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(with an appendix by Valentino Tosatti)

Variation of singular Kähler–Einstein metrics: Kodaira dimension zero

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Abstract. We study several questions involving relative Ricci-flat Kähler metrics for families of log Calabi–Yau manifolds. Our main result states that if $p : (X, B) \to Y$ is a Kähler fiber space such that $(X_y, B|_{X_y})$ is generically klt, $K_{X/Y} + B$ is relatively trivial and $p_*(m(K_{X/Y} + B))$ is Hermitian flat for some suitable integer m, then p is locally trivial. Motivated by questions in birational geometry, we investigate the regularity of the relative singular Ricci-flat Kähler metric corresponding to a family $p : (X, B) \to Y$ of klt pairs (X_y, B_y) such that $\kappa(K_{X_y} + B_y) = 0$. Finally, we disprove a folkore conjecture by exhibiting a one-dimensional family of elliptic curves whose relative (Ricci-)flat metric is not semipositive.

Keywords. Kähler fiber space, log Calabi–Yau manifolds, conic Kähler metrics, direct image of log pluricanonical bundles

Introduction

In this article we continue our study of fiberwise singular Kähler–Einstein metrics started in [19] in the following context.

Let $p : (X, B) \to Y$ be a Kähler fiber space, where B is an effective divisor such that $(X_y, B|_{X_y})$ is klt for all $y \in Y$ in the complement of some analytic subset of the base Y. We are interested here in the curvature and regularity properties of the metric induced on

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$K_{X/Y} + B$ by the canonical metrics on the fibers X_y under the hypothesis

$$\kappa(K_{X_y} + B_y) = 0,$$

where $B_y := B|_{X_y}$. The far-reaching goal we are pursuing here is a criterion for the birational equivalence of the fibers $(X_y, B|_{X_y})$ of p in a geometric context inspired by results due to E. Viehweg, Y. Kawamata and J. Kollár in connection with the C_{nm} conjecture. To this end, the fiberwise Kähler–Einstein metrics play a crucial role. Due to some technical difficulties – which we hope to overcome in a forthcoming paper – our most complete results are obtained under the more restrictive hypothesis $c_1(K_{X_y} + B_y) = 0$, i.e. in the absence of basepoints of the log-canonical bundles of the fibers.

Main results

Let $p : (X, B) \to Y$ be a proper, holomorphic fibration between two Kähler manifolds, where $B = \sum b_i B_i$ is an effective $\mathbb{Q}$-divisor on X whose coefficients $b_i \in (0, 1)$ are smaller than 1. We assume that there exists $Y^\circ \subset Y$ contained in the smooth locus of p such that $B|_{X_y}$ has snc support for $y \in Y^\circ$ and set $X^\circ := p^{-1}(Y^\circ)$. The fibers of p are assumed to satisfy

$$c_1(K_{X_y} + B|_{X_y}) = 0 \quad \text{for any } y \in Y^\circ.$$

If we fix a reference Kähler form ω on X, then we can construct a fiberwise Ricci-flat conic Kähler metric θ_y, i.e. a solution of the equation

$$\begin{cases} \operatorname{Ric}\theta_y = [B_y], \\ \theta_y \in [\omega_y]. \end{cases}$$

There exists a unique function $\varphi \in L^1_{\mathrm{loc}}(X^\circ)$ such that

$$\begin{cases} \theta_y = \omega_y + dd^c\varphi|_{X_y}, \\ \int_{X_y} \varphi\, \omega_y^n = 0. \end{cases}$$

The closed $(1,1)$-current $\theta^\circ_{\mathrm{KE}} := \omega + dd^c\varphi$ on X° is called a *relative Ricci-flat conic Kähler metric* in $[\omega]$. As we shall soon see, the current $\theta^\circ_{\mathrm{KE}}$ is not positive in general, which marks an important difference with the case of Kähler fiber spaces whose generic fiber is of (log) general type.

Nevertheless, we establish the following result (Theorem 1.2 for a complete version).

Theorem A. *Let $p : (X, B) \to Y$ be a map as above, and let ω be a fixed Kähler metric on X. Assume that the following conditions are satisfied:*

(i) *For $y \in Y^\circ$, the $\mathbb{Q}$-line bundle $K_{X_y} + B_y$ is numerically trivial.*

(ii) *For some m large enough, the line bundle $p_*(m(K_{X^\circ/Y^\circ} + B))$ is Hermitian flat with respect to the Narasimhan–Simha metric h on Y° (see (1.7)).*

Then we can construct a $(1,1)$-current $\theta^\circ_{\mathrm{KE}}$ such that the restriction θ_y of $\theta^\circ_{\mathrm{KE}}$ to X_y is a representative of $\{\omega\}|_{X_y}$ and solves $\operatorname{Ric}\theta_y = [B_y]$. Moreover:

(†) *$\theta^\circ_{\mathrm{KE}}$ is positive and it extends canonically to a closed positive current $\theta_{\mathrm{KE}} \in \{\omega\}$ on X.*

(‡) *The fibration $(X, B) \to Y$ is locally trivial over Y°. Moreover, if p is smooth in codimension 1 and $\operatorname{codim}_X(B \setminus X^\circ) > 1$, then p is locally trivial over the whole Y.*

The result above has many geometric applications, like for instance a Kähler version of a theorem of Ambro [1] (see Corollary 1.3 and its proof in Section 1.2).

Another striking consequence is the following positivity property of direct images of pluri-log canonical bundles (see Section 1.2 for a proof). It can be seen as a logarithmic version of Viehweg's $Q_{n,m}$-conjecture for families of log Calabi–Yau manifolds [53].

Corollary B. *Let $p : (X, B) \to Y$ be a fibration between two compact Kähler manifolds such that $c_1(K_{X_y} + B|_{X_y}) = 0$ for a generic $y \in Y$. Assume moreover that the logarithmic Kodaira–Spencer map*

$$T_Y \to \mathcal{R}^1 p_*(T_{X/Y}(-\log B)) \tag{0.1}$$

is generically injective. Then the bundle $p_*(m(K_{X/Y} + B))^{**}$ is big.

We remark that, based on Corollary B and some deep tools, Y. Deng [26] proved recently the hyperbolicity of bases of maximally variational smooth families of log Calabi–Yau pairs.

We are next interested in the following setting:

$$\kappa(K_{X_y} + B|_{X_y}) = 0,$$

which is more natural from the birational geometry point of view. The main result we establish in this context is a regularity theorem for the relative Kähler–Einstein metric. The point is that here we have no further assumptions on the basepoints of $K_{X_y} + B_y$ or the flatness of the direct image of some power of $K_{X/Y} + B$ (see the end of Section 3).

Theorem C. *In the above framework, let ω be a fixed Kähler metric on X and assume that for y generic the Kodaira dimension of $K_{X_y} + B_y$ equals zero. Let E be an effective $\mathbb{Q}$-divisor such that $K_{X_y} + B_y \sim_{\mathbb{Q}} E_y$. Then there exists a $(1,1)$-current $\theta^\circ_{\mathrm{KE}}$ whose restriction $\theta_y := \theta^\circ_{\mathrm{KE}}|_{X_y}$ is a representative of $\{\omega\}|_{X_y}$ and solves the equation $\operatorname{Ric}\theta_y = -[E_y] + [B_y]$. In addition, the local potentials of $\theta^\circ_{\mathrm{KE}}$ are Lipschitz on $X^\circ \setminus \operatorname{Supp}(B + E)$.*

One may wonder whether the assumptions concerning the flatness of the direct image of the bundle $m(K_{X/Y} + B)$ can be removed in Theorem A. Indeed, a folklore conjecture asserts that the form $\theta^\circ_{\mathrm{KE}}$ is *semipositive* provided that say $B = 0$ and $c_1(X_y) = 0$. By using the results in the Appendix, we show that this is simply wrong.

Theorem D. *There exists a smooth, proper fibration $p : X \to Y$ between Kähler manifolds such that $c_1(X_y) = 0$ for all $y \in Y$ and a Kähler form ω on X such that the relative Ricci-flat metric $\theta_{\mathrm{KE}} \in [\omega]$ is not semipositive.*

The example we exhibit is constructed from a special K3 surface admitting a non-isotrivial elliptic fibration as well as another transverse elliptic fibration. The construction is detailed in Section 3.

Previously known results

In connection with Theorem A, the statements obtained so far are based on two different types of techniques arising from algebraic geometry and complex differential geometry, respectively. One can profitably consult the articles [53], [34] and [33] for results aimed at the Iitaka conjecture. From the complex differential geometry side we refer to [4], [30], [6] and the references therein.

The folklore conjecture that we disprove in Theorem D arose from a result of Schumacher [45] who proved the semipositivity of the relative Kähler–Einstein metric for families of canonically polarized manifolds (see also the related works [2], [50]). He also implicitly conjectured that an analogous semipositivity result should hold for families of Calabi–Yau manifolds [45, p. 7], and this was explored in the thesis of Braun [10] and in the papers [11, 12] where positive partial results were obtained. The semipositivity question for θ_{KE} also appeared in the work [27] on the Kähler–Ricci flow.

Main steps of the proof

We next outline the proof of Theorems A, C and D.

• The first item of Theorem A is established by using two ingredients. The first one consists in showing that the conic Ricci-flat metric in $\{\omega_{X_y}\}$ on each fiber X_y is the normalized limit of the unique solution of the family of equations of the type

$$\operatorname{Ric}\rho_\varepsilon = -\rho_\varepsilon + \varepsilon\omega + [B] \tag{0.2}$$

on X_y where $\rho_\varepsilon \in \varepsilon\{\omega_{X_y}\}$. We show that $\omega_{\mathrm{KE}}^\circ|_{X_y}$ is obtained as the limit of $\frac{1}{\varepsilon}\rho_\varepsilon$ as $\varepsilon \to 0$. On the other hand, the main result of [28] shows that the family ρ_ε has psh variation for each positive $\varepsilon > 0$, and the result follows (the flatness of the direct image is crucial in order to be able to use [28]).

The argument for the second item of Theorem A is more involved. We use a different type of approximation of the conic Ricci-flat metric, by regularizing the volume element. Let τ_δ be the resulting family of metrics. The heart of the matter is to show that the horizontal lift with respect to τ_δ of any local holomorphic vector field on the base has a holomorphic limit as $\delta \to 0$. This is a consequence of the estimates in [29] combined with the PDE satisfied by the geodesic curvature of τ_δ [45]. Then we show that the geodesic curvature tends to a (positive) constant and as a consequence we finally infer that the horizontal lift of holomorphic vector fields with respect to $\omega_{\mathrm{KE}}^\circ$ is holomorphic and tangent to B.

• The equation $\operatorname{Ric}\omega = -[E] + [B]$ translates into a Monge–Ampère equation where the right hand side has poles and zeros. The poles are relatively manageable in the sense that they induce conic metrics, that is, we know relatively precisely the behavior of the complex Hessian of the solution. The zeros, however, are much more complicated to deal with for several reasons. First, it seems hard to produce a global degenerate model metric that should encode the behavior of the solution. Next, regularized solutions of the Kähler–Einstein equation do not satisfy a Ricci lower bound, hence it seems difficult to estimate their Sobolev constant.

In Proposition 2.1, we establish a uniform (weak) Sobolev inequality where the measure on the right hand side picks up zeros. Then we study the regularity of families of such metrics. Despite having a rather poor understanding of the fiberwise metrics, we are still able to analyze the first order derivatives of the potentials in the transverse directions, leading to an L^2 estimate, yet with respect to a more degenerate volume form (Theorem 2.6). This is however enough to deduce the Lipschitz variation of the potentials away from $\mathrm{Supp}(B+E)$.

• The counterexample provided by Theorem D is built from an elliptic fibration $p : X \to \mathbb{P}^1$ where X is a K3 surface. In the Appendix, it is shown that one can find such a fibration with the following properties: its singular fibers are irreducible and reduced, it is not isotrivial and it admits another transverse elliptic fibration. These properties allow us to find a semiample, p-ample line bundle $L \to X$ with numerical dimension 1. Then the relative Ricci-flat metric $\theta \in c_1(L)|_{X^\circ}$ cannot be semipositive, for otherwise one can show that it would extend to a positive current $\theta \in c_1(L)$ and as L is not big, results of Boucksom show that

$$\theta^2 \equiv 0 \quad \text{on } X^\circ.$$

Using horizontal lifts of θ, one can finally conclude that the foliation $\operatorname{Ker}\theta$ is holomorphic, induced by a local trivialization of the family. This contradicts the non-isotriviality of p. Passing from the relative Ricci-flat metric in $c_1(L)$ to one in a Kähler class can be done using a limiting process.

Organization of the paper

§1: We prove Theorem 1.2, and then derive successively Corollary 1.3 and Corollary B.

§2: We obtain transverse regularity results for families of Monge–Ampère equations corresponding to adjoint linear systems having basepoints. This leads to Theorem C.

§3: We prove Theorem 3.1 using results from the Appendix.

1. Relative Ricci-flat conic metrics

1.1. Setting

Let $p : X \to Y$ a holomorphic proper map of relative dimension n between Kähler manifolds. We denote by $Y^\circ \subset Y$ the set of regular values of p, and let $X^\circ := p^{-1}(Y^\circ)$ so that $p|_{X^\circ} : X^\circ \to Y^\circ$ is a smooth fibration. For $y \in Y^\circ$, one writes $X_y := p^{-1}(X_y)$, the fiber over y. Let B be an effective $\mathbb{Q}$-divisor on X that has coefficients in $(0,1)$ and whose support has snc. Our assumption throughout the current section will be that for each $y \in Y^\circ$ we have

$$c_1(K_{X_y} + B_y) = 0 \in H^{1,1}(X_y, \mathbb{Q}). \tag{1.1}$$

Thanks to the log abundance in the Kähler setting (Corollary 1.18), we know that $K_{X_y} + B_y$ is $\mathbb{Q}$-effective. Combining this with the Ohsawa–Takegoshi extension theorem in its Kähler version [18], one can assume that there exists $m \geq 1$ such that $m(K_{X_y} + B_y) \simeq \mathcal{O}_{X_y}$ for all $y \in Y^\circ$.

In this context the main result we obtain here shows that the flatness of the direct image $p_*(mK_{X/Y} + mB)$ implies the local isotriviality of the family $p : (X, B) \to Y$. By this we mean that there exists a holomorphic vector field v on X° whose flow identifies the pairs (X_y, B_y) and (X_w, B_w) provided that $y, w \in Y^\circ$ are close enough. This is the content of Theorem 1.2 below. Prior to stating our theorems in a formal manner, we need to recall a few notions and facts.

Given a point $y \in Y^\circ$, there exists a coordinate ball $U \subset Y^\circ$ containing y and a nowhere vanishing holomorphic section

$$\Omega \in H^0(X_U, m(K_{X/Y} + B)|_{X_U}) \tag{1.2}$$

by our assumption (1.1), where $X_U := p^{-1}(U)$.

If f_B is a local multivalued holomorphic function cutting out the $\mathbb{Q}$-divisor B, then the form $\frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{|f_B|^2}$ induces a volume element on the fibers of p over U. We fix a Kähler class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$. Up to renormalizing ω, one can assume that the constant function

$$Y^\circ \ni y \mapsto \int_{X_y} \omega^n$$

is identically equal to 1. We also define

$$V_y := \int_{X_y} \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{|f_B|^2};$$

this is a Hölder continuous function of $y \in Y^\circ$.

Let ρ_y be the unique positive current on X_y which is cohomologous to ω_y and satisfies

$$\rho_y^n = \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{V_y |f_B|^2}$$

(see [56]). One can write $\rho_y = \omega|_{X_y} + dd^c \varphi_y$, where the function φ_y is uniquely determined by the normalization

$$\int_{X_y} \varphi_y \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{|f_B|^2} = 0. \tag{1.3}$$

For each $y \in U \subset Y^\circ$, the current ρ_y is reasonably well understood: it has Hölder potentials, and it is quasi-isometric to a metric with conic singularities along B [29].

We next analyze its regularity properties in the "base directions"; this will allow us to derive a few interesting geometric consequences.

The function φ defined on X° by $\varphi(x) := \varphi_{p(x)}(x)$ is a locally bounded function on X° (by the family version of Kołodziej's estimates [25]), hence it induces a $(1,1)$-current

$$\rho := \omega + dd^c\varphi \tag{1.4}$$

on X°. Let $\Delta \subset Y^\circ$ be a small, 1-dimensional disk. If Δ is generic enough, then the inverse image $\mathcal{X} := p^{-1}(\Delta)$ is non-singular, and the restriction map $p : \mathcal{X} \to \Delta$ is a submersion. We denote by t a holomorphic coordinate on the disk Δ. Following [46] we next recall the expression of the *horizontal lift* of the local vector field $\frac{\partial}{\partial t}$. For the moment, this is a vector field v_ρ with distribution coefficients on the total space $\mathcal{X}$ given by the expression

$$v_\rho := \frac{\partial}{\partial t} - \sum_\alpha \rho^{\overline{\beta}\alpha}\rho_{t\overline{\beta}}\frac{\partial}{\partial z_\alpha}, \tag{1.5}$$

where the notations are as follows. We denote by $(z_1, \dots, z_n, t)$ a coordinate system centered at some point of $\mathcal{X}$, and $\rho_{t\overline{\beta}}$ is the coefficient of $dt \wedge d\overline{z}_\beta$. We denote by $(\rho^{\overline{\beta}\alpha})$ the entries of the inverse of the matrix $(\rho_{\alpha\overline{\beta}})$.

The reflexive hull of the direct image

$$\mathcal{F}_m := p_\star(m(K_{X/Y} + B))^{\star\star} \tag{1.6}$$

plays a key role in the study of the geometry of algebraic fiber spaces. It admits a positively curved singular metric whose construction we next recall (see [5, 42] and the references therein).

Let $\sigma \in H^0(U, \mathcal{F}_m|_U)$ be a local holomorphic section of the line bundle $\mathcal{F}_m$ defined over a small coordinate set $U \subset Y^\circ$. The expression

$$\|\sigma\|_y^2 := V_y^{m-1}\int_{X_y} \frac{|\sigma|^2}{|\Omega_y|^{2\frac{m-1}{m}}}e^{-\phi_B} \tag{1.7}$$

defines a metric h on $\mathcal{F}_m|_{Y^\circ}$. It is remarkable that this metric extends across the singularities of the map p, and it has semipositive curvature current; see [5,42] for more complete statements.

1.2. Main results

In this subsection we aim to prove the following results.

Theorem 1.1. *Let $p : (X, B) \to Y$ be a proper holomorphic map between Kähler manifolds as in (1.1). Assume moreover that the curvature of $\mathcal{F}_m$ with respect to the metric in (1.7) equals zero when restricted to Y°. Then the $(1,1)$-current ρ defined on X° by (1.4) is semipositive and it extends canonically to a closed positive current on X in the cohomology class $\{\omega\}$.*

For example, if we assume that Y is compact, then the curvature of $\mathcal{F}_m$ will automatically be zero if $c_1(\mathcal{F}_m) = 0$ thanks to the properties of the metric (1.7) discussed above [20, Thm. 5.2].

The word “canonically” in Theorem 1.1 means that the local potential φ of ρ is locally bounded above across $X \setminus X^\circ$.

We also prove the next statement.

Theorem 1.2. *Assume that the hypotheses in Theorem 1.1 are satisfied. Then p is locally trivial over Y°, that is, for every $y \in Y^\circ$, there exists a neighborhood $U \subset Y^\circ$ of y such that*

$$(p^{-1}(U), B) \simeq (X_y, B|_{X_y}) \times U.$$

Moreover, if p is smooth in codimension 1, then p is locally trivial over the whole Y provided that $\operatorname{codim}_{X \setminus X^\circ}(B \setminus X^\circ) > 0$.

In particular, under the assumptions in the “moreover” part of Theorem 1.2 the map p is automatically a locally isotrivial submersion.

As an application, we establish the following result; it partially generalizes to the Kähler case a theorem of F. Ambro [1].

Corollary 1.3. *Let $p : X \to Y$ be a fibration between two compact Kähler manifolds. Let B be a $\mathbb{Q}$-effective klt divisor on X with snc support.*

If $-(K_X + B)$ is nef, then $-K_Y$ is pseudo-effective. (1.8)

If $c_1(K_X + B) = 0$ and $c_1(Y) = 0$, then p is locally trivial, that is, for every $y \in Y$, there exists a neighborhood $U \subset Y$ of y such that

$$(p^{-1}(U), B) \simeq (X_y, B|_{X_y}) \times U.$$

In particular, if $c_1(K_X + B) = 0$, the Albanese map $p : X \to \operatorname{Alb}(X)$ is locally trivial. (1.9)

1.3. Proof of Theorem 1.1

We will proceed by approximation, mainly using the following lemma combined with the results in [28].

The next statement will enable us to reduce the problem to canonically polarized pairs.

Lemma 1.4. *Let X be a compact Kähler manifold and let B be an effective divisor such that (X, B) is klt. Assume that $c_1(K_X + B) = 0$. Let ω be the Kähler form on X. For every $\varepsilon > 0$, let $\rho_\varepsilon \in \varepsilon\{\omega\}$ be the unique twisted conic Kähler–Einstein metric such that*

$$\operatorname{Ric} \rho_\varepsilon = -\rho_\varepsilon + \varepsilon\omega + [B]. \tag{1.10}$$

Let $\rho \in \{\omega\}$ be the unique conic Kähler–Einstein metric such that $\operatorname{Ric} \rho = [B]$. *Then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \rho_\varepsilon = \rho$$

where the convergence is smooth outside $\operatorname{Supp}(B)$.

Proof. Let $m \in \mathbb{N}$ be such that $m(K_X + B)$ is effective. Let $\Omega \in H^0(X, m(K_X + B))$ be a holomorphic section normalized so that

$$\int_X \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2} = 1. \tag{1.11}$$

There exists a unique function φ_ε on X such that

$$\rho_\varepsilon = \varepsilon\omega + dd^c\varphi_\varepsilon, \tag{1.12}$$

$$\rho_\varepsilon^n = \varepsilon^n e^{\varphi_\varepsilon} \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2}. \tag{1.13}$$

Now, set

$$\psi_\varepsilon := \frac{1}{\varepsilon}\varphi_\varepsilon.$$

One has $\frac{1}{\varepsilon}\rho_\varepsilon = \omega + dd^c\psi_\varepsilon$ and

$$(\omega + dd^c\psi_\varepsilon)^n = e^{\varepsilon\psi_\varepsilon} \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2} \tag{1.14}$$

As $\frac{(\Omega\wedge\overline{\Omega})^{1/m}}{|f_B|^2}$ and $(\frac{1}{\varepsilon}\rho_\varepsilon)^n$ are probability measures and ψ_ε is ω-psh, Jensen's inequality yields $\int_X (\varepsilon\psi_\varepsilon) \frac{(\Omega\wedge\overline{\Omega})^{1/m}}{|f_B|^2} \le 0$, and therefore

$$\int_X \psi_\varepsilon \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2} \le 0. \tag{1.15}$$

As the measure $\frac{(\Omega\wedge\overline{\Omega})^{1/m}}{|f_B|^2}$ integrates every quasi-psh function, it follows from standard results in pluripotential theory that there exists a constant C such that

$$\sup_X \psi_\varepsilon \le C. \tag{1.16}$$

By (1.14)–(1.16) and Kołodziej's estimate [36], one gets

$$\operatorname{osc}_X \psi_\varepsilon \le C. \tag{1.17}$$

As $\frac{(\Omega\wedge\overline{\Omega})^{1/m}}{|f_B|^2}$ and $(\frac{1}{\varepsilon}\rho_\varepsilon)^n$ are probability measures again, (1.14) shows that

$$\inf_X \psi_\varepsilon \le 0 \le \sup_X \psi_\varepsilon.$$

Combining this information with (1.17), we obtain

$$\|\psi_\varepsilon\|_{L^\infty(X)} \le C. \tag{1.18}$$

Moreover, Jensen's inequality applied to the equation $\frac{(\Omega\wedge\overline{\Omega})^{1/m}}{|f_B|^2} = e^{-\varepsilon\psi_\varepsilon}(\frac{1}{\varepsilon}\rho_\varepsilon)^n$ yields

$$\int_X \psi_\varepsilon(\omega + dd^c\psi_\varepsilon)^n \ge 0. \tag{1.19}$$

From (1.14) and (1.18), we get uniform estimates at any order for ψ_ε outside B. If ψ is a subsequential limit of the family $(\psi_\varepsilon)_{\varepsilon>0}$, it will satisfy

$$(\omega + dd^c\psi)^n = \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2}.$$

Combining this information with (1.15) and (1.19), we find

$$\int_X \psi \, \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2} = 0.$$

Therefore ψ is uniquely determined, and the whole family $(\psi_\varepsilon)_{\varepsilon>0}$ converges to ψ. The lemma is thus proved. ∎

Proof of Theorem 1.1. We fix a reference Kähler form ω on X, and let U be some small topological open set of Y°. By hypothesis, the curvature of the bundle $\mathcal{F}_m|_U$ is identically zero. By using parallel transport, this is equivalent to the existence of a section

$$s \in H^0(X_U, mK_{X/Y} + mB|_{X_U}) \tag{1.20}$$

whose norm is a constant function on U, namely $\|s\|_h(y) = 1$ for every $y \in U$. Let

$$\Omega_y := s|_{X_y} \in H^0(X_y, mK_{X_y} + mB_y)$$

be the restriction of s to the fibers of p.

Since $c_1(K_{X_y} + B_y) + \varepsilon\omega|_{X_y}$ is a Kähler class for each $\varepsilon > 0$ and each $y \in Y^\circ$, there exists a unique φ_ε such that

$$(\varepsilon\omega + dd^c\varphi_\varepsilon)^n = \varepsilon^n e^{\varphi_\varepsilon} \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{|f_B|^2} \quad \text{on } X_y.$$

Since $y \in U$ is a regular value, this is equivalent to

$$\operatorname{Ric} \rho_{\varepsilon,y} = -\rho_{\varepsilon,y} + \varepsilon\omega + [B_y] \quad \text{on } X_y,$$

where $\rho_{\varepsilon,y} = \varepsilon\omega + dd^c\varphi_\varepsilon|_{X_y}$.

Next, the section s is holomorphic, hence the relative B-valued volume forms $(\Omega_y \wedge \overline{\Omega}_y)^{1/m}$ induce a metric with zero curvature on $K_{X/Y} + B$ over $p^{-1}(U)$. Because of that,

$$\rho_\varepsilon := \varepsilon\omega + dd^c\varphi_\varepsilon$$

coincides with the current studied in [28], and the content of the main theorem there is that ρ_ε is positive on $p^{-1}(U)$. Thanks to Lemma 1.4, ρ is the fiberwise weak limit on $p^{-1}(U)$ of the fiberwise twisted Kähler–Einstein metrics $\frac{1}{\varepsilon}\rho_\varepsilon$; moreover, the estimate (1.18) is uniform over U, so that ρ is actually the global weak limit of the metrics $\frac{1}{\varepsilon}\rho_\varepsilon$ on $p^{-1}(U)$. In particular, $\rho \geq 0$ on $p^{-1}(U)$, hence on X°.

As for the extension property, it is proved in [28] that ρ_ε extends canonically to the whole X as a positive current in $\{\varepsilon\omega\}$. This means that given any small neighborhood U

of a point $x \in X \setminus X^\circ$, one has $\sup_{U\cap X^\circ} \psi_\varepsilon < +\infty$. In other words, ψ_ε extends to an ω-psh function on X. Now, let us fix U as above. The family $(\widetilde{\psi}_\varepsilon)_{\varepsilon>0}$ of ω-psh functions on U defined by

$$\widetilde{\psi}_\varepsilon := \psi_\varepsilon - \sup_U \psi_\varepsilon$$

is relatively compact. In particular, one can find a sequence $\varepsilon_k \to 0$ and an ω-psh function $\widetilde{\psi}$ on U such that $\widetilde{\psi}_{\varepsilon_k} \to \widetilde{\psi}$ a.e. in U. Moreover, we know that $\psi_{\varepsilon_k} = \widetilde{\psi}_{\varepsilon_k} + \sup_U \psi_{\varepsilon_k}$ converges to the ω-psh function φ a.e. in $U \cap X^\circ$. This implies that $\sup_U \psi_{\varepsilon_k}$ converges as $k \to +\infty$. By the Hartogs lemma, this implies that $\sup_{U\cap X^\circ} \varphi < +\infty$, which was to be proved. ∎

1.4. Proof of Theorem 1.2

We will proceed in a few steps, roughly as follows.

• We start by approximating ρ by smoothing the volume element. Let τ_δ be the resulting $\mathscr{C}^\infty$ form. Then we have $\lim_\delta \tau_\delta = \rho$ in the weak sense.

• We next analyze the behavior of the geodesic curvature of τ_δ. The main tools are the Laplace equation satisfied by this quantity [45], and the $\mathscr{C}^2$ estimates for conic Monge–Ampère equations [29]. As a consequence, we first show that we can extract a limit of the horizontal lift v_δ (corresponding to τ_δ) which is *holomorphic* on the fibers of p. Afterwards we show that the geodesic curvature of τ_δ converges (on $\mathcal{X} \setminus \mathrm{Supp}(B)$) to a constant as $\delta \to 0$. Finally, we infer that v_δ converges to v_ρ uniformly on the complement of the divisor B.

• After completing the previous steps, we show that v_ρ is in fact holomorphic on the total space $\mathcal{X}$ by using a few arguments borrowed from [3].

• Finally, we show that v_ρ extends across the singular locus of p provided that X is compact and p is smooth in codimension 1.

1.4.1. Approximation. This is a fairly standard and widely used procedure, so we will be very brief.

By hypothesis, we have $B = \sum a_j B_j$ where $a_j \in (0, 1)$ and $\bigcup B_j$ has simple normal crossings. We consider a smooth metric $e^{-\phi_j}$ on the bundle associated to B_j; it induces a smooth metric $e^{-\phi_B} := e^{-\sum a_j\phi_j}$ on the $\mathbb{Q}$-line bundle associated to B. For any $\delta \geq 0$ we define the quantity $C_{\delta,y}$ by

$$e^{-C_{\delta,y}} = \int_{X_y} \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{a_j}}.$$

Here Ω is a section of $\mathcal{F}_m|_\Delta$ whose norm is 1 at each point, and f_j is a local holomorphic function cutting out B_j. The expression $\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{-a_j}$ is then a globally defined smooth metric on the $\mathbb{Q}$-line bundle associated to B. Finally, we let s_j be the canonical section of $\mathcal{O}_X(B_j)$, and we will denote by $|s_j|^2$ the squared norm of s_j with respect to $e^{-\phi_j}$.

Let us further define the smooth $(1,1)$-form

$$\tau_\delta = \omega + dd^c u_\delta \tag{1.21}$$

on X° such that $u_\delta|_{X_y}$ is a solution of

$$\begin{cases} (\omega + dd^c u_\delta)^n = e^{C_{\delta,y}} \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{a_j}}, \\ \int_{X_y} u_\delta \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{a_j}} = 0. \end{cases} \tag{1.22}$$

By the family version of Kołodziej's estimates [25], one can easily see that for any relatively compact subset $U \Subset Y^\circ$, there exists a constant $C > 0$ independent of $\delta \in (0,1)$ such that

$$\sup_{y \in U} \|u_\delta\|_{L^\infty(X_y)} \le C. \tag{1.23}$$

As a consequence, we get the following easy result (see (1.4) for the definition of ρ and φ).

Lemma 1.5. *When δ approaches zero, τ_δ converges weakly to ρ on X°. More precisely, $u_\delta \to \varphi$ in $L^1_{\mathrm{loc}}(X^\circ)$.*

Proof. The convergence $u_\delta \to \varphi$ in $L^1_{\mathrm{loc}}(X_y)$ follows from Kołodziej's stability theorem [37, Thm. 4.1] (one even gets uniform convergence). The convergence on the total space then follows from Lebesgue's dominated convergence theorem coupled with (1.23). ■

1.4.2. Uniformity properties of $(\tau_\delta)_{\delta>0}$. In this subsection we will only consider the restriction of our initial family of manifolds above a disk in the complex plane

$$p : \mathcal{X} \to \Delta \tag{1.24}$$

where we recall that $\Delta \subset Y^\circ$ is generic and $\mathcal{X} = p^{-1}(\Delta)$.

The coordinate on Δ will be denoted by t. We recall that the geodesic curvature of the form τ_δ is the function defined by the equality

$$\tau_\delta^{n+1} = c(\tau_\delta)\tau_\delta^n \wedge \sqrt{-1}\, dt \wedge d\bar{t}. \tag{1.25}$$

If v_δ is the horizontal lift of $\frac{\partial}{\partial t}$ with respect to τ_δ, then it is easy to verify that

$$c(\tau_\delta) = \langle v_\delta, v_\delta \rangle_{\tau_\delta}. \tag{1.26}$$

For each $\delta > 0$, the form τ_δ induces a metric h_δ on the relative canonical bundle $K_{\mathcal{X}/\Delta}$ as follows. Let $z_1, \dots, z_n, z_{n+1}$ be a coordinate system defined on the set $W \subset \mathcal{X}$. Recall that t is a coordinate on Δ. This data induces in particular a trivialization of $K_{\mathcal{X}/\Delta}$, with respect to which the weight of h_δ is given as follows:

$$e^{\Psi_\delta(z,t)} dz_1 \wedge \cdots \wedge dz_{n+1} = \tau_\delta^n \wedge \sqrt{-1}\, dt \wedge d\bar{t}. \tag{1.27}$$

The curvature of $(K_{\mathcal{X}/\Delta}, h_\delta)$ is the Hessian of the weight,

$$\Theta_\delta(K_{\mathcal{X}/\Delta})|_W = dd^c \Psi_\delta. \tag{1.28}$$

We have the following result, relating the various quantities defined above.

Lemma 1.6. *Let Δ''_δ be the Laplace operator corresponding to the metric $\tau_\delta|_{X_t}$. Then*

$$-\Delta''_\delta(c(\tau_\delta)) = |\bar\partial v_\delta|^2 - \Theta_\delta(K_{\mathcal{X}/\Delta})(v_\delta, v_\delta). \tag{1.29}$$

We will not prove Lemma 1.6 in detail because this type of result appears in many articles ([45] or [41]). The main steps are as follows: we have $\Psi_\delta = \log\det(g_{\alpha\overline{\beta}})$ where we denote $g_{\alpha\overline{\beta}} := \tau_{\delta,\alpha\bar\beta}$ and a few simple computations show that the Hessian of Ψ_δ evaluated in the v_δ-direction equals

$$\begin{aligned}\bar\partial\log\det(g_{\alpha\overline{\beta}})(v_\delta,\overline{v}_\delta) = {} & g^{\alpha\overline{\beta}}g_{t\bar t,\alpha\overline{\beta}} - g^{\alpha\overline{\gamma}}g^{\delta\overline{\beta}}g_{\gamma\overline{\delta},\bar t}g_{\alpha\overline{\beta},t} \\ & - g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\bar t}g^{\gamma\overline{\mu}}g_{t\overline{\mu}} - g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},t\overline{\gamma}}g^{\mu\overline{\gamma}}g_{\mu\bar t} \\ & + g^{\alpha\overline{\beta}}g_{\alpha\overline{\beta},\gamma\overline{\tau}}g^{\gamma\overline{\mu}}g^{\rho\overline{\tau}}g_{t\overline{\mu}}g_{\rho\bar t}.\end{aligned} \tag{1.30}$$

On the right hand side we recognize the beginning of $\Delta''_\delta(c(\tau_\delta))$ (cf. the 1st term), and in the end this gives (1.29). Again, we refer to [21, pp. 18–19] for a detailed account. ∎

Remark 1.7. Equation (1.29) can be seen as the analogue of the usual $\mathcal{C}^2$ estimates in "normal directions". By this we mean the following: the $\mathcal{C}^2$ estimates are derived by evaluating the Laplacian of the (log of the) sum of the eigenvalues of the solution metric with respect to the reference metric. Vaguely speaking, in (1.29) we compute the Laplacian of the normal eigenvalue.

The following result is an important step towards the proof of Theorem 1.2.

Proposition 1.8. *Let $t \in \Delta$ be fixed. For any sequence $\delta_j \to 0$, there exists a holomorphic vector field w on $X_t \setminus \mathrm{Supp}(B)$ such that, up to extracting a subsequence, the sequence $(v_{\delta_j}|_{X_t})_{j\geq 0}$ converges locally smoothly outside $\mathrm{Supp}(B)$ to the vector field w.*

Remark 1.9. At this point, it is not obvious that w is independent of the sequence δ_j and that it should coincide with the lift v of $\frac{\partial}{\partial t}$ with respect to $\rho|_{X^\circ\setminus\mathrm{Supp}(B)}$.

Before giving the proof of Proposition 1.8 we collect a few results concerning the family $(\tau_\delta)_{\delta>0}$ of forms, taken from [29] and [28].

(a) It follows from [29, §5.2] that $\tau_\delta|_{X_y}$ has "uniform regularized conic singularities" in the sense that if on a small coordinate open set $\Omega \subset \mathcal{X}$, the divisor B is given be $B = \sum_1^r a_j B_j$ where B_j is defined by $\{z_j = 0\}$, then there is a constant C independent of δ such that for any $y \in U$, we have

$$\begin{aligned} & C^{-1}\Big(\sum_{k=1}^r \frac{\sqrt{-1}\,dz_k\wedge d\overline{z}_k}{(|z_k|^2+\delta^2)^{a_k}} + \sum_{k\geq r+1}\sqrt{-1}\,dz_k\wedge d\overline{z}_k\Big) \\ & \qquad \leq \tau_\delta|_{X_y\cap\Omega} \leq C\Big(\sum_{k=1}^r \frac{\sqrt{-1}\,dz_k\wedge d\overline{z}_k}{(|z_k|^2+\delta^2)^{a_k}} + \sum_{k\geq r+1}\sqrt{-1}\,dz_k\wedge d\overline{z}_k\Big).\end{aligned} \tag{1.31}$$

(b) The estimates [28, (3.13), Prop. 4.1&4.2] go through for u_δ, that is, for any integer $k \geq 0$, there exists $C_k > 0$ independent of $\delta \in (0,1)$ such that

$$\sup_{t\in\Delta} \|\partial_t u_\delta\|_{\mathscr{C}^k(\Omega\cap X_t)} \leq C_k \tag{1.32}$$

and there exists a constant $C > 0$ such that the following global estimate holds:

$$\sup_{t\in\Delta} \int_{X_t} |v_\delta|^2_\omega \tau_\delta^n \leq C. \tag{1.33}$$

One also gets

$$\lim_{\delta\to 0} \sup_{t\in\Delta} \int_{X_t\cap\bigcup\{|s_j|^2<\delta\}} |v_\delta|^2_\omega \tau_\delta^n = 0. \tag{1.34}$$

Again, we will not reproduce the arguments for (1.32)–(1.34) here, but let us comment e.g. on (1.33) for the comfort of the reader. The main observation is that in local coordinates this amounts to obtaining a bound of $|\nabla^\delta(\partial_t u_\delta)|^2$ with respect to the volume element τ_δ^n on X_t. Here $|\cdot|^2$ is measured with respect to the reference metric ω, and ∇^δ is the gradient corresponding to τ_δ. By (1.31) this is smaller than $|\nabla^\delta(\partial_t u_\delta)|^2_\delta$ up to a uniform constant. This new quantity is controlled by taking the derivative of the Monge–Ampère equation satisfied by τ_δ in normal directions and integration by parts. Of course, the real proof is much more involved and we refer to *loc. cit.* for the details.

We see immediately that (1.33)–(1.34) imply the next statement.

Lemma 1.10. *One has*

$$\lim_{\delta\to 0} \sup_{t\in\Delta} \int_{X_t} \Big(\sum_j \frac{\delta^2}{|s_j|^2+\delta^2}\Big)|v_\delta|^2_\omega \tau_\delta^n = 0.$$

The proof of Lemma 1.10 is very elementary and we skip it.

Proof of Proposition 1.8. Recall that in local coordinates,

$$v_\delta = \frac{\partial}{\partial t} - \sum_{\alpha,\beta} \tau_\delta^{\bar\beta\alpha} \tau_{\delta,t\bar\beta} \frac{\partial}{\partial z_\alpha}.$$

By (1.32), the family $(v_\delta|_{X_t})_{\delta>0}$ is relatively compact in the $\mathscr{C}^\infty_{\mathrm{loc}}(X_t \smallsetminus \mathrm{Supp}(B))$ topology. Let δ_j be a sequence converging to zero such that $(v_{\delta_j}|_{X_t})_{j\geq 0}$ converges locally smoothly outside $\mathrm{Supp}(B)$ to a vector field w.

Now, the geodesic curvature $c(\tau_\delta)$ of τ_δ satisfies

$$-\Delta_{\tau_\delta} c(\tau_\delta) = \|\bar\partial v_\delta\|^2 - \Theta_\delta(K_{\mathcal{X}/\Delta})(v_\delta, \bar v_\delta) \tag{1.35}$$

by Lemma 1.6. In our setting (see (1.21) and the definition of τ_δ) the curvature term in (1.35) becomes

$$\frac{\partial^2 C_\delta(t)}{\partial t\partial\bar t} - \sum_j a_j \delta^2 \frac{\sqrt{-1}\,\langle\partial s_j, \partial s_j\rangle(v_\delta,\bar v_\delta)}{(|s_j|^2+\delta^2)^2} + \sum a_j\delta^2\frac{\Theta_j(v_\delta,\bar v_\delta)}{|s_j|^2+\delta^2} \tag{1.36}$$

where Θ_j is the curvature of the hermitian line bundle $(\mathcal{O}_X(B_j), e^{-\phi_j})$. Integrating (1.35) against τ_δ^n yields

$$\lim_{\delta\to 0}\sup_{t\in\Delta}\left(\int_{X_t}\|\bar\partial v_\delta\|^2\tau_\delta^n+\sum_j a_j\int_{X_t}\delta^2\frac{\sqrt{-1}\,\langle\partial s_j,\partial s_j\rangle(v_\delta,\bar v_\delta)}{(|s_j|^2+\delta^2)^2}\tau_\delta^n\right)=\lim_{\delta\to 0}\sup_{t\in\Delta}\frac{\partial^2 C_\delta(t)}{\partial t\partial\bar t}. \tag{1.37}$$

Indeed, thanks to Lemma 1.10 the third term in (1.36) vanishes as $\delta\to 0$.

We next show that

$$\lim_{\delta\to 0}\sup_{t\in\Delta}\frac{\partial^2 C_\delta(t)}{\partial t\partial\bar t}=0; \tag{1.38}$$

this will end the proof of Proposition 1.8. Recall that

$$C_\delta(t)=-\log\int_{X_t}\frac{(\Omega_y\wedge\overline{\Omega}_y)^{1/m}}{\prod_j(|f_j|^2+\delta^2e^{\phi_j})^{a_j}} \tag{1.39}$$

and since the norm of Ω is 1 at each point of Δ, we have

$$C_\delta(t)=-\log\left(1-\int_{X_t}\frac{\prod_j(|f_j|^2+\delta^2e^{\phi_j})^{a_j}-\prod_j|f_j|^{2a_j}}{\prod_j|f_j|^{2a_j}\prod_j(|f_j|^2+\delta^2e^{\phi_j})^{a_j}}(\Omega_y\wedge\overline{\Omega}_y)^{1/m}\right). \tag{1.40}$$

With the same notations as in (1.31), the restriction of the function under the integral sign in (1.40) to a coordinate set W_α reads

$$F_{\alpha,\delta}(z,t):=\frac{\prod_j(|z_j|^2+\delta^2e^{\phi_j})^{a_j}-\prod_j|z_j|^{2a_j}}{\prod_j|z_j|^{2a_j}\prod_j(|z_j|^2+\delta^2e^{\phi_j})^{a_j}} \tag{1.41}$$

and then the integral in (1.40) becomes

$$\sum_\alpha\int_{W_\alpha\cap X_t}\theta_\alpha F_{\alpha,\delta}(z,t)e^{f_\alpha}\omega^n \tag{1.42}$$

where θ_α is a partition of unity and the f_α are given smooth functions. If v is the horizontal lift of $\frac{\partial}{\partial t}$ with respect to the reference metric ω, then we have the usual formula

$$\frac{\partial}{\partial t}\sum_\alpha\int_{X_t}\theta_\alpha F_{\alpha,\delta}(z,t)e^{f_\alpha}\omega^n=\sum_\alpha\int_{X_t}v(\theta_\alpha F_{\alpha,\delta}(z,t)e^{f_\alpha})\omega^n. \tag{1.43}$$

Formula (1.39) shows that $\frac{\partial F_{\alpha,\delta}}{\partial t}$ converges to zero as $\delta\to 0$ because only the weights ϕ_j depend on t and the coefficients a_j are strictly smaller than 1. Indeed, we have

$$\frac{\partial F_{\alpha,\delta}}{\partial t}=\sum_j\frac{\delta^2e^{\phi_j}\partial_t\phi_j}{(|z_j|^2+\delta^2e^{\phi_j})^{1+a_j}}\frac{a_j}{\prod_{i\neq j}(|z_i|^2+\delta^2e^{\phi_i})^{a_i}} \tag{1.44}$$

and our claim follows since

$$\int_{(\mathbb{C},0)}\frac{\delta^2}{(|z|^2+\delta^2)^{1+a}}d\lambda(z)\to 0\quad\text{as }\delta\to 0\text{ for any }a<1.$$

For terms involving $\frac{\partial F_{\alpha,\delta}}{\partial z_i}$ we get the same conclusion (i.e. they tend to zero) by using integration by parts, as we explain next. The corresponding terms in (1.43) have the following shape:

$$\int_{X_t} \frac{\partial F_{\alpha,\delta}}{\partial z_i}(z)\tau_\alpha(z)\, d\lambda(z) \tag{1.45}$$

where τ_α is a smooth function with compact support in $W_\alpha \cap X_t$. The integral (1.45) is equal to

$$-\int_{X_t} \frac{\partial \tau_\alpha}{\partial z_i}(z) F_{\alpha,\delta}(z)\, d\lambda(z), \tag{1.46}$$

and this tends to zero by dominated convergence.

The same type of argument applies for the second order derivatives of $C_\delta(t)$; the claim (1.38) follows.

As $v_{\delta_j} \to w$ in the $\mathscr{C}^\infty_{\mathrm{loc}}(X_t \smallsetminus \mathrm{Supp}(B))$ topology as $j \to +\infty$, it follows from the identity (1.37) above that $w|_{X_t \smallsetminus \mathrm{Supp}(B)}$ is holomorphic. ∎

The next proposition is also important in the analysis of the uniformity properties of $(v_\delta)_{\delta>0}$.

Proposition 1.11. *Let $t \in \Delta$ be fixed. Then*

$$\lim_{\delta\to 0}\left(c(\tau_\delta) - \int_{X_t} c(\tau_\delta)\tau_\delta^n\right) = 0 \tag{1.47}$$

on $X_t \smallsetminus \mathrm{Supp}(B)$.

Proof. Let $G_\delta : X_t \times X_t \to \mathbb{R}$ be the Green function of (X_t, τ_δ). Let $x \in X_t \smallsetminus \mathrm{Supp}(B)$; by definition, one has

$$c(\tau_\delta)(x) - \int_{X_t} c(\tau_\delta)\tau_\delta^n = \int_{X_t} -\Delta_{\tau_\delta} c(\tau_\delta) \cdot G_\delta(x, \cdot)\tau_\delta^n. \tag{1.48}$$

Clearly, $\mathrm{Vol}(X_t, \tau_\delta) = \int_{X_t} \tau_\delta^n = \int_{X_t} \omega^n = 1$ is independent of δ. Moreover, by (1.31), there exists a constant $C_1 > 0$ independent of δ such that $\mathrm{diam}(X_t, \tau_\delta) \le C_2$. Therefore, it follows from [47, A.2] that

$$G(x, y) \ge -C_2 \tag{1.49}$$

for some $C_2 > 0$ independent of δ. Now recall that $G_\delta(x, y) = \int_0^{+\infty} G_\delta(x, y, s)\, ds$ where

$$G_\delta(x, y, s) \le \begin{cases} C_3 s^{-n} e^{-d_{\tau_\delta}(x,y)/(5s)} & \text{if } 0 < s < 1, \\ C_4 s^{-n} & \text{for any } 0 < s < +\infty, \end{cases}$$

where d_{τ_δ} is the geodesic distance induced by τ_δ on X_t. This follows respectively by [23, Thm. 16] and [47, p. 139] – recall that the Ricci curvature of τ_δ is uniformly bounded below thanks to (1.31). Integrating the above inequalities, one gets

$$G(x, y) \le C_3\, d_{\tau_\delta}(x, y)^{2-2n} \tag{1.50}$$

for some uniform $C_3 > 0$. Let $I_\delta(x) := c(\tau_\delta)(x) - \int_{X_t} c(\tau_\delta)\tau_\delta^n$, and let $C_4 > 0$ be large enough so that $\pm\Theta_\delta \le C_4\omega$. One has successively

$$\begin{aligned}
|I_\delta(x)| &= \left| \int_{X_t} -\Delta_{\tau_\delta} c(\tau_\delta) \cdot (G_\delta(x,\cdot) + C_2)\tau_\delta^n \right| \\
&\le \int_{X_t} \left(\|\bar\partial v_\delta\|^2 + C_4 \Big(\sum_j \frac{\delta^2}{|s_j|^2 + \delta^2} \Big) |v_\delta|_\omega^2 \right) \cdot (G_\delta(x,\cdot) + C_2)\tau_\delta^n \\
&\quad + \int_{X_t} \Big(\sum_j a_j \delta^2 \frac{\sqrt{-1}\,\langle \partial s_j, \partial s_j\rangle (v_\delta, \bar v_\delta)}{(|s_j|^2 + \delta^2)^2} \Big) \cdot (G_\delta(x,\cdot) + C_2)\tau_\delta^n \\
&\le C_5 \int_{X_t} \left(\|\bar\partial v_\delta\|^2 + \Big(\sum_j \frac{\delta^2}{|s_j|^2 + \delta^2} \Big) |v_\delta|_\omega^2 \right) \cdot d_{\tau_\delta}(x,\cdot)^{2-2n} \tau_\delta^n \\
&\quad + \int_{X_t} \Big(\sum_j a_j \delta^2 \frac{\sqrt{-1}\,\langle \partial s_j, \partial s_j\rangle (v_\delta, \bar v_\delta)}{(|s_j|^2 + \delta^2)^2} \Big) \cdot d_{\tau_\delta}(x,\cdot)^{2-2n} \tau_\delta^n
\end{aligned}$$

We claim that the right hand side converges to 0 when $\delta \to 0$, uniformly in x belonging to a fixed compact subset of $X_t \smallsetminus \mathrm{Supp}(B)$. To see this, it is enough to check that for any sequence $\delta_j \to 0$, one has $\lim_{j\to+\infty} I_{\delta_j}(x) = 0$ uniformly in x, up to extracting a subsequence. Thanks to Lemma 1.8, one can assume that v_{δ_j} converges locally smoothly to a holomorphic vector field w on $X_t \smallsetminus \mathrm{Supp}(B)$. Let us pick $\varepsilon > 0$.

By the estimates and observations above, one can find a small neighborhood $U_x \Subset X_t \smallsetminus \mathrm{Supp}(B)$ and a constant $C = C(x) > 0$ such that

(i) $|v_\delta|_\omega^2 \le C$, $\|\bar\partial v_{\delta_j}\|^2 \le \varepsilon$, and $|s_j|^2 \ge C^{-1}$ on U_x for any j;

(ii) $\int_{U_x} d_{\tau_\delta}(z,\cdot)^{2-2n}\tau_\delta^n \le C$;

(iii) $d_{\tau_\delta}(z,w)^{2-2n} \le C$ for any $w \notin U_x$.

The rest of the proof is easy: we split the integral into two pieces, on U_x and its complement.

• On the complement of U_x we use item (iii) so that we can replace the function $d_{\tau_\delta}(x,\cdot)^{2-2n}$ in the inequalities above by a constant independent of δ. The proof of Proposition 1.8 shows that the integral of the remaining terms tends to 0 as $\delta \to 0$.

• On U_x we are "far" from the support of B. Combined with items (i) and (ii) above, this finishes the proof of Proposition 1.11. ∎

In fact, Proposition 1.11 shows that the limit (1.47) is uniform on compact sets contained in the complement of the divisor B. We intend to couple this with the elliptic equation satisfied by $c(\tau_\delta)$ in order to obtain bounds for the derivatives of this function in fiber directions. To this end, we need the following statement.

Proposition 1.12. *There exists a constant $C > 0$ independent of $\delta > 0$ such that*

$$\left| \int_{X_t} c(\tau_\delta)\tau_\delta^n \right| \le C.$$

Proof. This statement can be seen as a by-product of the considerations in [28, (5.3) & Prop. 5.4]. Therefore we will content ourselves with highlighting the main steps.

To start with, we recall that the normalization of u_δ is

$$\int_{X_t} u_\delta \frac{(\Omega_y \wedge \overline{\Omega}_y)^{1/m}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{a_j}} = 0, \tag{1.51}$$

and this can be rewritten as

$$\int_{X_t} u_\delta e^{F_\delta} \omega_\delta^n = 0 \tag{1.52}$$

where ω_δ is a metric with conic singularities on X, whose multiplicities along the components of B are $1 > b_j \geq \max(a_j, 1/2)$ (notations as in (1.31)). Note that F_δ in (1.52) has an explicit expression, being the log of $\tau_\delta^n / \omega_\delta^n$.

Let V_δ be the horizontal lift of $\frac{\partial}{\partial t}$ with respect to ω_δ. By applying the $\frac{\partial^2}{\partial t \partial \bar{t}}$ operator in (1.52) we obtain

$$\begin{aligned} \int_{X_t} & V_\delta(\overline{V}_\delta(u_\delta)) e^{F_\delta} \omega_\delta^n \\ &= -\int_{X_t} V_\delta(u_\delta) \overline{V}_\delta(F_\delta) e^{F_\delta} \omega_\delta^n - \int_{X_t} \overline{V}_\delta(u_\delta) V_\delta(F_\delta) e^{F_\delta} \omega_\delta^n \\ &\quad - \int_{X_t} u_\delta V_\delta(\overline{V}_\delta(F_\delta)) e^{F_\delta} \omega_\delta^n - \int_{X_t} u_\delta |V_\delta(F_\delta)|^2 e^{F_\delta} \omega_\delta^n. \end{aligned} \tag{1.53}$$

Now the point is that, up to terms for which we have a uniform estimate already, the function $V_\delta(\overline{V}_\delta(u_\delta))$ is "the same" as $c(\tau_\delta)$. Hence the absolute value of the left hand side of (1.53) is equivalent to $|\int_{X_t} c(\tau_\delta)\tau_\delta^n|$.

The terms on the right hand side of (1.53) are uniformly bounded, as proved in the reference indicated at the beginning of the proof. ∎

We can now prove that the vector field v_ρ is holomorphic when restricted to the fibers of p.

Corollary 1.13. *Let $t \in \Delta$ be fixed. The family $(v_\delta|_{X_t})_{\delta>0}$ converges locally smoothly outside* $\mathrm{Supp}(B)$ *to the lift v of $\frac{\partial}{\partial t}$ with respect to $\rho|_{X^\circ \setminus \mathrm{Supp}(B)}$. In particular, $v|_{X_t \setminus \mathrm{Supp}(B)}$ is holomorphic.*

Proof. Combining Propositions 1.11 and 1.12, one sees that $c(\tau_\delta)$ is locally uniformly bounded on $X_t \setminus \mathrm{Supp}(B)$. Given the elliptic equation satisfied by $c(\tau_\delta)$, this implies local bounds of any order (in fiber directions).

Let $W \subset X$ be a coordinate open subset of $\mathcal{X}$ such that $W \cap \mathrm{Supp}(B) = \emptyset$. In local coordinates, this implies that

$$\frac{\partial^2 u_\delta}{\partial t \partial \bar{t}} \tag{1.54}$$

is bounded on W by a constant independent of δ. Since we already have at our disposal this type of bound for any other mixed second order derivative of u_δ, we infer that

$$|\Delta'' u_\delta| \le C_W \tag{1.55}$$

where Δ'' is the Laplace operator corresponding to the flat metric on W, and C_W is a constant independent of δ.

This implies that the global function u_δ admits $C^{1,\alpha}$ bounds locally on $\mathcal{X} \smallsetminus \mathrm{Supp}(B)$ for any $\alpha < 1$. By the Arzelà–Ascoli theorem and Lemma 1.5, this implies that u_δ converges to φ in $C^{1;\alpha}_{\mathrm{loc}}(\mathcal{X} \smallsetminus \mathrm{Supp}(B))$. In particular, φ is differentiable in t outside $\mathrm{Supp}(B)$, and on this locus, $\partial_t \varphi_t = \lim \partial_t u_\delta$ in the $\mathscr{C}^\alpha_{\mathrm{loc}}$ topology. Now, (1.32) shows that the convergence actually takes place in $\mathscr{C}^\infty_{\mathrm{loc}}(X_t \smallsetminus \mathrm{Supp}(B))$. In particular, outside $\mathrm{Supp}(B)$, $v_\rho|_{X_t}$ is the smooth limit of $v_\delta|_{X_t}$ as $\delta \to 0$. Corollary 1.13 is now a consequence of Proposition 1.8. ∎

Corollary 1.14. *Let $t \in \Delta$ be fixed. Then $dc(\tau_\delta)|_{X_t}$ converges locally uniformly to 0 on compact subsets of $X_t \smallsetminus \mathrm{Supp}(B)$.*

Proof. Let $K \Subset X_t \smallsetminus \mathrm{Supp}(B)$. By the proof of Corollary 1.13 and given (1.29), $c(\tau_\delta)|_K$ is bounded in L^∞ norm, hence in any $\mathscr{C}^k_{\mathrm{loc}}$ norm on K. This implies that the family $dc(\tau_\delta)|_K$ is relatively compact in the smooth topology, and the conclusion follows from Proposition 1.11. ∎

Lemma 1.15. *The vector field v on $\mathcal{X} \smallsetminus \mathrm{Supp}(B)$ is holomorphic and extends across* $\mathrm{Supp}(B)$.

Proof. This first assertion follows from a simple computation in [3, Lem. 2.5]. In our setting this yields, on $X_t \smallsetminus \mathrm{Supp}(B_t)$,

$$\bar\partial_t v_\delta \lrcorner \tau_\delta = \bar\partial c(\tau_\delta) - \sqrt{-1}\, \tau_\delta(\bar\partial v_\delta, \bar v_\delta). \tag{1.56}$$

Since on $X_t \smallsetminus \mathrm{Supp}(B_t)$, τ_δ and v_δ converge locally smoothly to ρ and v respectively, one deduces from Corollary 1.14 above that v is holomorphic (hence smooth) in t outside $\mathrm{Supp}(B)$.

For the second assertion, first observe that ${\tau_\delta}^n \wedge \sqrt{-1}\, dt \wedge d\bar t$ dominates a smooth volume form dV on $\mathcal{X}$. Therefore, it follows from (1.33) that

$$\int_{p^{-1}(U) \smallsetminus \mathrm{Supp}(B)} |v_\delta|^2_\omega \, dV \le C.$$

An application of the Fatou lemma gives

$$\int_{p^{-1}(U) \smallsetminus \mathrm{Supp}(B)} |v|^2_\omega \, dV < +\infty.$$

By the Hartogs theorem, v extends to a holomorphic vector field across $\mathrm{Supp}(B)$. ∎

Lemma 1.16. *The vector field v preserves ρ, hence its flow preserves B.*

Proof. On $\mathcal{X} \smallsetminus \mathrm{Supp}(B)$, we obtain

$$\mathcal{L}_v \rho = 0 \tag{1.57}$$

as a consequence of (1.56).

We next show that (1.57) extends in the sense of currents on $\mathcal{X}$. Indeed, if so then we claim that the flow of v produces biholomorphic maps $F_t : X_0 \to X_t$ such that F_0 is the identity and $F_t^* \omega_t = \omega_0$. It is for this equality that we need (1.57) to hold on $\mathcal{X}$ in the sense of currents: it gives

$$\frac{d}{dt} F_t^* \omega_t = 0 \tag{1.58}$$

in the weak sense on $\mathcal{X}$, but this is enough to conclude that $F_t^* \omega_t = \omega_0$.

If one pulls back the Kähler–Einstein equation satisfied by ω_t by F_t, one gets

$$\mathrm{Ric}\, F_t^* \omega_t = -F_t^* \omega_t + F_t^*[B_t]$$

where $[B_t] = \sum_k a_k [B_{t,k}]$ and $B_{t,k}$ are the irreducible components of $\mathrm{Supp}(B)$. Because $F_t^* \omega_t = \omega_0$, we obtain

$$F_t^*[B_t] = [B_0].$$

In particular, the local flow of v preserves $\mathrm{Supp}(B)$.

Let us now prove that $v \lrcorner \rho$ is zero on $\mathcal{X}$. First, observe that ρ being a positive current, its coefficients are locally defined complex measures. We claim that these measures put no mass on $\mathrm{Supp}(B)$.

Indeed, by e.g. [24, Prop. 1.14] the "mixed terms" of ρ are dominated by the trace of ρ (the sum of the diagonal coefficients). Therefore everything boils down to showing that if ω is a given smooth Kähler form on $\mathcal{X}$, then the positive measure $\rho \wedge \omega^n$ does not charge $\mathrm{Supp}(B)$. But it is easy to produce a family of cut-off functions χ_δ such that χ_δ tends to the characteristic function of $\mathrm{Supp}(B)$, and $\|\nabla\omega\chi_\delta\|_{L^2(\omega^{n+1})}$ and $\|\Delta_\omega \chi_\delta\|_{L^1(\omega^{n+1})}$ tend to 0. We refer e.g. to [15, §9] for this classical construction. Finally, let η be a smooth positive function with compact support on $\mathcal{X}$. One can assume that on $\mathrm{Supp}(\eta)$, $\rho = dd^c\psi$ admits a local (bounded) potential. Performing an integration by parts, one obtains

$$\begin{aligned}
\int_{\mathcal{X}} \eta \chi_\delta \rho \wedge \omega^n &= \int_{\mathcal{X}} \eta \chi_\delta \, dd^c\psi \wedge \omega^n \\
&= \int_{\mathcal{X}} \eta\psi \, dd^c \chi_\delta \wedge \omega^n + \int_{\mathcal{X}} \chi\psi \, dd^c\eta \wedge \omega^n + \int_{\mathcal{X}} \psi \, d\eta \wedge d^c\chi_\delta \wedge \omega^n \\
&\leq \|\psi\|_\infty \Big(\|\eta\|_\infty \cdot \|\Delta_\omega \chi_\delta\|_{L^1} + \|\Delta\eta\|_\infty \int_{\mathrm{Supp}(\chi_\delta)} \omega^{n+1} + \|\nabla\eta\|_{L^2} \cdot \|\nabla\chi_\delta\|_{L^2} \Big),
\end{aligned}$$

which tends to 0.

In conclusion, the coefficients of ρ and hence those of $v \lrcorner \rho$ are complex measures which do not charge B. As $v \lrcorner \rho = 0$ outside $\mathrm{Supp}(B)$, this identity extends across $\mathrm{Supp}(B)$, which is what we wanted to prove. ∎

If we sum up the results obtained so far, we can find near any $y \in Y^\circ$ a sufficiently small polydisk $U \subset Y^\circ$ with coordinates $(t_1, \dots, t_m)$ centered around y as well as holomorphic vector fields $v_1, \dots, v_m$ on $p^{-1}(U)$ lifting $\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_m}$ which are tangent to $\mathrm{Supp}(B)$. Up to shrinking U, one can assume that the flow of the vector fields $v_{\underline{a}} := \sum a_i v_i$ for $\underline{a} = (a_1, \dots, a_m) \in \mathbb{D}^m$ exists at least up to time 1. Here $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Then one has a holomorphic map $f : X_y \times \mathbb{D}^m \to p^{-1}(U)$ which sends $(x, \underline{a})$ to $\phi_1^{\underline{a}}(x)$ where $(\phi_t^{\underline{a}})_t$ is the flow of $v_{\underline{a}}$. It is easy to see that f is an isomorphism onto its image (see e.g. [39]).

To conclude the proof of Theorem 1.2, we need to show that v_ρ extends across the singular locus of p provided that X is compact and p is smooth in codimension 1. The argument goes as follows.

End of the proof of Theorem 1.2. Let n be the relative dimension of p and let $m := \dim Y$. Let $Y^\circ \subset Y$ be the smooth locus of p, and $X^\circ := p^{-1}(Y^\circ)$. Let $\Omega \in H^0(X, m(K_{X/Y} + B))$. Let $\rho = \omega + dd^c\psi$ be the positive current constructed in Theorem 1.1, and pick $y \in Y \smallsetminus Y^\circ$.

Let $x \in X$ be a generic point of $p^{-1}(y)$. Take a small neighborhood U of x, and set $D := p(U)$. As p is smooth in codimension 1, p is smooth on U. We can thus fix a coordinate system $(\underline{t}, z_1, \dots, z_n)$ in U such that $\underline{t}$ represents the horizontal directions and $\frac{\partial}{\partial z_i}$ is in the fiber direction. The notation $\underline{t}$ means that $\underline{t} = (t_1, \dots, t_m)$. There is a slight abuse of notation: the coordinate of the base is also $\underline{t}$. But as p is smooth on U, we just mean that $p_*(\frac{\partial}{\partial t_i}) = \frac{\partial}{\partial t_i}$, where the former is on X and the latter on Y. Finally, we set $p^*(\sqrt{-1}\, d\underline{t} \wedge \overline{d\underline{t}}) := \bigwedge_{k=1}^m \sqrt{-1}\, d\underline{t}_k \wedge \overline{d\underline{t}_k}$.

Let v_k be the holomorphic vector field on $X^\circ \cap p^{-1}(D)$ constructed in the proof of Theorem 1.2, attached to $\frac{\partial}{\partial t_k}$, where $1 \le k \le m$. We have

$$\rho^n \wedge p^*(\sqrt{-1}\, d\underline{t} \wedge \overline{d\underline{t}}) = \frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2} \wedge p^*(\sqrt{-1}\, d\underline{t} \wedge \overline{d\underline{t}}) \quad \text{on } U. \tag{1.59}$$

We know that $\iota_{v_k}\rho$ is proportional to $d\bar{t}_k$, from which it follows that

$$\iota_{v_1,\bar{v}_1} \cdots \iota_{v_m,\bar{v}_m}(\rho^n \wedge p^*(\sqrt{-1}\, d\underline{t} \wedge \overline{d\underline{t}})) = \rho^n. \tag{1.60}$$

Combining (1.59) and (1.60), one gets

$$\iota_{v_1,\bar{v}_1} \cdots \iota_{v_m,\bar{v}_m}\left[\frac{(\Omega \wedge \overline{\Omega})^{1/m}}{|f_B|^2} \wedge p^*(\sqrt{-1}\, d\underline{t} \wedge \overline{d\underline{t}})\right] = \rho^n.$$

One can find a Kähler form ω_X on X such that $\frac{(\Omega\wedge\overline{\Omega})^{1/m}}{|f_B|^2} \wedge p^*(\sqrt{-1}\, d\underline{t} \wedge \overline{d\underline{t}}) \ge \omega_X^{n+m}$. Since $\omega_X^m \wedge [\iota_{v_1,\bar{v}_1} \cdots \iota_{v_m,\bar{v}_m}(\omega_X^{n+m})] = (\prod_k |v_k|^2_{\omega_X}) \cdot \omega_X^{n+m}$ (maybe up to some constant), we eventually get

$$\int_{U\cap X^\circ} \Big(\prod_{k=1}^m |v_k|^2_{\omega_X}\Big) \cdot \omega_X^{n+m} \le \int_{U\cap X^\circ} \rho^n \wedge \omega_X^m$$

and the right hand side is finite, dominated by $\int_X \langle \rho^n \wedge \omega_X^m \rangle \le \{\omega\}^n \cdot \{\omega_X\}^m$ by [9, Prop. 1.6 & 1.20], because ρ is a closed, positive current on X in the cohomology class $\{\omega\}$.

As $|v_k|^2_{\omega_X}$ is uniformly bounded below by a positive constant on $p^{-1}(D) \cap X^\circ$, one deduces that $v_k \in L^2(p^{-1}(D) \cap X^\circ, \omega_X)$. By the Riemann extension theorem the holomorphic vector fields v_k extend to holomorphic vector fields on $p^{-1}(D)$ whose flow provides the expected trivialization. Indeed, the v_k are tangent to B on X°, hence they are tangent to B everywhere by the assumptions in Theorem 1.2. ■

As an application of Theorem 1.2 we can prove Corollary B.

Proof of Corollary B. Our proof follows the same line of arguments as in [34].

To reach a contradiction, assume that $\mathcal{F}_m$ is not big. In any case, this bundle can be endowed with a metric (used several times in the current subsection) with semipositive curvature form denoted by θ, and smooth on a Zariski open subset $V \subset Y$ as B is generically transverse to the fibers. Then we claim that

$$\theta|_V^{\dim(Y)} = 0 \tag{1.61}$$

at each point of V. Indeed, if (1.61) is not true, then there exists a point $y_0 \in V$ such that all the eigenvalues of θ_{y_0} are strictly positive. By the singular version of holomorphic Morse inequalities [8, Cor. 3.3] this implies that $\mathcal{F}_m$ is big, and we have assumed that this is not the case.

It follows that the kernel of θ is non-trivial at each point of V. Since $\theta|_V$ is smooth and closed, locally near each point of V its kernel defines a foliation whose leaves are analytic sets (see [34] and the references therein). We choose a smooth holomorphic disk Δ contained in such a leaf; the restriction of p to $p^{-1}(\Delta) := X_\Delta$ is a submersion, and the curvature of the direct image of the relative pluricanonical bundle is identically zero. By Theorem 1.2 the vector v_ρ is holomorphic. On the other hand, $\bar\partial v_\rho$ is a representative of the image of the tangent vector $\frac{\partial}{\partial t} \in T_\Delta$ by the map (0.1). Since by hypothesis this map is injective, we obtain a contradiction. ■

Proof of Corollary 1.3. Statement 1.8 is a direct consequence of [28] applied to the right hand side of the equality

$$-p^*(K_Y) = K_{X/Y} + (-K_X - B) + B. \tag{1.62}$$

By hypothesis the class $-c_1(K_X + B)$ is in the closure of the Kähler cone of X and one can use *loc. cit.*

Given Theorem 1.2, it is enough to prove that p is smooth in codimension 1. We use the following elegant argument due to Q. Zhang [59]. Assume that there exists some codimension 1 subvariety $D \subset X$ such that $p_*(D)$ is of codimension at least 2. Let $\tau : Y' \to Y$ be the composition of the blow-up of the closed analytic set $p_*(D)$ with a resolution of singularities of the resulting complex space. There exists an effective divisor $E_{Y'}$ whose support is contained in the τ-exceptional locus such that

$$K_{Y'} \sim E_{Y'}.$$

Let $p' : X' \to Y'$ be a resolution of indeterminacies of $X \dashrightarrow Y'$. As $c_1(K_X + B) = 0$, we have

$$(p')^*(-K_{Y'}) + E_{X'} \equiv_{\mathbb{Q}} K_{X'/Y'} + B',$$

where $E_{X'}$ is supported in the exceptional locus of $\pi : X' \to X$. By [28], $K_{X'/Y'} + B'$ is pseudo-effective. Therefore the direct image $\pi_*((p')^*(-K_{Y'}) + E_{X'}) = \pi_*(-E_{Y'})$ is pseudo-effective as well. However, by construction we have $\pi_*(E_{Y'}) \geq [D]$, and we obtain a contradiction.

We next prove that the map p is reduced in codimension 1. Let $E \subset Y$ be a divisor. Its p-inverse image can be written as

$$p^{-1}(E) = \sum_i a_i [D_i]$$

where $D_i \subset X$ are irreducible divisors. It is well known (see [20, Thm. 2.4] or [48]) that

$$K_{X/Y} + B \geq \sum_i (a_i - 1)_+ \cdot [D_i],$$

where $(a_i - 1)_+ := \max\{a_i - 1, 0\}$.

Therefore we must have $a_i = 1$ for every i, since by assumption $K_{X/Y} + B \equiv_{\mathbb{Q}} 0$. Corollary 1.3 is proved. ■

1.5. Log abundance in the Kähler setting

In this section, we briefly explain how to prove the log abundance for klt Kähler pairs (X, B) such that B has snc support. This is based on the following lemma, which is a consequence of [13] and [54, Cor. 1.4] (cf. also [16, Lem. 1.1] and [17] and the references therein). For the reader's convenience, we recall briefly the proof.[1]

After this paper was written, J. Wang [55, Thm. D] proved a slightly more general case of Corollary 1.18 below using similar arguments.

Lemma 1.17. *Let X be a compact Kähler manifold and let $\Delta = \sum a_i B_i$ be an effective klt $\mathbb{Q}$-divisor with simple normal crossing support. Assume that $\Delta \sim_{\mathbb{Q}} L_1$ for some $L_1 \in$* Pic(X). *For each integer $k \geq 0$, define $L_k := kL_1 - \lfloor k\Delta \rfloor$. Then for each k, i and q, the set*

$$V_i^q(L_k) = \{\lambda \in \mathrm{Pic}^\circ(X);\ h^q(X, K_X + L_k + \lambda) \geq i\}$$

is a finite union of translates of complex subtori of Pic$^\circ(X)$ *by torsion points.*

[1] We thank Botong Wang for telling us the following nice application of his result.

Proof. Let N be the minimal number such that $N \cdot a_i \in \mathbb{N}$ for every i. Let $\sigma : \widetilde{X} \to X$ be the N-cyclic cover of L_1 along the canonical section of NL_1. One can check that $\widetilde{X}$ has analytic quotient singularities [52, Lem. 2], hence rational singularities by e.g. [14, Prop. 4.1]. This implies in turn that for any resolution $\pi : \widehat{X} \to \widetilde{X}$, one has $\pi_* \mathcal{O}_{\widehat{X}}(K_{\widehat{X}}) = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ thanks to e.g. [35, Thm. 5.10] and $R^i \pi_* \mathcal{O}_{\widehat{X}}(K_{\widehat{X}}) = 0$ for $i > 0$ by Grauert–Riemenschneider vanishing. Moreover, one has $\sigma_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \mathcal{O}_X(K_X) \otimes \bigoplus_{k=0}^{N-1} L_k$ and $R^i \sigma_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = 0$ for $i > 0$ since σ is finite. Therefore, if we define $f := \sigma \circ \pi : \widehat{X} \to X$, we have

$$H^q(\widehat{X}, K_{\widehat{X}} + f^*\lambda) \simeq \bigoplus_{k=0}^{N-1} H^q(X, K_X + L_k + \lambda) \tag{1.63}$$

for any line bundle λ on X.

Let $g : \mathrm{Pic}^\circ(X) \to \mathrm{Pic}^\circ(\widehat{X})$ be the natural morphism induced by f and set

$$\begin{aligned} V_i^q(f) &:= \{\rho \in \mathrm{Pic}^\circ(X);\ h^q(\widehat{X}, K_{\widehat{X}} + f^*\rho) \geq i\}, \\ V_i^q &:= \{\rho \in \mathrm{Pic}^\circ(\widehat{X});\ h^q(\widehat{X}, K_{\widehat{X}} + \rho) \geq i\}. \end{aligned}$$

Then

$$V_i^q(f) = g^{-1}(V_i^q). \tag{1.64}$$

Thanks to [54], V_i^q is a finite union of torsion translates of complex subtori of $\mathrm{Pic}^\circ(\widehat{X})$. Together with (1.64), this shows that $V_i^q(f)$ has the same structure. Thanks to (1.63), we have

$$V_i^q(f) = \bigcup_{i_0+\cdots+i_{N-1}=i} \ \bigcap_{k=0}^{N-1} V_{i_k}^q(L_k), \tag{1.65}$$

where $V_i^q(L_k) := \{\rho \in \mathrm{Pic}^\circ(X);\ h^q(X, K_X + L_k + \rho) \geq i\}$. As $V_i^q(f)$ is the finite union of torsion translates of complex subtori, we infer from (1.65) that $V_i^q(L_k)$ has the same structure [16, Lem. 1.1]. ∎

Corollary 1.18. *Let (X, Δ) be a klt pair where X is compact Kähler and $\Delta = \sum a_i B_i$ is an effective $\mathbb{Q}$-divisor. If $c_1(K_X + \Delta) = 0 \in H^{1,1}(X, \mathbb{Q})$, then $K_X + \Delta$ is $\mathbb{Q}$-effective.*

Proof. Let $\pi : X' \to X$ be a log resolution of (X, Δ). Since $\mathrm{Pic}^\circ(X')$ is a torus and $c_1(K_X + \Delta) = 0$, we can find $L \in \mathrm{Pic}^\circ(X')$ such that $\pi^*(K_X + \Delta) \sim_{\mathbb{Q}} L$. We can also find a klt divisor Δ' on X' with normal crossing support such that

$$K_{X'} + \Delta' \equiv_{\mathbb{Q}} \pi^*(K_X + \Delta) + E$$

for some $\mathbb{Q}$-effective divisor E supported in the exceptional locus of π having no common component with Δ'. Let $m \geq 1$ be the smallest integer such that mE has integral coefficients. In particular, $m(K_{X'} + \Delta')$ is equivalent to some line bundle on X' by the formula above. Using the identity

$$m(K_{X'} + \Delta') = K_{X'} + \underbrace{\left(\Delta' + \frac{m-1}{m}\{E\}\right)}_{=:\Delta^+} + (m-1)(L + \lfloor E \rfloor)$$

we get a pair (X', Δ^+) such that

- Δ^+ has snc support and coefficients in $(0, 1) \cap \mathbb{Q}$,
- $\Delta^+ \sim M$ for some line bundle M on X',
- $K_{X'} + \Delta^+ + \rho$ is effective for some $\rho \in \mathrm{Pic}^\circ(X')$.

The first two properties are obvious, and the third follows from the identity $K_{X'} + \Delta^+ - L = mE - (m-1)\lfloor E \rfloor$. By applying Lemma 1.17 to $K_{X'} + \Delta^+$, we can assume that ρ is torsion, hence $h^0(X', r(K_{X'} + \Delta^+)) \geq 1$ for some integer $r \geq 1$ that we can choose so that $m \mid r$. By doing so, one can ensure that $r(K_{X'} + \Delta^+) = \pi^*(r(K_X + \Delta)) + F$ for some effective, integral π-exceptional divisor F. This implies that $h^0(X, r(K_X + \Delta)) \neq 0$. The corollary is proved. ∎

2. Transverse regularity of singular Monge–Ampère equations

In this section our main goal is to prove Theorem C. This will be achieved as a consequence of a few intermediate results which we state in a general setting.

The main source of difficulties in the proof of Theorem C is the fact that the set of basepoints of pluricanonical sections may be non-empty. The determinant of the metric adapted to this geometric setting *vanishes* along the said basepoints so in particular the Ricci curvature of this metric is not bounded from below. Unfortunately, under these circumstances we were not able to obtain a complete analogue of the Sobolev and Poincaré inequalities (which are needed for the study of the regularity properties of Monge–Ampère equations). We will therefore start this section with a weak version of these results.

2.1. *Weak Sobolev and Poincaré inequalities*

In this section we will derive a version of the usual Poincaré and Sobolev type inequalities which are needed in our context. As is well known, they play a crucial role in the regularity questions for the Monge–Ampère equations. The set-up is as follows: Let (X, ω) be a compact Kähler manifold of dimension n, and let

$$E := \sum_{\alpha \in I} e_\alpha E_\alpha, \qquad B := \sum_{\beta \in J} b_\beta B_\beta \tag{2.1}$$

be two effective divisors on X without common components such that $e_\alpha \in \mathbb{Q}_+$, $b_\beta \in [0, 1[$ and the support of $E + B$ is snc. We assume that the manifold X is covered by a fixed family $(\Omega_j)_j$ of coordinate sets such that

$$\Omega_j \cap \mathrm{Supp}(E + B) = (z_j^1 \cdots z_j^d = 0) \tag{2.2}$$

where (z_j) are coordinates on Ω_j.

Let σ_i, s_i be the canonical sections of the hermitian bundle $(\mathcal{O}(E_i), h_i)$ and $(\mathcal{O}(B_i), g_i)$ respectively, where h_i and g_i are non-singular reference metrics. For each positive $\varepsilon \geq 0$ and each multi-index q we introduce the volume element

$$d\mu_q^{(\varepsilon)} := \frac{\prod_{\alpha \in I} (\varepsilon^2 + |\sigma_\alpha|^2)^{q_\alpha}}{\prod_{\beta \in J} (\varepsilon^2 + |s_\beta|^2)^{b_\beta}} \, dV_\omega \tag{2.3}$$

where dV_ω is the volume element corresponding to the reference metric ω. Also, for each positive real number $p \leq 2$ we define the multi-index q_p whose components are

$$(1 - p/2)q_\alpha. \tag{2.4}$$

Then we have the following statements.

Proposition 2.1. *There exists a constant $C > 0$ independent of ε (but depending on everything else) such that for every smooth function f on X we have*

$$\left(\int_X |f|^{\frac{2np}{2n-p}} \, d\mu_q^{(\varepsilon)} \right)^{\frac{2n-p}{2np}} \leq C \left(\int_X |\nabla_\varepsilon f|^p \, d\mu_{q_p}^{(\varepsilon)} + \int_X |f|^p \, d\mu_{q_p}^{(\varepsilon)} \right)^{1/p} \tag{2.5}$$

where $1 \leq p < 2$ is a real number, and the gradient ∇_ε corresponds to the ε-regularization of a fixed metric with conic singularities along the divisor $\sum_{\beta \in J} b_\beta B_\beta$.

As we can see, there is an important difference between Proposition 2.1 and the standard *weighted Sobolev inequalities*: the volume element on the left hand side of (2.5) is not the same as the one on the right hand side.

In a similar vein, we have the next version of the Poincaré inequality.

Proposition 2.2. *There exists a constant $C > 0$ as above such that for any smooth function f on X we have*

$$\int_X |f - \mathrm{VM}_\mu(f)|^p \, d\mu_q^{(\varepsilon)} \leq C \int_X |\nabla_\varepsilon f|^p \, d\mu_{q_p}^{(\varepsilon)} \tag{2.6}$$

where $p \geq 1$ is a real number, and where we use the notation

$$\mathrm{VM}_\mu(f) := \int_X f \, d\mu_q^{(\varepsilon)}. \tag{2.7}$$

We first prove Proposition 2.1; the arguments to follow have been borrowed from the book [31, Chap. 15].

Proof of Proposition 2.1. We first assume that $B = 0$ because the arguments for the general case are practically identical.

A first remark is that it is enough to consider the local version of the statement, as follows. Let Ω be one of the domains covering (X, E) as in (2.2); we denote by $(z_1, \ldots, z_n)$ the corresponding coordinate system. We will assume that

$$\Omega = \prod_j (|z_j| < 1) \tag{2.8}$$

and that the function f has compact support in Ω.

In terms of this local setting, the quantity to be evaluated becomes

$$\int_{\Omega} |f|^{\frac{2np}{2n-p}} \prod_{\alpha=1}^{d} (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha} \, d\lambda \tag{2.9}$$

(since $b_i = 0$). Let $\mathbb{D} := (|t| < 1) \subset \mathbb{C}$ be the unit disk in the complex plane. We consider the function

$$F_\varepsilon(t) = \frac{(\varepsilon^2 + |t|^2)^{q/2}}{(1 + \varepsilon^2)^{q/2}} t \tag{2.10}$$

where $q > 0$ is a real number and $t \in \mathbb{D}$. It turns out that F_ε is a diffeomorphism and the square of the absolute value of its Jacobian $dF_\varepsilon \wedge d\overline{F}_\varepsilon$ satisfies the inequality

$$C^{-1}(\varepsilon^2 + |t|^2)^q \leq \frac{dF_\varepsilon \wedge d\overline{F}_\varepsilon}{dt \wedge d\bar{t}} \leq C(\varepsilon^2 + |t|^2)^q \tag{2.11}$$

where C is a constant independent of ε (it can be explicitly computed). Let G_ε be the inverse of F_ε. The implicit function theorem shows that

$$|dG_\varepsilon(t)| \leq \frac{C}{(\varepsilon^2 + |t|^2)^{q/2}}. \tag{2.12}$$

By the change of variables formula we have

$$\int_{\Omega} |f(z)|^{\frac{2np}{2n-p}} \prod_{\alpha=1}^{d} (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha} \, d\lambda \leq C \int_{\Omega} |\widetilde{f}(w)|^{\frac{2np}{2n-p}} \, d\lambda(w) \tag{2.13}$$

where by definition

$$\widetilde{f}(w) := f(G_\varepsilon(w_1), \ldots, G_\varepsilon(w_d), w_{d+1}, \ldots, w_n); \tag{2.14}$$

it is a function defined on the "same" polydisk Ω, and it has compact support.

Therefore, by the usual version of the Sobolev inequality we obtain

$$\left(\int_{\Omega} |\widetilde{f}(w)|^{\frac{2np}{2n-p}} \, d\lambda(w) \right)^{\frac{2n-p}{2n}} \leq C \int_{\Omega} |\nabla \widetilde{f}(w)|^p \, d\lambda(w). \tag{2.15}$$

We use (2.14) together with the change of coordinates $w_\alpha = F_\varepsilon(z_\alpha)$ for $\alpha = 1, \ldots, d$ to infer that

$$\int_{\Omega} |\nabla \widetilde{f}(w)|^p \, d\lambda(w) \leq C \int_{\Omega} |\nabla f(z)|^p \prod_{\alpha=1}^{d} (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha(1-p/2)} \, d\lambda. \tag{2.16}$$

In conclusion we have

$$\begin{aligned} \left(\int_{\Omega} |f(z)|^{\frac{2np}{2n-p}} \prod_{\alpha=1}^{d} (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha} \, d\lambda \right)^{\frac{2n-p}{2n}} & \\ \leq C \int_{\Omega} |\nabla f(z)|^p \prod_{\alpha=1}^{d} (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha(1-p/2)} \, d\lambda, & \end{aligned} \tag{2.17}$$

that is, we have established the local version of the inequality of Proposition 2.1. The general case follows by a partition of unity argument which we skip. ■

The same scheme of proof applies to Proposition 2.2: we will first show that the local version of the statement holds by using a change of coordinates and the classical version of the Poincaré inequality, and then we show that the global version (2.6) is true by applying a well-chosen covering of X.

Proof of Proposition 2.1. The inequality (2.6) is easily seen to follow provided that we are able to establish that

$$\int_{X\times X} |f(x)-f(y)|^p\, d\mu_q^{(\varepsilon)}(x)\, d\mu_q^{(\varepsilon)}(y) \le C\int_X |\nabla f|^p\, d\mu_{q_p}^{(\varepsilon)} \tag{2.18}$$

for any $1\le p\le 2$. This is very elementary and we will not provide any additional explanation.

Assume that we have a covering

$$X=\bigcup_i U_i \tag{2.19}$$

where each U_i is a coordinate open set. In order to obtain a bound as in (2.18), it would be enough to analyze the quantities

$$\int_{U_i\times U_j} |f(x)-f(y)|^p\, d\mu_q^{(\varepsilon)}(x)\, d\mu_q^{(\varepsilon)}(y) \tag{2.20}$$

for each couple of indices i, j, which we do next.

To start with, let Ω be one of the coordinate sets U_i; we will show that the following local version of (2.18) holds true:

$$\int_{\Omega\times \Omega} |f(x)-f(y)|^p\, d\mu_q^{(\varepsilon)}(x)\, d\mu_q^{(\varepsilon)}(y) \le C\int_\Omega |\nabla f|^p\, d\mu_{q_\alpha}^{(\varepsilon)}. \tag{2.21}$$

We proceed as in the previous proof: we have

$$\int_{\Omega\times \Omega} |f(x)-f(y)|^p\, d\mu_q^{(\varepsilon)}(x)\, d\mu_q^{(\varepsilon)}(y) \le C\int_{\Omega\times\Omega} |\widetilde{f}(z)-\widetilde{f}(w)|^p\, d\lambda(z,w) \tag{2.22}$$

by a change of coordinates as indicated in (2.10). Now we have

$$\widetilde{f}(z)-\widetilde{f}(w)=\int_0^1 \frac{d}{dt}\widetilde{f}((1-t)z+tw))\, dt \tag{2.23}$$

and it follows that

$$\begin{aligned}\int_{\Omega\times\Omega} &|\widetilde{f}(z)-\widetilde{f}(w)|^p\, d\lambda(z,w)\\ &\le C\int_0^1 dt\int_{\Omega\times\Omega} |\nabla\widetilde{f}((1-t)z+tw))|^p\, d\lambda(z,w),\end{aligned} \tag{2.24}$$

where the constant $C > 0$ in (2.24) depends on the diameter of Ω measured with respect to the Euclidean metric.

Then we invoke the usual trick: we split the integral in two, where the first part is

$$\int_0^{1/2} dt \int_{\Omega\times\Omega} |\nabla \widetilde{f}((1-t)z + tw))|^p \, d\lambda(z,w) \le C \int_\Omega |\nabla \widetilde{f}(z))|^p \, d\lambda(z) \tag{2.25}$$

with C only depending on the volume of Ω, up to a numerical constant. We have a similar estimate for the integral corresponding to the interval [1/2, 1], so all in all

$$\int_{\Omega\times\Omega} |\widetilde{f}(z) - \widetilde{f}(w)|^p \, d\lambda(z,w) \le C \int_\Omega |\nabla \widetilde{f}(z))|^p \, d\lambda(z). \tag{2.26}$$

Changing the coordinates back, the considerations in the proof of the weak Sobolev inequality show that (2.21) is proved.

The general case follows by choosing a covering (U_j) of X such that the following properties are satisfied:

(1) If $U_p \cap U_q \neq \emptyset$ and if at least one of them intersects the support of the divisor E, then the union $U_p \cup U_q$ is contained in a coordinate set endowed with coordinates adapted to (X, E) (as at the beginning of this section).

(2) If $U_p \cap U_q \neq \emptyset$ and if neither U_p nor U_q intersects $\mathrm{Supp}(E)$, then $U_p \cup U_q$ is contained in a coordinate ball disjoint from $\mathrm{Supp}(E)$.

(3) The $d\mu_q^{(\varepsilon)}$-volume of the coordinate sets containing $U_p \cup U_q$ in (1) and (2) is bounded from above and below by constants which are independent of ε.

It is clear that such a cover exists, and we fix one denoted by Λ for the rest of the proof. Note that this cover is independent of ε. Next, given any couple U_i, U_j of sets belonging to Λ, we consider a collection

$$\Xi_{ij} = (\Omega_1, \dots, \Omega_N) \tag{2.27}$$

of elements of Λ such that the following properties are satisfied:

(a) $\Omega_1 = U_i$ and $\Omega_N = U_j$, and all of the intermediate Ω's are elements of Λ.

(b) For any $r = 1, \dots, N-1$ we have $\Omega_r \cap \Omega_{r+1} \neq \emptyset$.

Again, there are many choices for such Ξ_{ij}, but we just pick one of them for each pair of indices (i, j).

We are now ready to analyze the quantities (2.20): for each couple (i, j) we consider the collection Ξ_{ij}. Given

$$(x_1, \dots, x_N) \in \Omega_1 \times \cdots \times \Omega_N \tag{2.28}$$

we have

$$|f(x_1) - f(x_N)|^\alpha \le C \sum_q |f(x_q) - f(x_{q+1})|^\alpha \tag{2.29}$$

for some numerical constant $C > 0$.

We now consider the expression

$$\int_{\Omega_1\times\cdots\times\Omega_N} |f(x_1) - f(x_N)|^p \, d\mu_q^{(\varepsilon)}(x_1)\dots d\mu_q^{(\varepsilon)}(x_N); \tag{2.30}$$

on the one hand, up to a constant this is simply (2.20). On the other hand, (2.30) is bounded from above by

$$C\sum_q \int_{\Omega_1\times\cdots\times\Omega_N} |f(x_q) - f(x_{q+1})|^p \, d\mu_q^{(\varepsilon)}(x_1)\dots d\mu_q^{(\varepsilon)}(x_N) \tag{2.31}$$

The last observation is that each term of the sum (2.31) is (2.21) – here we are using properties (1)–(3) and (a), (b) above – for which we have already shown the desired Poincaré inequality. This ends the proof of the case $B = 0$.

We will not detail the proof of the general statement, because the arguments are identical to the ones already given. The only change is that we work with geodesics with respect to the model conic metric

$$\sqrt{-1}\sum_{\alpha\in J} \frac{dz_\alpha \wedge d\overline{z}_\alpha}{(\varepsilon^2 + |z_\alpha|^2)^{b_\alpha}} + \sqrt{-1}\sum_{\alpha\notin J} dz_\alpha \wedge d\overline{z}_\alpha \tag{2.32}$$

instead of straight lines $(1-t)x + ty$. The same proof works because the Ricci curvature of the metric (2.32) is bounded from below by some constant independent of ε. For a complete treatment of this point we refer to [44, pp. 177–179]. ■

2.2. *Lie derivative of fiberwise Monge–Ampère equations*

In this subsection we consider the restriction of our initial family p to a generic disk contained in the base, together with a family of Monge–Ampère equations of its fibers. Let $\mathbb{D} \subset Y$ be a one-dimensional germ of submanifold contained in a coordinate set of Y, and let $\mathcal{X} := p^{-1}(\mathbb{D})$ (setting as in Theorem C).

The resulting map $p : \mathcal{X} \to \mathbb{D}$ will be a proper submersion, provided that $\mathbb{D}$ is generic. We recall that the total space $(\mathcal{X}, \omega)$ of p is a Kähler manifold. We denote by t a coordinate on the unit disk $\mathbb{D}$, and let

$$v = \frac{\partial}{\partial t} + v^\alpha \frac{\partial}{\partial z^\alpha} \tag{2.33}$$

be the local expression of a smooth vector field which projects into $\frac{\partial}{\partial t}$.

Another piece of data is the fiberwise Monge–Ampère equation

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi + f}\omega^n \tag{2.34}$$

on each $\mathcal{X}_t$. Here $\lambda \geq 0$ is a real number, and f is a smooth function on $\mathcal{X}$. We can write this globally as follows:

$$(\omega + dd^c\varphi)^n \wedge \sqrt{-1}\, dt \wedge d\overline{t} = e^{\lambda\varphi + f}\omega^n \wedge \sqrt{-1}\, dt \wedge d\overline{t} \tag{2.35}$$

on $\mathcal{X}$, where the meaning of dd^c and of φ is not the same as in (2.34), but…

We take the Lie derivative $\mathcal{L}_v$ of (2.35) with respect to the vector field v, and then restrict to a fiber $\mathcal{X}_t$. The Lie derivative of the left hand side of (2.35) equals

$$n\mathcal{L}_v(\omega + dd^c\varphi) \wedge (\omega + dd^c\varphi)^{n-1} \wedge \sqrt{-1}\, dt \wedge d\bar{t} \tag{2.36}$$

because $\mathcal{L}_v(\sqrt{-1}\, dt \wedge d\bar{t}) = 0$, given the expression (2.33).

The form $\omega + dd^c\varphi$ is closed, hence by the Cartan formula we have

$$\mathcal{L}_v(\omega + dd^c\varphi) = d(i_v \cdot (\omega + dd^c\varphi)) \tag{2.37}$$

where $i_v \cdot \omega$ is the contraction of ω with respect to the vector field v. We next evaluate the quantity

$$d(i_v \cdot dd^c\varphi) \wedge \sqrt{-1}\, dt \wedge d\bar{t} \tag{2.38}$$

by a pointwise computation. In local coordinates as in (2.33), we write

$$\begin{aligned} dd^c\varphi = {} & \varphi_{t\bar{t}}\sqrt{-1}\, dt \wedge d\bar{t} + \varphi_{t\bar{\alpha}}\sqrt{-1}\, dt \wedge dz^{\overline{\alpha}} + \varphi_{\beta\bar{t}}\sqrt{-1}\, dz^{\beta} \wedge d\bar{t} \\ & + \varphi_{\beta\bar{\alpha}}\sqrt{-1}\, dz^{\beta} \wedge dz^{\overline{\alpha}}; \end{aligned} \tag{2.39}$$

in the expression above we are using the Einstein convention. Then we have

$$d(i_v \cdot dd^c\varphi) \equiv (\varphi_{t\beta\bar{\alpha}} + \varphi_{\gamma\beta\bar{\alpha}}v^{\gamma} + \varphi_{\gamma\bar{\alpha}}v^{\gamma}_{\beta})dz^{\beta} \wedge dz^{\overline{\alpha}} \tag{2.40}$$

where $\equiv$ means that we are only considering the terms of type $(1, 1)$ which do not contain dt or its conjugate.

On the other hand, the coefficients of the Hessian of the function

$$v(\varphi) = \varphi_t + \varphi_{\gamma}v^{\gamma} \tag{2.41}$$

in the fiber direction are

$$v(\varphi)_{\beta\bar{\alpha}} = \varphi_{t\beta\bar{\alpha}} + \varphi_{\gamma\beta\bar{\alpha}}v^{\gamma} + \varphi_{\gamma\bar{\alpha}}v^{\gamma}_{\beta} + \varphi_{\gamma\beta}v^{\gamma}_{\bar{\alpha}} + \varphi_{\gamma}v^{\gamma}_{\beta\bar{\alpha}}. \tag{2.42}$$

The first three terms in the expression (2.42) are identical to those in (2.40). The last two terms can be expressed intrinsically as follows:

$$(\varphi_{\gamma\beta}v^{\gamma}_{\bar{\alpha}} + \varphi_{\gamma}v^{\gamma}_{\beta\bar{\alpha}})dz^{\beta} \wedge dz^{\overline{\alpha}} = \partial(\bar{\partial}v \cdot \varphi). \tag{2.43}$$

Here $\bar{\partial}v$ is a $(0, 1)$-form with values in $T_{\mathcal{X}_t}$ and so $\bar{\partial}v \cdot \varphi$ is a $(0, 1)$-form on $\mathcal{X}_t$.

On the other hand, if we denote by $\Delta_{\varphi} = \mathrm{Tr}_{\varphi}\sqrt{-1}\,\bar{\partial}$ the Laplace operator corresponding to the metric $\omega_{\varphi} := \omega + dd^c\varphi$ on the fibers of p, then we can rewrite equation (2.37) as

$$(\Delta_{\varphi}v(\varphi) - \mathrm{Tr}_{\varphi}\partial(\bar{\partial}v \cdot \varphi) + \Psi_{\varphi,v})\omega_{\varphi}^{n} \wedge \sqrt{-1}\, dt \wedge d\bar{t}. \tag{2.44}$$

Here Tr_{φ} is the trace with respect to ω_{φ} on $\mathcal{X}_t$, and we denote by $\Psi_{\varphi,v}$ the function on $\mathcal{X}$ such that

$$\Psi_{\varphi,v}\, \omega_{\varphi}^{n} \wedge \sqrt{-1}\, dt \wedge d\bar{t} = \mathcal{L}_v(\omega) \wedge \omega_{\varphi}^{n-1} \wedge \sqrt{-1}\, dt \wedge d\bar{t} \quad \text{on } \mathcal{X}. \tag{2.45}$$

As for the right hand side of (2.35), its Lie derivative reads

$$(\lambda v(\varphi) + v(f) + \Psi_v)\omega_\varphi^n \wedge \sqrt{-1}\, dt \wedge d\bar{t} \tag{2.46}$$

where (as before) the function Ψ_v is defined by the equality

$$\Psi_v\, \omega^n \wedge \sqrt{-1}\, dt \wedge d\bar{t} = \mathcal{L}_v(\omega) \wedge \omega^{n-1} \wedge \sqrt{-1}\, dt \wedge d\bar{t}. \tag{2.47}$$

In conclusion, for each $t \in \mathbb{D}$ we obtain

$$\Delta_\varphi v(\varphi) - \operatorname{Tr}_\varphi \partial(\bar{\partial} v \cdot \varphi) + \Psi_{\varphi,v} = \lambda v(\varphi) + v(f) + \Psi_v, \tag{2.48}$$

which is the identity we intended to obtain in this subsection.

2.3. *Regularity in transverse directions*

In this section we will apply the results above in order to analyze the transverse regularity of the solution of the equation

$$(\omega + dd^c\varphi_t)^n = e^{\lambda\varphi + f} \frac{\prod_{i\in I} |\sigma_i|^{2e_i}}{\prod_{j\in J} |s_j|^{2b_j}} \omega^n \tag{2.49}$$

on $\mathcal{X}_t$. Here $\lambda \geq 0$ is a real, and the parameters e_i, b_j are chosen as above. In the case $\lambda = 0$, the normalization we choose for the solution is

$$\int_{\mathcal{X}_t} \varphi_t \omega_{\varphi_t}^n = 0. \tag{2.50}$$

The function f in (2.49) is supposed to be smooth on the total space $\mathcal{X}$.

We consider the family of approximations of (2.49),

$$(\omega + dd^c\varphi_\varepsilon)^n = e^{\lambda\varphi_\varepsilon + f} \frac{\prod_{i\in I} (\varepsilon^2 + |\sigma_i|^2)^{e_i}}{\prod_{j\in J} (\varepsilon^2 + |s_j|^2)^{b_j}} \omega^n \tag{2.51}$$

on $\mathcal{X}_t$. By general results in MA theory, the function φ_ε obtained by glueing the fiberwise solutions of (2.51) is smooth. In the next subsections we will analyze the uniformity with respect to ε of several norms of φ_ε.

We recall the following important result whose origins can be found in [57].

Theorem 2.3. *For any strictly smaller disk $\mathbb{D}' \subset \mathbb{D}$ there exists a constant $C > 0$ such that*

$$\|\varphi_\varepsilon\|_{\mathcal{C}^1(\mathcal{X}_t)} \leq C \tag{2.52}$$

for all $t \in \mathbb{D}'$, where the $\mathcal{C}^1$ norm above is with respect to a fixed metric which is quasi-isometric to (2.32).

If $b_j = 0$, this is a consequence of [57], stating that

$$\omega + dd^c\varphi_\varepsilon \le C\omega|_{\mathcal{X}_t} \tag{2.53}$$

(cf. also the refinement later obtained in [40]). The conic case is much more involved and we refer to Theorem 2.7 and the few lines following that statement. Note that inequality (2.53) is still true provided that we replace the right hand side with $C\omega_{B,\varepsilon}|_{\mathcal{X}_t}$, where $\omega_{B,\varepsilon}$ is the regularization of a conic metric corresponding to (X, B) which is quasi-isometric to (2.32).

During the rest of the current subsection we assume that $\lambda = 0$, which is anyway what we need for the proof of Theorem C. We will explain along the way how to adapt our method to the case $\lambda > 0$.

2.3.1. Mean value of the t-derivative. Let v be a smooth $(1, 0)$-vector field on $\mathcal{X}$, which has the following properties.

(i) It is a lifting of $\frac{\partial}{\partial t}$, i.e.

$$dp(v) = \frac{\partial}{\partial t} \tag{2.54}$$

(with the usual abuse of notation).

(ii) We write v locally as in (2.33); then on Ω_j we have

$$|v^\alpha(z_j)| \le C|z_j^\alpha| \tag{2.55}$$

(we use the notations/conventions as in (2.2)) for all $\alpha = 1, \dots, d$. This means that v is a smooth section of the logarithmic tangent space of $(X, E_{\mathrm{red}} + B_{\mathrm{red}})$.

Such a vector field v is easy to construct by a partition of unity from local lifts of $\frac{\partial}{\partial t}$. We consider the coordinate sets Ω_j and the z_j adapted to the pair $(\mathcal{X}, B + E)$. Then the particular form of the transition implies (ii).

In this context we have the following statement.

Lemma 2.4. *There exists a constant $C > 0$ independent of ε such that*

$$\left| \int_{\mathcal{X}_t} v(\varphi_\varepsilon)\omega^n_{\varphi_\varepsilon} \right| \le C \quad \textit{for any } t \in \mathbb{D}'. \tag{2.56}$$

Proof. We consider a covering of $\mathcal{X}$ by coordinate sets $(U_i, (z_i, t))_i$ where the last coordinate t is given by the map p. The normalization condition (2.50) can be written as

$$\sum_i \int_{\|z_i\|<1} \theta_i(z_i, t)\varphi_\varepsilon(z_i, t) \frac{\prod_{\alpha\in I}(\varepsilon^2 + |z_i^\alpha|^2 e^{\phi_\alpha(z_i,t)})^{e_\alpha}}{\prod_{\beta\in J}(\varepsilon^2 + |z_i^\beta|^2 e^{\psi_\beta(z_i,t)})^{b_\beta}} e^{F_i(z_i,t)}\, d\lambda(z_i) = 0 \tag{2.57}$$

where θ_i is a partition of unity, $I \cap J = \emptyset$ and $e^{F_i(z_i,t)}\, d\lambda(z_i)$ is the volume element ω^n restricted to $\mathcal{X}_t$. We take the t-derivative of (2.57) and obtain

$$\sum_i \int_{\|z_i\|<1} \theta_i(z_i, t) \frac{\partial\varphi_\varepsilon(z_i, t)}{\partial t} \frac{\prod_{\alpha\in I}(\varepsilon^2 + |z_i^\alpha|^2 e^{\phi_\alpha(z_i,t)})^{e_\alpha}}{\prod_{\beta\in J}(\varepsilon^2 + |z_i^\beta|^2 e^{\psi_\beta(z_i,t)})^{b_\beta}} e^{F_i(z_i,t)}\, d\lambda(z_i) = O(1) \tag{2.58}$$

where $O(1)$ is uniform with respect to t, ε by the $\mathcal{C}^0$ estimates for φ_ε. Now by the construction of the vector v above the left hand side of (2.58) is the same as in (2.56), so the lemma follows. ■

2.3.2. L^2 bound of the t-derivative. We rewrite the relation corresponding to (2.48) in our setting; during the computations below, we denote

$$\tau := v(\varphi_\varepsilon) \tag{2.59}$$

and then

$$\begin{aligned}\Delta_{\varphi_\varepsilon}\tau - \operatorname{Tr}_{\varphi_\varepsilon}\partial(\bar\partial v\cdot\varphi_\varepsilon) + \Psi_{\varphi_\varepsilon,v} \\ = \lambda\tau + v(f) + \sum_j e_j v(\log(\varepsilon^2+|\sigma_j|^2)) - \sum_i b_i v(\log(\varepsilon^2+|s_i|^2)) + \Psi_v.\end{aligned} \tag{2.60}$$

This equality will be used in order to establish the following statement.

Proposition 2.5. *There exists a constant $C > 0$ such that*

$$\int_{\mathcal{X}_t} |\nabla_\varepsilon\tau|^2_\varepsilon\,\omega^n_{\varphi_\varepsilon} \le C\left(1 + \int_{\mathcal{X}_t}|\tau|\omega^n_{\varphi_\varepsilon}\right) \tag{2.61}$$

for any $\varepsilon > 0$. The operator ∇_ε is the gradient corresponding to the metric $\omega_{\varphi_\varepsilon}$.

Proof. In order to establish (2.61) we multiply (2.60) with $\overline{\tau}$ and then we integrate the result on $\mathcal{X}_t$ against the measure $\omega^n_{\varphi_\varepsilon}$. A few observations are in order.

- We have

$$\sup_{\mathcal{X}_t}\Big(|v(f)| + \Big|\sum_j e_j v(\log(\varepsilon^2+|\sigma_j|^2))\Big| + \Big|\sum_i b_i v(\log(\varepsilon^2+|s_i|^2))\Big| + |\Psi_v|\Big) \le C \tag{2.62}$$

  uniformly with respect to ε, by property (ii) of the vector field v and the definition (2.47) of the function Ψ_v.
- Since the constant λ is positive, the L^2 norm of $\sqrt{\lambda}\,\tau$ will be on the left hand side of (2.61), hence the presence of a strictly positive λ would reinforce the inequality we want to obtain.

The terms

$$\operatorname{Tr}_{\varphi_\varepsilon}\partial(\bar\partial v\cdot\varphi_\varepsilon), \quad \Psi_{\varphi_\varepsilon,v} \tag{2.63}$$

are somewhat troublesome, because we do not have an L^∞ bound for them. Nevertheless, we recall that we only intend to establish an inequality between L^p norms, and we will use integration by parts to deal with (2.63).

For the first term in (2.63) we argue as follows: integration by parts gives

$$\int_{\mathcal{X}_t} \overline{\tau}\partial(\bar\partial v\cdot\varphi_\varepsilon)\wedge\omega^{n-1}_{\varphi_\varepsilon} = -\int_{\mathcal{X}_t}\partial\overline{\tau}\wedge\bar\partial v\cdot\varphi_\varepsilon\wedge\omega^{n-1}_{\varphi_\varepsilon} \tag{2.64}$$

and then we use Cauchy–Schwarz: the L^2 norm of $\bar\partial\tau$ is what we are after, but on the right hand side we have it squared. The L^2 norm of $\bar\partial v\cdot\varphi_\varepsilon$ is completely under control, because it only involves the fiber-direction derivatives of φ_ε.

The second term is tamed in a similar manner. By definition of $\Psi_{\varphi_\varepsilon,v}$ we have

$$\int_{\mathcal{X}_t}\overline{\tau}\Psi_{\varphi_\varepsilon,v}\,\omega^n_{\varphi_\varepsilon}=\int_{\mathcal{X}_t}\overline{\tau}\mathcal{L}_v(\omega)\wedge\omega^{n-1}_{\varphi_\varepsilon} \tag{2.65}$$

and by the Cartan formula this is equal to

$$\int_{\mathcal{X}_t}\overline{\tau}d(i_v\cdot\omega)\wedge\omega^{n-1}_{\varphi_\varepsilon}=\int_{\mathcal{X}_t}\overline{\tau}\partial(i_v\cdot\omega)\wedge\omega^{n-1}_{\varphi_\varepsilon}. \tag{2.66}$$

By the Stokes formula the right hand side of (2.66) is equal to

$$\int_{\mathcal{X}_t}\partial\overline{\tau}\wedge(i_v\cdot\omega)\wedge\omega^{n-1}_{\varphi_\varepsilon} \tag{2.67}$$

and now things are getting much better, in the sense that the $(0,1)$-form $i_v\cdot\omega$ is clearly smooth, so its L^2 norm with respect to $\omega_{\varphi_\varepsilon}$ is dominated by $C\int_{\mathcal{X}_t}\omega\wedge\omega^{n-1}_{\varphi_\varepsilon}\le C'$ and we use the Cauchy–Schwarz inequality.

All in all, we infer the existence of two constants C_1 and C_2 such that

$$\int_{\mathcal{X}_t}|\nabla_\varepsilon\tau|^2\omega^n_{\varphi_\varepsilon}\le C_1\int_{\mathcal{X}_t}|\tau|\omega^n_{\varphi_\varepsilon}+C_2\Big(\int_{\mathcal{X}_t}|\nabla_\varepsilon\tau|^2\omega^n_{\varphi_\varepsilon}\Big)^{1/2} \tag{2.68}$$

for any $\varepsilon>0$. The inequality (2.61) follows. ■

Theorem 2.6. *There exists $N\in\mathbb{Z}_+$ and a positive constant C such that*

$$\int_{\mathcal{X}_t}|\tau|^2\,d\mu^{(\varepsilon)}_{Ne}\le C\quad\textit{for every }\varepsilon>0. \tag{2.69}$$

Proof. The arguments which will follow are absolutely standard, by combining the Sobolev and Poincaré inequalities with (2.61). Prior to this, we recall that

$$\omega_\varepsilon\le C\omega_{B,\varepsilon} \tag{2.70}$$

on each $\mathcal{X}_t$ for some constant C which is uniform with respect to ε and with respect to $t\in\mathbb{D}'$. On the right hand side of (2.70) we have $\omega_{B,\varepsilon}$ which stands for any metric quasi-isometric to (2.32). In particular, for any function f we have

$$|\nabla f|\le C|\nabla_\varepsilon f|_\varepsilon \tag{2.71}$$

where the symbols $|\cdot|$, ∇ and $|\cdot|_\varepsilon$, ∇_ε correspond to the metrics $\omega_{B,\varepsilon}$ and ω_ε respectively.

Now, the Poincaré inequality of Proposition 2.2 applied for $\alpha=1$ combined with Lemma 2.4 gives

$$\int_{\mathcal{X}_t}|\tau|\,d\mu^{(\varepsilon)}_e\le C\Big(1+\int_{\mathcal{X}_t}|\nabla\tau|\,d\mu^{(\varepsilon)}_{e/2}\Big). \tag{2.72}$$

On the other hand,

$$\int_{\mathcal{X}_t} |\nabla\tau|\, d\mu^{(\varepsilon)}_{e/2} \le C \int_{\mathcal{X}_t} |\nabla_\varepsilon\tau|_\varepsilon\, d\mu^{(\varepsilon)}_{e/2} \le C\left(\int_{\mathcal{X}_t} |\nabla_\varepsilon\tau|^2_\varepsilon\, d\mu^{(\varepsilon)}_{e}\right)^{1/2}$$
$$\le C + C\left(\int_{\mathcal{X}_t} |\tau|\, d\mu^{(\varepsilon)}_{e}\right)^{1/2}$$

where we have used Proposition 2.5 for the last inequality. When combined with (2.72), this implies

$$\int_{\mathcal{X}_t} |\tau|\, d\mu^{(\varepsilon)}_{e} \le C \quad \text{for any } \varepsilon > 0. \tag{2.73}$$

We next define the sequence of rational numbers

$$p_1 = 1, \quad p_{k+1} := \frac{2np_k}{2n - p_k} \tag{2.74}$$

as well as the sequence

$$q_1 = e, \quad q_{k+1} := \frac{2}{2 - p_k} q_k. \tag{2.75}$$

One can actually find a closed formula $p_k = \frac{2n}{2n-k+1}$ holding for $1 \le k \le 2n$. It also follows that $p_k < 2$ as long as $1 \le k \le n$, which is thus the range of integers for which q_{k+1} is defined; one can also check the formula $q_{k+1} = \frac{(2n)!(n-k)!}{n!(2n-k)!}\cdot q$. In particular, $q_{n+1} = \frac{(2n)!}{n!^2}\cdot q$. This is the factor N in the statement of the proposition.

We observe that for $k = 1, \ldots, n$ the components of q_k are positive rational numbers, greater than the respective components of q.

The Sobolev inequality of Proposition 2.1 gives

$$\left(\int_{\mathcal{X}_t} |\tau|^{p_{k+1}}\, d\mu^{(\varepsilon)}_{q_{k+1}}\right)^{1/p_{k+1}} \le C\left(\int_X |\nabla_\varepsilon\tau|^{p_k}_\varepsilon\, d\mu^{(\varepsilon)}_{q_k} + \int_{\mathcal{X}_t} |\tau|^{p_k}\, d\mu^{(\varepsilon)}_{q_k}\right)^{1/p_k} \tag{2.76}$$

We iterate (2.67) for $k = 1, \ldots, n$, and Proposition 2.6 is proved by observing that the following hold.

- We have $\int_{\mathcal{X}_t} |\nabla_\varepsilon\tau|^2_\varepsilon \omega^n_{\varphi_\varepsilon} \le C$, by Proposition 2.5 combined with (2.73) and the fact that the quotient of the two measures

$$\omega^n_{\varphi_\varepsilon},\ d\mu^{(\varepsilon)}_{e} \tag{2.77}$$

is uniformly bounded from above and below.

- For each $k = 1, \ldots, n$ we have

$$\int_{\mathcal{X}_t} |\nabla_\varepsilon\tau|^{p_k}_\varepsilon\, d\mu^{(\varepsilon)}_{q_k} \le C\left(\int_{\mathcal{X}_t} |\nabla_\varepsilon\tau|^2_\varepsilon\, d\mu^{(\varepsilon)}_{\frac{2}{p_k}q_k}\right)^{p_k/2} \le C \tag{2.78}$$

where the first inequality is simply Cauchy–Schwarz, and the second one is due to the fact that

$$d\mu^{(\varepsilon)}_{\frac{2}{p_k}q_k} \le C\omega^n_{\varphi_\varepsilon} \tag{2.79}$$

because $\frac{q_k}{p_k} \ge \frac{q}{2}$. This last inequality follows by induction given that $\frac{q_{k+1}}{p_{k+1}} = \frac{2n-p_k}{2n-np_k}\cdot\frac{q_k}{p_k}$.

∎

2.4. A gradient estimate in the conic case

Theorem 2.7. *Let (X, ω) be a compact Kähler manifold, and let $\omega_\varphi := \omega + dd^c\varphi$ be a Kähler metric satisfying*

$$\omega_\varphi^n = e^{\lambda\varphi + F}\omega^n$$

for some $F \in \mathscr{C}^\infty(X)$ and $\lambda \in \mathbb{R}$. Assume that there exists $C > 0$ and smooth functions Ψ, Φ such that:

(i) $\sup_X |\varphi| \le C$,

(ii) $\sup_X |\Psi| \le C$ *and for any $\delta > 0$, there exists C_δ such that*

  (a) $dd^c\Psi \ge \delta^{-1} d\Psi \wedge d^c\Psi - C_\delta\omega$,

  (b) $\Delta_\omega \Psi \ge \delta^{-1}|\nabla F|_\omega - C_\delta$,

(iii) $i\Theta_\omega(T_X) \ge -(C\omega + dd^c\Psi) \otimes \mathrm{Id}$,

(iv) $\omega_\varphi \le C\omega$.

Then there exists a constant $A > 0$ depending only on C and n such that $|\nabla\varphi|_\omega \le C$.

As a corollary of this result, the gradient estimate (2.52) in Theorem 2.3 holds.

Proof of Theorem 2.3. Let us rewrite (2.51) as

$$(\omega_\varepsilon + dd^c u_\varepsilon)^n = e^{\lambda u_\varepsilon + f_\varepsilon}\prod_{i\in I}(\varepsilon^2 + |\sigma_i|^2)^{e_i}\,\omega_\varepsilon^n$$

where the reference metric $\omega_\varepsilon \in \{\omega\}$ is an approximate conic metric along the divisor B, and u_ε differs from φ_ε by a function whose L^∞ norm as well as those of its gradient and complex Hessian are uniformly bounded with respect to ω_ε. Therefore it is sufficient to establish (2.52) for u_ε. We check successively that conditions (i)–(iv) are satisfied.

The bound (i) follows from Kołodziej's estimate. It is straighforward when $\lambda = 0$, and when $\lambda > 0$, it requires an additional step easily achieved with the Jensen inequality. Next, we choose $\Psi_\varepsilon := C(\sum_i (|\sigma_i|^2 + \varepsilon^2)^\rho + \sum_j (|s_j|^2 + \varepsilon^2)^\rho)$ for C large enough and $\rho > 0$ small enough. Condition (ii) (a) can be checked independently for each summand Ψ_ε^α of Ψ_ε in which case it follows from the fact that Ψ_ε^α is uniformly quasi-psh (hence $C\omega_\varepsilon$-psh). Condition (ii) (b) is an easy computation combined with [29, §5.2]. Condition (iii) is shown in [29, §4], while (iv) is the content of [29, Prop. 1]. To be more precise, *op. cit.* assumes an upper and lower bound on $f_\varepsilon + \sum e_i \log(|\sigma_i|^2 + \varepsilon^2)$ in order to get a two-sided inequality for ω_φ; however, one only needs an upper bound for the previous quantity if one only wishes to prove the one-sided inequality (iv). ∎

Proof of Theorem 2.7. Let $\beta := |\nabla\varphi|^2$ (computed with respect to ω) and $\alpha := \log\beta - \gamma\circ\varphi$ where γ is a function to be specified later. Without loss of generality, one can assume $\inf\varphi = 0$, and we set $\sup\varphi =: C_0$. We use the local notation $(g_{i\bar{j}})$ for ω. We work at a point $y \in X$ where $\alpha + 2\Psi$ attains its maximum, and we choose a system of geodesic coordinates for ω such that $g_{i\bar{j}}(y) = \delta_{i\bar{j}}$, $dg_{i\bar{j}}(y) = 0$, and $\varphi_{i\bar{j}}$ is diagonal. We write $u_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$ for the components of the metric ω_φ. As $\alpha_p = \beta_p/\beta - \gamma' \circ \varphi\, \varphi_p$ and $\alpha_p(y) = -2\Psi_p(y)$, one has

$$\frac{\beta_p}{\beta}(y) = (\gamma' \circ \varphi(y))\varphi_p(y) - 2\Psi_p(y). \tag{2.80}$$

Moreover, some computations show that

$$\begin{aligned} \alpha_{p\bar{p}} = & \frac{1}{\beta}\Big(R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\operatorname{Re}\sum_j u_{p\bar{p}j}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2 + \varphi_{p\bar{p}}^2\Big) \\ & - \frac{|\beta_p|^2}{\beta^2} - 2\lambda - \gamma''|\varphi_p|^2 - \gamma'\varphi_{p\bar{p}}. \end{aligned}$$

Therefore at y, one gets from (2.80) the following inequality:

$$\begin{aligned} \alpha_{p\bar{p}} \geq & \frac{1}{\beta}\Big(R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}} + 2\operatorname{Re}\sum_j u_{p\bar{p}j}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2 + \varphi_{p\bar{p}}^2\Big) \\ & - 2\lambda - \gamma''|\varphi_p|^2 - \gamma'\varphi_{p\bar{p}} - |\gamma'\varphi_p - 2\Psi_p|^2, \end{aligned} \tag{2.81}$$

so at y, the right hand side is non-positive.

Step 1. *The curvature term.* By assumption (iii), for all a, b we have $R_{j\bar{k}p\bar{q}}a_j\bar{a}_k b_p\bar{b}_q \geq -(C|a_j|^2 + \Psi_{j\bar{k}}a_j\bar{a}_k)|b|^2$ and by symmetry of the curvature tensor, we get $R_{j\bar{k}p\bar{q}}a_j\bar{a}_k b_p\bar{b}_q \geq -(C|b_p|^2 + \Psi_{p\bar{q}}b_p\bar{b}_q)|a|^2$. Applying that to $a = \nabla\varphi$ and b the vector with only the p-th component non-zero, equal to $\sqrt{u^{p\bar{p}}}$, we get

$$u^{p\bar{p}}R_{j\bar{k}p\bar{p}}\varphi_k\varphi_{\bar{l}} \geq -(Cu^{p\bar{p}} + u^{p\bar{p}}\Psi_{p\bar{p}})|\nabla\varphi|^2.$$

As a consequence,

$$\frac{1}{\beta}\sum_{p,j,k} u^{p\bar{p}}R_{j\bar{k}p\bar{p}}\varphi_j\varphi_{\bar{k}} \geq -C\sum_p u^{p\bar{p}} - \sum_p u^{p\bar{p}}\Psi_{p\bar{p}}. \tag{2.82}$$

Therefore, (2.81) becomes, at $y \in X$,

$$\begin{aligned} \Delta'(\alpha + \Psi) \geq & (\gamma' - C)\operatorname{tr}_{\omega_\varphi}\omega + \frac{1}{\beta}\sum_p u^{p\bar{p}}\Big(2\operatorname{Re}\sum_j u_{p\bar{p}j}\varphi_{\bar{j}} + \sum_j |\varphi_{jp}|^2\Big) \\ & - \gamma''|\nabla^\omega\varphi|^2_{\omega_\varphi} - n\gamma' - \sum_p u^{p\bar{p}}|\gamma'\varphi_p - 2\Psi_p|^2 - C. \end{aligned} \tag{2.83}$$

Step 2. *The gradient term.* The next term to analyze is

$$\frac{1}{\beta}\sum_p u^{p\bar{p}}\Big(2\operatorname{Re}\sum_j u_{p\bar{p}j}\varphi_{\bar{j}}\Big) = \frac{2}{\beta}\operatorname{Re}\sum_j F_j\varphi_{\bar{j}} \tag{2.84}$$

by [7, §1.13], and this term is dominated (in norm) by $2|\nabla F|\beta^{-1/2}$; moreover, at the point y, β can always be assumed to be larger than 1 so that our term is bigger than $-2|\nabla F|$.

In particular, at y one gets

$$\Delta'(\alpha+\Psi) \geq (\gamma'-C)\,\mathrm{tr}_{\omega_\varphi}\,\omega + \sum_p u^{p\bar{p}}\Big(\frac{1}{\beta}\sum_j |\varphi_{jp}|^2 - |\gamma'\varphi_p - 2\Psi_p|^2\Big)$$
$$-\gamma''|\nabla^\omega\varphi|^2_{\omega_\varphi} - n\gamma' - 2|\nabla F| - C \tag{2.85}$$

Step 3. *Using the second derivatives.* Recall that $\beta_p = \sum_j \varphi_{jp}\varphi_{\bar{j}} + \varphi_p(u_{p\bar{p}}-1)$. At y, $\beta_p/\beta - \gamma'\varphi_p = -2\Psi_p$, so that at this point,

$$\sum_j \varphi_{jp}\varphi_{\bar{j}} = (\gamma'\beta + 1 - u_{p\bar{p}})\varphi_p - 2\beta\Psi_p,$$

hence $|\sum_j \varphi_{jp}\varphi_{\bar{j}}| = \beta|(\gamma'\varphi_p - 2\Psi_p) + \beta^{-1}(1-u_{p\bar{p}})\varphi_p|$. By the Cauchy–Schwarz inequality, $|\sum_j \varphi_{jp}\varphi_{\bar{j}}|^2 \leq \beta\sum_j |\varphi_{jp}|^2$ and therefore

$$\frac{1}{\beta}\sum_j |\varphi_{jp}|^2 - |\gamma'\varphi_p - 2\Psi_p|^2 \geq |(\gamma'\varphi_p - 2\Psi_p) + \beta^{-1}(1-u_{p\bar{p}})\varphi_p|^2 - |\gamma'\varphi_p - 2\Psi_p|^2$$
$$\geq -2\beta^{-1}|1-u_{p\bar{p}}|\cdot|\gamma'\varphi_p - 2\Psi_p|\cdot|\varphi_p|$$

and by (iv), $|1-u_{p\bar{p}}| \leq C$, so that

$$\sum_p u^{p\bar{p}}\Big(\frac{1}{\beta}\sum_j |\varphi_{jp}|^2 - |\gamma'\varphi_p - 2\Psi_p|^2\Big) \geq -C(\mathrm{tr}_{\omega_\varphi}\,\omega + |\nabla\Psi|^2_{\omega_\varphi}).$$

Combining this last inequality with (2.85) we get, at y,

$$\begin{aligned} 0 &\geq \Delta'(\alpha+2\Psi) \\ &\geq (\gamma'-C)\,\mathrm{tr}_{\omega_\varphi}\,\omega - \gamma''|\nabla^\omega\varphi|^2_{\omega_\varphi} - n\gamma' + (\Delta'\Psi - C|\nabla\Psi|^2_{\omega_\varphi} - 2|\nabla F|_\omega) - C. \end{aligned}$$

As Ψ is quasi-psh and $\omega_\varphi \leq C\omega$, we have $\Delta'\Psi \geq C^{-1}\Delta\Psi - C\,\mathrm{tr}_{\omega_\varphi}\,\omega$ so by (ii) (b), $\Delta'\Psi \geq 4|\nabla F|_\omega - C(1+\mathrm{tr}_{\omega_\varphi}\,\omega)$. Using (ii) (a), one ends up with the following inequality at y:

$$(\gamma'-C)\,\mathrm{tr}_{\omega_\varphi}\,\omega - \gamma''|\nabla^\omega\varphi|^2_{\omega_\varphi} - n\gamma' \leq C.$$

Choosing $\gamma(t) = (C+1)t - \|\varphi\|_\infty^{-1}t^2$ enables to conclude just as in [7]. ∎

Proof of Theorem C. It is a combination of our preceding considerations. The equation which gives ω_{KE} fiberwise is of the same type as (2.49) (with $\lambda = 0$). We conclude by using Theorems 2.3 and 2.6. ∎

3. Existence of non-semipositive relative Ricci-flat Kähler metrics

Let $p : X \to Y$ be a holomorphic fibration between projective manifolds of relative dimension $n \geq 1$. Let Y° be the set of regular values, and let $X^\circ := p^{-1}(Y^\circ)$. We assume that for $y \in Y^\circ$, $c_1(K_{X_y}) = 0$, where $X_y := p^{-1}(y)$. Let L be a pseudo-effective, p-ample

$\mathbb{Q}$-line bundle on X. One can write $L = H + p^*M$ for some ample line bundle H on X and some line bundle M on Y. In particular, one can find a smooth $(1,1)$-form $\omega \in c_1(L)$ on X such that for any $y \in Y^\circ$, $\omega_y := \omega|_{X_y}$ is a Kähler form on X_y.

By Yau's theorem, there exists for any $y \in Y^\circ$ a unique function $\varphi_y \in \mathcal{C}^\infty(X_y)$ such that

(i) $\theta_y := \omega_y + dd^c\varphi_y$ is a Kähler form,

(ii) $\int_{X_y} \varphi_y \omega_y^n = 0$,

(iii) $\operatorname{Ric}\theta_y = -dd^c \log \omega_y^n = 0$.

Moreover, one can use the implicit function theorem to check that the dependence of φ_y on y is smooth, so that $\theta := \omega + dd^c\varphi$ is a well-defined smooth $(1,1)$-form on X° which is relatively Kähler. It is a folklore conjecture that the form θ is semipositive on X, say when L is globally ample. Building on the results in the Appendix, we are able to disprove this conjecture.

Theorem 3.1. *There exists a projective fibration $p : X \to Y$ as in the setting above and an ample line bundle L on X such that the relative Ricci-flat metric θ on X° associated with L is not semipositive.*

Remark 3.2. The counterexample is actually pretty explicit: X is a K3 surface and p is an elliptic fibration onto $Y = \mathbb{P}^1$.

Proof of Theorem 3.1. We proceed in three steps, arguing by contradiction. That is, we assume that the folklore conjecture recalled above is true for any such fibration $p : X \to Y$.

Step 1. *Choice of the fibration.* We consider a K3 surface X provided by Proposition A.3. Its (singular) fibers are irreducible and reduced. Moreover, X admits a semiample line bundle L which is p-ample and has numerical dimension 1. Indeed, L can be chosen as the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ by another elliptic fibration $q : X \to \mathbb{P}^1$. Moreover, one knows that p is not isotrivial, in the sense that two general fibers $X_y, X_{y'}$ of p are not isomorphic.

Step 2. *Reduction to the semiample case.* Let us pick an ample line bundle A on X, a Kähler form $\omega_A \in c_1(A)$, and consider the relative Ricci-flat form θ_ε on X° associated with the pair $(L + \varepsilon A, \omega + \varepsilon\omega_A)$. The line bundle L_ε is ample, hence our assumption implies that for any $\varepsilon > 0$, the relative Ricci-flat metric satisfies

$$\theta_\varepsilon \geq 0 \quad \text{on } X^\circ.$$

We are going to show that θ_ε converges weakly on X° to the current $\theta := \theta_0$. As a result, this will force θ to be semipositive on X°.

Let us write $\theta_\varepsilon = \omega + \varepsilon\omega_A + dd^c\varphi_\varepsilon$ where φ_ε is normalized such that for each $y \in Y^\circ$,

$$\int_{X_y} \varphi_\varepsilon(\omega + \varepsilon\omega_A) = 0.$$

If C_ε is the constant (converging to 0) defined by

$$e^{C_\varepsilon} = \frac{[X_y] \cdot c_1(L)}{[X_y] \cdot c_1(L + \varepsilon A)}$$

for any $y \in Y^\circ$, then

$$\omega + \varepsilon\omega_A + dd^c\varphi_\varepsilon = e^{C_\varepsilon} \cdot (\omega + dd^c\varphi) \quad \text{on } X_y.$$

The family $(\varphi_\varepsilon|_{X_y})_{\varepsilon,y}$ of potentials is normalized in a smooth way with respect to ε and y, and satisfies linear equations depending smoothly on the parameters as well. It is not difficult to see that the standard estimates hold uniformly in ε and y (as long as y varies in compact subsets of Y°), hence uniqueness imposes that $\varphi_\varepsilon \to \varphi$ smoothly in each X_y, locally uniformly in $y \in Y^\circ$. In particular, φ_ε converges weakly to φ in $L^1_{\mathrm{loc}}(X^\circ)$.

Step 3. *End of proof.* Thanks to Step 2, the relative Ricci-flat metric

$$\theta = \omega + dd^c\varphi$$

is semipositive on X°. Moreover, it follows from Proposition A.1 that φ is bounded above near $X \smallsetminus X^\circ$, hence θ extends to a semipositive current $\theta \in c_1(L)$ on the whole X. Let $\mathcal{F} \subset T_X$ be the holomorphic foliation induced by the fibration $q : X \to \mathbb{P}^1$. As the semipositive current θ is in the class of $c_1(L) = q^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)))$ and q has connected fibers, it follows that there exists a positive current $\gamma \in c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ such that $\theta = q^*\gamma$. In particular, if $X^1 \subset X$ denotes the locus where q is smooth and if $\Omega := X^\circ \cap X^1$, then $\mathcal{F}|_\Omega$ is contained in $\operatorname{Ker}\theta$ on Ω. As both foliations are smooth and have rank 1 on Ω, one has

$$\mathcal{F}|_\Omega = \operatorname{Ker}\theta|_\Omega. \tag{3.1}$$

Next, let us pick a trivializing open set $U \simeq \Delta \subset Y^\circ$, and let $V \in \mathcal{C}^\infty(X^\circ, T_X^{1,0})$ be the lift of $\frac{\partial}{\partial t}$ with respect to θ over U (see e.g. [28, Sect. 1.1]). One knows that in a trivializing chart (z,t) defined on a subset of $p^{-1}(U)$ such that $p(z,t) = t$, the vector field V can be written as

$$V = \frac{\partial}{\partial t} + a(z,t)\frac{\partial}{\partial z}$$

for some smooth function a. The function $c := \theta(V, V)$ satisfies the identity $\theta^2 = c\theta \wedge \sqrt{-1}\,dt \wedge d\bar{t}$, hence it vanishes identically on $p^{-1}(U)$, that is,

$$V \in \mathcal{C}^\infty(p^{-1}(U), \operatorname{Ker}\theta).$$

Thanks to (3.1), this shows that for any $x \in p^{-1}(U) \cap \Omega$, one has $\mathbb{C} \cdot V(x) = \mathcal{F}_x$. In particular, there exists a non-vanishing, smooth function f on $p^{-1}(U) \cap \Omega$ such that fV is holomorphic on $p^{-1}(U) \cap \Omega$. Now in local coordinates, this means that

$$0 = \bar{\partial}(fV) = \bar{\partial}f \otimes \frac{\partial}{\partial t} + \bar{\partial}(fa) \otimes \frac{\partial}{\partial z},$$

hence $\bar{\partial}f = 0$. As a result, the smooth vector field V on $p^{-1}(U)$ is holomorphic on $p^{-1}(U) \cap \Omega$, hence on the whole $p^{-1}(U)$. Therefore, its flow induces a local biholomorphism between any two near fibers. In particular, any two smooth fibers over U would be isomorphic, which contradicts the non-isotriviality of p. ∎

Valentino Tosatti
Appendix A

Let (X^n, ω_X) be a compact Kähler manifold, Y a compact Riemann surface, and $f : X \to Y$ a surjective holomorphic map with connected fibers. Let Y^0 be the locus of regular values for f, whose complement in Y is a finite set, and $X^0 = f^{-1}(Y^0)$, which is Zariski open in X, so that $f : X^0 \to Y^0$ is a proper holomorphic submersion. We will call the fibers over points in $Y \setminus Y^0$ the *singular fibers* of f.

Suppose that for every $y \in Y^0$ we have a smooth function ρ_y on the fiber $X_y = f^{-1}(y)$ which satisfies

$$\omega_X|_{X_y} + \sqrt{-1}\,\partial\bar{\partial}\rho_y \geq 0, \quad \int_{X_y} \rho_y(\omega_X|_{X_y})^n = 0. \tag{A.1}$$

Proposition A.1. *If all the singular fibers of f are reduced and irreducible, then there is a constant C such that*

$$\sup_{X_y} \rho_y \leq C \quad \text{for all } y \in Y^0.$$

Proof. Let $\omega_y = \omega_X|_{X_y}$, and g_y be its Riemannian metric, where in the following we fix any $y \in Y^0$. Thanks to (A.1), on X_y we have

$$\Delta_{g_y}\rho_y \geq -n + 1. \tag{A.2}$$

We have $\mathrm{Vol}(X_y, g_y) = c$, a constant independent of y, and the Sobolev constant of (X_y, g_y) has a uniform upper bound independent of y thanks to the Michael–Simon Sobolev inequality [38] (see the details e.g. in [49, Lemma 3.2]). Furthermore, $\mathrm{diam}(X_y, g_y) \leq C$, a constant independent of y, thanks to [49, Lemma 3.3].

So far we have not used the assumptions that all singular fibers are reduced and irreducible. This is used now to prove that the Poincaré constant of (X_y, g_y) also has a uniform upper bound independent of y, as shown by Yoshikawa [58] (see also the much clearer exposition in [43, Proposition 3.2]).

At this point we can use a classical argument of Cheng–Li [22], which is clearly explained in [47, Chapter 3, Appendix A, pp. 137–140], to deduce that the Green function $G_y(x, x')$ of (X_y, g_y), normalized by

$$\int_{X_y} G_y(x, x')\omega_y(x') = 0,$$

satisfies the bound

$$G_y(x, x') \geq -A \tag{A.3}$$

for all $y \in Y^0$ and all $x, x' \in X_y$, with a uniform constant A. The point of that argument is that A only depends on the constant in the Sobolev–Poincaré inequality, which here is controled uniformly, and on the dimension and on the bounds for the volume and diameter, which we have.

We can now apply Green's formula on X_y. Choose a point $x \in X_y$ such that $\rho_y(x) = \sup_{X_y} \rho_y$. Then, using the fact that ρ_y has average zero, together with (A.2) and (A.3), we obtain

$$\begin{aligned}\rho_y(x) &= -\int_{X_y} \Delta_{g_y}\rho_y(x')G_y(x,x')\omega_y(x') \\ &= -\int_{X_y} \Delta_{g_y}\rho_y(x')(G_y(x,x')+A)\omega_y(x') \\ &\le (n-1)\int_{X_y}(G_y(x,x')+A)\omega_y(x') \\ &\le (n-1)A\,\mathrm{Vol}(X_y,g_y). \qquad \blacksquare\end{aligned}$$

We now specialize to the setting where X is a K3 surface, $Y = \mathbb{P}^1$ and $f : X \to \mathbb{P}^1$ is an elliptic fibration. We further assume that ρ_y is chosen so that $\omega_X|_{X_y} + \sqrt{-1}\,\partial\bar\partial\rho_y > 0$ is the unique flat metric on X_y cohomologous to $\omega_X|_{X_y}$ (and we still assume that ρ_y has fiberwise average zero). In this case ρ_y varies smoothly in $y \in Y^0$, and so it defines a smooth function ρ on X^0. Thanks to (1.48), we conclude that

$$\sup_{X^0} \rho \le C.$$

This, together with the Grauert–Remmert extension theorem, immediately gives:

Corollary A.2. *In this setting, if $\omega_X + \sqrt{-1}\,\partial\bar\partial\rho \ge 0$ on X^0, then this extends to a closed positive current on all of X, in the class $[\omega_X]$.*

Lastly, we need the following examples:

Proposition A.3. *There exists a complex projective* K3 *surface X which admits two elliptic fibrations, one of which is non-isotrivial and has only reduced and irreducible singular fibers.*

Proof. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a general hypersurface of degree $(3,2)$. It is known that X has Picard number 2 [51, Section 5.8]. The projection to the $\mathbb{P}^1$ factor gives an elliptic fibration on X, which is clearly not isotrivial provided X is general.

To obtain the other fibration we compose the first fibration with the automorphism σ of X obtained as follows. Projecting X to the $\mathbb{P}^2$ factor shows that X is a double cover of $\mathbb{P}^2$ ramified along a sextic, and the covering involution of this cover is the σ that we want.

Explicitly, if we let $L = \mathcal{O}_{\mathbb{P}^2}(1)|_X$ and $M = \mathcal{O}_{\mathbb{P}^1}(1)|_X$, then the first elliptic fibration is defined by $|M|$ and the second by $|3L - M|$ (since $\sigma^*M = 3L - M$).

Lastly, we show that every elliptic fibration on X has only reduced and irreducible singular fibers. Given an elliptic fibration $f : X \to \mathbb{P}^1$, let $j : J \to \mathbb{P}^1$ be its Jacobian family [32, Section 11.4]. Then J is also an elliptic K3 surface, every fiber of j is isomorphic to the corresponding fiber of f, J has the same Picard number as X, but j always has a section. We can then apply the Shioda–Tate formula [32, Corollary 11.3.4] to j to obtain

$$2 = \rho(J) = 2 + \sum_{t \in \mathbb{P}^1} (r_t - 1) + \operatorname{rank} \mathrm{MW}(j),$$

where r_t is the number of irreducible components of the fiber J_t and $\mathrm{MW}(j)$ is the Mordell–Weil group of j. In particular we conclude that $r_t = 1$ for all t, i.e. all fibers of j (and therefore all fibers of f) are irreducible. Lastly, all fibers of f are reduced by [32, Proposition 3.1.6 (iii)]. ∎

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