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From non-defectivity to identifiability

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Abstract. A point p in a projective space is h-identifiable via a variety X if there is a unique way to write p as a linear combination of h points of X. Identifiability is important both in algebraic geometry and in applications. In this paper we propose an entirely new approach to study identifiability, connecting it to the notion of secant defect for any smooth projective variety. In this way we are able to improve the known bounds on identifiability and produce new identifiability statements. In particular, we give optimal bounds for some Segre and Segre–Veronese varieties and provide the first identifiability statements for Grassmann varieties.

Keywords. Tensor decomposition, Waring decomposition, identifiability, defective varieties

Introduction

The notion of identifiability or canonical form is ubiquitous in algebraic geometry and applications. We say that an element p of a projective space $\mathbb{P}^N$ is *h-identifiable* via a variety X if there is a unique way to write p as a linear combination of h elements of X. In the algebraic geometry setting this very often translates into the study of Cremona modifications associated to linear systems with prescribed singularities [25, 35], and has connections with geometric invariant theory, [27], using the dictionary of canonical forms. In the applied setup one usually considers a tensor space or a space of distributions, and the identifiability allows one to reconstruct a point of this set via a subset of special elements defined by rank conditions or other special requirements. For applications ranging from biology to Blind Signal Separation, data compression algorithms, quantum computing and analysis of mixture models [22–24, 28, 34], uniqueness of decompositions allows one to solve the problem once a solution is determined. For all these reasons it is interesting and often crucial to understand identifiability with respect to different projective varieties.

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Over a decade ago the notion of h-weakly defective varieties and birational projective geometry, with the maximal singularities method, have been connected to identifiability of polynomials [33]. This provided the first systematic study of identifiability for Veronese varieties. More recently with the work of Luca Chiantini and Giorgio Ottaviani [19], weakly defective varieties have been substituted by h-tangentially weakly defective varieties to study identifiability problems for tensor spaces. In both approaches, to provide identifiability, one has to check the behavior of special linear systems and quite often this is done by an ad hoc degeneration argument. As a consequence, identifiability has been proved in very few cases and very often the results obtained are not expected to be sharp [9, 12, 13, 19, 29].

In this paper, instead of looking at a specific projective variety $X \subset \mathbb{P}^N$ and studying its secant geometry, we aim to introduce a global approach to identifiability that can be applied to any projective variety. For this reason we do not use the standard methods of tensor analysis but provide, with birational projective geometry, a new viewpoint on identifiability. In this way we are able to recover the majority of identifiability results already known and prove infinitely many new identifiability statements, meeting the conjectural bounds in many cases.

Starting from the seminal paper [17], where the geometry of contact loci has been carefully studied, and the improvement presented in [8], we derive identifiability statements for non-secant defective varieties. With this new approach we are able to translate all the literature on defective varieties into identifiability statements.

To do it we first provide a bridge between secant defectivity and identifiability. The starting point of our analysis is the observation that, in all known examples, when a variety X is not h-identifiable then any element in $\sigma_{h+1}(X)$ has infinitely many decompositions [8, 11, 13, 18, 20]. Going back to the ideas in [33] we realize that the best way to use this observation is to set a connection between the abstract secant map and the tangential projection. This link is provided by the construction of a map from the Hilbert scheme of points of the contact loci of h-tangentially weakly defective varieties to a suitable Grassmannian. Under the right assumptions this map is proved to be of fiber type and it allows us to connect defectivity and non-identifiability.

One of the technical results we prove in this direction is the following conditional relation between identifiability and defectivity; we refer to Section 1 for the necessary definitions.

Theorem. *Let $X \subset \mathbb{P}^N$ be an irreducible reduced variety. Assume that $h > \dim X$, X is not $(h-1)$-tangentially weakly defective and it is not h-identifiable. Then X is $(h+1)$-defective.*

This, together with an improvement of the contact loci geometry studied in [17] and [8], leads us to derive identifiability from non-defectivity under a mild hypothesis.

Theorem. *Let $X \subset \mathbb{P}^N$ be a smooth variety. Assume that $\pi_k^X : \sigma_k(X) \to \mathbb{P}^N$ is generically finite and $k > 2\dim X$. Then X is $(k-1)$-identifiable.*

Let us stress that if π_k^X is not generically finite then X is not k-identifiable. Therefore the theorem gives an almost complete answer to the identifiability problem for any smooth variety, under a very mild numerical hypothesis on the codimension.

Next we apply our strategy to some special projective varieties. As already mentioned, identifiability issues are particularly interesting for tensor spaces. As a corollary we get the best asymptotic identifiability result so far for Segre, Segre–Veronese, and Grassmann varieties, that is, tensors and structured tensors (see Section 3). As a sample we state an application to binary tensors. This class of tensors is particularly interesting for quantum computing, being the geometrical version of qbits.

Theorem. *The Segre embedding of n copies of $\mathbb{P}^1$, with $n \geq 5$, is h-identifiable for any $h \leq \lfloor \frac{2^n}{n+1} \rfloor - 1$.*

Recall that the generic rank of the Segre embedding of $(\mathbb{P}^1)^n$ is $\lceil \frac{2^n}{n+1} \rceil$, therefore our result shows generic identifiability of all subgeneric binary tensors in the perfect case, that is, when $\frac{2^n}{n+1}$ is an integer, and all but the last one for the other values, as predicted by the conjecture posed in [12]. To show the flexibility of our method we deduce the identifiability of the Gaussian moment variety (see Example 40), studied in [4].

The paper is structured as follows. In Section 2 we study the geometry of contact locus and prove the main technical results about the connections between defectivity and non-identifiability. In the final section we apply our techniques to varieties that are meaningful for tensor decompositions.

1. Notation

We work over the complex field. A projective variety $X \subset \mathbb{P}^N$ is *non-degenerate* if it is not contained in any hyperplane.

Let $X \subset \mathbb{P}^N$ be a non-degenerate reduced variety. Let $X^{(h)}$ be the h-th symmetric product of X, that is, the variety parameterizing unordered sets of h points of X. Let $U_h^X \subset X^{(h)}$ be the smooth locus, given by sets of h distinct smooth points.

Definition 1. A point $z \in U_h^X$ represents a set of h distinct points, say $\{z_1, \ldots, z_h\}$. We say that a point $p \in \mathbb{P}^N$ is in the *span* of z, $p \in \langle z \rangle$, if it is a linear combination of the z_i.

With this in mind we define

Definition 2. The *abstract h-secant variety* is the variety

$$\sec_h(X) := \overline{\{(z, p) \in U_h^X \times \mathbb{P}^N \mid p \in \langle z \rangle\}} \subset X^{(h)} \times \mathbb{P}^N.$$

Let $\pi : X^{(h)} \times \mathbb{P}^N \to \mathbb{P}^N$ be the projection onto the second factor. The *h-secant variety* is

$$\sigma_h(X) := \pi(\sec_h(X)) \subset \mathbb{P}^N,$$

and $\pi_h^X := \pi_{|\sec_h(X)} : \sec_h(X) \to \mathbb{P}^N$ is the *h-secant map* of X.

The variety $\sec_h(X)$ has dimension $hn + h - 1$. If the variety X is irreducible and reduced we say that X is *h-defective* if

$$\dim \sigma_h(X) < \min\{\dim \sec_h(X), N\}.$$

Remark 3. If X is h-defective then the h-secant map is of fiber type.

Note that the image in $\mathbb{P}^N$ of a general point in $\sec_h(X)$ is a linear combination of h points of X. Thanks to the non-degeneracy assumption the image of a general point in $\sigma_h(X) \subsetneq \mathbb{P}^N$ is not a linear combination of fewer points on X.

The tricky part in studying secant varieties is the closure. Many different things can happen: the h points can group in non-reduced clusters or positive-dimensional intersection can appear. As a matter of fact, these special loci are really difficult to control and the main advantage to use birational geometry is the opportunity to get rid of them.

Definition 4. Let $X \subset \mathbb{P}^N$ be a non-degenerate subvariety. We say that a point $p \in \mathbb{P}^N$ has *rank h* with respect to X if $p \in \langle z \rangle$ for some $z \in U_h^X$, and $p \notin \langle z' \rangle$ for any $z' \in U_{h'}^X$ with $h' < h$.

Remark 5. With this in mind it is easy to produce examples of limits of rank h points with different rank. If we let one of the point degenerate to the span of the others we lower the rank. If we let two points collapse to one point, in general the rank may increase.

Definition 6. A point $p \in \mathbb{P}^N$ is *h-identifiable* with respect to $X \subset \mathbb{P}^N$ if p is of rank h and $(\pi_h^X)^{-1}(p)$ is a single point. The variety X is said to be *h-identifiable* if π_h^X is a birational map, that is, the general point of $\sigma_h(X)$ is h-identifiable.

It is clear, by the above remark, that when X is h-defective, or more generally when π_h^X is of fiber type, then X is not h-identifiable.

The next ingredient we need to introduce is the Terracini Lemma.

Theorem 7 (Terracini Lemma [16]). *Let $X \subset \mathbb{P}^N$ be an irreducible variety. Then*

- *for any $x_1, \dots, x_k \in X$ and $z \in \langle x_1, \dots, x_k \rangle$,*

$$\langle \mathbb{T}_{x_1} X, \dots, \mathbb{T}_{x_k} X \rangle \subseteq \mathbb{T}_z \sigma_k(X),$$

- *there is a dense open set $U \subset X^{(k)}$ such that*

$$\langle \mathbb{T}_{x_1} X, \dots, \mathbb{T}_{x_k} X \rangle = \mathbb{T}_z \sigma_k(X)$$

  for a general point $z \in \langle x_1, \dots, x_k \rangle$ with $(x_1, \dots, x_k) \in U$.

The Terracini Lemma yields a direct consequence of h-defectiveness. If X is h-defective then the general fiber of π_h^X has positive dimension. Therefore by Terracini the general hyperplane tangent at h points of X is singular along a positive-dimensional subvariety. This property does not characterize defective varieties.

Definition 8. Let $X \subset \mathbb{P}^N$ be a non-degenerate variety. The variety X is said to be *h-weakly defective* if the general hyperplane section singular along h general points is singular along a positive-dimensional subvariety.

There is a direct connection, proven in [16], between h-weak defectiveness and identifiability.

Theorem 9. *If X is not h-weakly defective then it is h-identifiable.*

The main problem is that it is quite hard in general to verify if a variety is h-weakly defective.

To overcome this problem, the notion of tangentially weakly defective varieties has been introduced [19]. Here we follow the notations of [8].

Let X be an irreducible and reduced non-degenerate variety. For a subset $A = \{x_1, \dots, x_h\} \subset X$ of general points we set

$$M_A := \Big\langle \bigcup_i \mathbb{T}_{x_i} X \Big\rangle.$$

By the Terracini Lemma the space M_A is the tangent space to $\sigma_h(X)$ at a general point in $\langle A \rangle$.

Definition 10. The *tangential h-contact locus* $\Gamma_h = \Gamma(A)$ is the closure in X of the union of all the irreducible components, which contain at least one point of A, of the locus of points of X where M_A is tangent to X. We will write $\gamma_h := \dim \Gamma(A)$. We say that X is *h-twd* (tangentially weakly defective) if $\gamma_h > 0$.

Remark 11. It is clear that if X is h-twd then it is h-weakly defective, it is $(h+1)$-twd and $\Gamma_h \subseteq \Gamma_{h+1}$. Using scrolls it is not too difficult to produce explicit examples of varieties that are h-weakly defective but are not h-twd; see also Remark 19.

For what follows it is useful to introduce also the notion of tangential projection.

Definition 12. Let $X \subset \mathbb{P}^N$ be a variety and $A = \{x_1, \dots, x_h\} \subset X$ a set of general points. The *h-tangential projection* (from A) of X is

$$\tau_h : X \dashrightarrow \mathbb{P}^M,$$

the linear projection from M_A, that is, by the Terracini Lemma, the projection from the tangent space of a general point $z \in \langle A \rangle$ of $\sigma_h(X)$ restricted to X.

2. Relation between twd and defectivity

We start by collecting properties of the tangential contact loci that will be useful for our purpose.

Theorem 13. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non-degenerate variety. Let $A \subset X$ be a set of h general points and Γ the associated contact locus. Assume that $\sigma_{h-1}(X) \subsetneq \mathbb{P}^N$. Then*

(a) *Γ is equidimensional and it is either irreducible* (*type I*) *or reduced* (*type II*) *with exactly h irreducible components, each of them containing a single point of A* [17, Proposition 3.9],

(b) $\langle\Gamma\rangle = \sigma_h(\Gamma)$ *and* $\sigma_i(\Gamma) \neq \langle\Gamma\rangle$ *for* $i < h$ [17, Proposition 3.9],

(c) *for* $z \in \langle\Gamma\rangle$ *general*, $\pi_h^X((\pi_h^X)^{-1}(z)) \subset \langle\Gamma\rangle$ [17, Proposition 3.9],

(d) *if we are in type I then* $\gamma_h > \gamma_{h-1}$ [8, Lemma 3.5],

(e) *if* $\gamma_h = \gamma_{h+1}$, $\sigma_{h+1}(X)$ *is not defective and does not fill up* $\mathbb{P}^N$ *then we are in type II and the irreducible components of both contact loci are linearly independent linear spaces* [8, Lemma 3.5],

(f) *if we are in type I and* $\sigma_{h+1}(X)$ *is not defective and does not fill up* $\mathbb{P}^N$ *then* Γ_{h+1} *is of type I.*

Proof. Points (a)–(e) are proved in the cited papers under the assumption that $\sigma_h(X) \subsetneq \mathbb{P}^N$. Points (a)–(d) are immediate when $\sigma_h(X) = \mathbb{P}^N$ and $\sigma_{h-1}(X) \subsetneq \mathbb{P}^N$.

We have only to prove point (f). Let $A = \{x_1, \ldots, x_h\}$ and $B = A \cup \{x_{h+1}\}$ be general sets in X. Assume that $\Gamma(B)$ is of type II. By definition $\Gamma(A) \subset \Gamma(B)$; on the other hand, by point (a) the irreducible component of $\Gamma(B)$ through x_1 does not contain x_2 and therefore it cannot contain $\Gamma(A)$. This contradiction proves the claim. ■

From the point of view of identifiability the notions of weak defectiveness and twd behave the same. The following proposition is well known to experts but we have not been able to find a written version of it.

Proposition 14 ([15]). *Let* $X \subset \mathbb{P}^N$ *be an irreducible, reduced, and non-degenerate variety. Assume that* X *is not* h*-twd. Then* X *is* h*-identifiable.*

Proof. Assume that X is not h-identifiable and let $z \in \sigma_h(X)$ be a general point. Let $z \in \langle x_1, \ldots, x_h\rangle$ for x_i general in X. The existence of a different decomposition yields a new set $\{y_1, \ldots, y_h\} \subset X$ such that $z \in \langle y_1, \ldots, y_h\rangle$. Moving the point z in the linear space $\langle x_1, \ldots, x_h\rangle$ yields a positive-dimensional contact locus. ■

Remark 15. We want to stress that h-identifiability is not equivalent to non-h-twd. In [21] and [11], examples are described of Segre and Grassmannian varieties that are h-identifiable but h-twd.

We aim to study the relation between twd and defectivity. The next lemma is a first step in this direction.

Lemma 16. *Let* $X \subset \mathbb{P}^N$ *be an irreducible, reduced, and non-degenerate variety of dimension* n,

$$\pi_k^X : \mathrm{sec}_k(X) \to \mathbb{P}^N$$

the k*-secant map,* $\tau_{k-1}^X : X \dashrightarrow \mathbb{P}^M$ *the* $(k-1)$*-tangential projection, and* $\Gamma := \Gamma(x_1, \ldots, x_k)$ *the* k*-contact locus associated to the general points* $x_1, \ldots, x_k$.

(i) *The map* π_k^X *is of fiber type if and only if* τ_{k-1}^X *is of fiber type.*

(ii) *Let* $\{x_1, \ldots, x_k, y_1, y_2\}$ *be general points. Then*

$$\dim(\Gamma(x_1, \ldots, x_k, y_1) \cap \Gamma(x_1, \ldots, x_k, y_2)) > 0$$

in a neighborhood of x_i *only if either* X *is* k*-twd or* π^X_{k+2} *has positive-dimensional fibers.*

(iii) *The map* $(\tau^X_{k-1})_{|\Gamma} : \Gamma \dashrightarrow \mathbb{P}^{\gamma_k}$ *is either of fiber type or dominant.*

Proof. (i) By the Terracini Lemma, π^X_k is of fiber type if and only if

$$\mathbb{T}_z\,\sigma_{k-1}(X) \cap \mathbb{T}_y X \neq \emptyset$$

for $y \in X$ general. This condition is clearly equivalent to τ^X_{k-1} being of fiber type.

(ii) Assume that X is not k-twd and $\dim(\Gamma(x_1,\dots,x_k,y_1) \cap \Gamma(x_1,\dots,x_k,y_2)) > 0$ in a neighborhood of x_i. Set

$$M_{A_i} = \langle \mathbb{T}_{x_1}X,\dots,\mathbb{T}_{x_k}X,\mathbb{T}_{y_i}X\rangle.$$

Since the variety X is not k-twd,

$$M_{A_1} \cap M_{A_2} \supsetneq \langle \mathbb{T}_{x_1}X,\dots,\mathbb{T}_{x_k}X\rangle.$$

In particular,

$$(M_{A_1} \cap M_{A_2}) \cap \mathbb{T}_{y_i}X \neq \emptyset,$$

and hence

$$\langle \mathbb{T}_{x_1}X,\dots,\mathbb{T}_{x_k}X,\mathbb{T}_{y_1}X\rangle \cap \mathbb{T}_{y_2}X \neq \emptyset.$$

This shows, by the generality of the points and point (i), that π^X_{k+2} is of fiber type.

(iii) Assume that $(\tau^X_{k-1})_{|\Gamma}$ is not of fiber type. Then by Theorem 13 (b) we have $\dim\langle\Gamma\rangle = k(\gamma_k+1)-1$. Hence $(\tau^X_{k-1})_{|\Gamma} = \tau^\Gamma_{k-1}$ and both maps are dominant onto $\mathbb{P}^{\gamma_k}$. ■

Next we prove a general statement for type II contact loci.

Lemma 17. *Let* $X \subset \mathbb{P}^N$ *be an irreducible, reduced, and non-degenerate variety. Assume that*

(a) X *is* k*-twd,*

(b) X *is not* $(k-1)$*-twd,*

(c) *the* k*-contact locus is of type II.*

Then π^X_{k+1} *is of fiber type.*

Proof. By Lemma 16 (i) it is enough to prove that τ^X_k is of fiber type. Then by projection it is enough to prove the latter for $k = 2$. Let $\{x_1, x_2, y\} \subset X$ be a set of general points and $\Gamma = \Gamma(x_1, x_2, y)$ the contact locus associated to $\{x_1, x_2, y\}$. To conclude the proof it is enough to prove that $\langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle \cap \mathbb{T}_x X \neq \emptyset$ for $x \in \Gamma$ a general point.

For a general point $p \in \Gamma$ we let

$$\Gamma^i_p \subset \Gamma(x_i, p)$$

be the irreducible component of the contact locus $\Gamma(x_i, p)$ through p. The contact locus is of type II, therefore $\Gamma_p^i \not\ni x_1, x_2$. Note that for a general point $x \in \Gamma_p^1$ we have $\mathbb{T}_x X \subset \langle \mathbb{T}_{x_1} X, \mathbb{T}_p X\rangle$. Then by semicontinuity for any point $w \in \Gamma_p^1$ there is a linear space of dimension n, say $A_w \subseteq \mathbb{T}_w X$, contained in the span.

Set

$$\mathbb{T}(\Gamma_p^1) = \langle A_w \rangle_{w\in\Gamma_p^1}.$$

We may assume that X is not 2-defective (otherwise there is nothing to prove), that is,

$$\mathbb{T}_{x_1} X \cap \mathbb{T}_{x_2} X = \emptyset, \tag{1}$$

and, since y is general,

$$\mathrm{codim}_{\mathbb{T}(\Gamma_y^1)}(\mathbb{T}(\Gamma_y^1) \cap \mathbb{T}_{x_1} X) = n+1. \tag{2}$$

The variety X is not 1-twd, so there are points $z \in \Gamma_y^1$ with $A_z \cap \mathbb{T}_{x_1} X \neq \emptyset$. Let $z \in \Gamma_y^1$ be a point with

$$A_z \cap \mathbb{T}_{x_1} X \neq \emptyset, \tag{3}$$

The contact locus is of type II, therefore $z \neq x_1$. We stress that this is the only point in the proof where we use the assumption that Γ is of type II.

If $A_z \cap \mathbb{T}_{x_2} X \neq \emptyset$, by (1) and (2) we have

$$\mathrm{codim}_{\mathbb{T}(\Gamma_y^1)}(\mathbb{T}(\Gamma_y^1) \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle) \leq n$$

and we conclude $\mathbb{T}_y X \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle \neq \emptyset$, that is, τ_2^X is of fiber type.

Assume that $A_z \cap \mathbb{T}_{x_2} X = \emptyset$. Then we consider the span $\langle A_z, \mathbb{T}_{x_2}\rangle$. By semicontinuity, to this linear space is associated a contact locus and we let Γ_z^2 be its irreducible component passing through z. As before we have

$$\mathrm{codim}_{\mathbb{T}(\Gamma_z^2)}(\mathbb{T}(\Gamma_z^2) \cap \mathbb{T}_{x_2}) = n+1,$$

and by (1) and (3) we conclude that

$$\mathrm{codim}_{\mathbb{T}(\Gamma_z^2)}(\mathbb{T}(\Gamma_z^2) \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle) \leq n.$$

This yields

$$A_w \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle \neq \emptyset \tag{4}$$

for any point $w \in \Gamma_z^2$. We have $z \neq y_1$, so the general choice of the points x_i and the assumption that X is not 2-defective ensure that

$$A_w \cap \mathbb{T}_{x_1} X = \emptyset \tag{5}$$

for general $w \in \Gamma_z^2$.

We let Γ_w^1 be the irreducible component through w of the contact locus associated to $\langle A_w, \mathbb{T}_{x_1} X\rangle$. Again $z \neq x_1$ and the general choice of the x_i ensure that $z \notin \Gamma_w^1$. In particular,

$$\Gamma_w^1 \neq \Gamma_z^2.$$

Set

$$S^2 := \bigcup_{v\in\Gamma^1_y \text{ general}} \Gamma^2_v.$$

Then Γ^2_z is in the closure of S^2 and, for $p \in \Gamma^1_y$ general, Γ^1_y is in the closure of

$$\bigcup_{w\in\Gamma^2_p \text{ general}} \Gamma^1_w.$$

Hence Γ^1_y is in the closure of

$$S^1 := \bigcup_{w\in\Gamma^2_z \text{ general}} \Gamma^1_w.$$

In particular, the general point of S^1 is a general point of X. By construction we have

$$\operatorname{codim}_{\mathbb{T}(\Gamma^1_w)}(\mathbb{T}(\Gamma^1_w) \cap \mathbb{T}_{x_1}X) \le n+1.$$

Equations (4) and (5) then give

$$\operatorname{codim}_{\mathbb{T}(\Gamma^1_w)}(\mathbb{T}(\Gamma^1_w) \cap \langle \mathbb{T}_{x_1}, \mathbb{T}_{x_2}\rangle) \le n,$$

and this concludes the proof. ∎

We are ready to prove our main result that connects twd and defectivity.

Theorem 18. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non-degenerate variety of dimension n. Assume that*

(a) *X is k-twd,*

(b) *X is not $(k-1)$-twd,*

(c) *$k > n$ and $N \ge (k+1)(n+1)-1$.*

Then π^X_{k+1} is of fiber type.

Proof. Thanks to Lemma 17 we may assume that the contact locus is of type I. By hypothesis the variety X is k-twd. Let $A = \{x_1, \dots, x_k\} \subset X$ be a set of general points and $\Gamma := \Gamma(A)$ the associated contact locus of dimension $\gamma > 0$. Let $z \in \langle A\rangle$ be a general point, $\tau_k := \tau^X_k : X \dashrightarrow \mathbb{P}^M$ the associated k-tangential projection, and $y \in X$ a general point. For a general set $Y := \{y_1, \dots, y_{k-1}\} \subset \Gamma$ let $\Gamma(Y \cup \{y\})$ be the contact locus associated to $\{y_1, \dots, y_k, y\}$.

Assume that π^X_{k+1} is not of fiber type. Then, by Lemma 16 (i), τ_k is not of fiber type, and by Lemma 16 (iii), $\tau_k(\Gamma(Y \cup \{y\})$ is a linear space of dimension γ through $z := \tau_k(y)$. This gives a map

$$\chi : \mathrm{Hilb}_{k-1}(\Gamma)_{\mathrm{red}} \dashrightarrow \mathbb{G}(\gamma-1, M-1).$$

The point z is smooth, hence all these linear spaces sit in $\mathbb{T}_z\tau_k(X)\cong\mathbb{P}^n$. In other words, we have a map

$$\chi:\mathrm{Hilb}_{k-1}(\Gamma)_{\mathrm{red}} \dashrightarrow \mathbb{G}(\gamma-1,n-1)\subset\mathbb{G}(\gamma-1,M-1).$$

Note that $\dim\mathbb{G}(\gamma-1,n-1)=\gamma(n-\gamma)$ and $\dim\mathrm{Hilb}_{k-1}(\Gamma)=(k-1)\gamma$. By hypothesis $k>n$ and $\gamma>0$, hence

$$(k-1)\gamma>\gamma(n-\gamma).$$

Then the map χ is of fiber type and fibers have dimension at least $\gamma(k-n+\gamma-1)$.

Let $[Y_1],[Y_2]\in\chi^{-1}([\Lambda])$ be general, for $[\Lambda]\in\chi(\mathrm{Hilb}_{k-1}(\Gamma)_{\mathrm{red}})\subset\mathbb{G}(\gamma-1,n-1)$ a general point. The variety X is not $(k-1)$-twd and we are assuming that π^X_{k+1} is not of fiber type; therefore, by Lemma 16 (ii),

$$\dim(\Gamma\cap\Gamma(Y_i\cup\{y\}))=0$$

in a neighborhood of y_i. Since the fiber of χ is positive-dimensional we have

$$\Gamma(Y_1\cup\{y\})\not\supset Y_2. \tag{6}$$

The contact loci are irreducible, so, by (6), we conclude that

$$\Gamma(Y_1\cup\{y\})\neq\Gamma(Y_2\cup\{y\}).$$

Therefore, by Lemma 16 (iii), the positive-dimensional fiber of χ induces a positive-dimensional fiber of τ_k and we derive, by Lemma 16 (i), the contradiction that π^X_{k+1} is of fiber type. ■

Remark 19. Both assumptions (b) and (c) alone are reasonable and not over-demanding. Unfortunately, their combination is quite restrictive and narrows the range of applications we are aiming at.

We believe the statement is not optimal with respect to assumption (c). But we are not sure if it is true, in full generality, without any assumption of this kind. On the other hand, we strongly believe that for many interesting varieties, like Segre, Grassmannian, Veronese and their combinations, twd can occur only one step before the secant map becomes of fiber type. This is not the case for weak defectiveness, as shown in [11]. In [11, Theorem 1.1 (a)] it is proven that $\mathbb{G}(2,7)$ is 2- and 3- weakly defective without being 3-defective. Note that this variety is 3-twd but not 2-twd.

The next result generalizes the main result of [8] and it allows to avoid the bottleneck introduced by conditions (b) and (c) of Theorem 18 in many interesting situations.

Lemma 20. *Let $X\subset\mathbb{P}^N$ be an irreducible, reduced, and non-degenerate variety. Assume that X is not 1-twd and π^X_{k+1} is generically finite, in particular X is not $(k+1)$-defective. If X is k-twd then $\gamma_k<\gamma_{k+1}$.*

Proof. The variety X is not 1-twd, so we may assume, without loss of generality, that

$$\gamma_{k-1} < \gamma_k = \gamma_{k+1}.$$

Then $\gamma_{k+1} < n$ and $\sigma_{k+1}(X) \subsetneq \mathbb{P}^N$, hence by Theorem 13 (e), the contact loci are of type II and linearly independent linear spaces. Fix a set $\{x_1, \ldots, x_k, y\} \subset X$ of general points and let

$$\Gamma(x_1, \ldots, x_k, y) = \bigcup_{i=1}^{k} P_i \cup P_y$$

be the contact locus. Moreover, the assumption $\gamma_k = \gamma_{k+1}$ and Theorem 13 (a) force

$$\Gamma(x_1, \ldots, x_{k-1}, y) = \bigcup_{i=1}^{k-1} P_i \cup P_y,$$

with the same P_i's. Then

$$\bigcap_{y \in X} \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X, \mathbb{T}_y X \rangle \supset \langle \mathbb{T}_z X \rangle_{z \in P_i, i=1,\ldots,k-1}$$

We are assuming that $\gamma_{k-1} < \gamma_k$, so $P_i \not\subset \Gamma(x_1, \ldots, x_{k-1})$, and we have a proper inclusion

$$\langle \mathbb{T}_z X \rangle_{z \in P_i, i=1,\ldots,k-1} \supsetneq \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X \rangle.$$

Set

$$M_{A_i} = \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X, \mathbb{T}_{y_i} X \rangle,$$

for general points $y_1, y_2 \in X$. Then we have

$$M_{A_1} \cap M_{A_2} \supset \langle \mathbb{T}_z X \rangle_{z \in P_i, i=1,\ldots,k-1} \supsetneq \langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X \rangle.$$

and we conclude that

$$(M_{A_1} \cap M_{A_2}) \cap \mathbb{T}_{y_i} \neq \emptyset.$$

This shows that

$$\langle \mathbb{T}_{x_1} X, \ldots, \mathbb{T}_{x_{k-1}} X, \mathbb{T}_{y_1} X \rangle \cap \mathbb{T}_{y_2} X \neq \emptyset.$$

Hence the k-tangential projection τ_k^X is of fiber type and by Lemma 16 we derive the contradiction that π_{k+1}^X is of fiber type. ∎

Remark 21. Let us recall that 1-twd varieties are classified in [26] and are essentially generalized developable varieties. In particular, they are ruled by linear spaces and, with the unique exception of linear spaces, they are singular.

We are ready to apply the above results to get non-tangential weak defectiveness and hence identifiability statements.

Corollary 22. *Let $X \subset \mathbb{P}^N$ be an irreducible, reduced, and non-degenerate variety that is not 1-twd, for instance a smooth variety or a variety that is not covered by linear spaces. Assume that π_k^X is generically finite and $k \geq \dim X$.*

Then X is not $(k - \dim X)$-twd and it is not $(k - \dim X + 1)$-twd if π_k^X is not dominant. If moreover either $k > 2\dim X$ or π_k^X is not dominant and $k \geq 2\dim X$ then X is not $(k-1)$-twd.

In all the above cases X is h-identifiable.

Proof. By hypothesis, π_h is generically finite for any $h \leq k$. Then by Theorem 20, if it is j-twd then

$$\gamma_j < \gamma_{j+1}.$$

The contact locus is a subvariety of X, hence $\gamma_{k-\dim X} = 0$. This proves the first statement.

If π_k^X is not dominant then the contact locus is a proper subvariety and we have $\gamma_{k-\dim X+1} = 0$.

Assume that $k \geq 2\dim X$. Then by the first part, X is not j-twd for some $j > \dim X$. Then we apply Theorem 18 recursively to conclude the proof. We derive identifiability by Proposition 14. ∎

Remark 23. The first part of Corollary 22 extends the bounds in [8] to non-1-twd varieties. The main novelty is the second part that allows one to derive identifiability from non-defectivity for large enough secant varieties.

3. Application to tensor and structured tensor spaces

As already mentioned, identifiability is particularly interesting for tensor spaces. In this section we use our main result to explicitly state identifiability of a variety of tensor spaces. For this we will consider Segre, Segre–Veronese and Grassmannian varieties and their h-twd properties.

We start with some notation.

Notation 24. The variety $\Sigma(d_1,\ldots,d_r;n_1,\ldots,n_r)$ is the Segre–Veronese embedding of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ in $\mathbb{P}^{\prod\binom{n_i+d_i}{n_i}-1}$ via the complete linear system $|\mathcal{O}(d_1,\ldots,d_r)|$.

When all d_i's are 1 we have the Segre embedding and we let

$$X_{n_1,\ldots,n_r} := \Sigma(1,\ldots,1;n_1,\ldots,n_r) \quad \text{and} \quad X_n^r := \Sigma(1,\ldots,1;n,\ldots,n) \cong (\mathbb{P}^n)^r.$$

The expected generic rank is

$$\mathrm{gr}(\Sigma(d_1,\ldots,d_r;n_1,\ldots,n_r)) = \left\lceil \frac{\prod\binom{n_i+d_i}{n_i}}{(\sum n_i)+1} \right\rceil.$$

Using the notations in [2] we define

$$s(\Sigma(d_1,\ldots,d_r;n_1,\ldots,n_r) := \left\lfloor \frac{\prod\binom{n_i+d_i}{n_i}}{(\sum n_i)+1} \right\rfloor.$$

For simplicity in the case $n_1 = \cdots = n_r = n$ and $d_1 = \cdots = d_r = 1$ we set

$$s_n^r := s(\Sigma(d_1, \ldots, d_r; n_1, \ldots, n_r))$$

The variety $\mathbb{G}(k,n)$ is the Grassmannian parameterizing k-planes in $\mathbb{P}^n$ embedded in $\mathbb{P}(\bigwedge^{k+1} V)$ via the Plücker embedding. The expected generic rank is

$$\mathrm{gr}(\mathbb{G}(k,n)) = \left\lceil \frac{\binom{n+1}{k+1}}{(n-k)(k+1)+1} \right\rceil.$$

Remark 25. Note that we always have

$$s(\Sigma(d_1, \ldots, d_r; n_1, \ldots, n_r) \geq \mathrm{gr}(\Sigma(d_1, \ldots, d_r; n_1, \ldots, n_r)) - 1,$$

and equality occurs only when $\frac{\prod \binom{n_i+d_i}{n_i}}{(\sum n_i)+1}$ is not an integer. In particular, for any $h < s(X)$ we have $\sigma_h(X) \subsetneq \mathbb{P}^N$.

The defectivity of Segre and Segre–Veronese varieties is in general very far from being completely understood [1, 2, 6], but it is still better understood than their identifiability. For the latter the best asymptotic bounds we are aware of are in [8].

We start by proving the theorem of the introduction.

Theorem 26. *Let $X = X_1^k \cong (\mathbb{P}^1)^k$. Then X is not h-twd, and hence is h-identifiable, in the following range:*

- $(k,h) = (2,1), (3,2), (4,2), (5,4), (6,9)$,
- $k \geq 7$, $h < s(X)$.

Proof. For $k \leq 5$ this is well known, and can be easily checked also via a direct computation with commutative algebra software. For $k = 6$ this has been checked in [12] by a computer aided computation. Let us fix $k \geq 7$. By [14, Theorem 4.1], X is never defective. In particular, the morphism π_h^X is generically finite for $h \leq s_1^k$. When $k \geq 7$ we have

$$2 \dim X = 2k < \frac{2^k}{k+1} - 1 < s_1^k,$$

and we can apply Corollary 22. ■

Remark 27. The theorem confirms Conjecture 1.2 in [12] when the generic rank is an integer, that is, $\frac{2^k}{k+1} \in \mathbb{N}$. For $k \leq 6$ the listed cases are the only identifiable ones.

For 3-factor Segre we plug [19] directly in Theorem 18 to get the following.

Theorem 28. *Let $X = X_n^3$. Then X is h-identifiable for $h < s(X)$.*

Proof. For $n \leq 7$ the statement is proved in [19, Theorem 1.2]. For $n > 7$, by [31], the variety X is not h-defective for $h \leq s_n^3$ and by the results in [19], X is not h-twd for $h = 3n$: see the table in [19, Theorem 1.2] . Thus we are in a position to apply Theorem 18 recursively to prove that X is not h-twd, and hence is identifiable, for $h < s_n^3$. ■

For general diagonal Segre varieties we have a similar statement using [2].

Theorem 29. *Let $X = X_n^k$ with $n \geq 2$ and $k \geq 4$. Let*

$$n \geq \delta(X) \equiv s_n^k \bmod (n+1).$$

Then X is not h-twd, and hence is h-identifiable, for $h < s(X) - \delta(X)$. In particular, when $\delta(X) = 0$, X is h-identifiable for all $h < s(X)$.

Proof. Using the notations in [19, Theorem 6.7] let α be the greatest integer such that $n + 1 \geq 2^\alpha$. First we prove the statement for all but finitely many cases.

Claim 1. *If*

$$(k,n) \notin \begin{Bmatrix} (k,6) \text{ with } k \leq 6, & (k,5) \text{ with } k \leq 5, \\ (k,4) \text{ with } k \leq 5, & (k,3) \text{ with } k \leq 4 \end{Bmatrix}$$

then X is h-identifiable for $h < s(X) - \delta(X)$.

Proof. By [2, Theorem 5.2] we know that X is not h-defective as long as $h \leq s(X) - \delta(X)$. The variety X_n^k is not h-twd for

$$h \leq 2^{(k-1)\alpha-(k-1)} = 2^{(k-1)(\alpha-1)}$$

by [19, Theorem 6.7]. Let us assume that $n \neq 2$. A short computation by hand shows that

$$2^{(k-1)(\alpha-1)} > \dim X = kn$$

for every (k,n) in the list. Then, using Theorem 18 recursively, we conclude the proof. ∎

For $n = 2$ it is easy to check that the inequality

$$s_2^k = \left\lfloor \frac{3^k}{2k+1} \right\rfloor - \delta(X_2^k) > 4k = 2 \dim X_2^k$$

is satisfied for every $k \geq 5$ and so we can use Corollary 22. When $(k,n) = (6,6), (5,6)$ we have the inequalities

$$s_6^6 - \delta(X_6^6) > 2 \cdot 36 = 2 \dim X_6^6,$$
$$s_6^5 - \delta(X_6^5) > 2 \cdot 30 = 2 \dim X_6^5.$$

Then we use Corollary 22.

For all the remaining cases we have $(n+1)^k \leq 15000$ and we may use the computation in [20, Theorem 1.1] to deduce the required identifiability. ∎

The next class of Segre varieties we treat in detail is given by

$$X[k,n] := \mathbb{P}^k \times (\mathbb{P}^n)^{k+1}.$$

For these varieties we have

$$\mathrm{gr}(X[k,n]) = \frac{(k+1)(n+1)^{k+1}}{(k+1)n + k + 1} = (n+1)^k.$$

In particular, $\mathrm{gr}(X[k,n]) = s(X[k,n])$ is always an integer, that is, $X[k,n]$ is always perfect. Thanks to this special condition we have the following.

Theorem 30. *Let $X = X[k,n]$ with n odd and $k > 1$. Then X is h-identifiable for $h < \mathrm{gr}(X)$.*

Proof. The proof is entirely similar to that of Theorem 29. Indeed, by [2, Theorem 5.11] we know that all these Segre varieties are non-defective. If

$$(k,n) \neq (4,1), (3,1), (2,1), (2,3), (2,5)$$

then

$$(n+1)^k > 2(k+kn+n) = 2\dim X$$

and we can use Corollary 22. For all the exceptional cases we have

$$(k+1)(n+1)^{k+1} \leq 15000,$$

hence we may apply [20, Theorem 1.1]. ∎

Remark 31. Defective Segre varieties are expected to be quite rare, besides the unbalanced ones: see the conjecture in [2]. This conjecture has been checked via a computer in many cases [20, 36]. For all these special values our argument gives identifiability, confirming the numerical computation in [20].

Next we apply the same strategy to Segre–Veronese varieties. For this class of varieties the defectivity results are much weaker, and so are our bounds. Again the special case of binary forms is more favorable. We start by recalling the terminology of [30].

Definition 32. We say that $(d_1,\ldots,d_r;n)$ is *special* if

$$(d_1,\ldots,d_r;n) = (2,2a;2a+1), (1,1,2a;2a+1), (2,2,2;7), (1,1,1,1;3)$$

for $a \geq 1$. Otherwise $(d_1,\ldots,d_r;n)$ is called *non-special*.

Theorem 33. *Let $X = \Sigma(d_1,\ldots,d_r;1,\ldots,1)$ with $r = \dim X$. Assume $(d_1,\ldots,d_r;n)$ is non-special and $r \geq 6$. Then X is h-identifiable for $h < s(X)$.*

Proof. Since $(d_1,\ldots,d_r;n)$ is non-special, by [30, Theorem 2.1] the variety X is not h-defective for $h \leq \mathrm{gr}(X)$. Thanks to Theorem 26 we may assume, without loss of generality, that $d_1 > 1$ and we have

$$s(X) = \left\lfloor \frac{(d_1+1)\cdots(d_r+1)}{r+1} \right\rfloor \geq \frac{3\cdot 2^{r-1}}{r+1} - 1.$$

In particular,

$$\frac{3\cdot 2^{r-1}}{r+1} - 1 > 2r = 2\dim X \quad \text{for every } r \geq 6.$$

The variety X is not 1-twd and so we conclude the proof by using Corollary 22. ∎

For general Segre–Veronese varieties we have the following.

Theorem 34. *Let $X := \Sigma(d_1,\dots,d_r;n_1,\dots,n_r)$ be a Segre–Veronese variety. Assume $r \geq 2$,*

$$n_1^{\lfloor \log_2(d-1)\rfloor} \geq 2(n_1+\dots+n_r),$$

and set $d = d_1+\dots+d_r$. Then X is h-identifiable for $h \leq n_1^{\lfloor \log_2(d-1)\rfloor}-1$.

Proof. By [6, Theorem 1.1], X is not h-defective for

$$h \leq n_1^{\lfloor \log_2(d-1)\rfloor} - (n_1+\dots+n_r)+1.$$

Under our numerical assumptions, $\sigma_h(X) \subsetneq \mathbb{P}^N$ and we may assume $h \geq 2\dim X$. Then we apply Corollary 22. ∎

Remark 35. For the Veronese variety of $\mathbb{P}^n$, that is, $\Sigma(d_1;n_1)$ it is easy, via Corollary 22 and [3], to re-prove the identifiability results of [33] and [21].

As in the Segre case, for special classes of Segre–Veronese varieties there are better non-defectivity results. Here we recall the notation in [1]. Let $X := \Sigma(1,2;m,n)$ be the Segre–Veronese variety $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$ in $\mathbb{P}^N$ where

$$N = (m+1)\binom{n+2}{2} - 1.$$

Let

$$r(m,n) = \begin{cases} m^3-2m & \text{if } m \text{ even and } n \text{ odd,} \\ \frac{(m-2)(m+1)^2}{2} & \text{otherwise,} \end{cases}$$

and

$$s(X) = \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor.$$

With this in mind we have the following.

Corollary 36. *Let $X = \Sigma(1,2;m,n)$. If $n > r(m,n)$ and*

$$\left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor \geq 2(m+n)$$

then X is not h-twd, and hence is h-identifiable, for $h < s(X)$.

Proof. In our range, X is not h-defective by [1, Theorem 1.1] and $\sigma_h(X) \subsetneq \mathbb{P}^N$. Moreover,

$$s(X) = \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor \geq 2(m+n) = 2\dim X$$

and we may apply Corollary 22. ∎

Let us now consider the case of $\mathbb{P}^m \times \mathbb{P}^n$ embedded with $\mathcal{O}(1,d)$ for $d \geq 3$.

Corollary 37. *Let $X = \Sigma(1, d; m, n)$ with $d \geq 3$ and $m, n \geq 1$. Let*

$$s(X) = \max\left\{ s \in \mathbb{N} \;\middle|\; s \text{ is a multiple of } m+1 \text{ and } s \leq \left\lfloor \frac{(m+1)\binom{n+d}{d}}{m+n+1} \right\rfloor \right\}.$$

If $s(X) > 2(m+n)$ then X is not h-twd, and hence is h-identifiable, for $h < s(X)$.

Proof. By [10, Theorem 2.3], X is not h-defective for $h \leq s(X)$ and $\sigma_h(X) \subsetneq \mathbb{P}^{(m+1)\binom{n+d}{d}-1}$.

As X is smooth, in particular it is not 1-twd. Since

$$s(X) > 2(m+n) = 2 \dim X$$

we can apply Corollary 22. ∎

Remark 38. Similar statements about subgeneric identifiability of $\mathbb{P}^n \times \mathbb{P}^1$ embedded with $\mathcal{O}(a, b)$ can be derived from Corollary 22 using the non-defectivity results of [7].

Finally, we consider Grassmannian varieties. For this class of tensor spaces very little is known about identifiability. To the best of our knowledge the following is the first non-computer-aided result for them.

Theorem 39. *Let $X = \mathbb{G}(k, n)$ with $2k + 1 \leq n$. Assume that*

$$\left\lfloor \left(\frac{n+1}{k+1} \right)^{\lfloor \log_2(k) \rfloor} \right\rfloor \geq 2(n-k)(k+1).$$

Then X is h-identifiable for

$$h \leq \left(\frac{n+1}{k+1} \right)^{\lfloor \log_2(k) \rfloor} - 1.$$

Proof. By [32, Theorem 5.4], in our numerical range X is not h-defective and $\sigma_h(X) \subsetneq \mathbb{P}^N$. Then we use Corollary 22. ∎

The technique we developed can be applied to many other classes of varieties, once their defectivity behavior is known. As a sample we conclude the paper with the following example.

Example 40. C. Améndola, J.-C. Faugère, K. Ranestad and B. Sturmfels [4, 5] studied the Gaussian moment variety

$$\mathcal{G}_{1,d} \subset \mathbb{P}^d$$

whose points are the vectors of all moments of degree $\leq d$ of a 1-dimensional Gaussian distribution. They proved that $\mathcal{G}_{1,d}$ is a surface for every d and $\sigma_h(\mathcal{G}_{1,d})$ always has the expected dimension. In [8, Example 5.8] it is shown that $\mathcal{G}_{1,d}$ is not uniruled by lines, in particular it is not 1-twd. As usual let

$$s(\mathcal{G}_{1,d}) = \left\lfloor \frac{d+1}{3} \right\rfloor \geq \operatorname{gr}(\mathcal{G}_{1,d}) - 1.$$

Then by Corollary 22, $\mathcal{G}_{1,d}$ is h-identifiable for $h < s(\mathcal{G}_{1,d})$ when $d \geq 14$.

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