contains all the $4p$th roots of unity.
- If w_∞ denotes the (unique) extension of v_∞ to L, then Lw_∞ contains F, hence is not p-closed (nor finite).

By Corollary 4.5, some definable valuation w on L is *not* a coarsening of w_∞. Let v be $w|K$. Then v is a coarsening of v_∞:

$$v(x) = v_\infty(x) + \Delta,$$

for some convex subgroup $\Delta < v_\infty K$. Coarsening w_∞ with respect to the same convex subgroup Δ, we get a coarsening w' of w_∞, whose restriction to K is v. But v is henselian, so $w = w'$, and w is coarser than w_∞, a contradiction. ∎

4.4. The non-saturated case

Theorem 4.8. *Let $(K, +, \cdot, \ldots)$ be a dp-minimal field, perhaps with additional structure. Let $\mathcal{O}$ be the intersection of all 0-definable valuation rings on K. Then*

(1) *$\mathcal{O}$ is itself a valuation ring, possibly trivial (i.e., $\mathcal{O}$ might equal K).*

(2) *Either $\mathcal{O} = K$, or K is unstable and $\mathcal{O}$ induces the canonical topology on K.*

(3) *$\mathcal{O}$ is henselian and defectless.*

(4) *The residue field of $\mathcal{O}$ is finite, algebraically closed, or real closed.*

Proof. The 0-definable valuation rings on K are henselian and pairwise comparable (Theorem 3.15 (2)), so their intersection $\mathcal{O}$ is a henselian valuation ring, as in the previous section.

If $\mathcal{O}$ is non-trivial, then there is at least one 0-definable non-trivial valuation ring $\mathcal{O}_1$ on K, and K is unstable. By Lemma 3.9, the valuation ring $\mathcal{O}_1$ induces the canonical topology on K. As $\mathcal{O}_1$ is a coarsening of $\mathcal{O}$, the valuation ring $\mathcal{O}$ also induces the canonical topology.

Let v be the valuation associated with $\mathcal{O}$.

Claim 4.9. *The valuation v is defectless.*

Proof. If Kv has characteristic 0, then henselian implies defectless. So suppose Kv has characteristic p. By Remark 4.6, there is some 0-definable valuation w with residue characteristic p. The place $K \to Kv$ splits as

$$K \to Kw \to Kv.$$

By Remark 2.10, both pieces are henselian. Then $K \to Kw$ is defectless by Theorem 2.11, and $Kw \to Kv$ is defectless by Lemma 2.5. By Remark 2.10, $K \to Kv$ is defectless. ∎

Finally, we need to show that the residue field Kv is finite, algebraically closed, or real closed.

Let $\mathbb{M} \succeq K$ be a sufficiently saturated elementary extension, and let $\mathcal{O}_\infty$ be the type-definable subring of $\mathbb{M}$ from the previous section—the intersection of the 0-definable valuation rings on $\mathbb{M}$. Then $\mathcal{O} = \mathcal{O}_\infty \cap K$.

There are three cases:

(1) If $\mathcal{O}_\infty$ has finite residue field, then $\mathcal{O}_\infty$ is definable, hence 0-definable. Then $\mathcal{O} \preceq \mathcal{O}_\infty$, so $\mathcal{O}$ has finite residue field.

(2) Next, suppose $\mathcal{O}_\infty$ has algebraically closed residue field. Then for every n, we have

$$\forall a_1, \dots, a_n \in \mathcal{O}_\infty \; \exists x : x^n + a_1 x^{n-1} + \cdots + a_n \in \mathfrak{m}_\infty.$$

By compactness, there is some 0-definable valuation ring $\mathcal{O}_n$ on $\mathbb{M}$ such that

$$\forall a_1, \dots, a_n \in \mathcal{O}_n \; \exists x : x^n + a_1 x^{n-1} + \cdots + a_n \in \mathfrak{m}_n.$$

This remains true in K, as $K \preceq \mathbb{M}$. Because $\mathcal{O}_n \supseteq \mathcal{O}$ and $\mathfrak{m}_n \subseteq \mathfrak{m}$, we get

$$\forall a_1, \dots, a_n \in \mathcal{O} \; \exists x : x^n + a_1 x^{n-1} + \cdots + a_n \in \mathfrak{m}.$$

Now if $a_1, \dots, a_n \in \mathcal{O}$ and $x^n + a_1 x^{n-1} + \cdots + a_n \in \mathfrak{m}$, then x is integral over $\mathcal{O}$. Thus $x \in \mathcal{O}$, as $\mathcal{O}$ is integrally closed. Therefore all degree n monic polynomials in $\mathcal{O}/\mathfrak{m}$ have roots.

(3) Next, suppose $\mathcal{O}_\infty$ has a real closed residue field. The unique extension of $\mathcal{O}_\infty$ to $\mathbb{M}(i)$ has algebraically closed residue field. Repeating the arguments we just gave with $K(i)$ and $\mathbb{M}(i)$ instead of K and $\mathbb{M}$, we see that the residue field of $K(i)$ with respect to $\mathcal{O}$ is algebraically closed. So the residue field of K is real closed or algebraically closed. ∎

Theorem 1.2 follows as a corollary.

Proof of Theorem 1.2. If K has a definable valuation, it is henselian by Theorem 3.15 (2). Otherwise, the valuation ring $\mathcal{O}$ in Theorem 4.8 is trivial, and K is the residue field, which is finite, algebraically closed, or real closed. ■

5. Dp-small fields

Recall the notion of dp-smallness (see §1.1). Dp-smallness is a stricter condition than dp-minimality. Like dp-minimality,[5] dp-smallness is preserved under reducts and under naming parameters. Guingona shows that VC-minimal structures are dp-small [10, Proposition 1.5].

Theorem 5.1. *Let K be a dp-small field. Then K is algebraically closed or real closed.*

Proof. We can and do take K to be sufficiently saturated. By [10, Theorem 1.6 (4)], the value group vK is divisible for any definable valuation v on K.

By Theorem 4.7, there is a henselian defectless valuation v_∞ on K whose valuation ring is the intersection of all definable valuation rings on K. The residue field of v_∞ is algebraically closed, real closed, or finite. In the finite case, v_∞ is definable, and by Theorem 2.11, the interval $[-v(p), v(p)]$ in the value group is finite, contradicting divisibility.

Therefore, the residue field Kv_∞ is algebraically closed or real closed. For K to be algebraically closed or real closed, it suffices to show that the value group $v_\infty K$ is divisible, by Ax–Kochen–Ershov in the real closed case and by defectlessness in the algebraically closed case.

Let ℓ be any prime. Let a be an element of $K^\times$. For each definable valuation $\mathcal{O}$ on K, the value group $K^\times/\mathcal{O}^\times$ is ℓ-divisible. So there are $b \in K^\times$ and $c \in \mathcal{O}^\times$ such that $a = b^\ell \cdot c$. The valuation ring $\mathcal{O}_\infty$ of v_∞ is the intersection of a small ordered set of $\mathcal{O}$'s, so by compactness, we can find $b \in K^\times$ and $c \in \mathcal{O}_\infty^\times$ such that $a = b^\ell \cdot c$. Then $v_\infty(a) = \ell \cdot v_\infty(b)$. So v_∞ has ℓ-divisible value group, for arbitrary ℓ. ■

We can also specialize this result to C-minimal and (weakly) o-minimal fields.

5.1. C-minimal fields

We will use the definition of C-minimality from Haskell and Macpherson [14], rather than the more general definition from Delon [5]. For us, a *C-relation* is a ternary relation $C(x; y, z)$ satisfying the following axioms:

(1) $\forall x, y, z : (C(x; y, z) \to C(x; z, y))$.

(2) $\forall x, y, z : (C(x; y, z) \to \neg C(y; x, z))$.

(3) $\forall x, y, z, w : (C(x; y, z) \to (C(w; y, z) \vee C(x; w, z)))$.

(4) $\forall x \, \forall y \neq x \, \exists z \neq y : C(x; y, z)$.

[5] And *unlike* VC-minimality…

The intuition is that $C(x; y, z)$ holds if y and z are closer to each other than either is to x, with respect to some ultrametric.

A *unary definable set* in a structure $(M, \ldots)$ is a definable subset of M^n with $n = 1$. A structure $(M, C, \ldots)$ is *C-minimal* if C is a C-relation and every unary definable set $D \subseteq M$ is definable without quantifiers in the reduct (M, C). We make two comments on the definition:

- In [5], Delon omits the density condition (4), which makes theories like RCF be C-minimal. But we will assume density, following Haskell and Macpherson [14].
- When the structure is a field, we will not assume any compatibility between the C-relation and field structure, *unlike* [14].

C-minimality implies VC-minimality and dp-smallness, so by Theorem 5.1, any C-minimal field is real closed or algebraically closed. We will show that the real closed case cannot occur, but we will first need to prove that infinite definable sets in C-minimal theories never admit definable total orders.

Definition 5.2. A *dorderable set* is a definable (not interpretable!) set admitting at least one definable total order.

Definition 5.3. A structure M *defines no total orders* if there are no infinite dorderable sets.

This condition can be checked on unary sets:

Lemma 5.4. *If M has no infinite dorderable sets $D \subseteq M^1$, then M defines no total orders.*

Proof. Let $\mathcal{C}$ be the class of definable sets X containing an infinite dorderable subset $D \subseteq X$.

Claim 5.5. *If $X \times Y \in \mathcal{C}$, at least one of X and Y is in $\mathcal{C}$.*

Proof. Given $D \subseteq X \times Y$ infinite and dorderable, consider the projection $\pi : D \to X$. Each fiber of π is dorderable and embeds definably into Y, so if some fiber of π is infinite, then $Y \in \mathcal{C}$. Otherwise, the fibers are all finite. Let $g : \pi(D) \to D$ pick out the least element of each fiber. We can pull the ordering on D back to $\pi(D)$ along g. Then the infinite subset $\pi(D)$ of X is dorderable, so $X \in \mathcal{C}$. ■

Consequently, if $M^1 \notin \mathcal{C}$, then $M^n \notin \mathcal{C}$, proving Lemma 5.4. ■

In the rest of this section, we will show that (dense) C-minimal structures define no total orders. For the sake of contradiction, suppose that there is an infinite dorderable unary set $X \subseteq M^1$.

Definition 5.6. A *tree-like set* is a non-empty finite set S such that for every ball B, $|S \cap B|$ is 0 or a power of 2.

Note that $|S| = 2^n$ for some n, because the entirety of M is a ball. We call n the *depth* of S.

Lemma 5.7. *Let B_1 and B_2 be disjoint balls. Let $S_i \subseteq B_i$ be a tree-like set of depth n. Then $S_1 \cup S_2$ is a tree-like set of depth $n+1$.*

Proof. Let B be any ball. If B intersects only B_i for $i = 1$ or 2, then $|B \cap S| = |B \cap S_i|$ has the desired form. Otherwise, B intersects both of B_i, hence contains both. So $B \cap S = S$ and $|B \cap S| = |S_1| + |S_2| = 2^{n+1}$. ∎

Lemma 5.8. *An infinite definable set X contains arbitrarily big tree-like sets.*

Proof. The C-minimal density assumption ensures that every infinite ball contains two disjoint infinite subballs. By induction on n, we see that

if B is an infinite ball, then B contains a tree-like set of depth n.

The density assumption also ensures that X contains an infinite ball. ∎

For any definable set D, the characteristic function χ_D of D can be written as

$$\chi_D = \sum_{i=1}^{m} a_i \cdot \chi_{B_i},$$

where the B_i's are balls and $a_i \in \{-1, 1\}$. This is an easy consequence of the swiss cheese decomposition. Call the least such m the *complexity* of D. By compactness, complexity is bounded in definable families.

If S is a tree-like set of depth n, then

$$|S \cap D| = \sum_{s \in S} \chi_D(s) = \sum_{s \in S} \sum_{i=1}^{m} a_i \cdot \chi_{B_i}(s) = \sum_{i=1}^{m} a_i \cdot |S \cap B_i| = \sum_{i=1}^{m} a_i' \cdot 2^{k_i},$$

for some $a_i' \in \{-1, 0, 1\}$ and some $k_i \in \{1, \dots, n\}$.

In particular, as D ranges through sets of complexity m, there are only $(3n)^m$ possibilities for $|S \cap D|$. On the other hand, as D ranges through the half-infinite intervals $(-\infty, a) \subseteq X$ (with respect to the ordering), the size $|S \cap D|$ should range through all the values in $\{0, 1, \dots, 2^n\}$.

Let m bound the complexity of the half-infinite intervals, let n be large enough that $2^n + 1 > (3n)^m$, and let $S \subseteq X$ be a tree-like set of depth n. Then we have a contradiction. We have shown

Proposition 5.9. (*Dense*) *C-minimal structures never define total orders.*

Corollary 5.10. (*Dense*) *C-minimal structures never eliminate imaginaries.*

Proof. Any C-minimal structure interprets the set of balls. Within this, the set of balls around a given point is totally ordered, and infinite under the density assumption. So any dense C-minimal structure interprets an infinite total order, but defines no infinite total order. ∎

Corollary 5.11. (*Dense*) *C-minimal fields are algebraically closed.*

Proof. Proposition 5.9 prevents dense C-minimal fields from being real-closed. ■

5.2. Weakly o-minimal fields

Weakly o-minimal fields turn out to be real closed. This will require a little bit of work. First we prove a lemma. Let $\mathbb{G}_{\mathrm{m}}$ denote the multiplicative group. Recall the various notions of connected component:

- $\mathbb{G}_{\mathrm{m}}^{0}$, the intersection of finite index definable subgroups.
- $\mathbb{G}_{\mathrm{m}}^{00}$, the smallest type-definable subgroup of bounded index.
- $\mathbb{G}_{\mathrm{m}}^{000}$, the smallest invariant[6] subgroup of bounded index.

The latter two groups exist by a theorem of Gismatullin [9]. On general grounds, $\mathbb{G}_{\mathrm{m}}^{000} \subseteq \mathbb{G}_{\mathrm{m}}^{00} \subseteq \mathbb{G}_{\mathrm{m}}^{0}$.

Lemma 5.12. *Let $\mathbb{M}$ be a sufficiently saturated dp-minimal field, not strongly minimal, possibly with extra structure. Then every infinitesimal type is multiplicatively stabilized by $\mathbb{G}_{\mathrm{m}}^{0}$.*

Proof. Let p be an infinitesimal type over $\mathbb{M}$, and take $a \in \mathbb{G}_{\mathrm{m}}^{0}$. Let $\psi(x; z)$ be a formula. We will show that $a^{-1} \cdot p$ and p have the same ψ-type.

Let $\phi(x; y, z)$ be the formula $\psi(x \cdot y; z)$. Every ψ-formula is a ϕ-formula, so it suffices to show that $a^{-1} \cdot p$ and p have the same ϕ-type. Moreover, the multiplicative group acts on ϕ-formulas and hence on ϕ-types.

For any $\alpha \in \mathbb{G}_{\mathrm{m}}$, the type $\alpha \cdot p$ is an infinitesimal type, because p is infinitesimal. By [18, Corollary 7.5], the orbit of p is small. Restricting to ϕ-types, we see that $p|\phi$ has a small orbit as well.

Because infinitesimal types are definable [18, Corollary 7.6], the multiplicative stabilizer of the ϕ-type $p|\phi$ is definable. Therefore the orbit is interpretable. Being bounded, it must be finite. So $p|\phi$ is stabilized by some finite-index subgroup of $\mathbb{G}_{\mathrm{m}}$. As $a \in \mathbb{G}_{\mathrm{m}}^{0}$, it follows that $a \cdot p|\phi = p|\phi$ as claimed. ■

Corollary 5.13. *Let $\mathbb{M}$ be a* (*sufficiently saturated*) *dp-minimal field. Then $\mathbb{G}_{\mathrm{m}}^{000}$ equals $\mathbb{G}_{\mathrm{m}}^{0}$, which in turn equals the intersection of the sets of nth powers as n ranges over positive integers.*

Proof. If $\mathbb{M}$ is strongly minimal, all these facts are well-known. Assume $\mathbb{M}$ is not strongly minimal. The fact that $\mathbb{G}_{\mathrm{m}}^{0}$ is $\bigcap_n (\mathbb{M}^\times)^n$ holds because $\mathbb{M}^\times/(\mathbb{M}^\times)^n$ is finite [18, Theorem 1.5]. The group $\mathbb{G}_{\mathrm{m}}^{000}$ exists by [9]. Because each non-zero infinitesimal type lives in a specific coset of $\mathbb{G}_{\mathrm{m}}^{000}$, the multiplicative stabilizer of any infinitesimal type must be contained in $\mathbb{G}_{\mathrm{m}}^{000}$. On the other hand, the stabilizer is $\mathbb{G}_{\mathrm{m}}^{0}$ by Lemma 5.12. So $\mathbb{G}_{\mathrm{m}}^{0} \subseteq \mathbb{G}_{\mathrm{m}}^{000} \subseteq \mathbb{G}_{\mathrm{m}}^{0}$. ■

[6] That is, $\mathrm{Aut}(\mathbb{M}/A)$-invariant for some small A.

Using Lemma 5.12, we can prove a rather strong and surprising result.

Theorem 5.14. *Let K be a dp-minimal algebraically closed field with extra structure. Then there is no infinite definable subset of K^n with a definable total ordering.*

Proof. We may replace K with a monster $\mathbb{M}$. Suppose some infinite definable set $D \subseteq \mathbb{M}^n$ admits a definable total ordering $<_D$. By Lemma 5.4, we may assume $n = 1$. Then D has non-empty interior by [18, Theorem 7.8]. Translating D, we may assume that 0 is in the interior of D. So all infinitesimal types over $\mathbb{M}$ live in D.

Let p be some non-zero infinitesimal type. Then p is multiplicatively stabilized by $\mathbb{G}_{\mathrm{m}}^{\mathbf{0}}$. Because $\mathbb{M}$ is algebraically closed, $\mathbb{G}_{\mathrm{m}}$ is divisible. This implies that it has no proper subgroups of finite index. Therefore $\mathbb{G}_{\mathrm{m}}^{\mathbf{0}} = \mathbb{G}_{\mathrm{m}}$, so $a \cdot p = p$ for any $a \in \mathbb{G}_{\mathrm{m}}$.

Let ω be some root of unity, other than 1. Then $\omega \cdot p = p$. As p is non-zero and $\omega \neq 1$, the type $p(x)$ must say $x \neq \omega \cdot x$. By totality of the ordering, we may assume $x <_D \omega \cdot x$ is in $p(x)$, reversing the order if necessary.

Now if a realizes p in some elementary extension of $\mathbb{M}$, then $\omega^i \cdot a \models \omega^i \cdot p = p$. In particular,

$$\omega^i \cdot a <_D \omega \cdot (\omega^i \cdot a),$$

for all i. By transitivity, the map $i \mapsto \omega^i \cdot a$ is strictly increasing, hence injective, contradicting the fact that ω is a root of unity. ∎

Because weakly o-minimal structures are VC-minimal and dp-small, we immediately get the following corollary, which was probably more easily proven by other means:

Corollary 5.15. *Weakly o-minimal fields are real-closed.*

Again, this is slightly more general than the result in [24], since we are not assuming that the weakly o-minimal ordering is a field ordering.

6. Quantifier elimination and dp-minimality

So far, we have assumed dp-minimality and obtained constraints. We now turn to the positive side of the classification—Theorem 1.3 (1), which asserts that certain theories are complete and dp-minimal.

Except in the cases of positive residue characteristic, completeness follows by the Ax–Kochen–Ershov principle, and dp-minimality follows from Chernikov and Simon's result, proven in [4]:

Fact 6.1. *A henselian valued field (K, v) with residue characteristic 0 is dp-minimal if and only if vK and Kv are dp-minimal.*

In this section, we will handle the remaining cases, which are

- Hahn series fields like $\mathbb{F}_p^{\mathrm{alg}}((t^\Gamma))$ (with Γ dp-minimal and p-divisible).
- Their mixed characteristic analogues.

Along the way, we will prove quantifier elimination results in §6.1.

We will make occasional use of the RV sort. If K is a valued field with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$, then $\mathrm{RV}(K)$ denotes the quotient $K^\times/(1+\mathfrak{m})$, and $\mathrm{rv} : K^\times \to \mathrm{RV}(K)$ is the natural map. We summarize the key facts about $\mathrm{RV}(K)$:

(1) rv is a homomorphism $K^\times \to \mathrm{RV}(K)$.

(2) $\mathrm{rv}(x-a) = \mathrm{rv}(y-a)$ if and only if some ball contains x and y but not a.

(3) $\mathrm{RV}(K)$ sits in a short exact sequence

$$1 \to Kv^\times \to \mathrm{RV}(K) \to vK \to 1,$$

where Kv is the residue field and vK is the value group.

(4) If L/K is an extension of valued fields, then L/K is immediate if and only if $\mathrm{RV}(L) = \mathrm{RV}(K)$.

(5) If L/K is an extension of valued fields and $\mathrm{rv}(x) \in \mathrm{RV}(L) \setminus \mathrm{RV}(K)$, then one of two things happens:
   - $v(x) \in vL \setminus vK$.
   - $v(x) \in vK$, so $v(x) = v(y)$ for some $y \in K^\times$. Then $v(x/y) = 0$ and

$$\mathrm{res}(x/y) \in \mathrm{res}(L) \setminus \mathrm{res}(K) = Lv \setminus Kv.$$

6.1. *A quantifier elimination result*

Fix a prime p. Recall the definition of rough p-divisibility (Definition 2.9). Let T_0 be the theory of henselian defectless fields (K, v) with $Kv \models \mathrm{ACF}_p$ and with p-divisible value group. Let T be the theory of henselian defectless roughly p-divisible fields (K, v) with $Kv \models \mathrm{ACF}_p$.

Every model of T_0 is a model of T, and the converse holds in equicharacteristic p. Models of T_0 are tame (in the sense of [23]), though models of T need not be—they are only “roughly” tame. Nevertheless, we will see that many of the good properties of T_0 extend to T.

Remark 6.2. If $M \models T_0$, then any finite field extension of M has degree prime to p. Indeed, if L/M is finite, then henselianity and defectlessness imply

$$[L : M] = |vL/vM| \cdot [Lv : Mv].$$

But Mv is algebraically closed, so $[Lv : Mv] = 1$. And vM is p-divisible, so $|vL/vM|$ is prime to p.

Remark 6.3. Let $(L, v)/(K, v)$ be an extension of valued fields. Suppose (L, v) is henselian and K is relatively separably closed in L. Then Kv is relatively separably closed in Lv.

Proof. Otherwise, take $\alpha \in (Lv \cap Kv^{\text{sep}}) \setminus Kv$. Let $\bar{f}(X)$ be the monic irreducible polynomial of α over Kv. Let $f(X)$ be a lift of $\bar{f}(X)$ to $K[X]$. By henselianity of L, there is a unique root a of $f(X)$ lying over α. Moreover, a is a simple root, so $a \in K^{\text{sep}}$. Therefore a and α are in K and Kv, respectively. ∎

Proposition 6.4. *Let (M, v) be a model of T.*

(1) *M is perfect.*

(2) *If $a \in M$ and $n \in \mathbb{N}$, then a is an nth power if and only if $v(a)$ is divisible by n.*

(3) *If K is relatively algebraically closed in M, then $K \models T$.*

Proof. First suppose that $(M, v) \models T_0$.

(1) If M has characteristic p, then M is perfect by Remark 6.2.

(2) One easily reduces to showing that if $v(a) = 0$, then a is an ℓth power for all primes ℓ. For $\ell \neq p$, this follows by henselianity and the fact that $\text{res}(a)$ is an ℓth power. For $\ell = p$, this follows by Remark 6.2.

(3) We will show $K \models T_0$. Note that K is henselian and perfect because it is relatively algebraically closed in M, which is henselian and perfect.[7] Then M/K is regular, so $\text{Gal}(M)$ surjects onto $\text{Gal}(K)$. Since p is prime to $\text{Gal}(M)$ (by Remark 6.2), p is also prime to $\text{Gal}(K)$. In other words, p does not divide the degree of any finite extension of K. It follows immediately that (K, v) is defectless, Kv is perfect, and vK is p-divisible. Also, Kv is separably closed in Mv by Remark 6.3, and so $Kv \models \text{ACF}_p$.

Next suppose $(M, v) \models T$ but $(M, v) \not\models T_0$. Then M has characteristic 0. Let v' be the coarsening of v with respect to the minimal convex subgroup of vM containing $v(p)$. (By rough p-divisibility, this convex subgroup is p-divisible.) Let v'' be the induced valuation on Mv'. Then (M, v') is a henselian field of residue characteristic 0, and (Mv', v'') is a model of T_0 of characteristic 0.

(1) M is perfect because it has characteristic 0.

(2) As before, one reduces to showing that if $v(a) = 0$, then a is an ℓth power. Because (M, v') is a henselian field of residue characteristic 0, and $v'(a) = 0$, the element a is an ℓth power if and only if its residue $\text{res}'(a) \in Mv'$ is an ℓth power. But $v''(\text{res}'(a)) = v(a) = 0$, so by the case of T_0 considered above, $\text{res}'(a)$ is an ℓth power.

(3) Applying Remark 6.3 to $(M, v')/(K, v')$, we see that Kv' is relatively algebraically closed in Mv'. As Mv' is a model of T_0, so is Kv', by the case of T_0 considered above. Therefore, the place $Kv' \to Kv$ is henselian and defectless, with p-divisible value group and algebraically closed residue field Kv. The place $K \to Kv'$ is henselian of residue characteristic 0 (hence defectless and roughly p-divisible). The composition $K \to Kv' \to Kv$ is therefore henselian, defectless, and roughly p-divisible. And Kv is algebraically closed. ∎

[7] Recall that a valued field is henselian and perfect if and only if it is definably closed in an ambient model of ACVF. If $\text{dcl}(M) = M$, then $\text{dcl}(K) \subseteq K^{\text{alg}} \cap \text{dcl}(M) = K^{\text{alg}} \cap M = K$.

Lemma 6.5. *Let K be a valued field. Suppose L and F are two immediate algebraic extensions of K which are models of T. Then L and F are isomorphic as valued fields over K.*

Proof. We may replace K with the perfection of its henselization, which embeds into both L and F. We then only need to show that L and F are conjugate (isomorphic as pure fields) over K.

First suppose that vK is p-divisible, so $L, F \models T_0$. Then K is Kaplansky, L and F are algebraically maximally complete, and the desired result follows by the uniqueness of maximal algebraic immediate extensions over Kaplansky fields.[8]

Otherwise, K, L, and F have characteristic 0. Let v' be the coarsening with respect to the convex subgroup generated by $v(p)$. As L and F are immediate, $v'L = v'K = v'F$. Then (K, v') is a henselian field with residue characteristic 0, and L, F are two unramified extensions.

By the structure theory of valued fields [7, Theorems 5.2.7 and 5.2.9], unramified extensions of K are exactly controlled by their residue fields. Consequently, L and F are isomorphic over K (as fields, or as valued fields) as long as Lv' and Fv' are isomorphic extensions of Kv'.

Note that Lv' and Fv' are immediate extensions of Kv'. Also, Lv' and Fv' are models of T_0. By the T_0 case considered above, Lv' and Fv' are isomorphic, and we are done. ■

Definition 6.6. Let M and N be valued fields. A *partial v-elementary map from M to N* is a valued field embedding $f : K \to N$ for some subfield $K \subseteq M$, such that the induced map $vf : vK \to vN$ is a partial elementary map from vM to vN. If $\operatorname{dom} f = M$, we call $f : M \to N$ a *v-elementary map*, or say that f is *total*.

Lemma 6.7. *If $M, N \models T$ and N is $|M|^+$-saturated, and f is a maximal partial v-elementary map from M to N, then f is total.*

Proof. Let K be the domain of f.

Claim 6.8. *K is henselian.*

Proof. Suppose not. As M and N are henselian, both contain the henselization of K. We can extend f to an isomorphism f' between the henselizations of K and $f(K)$. The henselization of K has the same value group as K, so $vf' = vf$ is still partial elementary. Then f' is a strictly larger v-elementary map, a contradiction. ■

[8] In this particular case, the fact that L and F are conjugate over K should follow more directly from the uniqueness-up-to-conjugacy of Hall subgroups in solvable groups. The extension K^{alg}/L is prime to p, because $L \models T_0$, and the extension L/K is purely of p-power degree, because it is an immediate extension. So $\operatorname{Gal}(L)$ should be a prime-to-p Hall subgroup of the pro-solvable group $\operatorname{Gal}(K)$.

Claim 6.9. *Let $P(X)$ be an irreducible polynomial over K of degree greater than* 1. *If $P(X)$ has a root in M, then it does* not *have a root in N.*

Proof. Otherwise, let α be a root of $P(X)$ in M and β be a root in N. By basic field theory, there is an embedding of fields $f' : K(\alpha) \to f(K)(\beta)$ extending f, sending α to β. This map f' must also be a map of valued fields, because there is a unique valuation on $K(\alpha)$ extending the valuation on K, by Claim 6.8.

We claim that f' is v-elementary. By saturation of vN, there is some elementary embedding $g : vM \to vN$ extending vf. The group homomorphism $g - vf'$ from $vK(\alpha) \to vN$ vanishes on vK, so it factors through the finite group $vK(\alpha)/vK$. As vN is torsion-free, $g - vf'$ vanishes on $vK(\alpha)$. Thus vf' is the restriction $g|vK(\alpha)$, so vf' is partial elementary, and f' is partial v-elementary. This contradicts the maximality of f. ∎

Claim 6.10. *Every element of $\mathcal{O}_K^\times$ is a pth power (in K). Consequently Kv is perfect.*

Proof. Take $a \in \mathcal{O}_K^\times$. Then $X^p - a$ has a root in both M and N, so it has one in K. ∎

Claim 6.11. *Kv is separably closed, hence algebraically closed.*

Proof. If not, let $\bar{f}(X) \in Kv[X]$ be a monic irreducible separable polynomial of degree greater than 1. Let $f(X) \in \mathcal{O}_K[X]$ be a monic polynomial lifting $\bar{f}(X)$. Then $f(X)$ is also irreducible in $K[X]$ with degree greater than 1. The fields Mv and Nv are algebraically closed, so $\bar{f}(X)$ has roots in both Mv and Nv. Henselianity lifts these roots to roots of $f(X)$ in M and N. This contradicts Claim 6.9. ∎

Say that an embedding of abelian groups $A \hookrightarrow B$ is *pure* if B/A is torsionless. If A, B are torsionless, this is equivalent to the condition that for every prime ℓ and every $a \in A$, if a is a multiple of ℓ in B, then a is a multiple of ℓ in A.

Claim 6.12. *vK is pure in vM and vN.*

Proof. Suppose γ is divisible by ℓ in one of vM or vN. As vf is partial elementary, γ is divisible by ℓ in both vM and vN. Take $a \in K$ with $v(a) = \gamma$. By Proposition 6.4 (2), the polynomial $X^\ell - a$ has a root in both M and N. By Claim 6.9, $X^\ell - a$ is not irreducible over K. Then a has an ℓth root in K, so $v(a) = \gamma$ is divisible by ℓ in vK. ∎

Claim 6.13. *K is relatively algebraically closed in M and N.*

Proof. Let K_M and K_N be the relative algebraic closures of K in both fields. By Proposition 6.4 (3), K_M and K_N are models of T. The value group extension $|vK_M/vK|$ is torsion, but vK is pure in vM, so the value group extension must be trivial. Similarly, $K_M v = Kv$ because $K_M v$ is algebraic over Kv, but Kv is algebraically closed.

Therefore K_M is an immediate algebraic extension of K. Similarly, K_N is an immediate algebraic extension of K. By Lemma 6.5, K_M and K_N are isomorphic over K. This contradicts Claim 6.9 unless $K_M = K = K_N$. ∎

Claim 6.14. *$Kv = Mv$.*

Proof. Otherwise, let t_M be an element of M whose residue is not in Kv. By saturation of N, we can find $t_N \in N$ with residue not in Kv.

Let f' be the map $K(t_M) \to f(K)(t_N)$ sending t_M to t_N and extending f. This is a map of valued fields, because there is a unique valuation on $K(t)$ making t have transcendental residue. (Modulo quantifier elimination in ACVF, this is the statement that there is a unique type $p(x)$ that lives in the closed unit ball, but not in any smaller subballs. The uniqueness of this type follows by C-minimality.) By Abhyankar's inequality,

$$\mathrm{tr.deg}(K(t_M)v/Kv) + \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} (vK(t_M)/vK)) \leq \mathrm{tr.deg}(K(t_M)/K) = 1.$$

Therefore $vK(t_M)/vK$ is torsion, and then $vK(t_M) = vK$ by Claim 6.12. Therefore $vf' = vf$ and the map f' is v-elementary, contradicting maximality of f. ∎

Claim 6.15. $vK = vM$.

Proof. Otherwise, take $\gamma_M \in vK \setminus vM$. Let g be an elementary embedding $vN \to vM$ extending vf. Let $\gamma_N = g(\gamma_M)$.

Let t_M (resp. t_N) be an element of M (resp. N) having valuation γ_M (resp. γ_N). The elements t_M and t_N are transcendental over K, so there is a map of fields $f' : K(t_M) \to f(K)(t_N)$ extending f and sending t_M to t_N. This is a map of valued fields, because γ_M and γ_N define the same cut in $\mathbb{Q} \otimes_{\mathbb{Z}} vK$, and there is a unique valuation on $K(t)$ making $v(t)$ land in this cut (again, this follows by C-minimality and quantifier elimination in ACVF).

We claim that f' is v-elementary, and that in fact vf' is $g|vK(t_M)$. By Abhyankar's inequality,

$$\frac{vK(t_M)}{vK + \mathbb{Z} \cdot v(t_M)}$$

is torsion. So it suffices to show that vf' and g agree on vK and $v(t_M)$. The former holds because f' extends f and g extends vf, and the latter holds by choice of t_N and γ_N. ∎

In summary, K is relatively algebraically closed in M and N, and M/K is an immediate extension. By Proposition 6.4 (3), K is itself a model of T. In particular, K is defectless.

Now take $a \in M \setminus K$. In $M^{\mathrm{alg}} \models$ ACVF, let $\mathcal{B}$ be the chain of K-definable balls containing a.

Claim 6.16. *No element of K^{alg} is in the intersection $\bigcap \mathcal{B}$.*

Proof. Suppose $a' \in K^{\mathrm{alg}} \cap \bigcap \mathcal{B}$. First suppose $a' \in K$. Then one can find $a'' \in K$ such that $\mathrm{rv}(a'' - a') = \mathrm{rv}(a - a')$, by immediacy of M/K. But then the minimal closed ball containing a and a'' is K-definable and fails to contain a', contradicting the choice of a'.

So $K \cap \bigcap \mathcal{B} = \emptyset$. Now take $a' \in K^{\mathrm{alg}} \cap \bigcap \mathcal{B}$ minimizing $[K(a') : K] =: d$. We claim $K(a')/K$ is an immediate extension. Indeed, any element of $K(a')$ can be written as $P(a')$, for some $P(X) \in K[X]$ of degree $d' < d$. Over K^{alg}, one factors $P(X)$ as

$$P(X) = \alpha(x - \beta_1)(x - \beta_2) \cdots (x - \beta_{d'}),$$

with $\alpha \in K$, $\beta_i \in K^{\mathrm{alg}}$. Then by choice of a', each of the β_i fails to be in $\bigcap \mathcal{B}$. This ensures that

$$\mathrm{rv}(a' - \beta_i) = \mathrm{rv}(a - \beta_i),$$

and so

$$\mathrm{rv}(P(a')) = \mathrm{rv}(\alpha) \cdot \prod_{i=1}^{d'} \mathrm{rv}(a' - \beta_i) = \mathrm{rv}(\alpha) \cdot \prod_{i=1}^{d'} \mathrm{rv}(a - \beta_i) = \mathrm{rv}(P(a)).$$

By immediacy of M/K, the right hand side is in $\mathrm{RV}(K)$. We have shown $K(a')/K$ is an immediate extension. This contradicts defectlessness of K. ■

By saturation of N, we can find some $a' \in N$ living in this intersection $\bigcap \mathcal{B}$. Let f' be the map $K(a) \to f(K)(a')$ extending f and sending a to a'. By C-minimality, quantifier elimination in ACVF, and Claim 6.16, there is a unique valuation on $K(t)$ making t live in each of the balls in $\mathcal{B}$. Consequently, f' preserves the valuation structure. Also, $vf' = vf$ because $vM = vK(a) = vK$. So f' is a strictly bigger v-elementary map, contradicting maximality. ■

Theorem 6.17. *Let M and N be models of T. Let f be a partial v-elementary map from M to N. Then f is a partial elementary map. In other words, if K is a common subfield of M and N, and if vM and vN are elementarily equivalent over vK, then M and N are elementarily equivalent over K. Consequently, T has quantifier elimination relative to the value group.*

Proof. If M and N are models of T, Zorn's lemma applies to partial v-elementary maps between M and N. So the previous lemma yields

Claim 6.18. *Let M and N be models of T, and N be $|M|^+$-saturated. Then every partial v-elementary map from M to N can be extended to a total v-elementary map from M to N.*

Now suppose M, N, and K are as in the statement of the theorem. Build a sequence $N_1, M_2, N_3, M_4, \ldots$ where

$$M \preceq M_2 \preceq M_4 \preceq \cdots, \qquad N \preceq N_1 \preceq N_3 \preceq \cdots,$$

and M_{i+1} is $|N_i|^+$-saturated and N_{i+1} is $|M_i|^+$-saturated.

By repeatedly applying the lemma, we can find partial v-elementary maps

$$M \to N_1 \to M_2 \to N_3 \to M_4 \to \cdots$$

extending the given embedding of K into N. These combine to yield an isomorphism between $\bigcup_i M_i$ and $\bigcup_i N_i$ over K. Then by Tarski–Vaught,

$$M \equiv_K \bigcup_i M_i \cong_K \bigcup_i N_i \equiv_K N.$$ ■

Recall the notation Int_p for the maximal p-divisible convex subgroup of an ordered abelian group (Definition 2.8).

Corollary 6.19. *Let Γ be an ordered abelian group. If $\Gamma = p \cdot \Gamma$, the theory of henselian defectless valued fields (K, v) of characteristic p with $Kv \models \mathrm{ACF}_p$ and $vK \equiv \Gamma$ is complete. If $a \in \mathrm{Int}_p \Gamma$, the theory of henselian defectless mixed characteristic fields with $(vK, v(p)) \equiv (\Gamma, a)$ and $Kv \models \mathrm{ACF}_p$ is complete.*

Proof. For pure characteristic p, let M_1 and M_2 be two models. Let K be $\mathbb{F}_p$. Then vM_1 and vM_2 are elementarily equivalent over vK, so M_1 and M_2 are elementarily equivalent.

For mixed characteristic, take K to be $\mathbb{Q}$ instead. If $(vM_1, v(p)) \equiv (vM_2, v(p))$, then vM_1 and vM_2 are elementarily equivalent over $v\mathbb{Q} = \mathbb{Z} \cdot v(p)$. ∎

6.2. *Dp-minimality*

Fact 6.20 ([17, proof of Proposition 5.1]). *Let $(\Gamma, \leq, +)$ be an ordered abelian group such that $\Gamma/n\Gamma$ is finite for all $n > 0$. Let M be the expansion of $(\Gamma, \leq, +, -)$ by the following:*

- *Constants naming a countable submodel of Γ.*
- *A unary predicate naming each coset of $n\Gamma$ in Γ.*
- *For each prime number p and $a \in \Gamma \setminus p\Gamma$, a unary predicate for the largest convex subgroup $H_{a,p} \leq \Gamma$ such that $a \notin p\Gamma + H_{a,p}$.*

Then M is a definitional expansion of $(\Gamma, \leq, +)$, and M has quantifier elimination.

In [17], this was proven in a context where Γ is $\aleph_1$-saturated. However, the general case follows by passing to an elementary extension.

Lemma 6.21. *Let Γ be an ordered abelian group such that $\Gamma/n\Gamma$ is finite for all $n > 0$. Then every definable subset of Γ is a boolean combination of cosets $\gamma + n\Gamma$ and definable cuts (upward-closed sets).*

Proof. This follows from Fact 6.20. If U is a coset of $n\Gamma$, then any atomic relation $t(x) \in U$ is equivalent to a union of cosets of $n\Gamma$, because the class of x in $\Gamma/n\Gamma$ determines the class of $t(x)$ in $\Gamma/n\Gamma$. Any atomic relation of the form $t_1(x) \leq t_2(x)$ or $t(x) \in H_{a,p}$ defines a convex set. Any definable convex set is a boolean combination of definable cuts. ∎

Let (K, v) be a valued field. We will call sets of the following forms *round sets* with center c:

$$c + a \cdot (K^\times)^n \quad \text{for some } a \in K^\times,$$
$$c + v^{-1}(\Xi) \quad \text{for some definable upward-closed subset } \Xi \subseteq vK \cup \{+\infty\}.$$

We will call sets of the first kind *angular sets* and sets of the second kind *ball-like sets*. The class of round sets is closed under affine transformations.

Proposition 6.22. *Let K be a henselian defectless roughly p-divisible field such that $Kv \models \mathrm{ACF}_p$ and $vK/n \cdot vK$ is finite for all $n \in \mathbb{N}$. Then every unary definable set in K is a finite boolean combination of round sets.*

Proof. We may replace K with an elementary extension. First pass to an extension in which every coset of $\bigcap_n n \cdot vK$ is represented. Then pass to a spherical completion (which is an elementary extension by quantifier elimination).

Now look at 1-types. It suffices to show that a 1-type is determined by which round sets contain it. Let a be a singleton from an elementary extension of K. By spherical completeness, some element of K is maximally close to a. Translating a, we may assume that element is 0. If $a = 0$, then the 1-type is determined by the assertion that $v(x) = \infty$. Otherwise, $a \notin K$, and $\mathrm{rv}(a)$ is new (not in $\mathrm{RV}(K)$). If $v(a)$ is new, then $\mathrm{tp}(v(a)/vK)$ implies $\mathrm{tp}(a/K)$. Indeed, if

$$v(a) \equiv_{vK} v(a'),$$

then

- By C-minimality and quantifier elimination in ACVF, there is an isomorphism of valued fields $K(a) \cong_K K(a')$ sending a to a'.
- The induced map on value groups is a partial elementary map, because $v(a) \equiv_{vK} v(a')$.
- By Theorem 6.17, the map $K(a) \to K(a')$ is a partial elementary map, and so $a \equiv_K a'$.

But by Lemma 6.21, $\mathrm{tp}(v(a)/vK)$ is implied by a collection of statements of the following forms:

- $v(a) + \gamma$ is divisible by n.
- $v(a)$ is greater than some cut.
- $v(a)$ is less than some cut.

Each of these is a round set or the complement of a round set, by Proposition 6.4 (2).

If, on the other hand, $v(a)$ is old (in vK), then we may rescale a so that $v(a) = 0$. Then $\mathrm{res}(a)$ is new (not in $\mathrm{res}(K) = Kv$), and $\mathrm{tp}(a/K)$ is the generic type of the closed unit ball, which is unique by a similar quantifier elimination argument. ∎

Definition 6.23. Fix a complete theory T. Let $\mathcal{B}$ be an ind-definable family of unary definable sets. In other words, there is a collection of formulas Φ and for any model K,

$$\mathcal{B}(K) = \{\phi(K; \vec{a}) : \phi(x; \vec{y}) \in \Phi,\ \vec{a} \in K^{|\vec{y}|}\}.$$

Say that $\mathcal{B}$ is a *unary basis* if it generates the family of all unary definable sets through boolean combinations. More precisely, if $K \models T$ and $D \subseteq K$ is K-definable, then D is in the boolean algebra generated by $\mathcal{B}(K)$.

Say that $\mathcal{B}$ is a *weak unary basis* if every unary definable set is a boolean combination of traces of externally definable sets in $\mathcal{B}$. In other words, if $K \models T$, then

$$\{K \cap D' : D' \in \mathcal{B}(K'),\ K' \succeq K\}$$

generates a boolean algebra containing all definable subsets of K.

In the setting of Proposition 6.22, round sets form a unary basis. Moreover, balls and angular sets form a weak unary basis, because every ball-like round set is the trace of an externally definable ball.

Lemma 6.24. *Let T be a complete theory with infinite models. Let $\mathcal{B}$ be a weak unary basis for T. Then T is not dp-minimal if and only if in some model of T, there are mutually indiscernible sequences*

$$\dots, X_{-1}, X_0, X_1, \dots, \qquad \dots, Y_{-1}, Y_0, Y_1, \dots$$

of sets from $\mathcal{B}$, and an element a such that

$$a \in X_0 \not\Leftrightarrow a \in X_1, \qquad a \in Y_0 \not\Leftrightarrow a \in Y_1.$$

Proof. If the given configuration occurs, it directly contradicts the characterization of dp-minimality in terms of mutually indiscernible sequences (one of the two sequences of sets must be a-indiscernible).

Conversely, suppose dp-minimality fails. Let $\mathcal{B}^+$ be the closure of $\mathcal{B}$ under boolean combinations. Let $\kappa = |T|^+$.

Claim 6.25. *There is an ict-pattern made of sets from $\mathcal{B}^+$, with two rows* (*depth* 2) *and κ-many columns.*

Proof. Take a mutually indiscernible ict-pattern of depth 2 and stretch the two sequences of sets to have length κ. So we have sets X_α and Y_β and elements $a_{\alpha,\beta}$ for $\alpha, \beta < \kappa$ such that

$$a_{\alpha,\beta} \in X_{\alpha'} \iff \alpha = \alpha', \qquad a_{\alpha,\beta} \in Y_{\beta'} \iff \beta = \beta'.$$

Let M be a small model defining the X's and Y's and containing the a's. In some $|M|^+$-saturated elementary extension $M^* \succeq M$, we can find sets X'_α and Y'_α from $\mathcal{B}^+(M^*)$ such that $X_\alpha \cap M = X'_\alpha \cap M$ and $Y_\alpha \cap M = Y'_\alpha \cap M$. As the $a_{\alpha,\beta}$ are in M,

$$a_{\alpha,\beta} \in X'_{\alpha'} \iff \alpha = \alpha', \qquad a_{\alpha,\beta} \in Y'_{\beta'} \iff \beta = \beta'.$$

Because $\kappa > |T|$, some length-κ subsequence of $\langle X'_\alpha \rangle_{\alpha<\kappa}$ is uniformly definable. Passing to this subsequence, and doing the same with $\langle Y'_\beta \rangle_{\beta<\kappa}$, we get an ict-pattern of depth 2 in M^*. ∎

Write the resulting pattern as

$$\langle f_i(B_i^1, B_i^2, \dots, B_i^{m_i}) \rangle_{i \in I}, \qquad \langle g_i(C_i^1, B_i^2, \dots, B_i^{\ell_i}) \rangle_{i \in I},$$

where the sets B and C are from $\mathcal{B}$ and f_i, g_i are boolean combinations. As $|I| > \aleph_0$, we can pass to a subsequence and arrange for f_i, m_i, g_i, ℓ_i to not depend on i. Then we can extract a mutually indiscernible array, with columns indexed by $\mathbb{Z}$. So we obtain an ict-pattern of the form

$$\dots, U_{-1}, U_0, U_1, \dots, \qquad \dots, V_{-1}, V_0, V_1, \dots,$$

where

$$U_i = f(B_i^1, \ldots, B_i^m), \qquad V_j = g(C_i^1, \ldots, C_i^\ell),$$

and the two sequences

$$\langle \ulcorner U_i \urcorner \ulcorner B_i^1 \urcorner \cdots \ulcorner B_i^m \urcorner \rangle_{i \in \mathbb{Z}}, \qquad \langle \ulcorner V_i \urcorner \ulcorner C_i^1 \urcorner \cdots \ulcorner C_i^\ell \urcorner \rangle_{i \in \mathbb{Z}}$$

are mutually indiscernible.

Because this is an ict-pattern, we can take an element $a = a_{00}$ such that for all $i \in \mathbb{Z}$,

$$a \in U_i \iff a \in V_i \iff i = 0.$$

As $a \in U_0$ and $a \notin U_1$, there must be some j such that $a \in B_0^j \not\Leftrightarrow a \in B_1^j$. Likewise, there must be some k such that $a \in C_0^k \not\Leftrightarrow a \in C_1^k$. Take $X_i = B_i^j$ and $Y_i = B_i^k$. Then the X_i's and Y_i's are mutually indiscernible, are sets in $\mathcal{B}$, and satisfy

$$a \in X_0 \not\Longleftrightarrow a \in X_1, \qquad a \in Y_0 \not\Longleftrightarrow a \in Y_1,$$

completing the proof of the lemma. ∎

Theorem 6.26. *Let (K, v) be a henselian defectless roughly p-divisible valued field with $vK/n \cdot vK$ finite for all $n \in \mathbb{N}$, and $Kv \models \mathrm{ACF}_p$. Then (K, v) is dp-minimal as a valued field.*

Proof. We may take (K, v) to be a monster model.

If dp-minimality failed, then by Lemma 6.24 there would exist an element a and two mutually indiscernible sequences of sets

$$\ldots, X_{-1}, X_0, X_1, \ldots, \qquad \ldots, Y_{-1}, Y_0, Y_1, \ldots$$

such that

$$a \in X_0 \not\Longleftrightarrow a \in X_1, \qquad a \in Y_0 \not\Longleftrightarrow a \in Y_1$$

and each X_i is a ball or an angular set. (Here we are using the fact that balls and angular sets form a weak unary basis, by Proposition 6.22.)

As v is henselian, there is a unique extension of v to K^{alg}. Consider the map χ from balls and angular sets in K to subsets of K^{alg} defined as follows:

- $\chi(B)$ is the ball in K^{alg} with the same center and radius as B if B is a ball in K.
- $\chi(B) = \{c\}$ if B is an angular set centered on c.

ACF and ACVF are dp-minimal, so (K^{alg}, v) is dp-minimal and one of the two sequences

$$\ldots, \chi(X_{-1}), \chi(X_0), \chi(X_1), \ldots, \qquad \ldots, \chi(Y_{-1}), \chi(Y_0), \chi(Y_1), \ldots$$

is a-indiscernible within (K^{alg}, v). Without loss of generality, $\langle \chi(X_i) \rangle_{i \in \mathbb{Z}}$ is a-indiscernible in K^{alg}.

If the X_i's are balls, then

$$a \in X_0 \iff a \in \chi(X_0) \iff a \in \chi(X_1) \iff a \in X_1,$$

a contradiction. So the X_i's are angular sets.

Write ξ_i for the center of X_i. Then $X_i - \xi_i$ is a coset of $(K^\times)^n$. As there are only finitely many of these cosets, the indiscernible sequence $\langle X_i - \xi_i \rangle_{i\in\mathbb{Z}}$ must be constant. So $X_i - \xi_i$ is some fixed coset of $(K^\times)^n$. By Proposition 6.4 (2), $X_i - \xi_i = v^{-1}(S)$ for some set $S \subseteq vK$. Therefore, whether $a \in X_i$ depends solely on $v(a - \xi_i)$. Consequently,

$$v(a - \xi_0) \neq v(a - \xi_1).$$

Now in K^{alg}, the sequence $\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots$ is a-indiscernible. So, perhaps after reversing the sequence, we have

$$\cdots < v(a - \xi_{-1}) < v(a - \xi_0) < v(a - \xi_1) < \cdots .$$

This in turn implies that

$$v(a - \xi_1) = v(\xi_2 - \xi_1) \quad \text{and} \quad v(a - \xi_0) = v(\xi_2 - \xi_0).$$

Whether an element x is in X_i depends only on $v(x - \xi_i)$, so

$$\xi_2 \in X_1 \iff a \in X_1 \not\Longleftrightarrow a \in X_0 \iff \xi_2 \in X_0.$$

But X_0 and X_1 have the same type over ξ_2 (the unique center of X_2), because the sequence $\langle X_i \rangle_{i\in\mathbb{Z}}$ is indiscernible *in K itself*. So we have a contradiction. ■

7. The classification of dp-minimal fields

We can now prove Theorem 1.3. By [4, Corollary 8], we have

Fact 7.1. *A henselian valued field (K, v) with residue characteristic 0 is dp-minimal if and only if vK and Kv are dp-minimal.*

Theorem 1.31 asserts that certain theories are complete and dp-minimal. Except in the case of positive residue characteristic, completeness follows by the Ax–Kochen–Ershov principle, and dp-minimality follows by Fact 7.1, using the dp-minimality of characteristic 0 local fields, plus the following characterization of dp-minimal ordered abelian groups:

Fact 7.2 ([17, Proposition 5.1]). *An ordered abelian group $(\Gamma, \leq)$ is dp-minimal if and only if $|\Gamma/n\Gamma|$ is finite for all $n \geq 1$.*

Characteristic 0 local fields are dp-minimal by [1, Corollary 7.8] in the non-archimedean case, and by VC-minimality in the case of $\mathbb{C}$ and $\mathbb{R}$.

For the remaining case of positive residue characteristic, Corollary 6.19 provides completeness, and Theorem 6.26 establishes dp-minimality. Finally, Theorem 1.3 (2) follows from a more general fact:

Theorem 7.3. *Let K be a sufficiently saturated dp-minimal field. Then there is a henselian defectless valuation v on K such that*

- *The residue field Kv is algebraically closed, real closed, or a local field of characteristic* 0*.*
- *The value group vK satisfies $|vK/n \cdot vK| < \aleph_0$ for all $n > 0$.*
- *If Kv has characteristic p, then vK is p-divisible.*
- *If v has mixed characteristic, then $v(p) \in \mathrm{Int}_p\, vK$.*

Proof. First we note that if v is *any* valuation on K, then $vK/n \cdot vK$ is finite for all n, because $K^\times/(K^\times)^n$ is finite by [18, Theorem 1.5]. For the other points, we break into cases.

Let v_∞ be the valuation from Theorem 4.7. First suppose that Kv_∞ is finite. Then v_∞ is definable. By Theorem 2.11, v_∞ has mixed characteristic and the interval $[-v_\infty(p), v_\infty(p)]$ is finite. Let Δ be the smallest convex subgroup of $v_\infty K$ containing $v_\infty(p)$. Then $\Delta \cong \mathbb{Z}$.

Let v be the coarsening of v_∞ by Δ. We get a decomposition of the henselian defectless place $K \to Kv_\infty$ as a composition

$$K \xrightarrow{v_\infty K/\Delta} Kv \xrightarrow{\Delta} Kv_\infty.$$

Because K is saturated and v_∞ is definable, countable chains of balls in (K, v_∞) have non-empty intersection, meaning that $K \to Kv_\infty$ satisfies the countable intersection property of Remark 2.10. Thus $Kv \to Kv_\infty$ also satisfies this condition. But the value group Δ of $Kv \to Kv_\infty$ is isomorphic to $\mathbb{Z}$, so the valuation $Kv \to \Delta$ is a mixed characteristic complete discrete valuation with a finite residue field. Therefore Kv is a characteristic 0 local field. So v is a henselian (and defectless) valuation on K, and its residue field Kv is local of characteristic 0. There is nothing else to show in this case, because v is equicharacteristic 0.

Otherwise, Kv_∞ is real closed or algebraically closed. In this case, we take $v = v_\infty$.

It remains to show that v_∞ is roughly p-divisible (see Definition 2.9) if Kv_∞ has characteristic p. By Remark 4.6 and the construction of v_∞, there is a definable valuation v_1, coarser than v_∞, such that Kv_1 has characteristic p. The place $K \to Kv_\infty$ decomposes as

$$K \to Kv_1 \to Kv_\infty,$$

where $K \to Kv_1$ is roughly p-divisible by Theorem 2.11, and $Kv_1 \to Kv_\infty$ is roughly p-divisible by Lemma 2.5. So the composition is roughly p-divisible by Remark 2.10. ■

8. Dp-minimal valued fields

The above results easily yield a sharp characterization of dp-minimal valued fields, which we give in the next two theorems:

Theorem 8.1. *Let (K, v) be a valued field with infinite residue field. Then (K, v) is dp-minimal* (*as a pure valued field*) *if and only if the following conditions all hold:*

(1) *The residue field Kv and value group vK are dp-minimal.*

(2) *The valuation v is henselian and defectless.*

(3) *In mixed characteristic, every element of $[-v(p), v(p)]$ is divisible by p.*

(4) *In pure characteristic p, the value group vK is p-divisible.*

Proof. First suppose (K, v) is dp-minimal. Both vK and Kv are dp-minimal because they are images of the dp-minimal field K. Theorem 3.15 (2) yields henselianity. Theorem 2.11 yields the divisibility conditions.

Conversely, suppose (K, v) satisfies conditions (1)–(4). These conditions are first-order, so we may assume (K, v) is sufficiently saturated. As Kv is a dp-minimal field, there is a place $Kv \to k$ which is henselian, defectless, roughly p-divisible, and with k algebraically closed or elementarily equivalent to a local field of characteristic 0. By Remark 2.10, the composition $K \to Kv \to k$ is also henselian, defectless, and roughly p-divisible.

Recall that an ordered abelian group Γ is dp-minimal if and only if $\Gamma/n \cdot \Gamma$ is finite for all $n > 0$ (Fact 7.2). If Γ is an ordered abelian group, and Δ is a convex subgroup, then Γ is dp-minimal if and only if Δ and Γ/Δ are.

Therefore, the value group of $K \to Kv \to k$ is dp-minimal because the value groups of $K \to Kv$ and $Kv \to k$ are.

In summary, the composite place $K \to k$ is henselian, defectless, and roughly p-divisible, its residue field is local of characteristic 0, or algebraically closed, and its value group Γ has the property that $\Gamma/n\Gamma$ is finite for all n. By Theorem 1.3 (1), the valued field $K \to k$ is dp-minimal. The original valued field $K \to Kv$ is a coarsening of $K \to k$, so it is definable in the Shelah expansion of $K \to Kv$ (the expansion by all externally definable sets). The Shelah expansion is still dp-minimal. Thus $K \to Kv$ is dp-minimal. ∎

Theorem 8.2. *Let (K, v) be a valued field with finite residue field. Then (K, v) is dp-minimal (as a valued field) if and only if the following conditions all hold:*

(1) *The value group vK is dp-minimal.*

(2) *The valuation v is henselian.*

(3) *The valuation is finitely ramified, in the sense that $[-v(p), v(p)]$ is finite. (In particular, K has characteristic 0 if v is non-trivial.)*

Proof. First suppose (K, v) is dp-minimal. Then henselianity follows by Theorem 3.15 (2), and dp-minimality of vK is immediate. Finite ramification follows by Theorem 2.11.

Conversely, suppose (K, v) satisfies conditions (1)–(3). These conditions are elementary, so we may assume K is saturated. If v is trivial, then K is finite, so it is dp-minimal. Otherwise, K has characteristic 0. Let w be the coarsening of v by the convex subgroup Δ generated by $v(p)$. As usual we get a decomposition $K \to Kw \to Kv$. The value group of $Kw \to Kv$ is Δ, which has rank 1 by finite ramification. By saturation of $K \to Kv$, the countable chain condition of Remark 2.10 holds in $K \to Kv$, hence in $Kw \to Kv$.

Thus $Kw \to Kv$ is spherically complete. Also, Δ is isomorphic to $\mathbb{Z}$. Thus $Kw \to Kv$ makes Kw into a complete mixed characteristic DVR with finite residue field. So Kw is a local field of characteristic 0.

Now $K \to Kw$ is henselian (because $K \to Kv$ is). As $K \to Kv$ has dp-minimal value group, so does $K \to Kw$ and $Kw \to Kv$. In particular, $K \to Kw$ makes K into a henselian valued field with dp-minimal value group and residue field local of characteristic 0. So $K \to Kw$ is dp-minimal by Theorem 1.3 (1a).

In characteristic 0 non-archimedean local fields, the valuation ring is always definable from the pure field language. Consequently, the dp-minimal structure $K \to Kw$ interprets $K \to Kw \to Kv$. Thus $K \to Kv$ is also dp-minimal. ∎

Question 8.3. *Do the above theorems remain true when* (K, v) *is expanded by additional structure on* vK *and* Kv (*preserving the dp-minimality of each*)*?*

9. Summary and future directions

We now know exactly which *pure* fields are dp-minimal, and we know a little bit about dp-minimal expansions of fields.

Here is a summary of what can be said about a dp-minimal field $(K, +, \cdot, \ldots)$, perhaps with other structure. Either K is strongly minimal (or finite), or all of the following facts are true:

- There is a definable V-topology on K [18, Theorem 1.3].
- With respect to this topology, there are only boundedly many infinitesimal types, which are all definable [18, Corollaries 7.5, 7.6].
- Any unary definable set has finite boundary [18, Theorem 1.3].
- Dp-rank of definable[9] sets is definable in families, and agrees with "geometric dimension" [18, Corollary 8.4].
- Any definable valuation ring is henselian and defectless, and any two definable valuation rings are comparable (Theorems 3.15 (2) and 2.11).
- For each n, $K^\times/(K^\times)^n$ is finite [18, Theorem 1.5].
- $(K^\times)^{000} = (K^\times)^0 = \bigcap_n (K^\times)^n$ (Corollary 5.13).
- Any finite extension of K is dp-minimal as a pure field (but not as an expansion of K, of course). This follows by inspecting the list of dp-minimal fields.
- There is at least one definable non-trivial valuation on K, unless K is finite, real closed, or algebraically closed (Theorem 4.7).

There are several obvious questions we have not addressed:

[9] One cannot hope to extend this to interpretable sets. For instance, $\exists^\infty$ is not eliminated in the value group of $\mathbb{Q}_p$.

Question 9.1. *If K is a dp-minimal field, is there always a definable valuation on K whose residue field is algebraically closed, real closed, or finite?*

Question 9.2. *Which unstable dp-minimal fields fail to define valuations?*

Question 9.3. *If $(K, +, \cdot, \dots)$ is a sufficiently saturated unstable dp-minimal field, not necessarily pure, and $\mathcal{O}_\infty$ is the valuation ring from Theorem 4.7, is the expansion of K by $\mathcal{O}_\infty$ still dp-minimal?*

Question 9.4. *Can any of the classifications be extended to fields of finite dp-rank?*

9.1. Defining the canonical valuation, or not

In general, the answer to Question 9.1 is *no*, though we can characterize the failure modes.

Proposition 9.5. *Let (K, v) be a sufficiently saturated valued field as in Theorem 1.3 (1). So (K, v) is dp-minimal, and Kv is elementarily equivalent to $\mathbb{F}_p^{\mathrm{alg}}$ or a characteristic 0 local field.*

Let $\mathcal{O}_\infty$ be the intersection of all valuation rings on K definable in the pure field language. Let w be the associated valuation.

- *If Kv is non-archimedean, then w is the composition of v with the canonical valuation on Kv.*
- *If Kv is real closed or algebraically closed, then w is the coarsening of v by the maximal convex divisible subgroup of vK.*

Proof. First we make a general observation.

Remark 9.6. Let $K \to k$ be a place. It cannot be the case that one of K or k is finite and the other is real closed or algebraically closed. Indeed, if K is finite then k is (obviously) finite. If K is algebraically closed or real closed, then k is algebraically closed or real closed (not respectively), by the Artin–Schreier theorem.

First suppose Kv is non-archimedean. Non-archimedean local fields define their valuation rings, so (K, v) interprets the canonical valuation on Kv. Let $K \to Kv \to Kv'$ be the composition. Then v' is a valuation on K, definable in (K, v), with finite residue field Kv'. We claim $w = v'$.

By Proposition 3.12 applied to the dp-minimal structure (K, v), the valuations v' and w must be comparable. So we either have a place map $Kw \to Kv'$ or $Kv' \to Kw$. By Theorem 4.7, Kw is finite, real closed, or algebraically closed. By Remark 9.6, Kw cannot be algebraically closed or real closed, so it is finite. Then $Kw \to Kv'$ or $Kv' \to Kw$ is trivial, and $w = v'$.

Next suppose Kv is real closed or algebraically closed. Let v' be the coarsening of v by the maximal divisible convex subgroup of vK. In the sequence

$$K \to Kv' \to Kv,$$

the value group of $Kv' \to Kv$ is divisible, and the value group of $K \to Kv'$ has no convex divisible subgroups. Also, $K \to Kv'$ is henselian, defectless, and roughly p-divisible. Because Kv' has a henselian defectless valuation with divisible value group and real or algebraically closed residue field, Kv' is itself real closed or algebraically closed.

So (K, v') is a model of one of the theories from Theorem 1.3 (1), though (K, v') need not be saturated. In the structure (K, v'), the only definable valuation on the residue field Kv' is the trivial one.[10]

By Proposition 3.12, w and v' must be comparable. If v' were strictly coarser than w, there would be a non-trivial valuation $Kv' \to Kw'$, definable in the structure (K, v').

So w is coarser than v', which is in turn coarser than v. Let Δ_w and $\Delta_{v'}$ be the convex subgroups of vK whose coarsenings yield w and v'. Then $\Delta_w \geq \Delta_{v'}$. We want to show $w = v'$, i.e., that $\Delta_w = \Delta_{v'}$. Otherwise, $\Delta_w > \Delta_{v'}$. As $\Delta_{v'}$ is the greatest convex divisible subgroup, Δ_w is *not* divisible. Then neither is $\Delta_w / \Delta_{v'}$. So the place $Kw \to Kv'$ has a value group that is not divisible.

Now Kv' is real closed or algebraically closed, so Kw is not finite by Remark 9.6. Therefore Kw is real closed or algebraically closed. But then any valuation on Kw has divisible value group. This contradicts the non-divisibility of the value group of $Kw \to Kv'$. ■

Theorem 9.7. *Let (K, v) be a dp-minimal valued field with residue field Kv algebraically closed or elementarily equivalent to a local field of characteristic* 0*. Suppose K is sufficiently saturated. The following are equivalent:*

- *There is a valuation w, definable in the pure field language, such that Kw is finite, real closed, or algebraically closed.*
- *Kv is non-archimedean or the maximal convex divisible subgroup of vK is definable in the structure $(vK, +, \leq)$.*

Proof. First suppose that Kv is non-archimedean. By Proposition 9.5, the canonical valuation on K (in the pure field language) has finite residue field (and so is definable by Theorem 4.7). So there is a valuation ring on K, definable in the pure field language, with finite residue field.

[10] One can show that in (K, v'), the induced structure on Kv' is the pure field structure. Pure models of RCF and ACF do not admit non-trivial valuations. But here is an alternative proof that Kv' fails to have a definable non-trivial valuation, relying on known results. By the completeness part of Theorem 1.3 (1), one can find a structure elementarily equivalent to (K, v') in which the residue field is either $\mathbb{F}_p^{alg}$, $\mathbb{Q}^{alg}$, or $\mathbb{Q}^{alg} \cap \mathbb{R}$, by explicitly constructing a model using Hahn series (or spherical completions in mixed characteristic). The field $\mathbb{F}_p^{alg}$ admits no non-trivial valuation. The non-trivial valuations on $\mathbb{Q}^{alg} \cap \mathbb{R}$ fail to be NIP, by [21, Theorems 11.3.1, 11.5.1]. Finally, in the case of $\mathbb{Q}^{alg}$, if some non-trivial valuation ring $\mathcal{O}$ on $\mathbb{Q}^{alg}$ happens to be definable, then using the Hahn series model one can show that $\sigma(\mathcal{O})$ is definable for any $\sigma \in \mathrm{Aut}(\mathbb{Q}^{alg})$. The p-adic valuation on $\mathbb{Q}$ is not henselian for any p, so one can find σ with $\sigma(\mathcal{O}) \neq \mathcal{O}$. Then by [21, Theorems 11.3.1, 11.5.1] again, the residue field fails to be NIP.

Next suppose that Kv is algebraically closed or real closed. If the maximal divisible convex subgroup of vK is definable, let v' be the coarsening, which is definable in (K, v). By the proposition, v' is the canonical valuation of Theorem 4.7 on the pure field K. Therefore the valuation ring of v' is type-definable in the pure field K, and definable in (K, v), hence definable in the pure field K, by saturation of (K, v). So v' is definable in the pure field language, and Kv' is finite, algebraically closed, or real closed, by Theorem 4.7.

Conversely, suppose that Kw is algebraically closed, real closed, or finite, for some w definable in the pure field language.

Now w is coarser than the canonical valuation on the pure field K, which is coarser than v by the proposition (applied to (K, v)). So there is a place $Kw \to Kv$. By Remark 9.6, Kw is algebraically closed or real closed.

Then we can apply the proposition to (K, w), seeing that the canonical valuation on K is a coarsening of w. So w and the canonical valuation on K are coarser than each other, hence equal.

Now by the proposition applied to (K, v), w is the coarsening of v by the maximal convex divisible subgroup of vK. This group must then be definable in (K, v), because w is definable. One can show[11] that (K, v) induces the pure ordered group structure on vK (with $v(p)$ named as a constant in mixed characteristic), and so this group is definable in vK. ■

Now let Γ be the lexicographic product

$$\mathbb{Z} \times \mathbb{Z}[1/2] \times \mathbb{Z}[1/2, 1/3] \times \mathbb{Z}[1/2, 1/3, 1/5] \times \cdots .$$

For each number n, all but finitely many of the factors are divisible by n, and in fact $\Gamma/n\Gamma$ is finite for all n. So $\mathbb{C}(\!(t^\Gamma)\!)$ is dp-minimal. But in a sufficiently saturated elementary extension of Γ, the maximal divisible convex subgroup of Γ is not definable. In fact, it is

[11] If Kv has characteristic 0, this follows by Ax–Kochen–Ershov quantifier elimination. Otherwise, (K, v) is a model of the theory T of §6.1. Assume (K, v) is equicharacteristic p for simplicity. By a compactness argument, it suffices to prove the following: if $\vec{\gamma}$ and $\vec{\delta}$ are two n-tuples in vK, and $\vec{\gamma}$ and $\vec{\delta}$ have the same type over $\emptyset$ in the pure structure $(vK, +, \leq)$, then $\vec{\gamma}$ and $\vec{\delta}$ have the same type over $\emptyset$ in (K, v). Dropping redundant elements from the tuple, we may assume that $\vec{\gamma}$ is $\mathbb{Q}$-linearly independent in vK. Then the same holds for $\vec{\delta}$. Let $\vec{s}$ and $\vec{t}$ be tuples in K lifting $\vec{\gamma}$ and $\vec{\delta}$, so $v(s_i) = \gamma_i$ and $v(t_i) = \delta_i$. From the $\mathbb{Q}$-linear independence, one can show that the $\vec{s}$ are algebraically independent over $\mathbb{F}_p$. The same holds for $\vec{t}$. Therefore there is an isomorphism of fields $g : \mathbb{F}_p(\vec{s}) \to \mathbb{F}_p(\vec{t})$ mapping $\vec{s}$ to $\vec{t}$. There is an automorphism σ of the pure structure $(vK, +, \leq)$ sending $\vec{\gamma}$ to $\vec{\delta}$. Using $\mathbb{Q}$-linear independence, one sees that $v(g(x)) = \sigma(v(x))$ for $x \in \mathbb{F}_p(\vec{s})$. (For example, it suffices to check the case where $x = P(\vec{s}) \in \mathbb{F}_p[\vec{s}]$, when $v(x)$ and $v(g(x))$ are determined explicitly by the coefficients of the polynomial P.) It follows that g is an isomorphism of valued fields, and even a partial v-elementary map from (K, v) to (K, v). By Theorem 6.17, g extends to an automorphism σ' of (K, v). Then σ' induces an automorphism of $(vK, +, \leq)$, which must send $\vec{\gamma}$ to $\vec{\delta}$. Therefore $\vec{\gamma}$ and $\vec{\delta}$ have the same type over $\emptyset$ in (K, v). In mixed characteristic, the proof is similar, except that $\mathbb{Q}$ takes the place of $\mathbb{F}_p$, and one arranges for $(v(p), \gamma_1, \ldots, \gamma_n)$ to be $\mathbb{Q}$-linearly independent rather than $\vec{\gamma}$ alone.

the intersection of the strictly decreasing sequence of definable subgroups:

$$\mathrm{Int}_2 \Gamma > \mathrm{Int}_3 \Gamma > \cdots .$$

So this gives a dp-minimal field in which no definable valuation ring has a residue field that is finite, algebraically closed, or real closed.

9.2. *Unstable dp-minimal fields that define no valuations*

We know that every unstable dp-minimal field K has a V-topology. This topology need not come from a definable valuation, as exhibited by RCF. On the other hand, if K admits no definable valuation rings, then the ring $\mathcal{O}_\infty$ in Theorem 4.7 is trivial, so K must be algebraically closed or real closed. So most dp-minimal fields admit a definable valuation, which determines the canonical topology (by Lemma 3.9).

A natural open question is then:

Question 9.8. *Are there dp-minimal unstable expansions of* ACF *which define no valuation rings?*

If the answer is *no*, the following conjecture is true:

Conjecture 9.9. *Let K be an unstable dp-minimal field. Then the canonical topology on K is induced by a definable ordering or a definable valuation.*

9.3. *Expanding by the canonical valuation*

In many cases, the answer to Question 9.3 is *yes*, because the canonical valuation is definable. (Whether this happens is more or less characterized by Proposition 9.5.) In the case of pure fields, we know that the answer to Question 9.3 is *yes*, at least under saturation assumptions—this is essentially the content of Theorem 7.3. When the residue field of $\mathcal{O}_\infty$ is real closed, Question 9.3 has been answered affirmatively by Jahnke [15, Theorem A].

9.4. *Finite dp-rank fields*

Strongly minimal fields are known to be algebraically closed by a theorem of Macintyre. The proof yields a stronger result: fields of finite Morley rank are algebraically closed.

In contrast, the classification of dp-minimal fields given here does not directly reveal anything about general fields of finite dp-rank. Nevertheless, it is natural to try generalizing the classification to fields of finite dp-rank.

In the years since dp-minimal fields were classified, a conjectural classification of "dp-finite" fields has emerged. By work of Anscombe, Dolich, Farré, Goodrick, Halevi, Hasson, Jahnke, and Sinclair,[12] the crux of the matter is the following conjecture:

[12] In detail, the analogue of §6 is Sinclair's thesis [26], which gives new examples of dp-finite fields, among other things. The analogue of Fact 7.2 are the three papers [6, 8, 11], which classify

Conjecture 9.10 (Shelah conjecture for dp-finite fields). *If K is a dp-finite field, then one of the following holds:*

- *K is finite.*
- *K is algebraically closed.*
- *K is real closed.*
- *K has a definable henselian valuation.*

For *positive characteristic* dp-finite fields, the Shelah conjecture is provable by a generalization of the techniques of [18] and the present paper. This has been carried out in [19, 20], building off work of Sinclair [26]. This yields a full classification of dp-finite fields of positive characteristic.

Several major hurdles arise in the generalization to dp-finite fields.

1. One needs a new definition of "infinitesimals" and the canonical topology. One can form the intersection

$$I_K \stackrel{?}{=} \bigcap \{X - X : X \text{ infinite and } K\text{-definable}\},$$

but it is often trivial. For example, if $\mathbb{M}$ is $\mathbb{C}$ expanded by a predicate for $\mathbb{R}$, then dp-rk$(K) = 2$, and for $X = \mathbb{R}$ and $Y = i\mathbb{R}$ we have

$$(X - X) \cap (Y - Y) = \{0\}.$$

Without adjusting the definition, the canonical topology would become the trivial discrete topology. A more natural choice is

$$I_K \stackrel{?}{=} \bigcap \{X - X : X \text{ is } K\text{-definable and } \text{dp-rk}(X) = \text{dp-rk}(\mathbb{M})\}.$$

This plays off the intuition that sets of full rank should have interior. However, in order to mimic the proofs of [18, §5], one needs to first consider the sets $X -_* X$, where

$$X -_* Y := \{\delta \in \mathbb{M} : \text{dp-rk}(X \cap (Y + \delta)) = \text{dp-rk}(\mathbb{M})\}.$$

Unfortunately, these sets $X -_* Y$ are not obviously definable, creating new problems.

2. After producing a good topology and a type-definable group of infinitesimals I_K, one cannot immediately get a valuation ring from I_K because the comparability Lemma 6.1 in [18] fails to hold in higher rank. Nevertheless, a valuation ring can be obtained by a complicated lattice-theoretic argument.

strongly dependent ordered abelian groups. The analogue of §3 is [13], which proves henselianity of definable valuations, *assuming* Conjecture 9.10. Everything is assembled together in [12], which uses an argument similar to §4 and §7 to control the residue field of the valuation in Conjecture 9.10, proving that the field must be on Sinclair's list. Moreover, the conjectural classification generalizes to strongly dependent fields.

3. One needs a new proof of henselianity. The argument used here relies in an essential way on the fact that definable sets have finite boundary [18, Theorem 7.8], which in turn holds because infinitesimal types are definable [18, Corollary 7.6]. Both facts fail already in $(\mathbb{C}, +, \cdot, \mathbb{R})$. Luckily, a variant of Proposition 3.7 works in positive characteristic dp-finite fields. But in characteristic 0, it remains unclear how to proceed.

Acknowledgments. The author would like to thank Nick Ramsey for informing him of Jahnke, Simon, and Walsberg's work on dp-minimal valued fields [17], the authors of [17] for discussing their work, Tom Scanlon for helping review an earlier version of this paper that was part of the author's dissertation, and Chen Meng for hosting the author at Fudan University when the paper was being prepared. The author would also like to thank an anonymous referee who provided many helpful comments, including a more direct proof of Remark 3.3.

Funding. Parts of this material are based upon work supported by the NSF under Grant No. DGE-1106400 and Award No. DMS-1803120. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

References

[1] Aschenbrenner, M., Dolich, A., Haskell, D., Macpherson, D., Starchenko, S.: Vapnik–Chervonenkis density in some theories without the independence property, I. Trans. Amer. Math. Soc. **368**, 5889–5949 (2016) Zbl 1423.03119 MR 3458402

[2] Becker, E.: Euklidische Körper und euklidische Hüllen von Körpern. J. Reine Angew. Math. **268/269**, 41–52 (1974) Zbl 0289.12103 MR 354625

[3] Bourbaki, N.: Algèbre commutative. Éléments de mathématique. Chapitres 5 à 7. Masson, Paris (1985) Zbl 0547.13002 MR 782297

[4] Chernikov, A., Simon, P.: Henselian valued fields and inp-minimality. J. Symbolic Logic **84**, 1510–1526 (2019) Zbl 1453.03030 MR 4045986

[5] Delon, F.: C-minimal structures without density assumption. In: Motivic Integration and its Interactions with Model Theory and Non-Archimedean Geometry. Vol. II, London Math. Soc. Lecture Note Ser. 384, Cambridge Univ. Press, Cambridge, 51–86 (2011) Zbl 1260.03072 MR 2885341

[6] Dolich, A., Goodrick, J.: A characterization of strongly dependent ordered Abelian groups. Rev. Colombiana Mat. **52**, 139–159 (2018) Zbl 1437.03125 MR 3909316

[7] Engler, A. J., Prestel, A.: Valued Fields. Springer Monogr. Math., Springer, Berlin (2005) Zbl 1128.12009 MR 2183496

[8] Farré, R.: Strong ordered abelian groups and dp-rank. arXiv:1706.05471v1 (2017)

[9] Gismatullin, J.: Model theoretic connected components of groups. Israel J. Math. **184**, 251–274 (2011) Zbl 1257.03060 MR 2823977

[10] Guingona, V.: On VC-minimal fields and dp-smallness. Arch. Math. Logic **53**, 503–517 (2014) Zbl 1354.03047 MR 3237864

[11] Halevi, Y., Hasson, A.: Strongly dependent ordered abelian groups and Henselian fields. Israel J. Math. **232**, 719–758 (2019) Zbl 1454.03043 MR 3990957

[12] Halevi, Y., Hasson, A., Jahnke, F.: A conjectural classification of strongly dependent fields. Bull. Symbolic Logic **25**, 182–195 (2019) Zbl 1477.03134 MR 3984972

[13] Halevi, Y., Hasson, A., Jahnke, F.: Definable V-topologies, Henselianity and NIP. J. Math. Logic **20**, art. 2050008, 33 pp. (2020) Zbl 1472.03035 MR 4128721

[14] Haskell, D., Macpherson, D.: Cell decompositions of C-minimal structures. Ann. Pure Appl. Logic **66**, 113–162 (1994) Zbl 0790.03039 MR 1262433

[15] Jahnke, F.: When does NIP transfer from fields to henselian expansions? arXiv:1607.02953v3 (2019)

[16] Jahnke, F., Koenigsmann, J.: Uniformly defining p-henselian valuations. Ann. Pure Appl. Logic **166**, 741–754 (2015) Zbl 1372.03077 MR 3344577

[17] Jahnke, F., Simon, P., Walsberg, E.: Dp-minimal valued fields. J. Symbolic Logic **82**, 151–165 (2017) Zbl 1385.03040 MR 3631280

[18] Johnson, W.: The canonical topology on dp-minimal fields. J. Math. Logic **18**, art. 1850007, 23 pp. (2018) Zbl 06996241 MR 3878469

[19] Johnson, W.: Dp-finite fields I(A): The infinitesimals. Ann. Pure Appl. Logic **172**, no. 6, art. 102947, 45 pp. (2021) Zbl 1473.03018 MR 4242905

[20] Johnson, W.: Dp-finite fields I(B): Positive characteristic. Ann. Pure Appl. Logic **172**, no. 6, art. 102949, 38 pp. (2021) Zbl 1477.03136 MR 4242906

[21] Johnson, W. A.: Fun with Fields. PhD thesis, Univ. of California, Berkeley, MI (2016) MR 3564042

[22] Kaplan, I., Scanlon, T., Wagner, F. O.: Artin–Schreier extensions in NIP and simple fields. Israel J. Math. **185**, 141–153 (2011) Zbl 1261.03120 MR 2837131

[23] Kuhlmann, F.-V.: Quantifier elimination for Henselian fields relative to additive and multiplicative congruences. Israel J. Math. **85**, 277–306 (1994) Zbl 0809.03028 MR 1264348

[24] Macpherson, D., Marker, D., Steinhorn, C.: Weakly o-minimal structures and real closed fields. Trans. Amer. Math. Soc. **352**, 5435–5483 (2000) Zbl 0982.03021 MR 1781273

[25] Simon, P.: A Guide to NIP Theories. Lecture Notes in Logic 44, Association for Symbolic Logic, Chicago, IL, and Cambridge Univ. Press, Cambridge (2015) Zbl 1332.03001 MR 3560428

[26] Sinclair, P.: Relationships between model theory and valuations on fields. PhD thesis, McMaster Univ. (2018)