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The Newman–Shapiro problem

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Abstract. We give a negative answer to the Newman–Shapiro problem on weighted approximation for entire functions formulated in 1966 and motivated by the theory of operators on the Fock space. There exists a function in the Fock space such that its exponential multiples do not approximate some entire multiples in the space. Furthermore, we establish several positive results under different restrictions on the function in question.

Keywords. Fock space, weighted approximation, Cauchy transform, Parseval-type relations

1. Introduction and the main results

Let $\mathcal{F} = \mathcal{F}_{(1)}$ be the classical Bargmann–Segal–Fock space, where

$$\mathcal{F}_{(\alpha)} = \left\{ F \in \mathrm{Hol}(\mathbb{C}) : \|F\|^2_{\mathcal{F}} = \frac{1}{\pi} \int_{\mathbb{C}} |F(z)|^2 e^{-\alpha\pi|z|^2}\, dm(z) < \infty \right\},$$

and m stands for the area Lebesgue measure. This space serves as a model of the phase space of a particle in quantum mechanics and so plays an important role in theoretical physics. Moreover, this space appears in time-frequency analysis, as a spectral model of $L^2(\mathbb{R})$ via the Bargmann transform (see, e.g., [15, Section 3.4]). Note also that the complex exponentials e_λ, where $e_\lambda(z) = e^{\lambda z}$, are the reproducing kernels of $\mathcal{F}$, i.e.,

$$\langle F, k_\lambda \rangle_{\mathcal{F}} = F(\lambda), \quad F \in \mathcal{F},$$

where $k_\lambda = \pi e_{\pi\bar{\lambda}}$.

In 1966, D. J. Newman and H. S. Shapiro [20] posed the following problem about the structure of the operator adjoint to the multiplication operator in Fock space. Let F be an entire function such that, for every $A > 0$,

$$|F(z)| \le C(A, F) \exp\left(\frac{\pi}{2}|z|^2 - A|z|\right), \quad z \in \mathbb{C}. \tag{1.1}$$

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This condition is equivalent to $e_\lambda \cdot F$ being in $\mathcal{F}$ for every $\lambda \in \mathbb{C}$. Now we can define the multiplication operator $M_F : G \mapsto FG$ on the linear span of the exponentials

$$\mathcal{L} = \mathrm{Span}\,\{e_\lambda : \lambda \in \mathbb{C}\}.$$

The natural domain of the operator M_F is

$$\mathcal{R}_F = \{G \in \mathcal{F} : FG \in \mathcal{F}\}.$$

Thus, we can consider the adjoint operator M_F^* as well as the operator adjoint to the restriction $M_F|_{\mathcal{L}}$, which we denote by $F^*(\frac{d}{dz})$ (following [20]). This notation is motivated by the fact that when $F = P$ is a polynomial, we have

$$P(\lambda)G(\lambda) = \langle M_P G, \pi e_{\pi\bar{\lambda}}\rangle = \langle G, P^*(d/dz)(\pi e_{\pi\bar{\lambda}})\rangle,$$

where $P^*(z) = \overline{P(\bar{z})}$ and $P^*(\frac{d}{dz})$ is understood in the usual sense as a differential operator. In this case it is easy to see that $M_P^* = P^*(\frac{d}{dz})$. The Newman–Shapiro problem (related to a much earlier work of E. Fischer [13]) is whether $M_F^* = F^*(\frac{d}{dz})$ for all F satisfying (1.1). In [20] (see also [22] and an extended unpublished manuscript [21]) Newman and Shapiro proved that this is the case when F is an exponential polynomial (i.e., $F = \sum_{k=1}^n P_k e_{\lambda_k}$, where P_k are polynomials and $\lambda_k \in \mathbb{C}$) and for some other special cases (i.e. F has no zeros or $F(z) = (\sin z)/z$). Moreover, they revealed some connections of this problem with weighted polynomial approximation in $\mathcal{F}$. More precisely, they proved the following result (to avoid inessential technicalities we assume that F has simple zeros only). Denote by $\mathcal{E}$ the space of all entire functions.

Theorem 1.1 ([20, Theorem 1], [22]). *For every F satisfying estimates (1.1) the following statements are equivalent:*

(1) $\overline{\mathrm{Span}}\,\{z^n F : n \ge 0\} = \mathcal{E}F \cap \mathcal{F}$;

(2) $M_F^* = F^*(\frac{d}{dz})$;

(3) $\mathrm{Ker}\, F^*(\frac{d}{dz}) = \overline{\mathrm{Span}}\,\{e_{\bar{\lambda}} : e_{\bar{\lambda}} \in \mathrm{Ker}\, F^*(\frac{d}{dz})\} = \overline{\mathrm{Span}}\,\{e_{\bar{\lambda}} : F(\lambda) = 0\}$.

The Newman–Shapiro problem has remained open since 1966. Several similar questions were studied, e.g., in [19] (see also [12, Chapter X.8]). For related questions on Toeplitz operators on the Fock space see [11] and the references therein.

It should be mentioned that the Newman–Shapiro problem is closely related to the spectral synthesis (hereditary completeness) problem for systems of reproducing kernels in the Fock space (or of Gabor-type expansions with respect to time-frequency shifts of the Gaussian). In the Paley–Wiener space setting, the spectral synthesis problem was solved in [2], whereas for the reproducing kernels of the Fock space the solution (in general, also negative) was recently given in [3].

In this article we prove that the answer to the Newman–Shapiro problem is in general negative and establish several positive results under different restrictions on the growth and regularity of the function F.

The original Newman–Shapiro problem is formulated for the Fock spaces on $\mathbb{C}^n$, $n \geq 1$. Here, we restrict ourselves to the case $n = 1$. The negative answer to the Newman–Shapiro problem in the case $n = 1$ means the negative answer for every $n \geq 1$. It seems that one should use different techniques to obtain positive results for $n > 1$.

Theorem 1.2. *For any $\alpha \in (1, 2)$, there exist entire functions F and G such that G, GF are in $\mathcal{F}$ and for every entire function h of order at most α we have $hF \in \mathcal{F}$, but*

$$GF \notin \overline{\operatorname{Span}}\{pF : p \in \mathcal{P}\} = \overline{\operatorname{Span}}\{e_\lambda F : \lambda \in \mathbb{C}\}.$$

Thus, the equivalent conditions of Theorem 1.1 do not hold for F. Here and later on, $\mathcal{P}$ is the space of the polynomials.

Next we establish that under more restrictive growth and regularity conditions on the function F the answer to the Newman–Shapiro problem becomes positive.

Given $\alpha \geq 0$, denote by $\mathcal{E}_{2,\alpha}$ the class of all entire functions of type at most α for order 2, that is,

$$\limsup_{|z|\to\infty} \frac{\log |F(z)|}{|z|^2} \leq \alpha.$$

Set $\mathcal{E}_2 = \bigcup_{\alpha<\infty} \mathcal{E}_{2,\alpha}$.

Given $F \in \mathcal{E}_2$, consider its indicator function for order 2,

$$h_F(\theta) = \limsup_{r\to\infty} \frac{\log |F(re^{i\theta})|}{r^2}, \quad \theta \in [0, 2\pi].$$

We say that $F \in \mathcal{E}_2$ is of *completely regular growth* if $\log |F(re^{i\theta})|/r^2$ converges uniformly in $\theta \in [0, 2\pi]$ to $h_F(\theta)$ as $r \to \infty$ and $r \notin E_F$ for some set $E_F \subset [0, \infty)$ of zero relative measure, that is,

$$\lim_{R\to\infty} \frac{E_F \cap [0, R]}{R} = 0.$$

Theorem 1.3. *Let $F \in \mathcal{E}$. Suppose that there exists $G \in \mathcal{E}_2$ of completely regular growth and $\alpha < 1$ such that $(FG \cdot \mathcal{E}) \cap \mathcal{F}_{(\alpha)} = FG \cdot \mathbb{C}$, and $\inf_{[0,2\pi]} h_G > 0$. Then $F \in \mathcal{F}_{(\gamma)}$ for every $\gamma \geq \alpha$, and*

$$\overline{\operatorname{Span}}\{e_\lambda F : \lambda \in \mathbb{C}\} = \mathcal{E}F \cap \mathcal{F}.$$

Thus, the equivalent conditions of Theorem 1.1 hold for such F.

The assumptions of the theorem mean that the zero set of F can be complemented by a set of positive angular density to a set Λ such that the system $\{k_\lambda\}_{\lambda\in\Lambda}$ is complete and minimal in $\mathcal{F}_{(\alpha)}$.

When the zero set of F is sufficiently regular and not very dense, we get the following result.

Corollary 1.4. *Let $F \in \mathcal{F}$ be of completely regular growth. Suppose that the upper Beurling–Landau density $D^+_{Z(F)}$ of the zero set $Z(F)$ of F (with multiplicities taken into account) is less than $1/\pi$:*

$$\limsup_{R\to\infty} \sup_{z\in\mathbb{C}} \frac{\operatorname{card}(Z(f) \cap D(z, R))}{\pi R^2} < \frac{1}{\pi}. \tag{1.2}$$

Then

$$\overline{\operatorname{Span}}\{e_\lambda F : \lambda \in \mathbb{C}\} = \mathcal{E}F \cap \mathcal{F}.$$

Here and later on $D(z,r)$ stands for the open disc centered at z of radius r.

Condition (1.2) is indispensable here as demonstrated by the example given in the proof of Theorem 1.2.

When we restrict the growth of F, there are no more regularity restrictions on the zeros:

Theorem 1.5. *There exists $\eta > 0$ such that if $F \in \mathcal{E}_{2,\eta}$, then*

$$\overline{\operatorname{Span}}\,\{e_\lambda F : \lambda \in \mathbb{C}\} = \mathcal{E} F \cap \mathcal{F}.$$

Thus, the situation here could be compared to that of the cyclicity/invertibility problem in the Bergman space. Invertibility does not imply cyclicity there [9], but it does if we impose additional growth restrictions. (Stronger lower estimates also imply cyclicity [7].) The main difference is that Bergman space cyclic/invertible functions f are zero free and one works with harmonic $\log|f|$, while in our situation Fock space functions have a lot of zeros, which makes the problem much more complicated.

The Fock space does not possess a Riesz basis of reproducing kernels. Instead, we have the system $\mathcal{K} = \{k_w\}_{w\in\mathcal{Z}_0}$ which is complete and minimal in $\mathcal{F}$. Here and later on, $\mathcal{Z} = \mathbb{Z} + i\mathbb{Z} \subset \mathbb{C}$ and $\mathcal{Z}_0 = \mathcal{Z} \setminus \{0\}$. Let σ be the Weierstrass sigma function associated to $\mathcal{Z}$, and $\sigma_0(z) = \sigma(z)/z$. For more information about these functions see Section 2. The system $\{g_w\}_{w\in\mathcal{Z}_0}$, where $g_w = \sigma_0/(\sigma_0'(w)(\cdot - w))$, is biorthogonal to $\mathcal{K}$. One of our main technical tools to get the completeness results is the following Parseval-type relation: if $F_1, F_2 \in \mathcal{F}$ and $\mu \in Z(F_2) \setminus \mathcal{Z}_0$, then

$$\sum_{w\in\mathcal{Z}_0} \langle F_2, k_w\rangle \cdot \langle g_w, F_1\rangle \left[\frac{1}{z-w} + \frac{1}{w-\mu}\right] = \frac{\langle F_2, F_1\rangle}{z} + \left\langle \frac{F_2}{\cdot - \mu}, F_1\right\rangle + o(|z|^{-1}), \quad |z| \to \infty,\ z \in \mathbb{C} \setminus \Omega,$$

for some thin set Ω. Furthermore, we study related continuous Cauchy transforms corresponding to pairs of Fock space functions, whose asymptotics gives their scalar product. In particular, given $F_1, F_2 \in \mathcal{F}$, we have

$$\begin{aligned}\frac{1}{\sigma_0(z)}\left\langle \frac{\sigma_0(z)F_1 - F_1(z)\sigma_0}{z - \cdot}, F_2\right\rangle &= \int_{\mathbb{C}} \frac{F_2(\zeta)\overline{F_1(\zeta)}}{z-\zeta} e^{-\pi|\zeta|^2}\, dm_2(\zeta) - \frac{F_2(z)}{\sigma_0(z)} \int_{\mathbb{C}} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}}{z-\zeta} e^{-\pi|\zeta|^2}\, dm_2(\zeta)\\ &= \frac{\langle F_2, F_1\rangle}{z} + o(|z|^{-1}), \quad |z| \to \infty,\ z \in \mathbb{C} \setminus \Omega,\end{aligned}$$

for some thin set Ω. Finally, we establish and use a number of uniqueness results on Fock space functions outside thin sets (thin lattice sets).

The plan of the paper is as follows. In Section 2 we introduce some notations and prove three uniqueness results for functions in the Fock space. Section 3 contains several auxiliary results on interpolation formulas and the scalar product in the Fock space. Theorem 1.3 together with some auxiliary lemmas is proved in Section 4. Theorem 1.5 and Corollary 1.4 are proved in Section 5. Finally, Theorem 1.2 is proved in Section 6 using techniques that are quite different from those in the previous part of the paper.

Notations

Throughout this paper the notation $U(x) \lesssim V(x)$ for functions $U, V \geq 0$ means that there is a constant C such that $U(x) \leq CV(x)$ for all x in the set in question. We write $U(x) \asymp V(x)$ if both $U(x) \lesssim V(x)$ and $V(x) \lesssim U(x)$.

2. Notations and some uniqueness results for the Fock space

In this section, after introducing some notations, we establish three uniqueness results for Fock space functions.

Given $\alpha \in \mathbb{C}$, the time-frequency shift operator $\mathcal{T}_\alpha$ given by

$$(\mathcal{T}_\alpha F)(z) = e^{\pi \bar{\alpha} z - \frac{\pi}{2}|\alpha|^2} F(z-\alpha)$$

is unitary on the Fock space $\mathcal{F}$.

Put $d\nu(z) = e^{-\pi|z|^2} dm_2(z)$.

Given $F \in \mathcal{E}$ we denote by $Z(F)$ its zero set.

It is known [17, Theorem 5, Chapter 3] that if $F, G \in \mathcal{E}_2$ and F is of completely regular growth, then

$$h_{FG} = h_F + h_G.$$

Together with $\mathcal{F}$ we consider its subspace

$$\mathcal{F}_0 = \{F \in \mathcal{E} : \mathcal{P}F \in \mathcal{F}\}.$$

Given $F \in \mathcal{F}_0$, denote

$$[F]_{\mathcal{F}} = \overline{\operatorname{Span}}\{\mathcal{P}F\}.$$

Following [3] we say that a measurable subset of $\mathbb{C}$ is *thin* if it is the union of a measurable set Ω_1 of zero (area) density,

$$\lim_{R\to\infty} \frac{m_2(\Omega_1 \cap D(0,R))}{R^2} = 0,$$

and a measurable set Ω_2 such that

$$\int_{\Omega_2} \frac{dm_2(z)}{(|z|+1)^2 \log(|z|+2)} < \infty.$$

The union of two thin sets is thin. If Ω is thin, then its lower density

$$\liminf_{R\to\infty} \frac{m_2(\Omega \cap D(0,R))}{R^2}$$

is zero. In particular, $\mathbb{C}$ is not thin. If Ω is thin, then its translations $z + \Omega$ are thin for all $z \in \mathbb{C}$.

We start with the following Liouville type result. Although we do not use it directly in the paper, it helps us to better understand how sparse thin sets are with respect to the small value sets of Fock space functions. The lemma was originally formulated in [3, Lemma 4.2]. A corrected proof is given in [4]. Here we give an alternative proof.

Lemma 2.1. *Let F be an entire function of finite order, bounded on $\mathbb{C} \setminus \Omega$ for some thin set Ω. Then F is a constant.*

Proof. Suppose that F is not a constant and that

$$\log |F(z)| = O(|z|^N), \quad |z| \to \infty, \tag{2.1}$$

for some $N < \infty$. We can find $w \in \mathbb{C}$ and $c \in \mathbb{R}$ such that the subharmonic function

$$u(z) = \log |F(z - w)| + c$$

is negative on $\mathbb{C} \setminus \widetilde{\Omega}$ for some open thin set $\widetilde{\Omega}$, and $u(0) = 1$. Given $R > 0$, consider the connected component O^R of $\widetilde{\Omega} \cap D(0, R)$ containing 0. By the theorem on harmonic estimation [16, VII.B.1], we have

$$1 = u(0) \le \omega(0, \partial O^R \cap \partial D(0, R), O^R) \cdot \max_{|z|=R} u(z),$$

where $\omega(z, E, O)$ is the harmonic measure at $z \in O$ of $E \subset \partial O$ with respect to O. By (2.1) we obtain

$$\omega(0, \partial O^R \cap \partial D(0, R), O^R) \ge aR^{-N}, \quad R \ge 1, \tag{2.2}$$

for some $a > 0$.

For some $R \ge 1$ to be chosen later we set

$$\varphi(z) = \begin{cases} \omega(z, \partial O^R \cap \partial D(0, R), O^R), & z \in O^R, \\ 1, & z \in \partial O^R \cap \partial D(0, R), \\ 0, & z \in (D(0, R) \setminus O^R) \cup (\partial D(0, R) \setminus \partial O^R), \end{cases}$$

$$\psi(r) = \max_{\partial D(0,r)} \varphi.$$

By the maximum principle, ψ increases on $[0, R]$.

We use the following radial version of Hall's lemma (attributed to Øksendal in [14, p. 125]): if E is a measurable subset of $D(0, 1) \setminus D(0, 1/2)$, then

$$\omega(0, E, D(0, 1) \setminus E) \ge \delta m_2(E)$$

for some absolute constant $\delta > 0$.

Let $0 < r < (1 + \varepsilon)r < R$ for some $\varepsilon \in (0, 1/2)$ and assume that

$$m_2\big(O^R \cap D(0, (1 + \varepsilon)r) \setminus D(0, r)\big) \le \frac{\pi\varepsilon^2}{8} r^2.$$

Then by Hall's lemma, applied in the discs

$$D(\zeta, \varepsilon r/2), \quad \zeta \in \partial D(0, (1+\varepsilon/2)r),$$

we obtain

$$\psi(r) \le (1-\beta)\psi((1+\varepsilon)r)$$

for some absolute constant $\beta > 0$. Choose $\varepsilon > 0$ such that $(1+\varepsilon)^{2N}(1-\beta) = 1$ and assume that $R = (1+\varepsilon)^M$ for some integer M. Put

$$\mathcal{N} = \left\{ n \ge 0 : m_2(\widetilde{\Omega} \cap D(0, (1+\varepsilon)^{n+1}) \setminus D(0, (1+\varepsilon)^n)) > \frac{\pi\varepsilon^2}{8}(1+\varepsilon)^{2n} \right\},$$

and set $\mathcal{N}_M^* = \mathbb{Z}_+ \cap [0, M) \setminus \mathcal{N}$. Then

$$\psi((1+\varepsilon)^n) \le (1+\varepsilon)^{-2N}\psi((1+\varepsilon)^{n+1}), \quad n \in \mathcal{N}_M^*.$$

By (2.2) we obtain

$$a(1+\varepsilon)^{-NM} \le \psi(1) \le (1+\varepsilon)^{-2N\,\mathrm{card}(\mathcal{N}_M^*)}\psi(R) = (1+\varepsilon)^{-2N\,\mathrm{card}(\mathcal{N}_M^*)},$$

and hence

$$M \ge 2\,\mathrm{card}(\mathcal{N}_M^*) - c, \quad M \ge 0.$$

In particular,

$$\mathrm{card}([3^s, 3^{s+1}) \cap \mathcal{N}) \ge 3^{s-2}, \quad s \ge s_0. \tag{2.3}$$

We have $\widetilde{\Omega} = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are open, and

$$\lim_{R\to\infty} \frac{m_2(\Omega_1 \cap D(0,R))}{R^2} = 0, \quad \int_{\Omega_2} \frac{dm_2(z)}{(|z|+1)^2 \log(|z|+2)} < \infty.$$

Furthermore,

$$\int_{\Omega_{2,n}} \frac{dm_2(z)}{(|z|+1)^2 \log(|z|+2)} \ge \frac{c}{n}, \quad n \ge n_0,\ n \in \mathcal{N},$$

for some $c > 0$, where $\Omega_{2,n} = \Omega_2 \cap D(0, (1+\varepsilon)^{n+1}) \setminus D(0, (1+\varepsilon)^n)$. Thus,

$$\sum_{n\in\mathcal{N}} \frac{1}{n} < \infty,$$

which contradicts (2.3). This completes the proof. ∎

We say that a subset A of the lattice $\mathcal{Z} = \mathbb{Z} + i\mathbb{Z}$ is *lattice thin* if for some (or equivalently for every) $c > 0$, the set

$$\bigcup_{w\in\mathcal{Z}} D(w, c)$$

is thin.

Let σ be the Weierstrass sigma function associated to $\mathcal{Z}$,

$$\sigma(z) = z \prod_{w\in\mathcal{Z}_0} \Big(1 - \frac{z}{w}\Big) e^{\frac{z}{w}+\frac{z^2}{2w^2}},$$

where $\mathcal{Z}_0 = \mathcal{Z} \setminus \{0\}$. Set $\sigma_0(z) = \sigma(z)/z$. Since

$$|\sigma(z)| \asymp \operatorname{dist}(z, \mathcal{Z}) e^{(\pi/2)|z|^2}, \quad z \in \mathbb{C},$$

we have $\sigma \in \mathcal{E}_{2,\pi/2}$, $h_\sigma \equiv \pi/2$ and $\mathcal{Z}_0$ is a uniqueness set for $\mathcal{F}$.

Lemma 2.2. *There exists $\beta \in (0, 1)$ such that if $S \in \mathcal{E}_2$ and $\mathcal{Z}_1$ is a subset of $\mathcal{Z}_0$ of lower density at least $1 - \beta$ satisfying*

$$\inf_{\partial D(z,\rho \log^{-2}(1+|z|))} |S| < 1$$

for every $\rho \in (0, 1)$ and $z \in \mathcal{Z}_1$, then S is a constant.

Proof. For every $z \in \mathcal{Z}_1$ put

$$\Delta_z = \{w \in D(z, \log^{-2}(1 + |z|)) : |S(w)| < 1\}.$$

Then every Δ_z contains a finite family of intervals $z + e^{i\theta_{k,z}} J_z^k$ with pairwise disjoint $J_z^k \subset \mathbb{R}_+$ of total length $(1/2)\log^{-2}(1 + |z|)$. Set

$$\Omega = \mathbb{C} \setminus \bigcup_{z\in\mathcal{Z}_1} \overline{\Delta_z}.$$

Given $z \in \mathbb{C}$ and $a > 0$, set

$$\mathcal{Z}_*^{z,a} = \{w \in \mathcal{Z}_1 : \overline{\Delta_w} \subset D(z, a) \setminus D(z, a/2)\}.$$

Next, given $\delta \in (0, 1)$ to be chosen later, if the lower density $\mathcal{Z}_1$ is at least $1 - \beta$ with $0 < \beta \le \beta(\delta) < 1$, then

$$\operatorname{card}(\mathcal{Z}_*^{z,r\delta}) \ge \frac{\pi}{4}\delta^2 r^2, \quad z \in \partial D(0, r),\ r > r(\delta). \tag{2.4}$$

Now we are going to prove that, under condition (2.4), we have

$$\omega\big(z, \partial\Omega \cap D(z, \delta|z|), D(z, \delta|z|) \cap \Omega\big) \ge \gamma, \quad |z| > r(\delta), \tag{2.5}$$

for some absolute constant $\gamma > 0$.

Given $z \in \mathbb{C}$, set $t = \delta|z|$ and

$$F = \Big\{w \in D(0, 1) : z + wt \in \bigcup_{w\in\mathcal{Z}_*^{z,r\delta}} \overline{\Delta_w}\Big\},$$

$$\mu = c \sum_{w\in\mathcal{Z}_*^{z,r\delta}} \mu_w = c \sum_{w\in\mathcal{Z}_*^{z,r\delta}} \sum_k \chi_{(w+e^{i\theta_{k,w}} J_w^k - z)/t}\, m,$$

where m is one-dimensional Lebesgue measure and c is a normalization constant such that μ is a probability measure.

Then, under condition (2.4), we have $c \asymp (\log^2 t)/t$ for large t, and the logarithmic energy of μ is estimated from below as follows:

$$\begin{aligned}
-I(\mu) &= -\int\int \log|\zeta_1-\zeta_2|\,d\mu(\zeta_1)\,d\mu(\zeta_2)\\
&= -c^2 \sum_{w\in\mathcal{Z}_*^{z,r\delta}} I(\mu_w)\\
&\quad - c^2 \sum_{w\in\mathcal{Z}_*^{z,r\delta}} \sum_{w_1\in\mathcal{Z}_*^{z,r\delta}\setminus\{w\}} \int\int \log|\zeta_1-\zeta_2|\,d\mu_w(\zeta_1)\,d\mu_{w_1}(\zeta_2)\\
&\le O(1) + \frac{\log^2 t}{t}\cdot \sup_{u\in D(0,1)} \sum_{w\in\mathcal{Z}_*^{z,r\delta}} \int \log\frac{1}{|\zeta-u|}\,d\mu_w(\zeta) = O(1).
\end{aligned}$$

Since $\operatorname{supp}\mu\subset F$, the logarithmic capacity of F is bounded below by an absolute constant $c>0$. Finally, [14, Theorem III.9.1] yields (2.5).

Put

$$\psi(r) = \max_{\partial D(0,r)} \log|S|, \quad r>0.$$

Since $S\in\mathcal{E}_2$, we have

$$\psi(r) = O(r^2), \quad r\to\infty. \tag{2.6}$$

Under condition (2.4), by the theorem on harmonic estimation [16, VII.B.1] and by (2.5), we obtain

$$\psi(r) \le \psi(r+\delta r)(1-\gamma), \quad r>r(\delta). \tag{2.7}$$

If δ is sufficiently small, $0<\delta\le(1-\gamma)^{-1/2}-1$, then (2.6) and (2.7) together imply that $\psi\le 0$, and hence S is a constant. This completes the proof. ∎

Lemma 2.3. *Let $F\in\mathcal{F}_0\cap\ell^\infty(\mathcal{Z})$. Then F is a constant.*

Proof. By the Lagrange interpolation formula, for every $k\ge 0$ and $z\in\mathbb{C}\setminus\mathcal{Z}_0$ we have

$$\frac{z^kF(z)}{\sigma(z)} = \sum_{w\in\mathcal{Z}} \frac{w^kF(w)}{\sigma'(w)(z-w)},$$

and hence

$$\left|\frac{z^kF(z)}{\sigma(z)}\right| = \left|\sum_{w\in\mathcal{Z}} \frac{w^kF(w)}{\sigma'(w)(z-w)}\right| \le \sum_{w\in\mathcal{Z}} \frac{|w|^k\cdot|F(w)|}{|\sigma'(w)|\cdot|z-w|}.$$

Therefore,

$$|F(z)| \lesssim |\sigma(z)|\cdot\min_{k\ge 0}\frac{1}{|z|^k}\sum_{w\in\mathcal{Z}}\frac{|w|^k}{|\sigma'(w)|}, \quad \operatorname{dist}(z,\mathcal{Z})>1/3.$$

Thus,

$$\begin{aligned}
|F(z)| &\lesssim e^{\frac{\pi}{2}|z|^2} \cdot \min_{k\ge 0} \frac{1}{|z|^k} \sum_{w\in\mathcal{Z}} |w|^k e^{-\frac{\pi}{2}|w|^2} \\
&\asymp e^{\frac{\pi}{2}|z|^2} \cdot \min_{k\ge 0} \frac{1}{|z|^k} \int_{\mathbb{C}} |w|^k e^{-\frac{\pi}{2}|w|^2}\, dm_2(w) \\
&= 2\pi e^{\frac{\pi}{2}|z|^2} \cdot \min_{k\ge 0} \frac{1}{|z|^k} \int_0^\infty r^{k+1} e^{-\frac{\pi}{2}r^2}\, dr \\
&\lesssim \min_{k\ge 0} \exp\Big[\frac{\pi}{2}|z|^2 - k\log|z| + \frac{k+1}{2}\log\frac{k+1}{\pi} - \frac{k+1}{2}\Big] \\
&\lesssim 1+|z|, \quad \operatorname{dist}(z,\mathcal{Z}) > 1/3.
\end{aligned}$$

It remains to use the Liouville theorem. ∎

3. Interpolation formulas and duality in the Fock space

In this section we establish several results on relations between interpolation formulas, expansions with respect to some fixed complete and minimal systems of reproducing kernels and their biorthogonal systems, and the scalar product in the Fock space.

Lemma 3.1. *Let $F_1, F_2 \in \mathcal{F}$ and $F_3 \in \mathcal{F}_0$. Then*

$$\begin{aligned}
\left|\int_{\mathbb{C}} \frac{F_2(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\, d\nu(\zeta)\right| &= o(1), && |z|\to\infty, \\
\left|\int_{\mathbb{C}} \frac{F_3(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\, d\nu(\zeta)\right| &= O((1+|z|)^{-1}), && |z|\to\infty, \\
\left|\int_{\mathbb{C}} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\, d\nu(\zeta)\right| &= O(|z|^{-1}\log^{1/2}|z|), && |z|\to\infty.
\end{aligned}$$

Proof. We use the fact that if $F \in \mathcal{F}$, then $|F(z)| = o(e^{\pi|z|^2/2})$ as $|z|\to\infty$. Furthermore, $F_2(\zeta)\overline{F_1(\zeta)}\, d\nu(\zeta) = \varphi(\zeta)\, dm_2(\zeta)$ with $\varphi \in L^1(\mathbb{C}) \cap C_0(\mathbb{C})$. Therefore, for every $R > 0$, we have

$$\begin{aligned}
\left|\int_{\mathbb{C}} \frac{F_2(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\, d\nu(\zeta)\right| &\le \left|\int_{\mathbb{C}\setminus D(z,R)} \frac{\varphi(\zeta)\, dm_2(\zeta)}{z-\zeta}\right| \\
&\quad + \left|\int_{D(z,R)} \frac{\varphi(\zeta)\, dm_2(\zeta)}{z-\zeta}\right| \\
&\lesssim \frac{\|\varphi\|_{L^1(\mathbb{C})}}{R} + R\cdot o(1), \quad |z|\to\infty.
\end{aligned}$$

The proof of the second inequality is analogous.

To prove the third inequality, we verify that

$$\begin{aligned}
&\left|\int_{\mathbb{C}} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\,d\nu(\zeta)\right| \\
&\qquad \le \left|\int_{\mathbb{C}\setminus D(z,1)} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}\,dm_2(\zeta)}{z-\zeta}\right| + \left|\int_{D(z,1)} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}\,dm_2(\zeta)}{z-\zeta}\right| \\
&\qquad \lesssim \|F_1\|_{L^2(\mathbb{C})}\biggl(\int_{\mathbb{C}\setminus D(z,1)} \frac{dm_2(\zeta)}{(1+|\zeta|^2)|z-\zeta|^2}\biggr)^{1/2} + o\Bigl(\frac{1}{|z|}\Bigr) \\
&\qquad = O(|z|^{-1}\log^{1/2}|z|), \quad |z|\to\infty. \qquad \blacksquare
\end{aligned}$$

Given $F \in \mathcal{F}$, $z \in \mathbb{C}$, set

$$\mathfrak{A}(F,z)(\zeta) = \frac{F(\zeta)\sigma_0(z) - F(z)\sigma_0(\zeta)}{z-\zeta}, \quad \zeta \in \mathbb{C}.$$

Then

$$\mathfrak{A}(F,z) = \sigma_0(z)\frac{F-F(z)}{z-\cdot} + F(z)\frac{\sigma_0(z)-\sigma_0}{z-\cdot} \in \mathcal{F}.$$

Given $F_1, F_2 \in \mathcal{F}$, set

$$\begin{aligned}
\mathcal{I}(F_1,F_2)(z) &= \frac{1}{\sigma_0(z)}\langle \mathfrak{A}(F_2,z), F_1\rangle \\
&= \int_{\mathbb{C}} \frac{F_2(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\,d\nu(\zeta) - \frac{F_2(z)}{\sigma_0(z)}\int_{\mathbb{C}} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\,d\nu(\zeta).
\end{aligned}$$

Then $\sigma_0 \cdot \mathcal{I}(F_1,F_2) \in \mathcal{E}$.

The following result is contained in [3, proof of Lemma 4.3].

Lemma 3.2. *Let $F_1, F_2 \in \mathcal{F}$. Then*

$$\mathcal{I}(F_1,F_2)(z) = \frac{\langle F_2, F_1\rangle}{z} + o(|z|^{-1}), \quad |z|\to\infty,\ z \in \mathbb{C}\setminus\Omega,$$

for some thin set Ω.

The system $\mathcal{K} = \{k_w\}_{w\in\mathcal{Z}_0}$ is a complete and minimal system in $\mathcal{F}$, and the system $\{g_w\}_{w\in\mathcal{Z}_0}$, where $g_w = \sigma_0/(\sigma_0'(w)(\cdot - w))$, is biorthogonal to $\mathcal{K}$ (see [3,6]).

Lemmas 2.3 and 4.1 of [3] give us the following result:

Lemma 3.3. *Let $F_1, F_2 \in \mathcal{F}$. Define*

$$c_w = \langle F_2, k_w\rangle \cdot \langle g_w, F_1\rangle = \frac{F_2(w)}{\sigma_0'(w)}\Bigl\langle \frac{\sigma_0}{\cdot - w}, F_1\Bigr\rangle, \quad w \in \mathcal{Z}_0.$$

Then

$$\sum_{w\in\mathcal{Z}_0} \frac{|c_w|^2}{\log(1+|w|)} < \infty, \tag{3.1}$$

and for every $\mu \in Z(F_2) \setminus \mathcal{Z}_0$ we have

$$\sum_{w\in\mathcal{Z}_0} c_w\left[\frac{1}{z-w}+\frac{1}{w-\mu}\right] = \mathcal{I}(F_1,F_2)(z)+\left\langle\frac{F_2}{\cdot-\mu},F_1\right\rangle,\quad z\in\mathbb{C}\setminus\mathcal{Z}_0,$$

with the series converging absolutely in $\mathbb{C}\setminus\mathcal{Z}_0$.

The following lemma establishes some relations between orthogonality in the Fock space and the corresponding discrete Cauchy transform.

Lemma 3.4. *Let $F_2, F_3 \in \mathcal{E}_2$ and $F_1, F_2F_3 \in \mathcal{F}$, and let F_3 be of completely regular growth with* $\inf_{[0,2\pi]} h_{F_3} = \eta > 0$. *Suppose that*

$$F_1 \perp \frac{F_2F_3}{\cdot-\lambda},\quad \lambda\in Z(F_3),$$

and define

$$d_w = \langle g_w, F_1\rangle = \frac{1}{\sigma_0'(w)}\left\langle\frac{\sigma_0}{\cdot-w},F_1\right\rangle,\quad w\in\mathcal{Z}_0.$$

Fix distinct $\lambda_1,\lambda_2\in Z(F_3)$ and set

$$\mathcal{C}(z)=\sum_{w\in\mathcal{Z}_0}\frac{d_wF_2(w)F_3(w)}{(z-w)(w-\lambda_1)(w-\lambda_2)}.$$

Then for every $\varepsilon>0$,

$$\mathcal{C}(z)=o(1),\quad |z|\to\infty,\ \mathrm{dist}(z,\mathcal{Z}_0)\ge\varepsilon. \tag{3.2}$$

Set $U=\sigma_0\cdot\mathcal{I}(F_1,F_2)$. Then $U\in\mathcal{E}_{2,\pi/2-\eta}$ and

$$\sigma_0\cdot\mathcal{C}=\frac{UF_3}{(\cdot-\lambda_1)(\cdot-\lambda_2)},$$
$$U(w)=d_w\sigma_0'(w)F_2(w),\quad w\in\mathcal{Z}_0.$$

Proof. By [3, Lemma 2.3], we have

$$|d_w|\lesssim e^{-(\pi/2)|w|^2}\log^{1/2}(2+|w|),\quad w\in\mathcal{Z}_0,$$

and hence

$$\sum_{w\in\mathcal{Z}_0}\frac{|F_2(w)F_3(w)|\cdot|d_w|}{|w|^2}<\infty.$$

This implies (3.2).

By the simple argument in [6, proof of Lemma 3.1], for any distinct $\lambda_1,\lambda_2,\lambda_3\in Z(F_3)$ we have

$$0=\left\langle\frac{F_2F_3}{(\cdot-\lambda_1)(\cdot-\lambda_2)(\cdot-\lambda_3)},F_1\right\rangle=\sum_{w\in\mathcal{Z}_0}\frac{d_wF_2(w)F_3(w)}{(w-\lambda_1)(w-\lambda_2)(w-\lambda_3)}.$$

Hence, for fixed $\lambda_1, \lambda_2 \in Z(F_3)$ we obtain

$$\mathcal{C}(z) = \sum_{w \in \mathcal{Z}_0} \frac{d_w F_2(w) F_3(w)}{(z-w)(w-\lambda_1)(w-\lambda_2)} = \frac{F_3(z) U(z)}{\sigma_0(z)(z-\lambda_1)(z-\lambda_2)}$$

for some entire function U. Next, since $\eta = \inf_{[0,2\pi]} h_{F_3} > 0$, we have $U \in \mathcal{E}_{2,\pi/2-\eta}$. Comparing the residues, we conclude that $U(w) = d_w \sigma_0'(w) F_2(w)$ for $w \in \mathcal{Z}_0$.

Finally, set

$$T = U - \sigma_0 \cdot \mathcal{I}(F_1, F_2).$$

Then $T \in \mathcal{E}$, and by Lemma 3.3, T vanishes on $\mathcal{Z}_0$. Set $\widetilde{T} = T/\sigma_0$. We have $\widetilde{T} \in \mathcal{E}$, and by Lemma 3.1, we deduce that $\widetilde{T}$ is of at most polynomial growth. Lemma 3.2 implies that $\widetilde{T} = o(1)$ as $|z| \to \infty$, $z \in \mathbb{C} \setminus \Omega$, for some thin set Ω, and hence $\widetilde{T} = 0$ and $T = 0$. ∎

Lemma 3.5. *Let $F_1, F_2 \in \mathcal{F}$, and suppose that $F_1 \not\perp F_2$, and for some $E \in \mathcal{E}$ and $P \in \mathcal{P}$ we have*

$$\mathcal{I}(F_1, F_2) = \frac{E}{\sigma_0} + P.$$

Then, given $\gamma > 0$, there exists a lattice thin set $\mathcal{Z}_1$ such that E has at least one zero in every disc $D(w, \gamma)$ with $w \in \mathcal{Z}_0 \setminus \mathcal{Z}_1$.

Proof. By Lemma 3.2,

$$\mathcal{I}(F_1, F_2)(z) = \frac{a + o(1)}{z}, \quad |z| \to \infty,\ z \in \mathbb{C} \setminus \Omega, \tag{3.3}$$

for some $a \neq 0$ and some thin set Ω. Set

$$I_1(z) = \int_{\mathbb{C}} \frac{F_2(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\, d\nu(\zeta), \quad I_2(z) = \int_{\mathbb{C}} \frac{\sigma_0(\zeta)\overline{F_1(\zeta)}}{z-\zeta}\, d\nu(\zeta).$$

so that $\mathcal{I}(F_1, F_2) = I_1 - F_2 I_2/\sigma_0$. By Lemma 3.1, for some $B < \infty$ we have

$$|I_2(z)|^2 \le B \frac{\log(2+|z|)}{1+|z|^2}, \quad z \in \mathbb{C}. \tag{3.4}$$

Let $\gamma \in (0, 1/2)$. Set

$$\mathcal{Z}_1 = \left\{ w \in \mathcal{Z}_0 : \sup_{D(w,\gamma)\setminus D(w,\gamma/2)} \left|\frac{F_2}{\sigma_0}\right|^2 \ge \frac{|a|^2}{100B \log(2+|z|)} \right\}.$$

If $w \in \mathcal{Z}_1$, then

$$\int_{D(w,1)} |F_2(\zeta)|^2\, d\nu(\zeta) \gtrsim \frac{1}{|w|^2 \log(1+|w|)},$$

and hence $\mathcal{Z}_1$ is lattice thin.

By (3.4), if $w \in \mathcal{Z} \setminus \mathcal{Z}_1$, then

$$\sup_{z \in D(w,\gamma)\setminus D(w,\gamma/2)} \left| z \frac{F_2(z)}{\sigma_0(z)} I_2(z) \right| \le \frac{|a|}{4}. \tag{3.5}$$

Next we use the fact that $\bar\partial I_1(z) = \pi F_2(z)\overline{F_1(z)}e^{-\pi|z|^2}$, and hence $\bar\partial I_1 \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C}) \cap C^\infty(\mathbb{C})$. Furthermore, ∂I_1 is the Beurling–Ahlfors transform [1, Chapter 4] of $\bar\partial I_1 \in L^2(\mathbb{C})$, and hence $\partial I_1 \in L^2(\mathbb{C})$. Set

$$\mathcal{Z}_2 = \Big\{ w \in \mathcal{Z}_0 : \int_{D(w,\gamma)\setminus D(w,\gamma/2)} |\nabla I_1(z)|^2\, dm_2(z) \ge \frac{|a|^2\gamma^2}{100|w|^2} \Big\}.$$

Since $\nabla I_1 \in L^2(\mathbb{C})$, the set $\mathcal{Z}_2$ is lattice thin. Furthermore, if $w \in \mathcal{Z} \setminus \mathcal{Z}_2$, then there exists $Q(w) \subset (\gamma/2, \gamma)$ such that $T(w) = \bigcup_{r\in Q(w)} \partial D(w,r)$ satisfies $m_2(T(w)) \ge \gamma^2$ and

$$\operatorname{osc}_{T(w)}(zI_1(z)) \le |a|/4.$$

Here and later on,

$$\operatorname{osc}_A(f) = \sup_{z_1,z_2\in A} |f(z_1) - f(z_2)|.$$

Now, if $w \in \mathcal{Z} \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2)$, then, by (3.5), we have

$$\operatorname{osc}_{T(w)}(z\mathcal{I}(F_1, F_2)(z)) \le |a|/2. \tag{3.6}$$

Set

$$\mathcal{Z}_3 = \{ w \in \mathcal{Z}_0 \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2) : |z\mathcal{I}(F_1, F_2)(z) - a| \ge 3|a|/4 \text{ for some } z \in T(w)\}.$$

By (3.3) and (3.6), the set $\mathcal{Z}_3$ is lattice thin.

Now, if $w \in \mathcal{Z}_0 \setminus (\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3)$ and $r = r(w) \in Q(w)$, then

$$|z\mathcal{I}(F_1, F_2)(z) - a| \le 3|a|/4, \quad z \in \partial D(w,r).$$

Thus the total change of the argument of E/σ_0 along $\partial D(w,r)$ is 0 (consider first the case $P = 0$, then the case $P \ne 0$), and hence E has one zero in $D(w,r)$.

Finally, E has at least one zero in every disc $D(w,\gamma)$ for $w \in \mathcal{Z}_0$ outside a lattice thin set. ∎

4. Proof of Theorem 1.3

We start this section with four lemmas dealing with the closed polynomial span $[F]_{\mathcal{F}}$ of $F \in \mathcal{F}$. Then we pass to the proof of the theorem.

Lemma 4.1. *Let $F \in \mathcal{E}$, and suppose that for every $A > 0$, the function F satisfies (1.1). Then*

$$\overline{\operatorname{Span}}\,\{e_\lambda F : \lambda \in \mathbb{C}\} = [F]_{\mathcal{F}}.$$

Proof. By (1.1), we have

$$\int_{\mathbb{C}} |F(z)|^2 \Big(\sum_{k\ge 0} \frac{|\lambda z|^k}{k!} \Big)^2 d\nu(z) < \infty, \quad \lambda \in \mathbb{C}.$$

Therefore,

$$e_\lambda F \in [F]_{\mathcal{F}}, \quad \lambda \in \mathbb{C}.$$

In the opposite direction, let $H \in \mathcal{F}$ be orthogonal to all $e_\lambda F$, $\lambda \in \mathbb{C}$. Set

$$a_n = \int_{\mathbb{C}} \zeta^n F(\zeta)\overline{H(\zeta)}\, d\nu(\zeta), \quad n \geq 0.$$

By (1.1), the series $\sum_{n\geq 0} a_n z^n/n!$ converges in the whole plane and equals the zero function. Hence, $a_n = 0$ for $n \geq 0$. By the Hahn–Banach theorem, we conclude that

$$\mathcal{P}F \subset \overline{\operatorname{Span}}\{e_\lambda F : \lambda \in \mathbb{C}\}. \qquad \blacksquare$$

Lemma 4.2. *Let $F \in \mathcal{F}_0$ and $H \in [F]_{\mathcal{F}}$. Then for every $\lambda \in Z(H) \setminus Z(F)$ we have*

$$\frac{H}{\cdot - \lambda} \in [F]_{\mathcal{F}}.$$

Proof. If $P \in \mathcal{P}$ and $\|H - PF\| < \varepsilon$, then

$$|P(\lambda)| \leq \varepsilon \cdot C(F, \lambda).$$

Hence,

$$\|H - (P - P(\lambda))F\| \leq \varepsilon \cdot C_1(F, \lambda),$$
$$\left\| \frac{H}{\cdot - \lambda} - \frac{P - P(\lambda)}{\cdot - \lambda} F \right\| \leq \varepsilon \cdot C_2(F, \lambda).$$

Thus, $H/(\cdot - \lambda)$ can be approximated by elements in $[F]_{\mathcal{F}}$. $\blacksquare$

Lemma 4.3. *Let $F \in \mathcal{F}_0$ and let $H \in \mathcal{E}_2$ be of completely regular growth. Suppose that $\inf_{[0,2\pi]} h_H > 0$, H has simple zeros, $Z(F) \cap Z(H) = \emptyset$, and $FH \in [F]_{\mathcal{F}}$. Next, let $W \in \mathcal{E}$ be such that $FW \in \mathcal{F}$ and*

$$FW \perp [F]_{\mathcal{F}}.$$

Then

$$H(z) \int_{\mathbb{C}} \frac{F(\zeta)\overline{F(\zeta)W(\zeta)}}{z - \zeta}\, d\nu(\zeta) = \int_{\mathbb{C}} \frac{F(\zeta)H(\zeta)\overline{F(\zeta)W(\zeta)}}{z - \zeta}\, d\nu(\zeta). \tag{4.1}$$

Proof. Denote

$$A(z) = \int_{\mathbb{C}} \frac{F(\zeta)\overline{F(\zeta)W(\zeta)}}{z - \zeta}\, d\nu(\zeta), \quad B(z) = \int_{\mathbb{C}} \frac{F(\zeta)H(\zeta)\overline{F(\zeta)W(\zeta)}}{z - \zeta}\, d\nu(\zeta).$$

Since $\bar\partial(AH - B) = 0$, $AH - B$ is an entire function; this function vanishes on $Z(H)$ because of Lemma 4.2. Denote $T = A - B/H$. We have $T \in \mathcal{E}$. Applying Lemma 3.1 to A and B we find that $\max_{\theta\in[0,2\pi]} |T(re^{i\theta})| \to 0$ as r tends to ∞, outside a set of r of zero relative measure, and hence $T = 0$. $\blacksquare$

Lemma 4.4. *Let $F \in \mathcal{F}_0$ and let $H \in \mathcal{E}_2$ be of completely regular growth. Suppose that $\eta = \inf_{[0,2\pi]} h_H > 0$, H has simple zeros, $Z(F) \cap Z(H) = \emptyset$, $(FH \cdot \mathcal{E}) \cap \mathcal{F} = FH \cdot \mathbb{C}$, and $FH \in [F]_{\mathcal{F}}$. Then*

$$[F]_{\mathcal{F}} = \mathcal{E} F \cap \mathcal{F}. \tag{4.2}$$

Proof. Without loss of generality, we can assume that F has infinitely many zeros. Shifting F and H by an operator $\mathcal{T}_\alpha$ if necessary, we can assume that $Z(FH) \cap \mathcal{Z}_0 = \emptyset$. Suppose that (4.2) does not hold and choose $V \in \mathcal{E} \setminus \{0\}$ such that $FV \in \mathcal{F}$ and

$$FV \perp [F]_{\mathcal{F}}.$$

By Lemma 4.2,

$$FV \perp \frac{FH}{\cdot - \lambda}, \qquad \lambda \in Z(H).$$

Set

$$a_w = F(w)V(w), \quad d_w = \frac{1}{\sigma_0'(w)} \left\langle \frac{\sigma_0}{\cdot - w}, FV \right\rangle, \quad w \in \mathcal{Z}_0.$$

Fix distinct $\lambda_1, \lambda_2 \in Z(H)$ and set $U = \sigma_0 \cdot \mathcal{I}(FV, F)$. By Lemma 3.4 (applied to $F_1 = FV$, $F_2 = F$, $F_3 = H$), we have $U \in \mathcal{E}_{2,\pi/2-\eta}$, and

$$\mathcal{C}_1(z) = \sum_{w \in \mathcal{Z}_0} \frac{d_w F(w) H(w)}{(z-w)(w-\lambda_1)(w-\lambda_2)} = \frac{H(z)U(z)}{\sigma_0(z)(z-\lambda_1)(z-\lambda_2)}.$$

Furthermore,

$$U(w) = d_w \sigma_0'(w) F(w), \quad w \in \mathcal{Z}_0,$$

and for every $\varepsilon > 0$,

$$\mathcal{C}_1(z) = o(1), \quad |z| \to \infty, \ \mathrm{dist}(z, \mathcal{Z}_0) \ge \varepsilon. \tag{4.3}$$

Fix $\mu \in Z(FV)$ and set

$$\mathcal{C}_2(z) = \sum_{w \in \mathcal{Z}_0} a_w d_w \left[\frac{1}{z-w} + \frac{1}{w-\mu} \right],$$

$$R = \frac{UV}{\sigma_0} - \mathcal{C}_2. \tag{4.4}$$

By Lemma 3.3 (applied to $F_1 = F_2 = FV$), the series $\mathcal{C}_2$ converges absolutely in $\mathbb{C} \setminus \mathcal{Z}_0$. By (3.1), for every $\varepsilon > 0$,

$$\mathcal{C}_2(z) = o(\log|z|), \quad |z| \to \infty, \ \mathrm{dist}(z, \mathcal{Z}_0) \ge \varepsilon. \tag{4.5}$$

Comparing the residues, we conclude that $R \in \mathcal{E}$.

Next, choose distinct $\mu_1, \mu_2 \in Z(F)$ and write the Lagrange interpolation formula

$$\mathcal{C}_3(z) = \sum_{w \in \mathcal{Z}_0} \frac{F(w)V(w)}{\sigma_0'(w)(z-w)(w-\mu_1)(w-\mu_2)} = \frac{F(z)V(z)}{\sigma_0(z)(z-\mu_1)(z-\mu_2)}.$$

Since $FV \in \mathcal{F}$, the series converges absolutely in $\mathbb{C} \setminus (\mathcal{Z}_0 \cup \{\mu_1, \mu_2\})$, and for every $\varepsilon > 0$,

$$\mathcal{C}_3(z) = o(1), \quad |z| \to \infty, \text{ dist}(z, \mathcal{Z}_0) \geq \varepsilon. \tag{4.6}$$

Now,

$$\begin{aligned}\mathcal{C}_1(z)\mathcal{C}_3(z) &= \frac{H(z)U(z)}{\sigma_0(z)(z-\lambda_1)(z-\lambda_2)} \cdot \frac{F(z)V(z)}{\sigma_0(z)(z-\mu_1)(z-\mu_2)} \\ &= \frac{F(z)H(z)}{\sigma_0(z)} \cdot \frac{R(z)+\mathcal{C}_2(z)}{(z-\lambda_1)(z-\lambda_2)(z-\mu_1)(z-\mu_2)}.\end{aligned}$$

By (4.3), (4.5), (4.6), and the maximum principle, FHR belongs to $\mathcal{P}\cdot\mathcal{F}$. Furthermore, $FH(R-1) \in \mathcal{P}\cdot\mathcal{F}$. Since the only entire multiples of FH in $\mathcal{F}$ are the constant ones, we conclude that R is a polynomial.

By Lemma 3.3 (applied to $F_1 = F_2 = FV$) and (4.4) we have

$$\frac{UV}{\sigma_0} = \mathcal{I}(FV, FV) + \int_{\mathbb{C}} \frac{|F(\zeta)V(\zeta)|^2}{\zeta - \mu}\, d\nu(\zeta) + R.$$

By Lemma 3.5 (applied to $F_1 = F_2 = FV$), we see that UV has at least one zero in every disc $D(w, 1/10)$ for $w \in \mathcal{Z}_0$ outside a lattice thin set. Since R is a polynomial, (4.4) and (4.5) show that $UV \in \mathcal{P}\cdot\mathcal{F}$. Since $U \in \mathcal{E}_{2,\pi/2-\eta}$, a subset of $Z(V)$ of positive lower density is contained in $\bigcup_{w\in\mathcal{Z}_0} D(w, 1/10)$. Repeating the above argument for $\mathcal{T}_{1/2}(F)$ and $\mathcal{T}_{1/2}(V)$, we obtain $U_1 \in \mathcal{E}_{2,\pi/2-\eta}$ such that $W = \mathcal{T}_{1/2}(V)U_1$ has at least one zero in every disc $D(w, 1/10)$ for $w \in \mathcal{Z}_0$ outside a lattice thin set. Since $W \in \mathcal{P}\cdot\mathcal{F}$, and a subset of $Z(W)$ of positive lower density is contained in $\mathbb{C} \setminus \bigcup_{w\in\mathcal{Z}_0} D(w, 1/10)$, we obtain a contradiction. Thus, relation (4.2) does hold. ∎

Proof of Theorem 1.3. Suppose that

$$\overline{\text{Span}}\,\{e_\lambda F : \lambda \in \mathbb{C}\} \neq \mathcal{E}F \cap \mathcal{F}.$$

Set $V(z) = G(z)\sigma((1-\alpha)^{1/2}z)$. We have $V \in \mathcal{E}_{2,q}$ for some $q \geq 1$. Furthermore, $FV \in \mathcal{F}$. Without loss of generality, we can assume that FV has simple zeros: otherwise, we can shift the zeros of F and G a bit without changing our hypothesis and conclusions. By Lemmas 4.1 and 4.4,

$$FV \notin [F]_{\mathcal{F}}.$$

Next, let

$$V_s(z) = V(sz), \quad 0 < s \leq 1.$$

Since $F \in \mathcal{E}_{2,\pi/2-\beta}$ for some $\beta > 0$, $FV \in \mathcal{F}$, and V is of completely regular growth with $\inf_{[0,2\pi]} h_V > 0$, we find that $FV_s \in \mathcal{F}$, $0 < s < 1$.

Let $0 < \eta < \eta_1 < \sqrt{\beta/q}$ and $0 < t \leq \eta$. Let P_n, $n \geq 0$, be the n-th partial sum of the Taylor series of V_t. Then

$$\sup_{z\in\mathbb{C}} |P_n(z) - V_t(z)| e^{-\eta_1^2 q|z|^2} \to 0, \quad n \to \infty.$$

Hence, $P_n F \to V_t F$ in $\mathcal{F}$ as $n \to \infty$. Thus,

$$FV_t \in [F]_{\mathcal{F}}, \quad 0 < t \le \eta.$$

Hence, there exist $\delta \in (0, \min(1-\eta, \eta^2(1-\alpha)/(2q)))$ and $s \in [\eta, 1-\delta]$ such that

$$FV_s \in [F]_{\mathcal{F}}, \quad FV_{s+\delta} \notin [F]_{\mathcal{F}}.$$

Once again, without loss of generality, we can assume that FV_s has simple zeros and $Z(FV_{s+\delta}) \cap \mathcal{Z}_0 = \emptyset$.

Choose $W \in \mathcal{E}$ such that

$$FW \in \mathcal{F}, \quad FW \perp [F]_{\mathcal{F}}, \quad FW \not\perp FV_{s+\delta}.$$

By Lemma 4.2,

$$FW \perp \frac{FV_s}{\cdot - \lambda}, \quad \lambda \in Z(V_s). \tag{4.7}$$

Set

$$\begin{aligned} a_w &= F(w)V_{s+\delta}(w), \\ d_w &= \frac{1}{\sigma_0'(w)} \Big\langle \frac{\sigma_0}{\cdot - w}, FW \Big\rangle, \quad w \in \mathcal{Z}_0. \end{aligned}$$

Furthermore, set

$$U = \sigma_0 \cdot \mathcal{I}(FW, F). \tag{4.8}$$

By (4.7) and by Lemma 3.4 (applied to $F_1 = FW$, $F_2 = F$, $F_3 = V_s$), we have $U \in \mathcal{E}_{2,\pi\alpha/2}$ and $U(w) = d_w \sigma_0'(w) F(w)$ for $w \in \mathcal{Z}_0$.

Fix $\mu \in Z(FV_{s+\delta}) \setminus \mathcal{Z}_0$ and define

$$\mathcal{C}(z) = \sum_{w \in \mathcal{Z}_0} a_w d_w \Big[\frac{1}{z-w} + \frac{1}{w-\mu} \Big],$$

$$S = \frac{UV_{s+\delta}}{\sigma_0} - \mathcal{C}. \tag{4.9}$$

Comparing the residues we see that $S \in \mathcal{E}$. By Lemma 3.3 (applied to $F_1 = FW$, $F_2 = FV_{s+\delta}$), the series defining $\mathcal{C}$ converges absolutely in $\mathbb{C} \setminus \mathcal{Z}_0$, and

$$\frac{UV_{s+\delta}}{\sigma_0} = \mathcal{I}(FW, FV_{s+\delta}) + \Big\langle \frac{FV_{s+\delta}}{\cdot - \mu}, FW \Big\rangle + S.$$

As in the proof of Lemma 4.4, we deduce that for every $\varepsilon > 0$,

$$\mathcal{C}_2(z) = o(\log|z|), \quad |z| \to \infty, \ \mathrm{dist}(z, \mathcal{Z}_0) \ge \varepsilon. \tag{4.10}$$

By (4.8),

$$\frac{UV_{s+\delta}}{\sigma_0} = V_{s+\delta} \cdot \mathcal{I}(FW, F).$$

Hence,

$$S(z) = -\int_{\mathbb{C}} \frac{F(\zeta)V_{s+\delta}(\zeta)\overline{F(\zeta)W(\zeta)}}{z-\zeta}\,d\nu(\zeta) + V_{s+\delta}(z)\int_{\mathbb{C}} \frac{F(\zeta)\overline{F(\zeta)W(\zeta)}}{z-\zeta}\,d\nu(\zeta) - \left\langle \frac{FV_{s+\delta}}{\cdot-\mu}, FW\right\rangle. \qquad (4.11)$$

By Lemma 4.3 applied with $H = V_s$ we have

$$S(z) = -\int_{\mathbb{C}} \frac{F(\zeta)V_{s+\delta}(\zeta)\overline{F(\zeta)W(\zeta)}}{z-\zeta}\,d\nu(\zeta) - \left\langle \frac{FV_{s+\delta}}{\cdot-\mu}, FW\right\rangle + \frac{V_{s+\delta}(z)}{V_s(z)}\int_{\mathbb{C}} \frac{F(\zeta)V_s(\zeta)\overline{F(\zeta)W(\zeta)}}{z-\zeta}\,d\nu(\zeta).$$

Therefore, $S \in \mathcal{E}_{2,q(s+\delta)^2-qs^2} \subset \mathcal{E}_{2,2q\delta}$. Hence, $\widetilde{S} = S_{(2q\delta)^{-1/2}} \in \mathcal{F}$.

Next, by (4.11) and by Lemma 3.1, S is bounded on $Z(V_{s+\delta})$, and hence $\widetilde{S}$ is bounded on a lattice of density at least $\eta^2(1-\alpha)/(2q\delta) > 1$. By Lemma 2.3, $\widetilde{S} = S$ is a constant. Thus,

$$\frac{UV_{s+\delta}}{\sigma_0} = \mathcal{I}(FW, FV_{s+\delta}) + \left\langle \frac{FV_{s+\delta}}{\cdot-\mu}, FW\right\rangle + S(0).$$

By (4.9) and (4.10), $UV_{s+\delta} \in \mathcal{P}\cdot\mathcal{F}$. By Lemma 3.5 (applied to FW and $FV_{s+\delta}$), we conclude that $UV_{s+\delta}$ has at least one zero in every disc $D(w, 1/10)$ for $w \in \mathcal{Z}_0$ outside a lattice thin set. However, shifting F and $V_{s+\delta}$ we see that a subset of $Z(V_{s+\delta})$ of positive density is contained in $\mathbb{C} \setminus \bigcup_{w\in\mathcal{Z}_0} D(w, 1/10)$. We get a contradiction, which completes the proof. ∎

5. Proof of Theorem 1.5 and Corollary 1.4

Proof of Corollary 1.4. Choose an integer N such that

$$\sup_{z\in\mathbb{C}} \operatorname{card}(Z(F)\cap Q_{z,N}) < N^2,$$

where $Q_{z,N}$ is the square centered at z of sidelength $\sqrt{\pi}\,N$. Let $M \in N\mathbb{N}$ be a number to be chosen later. Choose $\Lambda \subset \mathbb{C}$ disjoint from $Z(F)$ in such a way that

$$\operatorname{card}(\Gamma_z) = \alpha M^2, \quad z \in \mathcal{Z}^M = M\mathbb{Z}\times M\mathbb{Z},$$

for some $\alpha < 1$, where $\Gamma_z = (Z(F)\cup\Lambda)\cap Q_{z,M}$. Without loss of generality $Z(F)\cup\Lambda$ is disjoint from $\bigcup_{z\in\mathcal{Z}^M}\partial Q_{z,M}$.

We can choose large M and place the points of Λ in such a way that the measures $\alpha M^2\delta_z - \sum_{\lambda\in\Gamma_z}\delta_\lambda$, $z \in \mathcal{Z}^M$, have the first three moments equal to 0. Then, arguing as in [8, Section 4.4] (see also [18]), we conclude that the canonical product H corresponding to the set $Z(F)\cup\Lambda$ satisfies the estimate

$$\operatorname{dist}(w, Z(H))^B e^{-\alpha|w|^2/2} \lesssim |H(w)| \lesssim e^{-\alpha|w|^2/2}, \quad w \in \mathbb{C}, \qquad (5.1)$$

for some $B > 0$. Indeed, choose $w \in \mathbb{C}$ and $z \in \mathcal{Z}^M$, and denote $L(\zeta) = \log(1 - w/\zeta)$. We have $\alpha M^2 = \operatorname{card} \Gamma_z$ and

$$\Big|\sum_{\lambda\in\Gamma_z}\Big(\log\Big|1-\frac{w}{\lambda}\Big|-\log\Big|1-\frac{w}{z}\Big|\Big)\Big|$$
$$\le |L'(z)|\cdot\Big|\sum_{\lambda\in\Gamma_z}(z-\lambda)\Big|+\frac{1}{2}|L''(z)|\cdot\Big|\sum_{\lambda\in\Gamma_z}(z-\lambda)^2\Big|+C\cdot\Big(\frac{1}{|w-z|^3}+\frac{1}{|z|^3}\Big).$$

Since the first two sums on the right hand side are zero, summing over $z \in \mathcal{Z}^M$ we arrive at (5.1).

Choose $\lambda_1, \lambda_2 \in \Lambda$ and set $H_1 = H/[(\cdot - \lambda_1)(\cdot - \lambda_2)]$ and $G = H_1/F$. Then G is of completely regular growth, $h_G \equiv c > 0$, and $(FG \cdot \mathcal{E}) \cap \mathcal{F}_{(\alpha)} = FG \cdot \mathbb{C}$. It remains to apply Theorem 1.3. ■

Lemma 5.1. *There exists $K \in (1, \infty)$ such that if $\eta > 0$, $F \in \mathcal{E}_{2,\eta}$, $V \in \mathcal{E}$ and $FV \in \mathcal{F}$, then $V \in \mathcal{E}_{2,\pi/2+K\eta}$.*

Proof. See [17, Chapter I, Sections 8, 9]. ■

Proof of Theorem 1.5. Suppose that $\eta < \pi\beta/(8K)$, where $\beta \in (0, 1)$ is as in the statement of Lemma 2.2 and K is as in the statement of Lemma 5.1. Choose $\varepsilon \in (2K\eta/\pi, \beta/4)$ and set $H(z) = \sigma((1 - \varepsilon)^{1/2} z)$. Clearly, $FH \in [F]_{\mathcal{F}}$. Without loss of generality, we can assume that $Z(H) \cap Z(F) = \emptyset$.

Suppose that the claim of the theorem does not hold. Choose $V \in \mathcal{E}$ such that $FV \in \mathcal{F}$ and $FV \perp [F]$ and set $U = \sigma_0 \cdot \mathcal{I}(FV, F)$. By Lemma 3.4 (applied to $F_1 = FV$, $F_2 = F$, $F_3 = H$), we get $U \in \mathcal{E}_{2,\pi\varepsilon/2}$. Next, arguing as in the proof of Lemma 4.4, we obtain

$$\frac{UV}{\sigma_0} = \mathcal{I}(FV, FV) + S$$

for some $S \in \mathcal{E}$. Since $V \in \mathcal{E}_{2,\pi/2+K\eta}$, we have $S \in \mathcal{E}_{2,\pi\varepsilon}$.

Replacing F and V by $F_\alpha = \mathcal{T}_\alpha(F)$ and $V_\alpha = \mathcal{T}_\alpha(V)$, respectively, we find

$$\frac{U_\alpha V_\alpha}{\sigma_0} = \mathcal{I}(F_\alpha V_\alpha, F_\alpha V_\alpha) + S_\alpha.$$

If S_α and S_{α_1} are polynomials for some $\alpha, \alpha_1 \in \mathbb{C}$ such that $\alpha - \alpha_1 \notin \mathcal{Z}$, then we choose $\gamma > 0$ such that $D(\alpha - \alpha_1, \gamma) \cap \mathcal{Z} = \emptyset$. By Lemma 3.5 we find that both $U_\alpha V_\alpha$ and $U_{\alpha_1} V_{\alpha_1}$ have at least one zero in every disc $D(w, \gamma)$ for $w \in \mathcal{Z}$ outside a lattice thin set. Then the lower density of the set

$$\{w \in \mathcal{Z} : D(w + \alpha, \gamma) \cap Z(V) \neq \emptyset\}$$

is at least $3/4$. The same is true with α replaced by β. This contradicts the fact that the upper density of $Z(V)$ is at most $5/4$.

Thus, for some α_0 and every $\alpha \in \mathbb{C} \setminus (\alpha_0 + \mathcal{Z})$, S_α is not a polynomial. We can find $\alpha \notin \alpha_0 + \mathcal{Z}$ and $\mathcal{Z}_1 \subset \mathcal{Z}_0$ of lower density at least $1 - \varepsilon$ such that V_α has no zeros in

$\bigcup_{z\in\mathcal{Z}_1} D(z, \log^{-2}(1+|z|))$. Accordingly, for some $\mathcal{Z}_2 \subset \mathcal{Z}$ of lower density at least $1-2\varepsilon$ the function $U_\alpha V_\alpha$ has no zeros in $\bigcup_{z\in\mathcal{Z}_2} D(z, \log^{-2}(1+|z|))$. By Lemma 3.1, $U_\alpha V_\alpha/\sigma_0 - S_\alpha$ has at most polynomial growth. By the Rouché theorem, for some $C>0$ and $N<\infty$ we obtain

$$\inf_{\partial D(z,\rho\log^{-2}(1+|z|))} |S_\alpha| < C|z|^N$$

for every $0<\rho<1$ and every $z\in\mathcal{Z}_2$. Dividing S_α by several zeros (and possibly adding a polynomial), we arrive at the conditions of Lemma 2.2. Thus, we conclude that S_α is a polynomial. This contradiction completes the proof. ∎

6. Proof of Theorem 1.2

The main idea of the proof goes back to [10].

Step 1: Construction of F. We choose β such that

$$1<\alpha<\beta<2.$$

Consider the function

$$f(z)=\exp\Big(\frac{\pi}{2}z^2-z^\beta\Big),\qquad z\in\Omega=\{re^{i\theta}: r>0,\ |\theta|\le\pi/4\},$$

with the principal branch $z^\beta(1)=1$. The function f is bounded on $\partial\Omega$. Moreover,

$$\begin{aligned}
\log|f(re^{i\theta})| &= \frac{\pi}{2}\cos(2\theta)r^2-\cos(\beta\theta)r^\beta, && re^{i\theta}\in\Omega,\\
\log|f(x+iy)| &= \frac{\pi}{2}x^2-x^\beta+O(1), && x\to\infty,\ |y|\le 1,\\
\log|f(re^{\pm i\pi/4})| &= -\cos\Big(\frac{\pi\beta}{4}\Big)r^\beta, && r>0,\\
\log|f'(re^{\pm i\pi/4})| &= -\Big(\cos\Big(\frac{\pi\beta}{4}\Big)+o(1)\Big)r^\beta, && r\to\infty.
\end{aligned}$$

Next, set

$$f_1(z)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(w)\,dw}{z-w},\qquad z\in\mathbb{C}\setminus\overline{\Omega}.$$

It is well known that f_1 extends to an entire function. Indeed, f_1 is analytic in $\mathbb{C}\setminus\overline{\Omega}$. Put

$$f_1^R(z)=\frac{1}{2\pi i}\int_{\partial\Omega_R}\frac{f(w)\,dw}{z-w},\qquad z\in\mathbb{C}\setminus\overline{\Omega_R},$$
$$\Omega_R=\Omega\cap\{|z|>R\}.$$

The function f_1^R is an analytic continuation of f_1 to $\mathbb{C}\setminus\overline{\Omega_R}$. Thus, when $R\to\infty$, f_1 extends to an entire function.

By the Sokhotskiĭ–Plemelj theorem we get

$$|f_1(re^{i\theta})| = \exp\left[\frac{\pi}{2}\cos(2\theta)r^2 - \cos(\beta\theta)r^\beta\right] + O(1), \quad re^{i\theta} \in \Omega,\ r \to \infty, \tag{6.1}$$

$$|f_1(x+iy)| = \exp\left(\frac{\pi}{2}x^2 - x^\beta\right) + O(1), \quad x \to \infty,\ |y| \le 1. \tag{6.2}$$

We fix δ such that $\beta < \delta < 2$ and define

$$F(z) = f_1(e^{-i\pi/(2\delta)}z) + f_1(e^{i\pi/(2\delta)}z).$$

Claim 6.1. *For every entire function h of order at most α we have*

$$hF \in \mathcal{F}.$$

Proof. By (6.1),

$$\log|F(re^{i\theta}| \le \begin{cases} \frac{\pi}{2}(r^2 - r^\beta) + O(1), & \theta \in J, \\ \frac{\pi}{2}\cos(1/5)r^2, & \theta \notin J, \end{cases}$$

where

$$J = \left[-\frac{\pi}{2\delta} - \frac{1}{10}, -\frac{\pi}{2\delta} + \frac{1}{10}\right] \cup \left[\frac{\pi}{2\delta} - \frac{1}{10}, \frac{\pi}{2\delta} + \frac{1}{10}\right].$$

Therefore,

$$\int_{\mathbb{C}} |hF(z)|^2 e^{-\pi|z|^2}\, dm_2(z) < \infty.$$

■

Step 2: Key estimate. Choose $\gamma \in (\beta, \delta)$.

Claim 6.2. *Let $\mathcal{P}_1$ be the family of polynomials P such that*

$$\|PF\|_{\mathcal{F}} \le 1. \tag{6.3}$$

Then for some $C > 0$ we have

$$\sup_{P\in\mathcal{P}_1} |P(x)| \le C\exp(x^\gamma), \quad x \ge 0.$$

Proof. The estimates (6.2) and (6.3) yield

$$\int_{x>0,\,|y|\le 1} |P((x+iy)e^{i\pi/(2\delta)})|^2 e^{-2x^\beta}\, dx\, dy \le C, \quad P \in \mathcal{P}_1.$$

In the same way,

$$\int_{x>0,\,|y|\le 1} |P((x+iy)e^{-i\pi/(2\delta)})|^2 e^{-2x^\beta}\, dx\, dy \le C, \quad P \in \mathcal{P}_1.$$

By the Fubini theorem, for every $P \in \mathcal{P}_1$ we can find $y(P) \in [-1, 1]$ such that

$$\int_0^\infty |P((x \pm iy(P))e^{\pm i\pi/(2\delta)})|^2 e^{-2x^\beta}\, dx \le C_1.$$

Since point evaluations are locally uniformly bounded in the Fock space, by the maximum principle we obtain

$$\sup_{P\in\mathcal{P}_1,\,|z|\le 2} |P(z)| \le C_2.$$

Note that the lines $\{(x+iy(P))e^{i\pi/(2\delta)} : x\in\mathbb{R}\}$ and $\{(x-iy(P))e^{-i\pi/(2\delta)} : x\in\mathbb{R}\}$ intersect at the point $-y(P)/\sin\frac{\pi}{2\delta}$. Therefore, if we set

$$Q(z) = P\biggl(z - \frac{y(P)}{\sin\frac{\pi}{2\delta}}\biggr),$$

then we have

$$\int_0^\infty |Q(te^{\pm i\pi/(2\delta)})|^2 e^{-2t^\beta}\,dt \le C_3.$$

Put

$$Q_1(re^{i\theta}) = Q(r^{1/\delta}e^{i\theta/\delta})\exp\bigl[-\tfrac12 r^{\gamma/\delta}e^{i\theta\gamma/\delta}\bigr],\quad r\ge 0,\ |\theta|\le\pi/2.$$

Then Q_1 is bounded and analytic in the right half-plane, and

$$\int_{\mathbb{R}} |Q_1(iy)|^2\,dy \le C_4.$$

Therefore,

$$|Q_1(x)| \le C_5,\quad x\ge 1,$$

and as a result,

$$|P(x)| \le C_6\exp(x^\gamma),\quad x\ge 0. \qquad \blacksquare$$

Step 3: Construction of G. Next we fix σ and η such that $\delta<\eta<\sigma<2$. We consider the function

$$g(z) = \exp(z^\sigma),\quad z\in\Omega_1 = \biggl\{re^{i\theta} : r>0,\ |\theta|\le\frac{\pi}{2\eta}\biggr\},$$

with the principal branch $z^\sigma(1)=1$. Then

$$\begin{aligned}
\log|g(re^{i\theta})| &= \cos(\sigma\theta)r^\sigma, && re^{i\theta}\in\Omega_1,\\
\log|g(x)| &= x^\sigma, && x\ge 0,\\
\log|g(re^{\pm i\pi/(2\eta)})| &= \cos\biggl(\frac{\pi\sigma}{2\eta}\biggr)r^\sigma, && r>0,\\
\log|g'(re^{\pm i\pi/(2\eta)})| &= \biggl(\cos\biggl(\frac{\pi\sigma}{2\eta}\biggr)+o(1)\biggr)r^\sigma, && r\to\infty.
\end{aligned}$$

Set

$$G(z) = \frac{1}{2\pi i}\int_{\partial\Omega_1}\frac{g(w)\,dw}{z-w},\quad z\in\mathbb{C}\setminus\overline{\Omega_1}.$$

Then G extends to an entire function,

$$|G(x)| = \exp(x^\sigma) + O(1),\quad x\ge 0, \tag{6.4}$$

and

$$|G(re^{i\theta})| = \begin{cases} \exp(\cos(\sigma\theta)r^{\sigma}) + O(1), & \theta \in [-\frac{\pi}{2\sigma}, \frac{\pi}{2\sigma}], \\ O(1), & \theta \in [-\pi, \pi] \setminus [-\frac{\pi}{2\sigma}, \frac{\pi}{2\sigma}]. \end{cases} \tag{6.5}$$

Claim 6.3. $FG \in \mathcal{F}$.

Proof. By (6.1) and (6.5) we have

$$\begin{aligned} &\log|(FG)(re^{i\theta})| - \frac{\pi}{2}r^2 \\ &\quad \le \chi_{[\pi/(2\delta)-\pi/4, \pi/(2\delta)+\pi/4]}(\theta)\Big[\cos\Big(2\theta - \frac{\pi}{\delta}\Big)\cdot\frac{\pi}{2}r^2 - \cos\Big(\beta\theta - \frac{\beta\pi}{2\delta}\Big)\cdot r^{\beta}\Big]^{+} \\ &\qquad + \chi_{[-\pi/(2\delta)-\pi/4, -\pi/(2\delta)+\pi/4]}(\theta)\Big[\cos\Big(2\theta + \frac{\pi}{\delta}\Big)\cdot\frac{\pi}{2}r^2 - \cos\Big(\beta\theta + \frac{\beta\pi}{2\delta}\Big)\cdot r^{\beta}\Big]^{+} \\ &\qquad + \chi_{[-\pi/(2\sigma), \pi/(2\sigma)]}(\theta)(\cos(\sigma\theta)r^{\sigma}) - \frac{\pi}{2}r^2 + O(1), \quad r \to \infty,\ \theta \in [-\pi, \pi]. \end{aligned}$$

Hence, for some $\varepsilon = \varepsilon(\sigma, \delta) > 0$ and $d = d(\beta, \sigma, \delta) > 0$, we have

$$\begin{aligned} &\log|(FG)(re^{i\theta})| - \frac{\pi}{2}r^2 \\ &\qquad \le \begin{cases} -dr^{\beta}, & \theta \in J = [\frac{\pi}{2\delta} - \varepsilon, \frac{\pi}{2\delta} + \varepsilon] \cup [-\frac{\pi}{2\delta} - \varepsilon, -\frac{\pi}{2\delta} + \varepsilon], \\ -dr^2, & \theta \in [-\pi, \pi] \setminus J, \end{cases} \end{aligned}$$

and so $FG \in \mathcal{F}$. ∎

Step 4: End of the proof. Now, we argue as in [10]. Suppose that P_n are polynomials such that

$$P_n F \xrightarrow{\mathcal{F}} FG.$$

Then for some $C_1 > 0$ we have

$$\{P_n/C_1\}_{n\ge 1} \in \mathcal{P}_1,$$

and by Claim 6.2 we get

$$|P_n(x)| \le CC_1 \exp(x^{\gamma}), \quad n \ge 1,\ x \ge 0. \tag{6.6}$$

Since $P_n F$ tend to FG uniformly on compact subsets of the complex plane, (6.6) contradicts (6.4), and

$$FG \notin \mathrm{Clos}_{\mathcal{F}}\{\mathcal{P}F\}.$$

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References

[1] Astala, K., Iwaniec, T., Martin, G.: Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane. Princeton Math. Ser. 48, Princeton Univ. Press, Princeton, NJ (2009) Zbl 1182.30001 MR 2472875

[2] Baranov, A., Belov, Y., Borichev, A.: Hereditary completeness for systems of exponentials and reproducing kernels. Adv. Math. **235**, 525–554 (2013) Zbl 1263.42016 MR 3010068

[3] Baranov, A., Belov, Y., Borichev, A.: Summability properties of Gabor expansions. J. Funct. Anal. **274**, 2532–2552 (2018) Zbl 1402.46008 MR 3771835

[4] Baranov, A., Belov, Y., Borichev, A.: Summability properties of Gabor expansions. arXiv:1706.05685v2 (2018)

[5] Baranov, A., Belov, Y., Borichev, A.: The Young type theorem in weighted Fock spaces. Bull. London Math. Soc. **50**, 357–363 (2018) Zbl 1395.30004 MR 3830126

[6] Belov, Y.: Uniqueness of Gabor series. Appl. Comput. Harmon. Anal. **39**, 545–551 (2015) Zbl 1325.30026 MR 3398949

[7] Borichev, A. A.: Estimates from below and cyclicity in Bergman-type spaces. Int. Math. Res. Notices **1996**, 603–611 Zbl 0864.30040 MR 1405977

[8] Borichev, A., Gröchenig, K., Lyubarskii, Y.: Frame constants of Gabor frames near the critical density. J. Math. Pures Appl. (9) **94**, 170–182 (2010) Zbl 1202.94114 MR 2665416

[9] Borichev, A., Hedenmalm, H.: Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces. J. Amer. Math. Soc. **10**, 761–796 (1997) Zbl 0963.30033 MR 1446365

[10] Borichev, A. A.: The polynomial approximation property in Fock-type spaces. Math. Scand. **82**, 256–264 (1998) Zbl 0921.46021 MR 1646509

[11] Cichoń, D., Shapiro, H. S.: Toeplitz operators in Segal–Bargmann spaces of vector-valued functions. Math. Scand. **93**, 275–296 (2003) Zbl 1064.47026 MR 2009586

[12] Ehrenpreis, L.: Fourier Analysis in Several Complex Variables. Pure Appl. Math. 17, Wiley-Interscience, New York (1970) Zbl 0195.10401 MR 0285849

[13] Fischer, E.: Über die Differentiationsprozesse der Algebra. J. Reine Angew. Math. **148**, 1–78 (1918) JFM 46.1436.02 MR 1580952

[14] Garnett, J. B., Marshall, D. E.: Harmonic Measure. New Math. Monogr. 2, Cambridge Univ. Press, Cambridge (2005) Zbl 1077.31001 MR 2150803

[15] Gröchenig, K.: Foundations of Time-Frequency Analysis. Birkhäuser, Boston, MA (2001) Zbl 0966.42020 MR 1843717

[16] Koosis, P.: The Logarithmic Integral. I. Cambridge Stud. Adv. Math. 12, Cambridge Univ. Press, Cambridge (1988) Zbl 0665.30038 MR 961844

[17] Levin, B. J.: Distribution of Zeros of Entire Functions. Rev. ed., Transl. Math. Monogr. 5, Amer. Math. Soc., Providence, RI (1980) Zbl 0152.06703 MR 589888

[18] Lyubarskii, Y., Malinnikova, E.: On approximation of subharmonic functions. J. Anal. Math. **83**, 121–149 (2001) Zbl 0981.31002 MR 1828489

[19] Metzger, J. J.: Principal local ideals in weighted spaces of entire functions. Trans. Amer. Math. Soc. **165**, 149–158 (1972) Zbl 0235.46050 MR 296700

[20] Newman, D. J., Shapiro, H. S.: Certain Hilbert spaces of entire functions. Bull. Amer. Math. Soc. **72**, 971–977 (1966) Zbl 0201.46403 MR 205055

[21] Newman, D. J., Shapiro, H. S.: A Hilbert space of entire functions related to the operational calculus. Unpublished manuscript, 92 pp. (1964)

[22] Newman, D. J., Shapiro, H. S.: Fischer spaces of entire functions. In: Entire Functions and Related Parts of Analysis (La Jolla, CA, 1966), Proc. Sympos. Pure Math. 11, Amer. Math. Soc., Providence, RI, 360–369 (1968) Zbl 0191.41501 MR 0234012