J. Eur. Math. Soc. 25, 995–1044 (2023) DOI 10.4171/JEMS/1203

András Juhász · Ian Zemke

Concordance surgery and the Ozsváth–Szabó 4-manifold invariant

Received November 19, 2019

Abstract. We compute the effect of concordance surgery, a generalization of knot surgery defined using a self-concordance of a knot, on the Ozsváth–Szabó 4-manifold invariant. The formula involves the graded Lefschetz number of the concordance map on knot Floer homology. The proof uses the sutured Floer TQFT, and a version of sutured Floer homology perturbed by a 2-form.

Keywords. Heegaard Floer homology, 4-manifolds, concordance

1. Introduction

Let X be a smooth, connected, closed, and oriented 4-manifold with $b_2^+(X) \geq 2$. Suppose that $T \subseteq X$ is a smoothly embedded, homologically essential torus with trivial self-intersection, and let $K \subseteq S^3$ be a knot. Fintushel and Stern [4] defined the *knot surgery* operation on X, resulting in the 4-manifold X_K. This is obtained by gluing $X \setminus N(T)$ and $S^1 \times (S^3 \setminus N(K))$ via an orientation-reversing diffeomorphism of their boundaries that maps a meridian of T to a longitude of K. They showed that

$$SW(X_K) = \Delta_K(z) \cdot SW(X), \tag{1.1}$$

where SW denotes the Seiberg–Witten invariant, and $\Delta_K(z)$ is the symmetrized Alexander polynomial of K. The variable z corresponds to $\exp(2[T])$, where $[T]$ is the homology class induced by T in $H_2(X_K)$.

If $\pi_1(X \setminus T) = 1$, then X and X_K are simply connected and have the same intersection form, and are hence homeomorphic by Freedman's theorem. Note that every symmetric integral Laurent polynomial $p(z)$ satisfying $p(1) = \pm 1$ is the Alexander polynomial of a knot in S^3. Consequently, if $SW(X) \neq 0$, then we obtain infinitely many pairwise non-diffeomorphic smooth structures on X. When X is the K3 surface, $SW(X) = 1$, and hence we obtain a different smooth structure on X for every such Laurent polynomial.

András Juhász: Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK; juhasza@maths.ox.ac.uk

Ian Zemke: Department of Mathematics, Princeton University, Princeton, NJ 08544, USA; izemke@math.princeton.edu

Mathematics Subject Classification (2020): 57R58, 57R55, 57M27

Mark [20, Theorem 3.1] obtained a result analogous to equation (1.1) for the Ozsváth–Szabó 4-manifold invariant [25], which is expected to coincide with the Seiberg–Witten invariant. For a closed 4-manifold X with $b_2^+(X) \geq 2$, Ozsváth and Szabó's invariant takes the form of a map

$$\Phi_X \colon \mathrm{Spin}^c(X) \to \mathbb{F}_2.$$

We write $\Phi_{X,\mathfrak{s}}$ for the value of Φ_X on $\mathfrak{s}$. It is convenient to organize the invariants of different Spin^c structures into a single polynomial. Recall that $\mathrm{Spin}^c(X)$ is an affine space over $H^2(X)$, so the difference of two Spin^c structures is a well-defined cohomology class. If $\boldsymbol{b} = (b_1, \dots, b_n)$ is a basis of $H^2(X;\mathbb{R})$, we can arrange the 4-manifold invariant into the element

$$\Phi_{X;\boldsymbol{b}} := \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} \Phi_{X,\mathfrak{s}} \cdot z_1^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0) \cup b_1, [X] \rangle} \cdots z_n^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0) \cup b_n, [X] \rangle}$$

of the n-variable Novikov ring over $\mathbb{F}_2$, where $\mathfrak{s}_0$ is some choice of base Spin^c structure on X, and $i_* \colon H^2(X) \to H^2(X;\mathbb{R})$ is induced by the map of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$. If $H^2(X)$ is torsion-free, then $\Phi_{X;\boldsymbol{b}}$ completely encodes the map Φ_X. It is natural to view $\Phi_{X;\boldsymbol{b}}$ as a perturbed version of the mixed invariant; see Proposition 4.3.

Concordance surgery is a generalization of knot surgery due to Fintushel and Stern; see Akbulut [2, Section 2] and Tange [28]. Let K be a knot in a homology 3-sphere Y (note that Akbulut only considered the case $Y = S^3$). Given a self-concordance $\mathcal{C} = (I \times Y, A)$ from (Y, K) to itself, we can construct a 4-manifold $X_{\mathcal{C}}$ as follows. We glue the ends of A together to form a 2-torus $T_{\mathcal{C}}$ embedded in $S^1 \times Y$. After removing a neighborhood of $T_{\mathcal{C}}$, we get a 4-manifold $W_{\mathcal{C}}$ with boundary $\mathbb{T}^3$. Viewing $N(T)$ as $T \times D^2$, we pick any orientation-preserving diffeomorphism $\phi \colon \partial(X \setminus N(T)) \to \partial N(T_{\mathcal{C}})$ that sends $[\{p\} \times \partial D^2]$ to $[\{q\} \times \ell_K]$, where $p \in T$, $q \in S^1$, and ℓ_K is a longitude of K. We write $X_{\mathcal{C}}$ for any manifold constructed as the union

$$X_{\mathcal{C}} := (X \setminus N(T)) \cup_\phi W_{\mathcal{C}}.$$

Fintushel and Stern asked in the late 90s whether a formula similar to equation (1.1) relates $SW(X)$ and $SW(X_{\mathcal{C}})$; see Akbulut [2, Remark 2.2].

Our main result gives a formula relating the Ozsváth–Szabó 4-manifold invariants of X and $X_{\mathcal{C}}$ in terms of the graded Lefschetz number of the concordance map

$$\widehat{F}_{\mathcal{C}} \colon \widehat{HFK}(Y, K) \to \widehat{HFK}(Y, K)$$

defined by the first author [8]. This map preserves the Alexander and Maslov gradings [10, Theorem 5.18]. The graded Lefschetz number is the polynomial

$$\mathrm{Lef}_z(\mathcal{C}) := \sum_{i \in \mathbb{Z}} \mathrm{Lef}\big(\widehat{F}_{\mathcal{C}}|_{\widehat{HFK}(Y,K,i)} \colon \widehat{HFK}(Y,K,i) \to \widehat{HFK}(Y,K,i)\big) \cdot z^i.$$

We note that the concordance map $\widehat{F}_{\mathcal{C}}$ on knot Floer homology depends on some extra decorations that we are suppressing from the notation. Nonetheless, we will see that the graded Lefschetz number is independent of these decorations.

If $[T] \neq 0 \in H_2(X;\mathbb{R})$, then we can pick a basis $\boldsymbol{b} = (b_1, \dots, b_n)$ of $H^2(X;\mathbb{R})$ such that

$$\langle b_1, [T] \rangle = 1 \quad \text{and} \quad \langle b_i, [T] \rangle = 0 \quad \text{for } i > 1. \tag{1.2}$$

There are natural isomorphisms $H^2(X;\mathbb{R}) \cong H^2(X_{\mathcal{C}};\mathbb{R})$ and $\mathrm{Spin}^c(X) \cong \mathrm{Spin}^c(X_{\mathcal{C}})$. By a slight abuse of notation, we will use the same notation for corresponding second cohomology classes and Spin^c structures on X and $X_{\mathcal{C}}$. In particular, the base Spin^c structure $\mathfrak{s}_0$ on X corresponds to a base Spin^c structure $\mathfrak{s}_0$ on $X_{\mathcal{C}}$, and we define the 4-manifold invariants $\Phi_{X;\boldsymbol{b}}$ and $\Phi_{X_{\mathcal{C}};\boldsymbol{b}}$ using this correspondence. We now state our main result:

Theorem 1.1. *Let X be a closed, oriented 4-manifold such that $b_2^+(X) \geq 2$. Suppose that T is a smoothly embedded 2-torus in X with trivial self-intersection such that $[T] \neq 0$ in $H_2(X;\mathbb{R})$. Furthermore, let $\boldsymbol{b} = (b_1, \dots, b_n)$ be a basis of $H^2(X;\mathbb{R})$ satisfying equation (1.2). If $\mathcal{C}$ is a self-concordance of (Y, K), where Y is a homology 3-sphere, then*

$$\Phi_{X_{\mathcal{C}};\boldsymbol{b}} = \mathrm{Lef}_{z_1}(\mathcal{C}) \cdot \Phi_{X;\boldsymbol{b}}.$$

If $\mathcal{C}$ is the product concordance $(I \times Y, I \times K)$, then $\hat{F}_{\mathcal{C}}$ is the identity of $\widehat{HFK}(Y, K)$, so $\mathrm{Lef}_z(\mathcal{C})$ is the graded Euler characteristic of $\widehat{HFK}(Y, K)$, which is $\Delta_K(t)$. Hence, as a special case, we recover the formula of Mark [20, Theorem 3.1], i.e., the Heegaard Floer version of the Fintushel–Stern knot surgery formula.

When $\pi_1(X \setminus T) = 1$ and $Y = S^3$, the manifold $X_{\mathcal{C}}$ is homeomorphic to X. In contrast, we have the following corollary to Theorem 1.1, which we prove in Section 5.1:

Corollary 1.2. *If $\mathrm{Lef}_z(\mathcal{C}) \neq 1$ and $\Phi_{X;\boldsymbol{b}} \neq 0$, the 4-manifold $X_{\mathcal{C}}$ is not diffeomorphic to X.*

Since $\mathrm{Lef}_z(\mathcal{C})$ is always symmetric and satisfies $\mathrm{Lef}_z(\mathcal{C})(1) = \pm 1$, it is unclear whether, using concordance surgery, we obtain any smooth structures not arising from knot surgery. Nonetheless, in [12], we use the techniques of this paper to produce infinite families of exotic orientable surfaces in B^4.

We note that the proofs of the knot surgery formula (1.1) due to Fintushel and Stern for the Seiberg–Witten invariant, and to Mark for the Ozsváth–Szabó invariant, are based on the skein relation for the Alexander polynomial, and hence are only well-suited to knots in S^3. Our theorem applies to a more general setting, where K is allowed to be a null-homologous knot in an arbitrary homology 3-sphere Y. Our proof of Theorem 1.1 also extends to the situation where we consider a self-concordance $(W, \mathcal{C})$ of a pair (Y, K) such that W is an integer homology cobordism from Y to itself, though we restrict to the setting where $W = I \times Y$ to simplify the notation. The key technical advancement that led to this proof is our previous computation of the sutured Floer trace and cotrace cobordism maps [11, Theorem 1.1].

Our Theorem 1.1 could be used to construct exotic smooth structures on 4-manifolds with non-trivial fundamental group. Suppose that $\pi_1(X \setminus T) = 1$. If $\Phi_{X;\boldsymbol{b}} \neq 0$, and K and K' are knots in a homology 3-sphere Y such that X_K and $X_{K'}$ are homeomorphic and $\Phi_{X;\boldsymbol{b}} \cdot \Delta_K(z)$ and $\Phi_{X;\boldsymbol{b}} \cdot \Delta_{K'}(z)$ are not equivalent under the action of automorphisms of $H_2(X)$, then X_K and $X_{K'}$ are non-diffeomorphic 4-manifolds with fundamental group $\pi_1(Y)/\langle [K] \rangle$, where $\langle [K] \rangle$ is the normal subgroup of $\pi_1(Y)$ generated by K.

After proving Theorem 1.1, we give an account of the naturality and functoriality of the perturbed versions of sutured Floer homology and Heegaard Floer homology, since these are more subtle than in the unperturbed setting, and many details are only sketched in the literature.

Finally, we note that it might be possible to carry out our argument for the Seiberg–Witten invariant using the work of Zhenkun Li [16] to construct gluing and cobordism maps for Kronheimer and Mrowka's sutured monopole Floer homology [14]. A key technical step which has not yet been completed in this program is the computation of the induced maps by the trace and cotrace cobordisms, which we performed in the setting of sutured Floer homology in [11, Theorem 1.1].

1.1. Organization

In Sections 2 and 3, we give an overview of the construction of the perturbed Floer homology groups, and the perturbed cobordism maps, and we state the properties that are most relevant to the proof of Theorem 1.1. In Section 4, we give some background on the Ozsváth–Szabó 4-manifold invariant. In Section 5, we prove Theorem 1.1. In Sections 6 and 7, we give a proof of the naturality of the perturbed sutured Floer groups, the well-definedness of the cobordism maps, and also several useful properties.

2. Perturbing sutured Floer homology by a 2-form

Ozsváth and Szabó [21, Section 3.1] defined a version of Heegaard Floer homology for closed 3-manifolds perturbed by a second cohomology class, which we now extend to sutured manifolds. The unperturbed version of sutured Floer homology was defined by the first author [7], and its naturality was shown by Thurston and the authors [13].

Let Λ denote the Novikov ring over $\mathbb{F}_2$ in a single variable z. Its elements are formal sums $\sum_{x \in \mathbb{R}} n_x z^x$, where $n_x \in \mathbb{F}_2$, and the set

$$\{x \in (-\infty, c] : n_x \neq 0\}$$

is finite for every $c \in \mathbb{R}$. Note that Λ is a field.

Suppose that (M, γ) is a balanced sutured manifold, and ω is a closed 2-form on M. Then ω induces an action of $\mathbb{F}_2[H^1(M, \partial M)] \cong \mathbb{F}_2[H_2(M)]$ on Λ, via the formula

$$e^a \cdot z^x = z^{x + \int_a \omega}$$

for $x \in \mathbb{R}$ and $a \in H_2(M)$. We denote by Λ_ω the ring Λ viewed as a module over $\mathbb{F}_2[H^1(M, \partial M)]$.

For a sutured manifold (M, γ), equipped with a closed 2-form ω and a relative Spin^c structure $\underline{\mathfrak{s}}$, we write $SFH(M, \gamma, \underline{\mathfrak{s}}; \Lambda_\omega)$ for the perturbed sutured Floer homology, which we describe in this section. Using the terminology of Baldwin and Sivek [3], the most natural category for $SFH(M,\gamma,\underline{\mathfrak{s}};\Lambda_\omega)$ is the category of *projective transitive systems*. See Section 2.1 for a precise definition. We state the following version of naturality for perturbed sutured Floer homology:

Theorem 2.1. *Suppose (M,γ) is a sutured manifold, and ω is a closed 2-form on M.*

(1) *If $\underline{\mathfrak{s}}\in \mathrm{Spin}^c(M,\gamma)$, then $SFH(M,\gamma,\underline{\mathfrak{s}};\Lambda_\omega)$ forms a projective transitive system of Λ-modules, indexed by the set of pairs $(\mathcal{H},J)$, where $\mathcal{H}$ is an admissible diagram for (M,γ), and J is a generic almost complex structure.*

(2) *If $\omega=d\eta$ for a 1-form η, then $SFH(M,\gamma;\Lambda_\omega)$ (the sum over all Spin^c structures) forms a projective transitive system of Λ-modules, indexed by the set of pairs $(\mathcal{H},J)$, as above.*

We will prove Theorem 2.1 in Section 6, though we describe the construction of the perturbed groups in Section 2.2.

Remark 2.2. Our construction of $SFH(M,\gamma;\Lambda_\omega)$ gives neither a genuine transitive system when we restrict to a single Spin^c structure on M, nor a projective transitive system when we sum over all Spin^c structures. See Example 6.7 and Lemma 6.8 for counterexamples.

2.1. *Transitive systems and their morphisms*

Definition 2.3. Suppose that $\mathcal{C}$ is a category and I is a set. A *transitive system in $\mathcal{C}$, indexed by I*, is a collection of objects $(X_i)_{i\in I}$, as well as a distinguished morphism $\Psi_{i\to j}\colon X_i\to X_j$ for each $(i,j)\in I\times I$, such that

(1) $\Psi_{j\to k}\circ\Psi_{i\to j}=\Psi_{i\to k}$, and

(2) $\Psi_{i\to i}=\mathrm{id}_{X_i}$.

Example 2.4. Transitive systems in the following categories are important to our present paper.

(T-1) The category $\mathcal{C}=\mathcal{R}\text{-}\mathsf{Mod}$ of left modules over a ring $\mathcal{R}$. The morphism set $\mathrm{Hom}_{\mathcal{C}}(X_1,X_2)$ is equal to the set $\mathrm{Hom}_{\mathcal{R}}(X_1,X_2)$ of $\mathcal{R}$-module homomorphisms from X_1 to X_2.

(T-2) The projectivized category of Λ-modules $\mathcal{C}=\mathsf{P}(\Lambda\text{-}\mathsf{Mod})$. The objects are Λ-modules and the morphism set $\mathrm{Hom}_{\mathcal{C}}(X_1,X_2)$ is the projectivization of $\mathrm{Hom}_{\Lambda}(X_1,X_2)$ under the action of elements of Λ of the form $z^x\in\Lambda$.

(T-3) The homotopy category $\mathcal{C}=\mathsf{K}(\mathcal{R}\text{-}\mathsf{Mod})$ of chain complexes over the ring $\mathcal{R}$. The objects are chain complexes over $\mathcal{R}$. If X_1 and X_2 are two chain complexes, the set of $\mathcal{R}$-module homomorphisms $\mathrm{Hom}_{\mathcal{R}}(X_1,X_2)$ is a chain complex with differential $\partial_{\mathrm{Hom}}(f)=f\circ\partial_{X_1}-\partial_{X_2}\circ f$ for $f\in\mathrm{Hom}_{\mathcal{R}}(X_1,X_2)$. The morphism set $\mathrm{Hom}_{\mathcal{C}}(X_1,X_2)$ in $\mathcal{C}$ is the homology $H_*(\mathrm{Hom}_{\mathcal{R}}(X_1,X_2))$. Equivalently, $\mathrm{Hom}_{\mathcal{C}}(X_1,X_2)$ is the set of chain maps modulo chain homotopy.

(T-4) The projectivized homotopy category $\mathcal{C} = \mathsf{P}(\mathsf{K}(\Lambda\text{-}\mathsf{Mod}))$. The objects of $\mathcal{C}$ are chain complexes over Λ. The morphism set $\operatorname{Hom}_{\mathcal{C}}(X_1, X_2)$ is the projectivization of $H_*(\operatorname{Hom}_\Lambda(X_1, X_2))$ under the action of elements of Λ of the form z^x.

The categories in (T-1) and (T-3) are preadditive (i.e., the morphism sets are abelian groups), while the categories in (T-2) and (T-4) are not. In these latter categories, composition of projective morphisms is well-defined, though addition of morphisms is not.

Following the terminology of Baldwin and Sivek [3], we call a transitive system over one of the categories (T-2) and (T-4) a *projective transitive system*. In category (T-2), given morphisms $f, g \in \operatorname{Hom}_\Lambda(X_1, X_2)$, we will use the notation $f \doteq g$ if $f = z^x \cdot g$ for some $x \in \mathbb{R}$. Similarly, in case (T-4), given chain maps $\phi, \psi \in H_*(\operatorname{Hom}_\Lambda(X_1, X_2))$, we write $\phi \mathrel{\dot{\simeq}} \psi$ if $\phi \simeq z^x \cdot \psi$ for some $x \in \mathbb{R}$, where $\simeq$ denotes chain homotopy equivalence. If $\phi \mathrel{\dot{\simeq}} \psi$, we say ϕ and ψ are *projectively equivalent*. Finally, if X is a Λ-module and $a, b \in X$, we write $a \doteq b$ if $a = z^x \cdot b$ for some $x \in \mathbb{R}$.

There is a natural notion of morphism between transitive systems:

Definition 2.5. If $(C_i)_{i \in I}$ and $(D_j)_{j \in J}$ are two transitive systems in the category $\mathcal{C}$, a *morphism of transitive systems* is a collection of morphisms

$$F_{(i,j)} \colon C_i \to D_j$$

in $\mathcal{C}$ such that

$$\Psi_{j \to j'} \circ F_{(i,j)} \circ \Psi_{i' \to i} = F_{(i',j')}$$

for all $i, i' \in I$ and $j, j' \in J$.

Remark 2.6. If $f \colon C_{i_0} \to D_{j_0}$ is an element of $\operatorname{Hom}_{\mathcal{C}}(C_{i_0}, D_{j_0})$ for some fixed $i_0 \in I$ and $j_0 \in J$, then f induces a unique morphism $F_{(i,j)}$ of transitive systems from $(C_i)_{i \in I}$ to $(D_j)_{j \in J}$, given by

$$F_{(i,j)} = \Psi_{j_0 \to j} \circ f \circ \Psi_{i \to i_0}.$$

If $\mathcal{C}$ is a category, then the collection of transitive systems over $\mathcal{C}$ itself forms a category, for which we write $\mathcal{T}(\mathcal{C})$. Hence, we can define a transitive system of transitive systems over $\mathcal{C}$.

Remark 2.7. If $\mathcal{X} = ((X_{ij})_{j \in J_i})_{i \in I}$ is a transitive system in $\mathcal{T}(\mathcal{C})$, we may naturally view $\mathcal{X}$ as a transitive system over $\mathcal{C}$ indexed by $K := \bigcup_{i \in I} J_i$.

2.2. *The perturbed chain complexes*

In this section, we define the perturbed sutured Floer complexes. We use the cylindrical reformulation of Heegaard Floer homology, due to Lipshitz [17]. Suppose (M, γ) is a balanced sutured manifold with a closed 2-form ω. If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram, we pick an almost complex structure on $\Sigma \times I \times \mathbb{R}$ that is tamed by the split symplectic form. The surface Σ splits M into two sutured compression bodies, for which we write U_α and U_β. We let D_α and D_β be two choices of compressing disks for U_α

and U_β, equipped with radial foliations, such that D_α intersects Σ along $\boldsymbol{\alpha}$, and similarly for D_β.

For generators $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a homotopy class $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ of disks determines a 2-chain $\mathcal{D}(\phi)$ on Σ, which has boundary on $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$. We cone $\mathcal{D}(\phi)$ along the compressing disks D_α and D_β to obtain a 2-chain $\widetilde{\mathcal{D}}(\phi)$. We note that the 2-chain $\widetilde{\mathcal{D}}(\phi)$ depends on the choice of radial foliations on D_α and D_β. The 2-chain $\widetilde{\mathcal{D}}(\phi)$ is closed if and only if $\boldsymbol{x} = \boldsymbol{y}$.

We define

$$A_\omega(\phi) := \int_{\widetilde{\mathcal{D}}(\phi)} \omega.$$

When the choice of ω is clear from the context, we just write $A(\phi)$.

There is a map $H\colon \pi_2(\boldsymbol{x}, \boldsymbol{x}) \to H_2(M)$, obtained by coning off the periodic domain $\mathcal{D}(\phi)$ for $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{x})$; see [7, Definition 3.9]. In particular,

$$H(\phi) = [\widetilde{\mathcal{D}}(\phi)].$$

The chain complex $CF(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega)$ is the free Λ-module generated by intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ which satisfy $\mathfrak{s}(\boldsymbol{x}) = \underline{\mathfrak{s}}$. The differential is given by counting holomorphic curves in $\Sigma \times I \times \mathbb{R}$ via the formula

$$\partial \boldsymbol{x} := \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi) = 1}} (|\mathcal{M}(\phi)/\mathbb{R}| \bmod 2) \cdot z^{A(\phi)} \cdot \boldsymbol{y}$$

for $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The fact that $\partial^2 = 0$ follows by analyzing the ends of the 1-dimensional moduli spaces $\mathcal{M}(\phi)/\mathbb{R}$ for classes ϕ with Maslov index 2. We set

$$SFH(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega) := H_*(CF(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega), \partial).$$

The group $SFH(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega)$ also depends on J and the compressing disks, though we omit the extra data from the notation.

2.3. *Perturbed sutured cobordism maps*

In [8], the first author defined a notion of cobordism between sutured manifolds, and constructed functorial cobordism maps.

Definition 2.8. A *cobordism of sutured manifolds*

$$\mathcal{W} = (W, Z, [\xi])\colon (M_0, \gamma_0) \to (M_1, \gamma_1)$$

is a triple such that

(1) W is a compact, oriented 4-manifold with boundary,

(2) Z is a compact, codimension 0 submanifold with boundary of ∂W, and $\partial W \setminus \operatorname{int}(Z) = -M_0 \sqcup M_1$,

(3) $[\xi]$ is an equivalence class of positive contact structures on Z (see [8, Definition 2.3]) such that ∂Z is a convex surface with dividing set γ_i on ∂M_i for $i \in \{0, 1\}$.

In Section 7, we will define perturbed versions of the sutured manifold cobordism maps. If $\mathcal{W} = (W, Z, [\xi])$ is a sutured manifold cobordism from (M_0, γ_0) to (M_1, γ_1), and ω is a closed 2-form on W, then we will define a chain map

$$F_{\mathcal{W};\omega}: SFH(M_0, \gamma_0; \Lambda_{\omega|_{M_0}}) \to SFH(M_1, \gamma_1; \Lambda_{\omega|_{M_1}}),$$

which is only well-defined up to an ambiguity described in Proposition 2.9.

If $\mathcal{H}$ is a Heegaard diagram for (M, γ), we can view

$$SFH(\mathcal{H}; \Lambda_\omega) = \bigoplus_{\underline{\mathfrak{s}} \in \mathrm{Spin}^c(M,\gamma)} SFH(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega).$$

Consequently, there are inclusion and projection maps

$$i_{\underline{\mathfrak{s}}}: SFH(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega) \to SFH(\mathcal{H}; \Lambda_\omega) \quad \text{and} \quad \pi_{\underline{\mathfrak{s}}}: SFH(\mathcal{H}; \Lambda_\omega) \to SFH(\mathcal{H}, \underline{\mathfrak{s}}; \Lambda_\omega).$$

Proposition 2.9. *Suppose $\mathcal{W} = (W, Z, [\xi]): (M_0, \gamma_0) \to (M_1, \gamma_1)$ is a sutured manifold cobordism, and ω is a closed 2-form on W.*

(1) *If $\underline{\mathfrak{s}}_i \in \mathrm{Spin}^c(M_i, \gamma_i)$ for $i \in \{0, 1\}$, then the map*

$$\pi_{\underline{\mathfrak{s}}_1} \circ F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}_0}: SFH(M_0, \gamma_0, \underline{\mathfrak{s}}_0; \Lambda_{\omega|_{M_0}}) \to SFH(M_1, \gamma_1, \underline{\mathfrak{s}}_1; \Lambda_{\omega|_{M_1}})$$

is well-defined up to an overall factor of z^x, for $x \in \mathbb{R}$.

(2) *More generally, if $[\omega|_{M_1}] = 0$, then $F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}_0}$ is well-defined up to an overall factor of z^x. If $[\omega|_{M_0}] = 0$, then $\pi_{\underline{\mathfrak{s}}_1} \circ F_{\mathcal{W};\omega}$ is well-defined up to a factor of z^x. If $[\omega|_{M_0}] = 0$ and $[\omega|_{M_1}] = 0$, then the total map $F_{\mathcal{W};\omega}$ is well-defined up to an overall factor of z^x.*

The main idea of the construction is to incorporate the coning construction of Ozsváth and Szabó [21] at each step of the construction of the unperturbed sutured cobordism maps in [8]. In Section 7, we describe the construction in detail, and prove Proposition 2.9. We note that, to define the total cobordism map in of Proposition 2.9 (2), we use our formula for the sutured trace cobordism map [11, Theorem 1.1]; see Section 7.6. In Section 7.7, we will prove the following composition law for the perturbed sutured cobordism maps:

Proposition 2.10. *Suppose the sutured manifold cobordism $\mathcal{W} = (W, Z, [\xi])$ decomposes as $\mathcal{W}_2 \circ \mathcal{W}_1$, where*

$$\mathcal{W}_1 = (W_1, Z_1, [\xi_1]): (M_0, \gamma_0) \to (M_1, \gamma_1), \quad \mathcal{W}_2 = (W_2, Z_2, [\xi_2]): (M_1, \gamma_1) \to (M_2, \gamma_2).$$

Let ω be a closed 2-form on W, and write $\omega_1 = \omega|_{W_1}$ and $\omega_2 = \omega|_{W_2}$.

(1) *If $[\omega]$ restricts trivially to M_0, M_1, and M_2, then*

$$F_{\mathcal{W};\omega} \doteq F_{\mathcal{W}_2;\omega_2} \circ F_{\mathcal{W}_1;\omega_1}.$$

(2) *More generally, if $[\omega]$ restricts trivially to M_1 and M_2, and $\underline{\mathfrak{s}}_0 \in \mathrm{Spin}^c(M_0, \gamma_0)$, then*

$$F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}_0} \doteq F_{\mathcal{W}_2;\omega_2} \circ F_{\mathcal{W}_1;\omega_1} \circ i_{\underline{\mathfrak{s}}_0}.$$

Similar formulas hold if $[\omega]$ restricts trivially to both M_0 and M_1, or just to M_1.

2.4. Alexander gradings and perturbations on cylinders

We now state a simple formula for the sutured cobordism map for a perturbation of the identity cobordism of a knot complement, which we need for our proof of Theorem 1.1.

Suppose that K is a knot in an integer homology sphere Y. Let $Y(K)$ denote $Y \setminus N(K)$, decorated with two oppositely oriented meridional sutures. A sutured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $Y(K)$ is equivalent to a doubly-pointed diagram for (Y, K): To obtain a doubly-pointed diagram from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, we collapse each of the boundary components of Σ to a basepoint. We let w denote the point where K intersects Σ negatively, and z denotes the point where K intersects Σ positively. There is a tautological isomorphism

$$\widehat{HFK}(Y, K) \cong SFH(Y(K)),$$

since the generators and differential coincide.

The relative Alexander grading on $\widehat{HFK}(Y, K)$ is given as follows. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then we pick a class $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ (possibly going over w and z). The relative Alexander grading is given by the formula

$$A(\boldsymbol{x}, \boldsymbol{y}) = n_z(\phi) - n_w(\phi).$$

The relative Alexander grading admits an absolute lift, which can be specified by a symmetry requirement on $\widehat{HFK}(Y, K)$; see [22, Section 3.5].

Let $\mathcal{S}_K$ be a Seifert surface of K. Let

$$\omega_{\mathcal{S}_K} \in \Omega^2(I \times Y(K), \partial I \times Y(K))$$

be a closed 2-form dual to $\{1/2\} \times \mathcal{S}_K$ under Poincaré–Lefschetz duality

$$H^2(I \times Y(K), \partial I \times Y(K)) \cong H_2(I \times Y(K), I \times \partial Y(K)).$$

By definition, $\omega_{\mathcal{S}_K}$ vanishes on $\partial I \times Y(K)$.

Lemma 2.11. *Up to an overall factor of z^α, the map $F_{I \times Y(K); \omega_{\mathcal{S}_K}}$ is given by*

$$F_{I \times Y(K); \omega_{\mathcal{S}_K}}(z^x \cdot \boldsymbol{x}) = z^{x - A(\boldsymbol{x})} \cdot \boldsymbol{x},$$

where $A(\boldsymbol{x})$ denotes the Alexander grading.

We will prove Lemma 2.11 at the end of Section 7.2.

2.5. Changing the 2-form on W

We now state another result which will be helpful for proving Theorem 1.1:

Lemma 2.12. *Suppose that $\mathcal{W} = (W, Z, [\xi]) \colon (M_0, \gamma_0) \to (M_1, \gamma_1)$ is a sutured manifold cobordism, ω is a closed 2-form on W, and η is a 1-form that vanishes on a neighborhood of M_0 and M_1. If $[\omega]$ vanishes on $M_0 \cup M_1$, then*

$$F_{\mathcal{W}; \omega} \doteq F_{\mathcal{W}; \omega + d\eta}.$$

If $[\omega]$ is non-vanishing on M_0 and M_1, then the above equation holds when restricted to fixed Spin^c *structures on M_0 and M_1.*

We will prove Lemma 2.12 in Section 7.8.

3. Perturbed Heegaard Floer homology of closed 3-manifolds

We review some background on Heegaard Floer homology, due to Ozsváth and Szabó [23], [25]. To a closed 3-manifold Y with a Spin^c structure $\mathfrak{s}$, Ozsváth and Szabó assign $\mathbb{F}_2[U]$-modules $HF^-(Y,\mathfrak{s})$, $HF^\infty(Y,\mathfrak{s})$, and $HF^+(Y,\mathfrak{s})$ that fit into a long exact sequence

$$\cdots \xrightarrow{\delta} HF^-(Y,\mathfrak{s}) \to HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s}) \xrightarrow{\delta} HF^-(Y,\mathfrak{s}) \to \cdots . \tag{3.1}$$

There is also an $\mathbb{F}_2$-vector space $\widehat{HF}(Y,\mathfrak{s})$.

If W is a cobordism from Y_0 to Y_1, and $\mathfrak{s} \in \mathrm{Spin}^c(W)$ restricts to $\mathfrak{s}_0$ on Y_0 and to $\mathfrak{s}_1$ on Y_1, then there are maps

$$F^\circ_{W,\mathfrak{s}} \colon HF^\circ(Y_0,\mathfrak{s}_0) \to HF^\circ(Y_1,\mathfrak{s}_1)$$

for $\circ \in \{-,\infty,+,\wedge\}$ that commute with the maps in the long exact sequence (3.1).

If ω is a closed 2-form on Y, Ozsváth and Szabó [21] described an $\mathbb{F}_2[H^1(Y)]$-module denoted $HF^\circ(Y,\mathfrak{s};\Lambda_\omega)$, using the same coning procedure we described in Section 2.2. Similarly, if $\boldsymbol{\omega} = (\omega_1,\dots,\omega_n)$ is an n-tuple of closed 2-forms on Y, we can define the $\mathbb{F}_2[H^1(Y)]$-module $HF^\circ(Y,\mathfrak{s};\Lambda_{\boldsymbol{\omega}})$, which is also a $\Lambda_n[U]$-module, where Λ_n is the n-variable Novikov ring over $\mathbb{F}_2$. In this section, we focus on perturbing by a single 2-form, to simplify the notation.

Ozsváth and Szabó [21] defined perturbed versions of their cobordism maps (and more generally, fully twisted versions in [25]). The naturality and functoriality results described above for sutured Floer homology have analogues for the perturbed versions of the closed 3-manifold invariants, which we state here.

Theorem 3.1. (1) *Suppose Y is a closed 3-manifold with a chosen basepoint and a closed 2-form ω. If $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and $\circ \in \{-,\infty,+,\wedge\}$, then $HF^\circ(Y,\mathfrak{s};\Lambda_\omega)$ forms a projective transitive system of $\Lambda[U]$-modules, indexed by the set of pairs $(\mathcal{H}, J)$, where $\mathcal{H}$ is an $\mathfrak{s}$-admissible diagram of Y, and J is a generic almost complex structure.*

(2) *Suppose W is a connected, oriented cobordism from Y_0 to Y_1, with a chosen path connecting the basepoints of Y_0 and Y_1, a* Spin^c *structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$, and a closed 2-form ω on W. Then the cobordism map*

$$F^\circ_{W,\mathfrak{s};\omega} \colon HF^\circ(Y_0,\mathfrak{s}|_{Y_0};\Lambda_{\omega|_{Y_0}}) \to HF^\circ(Y_1,\mathfrak{s}|_{Y_1};\Lambda_{\omega|_{Y_1}})$$

due to Ozsváth and Szabó [21] is well-defined up to overall multiplication by z^x for $x \in \mathbb{R}$.

Ozsváth and Szabó's construction of the perturbed cobordism maps is similar to the construction we describe in Section 7 for sutured Floer homology. One important difference is how the maps are associated to Spin^c structures on W. If W is decomposed as $W_1 \cup W_2 \cup W_3$, where W_i is an index i handle cobordism, then the restriction map $\mathrm{Spin}^c(W) \to \mathrm{Spin}^c(W_2)$ is an isomorphism. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', w)$ is a triple for the 2-handle attachment, Ozsváth and Szabó [25, Section 8.1.4] define a map

$$\mathfrak{s}_w \colon \pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \to \mathrm{Spin}^c(W_2).$$

The map $F^\circ_{W,\mathfrak{s};\omega}$ counts only triangles with $\mathfrak{s}_w(\psi) = \mathfrak{s}|_{W_2}$. Note that this construction differs slightly from the Spin^c restricted versions of the perturbed sutured cobordism maps we gave in Section 2.3, which took the form $\pi_{\underline{\mathfrak{s}}_1} \circ F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}_0}$.

The Spin^c composition law is slightly subtle in the perturbed setting, since we are working in a projectivized category; see Example 2.4. The morphism sets in a projectivized category are not abelian groups, so sums of maps are not well-defined. Nonetheless, a Spin^c composition law can still be stated, as we now describe.

Suppose that $\mathfrak{S} \subseteq \mathrm{Spin}^c(W)$ is a subset of Spin^c structures. We suppose that each $\mathfrak{s} \in \mathfrak{S}$ has the same restriction to ∂W, unless $[\omega|_{\partial W}] = 0$. If $\circ \in \{-, \infty\}$, we must also assume that there are only finitely many $\mathfrak{s} \in \mathfrak{S}$ such that $F^\circ_{W,\mathfrak{s};\omega} \neq 0$. In this situation, we may define a cobordism map

$$F^\circ_{W,\mathfrak{S};\omega} \colon HF^\circ(Y_0; \Lambda_{\omega|_{Y_0}}) \to HF^\circ(Y_1; \Lambda_{\omega|_{Y_1}}),$$

which is well-defined up to multiplication by z^x for some $x \in \mathbb{R}$. The 2-handle portion of the map $F^\circ_{W,\mathfrak{S};\omega}$ counts triangles such that $\mathfrak{s}_w(\psi)$ is the restriction of an element of $\mathfrak{S}$.

By construction, we may find representatives of the maps $F^\circ_{W,\mathfrak{s};\omega}$ for $\mathfrak{s} \in \mathfrak{S}$ such that

$$F^\circ_{W,\mathfrak{S};\omega} \doteq \sum_{\mathfrak{s} \in \mathfrak{S}} F^\circ_{W,\mathfrak{s};\omega}.$$

The proof of the composition law given by Ozsváth and Szabó [25, Theorem 3.4] extends to give the following:

Proposition 3.2. *Suppose W is a cobordism which decomposes as $W_2 \circ W_1$. Suppose further that ω is a closed 2-form on W, and $\mathfrak{S}_1 \subseteq \mathrm{Spin}^c(W_1)$ and $\mathfrak{S}_2 \subseteq \mathrm{Spin}^c(W_2)$ are subsets as above. Let*

$$\mathfrak{S}(W, \mathfrak{S}_1, \mathfrak{S}_2) = \{\mathfrak{s} \in \mathrm{Spin}^c(W) : \mathfrak{s}|_{W_1} \in \mathfrak{S}_1 \text{ and } \mathfrak{s}|_{W_2} \in \mathfrak{S}_2\}.$$

Then

$$F^\circ_{W,\mathfrak{S}(W,\mathfrak{S}_1,\mathfrak{S}_2);\omega} \doteq F^\circ_{W_2,\mathfrak{S}_2;\omega|_{W_2}} \circ F^\circ_{W_1,\mathfrak{S}_1;\omega|_{W_1}}.$$

We have the following analogue of Lemma 2.12:

Lemma 3.3. *Suppose that $W \colon Y_0 \to Y_1$ is a cobordism of 3-manifolds, $\mathfrak{S} \subseteq \mathrm{Spin}^c(W)$ is a set of Spin^c structures as above, ω is a closed 2-form on W, and η is a 1-form that vanishes on a neighborhood of Y_0 and Y_1. If $[\omega]$ vanishes on $Y_0 \cup Y_1$, then*

$$F^\circ_{W,\mathfrak{S};\omega} \doteq F_{W,\mathfrak{S};\omega+d\eta}.$$

If $[\omega]$ *is non-vanishing on* Y_0 *and* Y_1*, then the above equation holds when restricted to fixed* Spin^c *structures on* Y_0 *and* Y_1*.*

Proof. This can be shown similarly to Lemma 2.12; see Section 7.8. ■

Lemma 3.4. *Let* W *be a cobordism from* Y_0 *to* Y_1*, and* ω *a closed* 2*-form on* W *that vanishes on* ∂W*. Furthermore, let* $\mathfrak{S} \subseteq \mathrm{Spin}^c(W)$ *be a set of* Spin^c *structures. If* $\circ \in \{-, \infty\}$*, we also assume there are only finitely many* $\mathfrak{s} \in \mathfrak{S}$ *for which* $F^\circ_{W,\mathfrak{s}} \neq 0$*. If* $\mathfrak{s}_0 \in \mathrm{Spin}^c(W)$ *is an arbitrary base* Spin^c *structure, then*

$$F^\circ_{W,\mathfrak{S};\omega} \doteq \sum_{\mathfrak{s}\in\mathfrak{S}} z^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup[\omega],[W,\partial W]\rangle} \cdot F^\circ_{W,\mathfrak{s}}. \tag{3.2}$$

We will prove Lemma 3.4 in Section 7.9.

Remark 3.5. As a consequence of Lemma 3.4, if ω is a closed 2-form on W that vanishes on ∂W, then $F^\circ_{W,\mathfrak{s};\omega} \doteq F^\circ_{W,\mathfrak{s}}$. We note that it is natural to normalize the perturbed maps in this situation by defining

$$F^\circ_{W,\mathfrak{s};\omega} := z^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup[\omega],[W,\partial W]\rangle} \cdot F^\circ_{W,\mathfrak{s}}$$

and

$$F^\circ_{W;\omega} = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(W)} F^\circ_{W,\mathfrak{s};\omega} = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(W)} z^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup[\omega],[W,\partial W]\rangle} \cdot F^\circ_{W,\mathfrak{s}}$$

for $\circ \in \{\wedge, +\}$. For $\circ \in \{-, \infty\}$, we may take this convention in the case when $F^\circ_{W,\mathfrak{s}}$ is non-vanishing for only finitely many $\mathfrak{s}$. It is straightforward to see that this normalization convention is compatible with the composition law.

4. Background on the Ozsváth–Szabó mixed invariants

For a closed 4-manifold X with $b_2^+(X) \geq 2$, Ozsváth and Szabó defined a map

$$\Phi_X \colon \mathrm{Spin}^c(X) \to \mathbb{F}_2.$$

We write $\Phi_{X,\mathfrak{s}}$ for the value of Φ_X on $\mathfrak{s}$. The map Φ_X is referred to as the *mixed invariant* of X, because it uses both HF^+ and HF^-.

The map Φ_X is defined by picking a connected, codimension 1 submanifold $N \subseteq X$ that cuts X into two pieces, W_1 and W_2, such that $b_2^+(W_i) > 0$, and such that the restriction map

$$H^2(X) \to H^2(W_1) \oplus H^2(W_2)$$

is an injection. Such a cut is called *admissible*. If we view W_1 as a cobordism from S^3 to N, and W_2 as a cobordism from N to S^3, the maps $F^\infty_{W_1,\mathfrak{s}|_{W_1}}$ and $F^\infty_{W_2,\mathfrak{s}|_{W_2}}$ vanish [25, Lemma 8.2]. Consequently, $F^-_{W_1,\mathfrak{s}_1}$ may be factored to have codomain

$$HF^-_{\mathrm{red}}(N, \mathfrak{s}|_N) := \ker(HF^-(N, \mathfrak{s}|_N) \to HF^\infty(N, \mathfrak{s}|_N)),$$

and $F^+_{W_2,\mathfrak{s}_2}$ may be factored to have domain

$$HF^+_{\mathrm{red}}(N,\mathfrak{s}|_N) := \operatorname{coker}(HF^\infty(N,\mathfrak{s}_N) \to HF^+(N,\mathfrak{s}|_N)).$$

The boundary map δ in the long exact sequence (3.1) induces an isomorphism between $HF^+_{\mathrm{red}}(N,\mathfrak{s}|_N)$ and $HF^-_{\mathrm{red}}(N,\mathfrak{s}|_N)$.

The invariant $\Phi_{X,\mathfrak{s}}$ is defined as the coefficient of the bottom-graded generator Θ_+ of $HF^+(S^3)$ in the expression

$$(F^+_{W_2,\mathfrak{s}|_{W_2}} \circ \delta^{-1} \circ F^-_{W_1,\mathfrak{s}|_{W_1}})(1),$$

where 1 denotes the top-graded generator of $HF^-(S^3) \cong \mathbb{F}_2[U]$. Ozsváth and Szabó prove that this is independent of the admissible cut N.

We now describe how to compute the mixed invariants using the perturbed cobordism maps. To do that, we will need the following two results:

Lemma 4.1. *Let X be a closed, oriented 4-manifold with $b_2^+(X) \geq 2$, and let $b \in H^2(X)$. Given an admissible cut $X = W_1 \cup_N W_2$, there is a closed 2-form ω on X such that*

(1) $[\omega] = b \in H^2(X;\mathbb{R})$, *and*

(2) $\omega|_N = 0$.

Proof. Choose $\varphi \in \Omega^2(X)$ such that $[\varphi] = b$. Since N gives an admissible cut, the coboundary map $H^1(N) \to H^2(X)$ is zero. This is Poincaré dual to the inclusion $H_2(N) \to H_2(X)$, so this is trivial as well. Hence, the restriction map from $H^2(X;\mathbb{R})$ to $H^2(N;\mathbb{R})$ is trivial. In particular, $[\varphi|_N] = 0$ in $H^2(N;\mathbb{R})$, and so there is a 1-form $\eta \in \Omega^1(N)$ such that $\varphi|_N = d\eta$.

Let $\nu(N)$ be a tubular neighborhood of N in X, and write $p\colon \nu(N) \to N$ for the projection. Choose a smooth function f on X that is 0 outside $\nu(N)$, and is 1 on a neighborhood of N contained in the interior of $\nu(N)$. We define

$$\omega := \varphi - d(f \cdot p^*\eta).$$

Then ω satisfies the required conditions. ∎

Lemma 4.2. *Let X be a closed, oriented 4-manifold with $b_2^+(X) > 1$, and let $X = W_1 \cup_N W_2$ be an admissible cut. If $\boldsymbol{\omega}$ is a tuple of closed 2-forms on X that vanish on N, then $F^-_{W_1,\mathfrak{t};\boldsymbol{\omega}|_{W_1}}$ and $F^+_{W_2,\mathfrak{u};\boldsymbol{\omega}|_{W_2}}$ are non-zero for only finitely many $\mathfrak{t} \in \mathrm{Spin}^c(W_1)$ and $\mathfrak{u} \in \mathrm{Spin}^c(W_2)$.*

Proof. By Lemma 3.4, it suffices to show this for the unperturbed maps $F^-_{W_1,\mathfrak{t}}$ and $F^+_{W_2,\mathfrak{u}}$. Note that $F^-_{W_1,\mathfrak{t}}$ has image in $HF^-_{\mathrm{red}}(N)$ for every $\mathfrak{t} \in \mathrm{Spin}^c(W_1)$. Let $d \in \mathbb{N}$ be such that $U^d \cdot HF^-_{\mathrm{red}}(N) = \{0\}$. If 1 is the generator of $HF^-(S^3)$, then

$$F^-_{W_1,\mathfrak{t}}(1) \notin U^d \cdot HF^-_{\mathrm{red}}(N) = \{0\}$$

only for finitely many $\mathfrak{t} \in \mathrm{Spin}^c(W_1)$ by [25, Theorem 3.3], and since $HF^-_{\mathrm{red}}(N,\mathfrak{s}) \neq 0$ only for finitely many $\mathfrak{s} \in \mathrm{Spin}^c(N)$. The same argument works for $F^+_{W_2,\mathfrak{u};\boldsymbol{\omega}|_{W_2}}$. ∎

Recall from the introduction that if $\boldsymbol{b}=(b_1,\dots,b_n)$ is a basis of $H^2(X;\mathbb{R})$, we define

$$\Phi_{X;\boldsymbol{b}} := \sum_{\mathfrak{s}\in \operatorname{Spin}^c(X)} \Phi_{X,\mathfrak{s}}\cdot z_1^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup b_1,[X]\rangle}\cdots z_n^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup b_n,[X]\rangle},$$

where $\mathfrak{s}_0\in \operatorname{Spin}^c(X)$ is a choice of base Spin^c structure. If $H^2(X)$ is torsion-free, then $\Phi_{X;\boldsymbol{b}}$ completely encodes the map $\mathfrak{s}\mapsto \Phi_{X,\mathfrak{s}}$. We now give a slight reformulation of $\Phi_{X;\boldsymbol{b}}$, which is well-suited for proving Theorem 1.1:

Proposition 4.3. *Suppose X is a closed, oriented 4-manifold with $b_2^+(X)>1$, and N is an admissible cut, dividing X into cobordisms W_1 and W_2. Suppose $\boldsymbol{b}=(b_1,\dots,b_n)$ is an n-tuple of classes in $H^2(X;\mathbb{R})$, represented by 2-forms $\boldsymbol{\omega}=(\omega_1,\dots,\omega_n)$ that vanish on N. Write $\boldsymbol{\omega}_1=\boldsymbol{\omega}|_{W_1}$ and $\boldsymbol{\omega}_2=\boldsymbol{\omega}|_{W_2}$. Then the maps $F^+_{W_2;\boldsymbol{\omega}_2}$ and $F^-_{W_1;\boldsymbol{\omega}_1}$ are well-defined, and satisfy*

$$\Phi_{X;\boldsymbol{b}} \doteq \langle (F^+_{W_2;\boldsymbol{\omega}_2}\circ\delta^{-1}\circ F^-_{W_1;\boldsymbol{\omega}_1})(1),\Theta_+\rangle. \tag{4.1}$$

Proof. Well-definedness of $F^-_{W_1;\boldsymbol{\omega}_1}$ and $F^+_{W_2;\boldsymbol{\omega}_2}$ follows from Lemma 4.2, so we focus on (4.1).

Let $\mathfrak{s}_0$ be a fixed element of $\operatorname{Spin}^c(X)$, and let $\mathfrak{t}_0=\mathfrak{s}_0|_{W_1}$ and $\mathfrak{u}_0=\mathfrak{s}_0|_{W_2}$. Since $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ vanish on N, we apply a straightforward adaptation of Lemma 3.4 from the single- to the multi-variable setting to obtain

$$\begin{aligned} F^-_{W_1;\boldsymbol{\omega}_1} &\doteq \sum_{\mathfrak{t}\in \operatorname{Spin}^c(W_1)} z_1^{\langle i_*(\mathfrak{t}-\mathfrak{t}_0)\cup[\omega_1],[W_1,\partial W_1]\rangle}\cdots z_n^{\langle i_*(\mathfrak{t}-\mathfrak{t}_0)\cup[\omega_n],[W_1,\partial W_1]\rangle}\cdot F^-_{W_1,\mathfrak{t}},\\ F^+_{W_2;\boldsymbol{\omega}_2} &\doteq \sum_{\mathfrak{u}\in \operatorname{Spin}^c(W_2)} z_1^{\langle i_*(\mathfrak{u}-\mathfrak{s}_0)\cup[\omega_1],[W_2,\partial W_2]\rangle}\cdots z_n^{\langle i_*(\mathfrak{u}-\mathfrak{u}_0)\cup[\omega_n],[W_2,\partial W_2]\rangle}\cdot F^+_{W_2,\mathfrak{u}}. \end{aligned} \tag{4.2}$$

Equation (4.1) is obtained by inserting (4.2) into the right-hand side of (4.1), and using the fact that if $\mathfrak{s}\in\operatorname{Spin}^c(X)$ restricts to $\mathfrak{t}\in \operatorname{Spin}^c(W_1)$ and $\mathfrak{u}\in\operatorname{Spin}^c(W_2)$, then

$$\begin{aligned} &\langle i_*(\mathfrak{t}-\mathfrak{t}_0)\cup[\omega_i],[W_1,\partial W_1]\rangle + \langle i_*(\mathfrak{u}-\mathfrak{u}_0)\cup[\omega_i],[W_2,\partial W_2]\rangle\\ &\qquad = \langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup[\omega_i],[X]\rangle. \qquad \blacksquare \end{aligned}$$

Remark 4.4. In light of Proposition 4.3, it is natural to view $\Phi_{X;\boldsymbol{b}}$ as a perturbed version of the mixed invariant.

5. Fintushel–Stern knot surgery and concordance surgery

Fintushel and Stern [4] described an operation on a 4-manifold X called *knot surgery*. Given a knot K in S^3 and an embedded torus T in X with zero self-intersection, we define the 4-manifold

$$X_0 := X\setminus N(T)$$

with boundary $\mathbb{T}^3$. A neighborhood of T can be identified with $T\times D^2$. We pick any orientation-preserving diffeomorphism $\phi\colon \partial(T\times D^2)\to S^1\times\partial N(K)$ such that

$\phi_*([\{p\} \times \partial D^2]) = [\{q\} \times \ell_K]$, where ℓ_K is a Seifert longitude on $\partial N(K)$, while $p \in T$ and $q \in S^1$. We let

$$X_K := X_0 \cup_\phi (S^1 \times (S^3 \setminus N(K)))$$

be the result of knot surgery on X using K and T. Note that there is some ambiguity in the choice of ϕ, so we write X_K for any 4-manifold constructed in this way. It is straightforward to see that $H^*(X_K)$ and $H^*(X)$ are canonically isomorphic.

Fintushel and Stern described a generalization of this operation called *concordance surgery*; see Akbulut [2]. Let K be a knot in a homology 3-sphere Y (note that Akbulut only considered $Y = S^3$). Given a self-concordance $\mathcal{C} = (I \times Y, A)$ from (Y, K) to itself, we can construct a 4-manifold $X_\mathcal{C}$ as follows. We take the annulus A, and glue its ends together to form a 2-torus $T_\mathcal{C}$ embedded in $S^1 \times Y$. The quotient map $I \times Y \to S^1 \times Y$ is given by $(t, y) \mapsto (e^{2\pi i t}, y)$ for $t \in I$ and $Y \in Y$. After removing a neighborhood of $T_\mathcal{C}$, we get a 4-manifold $W_\mathcal{C}$ with boundary $\mathbb{T}^3$. We pick any orientation-preserving diffeomorphism $\phi\colon \partial X_0 \to \partial N(T_\mathcal{C})$ that sends $[\{p\} \times \partial D^2]$ to $[\{1\} \times \ell_K]$. We write $X_\mathcal{C}$ for any manifold constructed as the union

$$X_\mathcal{C} := X_0 \cup_\phi W_\mathcal{C}.$$

It is easy to see that $H^*(X_\mathcal{C})$ and $H^*(X)$ are canonically isomorphic.

If $\mathcal{C} = (I \times Y, A)$ is a self-concordance of the knot K in Y, and a is a pair of parallel arcs on A connecting the two components of ∂A, then there is an induced map on knot Floer homology

$$\hat{F}_{\mathcal{C},a}\colon \widehat{HFK}(Y, K) \to \widehat{HFK}(Y, K),$$

described by the first author [8]. The map $\hat{F}_{\mathcal{C},a}$ preserves the Alexander and Maslov gradings according to Marengon and the first author [10, Theorem 5.18], and is non-vanishing when $Y = S^3$ by [9, Theorem 1.2].

Note that the group $\widehat{HFK}(Y, K)$ only becomes natural once we choose a pair P of basepoints on K, which we suppress from the notation. We require ∂a to be disjoint from P, and also to link ∂a. We define $\mathrm{Lef}_z(\mathcal{C})$ to be the polynomial

$$\mathrm{Lef}_z(\mathcal{C}) := \sum_{i \in \mathbb{Z}} \mathrm{Lef}\big(\hat{F}_{\mathcal{C},a}|_{\widehat{HFK}(Y,K,i)}\colon \widehat{HFK}(Y, K, i) \to \widehat{HFK}(Y, K, i)\big) \cdot z^i$$

for any pair of parallel arcs a connecting the two boundary components of $\mathcal{C}$. Although the map $\hat{F}_{\mathcal{C},a}$ depends on the arcs a, we have the following:

Lemma 5.1. *The graded Lefschetz number of $\hat{F}_{\mathcal{C},a}$ is independent of the choice of arcs a.*

Proof. Changing the arcs a by a proper isotopy that does not cross the basepoints P does not change the cobordism map $\hat{F}_{\mathcal{C},a}$. Hence, it suffices to show that the Lefschetz number is unchanged by applying a Dehn twist to a along one of the boundary components of the annulus A. The action of a Dehn twist on $\widehat{HFK}(Y, K)$ was computed by Sarkar [27] when $Y = S^3$, and by the second author [31, Theorem B] for a null-homologous knot in a general 3-manifold Y. If r_* denotes the action of a single Dehn twist, then

$$r_* = \mathrm{id} + \Phi\Psi,$$

where Φ and Ψ are two endomorphisms of $\widehat{HFK}(Y, K)$ that satisfy

$$\Phi^2 = \Psi^2 = 0, \quad \Phi\Psi = \Psi\Phi.$$

Since a Dehn twist on an annulus may be pulled to either boundary component, it follows that if a' differs from a by a single Dehn twist along one end of the annulus, then

$$\widehat{F}_{\mathcal{C},a'} = \widehat{F}_{\mathcal{C},a} \circ (\mathrm{id} + \Phi\Psi) = (\mathrm{id} + \Phi\Psi) \circ \widehat{F}_{\mathcal{C},a}.$$

Consequently, the map $\widehat{F}_{\mathcal{C},a} \circ (\Phi\Psi)$ is nilpotent, so has Lefschetz number 0 in each Alexander grading. ■

Lemma 5.2. *The graded Lefschetz number* $\mathrm{Lef}_z(\mathcal{C})$ *is symmetric with respect to the conjugation* $z \mapsto z^{-1}$.

Proof. The proof follows easily from the conjugation symmetry of the knot Floer homology groups [22, Proposition 3.10], as well as the corresponding symmetry of the knot cobordism maps [32, Theorem 1.3]. ■

If X is a closed, oriented 4-manifold with a smoothly embedded 2-torus T such that $[T] \neq 0 \in H_2(X;\mathbb{R})$, then we can pick a basis $\boldsymbol{b} = (b_1, \dots, b_n)$ of $H^2(X;\mathbb{R})$ such that

$$\langle b_1, [T]\rangle = 1 \quad \text{and} \quad \langle b_i, [T]\rangle = 0 \quad \text{for } i > 1. \tag{5.1}$$

This induces a basis of $H^2(X_{\mathcal{C}};\mathbb{R})$ that we also denote by $\boldsymbol{b}$. We restate our main theorem.

Theorem 1.1. *Let X be a closed, oriented 4-manifold such that $b_2^+(X) \geq 2$. Suppose that T is a smoothly embedded 2-torus in X with trivial self-intersection such that $[T] \neq 0$ in $H_2(X;\mathbb{R})$. Furthermore, let $\boldsymbol{b} = (b_1, \dots, b_n)$ be a basis of $H^2(X;\mathbb{R})$ satisfying (5.1). If $\mathcal{C}$ is a self-concordance of (Y, K), where Y is a homology 3-sphere, then*

$$\Phi_{X_{\mathcal{C}};\boldsymbol{b}} = \mathrm{Lef}_{z_1}(\mathcal{C}) \cdot \Phi_{X;\boldsymbol{b}}.$$

In order to prove Theorem 1.1, we need to perform several computations. Let $\mathcal{C}$ be a self-concordance of a knot K in the homology 3-sphere Y. On the torus $T_{\mathcal{C}} \subseteq S^1 \times Y$, we pick a pair of dividing curves, each intersecting $\{1\} \times K$ exactly once. Such dividing curves are determined up to Dehn twists about $\{1\} \times K$. The dividing set specifies an isotopically unique, positive, S^1-invariant contact structure $\xi_{\mathcal{C}}$ on $\mathbb{T}^3 = -\partial N(T_{\mathcal{C}})$, by the work of Lutz [19]. Note that this contact structure is positive with respect to the boundary orientation from $W_{\mathcal{C}}$.

Proposition 5.3. *Let $\omega_{\mathcal{C}}$ be a closed 2-form on the 4-manifold $W_{\mathcal{C}}$, Poincaré dual to $\{1\} \times \mathcal{S}_K$, where $\mathcal{S}_K$ is a Seifert surface for the knot K. If we view $W_{\mathcal{C}}$ as a cobordism from $-\mathbb{T}^3$ to $\emptyset$, and write $\tau_{\mathcal{C}} = \omega_{\mathcal{C}}|_{\partial W_{\mathcal{C}}}$, then*

$$\widehat{F}_{W_{\mathcal{C}};\omega_{\mathcal{C}}}(\widehat{c}(\xi_{\mathcal{C}};\tau_{\mathcal{C}})) \doteq \mathrm{Lef}_z(\mathcal{C})$$

as an element of $\widehat{HF}(\emptyset;\Lambda) \cong \Lambda$, where $\widehat{c}(\xi_{\mathcal{C}};\tau_{\mathcal{C}}) \in \widehat{HF}(-\mathbb{T}^3;\Lambda_{\tau_{\mathcal{C}}})$ is the contact class of $\xi_{\mathcal{C}}$ twisted by $\tau_{\mathcal{C}}$.

Proof. We consider the sutured manifold cobordism $\mathcal{W}_{\mathcal{C}} := (W_{\mathcal{C}}, \mathbb{T}^3, [\xi_{\mathcal{C}}])$ from the empty sutured manifold to itself. In Section 7, we define the sutured cobordism map as the composition of the contact gluing map for gluing $(\mathbb{T}^3, \xi_{\mathcal{C}})$ to the empty sutured manifold and perturbed by $\tau_{\mathcal{C}}$, followed by 4-dimensional 1-, 2-, and 3-handle maps. The composition of the handle maps is the perturbed cobordism map $\widehat{F}_{W_{\mathcal{C}};\omega_{\mathcal{C}}}$ induced by the cobordism $W_{\mathcal{C}}$ from $\mathbb{T}^3$ to $\emptyset$, as defined by Ozsváth and Szabó [25]. Since $\mathbb{T}^3$ is a closed 3-manifold, the gluing map sends the generator of $SFH(\emptyset;\Lambda) \cong \Lambda$ to the perturbed contact element $\widehat{c}(\xi_{\mathcal{C}};\tau_{\mathcal{C}})$. Consequently, the perturbed sutured cobordism map $F_{\mathcal{W}_{\mathcal{C}};\omega_{\mathcal{C}}}$ satisfies

$$F_{\mathcal{W}_{\mathcal{C}};\omega_{\mathcal{C}}}(1) \doteq \widehat{F}_{W_{\mathcal{C}};\omega_{\mathcal{C}}}(\widehat{c}(\xi_{\mathcal{C}};\tau_{\mathcal{C}})).$$

Let us write $Y(K)$ for the sutured manifold obtained by adding two meridional sutures to $Y \setminus N(K)$. We decompose $\mathcal{W}_{\mathcal{C}}$ as

$$\Cap_{Y(K)} \circ \mathrm{Id}_{Y(K)\sqcup -Y(K)} \circ (\mathcal{W}(\mathcal{C},a) \sqcup \mathrm{Id}_{-Y(K)}) \circ \Cup_{Y(K)},$$

where

- $\Cup_{Y(K)}$ is the cotrace cobordism from $\emptyset$ to $Y(K) \sqcup -Y(K)$,
- $\mathcal{W}(\mathcal{C},a)$ is the sutured manifold cobordism from $Y(K)$ to itself complementary to the decorated concordance $(\mathcal{C},a)$, and $\mathrm{Id}_{-Y(K)}$ is the identity cobordism of $-Y(K)$,
- $\mathrm{Id}_{Y(K)\sqcup -Y(K)}$ is the identity cobordism of $Y(K) \sqcup -Y(K)$, and
- $\Cap_{Y(K)}$ is the trace cobordism from $Y(K) \sqcup -Y(K)$ to $\emptyset$.

Since $\mathcal{W}_{\mathcal{C}}$ is a sutured cobordism from $\emptyset$ to $\emptyset$, it follows from Lemma 2.12 that replacing $\omega_{\mathcal{C}}$ with $\omega_{\mathcal{C}} + d\eta$ for a 1-form η only changes $F_{\mathcal{W}_{\mathcal{C}};\omega_{\mathcal{C}}}(1)$ by an overall factor of z^x. Hence, we may assume that the 2-form $\omega_{\mathcal{C}}$ restricts trivially to $\Cup_{Y(K)}$, $\mathcal{W}(\mathcal{C},a) \sqcup \mathrm{Id}_{-Y(K)}$, and $\Cap_{Y(K)}$. Its restriction ω' to $\mathrm{Id}_{Y(K)\sqcup -Y(K)}$ is Poincaré–Lefschetz dual to $\{1/2\} \times \mathcal{S}_K$ for a Seifert surface $\mathcal{S}_K \subseteq Y(K)$.

By Lemma 2.11, and since $\mathrm{Id}_{Y(K)\sqcup -Y(K)}$ is a disjoint union of two product cobordisms, we have

$$F_{\mathrm{Id}_{Y(K)\sqcup -Y(K)};\omega'}(\mathbf{x} \otimes \mathbf{y}) = z^{-A(\mathbf{x})} \cdot (\mathbf{x} \otimes \mathbf{y}),$$

up to an overall factor of z^x for some $x \in \mathbb{R}$. By [11, Theorem 1.1], we know that $\Cup_{Y(K)}$ and $\Cap_{Y(K)}$ induce the canonical cotrace and trace maps, respectively. It follows that

$$(F_{\Cap_{Y(K)};0} \circ F_{\mathrm{Id}_{Y(K)\sqcup -Y(K)};\omega'} \circ F_{\mathcal{W}(\mathcal{C},a)\sqcup \mathrm{Id}_{-Y(K)};0} \circ F_{\Cup_{Y(K)};0})(1)$$

is the graded Lefschetz number $\mathrm{Lef}_{z^{-1}}(\widehat{F}_{\mathcal{C},a})$. By Lemma 5.2, this coincides with the graded Lefschetz number $\mathrm{Lef}_z(\widehat{F}_{\mathcal{C},a})$, completing the proof. ∎

The special case of the unknot U and the trivial concordance $(I \times S^3, I \times U)$ is important. In this case, the dividing set on the torus $S^1 \times U \subseteq S^1 \times S^3$ determines an S^1-invariant, positive contact structure ξ_0 on $\mathbb{T}^3 = -\partial N(S^1 \times U)$. Consider the 4-manifold

$$W_0 = S^1 \times (S^3 \setminus N(U)) \cong S^1 \times S^1 \times D^2.$$

Corollary 5.4. *Let ω_0 be a closed 2-form on the 4-manifold W_0 such that $[\omega_0]$ is Poincaré dual to $\{(1,1)\} \times D^2$. If we view W_0 as a cobordism from $-\mathbb{T}^3$ to $\emptyset$, and write $\tau_0 = \omega_0|_{\partial W_0}$, then*

$$\widehat{F}_{W_0;\omega_0}(\widehat{c}(\xi_0;\tau_0)) \doteq 1$$

as an element of $\widehat{HF}(\emptyset;\Lambda) \cong \Lambda$.

A choice of dividing sets on $S^1 \times U$ in $S^1 \times S^3$ and $T_{\mathcal{C}}$ in $S^1 \times Y$ induces a diffeomorphism between $S^1 \times U$ and $T_{\mathcal{C}}$ that maps $\{1\} \times U$ to $\{1\} \times K$, well-defined up to isotopy. We can extend this diffeomorphism to a D^2-bundle map from $(S^1 \times U) \times D^2$ to $T_{\mathcal{C}} \times D^2$. We write $\mathbb{T}^3$ for both $-\partial N(S^1 \times U)$ and $-\partial N(T_{\mathcal{C}})$, identified via the restriction of such a diffeomorphism. Furthermore, the contact structures ξ_0 and $\xi_{\mathcal{C}}$ are identified by this diffeomorphism, and hence we will write ξ for both. Similarly, the 2-forms $\tau_0 = \omega_0|_{\mathbb{T}^3}$ and $\tau_{\mathcal{C}} = \omega_{\mathcal{C}}|_{\mathbb{T}^3}$ are identified, so we write $\tau \in \Omega^2(\mathbb{T}^3)$ for both.

Note that $\mathrm{Spin}^c(W_0) \cong \mathrm{Spin}^c(W_{\mathcal{C}}) \cong \mathbb{Z}$. We write $\mathfrak{t}_k \in \mathrm{Spin}^c(W_0)$ for the Spin^c structure with

$$c_1(\mathfrak{t}_k) = 2k \cdot PD[\{1\} \times \mathcal{S}_U],$$

where $\mathcal{S}_U$ is a Seifert surface for U in $S^3 \setminus N(U)$, and we are using Poincaré duality

$$H_2(W_0, \partial W_0) \cong H^2(W_0).$$

Similarly, we write $\mathfrak{t}'_k \in \mathrm{Spin}^c(W_{\mathcal{C}})$ for the Spin^c structure satisfying $c_1(\mathfrak{t}'_k) = 2k \cdot PD[\{1\} \times \mathcal{S}_K]$, where $\mathcal{S}_K$ is a Seifert surface for K in $Y \setminus N(K)$.

Corollary 5.5. *As maps from $HF^+(-\mathbb{T}^3;\Lambda_\tau)$ to $HF^+(\emptyset;\Lambda) \cong \Lambda$, we have*

$$F^+_{W_{\mathcal{C}},\mathfrak{t}'_0;\omega_{\mathcal{C}}} \doteq \mathrm{Lef}_z(\mathcal{C}) \cdot F^+_{W_0,\mathfrak{t}_0;\omega_0}.$$

Furthermore, $F^+_{W_0,\mathfrak{t}_k;\omega_0}$ and $F^+_{W_{\mathcal{C}},\mathfrak{t}'_k;\omega_{\mathcal{C}}}$ vanish for every $k \in \mathbb{Z} \setminus \{0\}$.

Proof. The contact element

$$c^+(\xi;\tau) \in HF^+(-\mathbb{T}^3;\Lambda_\tau)$$

was defined by Ozsváth and Szabó [24] as the image of $\widehat{c}(\xi;\tau)$ under the natural map

$$\iota_* \colon \widehat{HF}(-\mathbb{T}^3;\Lambda_\tau) \to HF^+(-\mathbb{T}^3;\Lambda_\tau).$$

Since ι_* commutes with the perturbed cobordism maps for W_0 and $W_{\mathcal{C}}$ on $\widehat{HF}$ and HF^+, we have

$$F^+_{W_{\mathcal{C}};\omega_{\mathcal{C}}}(c^+(\xi;\tau)) \doteq \mathrm{Lef}_z(\mathcal{C})$$

by Proposition 5.3, and

$$F^+_{W_0;\omega_0}(c^+(\xi;\tau)) \doteq 1$$

by Corollary 5.4. Hence $c^+(\xi;\tau) \neq 0$, and

$$F^+_{W_{\mathcal{C}};\omega_{\mathcal{C}}}(c^+(\xi,\tau)) \doteq \mathrm{Lef}_z(\mathcal{C}) \cdot F^+_{W_0;\omega_0}(c^+(\xi,\tau)). \tag{5.2}$$

Next, we use the well-known fact that if τ is any non-vanishing, closed 2-form on $-\mathbb{T}^3$, then

$$HF^+(-\mathbb{T}^3; \Lambda_\tau) \cong \Lambda,$$

and $HF^+(-\mathbb{T}^3; \Lambda_\tau)$ is supported in the torsion Spin^c structure on $-\mathbb{T}^3$; see Ai and Peters [1, Theorem 1.3], Jabuka and Mark [6, Theorem 10.1], Lekili [15, Theorem 14], and Wu [29]. It follows that $F^+_{W_{\mathcal{C}};\omega_{\mathcal{C}}}$ and $F^+_{W_0;\omega_0}$, whose domains are thus rank 1 over Λ, must be constant multiples of each other. Equation (5.2) and the fact that $c^+(\xi;\tau) \neq 0$ now establish that the ratio is $\mathrm{Lef}_z(\mathcal{C})$, up to an overall factor of z^x.

Finally, the maps in the Spin^c structures $\mathfrak{t}_k$ and $\mathfrak{t}'_k$ for $k \in \mathbb{Z} \setminus \{0\}$ vanish because they have trivial domain. In particular,

$$F^+_{W_{\mathcal{C}};\omega_{\mathcal{C}}} = F^+_{W_{\mathcal{C}},\mathfrak{t}'_0;\omega_{\mathcal{C}}} \quad \text{and} \quad F^+_{W_0;\omega_0} = F^+_{W_0,\mathfrak{t}_0;\omega_0},$$

completing the proof. ∎

Corollary 5.6. *If $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ is a collection of closed 2-forms on X satisfying*

$$\int_T \omega_1 = 1 \quad \textit{and} \quad \int_T \omega_i = 0 \quad \textit{for } i > 1,$$

and $\boldsymbol{\omega}' = (\omega'_1, \dots, \omega'_n)$ is the induced collection on $X_{\mathcal{C}}$ under the canonical isomorphism $H^2(X_{\mathcal{C}};\mathbb{R}) \cong H^2(X;\mathbb{R})$, then

$$F^+_{W_{\mathcal{C}},\mathfrak{t}'_0;\boldsymbol{\omega}'|W_{\mathcal{C}}} \doteq \mathrm{Lef}_{z_1}(\mathcal{C}) \cdot F^+_{W_0,\mathfrak{t}_0;\boldsymbol{\omega}|W_0},$$

and both maps vanish for all other Spin^c *structures.*

Proof. Let the 1-variable Novikov ring Λ act on the n-variable Novikov ring Λ_n in the variables $z_1, \dots, z_n$ via multiplication by the first variable. We write $\Lambda_{\boldsymbol{\omega}}$ for Λ_n viewed as a module over $\mathbb{F}_2[H_2(M)]$ via the formula $e^a \cdot z_i^x = z_i^{x+\int_a \omega_i}$ for $a \in H_2(M)$ and $x \in \mathbb{R}$. Since the classes $[\omega_2], \dots, [\omega_n]$ vanish on W_0 and $[\omega'_2], \dots, [\omega'_n]$ vanish on $W_{\mathcal{C}}$, arguing as in the proof of Lemma 4.1 we may assume the 2-forms $\omega_2, \dots, \omega_n$ and $\omega'_2, \dots, \omega'_n$ have been chosen to vanish on W_0 and $W_{\mathcal{C}}$. Hence, we see a canonical isomorphism

$$HF^+(-\mathbb{T}^3; \Lambda_{\boldsymbol{\omega}|_{-\mathbb{T}^3}}) \cong HF^+(-\mathbb{T}^3; \Lambda_\tau) \otimes_\Lambda \Lambda_n.$$

Immediately from the definitions, we see that, with respect to this decomposition,

$$F^+_{W_0,\mathfrak{t}_k;\boldsymbol{\omega}|W_0} = F^+_{W_0,\mathfrak{t}_k;\omega_1|W_0} \otimes \mathrm{id}_{\Lambda_n},$$

and similarly for $F^+_{W_{\mathcal{C}},\mathfrak{t}'_k;\boldsymbol{\omega}'|W_{\mathcal{C}}}$. The main result now follows from Corollary 5.5. ∎

We can now prove Theorem 1.1.

Proof of Theorem 1.1. As before, let $X_0 = X \setminus N(T)$. Since $b_2^+(X) \geq 2$, by analyzing the Mayer–Vietoris sequence for $X = X_0 \cup N(T)$ it is easy to see that $b_2^+(X_0) \geq 1$. Hence, there is a surface Q of positive self-intersection in the complement of T. Let N denote the boundary of a tubular neighborhood of Q. The manifold N is an admissible cut of X by [25, Example 8.4]. We write $W_1 = N(Q)$ and $W_2 = X \setminus \operatorname{int}(N(Q))$.

By Lemma 4.1, there are 2-forms $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ such that $[\omega_i] = b_i$ and $\omega_i|_N = 0$. Furthermore, we can arrange that $\omega_1|_{N(T)} = \omega_0$ and $\omega_i|_{N(T)} = 0$ for $i > 1$. We let $\boldsymbol{\omega}' = (\omega_1', \dots, \omega_n')$ be an n-tuple of forms on $X_{\mathcal{C}}$ such that $\omega_1'|_{N(T)} = \omega_{\mathcal{C}}$ and $\omega_i'|_{N(T)} = 0$ for $i > 1$, while $\omega_i'|_{X_0} = \omega_i|_{X_0}$ for $i \in \{1, \dots, n\}$.

By Proposition 4.3,

$$\Phi_{X;\boldsymbol{b}} = \langle (F^+_{W_2;\boldsymbol{\omega}|_{W_2}} \circ \delta^{-1} \circ F^-_{W_1;\boldsymbol{\omega}|_{W_1}})(1), \Theta_+ \rangle.$$

We now apply the composition law, Proposition 3.2, to the splitting $W_2 = W_0 \cup_{\mathbb{T}^3} W'$, where $W_0 = N(T)$ and $W' = W_2 \setminus \operatorname{int}(N(T))$, to obtain

$$F^+_{W_2;\boldsymbol{\omega}|_{W_2}} \doteq F^+_{W_0;\boldsymbol{\omega}|_{W_0}} \circ F^+_{W';\boldsymbol{\omega}|_{W'}}.$$

Similarly, if $W_2' := W_{\mathcal{C}} \cup_{\mathbb{T}^3} W'$, then

$$\Phi_{X_{\mathcal{C}};\boldsymbol{b}} = \langle (F^+_{W_2';\boldsymbol{\omega}'|_{W_2'}} \circ \delta^{-1} \circ F^-_{W_1;\boldsymbol{\omega}'|_{W_1}})(1), \Theta_+ \rangle,$$

where

$$F^+_{W_2';\boldsymbol{\omega}'|_{W_2'}} \doteq F^+_{W_{\mathcal{C}};\boldsymbol{\omega}'|_{W_{\mathcal{C}}}} \circ F^+_{W';\boldsymbol{\omega}'|_{W'}}.$$

By construction of $\boldsymbol{\omega}'$, we have $\boldsymbol{\omega}'|_{W_1} = \boldsymbol{\omega}|_{W_1}$ and $\boldsymbol{\omega}'|_{W'} = \boldsymbol{\omega}|_{W'}$. Hence, it follows from Corollary 5.6 that

$$\Phi_{X_{\mathcal{C}};\boldsymbol{b}} \doteq \operatorname{Lef}_{z_1}(\mathcal{C}) \cdot \Phi_{X;\boldsymbol{b}}. \tag{5.3}$$

Equality in (5.3) can be established using the conjugation symmetry of the Ozsváth–Szabó 4-manifolds invariants [25, Theorem 3.6]. ∎

5.1. *Concordance surgery and diffeomorphism types of 4-manifolds*

As an application of Theorem 1.1, we prove Corollary 1.2, which states that X and $X_{\mathcal{C}}$ are not diffeomorphic if $\Phi_{X;\boldsymbol{b}} \neq 0$ and $\operatorname{Lef}_z(\mathcal{C}) \neq 1$:

Proof of Corollary 1.2. Choose a basis $\boldsymbol{b} = (b_1, \dots, b_n)$ of $H_2(X; \mathbb{R})$ that is induced by a basis of $H^2(X)/\operatorname{Tors}$. In this situation, the invariant $\Phi_{X;\boldsymbol{b}}$ takes values in the integral group ring $\mathbb{F}[\mathbb{Z}^n]$. It is convenient to use the group ring notation

$$e^{(a_1,\dots,a_n)} := z_1^{a_1} \cdots z_n^{a_n},$$

where $(a_1, \dots, a_n) \in \mathbb{Z}^n$. If $\boldsymbol{b} = (b_1, \dots, b_n)$ is an n-tuple of cohomology classes, we abbreviate

$$\langle i_*(\mathfrak{s} - \mathfrak{s}_0) \cup \boldsymbol{b}, [X] \rangle := (\langle i_*(\mathfrak{s} - \mathfrak{s}_0) \cup b_1, [X] \rangle, \dots, \langle i_*(\mathfrak{s} - \mathfrak{s}_0) \cup b_n, [X] \rangle).$$

Performing a change of basis to Theorem 1.1, we obtain

$$\Phi_{X_{\mathcal{C}};\boldsymbol{b}} = \operatorname{Lef}_{e^{\langle \boldsymbol{b}, [T] \rangle}}(\mathcal{C}) \cdot \Phi_{X;\boldsymbol{b}}. \tag{5.4}$$

On the other hand, if $\phi\colon X_{\mathcal{C}} \to X$ were an orientation-preserving diffeomorphism, then

$$\Phi_{X,\mathfrak{s}} = \Phi_{X_{\mathcal{C}},\phi^*(\mathfrak{s})} \tag{5.5}$$

for all $\mathfrak{s}$. Hence

$$\begin{aligned}
\Phi_{X;\boldsymbol{b}} &:= \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} e^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup\boldsymbol{b},[X]\rangle}\cdot\Phi_{X,\mathfrak{s}} \\
&= \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} e^{\langle \phi^* i_*(\mathfrak{s}-\mathfrak{s}_0)\cup\phi^*(\boldsymbol{b}),[X_{\mathcal{C}}]\rangle}\cdot\Phi_{X_{\mathcal{C}},\phi^*(\mathfrak{s})} \\
&= \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X_{\mathcal{C}})} e^{\langle i_*(\mathfrak{s}-\phi^*(\mathfrak{s}_0))\cup\phi^*(\boldsymbol{b}),[X_{\mathcal{C}}]\rangle}\cdot\Phi_{X_{\mathcal{C}},\mathfrak{s}} \\
&\doteq \sum_{\mathfrak{s}\in\mathrm{Spin}^c(X_{\mathcal{C}})} e^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup\phi^*(\boldsymbol{b}),[X_{\mathcal{C}}]\rangle}\cdot\Phi_{X_{\mathcal{C}},\mathfrak{s}} \\
&= e^{M(\phi^*)^t}\cdot\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X_{\mathcal{C}})} e^{\langle i_*(\mathfrak{s}-\mathfrak{s}_0)\cup\boldsymbol{b},[X_{\mathcal{C}}]\rangle}\cdot\Phi_{X_{\mathcal{C}},\mathfrak{s}} \\
&= e^{M(\phi^*)^t}\cdot\Phi_{X_{\mathcal{C}};\boldsymbol{b}}.
\end{aligned} \tag{5.6}$$

Here, $M(\phi^*)$ denotes the element of $\mathrm{GL}_n(\mathbb{Z})$ induced by ϕ^* after identifying $H^2(X)/\mathrm{Tors}$ and $H^2(X_{\mathcal{C}})/\mathrm{Tors}$ with $\mathbb{Z}^n$ via the basis $\boldsymbol{b}$, and $M(\phi^*)^t$ denotes its transpose. Also, we are writing $e^{M(\phi^*)^t}$ for the endomorphism of $\mathbb{F}[\mathbb{Z}^n]$ given by $e^{M(\phi^*)^t}e^{\boldsymbol{a}} = e^{M(\phi^*)^t\cdot\boldsymbol{a}}$, where we view $\boldsymbol{a}$ as a column vector.

Equation (5.6) is justified as follows. The first equality is a definition. The second equality follows from (5.5), and the naturality of cohomology. The third equality follows from rearranging the sum. The fourth equality follows since $\Phi_{X;\boldsymbol{b}}$ is independent, up to overall multiplication by a monomial, of the choice of base Spin^c structure $\mathfrak{s}_0$. The fifth equality can be computed directly, and the final equality again holds by definition.

The ring $\mathbb{F}[\mathbb{Z}^n]$ is a UFD, since it is the localization of the polynomial ring $\mathbb{F}[z_1,\dots,z_n]$ at monomials. Furthermore, the units are exactly the monomials. The map $e^{M(\phi^*)^t}$ preserves the number of irreducible factors since

$$e^{M(\phi^*)^t}(f\cdot g) = (e^{M(\phi^*)^t}\cdot f)(e^{M(\phi^*)^t}\cdot g),$$

the map $e^{M(\phi^*)^t}$ sends monomials to monomials, and $e^{M(\phi^*)^t}$ is invertible.

In particular, if $\mathrm{Lef}_z(\mathcal{C}) \neq 1$ and $\Phi_{X;\boldsymbol{b}} \neq 0$, (5.4) implies that $\Phi_{X_{\mathcal{C}};\boldsymbol{b}}$ has more irreducible factors than $\Phi_{X;\boldsymbol{b}}$, while (5.6) implies they have the same number, a contradiction. ■

6. Naturality of perturbed sutured Floer homology

This section is devoted to defining transition maps on perturbed sutured Floer homology for naturality, and proving Theorem 2.1.

6.1. *Changing the* 2*-form*

We first describe transition maps for changing the 2-form by a boundary. Unlike the transition maps for changing the Heegaard diagrams, we usually do not want to view sutured Floer homology as a transitive system over closed 2-forms which represent the same cohomology class. Nonetheless, the transition maps for changing the 2-form are convenient to define.

Let $\mathcal{H}$ be an admissible diagram of the balanced sutured manifold (M,γ), and let ω and ω' be closed cohomologous 2-forms on M. Suppose η is a 1-form such that $d\eta = \omega' - \omega$. Then we may define a chain isomorphism

$$\Psi_{\omega\to\omega';\eta}\colon CF_J(\mathcal{H};\Lambda_\omega) \to CF_J(\mathcal{H};\Lambda_{\omega'})$$

via the formula

$$\Psi_{\omega\to\omega';\eta}(z^x\cdot \boldsymbol{x}) = z^{x+\int_{\gamma_{\boldsymbol{x}}}\eta}\cdot \boldsymbol{x},$$

where we obtain $\gamma_{\boldsymbol{x}}$ by connecting $\boldsymbol{x}$ to the centers of the disks D_α and D_β along radii. We orient $\gamma_{\boldsymbol{x}}$ from D_α to D_β. The map $\Psi_{\omega\to\omega';\eta}$ is a chain map by Stokes' theorem, and is an isomorphism since $\Psi_{\omega'\to\omega;-\eta}$ is its inverse.

Lemma 6.1. *When restricted to a single* Spin^c *structure, the map* $\Psi_{\omega\to\omega';\eta}$ *is independent of the* 1*-form* η *satisfying* $d\eta = \omega' - \omega$*, up to an overall factor of* z^x*.*

Proof. It is sufficient to show that if η is a closed 1-form, then $\Psi_{\omega\to\omega;\eta}$ is equal to overall multiplication by z^x for some $x\in\mathbb{R}$, when restricted to a single Spin^c structure. Hence, it is sufficient to show that if $\underline{\mathfrak{s}}(\boldsymbol{x}) = \underline{\mathfrak{s}}(\boldsymbol{y})$ and $d\eta = 0$, then

$$\int_{\gamma_{\boldsymbol{x}}}\eta = \int_{\gamma_{\boldsymbol{y}}}\eta.$$

The condition that $\underline{\mathfrak{s}}(\boldsymbol{x}) = \underline{\mathfrak{s}}(\boldsymbol{y})$ is equivalent to the condition that the integral 1-cycle $\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}$ is ∂S for some integral 2-chain S. By Stokes' theorem,

$$\int_{\gamma_{\boldsymbol{x}}-\gamma_{\boldsymbol{y}}}\eta = \int_S d\eta = 0,$$

completing the proof. ∎

In general, the map $\Psi_{\omega\to\omega';\eta}$ is not independent of η when working with multiple Spin^c structures at once, even if $[\omega] = [\omega'] = 0$; see Remark 7.3.

6.2. *Change of almost complex structure maps*

Suppose $\mathcal{H}$ is an admissible diagram of (M,γ). If J and J' are two cylindrical almost complex structures on $\Sigma\times I\times\mathbb{R}$, there is a standard Floer-theoretic construction that gives a transition map from $CF_J(\mathcal{H};\Lambda_\omega)$ to $CF_{J'}(\mathcal{H};\Lambda_\omega)$; see Lipshitz [17, Section 9]. Pick a generic almost complex structure $\widetilde{J}$ on $\Sigma\times I\times\mathbb{R}$ such that

$$\widetilde{J} = J \quad \text{on } \Sigma \times I \times (-\infty, a],$$
$$\widetilde{J} = J' \quad \text{on } \Sigma \times I \times [b, \infty),$$

where $a \ll 0$ and $b \gg 0$. Define

$$\Psi_{J \to J'} \colon CF_J(\mathcal{H}; \Lambda_\omega) \to CF_{J'}(\mathcal{H}; \Lambda_\omega)$$

via the formula

$$\Psi_{J\to J'}(z^x \cdot \boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x},\boldsymbol{y}) \\ \mu(\phi)=0}} (|\mathcal{M}_{\widetilde{J}}(\phi)| \bmod 2) \cdot z^{x + A_\omega(\phi)} \cdot \boldsymbol{y}.$$

Lemma 6.2. *The map $\Psi_{J\to J'}$ is a chain map, and is independent of $\widetilde{J}$, up to chain homotopy.*

Proof. The claim that $\Psi_{J\to J'}$ is a chain map is proven by counting the ends of the moduli spaces of index 1, $\widetilde{J}$-holomorphic curves. The claim that $\Psi_{J\to J'}$ is independent of $\widetilde{J}$ is proven by taking two generic choices $\widetilde{J}_0$ and $\widetilde{J}_1$, and connecting them via a path $(\widetilde{J}_z)_{t\in I}$. A chain homotopy between the map which counts $\widetilde{J}_0$-holomorphic curves and the map which counts $\widetilde{J}_1$-holomorphic curves is given by counting index -1 curves that are $\widetilde{J}_t$-holomorphic for some $t \in I$. ■

6.3. *Perturbed stabilization maps*

Suppose that $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram of (M, γ), and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$ is a stabilization of $\mathcal{H}$, i.e., there is a 3-ball B in $\operatorname{int}(M)$ such that

(1) $B \cap \Sigma$ is a disk and $B \cap \Sigma'$ is a punctured 2-torus that contains the curves α' and β', and is disjoint from $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$,

(2) $\Sigma \setminus B = \Sigma' \setminus B$, and

(3) α' and β' intersect transversely at a single point c.

The stabilization map

$$\sigma \colon CF(\mathcal{H}) \to CF(\mathcal{H}')$$

is given by $\sigma(\boldsymbol{x}) = \boldsymbol{x} \times c$. According to [23, Theorem 10.2], for a sufficiently stretched almost complex structure, the map σ is a chain map. See Lipshitz [17, Section 12] for the corresponding result in the cylindrical reformulation. We define the perturbed stabilization map

$$\sigma \colon CF(\mathcal{H}; \Lambda_\omega) \to CF(\mathcal{H}'; \Lambda_\omega)$$

via the formula $\sigma(z^x \cdot \boldsymbol{x}) = z^x \cdot (\boldsymbol{x} \times c)$.

Lemma 6.3. *For a sufficiently stretched almost complex structure, the perturbed stabilization map $\sigma \colon CF(\mathcal{H}; \Lambda_\omega) \to CF(\mathcal{H}'; \Lambda_\omega)$ is a chain map.*

Proof. If $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ is a class on $\mathcal{H}$, there is a unique class $\phi' \in \pi_2(\boldsymbol{x} \times c, \boldsymbol{y} \times c)$ whose domain agrees with ϕ on $\Sigma \setminus B$. The class ϕ' has the same Maslov index as ϕ.

Ozsváth and Szabó showed that if $\mu(\phi) = 1$, and if the almost complex structure on Σ' is sufficiently stretched, then

$$|\mathcal{M}(\phi)/\mathbb{R}| \equiv |\mathcal{M}(\phi')/\mathbb{R}| \bmod 2. \tag{6.1}$$

We note that the 2-chain $\widetilde{\mathcal{D}}(\phi')$ only differs from $\widetilde{\mathcal{D}}(\phi)$ in the 3-ball B. Furthermore, there is an integral 3-chain C_3 (a sum of solid tori) such that

$$\widetilde{\mathcal{D}}(\phi') + \partial C_3 = \widetilde{\mathcal{D}}(\phi).$$

Hence $A_\omega(\phi') = A_\omega(\phi)$, so (6.1) implies that σ is a chain map on the perturbed complex. ∎

6.4. Perturbed isotopy maps

Suppose that $(\phi_t)_{t\in I}$ is an isotopy of M satisfying $\phi_0 = \mathrm{id}_M$. For convenience, assume that ϕ_t is constant for t in a neighborhood of ∂I. If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram for (M, γ), write $\mathcal{H}'$ for the diagram obtained by pushing forward Σ along ϕ_1. Let J be a cylindrical almost complex structure on $\Sigma \times I \times \mathbb{R}$, and let J' denote its pushforward along ϕ_1. Given a choice of compressing disks D_α and D_β for $\mathcal{H}$, we use $\phi_1(D_\alpha)$ and $\phi_1(D_\beta)$ for $\mathcal{H}'$.

If $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is an intersection point on $\mathcal{H}$, let $\gamma_{\boldsymbol{x}}$ denote the 1-chain obtained by coning the points of $\boldsymbol{x}$ into U_α and U_β, and let $\Gamma_{\boldsymbol{x},\phi_t}$ denote the 2-chain in M obtained by sweeping out $\gamma_{\boldsymbol{x}}$ under ϕ_t. We define

$$(\phi_t)_*\colon CF_J(\mathcal{H}; \Lambda_\omega) \to CF_{J'}(\mathcal{H}'; \Lambda_\omega)$$

via the formula

$$z^x \cdot \boldsymbol{x} \mapsto z^{x + \int_{\Gamma_{\boldsymbol{x},\phi_t}} \omega} \cdot \phi_1(\boldsymbol{x}).$$

Stokes' theorem can be used to show that $(\phi_t)_*$ is a chain map. We define the transition map for the isotopy $(\phi_t)_{t\in I}$ from $\mathcal{H}$ to its image $\mathcal{H}'$ to be $(\phi_t)_*$.

Remark 6.4. As a special case of the above construction, when ϕ_t fixes the Heegaard surface pointwise for all t, the map $(\phi_t)_*$ induces a map for transitioning between collections of compressing disks that are related by an ambient isotopy fixing Σ pointwise. A similar construction gives a map for transitioning between collections of compressing disks that are instead only isotopic as maps from D^2 into Y, relative to ∂D^2. The construction also adapts to give a transition map for changing the choice of radial foliation on the disks.

The map $(\phi_t)_*$ depends only on ϕ_1, in the following sense:

Lemma 6.5. *Suppose that $(\phi_t)_{t\in I}$ and $(\psi_t)_{t\in I}$ are two isotopies of (M, γ) such that $\phi_0 = \psi_0 = \mathrm{id}_{(M,\gamma)}$, and $\phi_1 = \psi_1$. Then $(\phi_t)_* \doteq (\psi_t)_*$ on each* Spin^c *structure. If $[\omega] = 0$, then $(\phi_t)_* \doteq (\psi_t)_*$ on all of $CF_J(\mathcal{H}; \Lambda_\omega)$.*

Proof. Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two intersection points that represent the same Spin^c structure. This is equivalent to the condition that $\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}} = \partial S$ for some integral 2-chain S in M. The isotopies ϕ_t and ψ_t applied to S sweep out 3-chains C_{ϕ_t} and C_{ψ_t}. We have

$$\partial C_{\phi_t} = \Gamma_{\boldsymbol{x},\phi_t} - \Gamma_{\boldsymbol{y},\phi_t} - S + \phi_1(S), \tag{6.2}$$

and a similar formula holds for ∂C_{ψ_t}. Integrating $d\omega = 0$ on C_{ϕ_t} and C_{ψ_t}, and using (6.2) and Stokes' theorem, we obtain

$$\int_{\Gamma_{\boldsymbol{x},\phi_t}} \omega - \int_{\Gamma_{\boldsymbol{y},\phi_t}} \omega = \int_{\Gamma_{\boldsymbol{x},\psi_t}} \omega - \int_{\Gamma_{\boldsymbol{y},\psi_t}} \omega. \tag{6.3}$$

Equation (6.3) implies that $(\psi_t)_*$ and $(\phi_t)_*$ differ only by an overall factor of z^x when restricted to a single Spin^c structure.

Suppose now that $[\omega] = 0$, and let $\boldsymbol{x}$ and $\boldsymbol{y}$ be any two intersection points. Since $\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}$ is a 1-cycle, $(\Gamma_{\boldsymbol{x},\phi_t} - \Gamma_{\boldsymbol{y},\phi_t}) - (\Gamma_{\boldsymbol{x},\psi_t} - \Gamma_{\boldsymbol{y},\psi_t})$ is a 2-cycle, so ω integrates to zero over it, and (6.3) follows. ■

Let ϕ be an automorphism of (M,γ). If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram of (M,γ) with a cylindrical almost complex structure J on $\Sigma \times I \times \mathbb{R}$, and $\mathcal{H}' = \phi(\mathcal{H})$ and $J' = \phi_*(J)$ are their pushforwards, then there is a tautological chain isomorphism

$$\phi_*^{\text{taut}} \colon CF_J(\mathcal{H}; \Lambda_\omega) \to CF_{J'}(\mathcal{H}'; \Lambda_{\phi_*(\omega)}),$$

obtained by sending $z^x \cdot \boldsymbol{x}$ to $z^x \cdot \phi(\boldsymbol{x})$. If $\phi_*(\omega) = \omega$, we have the following relation between the tautological map and the map from naturality:

Lemma 6.6. *If $(\phi_t)_{t\in I}$ is an isotopy of (M,γ) such that $\phi_0 = \mathrm{id}$ and $(\phi_1)_*(\omega) = \omega$, then*

$$(\phi_t)_* \doteq (\phi_1)_*^{\text{taut}}$$

on each Spin^c *structure.*

Proof. By definition, $(\phi_t)_*(z^x \cdot \boldsymbol{x}) = z^{x+\int_{\Gamma_{\boldsymbol{x},\phi_t}} \omega} \cdot \boldsymbol{x}$, where $\Gamma_{\boldsymbol{x},\phi_t}$ is the 2-chain swept out by $\gamma_{\boldsymbol{x}}$ under ϕ_t. Hence, it is sufficient to show that if $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the same Spin^c structure, then

$$\int_{\Gamma_{\boldsymbol{x},\phi_t}} \omega = \int_{\Gamma_{\boldsymbol{y},\phi_t}} \omega.$$

As in the proof of Lemma 6.5, write S for a 2-chain such that $\partial S = \gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}}$. By (6.2), and since $d\omega = 0$, we have

$$\int_{\Gamma_{\boldsymbol{x},\phi_t}} \omega - \int_{\Gamma_{\boldsymbol{y},\phi_t}} \omega = \int_S \omega - \int_{\phi_1(S)} \omega.$$

Since $(\phi_1)_*(\omega) = \omega$, we have $\int_{\phi_1(S)} \omega = \int_{\phi_1(S)} (\phi_1)_*(\omega) = \int_S \omega$, and the result follows. ■

6.5. *Monodromy*

In this section, we give several examples which illustrate the existence of monodromy around loops of Heegaard diagrams.

Example 6.7. Suppose $D_{\alpha,t}$ for $t \in I$ is a path of compressing disks that moves just one of the compressing disks D_i. Further, assume that the center of D_i traces out a small loop in U_α that bounds a disk D_0. Following Remark 6.4, by modifying the transition maps for isotopies, the path $D_{\alpha,t}$ induces a transition map. Write $\gamma_{\boldsymbol{x},t}$ for the 1-chain obtained by coning $\boldsymbol{x}$ using $D_{\alpha,t}$, and write $\Gamma_{\boldsymbol{x}}$ for the 2-chain swept out by $\gamma_{\boldsymbol{x},t}$ for $t \in I$. Then $\Gamma_{\boldsymbol{x}} \cup D_0$ is a closed 2-chain, which is a boundary since $H_2(U_\alpha) = \{0\}$. Hence, the monodromy of the transition maps around the loop $D_{\alpha,t}$ is overall multiplication by

$$z^{\int_{\Gamma_{\boldsymbol{x}}} \omega} = z^{-\int_{D_0} \omega},$$

which may be non-zero.

We now show that the perturbed isotopy maps can have projectively non-trivial monodromy over loops of Heegaard diagrams if we consider multiple Spin^c structures simultaneously.

Lemma 6.8. *Suppose that $\mathcal{H}$ is an admissible diagram for (M,γ) and $(\phi_t)_{t\in I}$ is an isotopy of M such that $\phi_0 = \phi_1 = \mathrm{id}_{(M,\gamma)}$. Let*

$$f\colon H_1(M) \to H_2(M)$$

denote the composition $H_1(M) \to H_2(M \times S^1) \to H_2(M)$, where the first map is obtained via the cross product with the fundamental class of S^1, and the second map is induced by ϕ_t. If $\mathfrak{s}_0 \in \mathrm{Spin}^c(M,\gamma)$ is a fixed Spin^c structure, then the isotopy map (summed over all Spin^c structures)

$$(\phi_t)_*\colon CF(\mathcal{H};\Lambda_\omega) \to CF(\mathcal{H};\Lambda_\omega)$$

is projectively equivalent to the map

$$\boldsymbol{x} \mapsto z^{\int_{f(PD[\mathfrak{s}(\boldsymbol{x})-\mathfrak{s}_0])} \omega} \cdot \boldsymbol{x}.$$

Proof. As in the proof of Lemma 6.5, let $\Gamma_{\boldsymbol{x},\phi_t}$ denote the 2-chain obtained by sweeping out $\gamma_{\boldsymbol{x}}$ under ϕ_t. Let $\boldsymbol{x}_0$ be some fixed intersection point on $\mathcal{H}$, and let $\mathfrak{s}_0 = \mathfrak{s}(\boldsymbol{x}_0)$. If $\boldsymbol{x}$ is an arbitrary intersection point, then

$$PD[\mathfrak{s}(\boldsymbol{x}) - \mathfrak{s}_0] = \gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{x}_0}$$

by [7, Lemma 4.7]. The claim now follows from the computation

$$\int_{\Gamma_{\boldsymbol{x},\phi_t}} \omega - \int_{\Gamma_{\boldsymbol{x}_0,\phi_t}} \omega = \int_{\Gamma_{\boldsymbol{x},\phi_t} - \Gamma_{\boldsymbol{x}_0,\phi_t}} \omega = \int_{f(\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{x}_0})} \omega = \int_{f(PD[\mathfrak{s}(\boldsymbol{x}) - \mathfrak{s}_0])} \omega. \qquad \blacksquare$$

Example 6.9. Let $D \subseteq \mathbb{T}^2$ be a closed disk, and set $M = (\mathbb{T}^2 \setminus \operatorname{int}(D)) \times S^1$. Let the sutures $\gamma \subseteq \partial M$ be the images of two points in ∂D under the action of S^1. The S^1-action induces a loop ϕ_t of automorphisms of (M, γ) based at $\mathrm{id}_{(M,\gamma)}$. The map f is non-zero in this case, and hence $(\phi_t)_*$ is projectively non-trivial when considered over the whole chain complex by Lemma 6.8.

6.6. *Perturbed triangle maps*

Suppose $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram for (M, γ), and $\boldsymbol{\alpha}'$ is obtained from $\boldsymbol{\alpha}$ by a sequence of handleslides and isotopies. Suppose further that $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible. Then there is an unperturbed holomorphic triangle map

$$F_{\alpha',\alpha,\beta} \colon CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}) \otimes CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}).$$

Pick compressing disks $D_{\alpha'}$, D_α, and D_β for $\boldsymbol{\alpha}'$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, respectively. Note that since $U_\alpha = U_{\alpha'}$, the disks D_α and $D_{\alpha'}$ are compressing disks for the same handlebody. If $\psi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a homology class of triangles, we may cone the domain of ψ along the compressing disks to obtain a 2-chain $\widetilde{\mathcal{D}}(\psi)$ in M. By integrating ω over $\widetilde{\mathcal{D}}(\psi)$, we obtain a real number $A_\omega(\psi)$. Hence, we obtain a perturbed version of the triangle map

$$F_{\alpha',\alpha,\beta;\omega} \colon CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}; \Lambda_{\omega|_{U_\alpha}}) \otimes CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; \Lambda_\omega) \to CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}; \Lambda_\omega).$$

Some care is required in interpreting $CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}; \Lambda_{\omega|_{U_\alpha}})$, as its definition differs slightly from the other two complexes. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\alpha$ and $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, we cone the class ϕ in U_α, using the compressing disks D_α and $D_{\alpha'}$. We define $A_\omega(\phi)$ as the integral of ω over this 2-chain in U_α.

Since $H^2(U_\alpha) = 0$, we may write $\omega|_{U_\alpha} = d\eta$ for some 1-form $\eta \in \Omega^1(U_\alpha)$. There is a chain isomorphism

$$\Psi_{0 \to \omega|_{U_\alpha};\eta} \colon CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}) \otimes \Lambda \to CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}; \Lambda_{\omega|_{U_\alpha}}),$$

whose construction is analogous to the one in Section 6.1. The complex $CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha})$ contains a cycle $\Theta_{\alpha',\alpha}$ whose homology class is the top-graded generator of $SFH(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha})$. The cycle $\Theta_{\alpha',\alpha}$ is unique up to adding a boundary. We define

$$\Theta^\omega_{\alpha',\alpha} := \Psi_{0 \to \omega|_{U_\alpha};\eta}(\Theta_{\alpha',\alpha} \otimes 1_\Lambda) \in CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}; \Lambda_{\omega|_{U_\alpha}}). \tag{6.4}$$

A simple modification of Lemma 6.1 implies that $[\Theta^\omega_{\alpha',\alpha}]$ is independent of η, up to overall multiplication by z^x.

If the triple $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible, then the transition map

$$\Psi^\beta_{\alpha \to \alpha'} \colon CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; \Lambda_\omega) \to CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}; \Lambda_\omega)$$

is defined via the formula

$$\Psi^\beta_{\alpha \to \alpha'}(-) := F_{\alpha',\alpha,\beta;\omega}(\Theta^\omega_{\alpha',\alpha}, -).$$

If $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ is not admissible, we define $\Psi^{\beta}_{\alpha\to\alpha'}$ by picking a collection $\boldsymbol{\alpha}''$ such that the triples $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}'', \boldsymbol{\beta})$ and $(\Sigma, \boldsymbol{\alpha}'', \boldsymbol{\alpha}, \boldsymbol{\beta})$ are both admissible, and setting $\Psi^{\beta}_{\alpha\to\alpha'}$ to be the composition of the triangle maps for $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}'', \boldsymbol{\beta})$ and $(\Sigma, \boldsymbol{\alpha}'', \boldsymbol{\alpha}, \boldsymbol{\beta})$. A similar construction works for changes of the beta-curves.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ are two admissible diagrams, then we define a transition map

$$\Psi^{\beta\to\beta'}_{\alpha\to\alpha'} := \Psi^{\beta'}_{\alpha\to\alpha'} \circ \Psi^{\beta\to\beta'}_{\alpha}. \tag{6.5}$$

As in the unperturbed setting, the right-hand side is chain homotopic to $\Psi^{\beta\to\beta'}_{\alpha'} \circ \Psi^{\beta}_{\alpha\to\alpha'}$. A chain homotopy may be constructed by counting holomorphic quadrilaterals. More generally, an associativity argument gives the following:

Proposition 6.10. *The transition map $\Psi^{\beta\to\beta'}_{\alpha\to\alpha'}$ is well-defined up to chain homotopy and overall multiplication by z^x. Furthermore,*

$$\Psi^{\beta'\to\beta''}_{\alpha'\to\alpha''} \circ \Psi^{\beta\to\beta'}_{\alpha\to\alpha'} \mathrel{\dot{\simeq}} \Psi^{\beta\to\beta''}_{\alpha\to\alpha''}.$$

6.7. *Compatibility of the triangle and isotopy maps*

We now address compatibility of the maps induced by isotopies with the maps induced by counting holomorphic triangles.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be an admissible diagram, and $(\boldsymbol{\alpha}_t)_{t\in I}$ a small Hamiltonian isotopy with $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}$, which extends smoothly over $t \in \mathbb{R}$ and is constant outside I. Then there is a continuation map

$$\Gamma_{\alpha_t, J;\omega} \colon CF_J(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}; \Lambda_\omega) \to CF_J(\Sigma, \boldsymbol{\alpha}_1, \boldsymbol{\beta}; \Lambda_\omega)$$

that counts index 0, J-holomorphic curves with boundary on the cylinders

$$C_{\alpha_t} := \{(p, 0, t) : p \in \boldsymbol{\alpha}_t,\ t \in \mathbb{R}\} \quad \text{and} \quad C_\beta := \{(p, 1, t) : p \in \boldsymbol{\beta},\ t \in \mathbb{R}\},$$

weighted by their ω-area. The cylinder C_β is Lagrangian for the product symplectic form, while C_{α_t} is Lagrangian with respect to a symplectic form that has been deformed slightly near $\Sigma \times \{0\} \times \mathbb{R}$; see [18, (3.25)]. Finiteness of the counts contributing to $\Gamma_{\alpha_t, J;\omega}$ follows from the work of Ozsváth and Szabó [23, Lemma 7.4], using the admissibility assumption on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Compatibility of the triangle and continuation maps is given by the following lemma, adapted from the work of Lipshitz [17, Section 11]:

Lemma 6.11. *Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram for (M, γ), and $\boldsymbol{\alpha}'$ is obtained from $\boldsymbol{\alpha}$ by a small Hamiltonian isotopy $\boldsymbol{\alpha}_t$ (for some symplectic form on Σ) such that $|\alpha_i' \cap \alpha_j| = 2\delta_{ij}$, where δ_{ij} denotes the Kronecker delta. Let J denote a cylindrical almost complex structure on $\Sigma \times I \times \mathbb{R}$, and let $\Gamma_{\alpha_t, J;\omega} \colon CF_J(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; \Lambda_\omega) \to CF_J(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}; \Lambda_\omega)$ denote the continuation map. Then*

$$\Gamma_{\alpha_t, J;\omega}(-) \mathrel{\dot{\simeq}} F_{\alpha', \alpha, \beta;\omega}(\Theta^{\omega}_{\alpha', \alpha}, -).$$

Proof. The proof is an adaptation of the proof of the result in the unperturbed setting [17, Proposition 11.4]. Lipshitz's proof considers the moduli space of holomorphic *monogons* associated to the isotopy $\boldsymbol{\alpha}_t$, which are maps from a Riemann surface S to $\Sigma \times [0,\infty) \times \mathbb{R}$ that have punctures asymptotic to an intersection point $\boldsymbol{x} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\alpha}$, and have boundary mapping to the cylinder

$$C_{\alpha_t} := \{(p, 0, t) : p \in \boldsymbol{\alpha}_t, t \in \mathbb{R}\}.$$

Following Lipshitz's proof, a deformation of the almost complex structure on $\Sigma \times I \times \mathbb{R}$ gives a chain homotopy between $\Gamma_{\alpha_t, J;\omega}$ and the composition

$$F_{\alpha',\alpha,\beta;\omega}(M_{\alpha_t;\omega}(1), -),$$

where $M_{\alpha_t;\omega}$ is a map from Λ to $CF(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}; \Lambda_{\omega|U_\alpha})$ that sums over the count of index 0 monogons at all intersection points $\boldsymbol{x} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\alpha$. If $\boldsymbol{x} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_\alpha$ is an intersection point and $\phi \in \pi_2(\boldsymbol{x})$ is a class of monogons, then ϕ may be coned along a family of compressing disks D_{α_t} to obtain a 2-chain $\widetilde{\mathcal{D}}(\phi)$, on which we may integrate ω. According to [17, Lemma 11.8], there are no index 0 classes $\phi \in \pi_2(\boldsymbol{x})$ with holomorphic representatives unless $\boldsymbol{x} = \Theta_{\alpha',\alpha}$. Furthermore, a model computation involving a stabilized diagram of S^3 can be used to show that $M_{\alpha_t;\omega}(1) = z^x \cdot \Theta^{\omega}_{\alpha',\alpha}$ for some $x \in \mathbb{R}$. We refer the reader to [17, Proposition 11.4] for more details on the model computation. ∎

Next, we consider a diffeomorphism $\phi\colon \Sigma \to \Sigma$, which is near id_Σ, and is the time 1 flow of a Hamiltonian vector field for some symplectic form on Σ. Write ϕ_t for the time t flow of this Hamiltonian vector field. In particular, $\phi_1 = \phi$. By extending ϕ_t to an isotopy of M, we obtain an isotopy map $(\phi_t)_*$ on the perturbed Floer homology, as in Section 6.4.

Proposition 6.12. *Suppose* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *is an admissible diagram for a sutured manifold* (M, γ) *which is equipped with a closed* 2*-form* ω*, and* $\phi_t\colon \Sigma \to \Sigma$ *is the flow of a Hamiltonian vector field* (*for some symplectic form on* Σ) *as above. Write* $\boldsymbol{\alpha}_t = \phi_t(\boldsymbol{\alpha})$ *and* $\boldsymbol{\beta}_t = \phi_t(\boldsymbol{\beta})$*. Then the perturbed isotopy map* $(\phi_t)_*$ *satisfies*

$$(\phi_t)_* \mathrel{\dot{\simeq}} \Psi_{J \to \phi_*(J)} \circ \Psi^{\beta \to \beta_1}_{\alpha \to \alpha_1}.$$

Proof. The first step is to interpret the isotopy map $(\phi_t)_*$ as a continuation map. Consider the two cylinders C_{α_t} and C_{β_t}, where $\boldsymbol{\alpha}_t$ and $\boldsymbol{\beta}_t$ are the images of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ under ϕ_t. Let $\widetilde{J}$ denote the almost complex structure on $\Sigma \times I \times \mathbb{R}$ obtained by pushing forward a generic cylindrical almost complex structure J along the map $\Phi(x, s, t) = (\phi_t(x), s, t)$. For ϕ_t sufficiently small, $\widetilde{J}$ will be tamed by a product symplectic form, and achieve transversality at index 0 holomorphic curves with boundary on C_{α_t} and C_{β_t}. Hence, if $\Gamma_{\alpha_t,\beta_t,\widetilde{J};\omega}$ denotes the map that counts index 0, $\widetilde{J}$-holomorphic curves with boundary on C_{α_t} and C_{β_t}, we have

$$\Gamma_{\alpha_t,\beta_t,\widetilde{J};\omega}(\boldsymbol{x}) = (\phi_t)_*(\boldsymbol{x}). \tag{6.6}$$

We now consider a 1-parameter family of cylinders $C_{\alpha_t^\tau}$, $C_{\beta_t^\tau}$, and almost complex structures $\widetilde{J}^\tau$ for $\tau \in [0,\infty)$, as follows. The cylinder $C_{\alpha_t^\tau}$ is obtained by translating C_{α_t}

downward in the $\mathbb{R}$-direction by τ units. The cylinder $C_{\beta_t^\tau}$ coincides with C_{β_t} for all τ. The almost complex structure $\widetilde{J}^\tau$ is obtained by translating $\widetilde{J}$ upward in the $\mathbb{R}$-direction by τ units.

A chain homotopy H is defined by counting index -1, $\widetilde{J}^\tau$-holomorphic curves with boundary on $C_{\alpha_t^\tau}$ and $C_{\beta_t^\tau}$ for $\tau \in [0,\infty)$, weighted by their ω-area. Applying Gromov compactness to the moduli space of index 0, $\widetilde{J}^\tau$-holomorphic curves with boundary on $C_{\alpha_t^\tau}$ and $C_{\beta_t^\tau}$ for $\tau \in [0,\infty)$, we obtain

$$\Gamma_{\alpha_t,\beta_t,\widetilde{J};\omega} + \Psi_{J\to\phi_*(J)} \circ \Gamma_{\beta_t,J} \circ \Gamma_{\alpha_t,J} = \partial \circ H + H \circ \partial. \tag{6.7}$$

Indeed, at $\tau = 0$, we obtain $\Gamma_{\alpha_t,\beta_t,\widetilde{J}}$. At $\tau \to \infty$, we obtain $\Psi_{J\to\phi_*(J)} \circ \Gamma_{\beta_t,J} \circ \Gamma_{\alpha_t,J}$. The only other way a curve may break is for a family to split into an index -1 curve, giving H, and an index 1 curve, giving ∂. Combining (6.6) and (6.7) with Lemma 6.11, the result follows. ■

6.8. *Proof of naturality*

We now prove Theorem 2.1, naturality of the perturbed invariants:

Proof of Theorem 2.1. Our proof follows the framework of [13]. Suppose that (M,γ) is a balanced sutured manifold with a closed 2-form ω. We define a directed graph $\mathcal{G}_{(M,\gamma)}$ as follows. The vertices of $\mathcal{G}_{(M,\gamma)}$ consist of *isotopy diagrams* of (M,γ), i.e., tuples (Σ, A, B) consisting of an embedded Heegaard surface Σ, and isotopy classes A and B of attaching curves. If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram, we write $[\mathcal{H}]$ for the induced isotopy diagram.

If H_1 and H_2 are two isotopy diagrams, we define the set of edges in $\mathcal{G}_{(M,\gamma)}$ connecting H_1 and H_2 to be the union

$$\mathcal{G}_{(M,\gamma)}(H_1,H_2) := \mathcal{G}_\alpha(H_1,H_2) \cup \mathcal{G}_\beta(H_1,H_2) \cup \mathcal{G}_{\text{stab}}(H_1,H_2) \cup \mathcal{G}^0_{\text{diff}}(H_1,H_2) \tag{6.8}$$

of sets defined as follows. The set $\mathcal{G}_\alpha(H_1,H_2)$ consists of a single arrow if H_1 and H_2 share the same Heegaard surface, have isotopic beta-curves, and have alpha-curves that are related by a sequence of handleslides and isotopies; and $\mathcal{G}_\alpha(H_1,H_2)$ is empty otherwise. The set $\mathcal{G}_\beta(H_1,H_2)$ is defined similarly. The set $\mathcal{G}_{\text{stab}}(H_1,H_2)$ has a single arrow if H_1 and H_2 are related by a stabilization or destabilization, and is empty otherwise. Finally, $\mathcal{G}^0_{\text{diff}}(H_1,H_2)$ is the set of all automorphisms of (M,γ) which move H_1 to H_2, and are isotopic to the identity of (M,γ). Write $\mathcal{G}_\alpha$ for the union over all pairs (H_1,H_2) of $\mathcal{G}_\alpha(H_1,H_2)$, and define $\mathcal{G}_\beta$, $\mathcal{G}_{\text{stab}}$, and $\mathcal{G}^0_{\text{diff}}$ similarly.

If H is an isotopy diagram, write $SFH(H;\Lambda_\omega)$ for the projective transitive system of Λ-modules, indexed by pairs $(\mathcal{H}, J)$, where $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible Heegaard diagram with $[\mathcal{H}] = H$, and J is a generic almost complex structure on $\Sigma \times I \times \mathbb{R}$. The transition maps may be constructed using the holomorphic triangle maps, as in Section 6.6, as well as change of almost complex structure maps from Section 6.2. Propositions 6.10 and 6.12 imply that this gives a projective transitive system of Λ-modules.

We consider the following cycles in $\mathcal{G}_{(M,\gamma)}$:

(L-1) A loop formed by a stabilization followed by a destabilization.

(L-2) A rectangular subgraph

$$\begin{array}{ccc} H_1 & \xrightarrow{e} & H_2 \\ f\downarrow & & \downarrow g \\ H_3 & \xrightarrow{h} & H_4 \end{array}$$

of $\mathcal{G}_{(M,\gamma)}$, where one of the following holds:

(R-1) Both $e, h \in \mathcal{G}_\alpha$ and $f, g \in \mathcal{G}_\beta$.

(R-2) Either $e, h \in \mathcal{G}_\alpha$, or $e, h \in \mathcal{G}_\beta$. Furthermore, $f, g \in \mathcal{G}_{\text{stab}}$.

(R-3) Either $e, h \in \mathcal{G}_\alpha$, or $e, h \in \mathcal{G}_\beta$. Furthermore, $f, g \in \mathcal{G}^0_{\text{diff}}$.

(R-4) The edges e, f, g, h are all in $\mathcal{G}_{\text{stab}}$. Furthermore, e and h correspond to stabilizing in a 3-ball B, while f and g correspond to stabilizing in a 3-ball B', and $B \cap B' = \emptyset$.

(R-5) Both $e, h \in \mathcal{G}_{\text{stab}}$, while $f, g \in \mathcal{G}^0_{\text{diff}}$. Furthermore, f and g may be induced by the same diffeomorphism ϕ of (M,γ), and the stabilization 3-ball for e is pushed forward to the stabilization 3-ball for h by ϕ.

(L-3) A loop formed by an edge in $\mathcal{G}^0_{\text{diff}}(H, H)$.

(L-4) A simple handleswap loop; see Figure 6.1, which is [13, Figure 4], for an illustration.

Commutativity of the transition maps along the loops (L-1)–(L-4) corresponds to the axioms for a *strong Heegaard invariant* [13, Definition 2.32]. According to [13, Theorem 2.38], it suffices to prove that the perturbed transition maps have no monodromy around loops (L-1)–(L-4).

As in Remark 2.7, to define a projectively transitive system indexed by all pairs $(\mathcal{H}, J)$, it is sufficient to define a morphism of transitive systems for each edge of $\mathcal{G}_{(M,\gamma)}$, and show that there is only projective monodromy around loops (L-1)–(L-4).

We define chain maps for edges in $\mathcal{G}_\alpha(H_1, H_2)$ and $\mathcal{G}_\beta(H_1, H_2)$ to be triangle maps, as described in Section 6.6. Chain maps for stabilizations are described in Section 6.3. Maps for edges in $\mathcal{G}^0_{\text{diff}}(H_1, H_2)$ are defined in Section 6.4. It is straightforward to see that these chain maps induce morphisms of transitive systems between the transitive systems associated to each isotopy diagram.

The main subtlety compared to the unperturbed setting is that the map associated to a diffeomorphism ϕ in $\mathcal{G}^0_{\text{diff}}(H_1, H_2)$ is defined with an auxiliary choice of an isotopy ϕ_t connecting ϕ to $\text{id}_{(M,\gamma)}$. The induced map ϕ is only well-defined as a projective map when restricted to each Spin^c structure by Lemma 6.5, or when $[\omega] = 0$. See Remark 6.8 for an example illustrating the subtlety.

We now verify that the monodromy around loops (L-1)–(L-4) is of projective type. The monodromy around loops of type (L-1) is clearly trivial. Similarly to the unperturbed setting, associativity of the holomorphic triangle maps, Proposition 6.10, implies

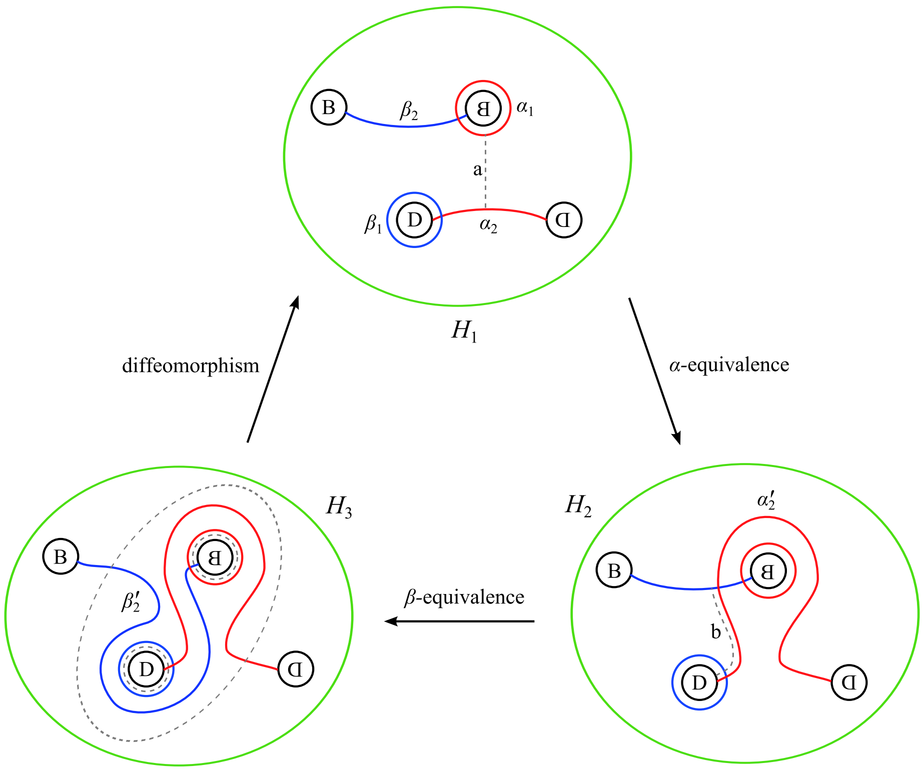
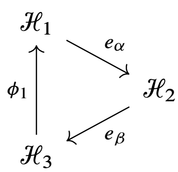


Fig. 6.1. A simple handleswap, which is a loop of diagrams consisting of an α-handleslide, a β-handleslide, and a diffeomorphism. The green curve is the boundary of the punctured genus 2 surface P that is obtained by identifying the circles marked with corresponding letters (namely, B and D). We draw the α-curves in red and the β-curves in blue.

that loops of type (L-2) induce projectively trivial monodromy. Loops of type (L-3) induce projectively trivial monodromy by Lemma 6.5 and Proposition 6.12, when restricted to individual Spin^c structures, or when $[\omega] = 0$. The main claim follows once we verify that there is only projective monodromy around simple handleswap loops (L-4), which is verified in Lemma 6.13 below. ∎

Lemma 6.13. *Suppose* (M, γ) *is a balanced sutured manifold, with a closed* 2*-form* ω*, and* $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(M, \gamma)$*. Suppose further that*



is a simple handleswap loop, where $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are admissible diagrams of (M,γ), and $e_\alpha \in \mathcal{G}_\alpha$, $e_\beta \in \mathcal{G}_\beta$, and $(\phi_t)_{t\in I}$ is an isotopy with $\phi_0 = \mathrm{id}_{(M,\gamma)}$. Then

$$(\phi_t)_* \circ \Psi_{e_\beta} \circ \Psi_{e_\alpha} \mathrel{\dot{\simeq}} \mathrm{id}_{CF(\mathcal{H}_1,\underline{\mathfrak{s}};\Lambda_\omega)}\,.$$

The same statement holds for the total complex $CF(\mathcal{H}_1;\Lambda_\omega)$ if $[\omega]=0$.

Proof. By definition, the diagrams $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are all 2-fold stabilizations of a fixed diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. If $i \in \{1,2,3\}$, write $\mathcal{H}_i' = (\Sigma_0, \boldsymbol{\alpha}_i', \boldsymbol{\beta}_i', p_0)$ for the genus 2 portion of $\mathcal{H}_i$ in the handleswap region. With this notation, we think of $\mathcal{H}_i$ as $\mathcal{H} \# \mathcal{H}_i'$, where the connected sum is taken at $p_0 \in \Sigma_0$ and a point $p \in \Sigma$. The diagrams $\mathcal{H}_i'$ are all genus 2 diagrams for S^3. Note that

$$\boldsymbol{\beta}_2' = \boldsymbol{\beta}_1' \quad \text{and} \quad \boldsymbol{\alpha}_3' = \boldsymbol{\alpha}_2'.$$

The map Ψ_{e_α} may be computed as the composition of a triangle map for an alpha-handleslide, followed by a continuation map to move the alpha-curves on $\mathcal{H}$ back to their original position. Similarly, the map Ψ_{e_β} may be computed as the composition of the triangle map for a beta-handleslide, followed by a continuation map to move the beta-curves on $\mathcal{H}$ back to their original position. The map $(\phi_t)_*$ is the isotopy map described in Section 6.4.

For a sufficiently stretched almost complex structure $J(T)$ along the connected sum tube of $\Sigma \# \Sigma_0$, the proof of stabilization invariance implies that the unperturbed complex for $\mathcal{H}_i$ decomposes as a tensor product:

$$CF_{J(T)}(\mathcal{H}_i) \cong CF_J(\mathcal{H}) \otimes_{\mathbb{F}_2} \langle \boldsymbol{c}_i \rangle, \tag{6.9}$$

where $\{\boldsymbol{c}_i\} = \mathbb{T}_{\alpha_i'} \cap \mathbb{T}_{\beta_i'}$, and $\langle \boldsymbol{c}_i \rangle$ denotes the 1-dimensional vector space over $\mathbb{F}_2$ generated by $\boldsymbol{c}_i$ for $i \in \{1,2,3\}$.

In the unperturbed setting, handleswap invariance [13, Theorem 9.30] is proven by showing

$$\Psi_{e_\alpha} = (\Gamma_{\alpha_t,J} \circ \Psi^{\beta}_{\alpha\to\alpha^H}) \otimes (\boldsymbol{c}_1 \mapsto \boldsymbol{c}_2) \tag{6.10}$$

with respect to the chain isomorphism of (6.9), where $\boldsymbol{\alpha}^H$ is a small Hamiltonian translate of $\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}_t$ is a Hamiltonian isotopy moving $\boldsymbol{\alpha}^H$ back to $\boldsymbol{\alpha}$. A similar tensor product description holds for the unperturbed version of Ψ_{e_β}.

For the perturbed versions, an extension of Lemma 6.3 to genus 2 stabilizations gives an analog of (6.9) for the perturbed setting, namely

$$CF_{J(T)}(\mathcal{H}_i,\underline{\mathfrak{s}};\Lambda_\omega) \cong CF_J(\mathcal{H},\underline{\mathfrak{s}};\Lambda_\omega) \otimes_{\mathbb{F}_2} \langle \boldsymbol{c}_i \rangle. \tag{6.11}$$

We now show that a similar tensor product decomposition as in (6.10) holds for the perturbed versions of Ψ_{e_α} and Ψ_{e_β}.

Firstly, if $\psi \# \psi_0$ is a class of triangles on $(\Sigma \# \Sigma_0, \boldsymbol{\alpha}^H \cup \boldsymbol{\alpha}_2', \boldsymbol{\alpha} \cup \boldsymbol{\alpha}_1', \boldsymbol{\beta} \cup \boldsymbol{\beta}_1')$, then

$$A_\omega(\psi \# \psi_0) = A_\omega(\psi) + A_\omega(\psi_0). \tag{6.12}$$

According to the proof of [13, Proposition 9.31], for a sufficiently stretched almost complex structure, all index 0 triangles $\psi \# \psi_0$ that are counted by $\Psi^{\beta\cup\beta_1'}_{\alpha\cup\alpha_1'\to\alpha^H\cup\alpha_2'}$ have $\mu(\psi)=0$. Furthermore, if $\mu(\psi)=0$, then

$$|\mathcal{M}(\psi)| \equiv \sum_{\substack{\psi_0\in\pi_2(\Theta_{\alpha_2',\alpha_1'},\boldsymbol{c}_1,\boldsymbol{c}_2)\\ n_{p_0}(\psi_0)=n_p(\psi)}} |\mathcal{M}(\psi \# \psi_0)| \bmod 2. \tag{6.13}$$

Next, we claim that $A_\omega(\psi_0)$ is independent of the triangle class ψ_0 in $\pi_2(\Theta_{\alpha_2',\alpha_1'}, \boldsymbol{c}_1, \boldsymbol{c}_2)$. This is established by observing that any two classes in $\pi_2(\Theta_{\alpha_2',\alpha_1'}, \boldsymbol{c}_1, \boldsymbol{c}_2)$ differ by a sum of doubly periodic domains. Doubly periodic domains on $\mathcal{H}_i'$ cone to closed 2-chains in $C_2(S^3)$, and hence do not affect the ω-area, so $A_\omega(\psi_0)$ is independent of the triangle class. A similar claim holds for triangles in $\pi_2(\boldsymbol{c}_2, \Theta_{\beta_1',\beta_3'}, \boldsymbol{c}_3)$.

Combining (6.12), (6.13), and the independence of $A_\omega(\psi_0)$ from ψ_0, we find that the perturbed transition maps satisfy

$$\begin{aligned}&(\phi_t)_* \circ \Psi_{e_\beta} \circ \Psi_{e_\alpha}\\ &\doteq (\phi_t)_* \circ \big((\Gamma_{\beta_t} \circ \Psi_\alpha^{\beta\to\beta^H}) \otimes (\boldsymbol{c}_2 \mapsto \boldsymbol{c}_3)\big) \circ \big((\Gamma_{\alpha_t} \circ \Psi^\beta_{\alpha\to\alpha^H}) \otimes (\boldsymbol{c}_1 \mapsto \boldsymbol{c}_2)\big),\end{aligned} \tag{6.14}$$

with respect to the tensor product decomposition from (6.11).

Since the isotopy ϕ_t is supported in the 3-ball of the handleswap, it follows that

$$(\phi_t)_* \doteq \mathrm{id}_{CF(\mathcal{H},\mathfrak{s};\Lambda_\omega)} \otimes (\boldsymbol{c}_3 \mapsto \boldsymbol{c}_1). \tag{6.15}$$

Furthermore, by Lemma 6.11,

$$\Gamma_{\beta_t} \circ \Psi_\alpha^{\beta\to\beta^H} \mathrel{\dot{\simeq}} \mathrm{id}_{CF(\mathcal{H},\mathfrak{s};\Lambda_\omega)} \quad \text{and} \quad \Gamma_{\alpha_t} \circ \Psi^\beta_{\alpha\to\alpha^H} \mathrel{\dot{\simeq}} \mathrm{id}_{CF(\mathcal{H},\mathfrak{s};\Lambda_\omega)}\,. \tag{6.16}$$

Combining (6.14)–(6.16) yields the main statement. ∎

7. Perturbed sutured cobordism maps

In this section, we define the perturbed sutured cobordism maps, and prove that they are well-defined in Proposition 2.9. Furthermore, we prove the composition law, Proposition 2.10, the effect of changing the 2-form on the cobordism, Lemma 2.12, and finally compare the perturbed and unperturbed maps when the 2-form vanishes on the boundary in Lemma 3.4.

7.1. The perturbed contact gluing map

We now describe a perturbed version of the Honda–Kazez–Matić contact gluing map [5]. Suppose (M,γ) is a sutured submanifold of (M',γ') (i.e., M is a submanifold with boundary of M' such that $M \subseteq \mathrm{int}(M')$), ω and ω' are closed 2-forms on M and M', respectively, such that $\omega=\omega'|_M$, and ξ is a co-oriented contact structure on $M'\setminus \mathrm{int}(M)$.

Let $\underline{\mathfrak{s}}$ be a Spin^c structure on M represented by a non-vanishing vector field v, and let $\underline{\mathfrak{s}}'$ be the Spin^c structure on M' obtained by gluing v to $\xi^\perp$. We will define a gluing map

$$\Phi_{\xi;\omega}\colon SFH(-M,-\gamma,\underline{\mathfrak{s}};\Lambda_\omega) \to SFH(-M',-\gamma',\underline{\mathfrak{s}}';\Lambda_{\omega'})$$

by adapting the construction from the unperturbed setting. Our description will use the reformulation of the gluing map given in [11] using contact handles. See [11, Definition 3.11] for background on contact handles in this setting.

Remark 7.1. We require that M' should have no closed components, though we allow $M' \setminus \operatorname{int}(M)$ to have what Honda, Kazez, and Matić refer to as *isolated components*, which are components of $M' \setminus \operatorname{int}(M)$ that are disjoint from $\partial M'$. These are permitted since the construction from [11] had a contact 3-handle map, which was not present in [5].

On Heegaard diagrams, adding a contact 0-handle has the effect of adding a disk D to the Heegaard surface, with no alpha- or beta-curves. The contact 0-handle map is the canonical chain isomorphism between $CF(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ and $CF(\Sigma \sqcup D,\boldsymbol{\alpha},\boldsymbol{\beta})$. This extends to the perturbed setting via the formula

$$\Phi_{\xi;\omega}(z^x \cdot \boldsymbol{x}) = z^x \cdot \boldsymbol{x}$$

for any closed 2-form on the 0-handle.

Adding a contact 1-handle has the effect of attaching a band to the boundary of the Heegaard surface. The contact 1-handle map is the canonical chain isomorphism between $CF(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ and $CF(\Sigma\cup B,\boldsymbol{\alpha},\boldsymbol{\beta})$, which extends to a map on the perturbed complexes with no complications.

The contact 2-handle map is slightly more involved. The effect on diagrams is to add a band and a pair of new curves, α and β, which have a single intersection point c in the band. See [11, Figure 3.11] for the precise configuration. The contact 2-handle map is defined via the formula

$$\Phi_{\xi;\omega}(z^x \cdot \boldsymbol{x}) = z^x \cdot \boldsymbol{x} \times c.$$

To see that this is a chain map on the perturbed complexes, note that all disks counted by $\partial(\boldsymbol{x}\times c)$ have homology class of the form $\phi \mathbin{\#} e_c$, where $\phi \in \pi_2(\boldsymbol{x},\boldsymbol{y})$ is a homology class, and e_c is the constant class at c. However,

$$A_{\omega'}(\phi \mathbin{\#} e_c) = A_\omega(\phi).$$

Hence, the contact 2-handle map is a chain map on the perturbed complexes.

Finally, a contact 3-handle is attached along a boundary component $S^2 \subseteq \partial M$ which is a 2-sphere with a single suture s. Then pick a diagram $(\Sigma,\boldsymbol{\alpha}\cup\{\alpha_0\},\boldsymbol{\beta}\cup\{\beta_0\})$, where α_0 and β_0 are parallel to the boundary component of Σ corresponding to s, and intersect each other in a pair of points. The contact 3-handle map is obtained by filling $s \subseteq \partial\Sigma$ with a disk D, and setting

$$\Phi_{\xi;\omega}(z^x\cdot\boldsymbol{x}\times\theta) = \begin{cases} z^x\cdot\boldsymbol{x} & \text{if } \theta=\theta^-,\\ 0 & \text{if } \theta=\theta^+,\end{cases}$$

where $\{\theta^+, \theta^-\} = \alpha_0 \cap \beta_0$, with relative grading $\mu(\theta^+, \theta^-) = 1$ induced by the Maslov index on $(-\Sigma, \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$. (The formula is the same as the 4-dimensional 3-handle map). Note that the contact 3-handle map is only defined if ∂M has at least one other boundary component. Furthermore, either we must choose $(\Sigma, \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$ so that α_0 and β_0 are adjacent to another component of $\partial\Sigma$, or we must stretch the almost complex structure along a circle bounding α_0 and β_0. We focus on the case when $(\Sigma, \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$ has been chosen so that $\partial\Sigma$ has an additional boundary component adjacent to α_0 and β_0. (The more general case requires using a holomorphic degeneration argument [26, Proposition 6.5], but follows similarly.) In this situation, an index 1 class on $(-\Sigma, \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$ with holomorphic representatives has one of the following forms:

- $\phi \# e_\theta$, where ϕ is an index 1 class on $(-\Sigma \cup D, \boldsymbol{\alpha}, \boldsymbol{\beta})$, with zero multiplicity on D, and e_θ is the constant class at $\theta \in \alpha_0 \cap \beta_0$,
- $e_{\boldsymbol{x}} \# \phi_0$, where ϕ_0 is one of the two bigons between α_0 and β_0.

To see that the contact 3-handle map is a chain map, it suffices to show that the two bigons have the same ω-area. The difference of the bigons is a periodic domain, which cones to a 2-sphere bounding the S^2 boundary component of ∂M which is filled in by the 3-handle. Since ω extends over the contact 3-handle, ω must integrate to zero on this 2-sphere, and hence have equal area on the cones of the two bigons.

As in the unperturbed case, the composition of the contact handle maps for a canceling pair of contact i and $i+1$ handles coincides with the transition map from naturality (up to an overall factor of z^x); see [11, Figures 3.13, 3.14]. By following our contact handle proof of invariance of the contact gluing map in the unperturbed case [11, Theorem 3.14], it follows that the perturbed contact gluing map is well-defined up to an overall factor of z^x, when restricted to each Spin^c structure on (M, γ). Furthermore, if $[\omega'] = 0$, then the gluing map is well-defined on all Spin^c structures, up to an overall factor of z^x.

7.2. *Perturbed maps for cylinders*

We now define the 4-dimensional cobordism maps for $W = I \times M$, equipped with a closed 2-form ω.

Recall that a sutured manifold cobordism is called *special* if it is a product along the boundary, with an I-invariant contact structure compatible with the dividing sets; see [8, Definition 5.1]. Suppose that $\mathcal{W} = (W, Z, [\xi])\colon (M_0, \gamma_0) \to (M_1, \gamma_1)$ is a special cobordism which is equipped with a Morse function f with no critical points, and let v be a gradient-like vector field for f.

To define the map for $\mathcal{W}$, we first pick an admissible diagram $\mathcal{H}_0 = (\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ for (M_0, γ_0). The flow of v induces a diffeomorphism between M_0 and M_1, and we write $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ for the pushforward of $\mathcal{H}_0$ under this diffeomorphism. If $\boldsymbol{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$, we write $v_*(\boldsymbol{x}) \in \mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}$ for the corresponding intersection point. Write $\Gamma_{\boldsymbol{x}}$ for the 2-chain traced out by the flow of v applied to $\gamma_{\boldsymbol{x}} \subseteq M_0$.

We define the perturbed cylinder map

$$F_{\mathcal{W};\omega,(f,v)}\colon CF(\mathcal{H}_0;\Lambda_{\omega|_{M_0}}) \to CF(\mathcal{H}_1;\Lambda_{\omega|_{M_1}})$$

via the formula

$$F_{\mathcal{W};\omega,(f,v)}(z^x \cdot \boldsymbol{x}) = z^{x+\int_{\Gamma_{\boldsymbol{x}}}\omega} \cdot v_*(\boldsymbol{x}). \tag{7.1}$$

As in Remark 2.6, for a choice of diagram $\mathcal{H}_0$ of (M_0,γ_0) and $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(M_0,\gamma_0)$, (7.1) gives a morphism of transitive systems from $CF(M_0,\gamma_0,\underline{\mathfrak{s}};\Lambda_{\omega|_{M_0}})$ to $CF(M_1,\gamma_1,v_*(\underline{\mathfrak{s}});\Lambda_{\omega|_{M_1}})$.

Lemma 7.2. *Suppose that $\mathcal{W} = (W,Z,[\xi])$ is a special cobordism with a Morse function f with no critical points and gradient-like vector field v.*

(1) *The map $F_{\mathcal{W},\omega;(f,v)}$ is a chain map.*

(2) *The induced morphism of transitive systems is independent of the choice of Heegaard diagram $\mathcal{H}_0$ for (M_0,γ_0).*

(3) *The induced morphism of transitive systems is independent of v.*

Proof. Claim (1), that $F_{\mathcal{W},\omega;(f,v)}$ is a chain map, follows from Stokes' theorem.

We now consider claim (2), that the morphism induced by $F_{\mathcal{W},\omega;(f,v)}$ is independent of $\mathcal{H}_0$. This amounts to showing that the maps $F_{\mathcal{W},\omega;(f,v)}$ commute with the transition maps for changing diagrams, up to an overall factor of z^x. We focus on the case when we have two diagrams for (M_0,γ_0) that are related by a single beta-handleslide or isotopy. We leave verification of claim (2) for other Heegaard moves to the reader.

Suppose that $(\Sigma_0,\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0,\boldsymbol{\beta}_0')$ is an admissible Heegaard triple for a beta-handleslide or isotopy in (M_0,γ_0). Set $\mathcal{H}_0 = (\Sigma_0,\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0)$ and $\mathcal{H}_0' = (\Sigma_0,\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0')$. Let $\mathcal{H}_1$ and $\mathcal{H}_1'$ denote their images in M_1 under the flow of v.

It is sufficient to consider the claim when the top-graded generator of $SFH(\Sigma_0,\boldsymbol{\beta}_0,\boldsymbol{\beta}_0')$ is represented by a single intersection point $\Theta_{\beta_0,\beta_0'} \in \mathbb{T}_{\beta_0} \cap \mathbb{T}_{\beta_0'}$, since a general beta-isotopy or handleslide may be decomposed into a sequence of beta-isotopies and handleslides which each satisfy this condition.

Let $\psi \in \pi_2(\boldsymbol{x},\Theta_{\beta_0,\beta_0'},\boldsymbol{z})$ be a homology class of triangles, where $\boldsymbol{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$ and $\boldsymbol{z} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0'}$. Let $\Phi\colon I \times M_0 \to W$ denote the flow of $v/v(f)$. Let $C_3 \subseteq W$ be the 3-chain $\Phi(I \times \widetilde{\mathcal{D}}(\psi))$, where $\widetilde{\mathcal{D}}(\psi) \subseteq M_0$ is the 2-chain constructed in Section 6.6. Since

$$\partial C_3 = \Phi(\{1\} \times \widetilde{\mathcal{D}}(\psi)) - \Phi(\{0\} \times \widetilde{\mathcal{D}}(\psi)) + \Gamma_{\boldsymbol{z}} - \Gamma_{\boldsymbol{x}} - \Gamma_{\Theta_{\beta_0,\beta_0'}}, \tag{7.2}$$

it follows that ω evaluates trivially on the sum of the 2-chains on the right-hand side of (7.2). The quantities $\int_{\Phi(\{0\}\times\widetilde{\mathcal{D}}(\psi))}\omega$ and $\int_{\Phi(\{1\}\times\widetilde{\mathcal{D}}(\psi))}\omega$ are the area contributions of $\Psi_{\mathcal{H}_0\to\mathcal{H}_0'}(\boldsymbol{x})$ and $\Psi_{\mathcal{H}_1\to\mathcal{H}_1'}(v_*(\boldsymbol{x}))$, respectively. The quantity $\int_{\Gamma_{\boldsymbol{z}}}\omega$ is the area contribution of $F_{\mathcal{W};\omega,(f,v)}(\boldsymbol{z})$, and $\int_{\Gamma_{\boldsymbol{x}}}\omega$ is the area contribution of $F_{\mathcal{W};\omega,(f,v)}(\boldsymbol{x})$. Hence

$$F_{\mathcal{W};\omega,(f,v)}(\Psi_{\mathcal{H}_0\to\mathcal{H}_0'}(\boldsymbol{x})) = z^{-\int_{\Gamma_{\Theta_{\beta_0,\beta_0'}}}\omega} \cdot \Psi_{\mathcal{H}_1\to\mathcal{H}_1'}(F_{\mathcal{W};\omega,(f,v)}(\boldsymbol{x})).$$

Since $\int_{\Gamma_{\Theta_{\beta_0,\beta_0'}}}\omega$ is independent of $\boldsymbol{x}$ and $\boldsymbol{z}$, the result follows.

We now consider claim (3), independence from the gradient-like vector field. Any two v may be connected by a 1-parameter family $(v_t)_{t\in I}$. As before, let $\mathcal{H}_0=(\Sigma_0,\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0)$ denote a diagram for (M_0,γ_0). For $t\in I$, let $\Phi_t\colon I\times M_0\to W$ denote the flow of $v_t/v_t(f)$.

Write $\phi_t\colon M_1\to M_1$ for the diffeomorphism $(\Phi_t\circ\Phi_0^{-1})|_{M_1}$. Claim (3) amounts to showing

$$F_{\mathcal{W};\omega,(f,v_1)}\doteq(\phi_t)_*\circ F_{\mathcal{W};\omega,(f,v_0)},\tag{7.3}$$

where $(\phi_t)_*$ denotes the isotopy map from Section 6.4.

Let $\Gamma_{\boldsymbol{x},t}$ denote the 2-chain $\Phi_t(I\times\gamma_{\boldsymbol{x}})\subseteq W$, and let $\Gamma'_{\boldsymbol{x}}\subseteq M_1$ denote the 2-chain swept out by $\Phi_t(\{1\}\times\gamma_{\boldsymbol{x}})$ as t ranges over I. Equation (7.3) amounts to showing that

$$\int_{\Gamma_{\boldsymbol{x},1}}\omega-\int_{\Gamma_{\boldsymbol{x},0}}\omega-\int_{\Gamma'_{\boldsymbol{x}}}\omega\tag{7.4}$$

is independent of $\boldsymbol{x}$.

Write $\widehat{\Phi}\colon I\times I\times M_0\to W$ for the map $\widehat{\Phi}(t,s,x)=\Phi_t(s,x)$. Let C_3 be the 3-chain defined by applying $\widehat{\Phi}$ to $I\times I\times\gamma_{\boldsymbol{x}}$. The expression (7.4) is equal to $\int_{\partial(I\times I)\times\gamma_{\boldsymbol{x}}}\widehat{\Phi}^*(\omega)$. Since $\int_{C_3}d\omega=0$, Stokes' theorem implies that (7.4) is equal to $\int_{I\times I\times\partial\gamma_{\boldsymbol{x}}}\widehat{\Phi}^*(\omega)$. Since $\partial\gamma_{\boldsymbol{x}}$ is independent of $\boldsymbol{x}$, it follows that the quantity (7.4) is also independent of $\boldsymbol{x}$, completing the proof. ■

We are now ready to prove Lemma 2.11.

Proof of Lemma 2.11. By construction, $F_{I\times Y(K);\omega_{\mathcal{S}_K}}$ sends $z^x\cdot\boldsymbol{x}$ to $z^{x+\int_{\Gamma_{\boldsymbol{x}}}\omega_{\mathcal{S}_K}}\cdot\boldsymbol{x}$, where $\Gamma_{\boldsymbol{x}}=I\times\gamma_{\boldsymbol{x}}$. It is sufficient to show that

$$\int_{\Gamma_{\boldsymbol{x}}-\Gamma_{\boldsymbol{y}}}\omega_{\mathcal{S}_K}=-A(\boldsymbol{x},\boldsymbol{y}),\tag{7.5}$$

where $A(\boldsymbol{x},\boldsymbol{y})$ is the relative Alexander grading.

Since $\omega_{\mathcal{S}_K}$ is the Poincaré–Lefschetz dual of $\{1/2\}\times\mathcal{S}_K$, we have

$$\int_{\Gamma_{\boldsymbol{x}}-\Gamma_{\boldsymbol{y}}}\omega_{\mathcal{S}_K}=\#((\gamma_{\boldsymbol{x}}-\gamma_{\boldsymbol{y}})\cap\mathcal{S}_K).$$

If $\phi\in\pi_2(\boldsymbol{x},\boldsymbol{y})$ is a class of disks, then, by definition,

$$A(\boldsymbol{x},\boldsymbol{y})=n_z(\phi)-n_w(\phi).$$

On the other hand,

$$\partial\widetilde{\mathcal{D}}(\phi)=\gamma_{\boldsymbol{y}}-\gamma_{\boldsymbol{x}}.$$

Using the Leibniz rule for intersections, we have

$$\#((\gamma_{\boldsymbol{x}}-\gamma_{\boldsymbol{y}})\cap\mathcal{S}_K)=-\#(\partial\widetilde{\mathcal{D}}(\phi)\cap\mathcal{S}_K)=-\#(\widetilde{\mathcal{D}}(\phi)\cap\partial\mathcal{S}_K).\tag{7.6}$$

Since $\partial\mathcal{S}_K=K$, (7.6) gives

$$\#((\gamma_{\boldsymbol{x}}-\gamma_{\boldsymbol{y}})\cap\mathcal{S}_K)=-\#(\widetilde{\mathcal{D}}(\phi)\cap K),$$

which is $-(n_z(\phi) - n_w(\phi)) = -A(\boldsymbol{x}, \boldsymbol{y})$, because, by convention, K intersects Σ positively at z and negatively at w. ∎

Remark 7.3. In Lemma 6.1, we described a transition map $\Psi_{\omega\to\omega';\eta}$ for changing between cohomologous closed 2-forms ω and ω' when $d\eta = \omega' - \omega$, though the map was only independent of η when restricted to a fixed Spin^c structure. Lemma 2.11 is a perfect example of why this is important. The 2-form ω_{S_K} is a boundary on $I \times Y(K)$. Write $\omega_{S_K} = d\eta$, and write $\eta_i := \eta|_{\{i\}\times Y(K)}$. Note that ω_{S_K} restricts trivially to $\{i\} \times Y(K)$ for $i \in \{0, 1\}$. An easy Stokes' theorem argument shows that the diagram

$$\begin{array}{ccc} SFH(Y(K);\Lambda_0) & \xrightarrow{\Psi_{0\to d\eta_0;\eta_0}} & SFH(Y(K);\Lambda_{d\eta_0}) \\ {\scriptstyle F_{I\times Y(K);\omega_{S_K}}}\big\downarrow & & \big\downarrow{\scriptstyle F_{I\times Y(K);0}} \\ SFH(Y(K);\Lambda_0) & \xleftarrow{\Psi_{d\eta_1\to 0;-\eta_1}} & SFH(Y(K);\Lambda_{d\eta_1}) \end{array} \tag{7.7}$$

commutes up to an overall factor of z^x. Hence $F_{I\times Y(K);\omega_{S_K}} \doteq \Psi_{0\to 0;\eta_0-\eta_1}$, but this does not imply that $F_{I\times Y(K);\omega_{S_K}} \doteq \mathrm{id}$, since Lemma 6.1 only applies if we restrict to a single Spin^c structure.

7.3. *Perturbed* 1*-handle and* 3*-handle maps*

We now describe the cobordism maps for 1-handles and 3-handles. We focus on 1-handles, since the 3-handle maps are algebraically dual.

Suppose that

$$\mathcal{W}_1 = (W_1, Z_1, [\xi_1]) \colon (M_0, \gamma_0) \to (M_1, \gamma_1)$$

is a special cobordism with a Morse function f that has a single index 1 critical point p_0. Let v be a gradient-like vector field for f. We use f and v as auxiliary data to construct the cobordism map for $\mathcal{W}_1$.

The stable manifold of v at p_0 intersects M_0 in two points, p_1 and p_2. Let $\mathcal{H}_0 = (\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ be an admissible diagram for (M_0, γ_0) such that $p_1, p_2 \in \Sigma_0 \setminus (\boldsymbol{\alpha}_0 \cup \boldsymbol{\beta}_0)$. Let D_1 and D_2 be two small disks in Σ_0, centered at p_1 and p_2. The flow of v induces an embedding of $\Sigma_0 \setminus (D_1 \cup D_2)$ into M_1.

A Heegaard diagram $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ for (M_1, γ_1) is constructed as follows. The surface Σ_1 is obtained by connecting the boundary components of the image of $\Sigma_0 \setminus (D_1 \cup D_2)$ under the flow of v with an annulus in the 1-handle region. The attaching curves $\boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1$ are given by $\boldsymbol{\alpha}_1 \cup \{\alpha\}$ and $\boldsymbol{\beta}_1 \cup \{\beta\}$, where α and β are contained in the 1-handle annulus, intersect transversely, are homologically essential therein, and satisfy $|\alpha \cap \beta| = 2$. Write $\alpha \cap \beta = \{\theta^+, \theta^-\}$, where θ^+ has the larger relative Maslov grading.

If $\boldsymbol{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$, write $v_*(\boldsymbol{x})$ for the corresponding tuple of points on Σ_1. A set of compressing disks in M_0 may be pushed forward under the flow of v. By adding two disks in the 1-handle region, we naturally obtain a set of compressing disks in (M_1, γ_1). If $\boldsymbol{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$, write $\Gamma_{\boldsymbol{x}} \subseteq W_1$ for the 2-chain traced out by applying the flow of v to

$\gamma_{\boldsymbol{x}} \subseteq M_0$. We define the perturbed 1-handle map $F_{\mathcal{W}_1;\omega,(f,v)}$ as

$$F_{\mathcal{W}_1;\omega,(f,v)}(z^x \cdot \boldsymbol{x}) := z^{x+\int_{\Gamma_{\boldsymbol{x}}} \omega} \cdot v_*(\boldsymbol{x}) \times \theta^+.$$

Lemma 7.4. *Suppose that $\mathcal{W}_1 = (W_1, Z_1, [\xi_1])\colon (M_0, \gamma_0) \to (M_1, \gamma_1)$ is a special cobordism and (f, v) is a Morse function and gradient-like vector field on W_1 with a single index 1 critical point.*

(1) *For an almost complex structure sufficiently stretched on the two boundary components of the 1-handle annulus, the map $F_{\mathcal{W}_1;\omega,(f,v)}$ is a chain map.*

(2) *The morphism of transitive systems induced by $F_{\mathcal{W}_1;\omega,(f,v)}$ is independent of the Heegaard diagram for (M_0, γ_0).*

(3) *The morphism of transitive systems induced by $F_{\mathcal{W}_1;\omega,(f,v)}$ is independent of v.*

Proof. The proof of claim (1), that $F_{\mathcal{W}_1;\omega,(f,v)}$ is a chain map, relies on the same holomorphic degeneration argument used in the unperturbed setting. See [25, Section 4.3] for the original proof, as well as [8, Section 7], or [30, Section 8] for versions of the proof in several related contexts. In the perturbed setting, one must also check that the cones of the two bigons in the 1-handle region are assigned the same ω-area. Note that the difference between these two bigon classes is a periodic domain, which cones off to a 2-sphere S that is homotopic to the belt sphere of the 4-dimensional 1-handle. Since ω is defined on all of W (in particular, on the co-core of the 1-handle), we must have $\int_S \omega = 0$.

To prove claim (2), that the morphism of transitive systems induced by $F_{\mathcal{W}_1,\omega;(f,v)}$ is independent of the Heegaard diagram $\mathcal{H}_0$, one repeats the standard proof of the well-definedness of the 1-handle maps [25, Theorem 4.10], while keeping track of areas as in the proof of Lemma 7.2.

Claim (3), independence from v, is proven as follows. Suppose that $(v_t)_{t\in I}$ is a path of gradient-like vector fields. We can pick an isotopy ϕ_t of M_0, and an admissible diagram $(\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ for (M_0, γ_0) such that the stable manifold of the critical point of f is contained in $\phi_t(\Sigma_0 \setminus (\boldsymbol{\alpha}_0 \cup \boldsymbol{\beta}_0))$ for all t. We can choose an isotopy ψ_t of (M_1, γ_1) such that the image of $\phi_t(\Sigma_0)$ under the flow of v_t coincides with $\psi_t(\Sigma_1)$ outside the 1-handle region. Write $(\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')$ for the image of $(\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ under ϕ_1, and write $(\Sigma_1', \boldsymbol{\alpha}_1', \boldsymbol{\beta}_1')$ for the image of $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ under ψ_t.

It suffices to show that the following diagram commutes, up to overall multiplication by z^x:

$$\begin{array}{ccc} CF(\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0; \Lambda_{\omega|_{M_0}}) & \xrightarrow{(\phi_t)_*} & CF(\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0'; \Lambda_{\omega|_{M_0}}) \\ {\scriptstyle F_{\mathcal{W}_1;\omega,(f,v_0)}}\downarrow & & \downarrow{\scriptstyle F_{\mathcal{W}_1;\omega,(f,v_1)}} \\ CF(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1; \Lambda_{\omega|_{M_1}}) & \xrightarrow{(\psi_t)_*} & CF(\Sigma_1', \boldsymbol{\alpha}_1', \boldsymbol{\beta}_1'; \Lambda_{\omega|_{M_1}}) \end{array} \tag{7.8}$$

We define

$$\widehat{\Phi}\colon I \times I \times \gamma_{\boldsymbol{x}} \to W_1,$$

where $\widehat{\Phi}(t,s,x)$ is the time s flow of $\phi_t(x)$ under $v_t/v_t(f)$. Consider the 3-chain $C_3 = \Phi(I \times I \times \gamma_{\boldsymbol{x}})$ in W_1. Then we have

$$\partial C_3 = \widehat{\Phi}(\partial(I \times I) \times \gamma_{\boldsymbol{x}}) + \widehat{\Phi}(I \times I \times \partial\gamma_{\boldsymbol{x}}). \tag{7.9}$$

Write $\gamma_{\theta^+} \subseteq M_1$ for the 1-chain obtained by coning θ^+ into the two handlebodies, and let $\Gamma_{\theta^+,\psi_t} \subseteq M_1$ denote the 2-chain swept out by the family $(\psi_t(\gamma_{\theta_+}))_{t\in I}$. By definition, the difference in area contributions from the two length 2 paths in (7.8) is

$$\int_{\partial(I\times I)\times\gamma_{\boldsymbol{x}}} \widehat{\Phi}^*(\omega) + \int_{\Gamma_{\theta^+,\psi_t}} \omega. \tag{7.10}$$

Applying Stokes' theorem to (7.9), we see that the sum (7.10) is equal to

$$-\int_{I\times I\times\partial\gamma_{\boldsymbol{x}}} \widehat{\Phi}^*(\omega) + \int_{\Gamma_{\theta^+,\psi_t}} \omega,$$

which is independent of $\boldsymbol{x}$. It follows that (7.8) commutes up to an overall factor of z^x, completing the proof. ∎

The perturbed 3-handle maps are dual to the 1-handle maps. We leave the details of the definition to the reader.

7.4. *Perturbed 2-handle maps*

Suppose that

$$\mathcal{W}_2 = (W_2, Z_2, [\xi_2])\colon (M_0,\gamma_0) \to (M_1,\gamma_1)$$

is a special cobordism equipped with a Morse function f and gradient-like vector field v such that f has only index 2 critical points, and the stable and unstable manifolds of v are transverse.

Let $\mathbb{S}_1 \subseteq M_0$ denote the intersection of the stable manifolds of (f,v) and M_0. Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}')$ be a Heegaard triple subordinate to a bouquet for $\mathbb{S}_1$; see [8, Definition 6.3]. Let

$$\mathcal{W}_{\alpha,\beta,\beta'} = (W_{\alpha,\beta,\beta'}, Z_{\alpha,\beta,\beta'}, [\xi_{\alpha,\beta,\beta'}])$$

be the associated sutured manifold cobordism, as described in [11, Section 7]. The 4-manifold $W_{\alpha,\beta,\beta'}$ is defined as follows. If Δ denotes a triangle with edges e_α, e_β, and $e_{\beta'}$, then

$$W_{\alpha,\beta,\beta'} := (\Sigma \times \Delta) \cup (U_\alpha \times e_\alpha \cup U_\beta \times e_\beta \cup U_{\beta'} \times e_{\beta'}),$$

where U_α, U_β, and $U_{\beta'}$ are the sutured compression bodies corresponding to $(\Sigma,\boldsymbol{\alpha})$, $(\Sigma,\boldsymbol{\beta})$, and $(\Sigma,\boldsymbol{\beta}')$, respectively. We view the 4-manifold $W_{\alpha,\beta,\beta'}$ as having three sutured manifold boundary components, M_0, $M_{\beta,\beta'}$, and M_1.

From our choice of (f,v), we obtain an embedding

$$\Phi_{(f,v)}\colon W_{\alpha,\beta,\beta'} \to W_2,$$

which is well-defined up to isotopy, as follows. Let $\{b_1, \dots, b_k\} \subseteq (0,1)$ be the critical values of f, and let $\epsilon > 0$ be chosen such that $\epsilon < b_i < 1-\epsilon$ for $i \in \{1, \dots, k\}$. Let $N(\Sigma)$ be a product neighborhood of Σ in M_0. We can view M_0 as $U_\alpha \cup N(\Sigma) \cup -U_\beta$. We can correspondingly view $W_{\alpha,\beta,\beta'}$ as

$$(N(\Sigma) \times I) \cup (U_\alpha \times I) \cup (-U_\beta \times [0,\epsilon]) \cup (-U_{\beta'} \times [1-\epsilon, 1]).$$

The embedding $\Phi_{(f,v)}$ sends a point $(x,t) \in U_\beta \times [0,\epsilon]$ to the point $z \in W_2$ which is in the flow line of v over $x \in U_\beta \subseteq M$ and has $f(z) = t$. The embeddings on the other portions of $W_{\alpha,\beta,\beta'}$ are defined similarly. See Figure 7.1 for a schematic. We note also that the boundary component $M_{\beta,\beta'} \subseteq \partial \mathcal{W}_{\alpha,\beta,\beta'}$ may be naturally filled in with a sutured manifold cobordism $\mathcal{W}_{\beta,\beta'}$ to obtain the sutured 2-handle cobordism $\mathcal{W}_2$. A description of $\mathcal{W}_{\beta,\beta'}$ may be found in [8, Proposition 6.6] (see also [11, Section 8]).

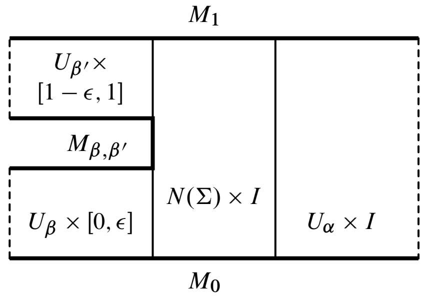


Fig. 7.1. The triple cobordism $W_{\alpha,\beta,\beta'}$.

A homology class $\psi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ induces a coned-off singular 2-chain $\widetilde{\mathcal{D}}(\psi)$ in $W_{\alpha,\beta,\beta'}$ as follows. Firstly, the class ψ induces a singular 2-chain $\mathcal{D}_0(\psi)$ in $\Sigma \times \Delta$, which has boundary on $(\boldsymbol{\alpha} \times e_\alpha) \cup (\boldsymbol{\beta} \times e_\beta) \cup (\boldsymbol{\beta}' \times e_{\beta'})$, where $\partial \Delta = e_\alpha \cup e_\beta \cup e_{\beta'}$. The 2-chain $\mathcal{D}_0(\psi)$ is determined, up to addition of a boundary, by the property that its projection to Σ is the domain of ψ, and that the projection onto Δ is degree d, where $d = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = |\boldsymbol{\beta}'|$. We pick compressing disks D_α, D_β, and $D_{\beta'}$, and we let c_α, c_β, and $c_{\beta'}$ denote the sets of center points of these compressing disks, respectively. We cone $\mathcal{D}_0(\psi)$ into $U_\alpha \times e_\alpha$, $U_\beta \times e_\beta$, and $U_{\beta'} \times e_{\beta'}$ to obtain a 2-chain $\widetilde{\mathcal{D}}(\psi)$ in $W_{\alpha,\beta,\beta'}$ that has boundary

$$-\gamma_{\boldsymbol{x}} - \gamma_{\boldsymbol{y}} + \gamma_{\boldsymbol{z}} + c_\alpha \times e_\alpha + c_\beta \times e_\beta + c_{\beta'} \times e_{\beta'}.$$

We define $A_\omega(\psi)$ to be the integral of $\Phi^*_{(f,v)}(\omega)$ over $\widetilde{\mathcal{D}}(\psi)$. We write $(M_{\beta,\beta'}, \gamma_{\beta,\beta'})$ for the sutured manifold defined by the diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}')$, and $\omega_{\beta,\beta'} = \omega|_{M_{\beta,\beta'}}$.

By counting index 0 holomorphic triangles weighted with $z^{A_\omega(\psi)}$, we obtain a perturbed triangle map

$$F_{\alpha,\beta,\beta';\omega} \colon CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}; \Lambda_{\omega|_{M_0}}) \otimes CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}'; \Lambda_{\omega_{\beta,\beta'}}) \to CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}'; \Lambda_{\omega|_{M_1}}). \quad (7.11)$$

Finally, the perturbed 2-handle map is given by the formula

$$F_{\mathcal{W}_2;\omega,(f,v)}(z^x \cdot \boldsymbol{x}) = z^x \cdot F_{\alpha,\beta,\beta';\omega}(\boldsymbol{x} \otimes \Theta^{\omega_{\beta,\beta'}}_{\beta,\beta'}), \tag{7.12}$$

where $\Theta^{\omega_{\beta,\beta'}}_{\beta,\beta'} \in CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}'; \Lambda_{\omega_{\beta,\beta'}})$ is defined analogously to (6.4).

The domain and codomain of $F_{\mathcal{W}_2;\omega,(f,v)}$ do not form projective transitive systems unless either we restrict to a single Spin^c structure on (M_0,γ_0) and (M_1,γ_1), or $[\omega]|_{M_i}=0$ for $i \in \{0,1\}$. However, if we fix $\underline{\mathfrak{s}}_0 \in \mathrm{Spin}^c(M_0,\gamma_0)$ and $\underline{\mathfrak{s}}_1 \in \mathrm{Spin}^c(M_1,\gamma_1)$, we obtain a morphism of projective transitive systems

$$\pi_{\underline{\mathfrak{s}}_1} \circ F_{\mathcal{W}_2;\omega,(f,v)} \circ i_{\underline{\mathfrak{s}}_0} \colon CF(M_0,\gamma_0,\underline{\mathfrak{s}}_0;\Lambda_{\omega|_{M_0}}) \to CF(M_1,\gamma_1;\underline{\mathfrak{s}}_1,\Lambda_{\omega|_{M_1}}).$$

Lemma 7.5. *Suppose that $\mathcal{W}_2\colon (M_0,\gamma_0) \to (M_1,\gamma_1)$ is a special cobordism with a Morse function f and gradient-like vector field v with only index 2 critical points, which is Morse–Smale. Let $\mathbb{S}_1$ denote the corresponding framed link in M_0.*

(1) *The morphism of transitive systems induced by $F_{\mathcal{W}_2;\omega,(f,v)}$ is independent of the choice of bouquet for $\mathbb{S}_1$, or the Heegaard triple subordinate to it.*

(2) *The morphism of transitive systems $F_{\mathcal{W}_2;\omega,(f,v)}$ is independent of v.*

Proof. The proof of claim (1) is similar to the original proof given by Ozsváth and Szabó [25, Proposition 4.6, Lemma 4.8], and follows from associativity of the perturbed holomorphic triangle maps. See also [8, Theorem 6.9] for a more detailed explanation of the argument in the sutured setting.

Independence from v, claim (2), is proven as follows. The space of gradient-like vector fields of f is connected. Suppose $(v_t)_{t\in I}$ is a path of gradient-like vector fields. Let $\mathbb{S}_1^t$ denote the intersection of the stable manifolds of v_t with M_0. Generically, v_t is Morse–Smale at all but finitely many t, at which time a handleslide amongst two of the components of $\mathbb{S}_1^t$ occurs.

We break I into two types of subintervals: $[a,b]$, where (f,v_t) is Morse–Smale for all $t \in [a,b]$; and $[t_0-\epsilon, t_0+\epsilon]$, where $\epsilon > 0$ is small, and a handleslide occurs at t_0.

For the first type of subinverval $[a,b]$, let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}')$ be subordinate to a bouquet for $\mathbb{S}_1^a$. Let $(\phi_t)_{t\in[a,b]}$ be an isotopy of M_0 such that $\phi_a = \mathrm{id}_{M_0}$ and the diagram

$$(\Sigma_t,\boldsymbol{\alpha}_t,\boldsymbol{\beta}_t,\boldsymbol{\beta}'_t) := \phi_t(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}') \subseteq M_0$$

is subordinate to $\mathbb{S}_1^t$.

Using the abbreviation Φ_t for $\Phi_{(f,v_t)}$, we obtain a family $(\Phi_t)_{t\in[a,b]}$ of embeddings of $W_{\alpha,\beta,\beta'}$ into W_2. Let $\psi_t\colon M_1 \to M_1$ denote the map $(\Phi_t \circ \Phi_a^{-1})|_{M_1}$. We claim that the following diagram commutes up to an overall factor of z^x:

$$\begin{array}{ccc} CF(\Sigma_a,\boldsymbol{\alpha}_a,\boldsymbol{\beta}_a;\Lambda_{\omega|_{M_0}}) & \xrightarrow{(\phi_t)_*} & CF(\Sigma_b,\boldsymbol{\alpha}_b,\boldsymbol{\beta}_b;\Lambda_{\omega|_{M_0}}) \\ {\scriptstyle F_{\mathcal{W}_2;\omega,(f,v_a)}}\downarrow & & \downarrow{\scriptstyle F_{\mathcal{W}_2;\omega,(f,v_b)}} \\ CF(\Sigma_a,\boldsymbol{\alpha}_a,\boldsymbol{\beta}'_b;\Lambda_{\omega|_{M_1}}) & \xrightarrow{(\psi_t)_*} & CF(\Sigma_b,\boldsymbol{\alpha}_b,\boldsymbol{\beta}'_b;\Lambda_{\omega|_{M_1}}) \end{array} \tag{7.13}$$

Suppose $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\beta,\beta'}, \boldsymbol{z})$ is a homology class of triangles on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$, where $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\boldsymbol{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$. Write $\Gamma_{\boldsymbol{x},\phi_t} \subseteq M_0$ and $\Gamma_{\boldsymbol{z},\psi_t} \subseteq M_1$ for the 2-chains swept out by $\gamma_{\boldsymbol{x}}$ and $\gamma_{\boldsymbol{z}}$ by ϕ_t and ψ_t for $t \in [a,b]$, respectively. Commutativity of (7.13) up to an overall factor of z^x amounts to showing that the integral of ω over

$$\Phi_a(\widetilde{\mathcal{D}}(\psi)) - \Phi_b(\widetilde{\mathcal{D}}(\psi)) + \Gamma_{\boldsymbol{z},\psi_t} - \Gamma_{\boldsymbol{x},\phi_t} \tag{7.14}$$

is independent of ψ, $\boldsymbol{x}$, and $\boldsymbol{z}$.

The family Φ_t induces a map $\widehat{\Phi}\colon [a,b] \times W_{\alpha,\beta,\beta'} \to W_2$, and we let $C_3 \subseteq W_2$ be the 3-chain $\widehat{\Phi}([a,b] \times \widetilde{\mathcal{D}}(\psi))$. Stokes' theorem applied to ∂C_3 implies that the integral of ω over the 2-chain in (7.14) is equal to the integral of ω over

$$\Gamma_{\Theta_{\beta,\beta'},\Phi_t} + C_{\alpha,\beta,\beta'}, \tag{7.15}$$

where $\Gamma_{\Theta_{\beta,\beta'},\Phi_t}$ is the 2-chain $\widehat{\Phi}([a,b] \times \gamma_{\Theta_{\beta,\beta'}})$, and $C_{\alpha,\beta,\beta'}$ is defined as follows. Let $c_\alpha \subseteq U_\alpha$ be the union of the centers of the alpha compressing disks, and let e_α denote the alpha side of the triangle Δ used to build $W_{\alpha,\beta,\beta'}$. Let c_β, $c_{\beta'}$, e_β, and $e_{\beta'}$ be defined similarly. Then $C_{\alpha,\beta,\beta'}$ is the image under $\widehat{\Phi}$ of $[a,b] \times (c_\alpha \times e_\alpha \cup c_\beta \times e_\beta \cup c_{\beta'} \times e_{\beta'})$. Since the sum (7.15) is independent of $\boldsymbol{x}$, $\boldsymbol{z}$, and ψ, it follows that the diagram (7.13) commutes up to an overall factor of z^x.

Next, we consider the case when the subinterval of I is of the form $[t_0 - \epsilon, t_0 + \epsilon]$, where a handleslide amongst the components of $\mathbb{S}_1^t$ occurs at $t = t_0$. Adapting the proof of Ozsváth and Szabó [25, Lemma 4.14], we may pick a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ subordinate to a bouquet for $\mathbb{S}_1^{t_0-\epsilon}$ such that there are attaching curves $\bar{\boldsymbol{\beta}}$ and $\bar{\boldsymbol{\beta}}'$ on Σ, where $\bar{\boldsymbol{\beta}}$ is obtained from $\boldsymbol{\beta}$ and $\bar{\boldsymbol{\beta}}'$ is obtained from $\boldsymbol{\beta}'$ via a sequence of handleslides and isotopies, and $(\Sigma, \boldsymbol{\alpha}, \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}}')$ is subordinate to a bouquet for $\mathbb{S}_1^{t_0+\epsilon}$. The 4-manifold $W_{\alpha,\beta,\beta'}$ is unchanged by isotopies and handleslides of the attaching curves. A straightforward associativity argument shows that the two morphisms constructed with the embedding $\Phi_{t_0-\epsilon}$ and either of the triples $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ or $(\Sigma, \boldsymbol{\alpha}, \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}}')$ coincide. Similarly, the previous argument shows that the two morphisms computed using the triple $(\Sigma, \boldsymbol{\alpha}, \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}}')$ and either of the embeddings $\Phi_{t_0-\epsilon}$ or $\Phi_{t_0+\epsilon}$ coincide, completing the proof. ∎

7.5. *Defining the* Spin^c *restricted cobordism maps*

In this section, we define the Spin^c restricted versions of the perturbed sutured cobordism maps. Suppose that

$$\mathcal{W} = (W, Z, [\xi])\colon (M_0, \gamma_0) \to (M_1, \gamma_1)$$

is a cobordism of sutured manifolds equipped with a closed 2-form ω on W. We remove a collection of tight 3-balls from Z, adding them to M_0 or M_1, so that $M_0 \cup Z$ has no closed components, and so that each component of W intersects M_0 and M_1 non-trivially.

We can decompose $\mathcal{W}$ as $\mathcal{W}^s \circ \mathcal{W}^\partial$, where $\mathcal{W}^\partial$ consists of $I \times (M_0 \cup Z)$, viewed as a cobordism from M_0 to $M_0 \cup Z$, and $\mathcal{W}^s$ consists of W, viewed as a special cobordism from $M_0 \cup Z$ to M_1.

We choose a self-indexing Morse function f on $\mathcal{W}^s$, with no index 0 or 4 critical points, and a gradient-like vector field v for f. The pair (f, v) induces a decomposition

$$\mathcal{W}^s = \mathcal{W}_3 \circ \mathcal{W}_2 \circ \mathcal{W}_1,$$

where $\mathcal{W}_i = (W_i, Z_i, [\xi_i])$ is a special cobordism that contains the index i critical points of f.

Suppose $\underline{\mathfrak{s}}_0 \in \mathrm{Spin}^c(M_0, \gamma_0)$ and $\underline{\mathfrak{s}}_1 \in \mathrm{Spin}^c(M_1, \gamma_1)$. The Spin^c structure $\underline{\mathfrak{s}}_1$ extends uniquely over $\mathcal{W}_3$. Write $\underline{\mathfrak{u}}$ for its restriction to the incoming boundary of $\mathcal{W}_3$. We define

$$\begin{aligned}\pi_{\underline{\mathfrak{s}}_1} \circ F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}_0} \\ := F_{\mathcal{W}_3;\omega|_{W_3}} \circ \pi_{\underline{\mathfrak{u}}} \circ F_{\mathcal{W}_2;\omega|_{W_2}} \circ F_{\mathcal{W}_1;\omega|_{W_1}} \circ \Phi_{\xi;\omega|_{M_0 \cup Z}} \circ i_{\underline{\mathfrak{s}}_0} \qquad (7.16)\end{aligned}$$

where we have suppressed the dependence of the map $F_{\mathcal{W}_i;\omega|_{W_i}}$ on the Morse function $f|_{W_i}$. There is no dependence on the gradient-like vector field $v|_{W_i}$ according to Lemmas 7.2, 7.4, and 7.5.

We now prove that the Spin^c restricted perturbed cobordism maps are well-defined:

Proof of Proposition 2.9 (1). The proof is similar to the proof of the corresponding claim in the unperturbed setting; see [25, Section 4.4] and [8, Theorem 8.2]. Given two Morse functions f_0 and f_1 on W, viewed as a special cobordism from $M_0 \cup Z$ to M_1, one may pick a generic path $(f_t)_{t\in I}$ of smooth functions that are Morse at all but finitely many t and connect f_0 to f_1. Furthermore, using Cerf theory, one may assume that there are no index 0 or 4 critical points, and that critical points of index i for $i \in \{2, 3\}$ have values greater than the values of critical points of index less than i. Furthermore, at the finitely many t where f_t fails to be Morse, an index 1/2 or 2/3 birth-death singularity occurs.

If f_t is Morse for every $t \in [a, b] \subseteq [0, 1]$, the decompositions of $\mathcal{W}^s$ as $\mathcal{W}_1 \circ \mathcal{W}_2 \circ \mathcal{W}_3$ corresponding to f_a and f_b are isotopic, so adaptations of Lemmas 7.4 and 7.5 show that the composition is unchanged, up to an overall factor of z^x.

Invariance under index 1/2 birth-death follows from Ozsváth and Szabó's holomorphic triangle computation [25, Lemma 4.16], with extra attention paid to areas. Invariance under index 2/3 birth-deaths follows by the same argument. ■

7.6. *Defining the total cobordism map*

In this section, we define the total perturbed cobordism map $F_{\mathcal{W};\omega}$, when $[\omega]$ restricts trivially to M_0 and M_1. This addresses part (2) of Proposition 2.9.

As a first step, if $[\omega]$ restricts trivially to M_1, and $\underline{\mathfrak{s}}_0 \in \mathrm{Spin}^c(M_0, \gamma_0)$, we may define the partially Spin^c restricted map $F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}_0}$ by omitting $\pi_{\underline{\mathfrak{u}}}$ from (7.16).

This strategy does not extend to the case when $[\omega]|_{M_0} = 0$, since we also need $[\omega]|_{M_0 \cup Z} = 0$ for the gluing map to be well-defined. Instead, when $[\omega]$ restricts trivially to M_0 and M_1, we make an alternative construction. Pick an open collar neighborhood $N \subseteq W$ of M_0. Set

$$\mathcal{N} = (\overline{N}, Z \cap \overline{N}, [\xi|_{\overline{N}}]),$$

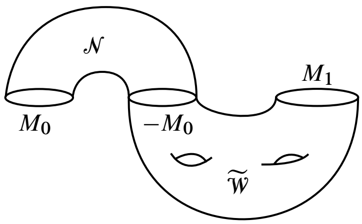


Fig. 7.2. Decomposing $\mathcal{W}$ into $\mathcal{N}$ and $\widetilde{\mathcal{W}}$.

which we view as a sutured manifold cobordism from $(M_0, \gamma_0) \cup (-M_0, \gamma_0)$ to the empty set. Let us write

$$\widetilde{\mathcal{W}} = (W \setminus N, Z \setminus N, [\xi|_{Z \setminus N}]).$$

We view $\widetilde{\mathcal{W}}$ as a cobordism from the empty set to $(-M_0, \gamma_0) \cup (M_1, \gamma_1)$. See Figure 7.2.

The previous case gives a map

$$F_{\widetilde{\mathcal{W}};\omega|_{W\setminus N}} \colon \Lambda \to SFH(-M_0, \gamma_0; \Lambda_{\omega|_{M_0}}) \otimes SFH(M_1, \gamma_1; \Lambda_{\omega|_{M_1}}). \tag{7.17}$$

Implicitly, we are precomposing with the map $i_{\mathfrak{s}_0}$, where $\mathfrak{s}_0$ is the unique Spin^c structure on the empty set. We define the total cobordism map $F_{\mathcal{W};\omega}$ via the formula

$$F_{\mathcal{W};\omega} := (F_{\mathcal{N},\omega|_{\overline{N}}} \otimes \mathrm{id}_{SFH(M_1)}) \circ (\mathrm{id}_{SFH(M_0)} \otimes F_{\widetilde{\mathcal{W}};\omega|_{W\setminus N}}). \tag{7.18}$$

If $[\omega]$ restricts trivially to $M_0 \cup Z$ and M_1, then we may also define the total perturbed cobordism map by removing the projections and inclusions of Spin^c structures from (7.16). We claim that this more direct construction coincides with the construction given in (7.18). To see this, we note that if $\mathcal{W} = (W, Z, [\xi])$ is a sutured manifold cobordism which decomposes as the composition of two cobordisms, $\mathcal{W}_1 = (W_1, Z_1, [\xi_1])$ and $\mathcal{W}_2 = (W_2, Z_2, [\xi_2])$, and ω is a 2-form such that $[\omega|_{M_0 \cup Z}] = 0$ and $[\omega|_{M_1}] = 0$, then the original proof of the sutured cobordism composition law [8, Theorem 11.3] (see also [25, Theorem 3.4]) adapts to show that

$$F_{\mathcal{W};\omega} \doteq F_{\mathcal{W}_2;\omega_2} \circ F_{\mathcal{W}_1;\omega_1}, \tag{7.19}$$

where the maps are defined using the construction in (7.16). When $[\omega]$ restricts trivially to $M_0 \cup Z$ and M_1, the right-hand side of (7.18) may be interpreted as a composition satisfying these hypotheses, so the composition law of (7.19) implies that (7.18) coincides with the construction obtained by removing the Spin^c restrictions from (7.16).

7.7. The composition law

We now sketch a proof of the composition law, Proposition 2.10.

Proof of Proposition 2.10. We focus on part (1), as part (2) follows from a simple modification. Assume, as in the statement, that $\mathcal{W} = (W, Z, [\xi])$ is a sutured cobordism from (M_0, γ_0) to (M_2, γ_2), which decomposes into $\mathcal{W}_1 = (W_1, Z_1, [\xi_1])$ and $\mathcal{W}_2 = (W_2, Z_2, [\xi_2])$

that meet along a sutured manifold (M_1, γ_1). We are interested in the case when $[\omega]$ restricts trivially to M_0, M_1, and M_2.

As a first step, we claim that, via the same argument that gives (7.19), if $[\omega]$ restricts trivially to $M_1 \cup Z_2$ and M_2, then

$$F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}} \doteq F_{\mathcal{W}_2;\omega_2} \circ F_{\mathcal{W}_1;\omega_1} \circ i_{\underline{\mathfrak{s}}}, \tag{7.20}$$

where $\underline{\mathfrak{s}} \in \mathrm{Spin}^c(M_0, \gamma_0)$, and the maps $F_{\mathcal{W};\omega} \circ i_{\underline{\mathfrak{s}}}$, $F_{\mathcal{W}_2;\omega_2}$, and $F_{\mathcal{W}_1;\omega_1} \circ i_{\underline{\mathfrak{s}}}$ are defined using the appropriate modification of (7.16).

We now claim that the restricted composition law stated in (7.20) implies the full version of part (1) of Proposition 2.10. We recall that the full version of Proposition 2.10 involves the maps defined in (7.18). Following the construction of Section 7.6, we decompose $\mathcal{W}_1$ into sutured manifold cobordisms $\mathcal{N}_1$ and $\widetilde{\mathcal{W}}_1$, and we decompose $\mathcal{W}_2$ into $\mathcal{N}_2$ and $\widetilde{\mathcal{W}}_2$. We give $\mathcal{W}$ the analogous decomposition into $\mathcal{N}_1$ and $\widetilde{\mathcal{W}} := \widetilde{\mathcal{W}}_1 \cup \mathcal{N}_2 \cup \widetilde{\mathcal{W}}_2$; see Figure 7.3.

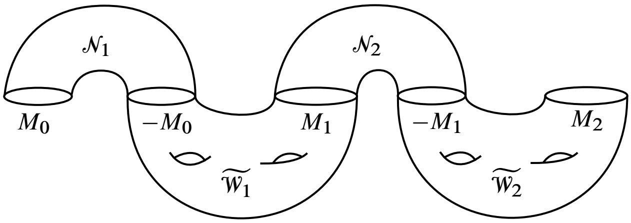


Fig. 7.3. Decomposing $\mathcal{W} = \mathcal{W}_2 \circ \mathcal{W}_1$ into $\mathcal{N}_1$ and $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}_1 \cup \mathcal{N}_2 \cup \widetilde{\mathcal{W}}_2$.

Using the definition from (7.18), we have

$$\begin{aligned} F_{\mathcal{W}_2;\omega_2} \circ F_{\mathcal{W}_1;\omega_1} := (F_{\mathcal{N}_2,\omega|_{\overline{N}_2}} \otimes \mathrm{id}_{SFH(M_2)}) \circ (\mathrm{id}_{SFH(M_1)} \otimes F_{\widetilde{\mathcal{W}}_2;\omega|_{W_2 \setminus N_2}}) \\ \circ (F_{\mathcal{N}_1,\omega|_{\overline{N}_1}} \otimes \mathrm{id}_{SFH(M_1)}) \circ (\mathrm{id}_{SFH(M_0)} \otimes F_{\widetilde{\mathcal{W}}_1;\omega|_{W_1 \setminus N_1}}). \end{aligned} \tag{7.21}$$

By commuting tensor factors, we see that the right-hand side of (7.21) coincides with the composition of $F_{\mathcal{N}_1;\omega|_{\overline{N}_1}} \otimes \mathrm{id}_{SFH(M_2)}$ and

$$\begin{aligned} (\mathrm{id}_{SFH(M_0)} \otimes \mathrm{id}_{SFH(-M_0)} \otimes F_{\mathcal{N}_2;\omega|_{\overline{N}_2}} \otimes \mathrm{id}_{SFH(M_2)}) \\ \circ (\mathrm{id}_{SFH(M_0)} \otimes F_{\widetilde{\mathcal{W}}_1;\omega|_{W_1 \setminus N_1}} \otimes F_{\widetilde{\mathcal{W}}_2;\omega|_{W_2 \setminus N_2}}). \end{aligned} \tag{7.22}$$

The hypotheses stated for the restricted version of the composition law from (7.20) are satisfied for decomposing $\widetilde{\mathcal{W}}$ into the composition of $\mathsf{Id}_{-M_0} \sqcup \mathcal{N}_2 \sqcup \mathsf{Id}_{M_2}$ and $\widetilde{\mathcal{W}}_1 \sqcup \widetilde{\mathcal{W}}_2$ (note that we are implicitly precomposing with $i_{\underline{\mathfrak{s}}_0}$, where $\underline{\mathfrak{s}}_0$ is the unique Spin^c structure on the empty set). Hence (7.22) coincides with $\mathrm{id}_{SFH(M_0)} \otimes F_{\widetilde{\mathcal{W}};\omega|_{W \setminus N_1}}$. It follows that (7.21) coincides with

$$(F_{\mathcal{N}_1,\omega|_{N_1}} \otimes \mathrm{id}_{SFH(M_2)}) \circ (\mathrm{id}_{SFH(M_0)} \otimes F_{\widetilde{\mathcal{W}};\omega|_{W \setminus N_1}}),$$

which is the definition of $F_{\mathcal{W};\omega}$ in (7.18). ∎

7.8. *Changing the 2-form on W*

We now prove Lemma 2.12.

Proof of Lemma 2.12. We investigate the diagram (7.7) from Remark 7.3. Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_n$ is a sequence of sutured Heegaard diagrams such that

- $\mathcal{H}_1$ is a diagram for (M_0, γ_0) and $\mathcal{H}_n$ is a diagram for (M_1, γ_1),
- $\mathcal{H}_{i+1}$ is obtained from $\mathcal{H}_i$ by either an elementary Heegaard move, the contact gluing map, or is the result of applying a 1-handle, 2-handle, or 3-handle map.

Consider the case when $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ are diagrams for the boundaries of the 2-handle submanifold $\mathcal{W}_2 = (W_2, Z_2, [\xi_2])$ of $\mathcal{W}$. Furthermore, assume $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ are subdiagrams of a triple which is subordinate to a bouquet for a framed link in the incoming boundary of W_2. Write $\widehat{\omega}_2$ for the restriction of ω to $\mathcal{W}_2$. Write ω_i and ω_{i+1} for the restrictions of ω to the manifolds defined by $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$, respectively. Define $\widehat{\eta}_2$, η_i, and η_{i+1} similarly. An argument using Stokes' theorem implies that the following diagram commutes up to an overall factor of z^x:

$$\begin{array}{ccc} CF(\mathcal{H}_i;\Lambda_{\omega_i}) & \xrightarrow{\Psi_{\omega_i\to\omega_i+d\eta_i;\eta_i}} & CF(\mathcal{H}_i;\Lambda_{\omega_i+d\eta_i}) \\ {\scriptstyle F_{\mathcal{W}_2;\widehat{\omega}_2}}\downarrow & & \downarrow{\scriptstyle F_{\mathcal{W}_2;\widehat{\omega}_2+d\widehat{\eta}_2}} \\ CF(\mathcal{H}_{i+1},\Lambda_{\omega_i}) & \xrightarrow{\Psi_{\omega_{i+1}+d\eta_{i+1};\eta_{i+1}}} & CF(\mathcal{H}_{i+1};\Lambda_{\omega_{i+1}+d\eta_{i+1}}) \end{array}$$

In an analogous manner, we may relate $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ by a similar commutative square when $\mathcal{H}_{i+1}$ is obtained from $\mathcal{H}_i$ by an elementary Heegaard move, or a 1-handle or 3-handle attachment. Stacking the $n-1$ projectively commutative squares, we see that the square

$$\begin{array}{ccc} CF(\mathcal{H}_1;\Lambda_{\omega_1}) & \xrightarrow{\Psi_{\omega_1\to\omega_1+d\eta_1;\eta_1}} & CF(\mathcal{H}_1;\Lambda_{\omega_1+d\eta_1}) \\ {\scriptstyle F_{\mathcal{W};\omega}}\downarrow & & \downarrow{\scriptstyle F_{\mathcal{W};\omega+d\eta}} \\ CF(\mathcal{H}_n,\Lambda_{\omega_n}) & \xrightarrow{\Psi_{\omega_n\to\omega_n+d\eta_n;\eta_n}} & CF(\mathcal{H}_n;\Lambda_{\omega_n+d\eta_n}) \end{array}$$

commutes, up to an overall factor of z^x. Since $\eta|_{M_0} = \eta_1 = 0$ and $\eta|_{M_1} = \eta_n = 0$, the maps $\Psi_{\omega_1\to\omega_1+d\eta_1;\eta_1}$ and $\Psi_{\omega_n\to\omega_n+d\eta_n;\eta_n}$ are the identity, completing the proof. ∎

7.9. *Perturbed and unperturbed cobordism maps*

We are finally ready to prove Lemma 3.4.

Proof of Lemma 3.4. Let us write $W = W_1 \cup W_2 \cup W_3$, where W_i is the i-handle part of W. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', w)$ be a triple subordinate to a bouquet for the 2-handles of W, and write $W_{\alpha,\beta,\beta'}$ for the corresponding portion of W_2. In particular, $W_0 := W_2 \setminus \operatorname{int}(W_{\alpha,\beta,\beta'})$ is a boundary connected sum of copies of $S^1 \times D^3$. As $H^2(W_1, Y_0; \mathbb{R}) = 0$ and $H^2(W_3, Y_1; \mathbb{R}) = 0$, the restriction maps $H^2(W_1; \mathbb{R}) \to H^2(Y_0; \mathbb{R})$ and $H^2(W_3; \mathbb{R}) \to H^2(Y_1; \mathbb{R})$ are both injective. Furthermore, $H^2(W_0; \mathbb{R}) = 0$. Hence, since $\omega|_{\partial W} = 0$, we have $[\omega|_{W_0}] = 0$, $[\omega|_{W_1}] = 0$, and $[\omega|_{W_3}] = 0$. So there is a 1-form η on W such that

$\eta|_{\partial W} = 0$, and $\omega - d\eta$ vanishes on $W \setminus \mathrm{int}(W_{\alpha,\beta,\beta'})$; compare the proof of Lemma 4.1. By Lemma 3.3, we have

$$F^{\circ}_{W,\mathfrak{S};\omega} \doteq F^{\circ}_{W,\mathfrak{S};\omega-d\eta}.$$

Hence, we may assume that ω vanishes on W_0, W_1, and W_3. With this assumption, the maps $F^{\circ}_{W_1,\mathfrak{S}|_{W_1};\omega|_{W_1}}$ and $F^{\circ}_{W_3,\mathfrak{S}|_{W_3};\omega|_{W_3}}$ are unperturbed. Furthermore,

$$\langle i_*(\mathfrak{s} - \mathfrak{s}_0) \cup [\omega], [W, \partial W]\rangle = \langle i_*(\mathfrak{s}|_{W_2} - \mathfrak{s}_0|_{W_2}) \cup [\omega|_{W_2}], [W_2, \partial W_2]\rangle.$$

So, without loss of generality, we can assume that $W = W_2$.

Let $\boldsymbol{x}, \boldsymbol{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}$. Furthermore, let $\psi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}, \Theta_{\beta,\beta'})$ and $\psi' \in \pi_2(\boldsymbol{x}', \boldsymbol{y}', \Theta_{\beta,\beta'})$ be homology classes of triangles, where $\Theta_{\beta,\beta'} \in \mathbb{T}_\beta \cap \mathbb{T}_{\beta'}$. Note that

$$HF^{\circ}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}'; \Lambda_{\omega|_{\partial W_0}}) = HF^{\circ}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}') \otimes \Lambda,$$

since $\omega|_{\partial W_0} = 0$. Then, the coned-off domain $\widetilde{\mathcal{D}}(\psi) - \widetilde{\mathcal{D}}(\psi')$ represents the Poincaré dual of $\mathfrak{s}_w(\psi) - \mathfrak{s}_w(\psi') \in H^2(W_2)$. Hence

$$A_\omega(\psi) - A_\omega(\psi') = \int_{\widetilde{\mathcal{D}}(\psi)} \omega - \int_{\widetilde{\mathcal{D}}(\psi')} \omega = \langle i_*(\mathfrak{s}_w(\psi) - \mathfrak{s}_w(\psi')) \cup [\omega], [W, \partial W]\rangle,$$

and (3.2) follows. ∎

Acknowledgements. We would like to thank Ciprian Manolescu, Thomas Mark, and Zoltán Szabó for helpful discussions, and Ronald Fintushel and Ronald Stern for their comments on the history of this problem. We would also like to thank an anonymous referee for a very careful reading and helpful suggestions.

Funding. The first author was supported by a Royal Society Research Fellowship, and the second author by an NSF Postdoctoral Research Fellowship (DMS-1703685). This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 674978).

References

[1] Ai, Y., Peters, T. D.: The twisted Floer homology of torus bundles. Algebr. Geom. Topol. **10**, 679–695 (2010) Zbl 1209.57006 MR 2606797

[2] Akbulut, S.: Variations on Fintushel–Stern knot surgery on 4-manifolds. Turkish J. Math. **26**, 81–92 (2002) Zbl 1007.57019 MR 1892802

[3] Baldwin, J. A., Sivek, S.: A contact invariant in sutured monopole homology. Forum Math. Sigma **4**, art. e12, 82 pp. (2016) Zbl 1355.53068 MR 3510331

[4] Fintushel, R., Stern, R. J.: Knots, links, and 4-manifolds. Invent. Math. **134**, 363–400 (1998) Zbl 0914.57015 MR 1650308

[5] Honda, K., Kazez, W., Matić, G.: Contact structures, sutured Floer homology and TQFT. arXiv:0807.2431 (2008)

[6] Jabuka, S., Mark, T. E.: Product formulae for Ozsváth–Szabó 4-manifold invariants. Geom. Topol. **12**, 1557–1651 (2008) Zbl 1156.57026 MR 2421135

[7] Juhász, A.: Holomorphic discs and sutured manifolds. Algebr. Geom. Topol. **6**, 1429–1457 (2006) Zbl 1129.57039 MR 2253454

[8] Juhász, A.: Cobordisms of sutured manifolds and the functoriality of link Floer homology. Adv. Math. **299**, 940–1038 (2016) Zbl 1358.57021 MR 3519484

[9] Juhász, A., Marengon, M.: Concordance maps in knot Floer homology. Geom. Topol. **20**, 3623–3673 (2016) Zbl 1364.57013 MR 3590358

[10] Juhász, A., Marengon, M.: Computing cobordism maps in link Floer homology and the reduced Khovanov TQFT. Selecta Math. (N.S.) **24**, 1315–1390 (2018) Zbl 1397.57026 MR 3782423

[11] Juhász, A., Zemke, I.: Contact handles, duality, and sutured Floer homology. Geom. Topol. **24**, 179–307 (2020) Zbl 1467.57023 MR 4080483

[12] Juhász, A., Miller, M., Zemke, I.: Transverse invariants and exotic surfaces in the 4-ball. Geom. Topol. **25**, 2963–3012 (2021) Zbl 07442196 MR 4347309

[13] Juhász, A., Thurston, D., Zemke, I.: Naturality and mapping class groups in Heegaard Floer homology. Mem. Amer. Math. Soc. **273**, no. 1338, v+174 pp. (2021) MR 4337438

[14] Kronheimer, P., Mrowka, T.: Knots, sutures, and excision. J. Differential Geom. **84**, 301–364 (2010) Zbl 1208.57008 MR 2652464

[15] Lekili, Y.: Heegaard–Floer homology of broken fibrations over the circle. Adv. Math. **244**, 268–302 (2013) Zbl 1306.57012 MR 3077873

[16] Li, Z.: Gluing maps and cobordism maps for sutured monopole Floer homology. Algebr. Geom. Topol. **20**, 2553–2588 (2020) Zbl 1462.57036 MR 4171573

[17] Lipshitz, R.: A cylindrical reformulation of Heegaard Floer homology. Geom. Topol. **10**, 955–1096 (2006) Zbl 1130.57035 MR 2240908

[18] Lipshitz, R., Ozsváth, P. S., Thurston, D. P.: Bordered Floer homology and the spectral sequence of a branched double cover II: the spectral sequences agree. J. Topol. **9**, 607–686 (2016) Zbl 1345.57023 MR 3509974

[19] Lutz, R.: Structures de contact sur les fibrés principaux en cercles de dimension trois. Ann. Inst. Fourier (Grenoble) **27**, ix, 1–15 (1977) Zbl 0328.53024 MR 478180

[20] Mark, T. E.: Knotted surfaces in 4-manifolds. Forum Math. **25**, 597–637 (2013) Zbl 1270.57065 MR 3062865

[21] Ozsváth, P., Szabó, Z.: Holomorphic disks and genus bounds. Geom. Topol. **8**, 311–334 (2004) Zbl 1056.57020 MR 2023281

[22] Ozsváth, P., Szabó, Z.: Holomorphic disks and knot invariants. Adv. Math. **186**, 58–116 (2004) Zbl 1062.57019 MR 2065507

[23] Ozsváth, P., Szabó, Z.: Holomorphic disks and topological invariants for closed three-manifolds. Ann. of Math. (2) **159**, 1027–1158 (2004) Zbl 073.57009 MR 2113019

[24] Ozsváth, P., Szabó, Z.: Heegaard Floer homology and contact structures. Duke Math. J. **129**, 39–61 (2005) Zbl 1083.57042 MR 2153455

[25] Ozsváth, P., Szabó, Z.: Holomorphic triangles and invariants for smooth four-manifolds. Adv. Math. **202**, 326–400 (2006) Zbl 1099.53058 MR 2222356

[26] Ozsváth, P., Szabó, Z.: Holomorphic disks, link invariants and the multi-variable Alexander polynomial. Algebr. Geom. Topol. **8**, 615–692 (2008) Zbl 1144.57011 MR 2443092

[27] Sarkar, S.: Moving basepoints and the induced automorphisms of link Floer homology. Algebr. Geom. Topol. **15**, 2479–2515 (2015) Zbl 1331.57015 MR 3426686

[28] Tange, M.: On the diffeomorphisms for Akbulut's knot concordance surgery. J. Knot Theory Ramifications **14**, 539–563 (2005) Zbl 1083.57043 MR 2162113

[29] Wu, Z.: Perturbed Floer homology of some fibered three-manifolds. Algebr. Geom. Topol. **9**, 337–350 (2009) Zbl 1161.53364 MR 2482081

[30] Zemke, I.: Graph cobordisms and Heegaard Floer homology (2015)

[31] Zemke, I.: Quasistabilization and basepoint moving maps in link Floer homology. Algebr. Geom. Topol. **17**, 3461–3518 (2017) Zbl 1387.57020 MR 3709653

[32] Zemke, I.: Connected sums and involutive knot Floer homology. Proc. London Math. Soc. (3) **119**, 214–265 (2019) Zbl 07094288 MR 3957835