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Chern classes in precobordism theories

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Abstract. We construct Chern classes of vector bundles in the universal precobordism theory of Annala–Yokura over an arbitrary Noetherian base ring of finite Krull dimension. As an immediate corollary, we show that the Grothendieck ring of vector bundles can be recovered from the universal precobordism ring, and that we can construct candidates for Chow rings satisfying an analogue of the classical Grothendieck–Riemann–Roch theorem. We also strengthen the weak projective bundle formula of Annala–Yokura to the case of arbitrary projective bundles.

Keywords. Algebraic cobordism, derived algebraic geometry, Chern classes, projective bundle formula

1. Introduction

Algebraic cobordism Ω^* is a still conjectural cohomology theory of schemes playing the role of complex cobordism in algebraic geometry: Ω^* is supposed to provide the universal oriented (co)homology theory of schemes – a theory that is more refined than either the theory of algebraic cycles or algebraic K-theory, and which can, in fact, be used to recover the two more classical theories in a completely formal fashion. Of course, we do not expect such a theory to be easily computable: even computing $\Omega^*(\mathrm{Spec}(k))$ should amount to classifying all nice enough projective (derived) k-schemes up to cobordism, and the expected answer is Lazard's universal ring $\mathbb{L}$ rather than the integers. Instead of being computable, one could say that the role of algebraic cobordism is analogous to the role of the Grothendieck ring of varieties in the study of Euler characteristic: because the theory is so general, and because essentially the only tools we have at our disposal when studying algebraic cobordism are geometric constructions, the proofs of various formulas in algebraic cobordism should contain the geometric essence of "the" correct argument.

The first model of algebraic cobordism was the cohomology theory represented by the motivic Thom spectrum *MGL* in the stable motivic homotopy category, and it was constructed by Voevodsky for his original approach to the proof of the Milnor conjectures.

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This approach has the advantage of immediately giving rise to a bigraded cohomology theory, i.e., one immediately obtains the higher algebraic cobordism groups as well. There are, however, two serious drawbacks: it is not clear that this model has a geometric description, and moreover any theory obtained from motivic homotopy theory is $\mathbb{A}^1$-invariant. In the light of the latter issue, we point out that it seems likely that *MGL* represents the homotopy algebraic cobordism rather than algebraic cobordism, much like the motivic K-theory spectrum *KGL* represents the homotopy K-theory rather than algebraic K-theory.

Another model of algebraic cobordism was constructed by Levine and Morel in their seminal work [7]. The authors give a geometric construction of algebraic bordism Ω_* for algebraic k-schemes, where k is a field of characteristic 0. Note that Ω_* is an oriented Borel–Moore homology theory rather than an oriented cohomology theory, but by a version of Poincaré duality $\Omega_*(X)$ should recover the algebraic cobordism of X (up to changing grading conventions) whenever X is smooth. In particular, Levine and Morel give a geometric model for algebraic cobordism only for smooth varieties over a field of characteristic 0. They show that the theory Ω_* has essentially all the properties expected from it, but the construction also has some serious drawbacks. First of all, it only gives the "geometric" or "zeroth" part of algebraic cobordism, and hence sheds no light on the geometric nature of higher bordism groups. Moreover, the techniques are specific to characteristic 0, as resolution of singularities and weak factorization are used throughout the text. The groups $\Omega_*(X)$ were given an even simpler description by Levine and Pandharipande [8], and Levine [6] proved that Ω_* recovers the geometric part of the Borel–Moore homology theory represented by the motivic spectrum *MGL*.

A paradigm shift appeared in the work of Lowrey and Schürg [9]. They used derived algebraic geometry developed by Toën and Vezzosi [14, 15] and by Lurie [10–12] to construct derived bordism groups $d\Omega_*(X)$ for all quasi-projective derived k-schemes, where k is a field. They also showed that when k has characteristic 0, then $d\Omega_*(X)$ recovers in a natural fashion the algebraic bordism group $\Omega_*(\tau_0(X))$ of Levine–Morel, where $\tau_0(X)$ is the underlying classical scheme of X. Even though Lowrey and Schürg were not able to prove that their model has any of the expected properties in any case that was not known previously, their construction has the advantage that the the existence of lci pullbacks, which is by far the most difficult and subtle part of [7], is completely trivial, as they can be defined using derived fibre product. The results of Lowrey and Schürg have also served as an inspiration for some recent work in motivic homotopy theory: in [4] Elmanto et al. show, roughly speaking, that the motivic spectrum *MGL* can be represented by the moduli stack of finite quasi-smooth morphisms.

This paper fits into the the author's program of constructing a well behaved theory of algebraic cobordism as generally as possible, and proving that it satisfies all the expected properties. The first steps were taken in [1], where the author, inspired by the universal bivariant theories of Yokura [16], extended the groups of Lowrey and Schürg to a bivariant theory Ω^* called bivariant derived algebraic cobordism, which, as a special case, gave geometric models for the algebraic cobordism rings $\Omega^*(X)$ even when X is singular. Moreover, using the construction of Chern class operators for Ω_* from [7], the author was able to show that if X is quasi-projective over a field of characteristic 0, then one can recover $K^0(X)$ from $\Omega^*(X)$ in the expected way, proving also that $\Omega^*(X)$ is not $\mathbb{A}^1$-invariant, and therefore cannot be represented by a motivic spectrum. The second step was taken by Annala–Yokura [3] who constructed universal precobordism theory $\underline{\Omega}^*$ as a bivariant and derived analogue of [8], and managed to prove the projective bundle formula for trivial projective bundles. The results of this paper constitute the third, and thus far the largest, step forward in the program.

Summary of results

Let us now summarize the contributions of this article. We fix a Noetherian base ring A of finite Krull dimension, and for each quasi-projective derived A-scheme X we denote by $\underline{\Omega}^*(X)$ the universal precobordism ring of X as constructed in [3, Section 6.1]. These rings serve only as an approximation of the "zeroth part" of algebraic cobordism of X: we have reasons to believe that more relations have to be imposed on $\underline{\Omega}^*(X)$ before it deserves to be called algebraic cobordism (see [2] for more details). However, since the results we prove for $\underline{\Omega}^*$ will also hold for any nice enough quotient theory of $\underline{\Omega}^*$, we do not need to worry about the potential missing relations in this paper.

Section 3 is dedicated to proving that the Euler classes of line bundles satisfy a formal group law (Theorem 3.4), which is necessary for nearly all subsequent computations in the article. This result also allows us to characterize $\underline{\Omega}^*$ by a simple universal property (Theorem 3.11): it is the universal oriented cohomology theory on quasi-projective derived A-schemes in the sense of Definition 3.11.

Section 4 is the technical heart of the paper, and it is dedicated to constructing and studying Chern classes

$$c_i(E) \in \underline{\Omega}^i(X),$$

where E is a vector bundle on the quasi-projective derived A-scheme X. The construction is performed in Section 4.1, and the last two subsections are dedicated to showing that the Chern classes satisfy the expected properties:

Theorem (Theorems 4.7 and 4.15). *Given a rank r vector bundle E on a quasi-projective derived A-scheme X, let us denote by*

$$c(E) := 1_X + c_1(E) + \cdots + c_r(E) \in \underline{\Omega}^*(X)$$

its total Chern class. Then

(1) naturality: *if $f : Y \to X$ is a morphism of quasi-projective derived A-schemes, then*

$$f^*(c_i(E)) = c_i(f^*E) \in \underline{\Omega}^i(Y);$$

(2) normalization: $c_r(E) = e(E) \in \underline{\Omega}^r(X)$, *where $e(E)$ is the Euler class of E;*

(3) Whitney sum formula: *if $E' \to E \to E''$ is a cofibre sequence of vector bundles on X, then*

$$c(E) = c(E') \bullet c(E'') \in \underline{\Omega}^*(X).$$

Since we prove the projective bundle formula in Section 5, we have to come up with a non-standard construction of Chern classes. Our construction is based on carefully applying the derived blowups of Khan–Rydh from [5] in order to produce a natural instance where E has a filtration with line bundles as graded pieces, and then defining the Chern classes by mimicking the splitting principle. Naturality and normalization are easy consequences of the definition, but establishing the Whitney sum formula, which is the hardest part of Section 4, requires further analysis of these classes and is done at the end of Section 4.3.

Section 5 is dedicated to applications. The following two theorems, the main results of Sections 5.1 and 5.2 respectively, are standard consequences of the theory of Chern classes as established in Section 4.

Theorem (Theorem 5.4). *Let X be a quasi-projective derived A-scheme. Then the natural map*

$$\eta_K : \mathbb{Z}_{\mathrm{m}} \otimes_{\mathbb{L}} \underline{\Omega}^*(X) \cong \underline{\Omega}^0_{\mathrm{m}}(X) \to K^0(X)$$

defined by

$$[V \xrightarrow{f} X] \mapsto [f_* \mathcal{O}_V]$$

is an isomorphism of rings, which commutes with pullbacks and Gysin pushforwards. Here $\mathbb{Z}_{\mathrm{m}}$ is the integers considered as an $\mathbb{L}$-algebra via the formal group law $x + y - xy$.

Theorem (Theorem 5.9). *Let X be a quasi-projective derived A-scheme. Then there exists a Chern character morphism*

$$\mathrm{ch}_{\mathrm{a}} : K^0(X) \to \mathbb{Q} \otimes \underline{\Omega}^*_{\mathrm{a}}(X)$$

which is a ring homomorphism, commutes with pullbacks, and commutes with Gysin pushforwards up to a twist by a Todd class. Moreover, $\mathbb{Q} \otimes \mathrm{ch}_{\mathrm{a}}$ is an isomorphism.

Note that the main role of Theorem 5.4 is to serve as a sanity check: algebraic K-theory is the only oriented cohomology theory whose correct definition is known for all schemes, and therefore it provides a useful measure against which other theories can be compared. Moreover, it seems reasonable to demand that any serious construction attempting to define algebraic cobordism should at the very least satisfy an analogue of Theorem 5.4.

The most interesting application in Section 5 is the following projective bundle formula proven in Section 5.3.

Theorem (Theorem 5.11). *Let X be a quasi-projective derived A-scheme and let E be a vector bundle of rank r on X. Then*

$$\underline{\Omega}^*(\mathbb{P}(E)) \cong \underline{\Omega}^*(X)[t]/\big(c_r(E^\vee) - c_{r-1}(E^\vee)t + \cdots + (-1)^r t^r\big)$$

where $t \in \underline{\Omega}^1(\mathbb{P}(E))$ is the first Chern class of $\mathcal{O}(1)$.

Note that the standard proof of the projective bundle formula for algebraic K-theory uses properties of categories of vector bundles and is therefore specific to K-theory, while the standard argument for Chow groups uses homotopy invariance and localization, and hence is not applicable to our situation (localization is not known to hold, and $\mathbb{A}^1$-invariance is known to fail because of Theorem 5.4). Our argument is based on embedding $\underline{\Omega}^*(\mathbb{P}(E))$ into the cobordism ring with line bundles $\underline{\Omega}^{*,1}(X)$, which is a free $\underline{\Omega}^*(X)$-module with an explicit basis by [3, Theorems 6.12 and 6.13], and this reduces the problem to linear algebra over $\underline{\Omega}^*(X)$. Note that the arguments use the properties of Chern classes proven in Sections 3 and 4. We then end by quickly showing that essentially the same arguments prove a bivariant projective bundle formula for $\mathbb{B}^*$, where $\mathbb{B}^*$ is a precobordism theory in the sense of [3, Section 6.1].

Conventions

Our derived A-schemes are locally modelled by simplicial A-algebras. A quasi-projective A-scheme X over a Noetherian ring A of finite Krull dimension is implicitly assumed to be Noetherian, i.e., the homotopy sheaves $\pi_i(\mathcal{O}_X)$ should be coherent sheaves on the truncation $\tau_0(X)$. If a diagram is said to commute, it is to be understood that it commutes coherently up to homotopy. An *inclusion* or *embedding* of vector bundles $E \hookrightarrow F$ is a morphism whose dual $F^\vee \to E^\vee$ is surjective (that is, the dual is surjective on π_0).

2. Background

In this section, we are going to recall definitions and results that are necessary for the main part of the paper. As a rule, only results that are new are proven.

2.1. Projective bundles

Let X be a derived scheme and let E be a vector bundle on X. We define the *projective bundle* $\pi : \mathbb{P}_X(E) \to X$ as the derived X-scheme such that given an X-scheme Y, the space of morphisms $Y \to \mathbb{P}_X(E)$ is canonically equivalent to the space (∞-groupoid) of surjections (i.e., morphisms inducing a surjection on π_0)

$$E^\vee|_Y \to \mathscr{L}^\vee$$

where $\mathscr{L}$ is a line bundle on Y. It is known that $\mathbb{P}_X(E)$ exists as a derived scheme and that the structure morphism π is smooth. Moreover, the universal property induces a canonical surjection $E^\vee \to \mathcal{O}_{\mathbb{P}_X(E)}(1)$ to the *anticanonical line bundle* of $\mathbb{P}_X(E)$. The derived pushforward to X of the canonical surjection is equivalent to the identity morphism of E. If it causes no confusion, we will denote the projective bundle simply by $\mathbb{P}(E)$ and the anticanonical line bundle by $\mathcal{O}(1)$.

Note that the whole discussion of the last paragraph can be dualized so that $\mathbb{P}(E)$ represents embeddings $\mathscr{L} \hookrightarrow E|_Y$, and so that there is a canonical embedding $\mathcal{O}(-1) \hookrightarrow E$

on $\mathbb{P}(E)$. It is then easy to see that given an embedding $E \hookrightarrow F$ of vector bundles on X, we get the induced *linear embedding* $i : \mathbb{P}(E) \hookrightarrow \mathbb{P}(F)$. It is known that i is a quasi-smooth embedding of virtual codimension $\mathrm{rank}(F) - \mathrm{rank}(E)$. For the details, the reader may consult [3, Section 2.8].

Warning 2.1. Note that there are two conventions for projective bundles, so that what we call $\mathbb{P}(E)$ here, might somewhere else be called $\mathbb{P}(E^\vee)$. Our convention is closer to the one standard in intersection theory.

2.2. *Derived vanishing loci*

Given a global section s of a vector bundle E of rank r on a derived scheme X, we can form the X-scheme called the *derived vanishing locus* $i : V(s) \hookrightarrow X$ of s in X as the homotopy Cartesian product

$$\begin{array}{ccc} V(s) & \xrightarrow{i} & X \\ \downarrow & & \downarrow s \\ X & \xrightarrow{0} & E \end{array}$$

Note that i is a quasi-smooth embedding of virtual codimension r. By the defining property of homotopy Cartesian products, $V(s)/X$ has the following functor of points description: the space of morphisms over X from Y to $V(s)$ is canonically equivalent to the space of paths $s|_Y \sim 0$ in the space $\Gamma(Y; E|_Y)$ of global sections. This allows us to make the following observation.

Lemma 2.2. *In the derived vanishing locus $i : V(s) \hookrightarrow X$, there is a canonical path $\alpha : i^*s \sim 0$ which is natural under pullbacks of the data (E, s) in the obvious way.*

Proof. Indeed, α is the path corresponding to the identity morphism $V(s) \to V(s)$ over X. The other claims follow trivially. ∎

Our next goal is to prove a kind of transitivity result for derived vanishing loci (Proposition 2.4).

Construction 2.3. Suppose now X is a derived scheme, and

$$E' \to E \to E'' \tag{2.1}$$

is a cofibre sequence of vector bundles. If s is a global section of E, then it maps to a global section s'' of E'; denote the derived vanishing locus of s'' by Z_1. The pullback of (2.1) induces a (homotopy) fibre sequence

$$\Gamma(Z_1; E') \to \Gamma(Z_1; E) \to \Gamma(Z_1; E'') \tag{2.2}$$

of spaces of global sections. The natural path $\alpha : s''|_{Z_1} \sim 0$ in $\Gamma(Z_1; E'')$ given by

Lemma 2.2 allows us to lift s to a natural element (unique up to homotopy) $s' \in \Gamma(Z_1; E')$; denote the derived vanishing locus of s' by Z_2. The following proposition should be thought of as some kind of transitivity result for derived vanishing loci. Notice how this construction is natural under pullbacks of the data (E, E', E'', s).

Proposition 2.4. *Let X, s and Z_2 be as in Construction 2.3. Then there exists an equivalence*

$$Z_2 \simeq Z,$$

where Z is the derived vanishing locus of s in X, which is natural under pullbacks of the data (E, E', E'', s).

Proof. In order to get a morphism $Z_2 \to Z$, we need to find a path $s|_{Z_2} \sim 0$ in $\Gamma(Z_2; E)$. But this is easy: there is a natural path $\alpha : s''|_{Z_2} \sim 0$ in $\Gamma(Z_2; E'')$ which lifts to a natural path $\widetilde{\alpha} : s|_{Z_2} \sim s'|_{Z_2}$ (this is essentially how s' was constructed). Recalling that Z_2 was defined as the vanishing locus of s', we obtain a natural path $\beta : s'|_{Z_2} \sim 0$ in $\Gamma(Z_2; E')$, which maps to give a natural path $s'|_{Z_2} \sim 0$ in $\Gamma(Z_2; E)$ denoted abusively by β. The desired natural path $s|_{Z_2} \sim 0$ is then given by the composition of $\widetilde{\alpha}$ and β, and we get a natural map $\psi : Z_2 \to Z$.

It remains to show that ψ is an equivalence. Because everything is natural under pullbacks, we can check the equivalence locally on X, and therefore we can reduce to the case where all the vector bundles are trivial, and (2.1) is the standard split exact sequence

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n+m} \to \mathcal{O}_X^{\oplus m}.$$

In this situation s corresponds to an $(n + m)$-tuple $(s_1, \ldots, s_{n+m})$ of functions of X, s'' corresponds to $(s_{n+1}, \ldots, s_{n+m})$ and s' corresponds to $(s_1, \ldots, s_n)$, and the equivalence follows from the equivalence of the derived intersection $V(s_1, \ldots, s_n) \cap V(s_{n+1}, \ldots, s_{n+m})$ with $V(s_1, \ldots, s_{n+m})$. ■

The following result shows that all linear embeddings of projective bundles are vanishing loci of sections of vector bundles.

Proposition 2.5. *Let X be a derived scheme, and let*

$$E' \xrightarrow{i} E \xrightarrow{f} E'' \tag{2.3}$$

be a cofibre sequence of vector bundles on X. Consider the vector bundle $E(1)$ on $\mathbb{P}(E)$ and the section $s \in \Gamma(\mathbb{P}(E); E(1))$ corresponding via the natural identifications

$$\Gamma(\mathbb{P}(E); E(1)) \simeq \Gamma(X; E^\vee \otimes E) \simeq \mathrm{Hom}_X(E, E)$$

to the identity morphism. Then the derived vanishing locus of $f_*(s) \in \Gamma(\mathbb{P}(E); E''(1))$ is equivalent to the linear embedding $\mathbb{P}(E') \hookrightarrow \mathbb{P}(E)$ over $\mathbb{P}(E)$.

Proof. Consider the diagram

$$\begin{array}{ccccc} \Gamma(\mathbb{P}(E);E(1)) & \xrightarrow{f_*} & \Gamma(\mathbb{P}(E);E''(1)) & \xrightarrow{i^*} & \Gamma(\mathbb{P}(E');E''(1)) \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \Gamma(X;E^\vee\otimes E) & \xrightarrow{(1\otimes f)_*} & \Gamma(X;E^\vee\otimes E'') & \xrightarrow{(i^\vee\otimes 1)_*} & \Gamma(X;E'^\vee\otimes E'') \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Hom}_X(E,E) & \xrightarrow{f\circ} & \mathrm{Hom}_X(E,E'') & \xrightarrow{\circ i} & \mathrm{Hom}_X(E',E'') \end{array}$$

It is clear that the two squares on the left commute. To see why the upper right square must commute, we recall that the linear embedding $i : \mathbb{P}(E') \hookrightarrow \mathbb{P}(E)$ is induced by the surjection

$$E^\vee \xrightarrow{i^\vee} E'^\vee \to \mathcal{O}(1)$$

on $\mathbb{P}(E)$, proving the commutativity of

$$\begin{array}{ccc} \Gamma(\mathbb{P}(E);\mathcal{O}(1)) & \xrightarrow{i^*} & \Gamma(\mathbb{P}(E');\mathcal{O}(1)) \\ \downarrow\simeq & & \downarrow\simeq \\ \Gamma(X;E^\vee) & \xrightarrow{i^\vee_*} & \Gamma(X;E'^\vee) \end{array}$$

The commutativity of the bottom right square follows from the dual statement that

$$\begin{array}{ccc} \Gamma(X,E^\vee\otimes E'') & \xrightarrow{(i^\vee\otimes 1)_*} & \Gamma(X,E'^\vee\otimes E'') \\ \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Hom}_X(E''^\vee,E^\vee) & \xrightarrow{i^\vee\circ} & \mathrm{Hom}_X(E''^\vee,E'^\vee) \end{array}$$

commutes.

But now it is clear that there exists a path $i^*(f_*(s)) \sim 0$ inside $\Gamma(\mathbb{P}(E');E''(1))$ since it corresponds to the morphism $f \circ i$, and therefore has a preferred nullhomotopy by the exactness of (2.3). Therefore we get a morphism $\mathbb{P}(E') \to V(i^*(f_*(s)))$ over $\mathbb{P}(E)$. Note that both derived schemes have the same virtual codimension in $\mathbb{P}(E)$. Therefore one can conclude, after checking locally that $V(i^*(f_*(s)))$ is smooth over X (a standard argument we are not going to repeat here), that the map $\mathbb{P}(E') \to V(i^*(f_*(s)))$ must be an equivalence as a quasi-smooth embedding of virtual codimension 0 (recall that an X-morphism between smooth X-schemes is quasi-smooth). ■

2.3. *Derived blowups*

Let us recall the definition of a derived blowup from [5, Section 4.1]. Namely, given a derived scheme X and a quasi-smooth embedding $j : Z \hookrightarrow X$, the *derived blowup of X along Z* is the derived scheme $\pi : \mathrm{Bl}_Z(X) \to X$ such that, given another X-scheme $\pi_Y : Y \to X$, the space of morphisms $Y \to \mathrm{Bl}_Z(X)$ over X is canonically equivalent to the space of commuting diagrams

$$\begin{array}{ccc} D & \xrightarrow{i_D} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ Z & \xrightarrow{j} & X \end{array}$$

such that

(1) i_D is a quasi-smooth embedding of virtual codimension 1;

(2) the above square truncates to a Cartesian square of schemes;

(3) the canonical morphism $g^*\mathcal{N}^\vee_{Z/X} \to \mathcal{N}^\vee_{D/Y}$ of conormal bundles is surjective.

Let us recall some of the basic properties of blowups that are going to be useful for us.

Theorem 2.6 (cf. [5, Theorem 4.1.5]). *Let $Z \hookrightarrow X$ be a quasi-smooth immersion. Then*

(1) *the blowup $\pi : \mathrm{Bl}_Z(X) \to X$ is natural under pullbacks;*

(2) *if $X \hookrightarrow Y$ is a quasi-smooth immersion, then there exists a canonical quasi-smooth immersion $\mathrm{Bl}_Z(X) \hookrightarrow \mathrm{Bl}_Z(Y)$ called the* strict transform*;*

(3) *if both Z and X are classical, $\pi : \mathrm{Bl}_Z(X) \to X$ is the classical blowup of X along Z.*

The following proposition gives an explicit presentation for the derived blowup of X at Z in the case where Z is the derived vanishing locus of a global section s of a vector bundle E.

Proposition 2.7. *Let X be a derived scheme, E be a vector bundle on X and s a global section of E. Consider the natural cofibre sequence*

$$\mathcal{O}(-1) \to E \to Q \tag{2.4}$$

of vector bundles on $\mathbb{P}(E)$, and denote by s'' the image of s under the composition

$$\Gamma(X; E) \simeq \Gamma(\mathbb{P}(E); E) \to \Gamma(\mathbb{P}(E); Q).$$

If we denote by Z the derived vanishing locus of s in X and by Y the derived vanishing locus of s'' in $\mathbb{P}(E)$, then there is an equivalence

$$Y \simeq \mathrm{Bl}_Z(X)$$

of derived schemes which is natural under pullbacks of the data (E, s).

Proof. Our first task is to find a natural map $Y \to \mathrm{Bl}_Z(X)$. By composing with the projection $\pi : \mathbb{P}(E) \to X$, we get a natural map $Y \to X$. Moreover, the cofibre sequence (2.4) yields as in Construction 2.3 a natural global section s' of $\mathcal{O}_Y(-1)$ whose vanishing locus $D \hookrightarrow Y$ is naturally identified with the pullback of $Z \hookrightarrow X$ via π. Therefore we get a homotopy commutative square

$$\begin{array}{ccc} D & \hookrightarrow & Y \\ {\scriptstyle \pi|_Z}\downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array} \tag{2.5}$$

which comes from a diagram

$$\begin{array}{ccccc} D & \xhookrightarrow{i_D} & Y & \hookrightarrow & \mathbb{P}(E) \\ {\scriptstyle \pi|_Z}\downarrow & & & & \downarrow \\ Z & & \hookrightarrow & & X \end{array}$$

whose outer square is homotopy Cartesian. We therefore deduce that

(1) (2.5) truncates to a Cartesian square of classical schemes;

(2) the induced map $\pi|_Z^* \mathcal{N}^\vee_{Z/X} \to \mathcal{N}^\vee_{D/Y}$ on conormal bundles is surjective,

so that by the universal property of derived blowups (see [5]), we obtain a map $\phi : Y \to \mathrm{Bl}_Z(X)$. As the whole construction was natural under derived pullbacks, also the morphism ϕ is.

We prove that ϕ is an equivalence in the usual way: by naturality, we can check this locally, and therefore we can use naturality again to reduce to the situation where we blow up the section $s = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ of the trivial vector bundle $\mathcal{O}^{\oplus n}_{\mathbb{A}^n}$ on $\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n])$. Note that (2.4) is equivalent to the twisted Euler sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus n} \to T_{\mathbb{P}^{n-1}_{\mathbb{A}^n}/\mathbb{A}^n}(-1) \to 0$$

on $\mathbb{P}^{n-1}_{\mathbb{A}^n}$. Denoting by y_i the dual basis for e_i, we see that the section s'' corresponds to $x_1 \partial_{y_1} + x_2 \partial_{y_2} + \cdots + x_n \partial_{y_n}$, whose vanishing locus consists clearly of exactly those points $((x_1, \ldots, x_n), [y_1 : \cdots : y_n])$ with $[y_1 : \cdots : y_n] = [x_1 : \cdots : x_n]$ whenever the latter is well defined, and of arbitrary $((0, \ldots, 0), [y_1 : \cdots : y_n])$ otherwise. Hence the vanishing locus is just the blowup of $\mathbb{A}^n$ at the origin and we are done. ■

Example 2.8. It might be enlightening (but it will certainly be useful later) to consider the special case of Proposition 2.7 where the section s vanishes nowhere. Since blowing up along an empty centre does not do anything, we obtain the commutative triangle

$$\begin{array}{ccc} X & \xhookrightarrow{i} & \mathbb{P}(E) \\ & {\scriptstyle \mathrm{Id}} \searrow & \downarrow \\ & & X \end{array}$$

where i identifies X as the vanishing locus of $s' \in \Gamma(\mathbb{P}(E); Q)$. We claim that

$$0 \to i^* \mathcal{O}_{\mathbb{P}(E)} \xrightarrow{i^* s} i^* E \to i^* Q \to 0 \tag{2.6}$$

is exact. Indeed, $i^* s$ is an embedding of vector bundles since s was assumed not to vanish anywhere, and since the composition $i^* \mathcal{O}_{\mathbb{P}(E)} \to i^* Q$ is canonically homotopic to 0 by Lemma 2.2, the exactness follows from rank considerations. As the i-pullback of the canonical cofibre sequence (2.4) is a cofibre sequence of vector bundles, and since it shares the right side with (2.6), it follows from the universal property of $\mathbb{P}(E)$ that i is equivalent to the linear embedding $s : \mathbb{P}(\mathcal{O}_X) \to \mathbb{P}(E)$.

Of course, if s has a non-empty vanishing locus, then the linear embedding $s : \mathbb{P}(\mathcal{O}_X) \to \mathbb{P}(E)$ does not make sense. In that situation, we will have to first modify our space by taking the blowup, and then, on the modified space, there exists a canonical inclusion $\mathcal{O}(\mathcal{E}) \to E$ (where $\mathcal{E}$ is the exceptional divisor) that allows us to define a map to $\mathbb{P}(E)$. In fact, one can show that blowing up is the "most economic way" of making s equivalent to $s_1 \otimes s_2$, where $s_2 : \mathscr{L} \hookrightarrow E$ is an embedding and s_1 is a section of $\mathscr{L}$.

Next we are going to show that a linear embedding $\mathbb{P}(E') \hookrightarrow \mathbb{P}(E' \oplus E'')$ is almost a retraction. Let us begin with more general considerations: suppose X is a derived scheme and

$$E' \to E \to E''$$

a cofibre sequence of vector bundles on X. As $\mathbb{P}(E')$ is the derived vanishing locus of a section s'' of $E''(1)$ on $\mathbb{P}(E)$, Proposition 2.7 yields a natural surjection

$$E''^{\vee}(-1) \to \mathcal{O}(-\mathcal{E})$$

of vector bundles on $\mathrm{Bl}_{\mathbb{P}(E')}(\mathbb{P}(E))$. We can then twist this surjection to obtain a surjection

$$E''^{\vee} \to \mathcal{O}(1 - \mathcal{E}) \tag{2.7}$$

that gives rise to a morphism $\rho : \mathrm{Bl}_{\mathbb{P}(E')}(\mathbb{P}(E)) \to \mathbb{P}(E'')$.

Proposition 2.9. *Let X be a derived scheme, and let*

$$E' \to E' \oplus E'' \to E''$$

be a split cofibre sequence of vector bundles on X. Then the morphism ρ, constructed as above, expresses $\mathbb{P}(E'')$ as a retract (up to homotopy) of $\mathrm{Bl}_{\mathbb{P}(E')}(\mathbb{P}(E' \oplus E''))$.

Remark 2.10. Of course, the last claim should also hold in the ∞-categorical sense and not only up to homotopy. But we have decided to restrict the generality in order to get away with a simpler proof.

Proof of Proposition 2.9. The only thing that is not obvious is that ρ expresses $\mathbb{P}(E'')$ as a retract of the blowup. First of all, since $\mathbb{P}(E')$ and $\mathbb{P}(E'')$ do not meet inside $\mathbb{P}(E' \oplus E'')$, we can form the homotopy Cartesian square

$$\begin{array}{ccc} \mathbb{P}(E'') & \xrightarrow{i'} & \mathrm{Bl}_{\mathbb{P}(E')}(\mathbb{P}(E' \oplus E'')) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow \\ \mathbb{P}(E'') & \xrightarrow{i} & \mathbb{P}(E' \oplus E'') \end{array}$$

providing the map i', which we claim to satisfy $\rho \circ i' \simeq \mathrm{Id}$.

Note that we have to show that the i'-pullback of the surjection (2.7) is equivalent to the canonical surjection $E''^{\vee} \to \mathcal{O}(1)$. But since

(1) i factors through the open subset $U := \mathbb{P}(E' \oplus E'') \setminus \mathbb{P}(E')$, and

(2) the restriction $E''^{\vee} \to \mathcal{O}_U(1)$ of (2.7) to U corresponds to $s''^{\vee}|_U : E''^{\vee}(-1) \to \mathcal{O}_U$ by Example 2.8,

we conclude that the pullback of (2.7) corresponds to the global section $i^*(s''^\vee)$ of $E''^\vee(-1)$ on $\mathbb{P}(E'')$. By unwinding the definitions (much as in the proof of Proposition 2.5), one sees that $i^*(s''^\vee)$ corresponds to the identity morphism $E'' \to E''$ on X, identifying the corresponding surjection $E''^\vee \to \mathcal{O}(1)$ on $\mathbb{P}(E'')$ with the canonical one. ∎

2.4. *Precobordism theories*

Let us recall from [3] that a *precobordism theory* $\mathbb{B}^*$ is a bivariant theory (on the homotopy category of quasi-projective derived schemes over a Noetherian ring A) satisfying certain axioms. This means that we get an Abelian group $\mathbb{B}^*(X \xrightarrow{f} Y)$ for any morphism $f : X \to Y$, a bilinear *bivariant product* $\bullet$ associated to compositions of morphisms, *bivariant pullbacks* associated to derived Cartesian squares, and *bivariant pushforwards* associated to the factoring of f through a projective morphism $g : X \to X'$. The structure of bivariant theory makes $(\mathbb{B}^*(X), \bullet) := (\mathbb{B}^*(X \xrightarrow{\mathrm{Id}} X), \bullet)$ (commutative) rings, contravariantly functorial in X. The choice of an *orientation* for $\mathbb{B}^*$ gives rise to *Gysin pushforward morphisms* $f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y)$ Since we are mostly interested in cohomology rings in this paper, we are not going to recall the bivariant formalism in greater detail here. The interested reader can consult [1] or [3] for further details.

What we are going to need is a detailed understanding of the cohomology theory associated to the universal precobordism. It is recorded in the following construction, which is just the cohomological restriction of the bivariant construction of [3, Section 6.1].

Construction 2.11 (Universal precobordism rings). Let A be a Noetherian ring. For any quasi-projective derived A-scheme X, we define the *ring of cobordism cycles* $\mathcal{M}^*_+(X)$ so that the degree d part $\mathcal{M}^d_+(X)$ is the free Abelian group generated by equivalence classes $[V \xrightarrow{f} X]$ with f projective and quasi-smooth of relative virtual dimension $-d$ (modulo, of course, the relation that disjoint union of cycles corresponds to addition). The (commutative) ring structure is given by the homotopy fibre product over X, with the class of the identity morphism serving as the unit element.

We can also define operations on $\mathcal{M}^*_+$. For an arbitrary map $g : X \to Y$, the *pullback morphism* $g^* : \mathcal{M}^*_+(Y) \to \mathcal{M}^*_+(X)$ is defined by linearly extending

$$[V \to Y] \mapsto [V \times_Y X \to X].$$

Note that g^* acts by ring homomorphisms and preserves degrees. For every projective $g : X \to Y$ that is also quasi-smooth (of relative virtual dimension $-d$), we can define the *Gysin pushforward morphism* $g_! : \mathcal{M}^*_+(X) \to \mathcal{M}^{*+d}_+(Y)$ by linearly extending

$$[V \xrightarrow{f} X] \mapsto [V \xrightarrow{g \circ f} Y].$$

Note that $g_!$ preserves addition, but need not preserve multiplication. Both the pullbacks and pushforwards are functorial in the obvious sense.

There are now two equivalent sets of relations one can enforce on the rings $\mathcal{M}_+^*$ to obtain the *universal precobordism rings* $\underline{\Omega}^*$: either

(1) *double point relations*: given a projective quasi-smooth morphism $W \to \mathbb{P}^1 \times X$, let W_0 be the fibre over 0 and suppose that the fibre W_∞ over ∞ is the sum of two virtual Cartier divisors A and B inside W; we then require that

$$[W_0 \to X] - [A \to X] - [B \to X] + [\mathbb{P}_{A \times_W B}(\mathcal{O}(A) \oplus \mathcal{O}) \to X] = 0 \tag{2.8}$$

(the linear combinations of elements of the above form clearly form ideals that are stable under the operations of $\mathcal{M}_+^*$, and therefore the quotient theory makes sense);

or

(2a) *homotopy fibre relations*: given a projective quasi-smooth morphism $W \to \mathbb{P}^1 \times X$, let W_0 and W_∞ be the fibres over 0 and ∞ respectively; then we require that

$$[W_0 \to X] - [W_\infty \to X] = 0$$

(again, linear combinations of these form ideals stable ubder operations of $\mathcal{M}_+^*$, making the quotient theory sensible);

(2b) given line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on X, we require that

$$\begin{aligned} c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) = c_1(\mathscr{L}_1) + c_1(\mathscr{L}_2) - c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_2) \bullet [\mathbb{P}_1 \to X] \\ - c_1(\mathscr{L}_1) \bullet c_1(\mathscr{L}_2) \bullet c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) \bullet ([\mathbb{P}_2 \to X] - [\mathbb{P}_3 \to X]), \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \mathbb{P}_1 &:= \mathbb{P}_X(\mathscr{L}_1 \oplus \mathcal{O}), \\ \mathbb{P}_2 &:= \mathbb{P}_X(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2) \oplus \mathcal{O}), \\ \mathbb{P}_3 &:= \mathbb{P}_{\mathbb{P}_X(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2))}(\mathcal{O}(-1) \oplus \mathcal{O}), \end{aligned}$$

and $c_1(\mathscr{L})$ is the *first Chern class* (the *Euler class*) of the line bundle $\mathscr{L}$ (equivalent to $[V(s) \hookrightarrow X]$ for any global section s of $\mathscr{L}$); as this equation is not stable under pushforwards, one needs to take all the generated relations before taking the quotient;

the equivalence of these two sets of relations was proven in [3] (following the original proof in [8]). The two slightly different sets of relations, (1) and (2a)–(2b), will both be convenient for us at certain points of the paper.

We will also need to recall some of the formal properties of $\underline{\Omega}^*$ that are helpful in performing computations.

Theorem 2.12. *The operations of $\underline{\Omega}^*$ satisfy the following basic properties.*

(1) Push-pull formula: *if the square*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is homotopy Cartesian with f projective and quasi-smooth (of pure relative virtual dimension), then $f'_! g'^*(\alpha) = g^* f_!(\alpha)$ for all $\alpha \in \underline{\Omega}^*(X)$.

(2) Projection formula: *if $f : X \to Y$ is projective and quasi-smooth (of pure relative virtual dimension), then $f_!(f^*(\alpha) \bullet \beta) = \alpha \bullet f_!(\beta)$ for all $\alpha \in \underline{\Omega}^*(Y)$ and $\beta \in \underline{\Omega}^*(X)$.*

Proof. Of course, this does not have anything to do with the definition of $\underline{\Omega}^*$. All the formulas hold already for $\mathcal{M}^*_+$ because it is part of a bivariant theory with an orientation (the formulas can also be checked easily on the level of cycles). ∎

The following simple observations will also be useful.

Corollary 2.13. *Let $f : X \to Y$ be a projective quasi-smooth morphism between derived schemes such that there exists a class $\eta_X \in \underline{\Omega}^*(X)$ with $f_!(\eta_X) = 1$. Then the pullback morphism $\pi^* : \underline{\Omega}^*(Y) \to \underline{\Omega}^*(X)$ is injective.*

Proof. Indeed, it follows from the projection formula that

$$f_!(\pi^*(\alpha) \bullet \eta_X) = \alpha \bullet f_!(\eta_X) = \alpha$$

for all $\alpha \in \underline{\Omega}^*(Y)$, proving the claim. ∎

Proposition 2.14. *Let E and F be vector bundles on a derived scheme X. Then the embedding $i : \mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus F)$ gives rise to a split injective Gysin pushforward morphism $i_! : \underline{\Omega}^*(\mathbb{P}(E)) \to \underline{\Omega}^*(\mathbb{P}(E \oplus F))$.*

Proof. By Proposition 2.9 we can form a homotopy Cartesian diagram

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{i'} & \mathrm{Bl}_{\mathbb{P}(F)}(\mathbb{P}(E \oplus F)) \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}(E) & \xrightarrow{i} & \mathbb{P}(E \oplus F) \end{array}$$

such that i' admits a partial inverse ρ, so $\rho \circ i' \simeq \mathrm{Id}_{\mathbb{P}(E)}$. As $i'_!$ is injective by functoriality, we can use the push-pull formula to conclude that also $\pi^* i_!$ is injective, and therefore $i_!$ must be injective. ∎

Precobordism with bundles. We can also define a slight variant of the above theory called the *universal precobordism ring with vector bundles*. Namely, we start with the group completions $\mathcal{M}^{d,r}_+(X)$ of the Abelian monoid on cycles

$$[V \to X, E],$$

where $V \to X$ is quasi-smooth and projective of relative virtual dimension $-d$, E is a vector bundle of rank r on V and the monoid structure is given by disjoint union. One defines pullbacks and pushforwards for $\mathcal{M}^{*,*}_+$ in the obvious way. There are now two product structures that make sense:

$$[V_1 \to X, E_1] \bullet_\oplus [V_2 \to X, E_2] := [V_1 \times_X V_2 \to X, E_1 \oplus E_2],$$
$$[V_1 \to X, E_1] \bullet_\otimes [V_2 \to X, E_2] := [V_1 \times_X V_2 \to X, E_1 \otimes E_2].$$

Note that the restricted group $\mathcal{M}_+^{*,1}(X)$ with line bundles is a ring with multiplication $\bullet_\otimes$.

The rings $\underline{\Omega}^{*,*}$ are obtained by enforcing the equality (cf. (2.8))

$$[W_0 \to X, E|_{W_0}] - [A \to X, E|_A] - [B \to X, E|_B] + [\mathbb{P}_{A \times_W B}(\mathcal{O}(A) \oplus \mathcal{O}) \to X, E|_{A \times_W B}] = 0,$$

where E is any vector bundle on W. Of course one could also arrive at the same rings by enforcing the homotopy fibre relation and (2.9). Note that we have natural embeddings of theories

$$\underline{\Omega}^*(X) \hookrightarrow \underline{\Omega}^{*,0}(X) \quad \text{and} \quad \underline{\Omega}^*(X) \hookrightarrow \underline{\Omega}^{*,1}(X)$$

defined by equipping the cycle with a rank 0 vector bundle or a trivial line bundle respectively. When it causes no confusion, we often omit the target X from cycles and the subscripts from the products $\bullet_\oplus$ and $\bullet_\otimes$ in order to simplify the notation.

3. The formal group law of precobordism theory

The purpose of this section is to prove that the Euler class of $\mathscr{L}_1 \otimes \mathscr{L}_2$ can be computed easily from those of $\mathscr{L}_1$ and $\mathscr{L}_2$. Recall first that in [3, Section 6.3], we were able to find coefficients $a_{i,j} \in \underline{\Omega}^*(\mathrm{pt})$ of a formal group law

$$F(x, y) = \sum_{i,j} a_{ij} x^i y^j$$

by letting a_{ij} be such that

$$e(\mathcal{O}(1,1)) = \sum_{i,j} a_{ij} e(\mathcal{O}(1,0))^i \bullet e(\mathcal{O}(0,1))^j \in \underline{\Omega}^*(\mathbb{P}^n \times \mathbb{P}^m)$$

for any (and hence all) $n \geq i$ and $m \geq j$. It follows from the projective bundle formula for trivial projective bundles [3, Theorem 6.22] that this is well defined. Moreover, it is a trivial consequence that given globally generated line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ on a quasi-projective derived scheme X, one has

$$e(\mathscr{L}_1 \otimes \mathscr{L}_2) = \sum_{i,j} a_{ij} e(\mathscr{L}_1)^i \bullet e(\mathscr{L}_2)^j \in \underline{\Omega}^*(X). \tag{3.1}$$

We can now state more precisely the main goal of this section: we want to show that (3.1) holds for arbitrary $\mathscr{L}_1$ and $\mathscr{L}_2$, not just globally generated ones (Theorem 3.4). Traditionally, there are two ways to reduce the general case to the globally generated one:

(1) using properties of formal group laws, it is possible to define new Euler classes $\tilde{e}(\mathscr{L})$ in such a way that $\tilde{e}(\mathscr{L}) = e(\mathscr{L})$ whenever $\mathscr{L}$ is globally generated and (3.1) holds for arbitrary $\mathscr{L}_1, \mathscr{L}_2$;

(2) using Jouanolou's trick, one can find an affine scheme Y that is a torsor for a vector bundle on X; then, assuming that the cohomology theory satisfies a strong enough form of homotopy invariance, we can check (3.1) by pulling back to Y, where the line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ become globally generated.

Since we do not want to redefine Euler classes, and since $\underline{\Omega}^*$ does not satisfy homotopy invariance, we are forced to do something different. Luckily, everything boils down to explicit computations.

3.1. Formal group law for arbitrary line bundles

Our strategy is essentially to show that one can derive the formal group law directly from (2.9) without knowing the projective bundle formula beforehand. To do this, we need to be able to express the classes $[\mathbb{P}_i]$ as power series in $e(\mathscr{L}_1)$ and $e(\mathscr{L}_2)$ whose coefficients do not depend on $X, \mathscr{L}_1$ or $\mathscr{L}_2$. From now on, X is assumed to be quasi-projective over a Noetherian ring A of finite Krull dimension and $\mathscr{L}_1, \mathscr{L}_2$ are line bundles on X.

It will be useful to take advantage of the universal precobordism rings equipped with line bundles $\underline{\Omega}^{*,1}(X)$ defined in [3], where "1" stands for the rank, which is 1 in the case of line bundles. Recall that the groups $\underline{\Omega}^{*,1}(X)$ are generated by cycles of the form $[V \to X; \mathscr{L}]$ with $V \to X$ quasi-smooth and projective and $\mathscr{L}$ a line bundle on V, and these cycles are subject to double point cobordism relations (see [3, Definition 6.6]). The groups $\underline{\Omega}^{*,1}(X)$ form an oriented cohomology theory, and the ring structure is given by linearly extending the formula

$$[V \to X; \mathscr{L}] \bullet [W \to X; \mathscr{M}] = [V \times_X W \to X; \mathscr{L} \boxtimes \mathscr{M}] \in \underline{\Omega}^{*,1}(X).$$

Note that the rings $\underline{\Omega}^{*,1}(X)$ are $\underline{\Omega}^*(X)$-algebras via the canonical inclusion

$$[V \to X] \mapsto [V \to X; \mathscr{O}].$$

The following result was proven in [3].

Lemma 3.1 (cf. [3, Lemma 6.18]). *We have the equality*

$$[X, \mathscr{L}] = \sum_{i \geq 0} e(\mathscr{L})^i \bullet (\beta_i - [\mathbb{P}(\mathscr{L} \oplus \mathscr{O})] \bullet \beta_{i-1}) \in \underline{\Omega}^{*,1}(X),$$

where

$$\begin{aligned} (P_0, \mathcal{M}_0) &:= (\mathrm{Spec}(A), \mathscr{O}), & P_{i+1} &:= \mathbb{P}_{P_i}(\mathcal{M}_i \oplus \mathscr{O}), \\ \mathcal{M}_{i+1} &:= \mathcal{M}_i(1), & \beta_i &:= \pi^*[P_i, \mathcal{M}_i] \in \underline{\Omega}^*(X), \end{aligned}$$

and $P_{-1} = \emptyset$.

Proof. This is what you get when you apply the relation of [3, Lemma 6.18] infinitely many times to kill all the T_i. ∎

As an easy consequence, we get the following formula taking care of the class $[\mathbb{P}_1]$.

Lemma 3.2. *We have the equality*

$$[\mathbb{P}(\mathscr{L} \oplus \mathcal{O})] = \frac{\sum_{i \geq 0} \pi^*[P_{i+1}] \bullet e(\mathscr{L})^i}{\sum_{i \geq 0} \pi^*[P_i] \bullet e(\mathscr{L})^i} \in \underline{\Omega}^*(X),$$

where P_i are as in Lemma 3.1.

Proof. It is clear that we have a well defined $\underline{\Omega}^*$-linear transformation $\mathbb{P} : \underline{\Omega}^{*,1}(X) \to \underline{\Omega}^{*-1}(X)$ defined by the formula

$$[V \to X, \mathscr{L}] \mapsto [\mathbb{P}_V(\mathscr{L} \oplus \mathcal{O}) \to X].$$

Applying this transformation to Lemma 3.1, and noting that

$$\mathbb{P}(\beta_i) = \mathbb{P}\pi^*[P_i, \mathcal{M}_i] = \pi^*[\mathbb{P}_{P_i}(\mathcal{M}_i \oplus \mathcal{O})] = \pi^*[P_{i+1}],$$

we conclude that

$$[\mathbb{P}(\mathscr{L} \oplus \mathcal{O})] = \sum_{i \geq 0} e(\mathscr{L})^i \bullet \big(\pi^*[P_{i+1}] - [\mathbb{P}(\mathscr{L} \oplus \mathcal{O})] \bullet \pi^*[P_i]\big).$$

The claim follows by solving for $[\mathbb{P}(\mathscr{L} \oplus \mathcal{O})]$. ∎

It is then easy to find formulas for the classes $[\mathbb{P}_2]$ and $[\mathbb{P}_3]$.

Lemma 3.3. *We have the equalities*

$$[\mathbb{P}(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2) \oplus \mathcal{O})] = \sum_{i,j} \gamma_{ij} e(\mathscr{L}_1)^i \bullet e(\mathscr{L}_2)^j, \tag{3.2}$$

$$[\mathbb{P}_{\mathbb{P}_X(\mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2))}(\mathcal{O}(-1) \oplus \mathcal{O})] = \sum_{i,j} \phi_{ij} e(\mathscr{L}_1)^i \bullet e(\mathscr{L}_2)^j \tag{3.3}$$

in $\underline{\Omega}^*(X)$ for some $\gamma_{ij}, \phi_{ij} \in \underline{\Omega}^*(\mathrm{pt})$ not depending on X, $\mathscr{L}_1$ or $\mathscr{L}_2$.

Proof. The proof is quite easy. Denote by $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ the $\underline{\Omega}^*$-linear natural transformations from $\underline{\Omega}^{i,j}$ to $\underline{\Omega}^{i-j}$ defined by the formulas

$$[V \to X, E] \mapsto [\mathbb{P}_V(E \oplus \mathcal{O}) \to X],$$
$$[V \to X, E] \mapsto [\mathbb{P}_{\mathbb{P}_V(E)}(\mathcal{O}(-1) \oplus \mathcal{O}) \to X]$$

respectively. The desired formula is obtained in each case by applying Lemma 3.1 to the right hand side of

$$[X, \mathscr{L}_1 \oplus (\mathscr{L}_1 \otimes \mathscr{L}_2)] = [X, \mathscr{L}_1] \bullet_{\otimes} ([X, \mathcal{O}] \bullet_{\oplus} [X, \mathscr{L}_2]),$$

replacing the instances of $\underline{\Omega}(\mathscr{L} \oplus \mathcal{O})$ by power series using Lemma 3.2, and finally applying either $\mathbb{P}$ or $\widetilde{\mathbb{P}}$. ∎

We are now ready to prove the main theorem of this section.

Theorem 3.4. *Suppose* $\pi : X \to \mathrm{Spec}(A)$ *is a quasi-projective derived scheme over a Noetherian ring* A *of finite Krull dimension, and suppose* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are line bundles on* X*. Then*

$$e(\mathscr{L}_1 \otimes \mathscr{L}_2) = \sum_{i,j} a_{ij} e(\mathscr{L}_1)^i \bullet e(\mathscr{L}_2)^j \in \underline{\Omega}^*(X)$$

where $a_{ij} \in \underline{\Omega}^*(\mathrm{pt})$ *are the coefficients of the formal group law* (3.1) *on globally generated line bundles constructed in* [3].

Proof. First of all, we can apply (2.9) to conclude that

$$e(\mathscr{L}_1 \otimes \mathscr{L}_2) = \frac{e(\mathscr{L}_1) + e(\mathscr{L}_2) - e(\mathscr{L}_1) \bullet e(\mathscr{L}_2) \bullet [\mathbb{P}_1 \to X]}{1 + e(\mathscr{L}_1) \bullet e(\mathscr{L}_2) \bullet ([\mathbb{P}_2 \to X] - [\mathbb{P}_3 \to X])} \in \underline{\Omega}^*(X).$$

Lemmas 3.2 and 3.3 allow us to replace the classes $[\mathbb{P}_i \to X]$ by natural power series in $e(\mathscr{L}_1)$ and $e(\mathscr{L}_2)$, whose coefficients do not depend on $X, \mathscr{L}_1$ or $\mathscr{L}_2$, and therefore

$$e(\mathscr{L}_1 \otimes \mathscr{L}_2) = \sum_{i,j} a'_{ij} e(\mathscr{L}_1)^i \bullet e(\mathscr{L}_2)^j$$

for some $a'_{ij} \in \underline{\Omega}^*(\mathrm{pt})$ and for all X, $\mathscr{L}_1$ and $\mathscr{L}_2$. But since the coefficients a_{ij} of (3.1) were defined by computing the class $e(\mathcal{O}(1,1))$ on $\underline{\Omega}^*(\mathbb{P}^n \times \mathbb{P}^m)$ as n and m tend to infinity, we see that $a'_{ij} = a_{ij}$ for all $i, j \geq 0$, proving the claim. ■

3.2. *Universal property of the universal precobordism*

The construction of the universal precobordism rings $\underline{\Omega}^*(X)$ in terms of free generators and relations gives them a universal property. Since the cohomological universal property is not explicitly written down anywhere, we will write down a nice universal property in this subsection (see Theorem 3.11). In order to state the theorem, we need to make some preliminary definitions. Throughout this section, A will be a Noetherian ring of finite Krull dimension, $d\mathcal{S}ch_A$ will be the homotopy category of quasi-projective derived schemes over A, and $\mathcal{R}^*$ the category of commutative graded rings (not graded commutative).

Definition 3.5. A functor $\mathcal{F} : d\mathcal{S}ch_A^{\mathrm{op}} \to \mathcal{R}^*$ is *additive* if the natural inclusions $X \hookrightarrow X \amalg Y$ and $Y \hookrightarrow X \amalg Y$ induce an isomorphism

$$\mathcal{F}(X \amalg Y) \xrightarrow{\cong} \mathcal{F}(X) \times \mathcal{F}(Y)$$

of graded rings.

The main definition we are going to work with is the following:

Definition 3.6. An *oriented cohomology theory* $\mathbb{H}^*$ on $d\mathcal{S}ch_A$ consists of

$(O1)$ an additive functor $\mathbb{H}^* : d\mathcal{S}ch_A^{\mathrm{op}} \to \mathcal{R}^*$;

($O2$) for every quasi-smooth projective morphism $f : X \to Y$ of pure relative dimension, a *Gysin pushforward morphism*

$$f_! : \mathbb{H}^*(X) \to \mathbb{H}^*(Y),$$

which is required to be a morphism of $\mathbb{H}^*(Y)$-modules, and which is allowed not to preserve the grading.

Note that such a data allows one to define the *Euler class* $e(\mathscr{L})$ of a line bundle $\mathscr{L}$ on X as $s_0^* s_{0!}(1_X)$, where $s_0 : X \to \mathscr{L}$ is the zero section. This data is required to satisfy the following compatibility conditions:

(Fun) the Gysin pushforwards are required to be functorial;

(PP) given a homotopy Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with f projective and quasi-smooth (of pure relative virtual dimension), we have $f'_! g'^*(\alpha) = g^* f_!(\alpha)$ for all $\alpha \in \mathbb{H}^*(X)$;

($Norm$) if $\mathscr{L}$ is a line bundle on X and $i : D \hookrightarrow X$ is the inclusion of a derived vanishing locus of a global section of $\mathscr{L}$, then

$$e(\mathscr{L}) = i_!(1_D);$$

(FGL) the Euler classes of line bundles are nilpotent and there exist elements $b_{ij} \in \mathbb{H}^*(\mathrm{pt})$ for $i, j \geq 0$ such that

$$e(\mathscr{L}_1 \otimes \mathscr{L}_2) = \sum_{i,j} b_{ij} e(\mathscr{L}_1)^i \bullet e(\mathscr{L}_2)^j \quad \text{for all } X, \mathscr{L}_1 \text{ and } \mathscr{L}_2.$$

Example 3.7. Both K^0 and $\underline{\Omega}^*$ are oriented cohomology theories on $dSch_A$. Note that we have to regard $K^0(X)$ as a graded ring concentrated in degree 0, and therefore we do not want to make restrictions on how Gysin pushforwards affect the grading.

Remark 3.8. Why does this definition deserve the name "oriented cohomology theory"? It does not at all look similar to the definition used by Levine and Morel [7]. For example, here the formal group law is one of the axioms, whereas in [7] it was a consequence of the projective bundle formula (which is not an axiom for us). But as we already noted in Section 3.1, the failure of the homotopy property seems to make it impossible to deduce the validity of formal group laws for line bundles that are not globally generated. Hence, a Levine–Morel style characterization does not seem to produce a useful notion in this generality.

Of course, it will turn out that $\underline{\Omega}^*$ is the universal oriented cohomology theory. Let us start with an easy lemma.

Lemma 3.9. *Suppose $\mathbb{H}^*$ is an oriented cohomology theory on $d\mathcal{S}ch_A$. Then there is a unique natural transformation $\eta' : \mathcal{M}^*_+ \to \mathbb{H}^*$ commuting with pullbacks and pushforwards and preserving the ring structure.*

Proof. Indeed, such a transformation must preserve fundamental classes, and therefore

$$\eta([V \xrightarrow{f} X]) = \eta'(f_!(1_V)) = f_!(\eta'(1_V)) = f_!(1_V^{\mathbb{H}^*}),$$

where $1_V^{\mathbb{H}^*} \in \mathbb{H}^*(V)$ is the multiplicative unit. The morphisms η' preserve addition since $\mathbb{H}^*$ was assumed to be additive. They commute with pushforwards by (*Fun*) and with pullbacks by (*PP*). To prove that η' preserves multiplication, we compute

$$\begin{aligned}\eta'([V_1 \xrightarrow{f_1} X]) \bullet \eta'([V_2 \xrightarrow{f_2} X]) &= f_{1!}(1_{V_1}^{\mathbb{H}^*}) \bullet \eta'([V_2 \xrightarrow{f_2} X]) \\ &= f_{1!}(f_1^*(\eta'([V_2 \xrightarrow{f_2} X]))) \quad \text{(linearity of } f_!) \\ &= \eta'(f_{1!}(f_1^*([V_2 \xrightarrow{f_2} X]))) \\ &= \eta'([V_1 \xrightarrow{f_1} X] \bullet [V_2 \xrightarrow{f_2} X]).\end{aligned}$$

We have shown that η' satisfies all the desired properties, so we are done. ∎

In order to show that the above morphisms η descend the double point cobordism relations, we need the following result (cf. [8, Lemma 3.3]).

Lemma 3.10. *Let $\mathbb{H}^*$ be an oriented cohomology theory, and let $X \in d\mathcal{S}ch_A$. If $\mathscr{L}$ is a line bundle on X, then*

$$\pi_!(1_{\mathbb{P}(\mathscr{L}\oplus\mathcal{O})}) = -\sum_{i,j\geq 1} b_{ij} e(\mathscr{L})^{i-1} \bullet e(\mathscr{L}^\vee)^{j-1},$$

where b_{ij} are as in (FGL) and π is the natural projection $\mathbb{P}(\mathscr{L} \oplus \mathcal{O}) \to X$.

Proof. The proof is the same as in [8] except that all blowups should be derived. ∎

We are finally ready to prove the main result of this subsection: $\underline{\Omega}^*$ is the universal oriented cohomology theory.

Theorem 3.11. *Suppose $\mathbb{H}^*$ is an oriented cohomology theory on $d\mathcal{S}ch_A$. Then there is a unique natural transformation $\eta : \underline{\Omega}^* \to \mathbb{H}^*$ commuting with pullbacks and pushforwards and preserving the ring structure.*

Proof. We only have to show that the natural transformation η' descends the double point cobordism relations (2.8). This boils down to showing that

$$\eta'\big([\mathbb{P}_{A\times_W B}(\mathcal{O}(A) \oplus \mathcal{O}) \to X]\big) = -\sum_{i,j\geq 1} b_{ij} e(\mathcal{O}(A))^i \bullet e(\mathcal{O}(B))^j \in \mathbb{H}^*(W).$$

Denote by i the embedding $A \times_W B \hookrightarrow W$. We note that $i_!(1_{A \times_W B})$ is just the product $e(\mathcal{O}(A)) \bullet e(\mathcal{O}(B))$, and the restrictions of $\mathcal{O}(A)$ and $\mathcal{O}(B)$ to $A \times_W B$ are duals of each other. We can now compute that

$$\begin{aligned}\eta'\big([\mathbb{P}_{A \times_W B}(\mathcal{O}(A) \oplus \mathcal{O}) \to X]\big)& \\ &= -i_!\Big(\sum_{i,j \geq 1} b_{ij} e(\mathcal{O}(A))^{i-1} \bullet e(\mathcal{O}(-A))^{j-1}\Big) \quad \text{(Lemma 3.10)} \\ &= -i_!\Big(\sum_{i,j \geq 1} b_{ij} e(\mathcal{O}(A))^{i-1} \bullet e(\mathcal{O}(B))^{j-1}\Big) \\ &= \sum_{i,j \geq 1} b_{ij} e(\mathcal{O}(A))^{i} \bullet e(\mathcal{O}(B))^{j} \quad \text{(linearity of } i_!\text{)},\end{aligned}$$

which finishes the proof. ∎

4. Chern classes in precobordism rings

The purpose of this section is to construct *Chern classes* for the precobordism rings $\underline{\Omega}^*(X)$. More precisely, given a quasi-projective derived scheme X over a Noetherian ring A and a vector bundle E on X of rank r, we are going to construct in Section 4.1 natural classes

$$c_i(E) \in \underline{\Omega}^i(X)$$

for $1 \leq i \leq r$, essentially by applying the Whitney sum formula to a natural filtration of E on a derived X-scheme naturally associated to the pair (X, E). This is enough to prove the first basic properties of Chern classes in Section 4.2, but in order to prove that the $c_i(E)$ have all the expected properties, we must obtain more flexible ways to compute the Chern classes. This is the most subtle part of the theory, and it is done in Section 4.3 by proving the splitting principle.

Before beginning the construction, we recall that given a vector bundle E of rank r on a derived scheme X, we can define its *Euler class* (or *top Chern class*) $e(E) \in \underline{\Omega}^r(X)$ as the cycle class $[V(s) \to X]$ of the inclusion of the derived vanishing locus of any global section s of E (this class does not depend on s). The following easy results about Euler classes will be used in the construction of Chern classes.

Lemma 4.1. *Let X be a quasi-projective derived scheme over a Noetherian ring A, and let*

$$E' \to E \to E''$$

be a cofibre sequence of vector bundles of rank r', r, r'' respectively on X. Then

$$e(E) = e(E') \bullet e(E'') \quad \text{in } \underline{\Omega}^r(X).$$

Proof. Let s be a global section of E, which maps to a global section s'' of E''. Denote the inclusion $V(s'') \hookrightarrow X$ by i. As $e(E'') = i_!(1_{V(s'')})$, we can use the projection formula to conclude that

$$e(E') \bullet e(E'') = e(E') \bullet i_!(1_{V(s'')}) = i_!(e(i^*E')).$$

We can now use the canonical section s' of i^*E' as in Construction 2.4 to see that $e(i^*E')$ is represented by $[V(s) \to V(s'')]$, where $V(s)$ is the vanishing locus of s in X. Hence $i_!(e(i^*E')) = [V(s) \to X]$ and the claim follows. ■

Lemma 4.2. *Let X be a quasi-projective derived scheme over a Noetherian ring A, and let E be a vector bundle of rank r on X. Then the Euler class $e(E)$ is a nilpotent element of $\underline{\Omega}^*(X)$.*

Proof. The case $r = 1$ is [3, Lemma 6.2]. Note that if a section s of $E^\vee \otimes \mathscr{L}$ does not vanish on some open subset $U \subset X$, then the naturally associated map $s' \in \mathrm{Hom}_X(E, \mathscr{L})$ is surjective on U. As X is quasi-projective and A is Noetherian, we can find a line bundle $\mathscr{L}$ and global sections $s_1, \dots, s_n \in \Gamma(X; E^\vee \otimes \mathscr{L})$ such that the total vanishing locus of the sections s_i is empty. It follows that they induce a surjective morphism $E^{\oplus n} \to \mathscr{L}$ of vector bundles, and therefore we must have

$$e(E)^n = e(E^{\oplus n}) = \alpha \bullet e(\mathscr{L})$$

for some $\alpha \in \underline{\Omega}^{rn-1}(X)$ by Lemma 4.1. The claim now follows from the nilpotence of $e(\mathscr{L})$ and the commutativity of the product $\bullet$. ■

Remark 4.3. There are much more natural proofs of Lemma 4.2, for example by following ideas of [3, Section 6.2] and reducing to the globally generated case by deforming the vector bundle $E \otimes \mathscr{L}$ to E. Such an approach would have the advantages of being much more natural and yielding a precise formula for $e(E)$ in terms of $e(E)$, $e(\mathscr{L})$ and some geometric data, but it would be considerably longer.

4.1. Construction of Chern classes

In this section, we are going to construct the Chern classes. The key fact we are going to need is the following observation:

Lemma 4.4. *Let X be a derived scheme, E a vector bundle on X and $s \in \Gamma(X; E)$ a global section of E. Now the canonical equivalence of Proposition 2.7 and the canonical surjection $E^\vee \to \mathcal{O}(1)$ on $\mathbb{P}(E)$ yield a natural surjection*

$$E^\vee \to \mathcal{O}(-\mathcal{E}) \tag{4.1}$$

of vector bundles on $\mathrm{Bl}_Z(X)$, where $\mathcal{E}$ is the exceptional divisor.

Proof. The only thing that has not yet been explicitly stated is that the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ restricts to $\mathcal{O}(-\mathcal{E})$ on the blowup $\mathrm{Bl}_Z(X)$. But this is easy as $\mathcal{E}$ is by construction the vanishing locus of a global section of restriction of $\mathcal{O}(-1)$. ■

Another result we are going to need is the following.

Lemma 4.5. *Let X be a quasi-projective derived scheme over a Noetherian ring A of finite Krull dimension, and let E be a vector bundle on X. Let $s \in \Gamma(X; E)$ be a global section of E, and let $Z \hookrightarrow X$ be the inclusion of the derived vanishing locus of s. Denote by $\widetilde{X}$ the derived blowup $\mathrm{Bl}_Z(X)$. Then*

$$1_X = \sum_{i=0}^{\infty} e(E)^i \bullet [\mathbb{P}_X(E \oplus \mathcal{O}) \to X]^i \bullet \big([\widetilde{X} \to X] - [\mathbb{P}_{\mathcal{E}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}) \to X]\big) \in \underline{\Omega}^*(X),$$

where $\mathcal{E}$ is the exceptional divisor of $\widetilde{X}$.

Proof. Consider $W := \mathrm{Bl}_{\infty \times Z}(\mathbb{P}^1 \times X) \to \mathbb{P}^1 \times X$ and note that the fibre of W over ∞ is the sum of the strict transform $\mathrm{Bl}_Z(X)$ and the exceptional divisor $\mathcal{E}' \simeq \mathbb{P}_Z(E \oplus \mathcal{O})$, as virtual Cartier divisors on W intersecting in the exceptional divisor $\mathcal{E}$ of $\mathrm{Bl}_Z(X)$. The morphism $W \to \mathbb{P}^1 \times X$ is therefore a derived double point degeneration over X realizing the relation

$$\begin{aligned} 1_X &= [\widetilde{X} \to X] + [\mathbb{P}_Z(E \oplus \mathcal{O}) \to X] - [\mathbb{P}_{\mathcal{E}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}) \to X] \\ &= [\widetilde{X} \to X] + e(E) \bullet [\mathbb{P}(E \oplus \mathcal{O}) \to X] - [\mathbb{P}_{\mathcal{E}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}) \to X] \end{aligned}$$

in $\underline{\Omega}^*(X)$. Moreover, $\mathbb{P}_W(E \oplus \mathcal{O}) \to \mathbb{P}^1 \times X$ realizes the relation

$$\begin{aligned} [\mathbb{P}_X(E \oplus \mathcal{O}) \to X] = [\mathbb{P}_{\widetilde{X}}(E \oplus \mathcal{O}) \to X] + e(E) \bullet [\mathbb{P}_X(E \oplus \mathcal{O}) \to X]^2 \\ - [\mathbb{P}_X(E \oplus \mathcal{O}) \to X] \bullet [\mathbb{P}_{\mathcal{E}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}) \to X], \end{aligned}$$

and more generally, the n-fold derived fibre product of $\mathbb{P}_W(E \oplus \mathcal{O})$ over W realizes the relation

$$\begin{aligned} [\mathbb{P}_X(E \oplus \mathcal{O}) \to X]^n = [\mathbb{P}_X(E \oplus \mathcal{O}) \to X]^n \bullet [\widetilde{X} \to X] + e(E) \bullet [\mathbb{P}_X(E \oplus \mathcal{O}) \to X]^{n+1} \\ - [\mathbb{P}_X(E \oplus \mathcal{O}) \to X]^n \bullet [\mathbb{P}_{\mathcal{E}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}) \to X]. \end{aligned}$$

Combining these these relations and remembering that Euler classes are nilpotent, we obtain the desired equality. ■

We can now begin our construction of Chern classes. The idea is to mimic the computation of Chern classes using the splitting principle (which we will establish later in Section 4.3) in a case that is manifestly natural under pullbacks.

Construction 4.6 (of $c_i(E)$). Suppose X is a quasi-projective derived scheme over a Noetherian ring A of finite Krull dimension, and let E be a vector bundle of rank r on X. We are going to construct

(1) a projective quasi-smooth morphism $\pi_{X,E} : \widetilde{X}_E \to X$ of relative virtual dimension 0 that is moreover natural under pullbacks in the obvious sense;

(2) a natural filtration $E_\bullet$ of $\pi_{X,E}^* E$ by vector bundles with line bundles $\mathscr{L}_1, \dots, \mathscr{L}_r$ as the associated graded pieces;

(3) a class $\eta_{X,E} \in \underline{\Omega}^0(\widetilde{X}_E)$ pushing forward to $1_X \in \underline{\Omega}^0(X)$ that is natural in the sense that given $f : Y \to X$, if f' is as in the homotopy Cartesian square (by the first item)

$$\begin{array}{ccc} \widetilde{Y}_{f^*E} & \xrightarrow{f'} & \widetilde{X}_E \\ {\scriptstyle \pi_{Y,f^*E}}\downarrow & & \downarrow{\scriptstyle \pi_{X,E}} \\ Y & \xrightarrow{f} & X \end{array}$$

we have $f'^*(\eta_{X,E}) = \eta_{Y,f^*E}$.

After finding such data, we can define the *Chern classes*

$$c_i(E) := \pi_{X,E!}\big(s_i(e(\mathscr{L}_1), \dots, e(\mathscr{L}_r)) \bullet \eta_{X,E}\big) \tag{4.2}$$

where s_i is the ith elementary symmetric polynomial, and the *total Chern class*

$$c(E) := 1 + c_1(E) + \cdots + c_r(E). \tag{4.3}$$

Note that the existence of $\eta_{X,E}$ implies that the pullback morphism

$$\pi^*_{X,E} : \underline{\Omega}^*(X) \to \underline{\Omega}^*(\widetilde{X}_E)$$

is an injection by Corollary 2.13. In a sense, we are merely pretending that Chern classes can be computed using the Whitney sum formula (which we will establish later), but we use the formula only on $\widetilde{X}_E$ so that the Chern classes are well defined.

In order to perform the desired construction, we are going to proceed by induction on the rank r. For the base case $r = 1$, we set $\pi_{X,\mathscr{L}}$ to be the identity morphism $X \to X$, the filtration to be the trivial filtration $0 \subset \mathscr{L}$, and the class $\eta_{X,\mathscr{L}} \in \underline{\Omega}^0(X)$ to be 1_X.

Suppose then that $r > 1$, we have performed the desired construction for all derived schemes Y and all vector bundles F of rank at most $r - 1$, and let E be a rank r vector bundle on a derived scheme X. Let $Z \hookrightarrow X$ be the derived vanishing locus of the zero section of E, and denote by $\pi'_{X,E}$ the structure morphism $\mathrm{Bl}_Z(X) \to X$. Note that by Lemma 4.4, there is a natural cofibre sequence

$$\mathcal{O}(\mathcal{E}) \to E \to Q \tag{4.4}$$

of vector bundles on $\mathrm{Bl}_Z(X)$, which provides the first step of the desired filtration. Moreover, by Lemma 4.5, the class

$$\eta'_{X,E} := \frac{1_{\mathrm{Bl}_Z(X)} - [\mathbb{P}_{\mathcal{E}}(\mathcal{O}(\mathcal{E}) \oplus \mathcal{O}) \to \mathrm{Bl}_Z(X)]}{1_{\mathrm{Bl}_Z(X)} - e(E) \bullet [\mathbb{P}_{\mathrm{Bl}_Z(X)}(E \oplus \mathcal{O}) \to \mathrm{Bl}_Z(X)]} \in \underline{\Omega}^*(\mathrm{Bl}_Z(X))$$

pushes forward to $1_X \in \underline{\Omega}^*(X)$. Note that both $\pi'_{X,E}$ and $\eta'_{X,E}$ are natural under pullbacks. We can then apply the inductive argument to the pair $(\mathrm{Bl}_Z(X), Q)$ to obtain a natural map $\pi_{\mathrm{Bl}_Z(X),Q} : \widetilde{\mathrm{Bl}_Z(X)}_Q \to \mathrm{Bl}_Z(X)$, a natural filtration of the vector bundle $\pi^*_{\mathrm{Bl}_Z(X),Q} Q$, and a natural class $\eta_{\mathrm{Bl}_Z(X),Q} \in \underline{\Omega}^0(\widetilde{\mathrm{Bl}_Z(X)}_Q)$ having the desired properties. We now set

$$\pi_{X,E} : \widetilde{X}_E := \widetilde{\mathrm{Bl}_Z(X)}_Q \xrightarrow{\pi_{\mathrm{Bl}_Z(X),Q}} \mathrm{Bl}_Z(X) \xrightarrow{\pi'_{X,E}} X \tag{4.5}$$

and

$$\eta_{X,E} := \eta_{\mathrm{Bl}_Z(X),Q} \bullet \pi^*_{\mathrm{Bl}_Z(X),Q}(\eta'_{X,E}). \tag{4.6}$$

Moreover, we obtain a natural filtration $E_\bullet$ of E on $\widetilde{X}_E$ by setting E_1 to be the pullback of the canonical inclusion $\mathcal{O}(\mathcal{E}) \to E$ to $\widetilde{X}_E$, and then pulling back the natural filtration of Q along the surjection $E \to Q$. To check that $\eta_{X,E}$ pushes forward to 1_X, we compute that

$$\begin{aligned}
\pi_{X,E!}(\eta_{X,E}) &= \pi_!\big(\pi_{\mathrm{Bl}_Z(X),Q!}(\eta_{\mathrm{Bl}_Z(X),Q} \bullet \pi^*_{\mathrm{Bl}_Z(X),Q}(\eta'_{X,E}))\big) \\
&= \pi_!\big(\pi_{\mathrm{Bl}_Z(X),Q!}(\eta_{\mathrm{Bl}_Z(X),Q}) \bullet \eta'_{X,E}\big) \quad \text{(projection formula)} \\
&= \pi_!(1_{\mathrm{Bl}_Z(X)} \bullet \eta'_{X,E}) \quad \text{(induction)} \\
&= \pi_!(\eta'_{X,E}) = 1_X.
\end{aligned}$$

We can therefore define Chern classes using formula (4.2).

4.2. *Basic properties of Chern classes*

Let us start by showing that Chern classes have many desirable properties.

Theorem 4.7. *Let X be a quasi-projective derived A-scheme for a Noetherian ring A, and let E be a vector bundle of rank r on X. Now the Chern classes of Construction 4.6 satisfy the following basic properties:*

(1) Naturality: *if $f : Y \to X$ is a quasi-projective map of derived schemes, then*

$$f^* c_i(E) = c_i(f^* E) \in \underline{\Omega}^r(Y).$$

(2) Normalization: $c_r(E) = e(E) \in \underline{\Omega}^r(X)$.

Proof. (1) This follows immediately from Construction 4.6, since it is completely natural under pullbacks.

(2) By the definition (4.2) of the top Chern class, we have

$$\begin{aligned}
c_r(E) &= \pi_{X,E!}\big(e(\mathscr{L}_1) \bullet \cdots \bullet e(\mathscr{L}_r) \bullet \eta_{X,E}\big) \\
&= \pi_{X,E!}(e(E) \bullet \eta_{X,E}) \quad \text{(Lemma 4.1)} \\
&= e(E) \bullet \pi_{X,E!}(\eta_{X,E}) \quad \text{(projection formula)} \\
&= e(E),
\end{aligned}$$

proving the claim. ∎

Next we are going to show that the classical formula satisfied by the Chern classes of $\mathcal{O}(1)$ in the cohomology ring of $\mathbb{P}(E)$ holds in our precobordism rings $\underline{\Omega}^*(\mathbb{P}(E))$ as well. This result will be used later in the proof of the Projective Bundle Formula (Theorem 5.11). The proof follows closely the proofs of [7, Lemmas 4.1.18 and 4.1.19].

Theorem 4.8. *Let X be a quasi-projective derived scheme over a Noetherian ring A of finite Krull dimension and let E be a vector bundle of rank r on X. Then, in $\underline{\Omega}^*(\mathbb{P}(E))$,*

$$0 = e(E(1)) = \sum_{i=0}^{r} (-1)^i c_{r-i}(E) \bullet c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))^i, \tag{4.7}$$

$$0 = e(E^\vee(-1)) = \sum_{i=0}^{r} (-1)^i c_{r-i}(E^\vee) \bullet c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^i. \tag{4.8}$$

Proof. We start by considering the vector bundle $E(1)$ on $\mathbb{P}(E)$. By the basic properties of projective bundles, we have a natural identification

$$\Gamma(\mathbb{P}(E); E(1)) \simeq \mathrm{Hom}_X(E, E);$$

let s be the global section of $E(1)$ corresponding to the identity morphism $E \to E$. It is easy to verify locally that the derived vanishing locus of s is empty so that $c_r(E(1)) = 0$. The triviality of $e(E^\vee(-1))$ then follows from Lemma 4.9, and formulas (4.7) and (4.8) follow from Lemma 4.10. ■

Lemma 4.9. *Let E be a vector bundle on a quasi-projective derived scheme X over a Noetherian ring A of finite Krull dimension. If $e(E) = 0$, then $e(E^\vee) = 0$.*

Proof. By naturality, $e(E^\vee) = c_r(E)$. Consider the map $\pi_{X,E^\vee} : \widetilde{X}_{E^\vee} \to X$, and recall that $E^\vee$ has a natural filtration on $\widetilde{X}_{E^\vee}$ with line bundles $\mathscr{L}_i$ as associated graded pieces. By dualizing, we obtain a natural filtration of E on $\widetilde{X}_{E^\vee}$, with associated graded pieces $\mathscr{L}_i^\vee$. Since $\underline{\Omega}^*$ has a formal group law, we have

$$e(\mathscr{L}_i) = e(\mathscr{L}_i^\vee) \bullet \sum_j a_j e(\mathscr{L}_i^\vee)^j =: e(\mathscr{L}_i^\vee) \bullet \rho(\mathscr{L}_i^\vee) \tag{4.9}$$

for some $a_j \in \underline{\Omega}^*(\mathrm{Spec}(A))$ (and even $a_0 = -1$).

We can then compute that

$$\begin{aligned} c_r(E^\vee) &= \pi_{X,E^\vee !}\big(e(\mathscr{L}_1) \bullet \cdots \bullet e(\mathscr{L}_r) \bullet \eta_{X,E^\vee}\big) \\ &= \pi_{X,E^\vee !}\big(e(\mathscr{L}_1^\vee) \bullet \rho(\mathscr{L}_1^\vee) \bullet \cdots \bullet e(\mathscr{L}_r^\vee) \bullet \rho(\mathscr{L}_r^\vee) \bullet \eta_{X,E^\vee}\big) \quad \text{(by (4.9))} \\ &= \pi_{X,E^\vee !}\big(e(E) \bullet \rho(\mathscr{L}_1^\vee) \bullet \cdots \bullet \rho(\mathscr{L}_r^\vee) \bullet \eta_{X,E^\vee}\big) \quad \text{(Lemma 4.1)} \\ &= \pi_{X,E^\vee !}\big(0 \bullet \rho(\mathscr{L}_1^\vee) \bullet \cdots \bullet \rho(\mathscr{L}_r^\vee) \bullet \eta_{X,E^\vee}\big) = 0, \end{aligned}$$

which proves the claim. ■

Lemma 4.10 (cf. [7, Lemma 4.1.18]). *Let E be a vector bundle on a quasi-projective derived scheme X over a Noetherian ring A of finite Krull dimension. If $e(E \otimes \mathscr{L}) = 0$, then*

$$\sum_{i=0}^{r} (-1)^i c_{r-i}(E) \bullet c_1(\mathscr{L}^\vee)^i = 0.$$

Proof. Let $\mathscr{L}_i$ be the line bundles associated to the natural filtration of E on $\widetilde{X}_E$, and let F denote the formal group law of the theory $\underline{\Omega}^*$. We now have

$$\big(F(e(\mathscr{L}_1 \otimes \mathscr{L}), e(\mathscr{L}^\vee)) - e(\mathscr{L}^\vee)\big) \bullet \cdots \bullet \big(F(e(\mathscr{L}_r \otimes \mathscr{L}), e(\mathscr{L}^\vee)) - e(\mathscr{L}^\vee)\big) = 0$$

since the product is clearly divisible by $e(\mathscr{L}_1 \otimes \mathscr{L}) \bullet \cdots \bullet e(\mathscr{L}_r \otimes \mathscr{L}) = e(E \otimes \mathscr{L}) = 0$. But as

$$F(e(\mathscr{L}_1 \otimes \mathscr{L}), e(\mathscr{L}^\vee)) = e(\mathscr{L}_1),$$

we have shown that

$$(e(\mathscr{L}_1) - e(\mathscr{L}^\vee)) \bullet \cdots \bullet (e(\mathscr{L}_r) - e(\mathscr{L}^\vee)) = 0 \in \underline{\Omega}(\widetilde{X}_E),$$

and the desired formula follows from the definition (4.2) of Chern classes after pushing forward along $\pi_{X,E}$ and applying the projection formula. ∎

4.3. *Splitting principle and further properties of Chern classes*

The purpose of this section is to derive further desirable properties of Chern classes that are going to be necessary in Section 5. The main tool is going to be the following theorem:

Theorem 4.11 (Splitting principle). *Suppose X is a derived scheme over a Noetherian ring A of finite Krull dimension, and suppose E is a vector bundle of rank r on X. If E has a filtration with line bundles $\mathscr{L}_i$ as the associated graded pieces* (*we say that E* has a splitting by $\mathscr{L}_i$), *then*

$$c_i(E) = s_i(e(\mathscr{L}_1), \dots, e(\mathscr{L}_r)),$$

where s_i is the ith elementary symmetric polynomial.

The above theorem will follow easily from Theorem 4.8 after we prove the following lemma.

Lemma 4.12. *Let X be a quasi-projective derived scheme over a Noetherian ring A of finite Krull dimension, and let E be a vector bundle of rank r on X. Then the morphism*

$$\mathscr{P}roj : \bigoplus_{i=0}^{r-1} \underline{\Omega}^{*-i}(X) \to \underline{\Omega}^*(\mathbb{P}(E))$$

defined by

$$(\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \mapsto \sum_{i=0}^{r-1} \alpha_i \bullet e(\mathcal{O}(1))^i$$

is an injection.

Remark 4.13. Of course, later we will show that the above map is surjective as well, giving rise to the projective bundle formula (Theorem 5.11). Unfortunately, the injectivity part seems to be necessary for the proof of the splitting principle, and that in turn seems to be necessary to conclude that all Chern classes are nilpotent. Nilpotence of Chern classes is used in the proof of Theorem 5.11.

Proof of Lemma 4.12. Note that we can assume that E has a splitting by some line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_r$ on X, by Corollary 2.13 the pullback morphism $\underline{\Omega}^*(X) \to \underline{\Omega}(\widetilde{X}_E)$ is an injection (see Construction 4.6) and moreover the pullback of E splits on $\widetilde{X}_E$.

Since X is quasi-projective we can find a cofibre sequence

$$\mathscr{L} \otimes E \to \mathscr{O}_X^{\oplus n+1} \to F$$

of vector bundles on X inducing a linear embedding $i : \mathbb{P}(E) \hookrightarrow \mathbb{P}^n_X$ with the property that

$$i^* \mathscr{L}(1) \simeq \mathscr{O}(1). \tag{4.10}$$

Of course, the advantage of this embedding is that the structure of $\underline{\Omega}^*(\mathbb{P}^n_X)$ is understood: by [3, Theorem 6.22], $\underline{\Omega}^*(\mathbb{P}^n_X) \cong \underline{\Omega}^*(X)[t]/(t^{n+1})$, where $t = e(\mathscr{O}(1))$.

By Proposition 2.5, $\mathbb{P}(E)$ is the derived vanishing locus of a section of $F(1)$. Without loss of generality we may assume that F has a splitting by line bundles $\mathscr{M}_1, \ldots, \mathscr{M}_s$ on X (note that $r + s = n + 1$) so that

$$\begin{aligned} i_!(1_{\mathbb{P}(E)}) &= e(F(1)) \\ &= F\big(e(\mathscr{M}_1), e(\mathscr{O}(1))\big) \bullet \cdots \bullet F\big(e(\mathscr{M}_s), e(\mathscr{O}(1))\big) \quad \text{(Proposition 4.1)}, \end{aligned}$$

where F is the formal group law of $\underline{\Omega}^*$. Expanding the right hand side, we see that

$$i_!(1_{\mathbb{P}(E)}) = \sum_{j=0}^{n} \beta_j \bullet e(\mathscr{O}(1))^j \tag{4.11}$$

where $\beta_j \in \underline{\Omega}^*(X)$ is nilpotent for $j \neq n - s = r$ and a unit for $j = n - s = r$. Using this, we can conclude that for $l \leq r$,

$$\begin{aligned} i_!(e(\mathscr{O}(1))^l) &= e(\mathscr{L}(1))^l \bullet i_!(1_{\mathbb{P}(E)}) \quad \text{(projection formula and (4.10))} \\ &= F(e(\mathscr{L}), e(\mathscr{O}(1)))^l \bullet e(F(1)) \\ &= \sum_{j=0}^{n} \beta_{l,j} \bullet e(\mathscr{O}(1))^j \quad \text{(by (4.11))} \end{aligned}$$

where $\beta_{l,j} \in \underline{\Omega}^*(X)$ is nilpotent for $j \neq r - l$ and a unit for $j = r - l$. In other words, the elements

$$i_!(e(\mathscr{O}(1))^l) \in \underline{\Omega}^*(\mathbb{P}^n_X)$$

are linearly independent over $\underline{\Omega}^*(X)$ when l ranges from 0 to n.

We have shown that the morphism of $\underline{\Omega}^*(X)$-modules

$$i_! \circ \mathscr{P}roj : \bigoplus_{i=0}^{r-1} \underline{\Omega}^{*-i}(X) \to \underline{\Omega}^*(\mathbb{P}^n_X)$$

is injective. But this implies the injectivity of $\mathscr{P}roj$, proving the claim. ∎

We can now prove the splitting principle:

Proof of Theorem 4.11. By Theorem 4.8, we have

$$\sum_{i=0}^{r}(-1)^i c_{r-i}(E) \bullet e(\mathcal{O}(1))^i = 0 \in \underline{\Omega}^*(\mathbb{P}(E^\vee)).$$

On the other hand, since E has a splitting by line bundles $\mathscr{L}_i$, we can argue as in the proof of Lemma 4.10 to find that

$$\sum_{i=0}^{r}(-1)^i s_{r-i}(e(\mathscr{L}_1),\dots,e(\mathscr{L}_r)) \bullet e(\mathcal{O}(1))^i = 0 \in \underline{\Omega}^*(\mathbb{P}(E^\vee)).$$

The left hand sides of both formulas share the term $(-1)^r e(\mathcal{O}(1))^r$, so we conclude that

$$\sum_{i=0}^{r-1}(-1)^i c_{r-i}(E) \bullet e(\mathcal{O}(1))^i = \sum_{i=0}^{r-1}(-1)^i s_{r-i}(e(\mathscr{L}_1),\dots,e(\mathscr{L}_r)) \bullet e(\mathcal{O}(1))^i.$$

It then follows from Lemma 4.12 that

$$c_i(E) = s_i(e(\mathscr{L}_1),\dots,e(\mathscr{L}_r)),$$

which is exactly what we wanted. ■

As an immediate application we prove the following properties of Chern classes, which will be useful later.

Theorem 4.14 (Nilpotence of Chern classes). *Let X be a quasi-projective derived scheme over a Noetherian ring A of finite Krull dimension and suppose E is a vector bundle on X. Then the Chern classes $c_i(E) \in \underline{\Omega}^*(X)$ are nilpotent.*

Proof. Since Euler classes are nilpotent, the theorem follows trivially from Theorem 4.11 whenever E splits. On the other hand, the pullback morphism

$$\pi^*_{X,E} : \underline{\Omega}^*(X) \to \underline{\Omega}^*(\widetilde{X}_E)$$

(see Construction 4.6) is injective by Corollary 2.13, and as E splits on $\widetilde{X}_E$, the claim follows. ■

Theorem 4.15 (Whitney sum formula). *Let X be a quasi-projective derived scheme over a Noetherian ring A of finite Krull dimension and let*

$$E' \to E \to E''$$

be a cofibre sequence of vector bundles on X. Then

$$c(E) = c(E') \bullet c(E'') \in \underline{\Omega}^*(X).$$

Proof. Again, the theorem follows trivially from Theorem 4.11 whenever both E' and E'' split. Applying Construction 4.6 first to (X, E') and then to $(\widetilde{X}_{E'}, E'')$, we obtain a map $\pi : \widetilde{X} \to X$ from an X-scheme $\widetilde{X}$ on which both E' and E'' split, and a class $\eta \in \underline{\Omega}^*(\widetilde{X})$ pushing forward to 1_X. Hence π^* is injective by Corollary 2.13, and the desired formula in $\underline{\Omega}^*(X)$ follows from the analogous formula in $\underline{\Omega}^*(\widetilde{X})$. ■

5. Applications

The purpose of this section is to use the theory of Chern classes to study the relationship between the universal precobordism theory $\underline{\Omega}^*(X)$, K-theory and intersection theory, as well as strengthen the weak projective bundle theorem from [3]. In Section 5.1 we will show that the algebraic K-theory ring $K^0(X)$ can be recovered in a simple way from the universal precobordism ring $\underline{\Omega}^*(X)$. In Section 5.2, we will study a candidate theory for the Chow cohomology of X (the *universal additive precobordism*, see Definition 5.1 below), and show that the classical Grothendieck–Riemann–Roch theorem generalizes to this setting.

First, we need some preliminary definitions. Recall (from Section 2.4) that for a fixed Noetherian ground ring A of finite Krull dimension, *precobordism theories* are defined as certain quotients of rings $\mathcal{M}_+^*$ of cobordism cycles. For the following constructions, it is convenient to think about the second set of relations defining $\underline{\Omega}^*$.

Definition 5.1. The *universal additive precobordism* $\underline{\Omega}_{\mathrm{a}}^*$ is the quotient theory obtained from $\mathcal{M}_+^*$ by enforcing the homotopy fibre relation, as well as the relation

$$e(\mathscr{L} \otimes \mathscr{L}') = e(\mathscr{L}) + e(\mathscr{L}')$$

on Euler classes of line bundles.

Definition 5.2. The *universal multiplicative precobordism* $\underline{\Omega}_{\mathrm{m}}^0$ is the quotient theory obtained from $\mathcal{M}_+^*$ by enforcing the homotopy fibre relation, as well as the relation

$$e(\mathscr{L} \otimes \mathscr{L}') = e(\mathscr{L}) + e(\mathscr{L}') - e(\mathscr{L}) \bullet e(\mathscr{L}')$$

on Euler classes of line bundles. Note that the latter relation does not respect the grading of $\mathcal{M}_+^*$, and therefore we do not get a natural grading on $\underline{\Omega}_{\mathrm{m}}^0$.

Proposition 5.3. *Consider the precobordism rings $\underline{\Omega}^*(X \to Y)$ as $\mathbb{L}$-algebras via the morphism $\mathbb{L} \to \underline{\Omega}^*(\mathrm{pt})$ classifying the formal group law. Let $\mathbb{L} \to \mathbb{Z}_{\mathrm{a}}$ and $\mathbb{L} \to \mathbb{Z}_{\mathrm{m}}$ be the ring homomorphisms classifying respectively the additive and the multiplicative formal group laws on the integers. Then we have natural equivalences*

$$\mathbb{Z}_{\mathrm{a}} \otimes_{\mathbb{L}} \underline{\Omega}^* \cong \underline{\Omega}_{\mathrm{a}}^* \quad \textit{and} \quad \mathbb{Z}_{\mathrm{m}} \otimes_{\mathbb{L}} \underline{\Omega}^* \cong \underline{\Omega}_{\mathrm{m}}^0.$$

Proof. Proofs of the claims are essentially the same, so let us prove the first one. We first note that by construction

$$e(\mathscr{L}_1 \otimes \mathscr{L}_2) = e(\mathscr{L}_1) + e(\mathscr{L}_2) \in \mathbb{Z}_{\mathrm{a}} \otimes_{\mathbb{L}} \underline{\Omega}_{\mathrm{a}}^*(X)$$

and therefore we obtain a well defined morphism $\psi : \underline{\Omega}_{\mathrm{a}}^*(X) \to \mathbb{Z}_{\mathrm{a}} \otimes_{\mathbb{L}} \underline{\Omega}^*(X)$ given by identity on the level of cycles. To finish the proof, we need the inverse to be well defined, and by Theorem 3.11 this is essentially equivalent to showing that $\underline{\Omega}_{\mathrm{a}}^*(X)$ is an oriented cohomology theory in the sense of Definition 3.6. The nontrivial axioms to check are (*Norm*) and (*FGL*).

(*Norm*) Applying (*PP*) to the definition of Euler class, we see that $e(\mathscr{L}) = i_{0!}(1_{D_0})$, where $i_0 : D_0 \hookrightarrow X$ is the inclusion of the derived vanishing locus of the zero section of $\mathscr{L}$. But it follows from the homotopy fibre relation that $i_{0!}(1_{D_0}) = i_!(1_D)$ for any inclusion $i : D \hookrightarrow X$ of a virtual Cartier divisor D in the linear system of $\mathscr{L}$. This proves the claim.

(*FGL*) By the above, $e(\mathscr{L})$ is nilpotent for $\mathscr{L}$ globally generated. A general line bundle $\mathscr{L}$ can be expressed as $\mathscr{L}_1 \otimes \mathscr{L}_2^\vee$ with $\mathscr{L}_i$ globally generated, and therefore $e(\mathscr{L}) = e(\mathscr{L}_1) - e(\mathscr{L}_2)$ is nilpotent for general $\mathscr{L}$. Moreover, the Euler classes satisfy a formal group law by construction, so we are done.

Since $\underline{\Omega}_\mathrm{a}(X)$ is an oriented cohomology theory, and since it satisfies the additive formal group law, we obtain a well defined morphism $\eta : \mathbb{Z}_\mathrm{a} \otimes_\mathbb{L} \underline{\Omega}^*(X) \to \underline{\Omega}_\mathrm{a}^*(X)$ commuting with pullbacks and pushforwards and preserving the identity elements. Hence, on the level of cycles, η is the identity, and therefore η and ψ are inverses of each other. ∎

5.1. *Conner–Floyd theorem*

The purpose of this section is to prove the following theorem:

Theorem 5.4. *Suppose we are working over a Noetherian base ring A of finite Krull dimension. Then the natural map*

$$\eta_K : \mathbb{Z}_\mathrm{m} \otimes_\mathbb{L} \underline{\Omega}^*(X) \cong \underline{\Omega}_\mathrm{m}^0(X) \to K^0(X)$$

sending $[V \xrightarrow{f} X]$ to $[f_*\mathcal{O}_V]$ is a natural isomorphism of rings commuting with pullbacks and Gysin pushforwards.

Note that even though the most interesting part of the above theorem is that it generalizes [1, Theorem 4.6] to a more general base ring, it actually also (slightly) generalizes the theorem in characteristic 0 since we have potentially fewer relations. Moreover, it is self-contained in that it does not use any previous results of [7] or [9]. As usual, the hardest part of the theorem is the construction of Chern classes, which is already taken care of. As the proof is essentially the same as in [1] (which is a derived version of the arguments in [7, Section 4.2]), we will merely outline the steps here, and refer to *loc. cit.* for the details.

The idea is to show that the morphism $\mathrm{ch}_\mathrm{m} : K^0(X) \to \underline{\Omega}_\mathrm{m}^0(X)$, which is defined by the formula

$$\mathrm{ch}_\mathrm{m}[E] = \mathrm{rank}(E) - c_1(E^\vee), \tag{5.1}$$

gives an inverse to η_K. This morphism is a homomorphism of rings: additivity follows from Theorem 4.15 and multiplicativity can be shown as in [1, Section 4.1]. Note that η_K clearly preserves Chern classes of line bundles as it is compatible with pullbacks and Gysin pushforwards, and it therefore follows from the splitting principle that η_K preserves all Chern classes, and therefore one can compute easily that $\eta_K \circ \mathrm{ch}_\mathrm{m}$ is the identity. Our next task is to show that ch_m commutes with Gysin pushforwards along projective quasi-smooth morphisms, which is the hardest part of showing that also $\mathrm{ch}_\mathrm{m} \circ \eta_K$ is the identity. We are going to need several lemmas for this.

Lemma 5.5. *The map* $\mathrm{ch}_\mathrm{m} : K^0 \to \underline{\Omega}^0_\mathrm{m}$ *preserves Chern classes.*

Proof. Proceed as in [1, proof of Lemma 4.1] and set $\beta = 1$. ■

Next we take care of a toy case.

Lemma 5.6. *Let E be a vector bundle on X and let $s : X \simeq \mathbb{P}(\mathcal{O}_X) \hookrightarrow \mathbb{P}(\mathcal{O}_X \oplus E)$ be the natural quasi-smooth inclusion. Then* ch_m *commutes with* $s_!$.

Proof. Follows from the previous lemma as in [1, proof of Lemma 4.2]. ■

Proposition 5.7. *The map* ch_m *commutes with Gysin pushforwards along arbitrary projective quasi-smooth morphisms.*

Proof. Proved in the same way as [1, Lemma 4.4]. ■

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.4. We already know that $\eta_\mathrm{m} \circ \mathrm{ch}_K$ is the identity transformation, so it is enough to show that $\mathrm{ch}_\mathrm{m} \circ \eta_K$ is. As both ch_m and η_K preserve the identity element and commute with Gysin pushforwards, the same is true for the composition $\mathrm{ch}_K \circ \eta_K$. But then it must be the identity on generators: indeed,

$$\begin{aligned}
\mathrm{ch}_K \circ \eta_K([V \xrightarrow{f} X]) &= \mathrm{ch}_K \circ \eta_K(f_!(1_V)) \\
&= f_!(\mathrm{ch}_K \circ \eta_K(1_V)) \quad \text{(Proposition 5.7)} \\
&= f_!(1_V) = [V \xrightarrow{f} X],
\end{aligned}$$

and we conclude that $\mathrm{ch}_K \circ \eta_K$ is the identity. ■

As an immediate consequence, we can prove the following analogue of a classical theorem of Levine–Morel.

Corollary 5.8. *Suppose we are working over a Noetherian base ring A of finite Krull dimension. Then the Grothendieck rings of vector bundles on quasi-projective A-schemes X give the universal oriented cohomology theory (in the sense of Definition 3.6) satisfying the multiplicative formal group law $F(x, y) = x + y - xy$.*

5.2. Intersection theory and Riemann–Roch

Recall that we can define a *Chern character* map $\mathrm{ch}_\mathrm{a} : K^0(X) \to \mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}(X)$ by the formula

$$[\mathscr{L}] \mapsto e^{-c_1(\mathscr{L})}$$

on line bundles and then extending formally to all vector bundles by using the splitting principle, and requiring ch_a to be additive. Similarly, we may define the *Todd class* $\mathrm{Td} : K^0(X) \to \mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}(X)$ by the formula

$$[\mathscr{L}] \mapsto \frac{c_1(\mathscr{L})}{e^{-c_1(\mathscr{L})} - 1}$$

on line bundles, and extending formally to all vector bundles by the splitting principle, and requiring that $\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \bullet \mathrm{Td}(F)$. Note for X quasi-projective, both classes are well defined on K-theory classes of perfect complexes, since $K^0(X)$ has a presentation as the Grothendieck group of vector bundles modulo exact sequences.

The purpose of this section is to prove the following theorem.

Theorem 5.9. *The maps*

$$\mathrm{ch}_\mathrm{a} : K^0(X) \to \mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}(X)$$

are ring homomorphisms which commute with pullbacks. Moreover, given $f : X \to Y$ quasi-smooth and projective, we have

$$f_!(\mathrm{ch}_\mathrm{a}(\alpha) \bullet \mathrm{Td}(\mathbb{L}_f)) = \mathrm{ch}_\mathrm{a}(f_!(\alpha)) \tag{5.2}$$

for all $\alpha \in K^0(X)$, where $\mathbb{L}_f$ is the relative cotangent complex of $X \to Y$. Finally, the induced map

$$\mathrm{ch}_\mathrm{a} : \mathbb{Q} \otimes K^0(X) \to \mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}(X)$$

is an isomorphism.

The proof is the same as in [1], but we have decided to write it down in a more explicit form here avoiding the bivariant formalism. Of course, the techniques are formally the same as those used to prove similar results in the context of oriented Borel–Moore homology theories (see e.g. [7, Example 4.1.24]), which are based on Quillen's idea from [13] of twisting cohomology theories.

Proof of Theorem 5.9. Let us start by *twisting* the theory $\mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}$. We define a new oriented cohomology theory $\mathbb{H}^0_\mathrm{t}$ by setting

$$\mathbb{H}^0_\mathrm{t}(X) := \mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}$$

and

$$f^\mathrm{t}_!(-) := f_!(- \bullet \mathrm{Td}(\mathbb{L}_f))$$

for any $f : X \to Y$ projective and quasi-smooth. The pullbacks of $\mathbb{H}^0_\mathrm{t}$ are by definition the pullbacks of $\mathbb{Q} \otimes \underline{\Omega}^*_\mathrm{a}$. Note that the pushforwards no longer respect degrees, and therefore we do not get a natural grading on $\mathbb{H}^0_\mathrm{t}$. To prove that $\mathbb{H}^0_\mathrm{t}$ is an oriented cohomology theory, we need to check that it satisfies the axioms in Definition 3.6.

(*Fun*) This is a computation: given $\alpha \in \underline{\Omega}^*(X)$,

$$\begin{aligned}(f \circ g)^{\mathrm{t}}_!(\alpha) &= f_!\big(g_!(\alpha \bullet \mathrm{Td}(\mathbb{L}_{f\circ g}))\big) \\ &= f_!\big(g_!(\alpha \bullet \mathrm{Td}(\mathbb{L}_g) \bullet g^*(\mathrm{Td}(\mathbb{L}_f)))\big) \\ &= f_!\big(g_!(\alpha \bullet \mathrm{Td}(\mathbb{L}_g)) \bullet \mathrm{Td}(\mathbb{L}_f)\big) \quad \text{(projection formula)} \\ &= f^{\mathrm{t}}_!(g^{\mathrm{t}}_!(\alpha)),\end{aligned}$$

proving the claim.

(*PP*) Follows easily from the fact that Todd classes and cotangent compexes are stable under derived pullbacks.

(*Norm*) Follows from (*PP*) and the homotopy fibre relation as in the proof of Proposition 5.3.

(*FGL*) Note that the cotangent complex of the zero section $s_0 : X \to \mathscr{L}$ is $\mathscr{L}^\vee[1]$. Let us denote by π the natural projection $\mathscr{L} \to X$. We can now compute the twisted Chern class for a line bundle $\mathscr{L}$ on X in terms of the untwisted Chern classes (satisfying the formal group law):

$$\begin{aligned}c^{\mathrm{t}}_1(\mathscr{L}) &= s_0^* s^{\mathrm{t}}_{0!}(1_X) = s_0^* s_{0!}(1_X \bullet \mathrm{Td}(\mathscr{L}^\vee)^{-1}) \\ &= s_0^*(s_{0!}(1_X) \bullet \pi^*\mathrm{Td}(\mathscr{L}^\vee)^{-1}) \quad \text{(projection formula)} \\ &= s_0^*(s_{0!}(1_X)) \bullet \mathrm{Td}(\mathscr{L}^\vee)^{-1} = c_1(\mathscr{L}) \bullet \mathrm{Td}(\mathscr{L}^\vee)^{-1} \\ &= c_1(\mathscr{L}) \bullet \frac{e^{-c_1(\mathscr{L}^\vee)} - 1}{c_1(\mathscr{L}^\vee)} = 1 - e^{c_1(\mathscr{L})}. \end{aligned} \tag{5.3}$$

Therefore

$$\begin{aligned}c^{\mathrm{t}}_1(\mathscr{L}_1 \otimes \mathscr{L}_2) &= 1 - e^{c_1(\mathscr{L}_1 \otimes \mathscr{L}_2)} = 1 - e^{c_1(\mathscr{L}_1)} \bullet e^{c_1(\mathscr{L}_2)} \\ &= (1 - e^{c_1(\mathscr{L}_1)}) + (1 - e^{c_1(\mathscr{L}_2)}) - (1 - e^{c_1(\mathscr{L}_1)}) \bullet (1 - e^{c_1(\mathscr{L}_2)}) \\ &= c^{\mathrm{t}}_1(\mathscr{L}_1) + c^{\mathrm{t}}_1(\mathscr{L}_2) - c^{\mathrm{t}}_1(\mathscr{L}_1) \bullet c^{\mathrm{t}}_1(\mathscr{L}_2),\end{aligned}$$

and $\mathbb{H}^0_{\mathrm{t}}$ satisfies the multiplicative formal group law. Nilpotency of Chern classes follows as in Proposition 5.3.

The universal property of K-theory (Corollary 5.8) therefore gives a morphisms of rings $\mathrm{ch}'_{\mathrm{a}} : K^0(X) \to \mathbb{Q} \otimes \underline{\Omega}^*_{\mathrm{a}}(X)$ commuting with pullbacks and satisfying formula (5.2). To prove that $\mathrm{ch}'_{\mathrm{a}}$ coincides with ch_{a} defined earlier, we note that $\mathrm{ch}'_{\mathrm{a}}$ must send Chern classes in K-theory to twisted Chern classes in the target, and compute

$$\begin{aligned}\mathrm{ch}'_{\mathrm{a}}([\mathscr{L}]) &= \mathrm{ch}'_{\mathrm{a}}(1 - c_1(\mathscr{L}^\vee)) = 1 - c^{\mathrm{t}}_1(\mathscr{L}^\vee) = e^{c_1(\mathscr{L}^\vee)} \\ &= e^{-c_1(\mathscr{L})} = \mathrm{ch}_{\mathrm{a}}([\mathscr{L}]).\end{aligned}$$

We have therefore proven everything except equivalence with rational coefficients.

In order to finish the proof, it is useful to consider the presentation $\underline{\Omega}^0_{\mathrm{m}}$ for K-theory (see Theorem 5.4). We want to find morphisms

$$\phi : \mathbb{Q} \otimes \underline{\Omega}^*_{\mathrm{a}}(X) \to \mathbb{Q} \otimes \underline{\Omega}^0_{\mathrm{m}}(X)$$

giving inverses to $\mathrm{ch_a}$. To do this, consider the Todd classes in $\mathbb{Q} \otimes \underline{\Omega}^0_{\mathrm{m}}(X)$ defined on line bundles by the formula

$$\mathrm{Td}'(\mathscr{L}) := \frac{c_1(\mathscr{L}^\vee)}{\log(1 - c_1(\mathscr{L}^\vee))}$$

and the theory $\mathbb{H}^*_{\mathrm{t}'}$ obtained from $\mathbb{Q} \otimes \underline{\Omega}^0_{\mathrm{m}}(X)$ by a twisting construction as above. One then computes that

$$\begin{aligned} c_1^{\mathrm{t}'}(\mathscr{L}) &= c_1(\mathscr{L}) \bullet \mathrm{Td}'(\mathscr{L}^\vee)^{-1} \\ &= c_1(\mathscr{L}) \bullet \frac{\log(1 - c_1(\mathscr{L}))}{c_1(\mathscr{L})} \\ &= \log(1 - c_1(\mathscr{L})) \end{aligned} \tag{5.4}$$

and therefore

$$\begin{aligned} c_1^{\mathrm{t}'}(\mathscr{L}_1 \otimes \mathscr{L}_2) &= \log(1 - c_1(\mathscr{L}_1 \otimes \mathscr{L}_2)) = \log\big((1 - c_1(\mathscr{L}_1)) \bullet (1 - c_1(\mathscr{L}_2))\big) \\ &= \log\big(1 - c_1(\mathscr{L}_1)) + \log(1 - c_1(\mathscr{L}_2)) \\ &= c_1^{\mathrm{t}'}(\mathscr{L}_1) + c_1^{\mathrm{t}'}(\mathscr{L}_2) \end{aligned}$$

so that $\mathbb{H}^*_{t'}$ is an oriented cohomology theory satisfying the additive formal group law. The universal property of $\underline{\Omega}^*_{\mathrm{a}}$ induces unique morphisms

$$\phi : \mathbb{Q} \otimes \underline{\Omega}^*_{\mathrm{a}}(X) \to \mathbb{Q} \otimes \underline{\Omega}^0_{\mathrm{m}}(X)$$

which are defined on the level of cycles as

$$[V \xrightarrow{f} X] \mapsto f_!(1_V \bullet \mathrm{Td}'(\mathbb{L}_f)) \in \mathbb{Q} \otimes \underline{\Omega}^0_{\mathrm{m}}(X).$$

Note that also the morphism $\mathrm{ch_a}$ has a cycle level description as

$$[V \xrightarrow{f} X] \mapsto f_!(1_V \bullet \mathrm{Td}(\mathbb{L}_f)) \in \mathbb{Q} \otimes \underline{\Omega}^*_{\mathrm{a}}(X).$$

These are inverses of each other. This is a computation, but before doing it, we note that we have to be slightly careful, since the Chern classes will be different in different theories. The first composition gives

$$\mathrm{ch_a}(\phi([V \xrightarrow{f} X])) = \mathrm{ch_a}(f_!(1_V \bullet \mathrm{Td}'(\mathbb{L}_f))) = f_!\big(1_V \bullet \mathrm{Td}'(\mathbb{L}_f)^{\mathrm{t}} \bullet \mathrm{Td}(\mathbb{L}_f)\big)$$

and the second one gives

$$\phi(\mathrm{ch_a}([V \xrightarrow{f} X])) = \phi(f_!(1_V \bullet \mathrm{Td}(\mathbb{L}_f))) = f_!\big(1_V \bullet \mathrm{Td}(\mathbb{L}_f)^{\mathrm{t}'} \bullet \mathrm{Td}'(\mathbb{L}_f)\big)$$

where the superscripts t, t' indicate the necessity of taking twists. We are done if we can show that $\mathrm{Td}'(\mathbb{L}_f)^{\mathrm{t}} \bullet \mathrm{Td}(\mathbb{L}_f) = 1$ and $\mathrm{Td}(\mathbb{L}_f)^{\mathrm{t}'} \bullet \mathrm{Td}'(\mathbb{L}_f)$.

But this reduces via the splitting principle to checking the identity for line bundles. We compute that

$$\begin{aligned}
\mathrm{Td}'(\mathscr{L})^{\mathrm{t}} \bullet \mathrm{Td}(\mathscr{L}) &= \frac{c_1^{\mathrm{t}}(\mathscr{L}^\vee)}{\log(1 - c_1^{\mathrm{t}}(\mathscr{L}^\vee))} \bullet \frac{c_1(\mathscr{L})}{e^{-c_1(\mathscr{L})} - 1} \\
&= \frac{1 - e^{c_1(\mathscr{L}^\vee)}}{\log(e^{c_1(\mathscr{L}^\vee)})} \bullet \frac{c_1(\mathscr{L})}{e^{-c_1(\mathscr{L})} - 1} \qquad \text{(by (5.3))} \\
&= \frac{1 - e^{-c_1(\mathscr{L})}}{-c_1(\mathscr{L})} \bullet \frac{c_1(\mathscr{L})}{e^{-c_1(\mathscr{L})} - 1} = 1
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Td}(\mathscr{L})^{\mathrm{t}'} \bullet \mathrm{Td}'(\mathscr{L}) &= \frac{c_1^{\mathrm{t}'}(\mathscr{L})}{e^{-c_1^{\mathrm{t}'}(\mathscr{L})} - 1} \bullet \frac{c_1(\mathscr{L}^\vee)}{\log(1 - c_1(\mathscr{L}^\vee))} \\
&= \frac{\log(1 - c_1(\mathscr{L}))}{e^{-\log(1 - c_1(\mathscr{L}))} - 1} \bullet \frac{c_1(\mathscr{L}^\vee)}{\log(1 - c_1(\mathscr{L}^\vee))} \qquad \text{(by (5.4))} \\
&= -\frac{\log(1 - c_1(\mathscr{L}^\vee))}{e^{\log(1 - c_1(\mathscr{L}^\vee))} - 1} \bullet \frac{c_1(\mathscr{L}^\vee)}{\log(1 - c_1(\mathscr{L}^\vee))} \\
&= -\frac{\log(1 - c_1(\mathscr{L}^\vee))}{-c_1(\mathscr{L}^\vee)} \bullet \frac{c_1(\mathscr{L}^\vee)}{\log(1 - c_1(\mathscr{L}^\vee))} = 1,
\end{aligned}$$

which shows that ϕ and ch_{a} are inverses of each other and finishes the proof. ■

Remark 5.10. We note that for X smooth over a field, one can define a morphism

$$\eta_{\mathrm{a}} : \underline{\Omega}_{\mathrm{a}}^*(X) \to \mathrm{CH}^*(X)$$

by the formula

$$[V \xrightarrow{f} X] \mapsto f_!([V]^{\mathrm{vir}}),$$

where $[V]^{\mathrm{vir}}$ is the virtual fundamental class of the quasi-smooth derived scheme V. If k has characteristic 0, then η_{a} is a surjection because pushforwards of virtual fundamental classes of quasi-smooth derived schemes generate $\mathrm{CH}^*(X)$ by [7, Theorem 4.5.1], but we are unable to conclude that η_{a} is an isomorphism. Note that this is in contrast with results of [1, Section 4.2], because in characteristic 0 there exist natural isomorphisms

$$\mathbb{Z}_{\mathrm{a}} \otimes_{\mathbb{L}} \Omega^*(X) \cong \mathrm{CH}^*(X)$$

for smooth varieties X, where Ω^* is the bivariant derived algebraic cobordism of [1]. The theories $\underline{\Omega}^*$ and Ω^* are compared in [2], where it is shown that Ω^* is a quotient of $\underline{\Omega}^*$ by the so called snc relations.

One can check that η_{a} preserves Chern classes, and therefore the usual Chern character morphism factors as

$$K^0(X) \xrightarrow{\mathrm{ch}_{\mathrm{a}}} \mathbb{Q} \otimes \underline{\Omega}_{\mathrm{a}}^*(X) \xrightarrow{\mathbb{Q} \otimes \eta_{\mathrm{a}}} \mathbb{Q} \otimes \mathrm{CH}^*(X)$$

whenever X is smooth over a field. Combining the rational isomorphism results of the classical Grothendieck–Riemann–Roch and Theorem 5.9, it follows that $\mathbb{Q} \otimes \eta_{\mathrm{a}}$ is an isomorphism. In other words, in the case of smooth varieties, Theorem 5.9 recovers the usual Grothendieck–Riemann–Roch theorem.

5.3. *Projective bundle formula*

The purpose of this section is to generalize the projective bundle formula for trivial projective bundles [3, Theorem 6.22] to general projective bundles. More precisely, we want to prove the following theorem:

Theorem 5.11 (Projective bundle formula). *Let X be a quasi-projective derived scheme over a Noetherian ring A and let E be a vector bundle of rank r on X. Then*

$$\underline{\Omega}^*(\mathbb{P}(E)) \cong \underline{\Omega}^*(X)[t]/\big(c_r(E^\vee) - c_{r-1}(E^\vee)t + \cdots + (-1)^r t^r\big)$$

where $t \in \underline{\Omega}^1(\mathbb{P}(E))$ is the first Chern class of $\mathcal{O}(1)$.

Our strategy is to first embed $\underline{\Omega}^*(\mathbb{P}(E))$ into $\underline{\Omega}^{*,1}(X)$, where $\underline{\Omega}^{*,1}$ is the precobordism of line bundles from [3, Section 6]. Let us recall that by [3, Theorems 6.12 and 6.13], $\underline{\Omega}^{*,1}(X)$ is a free $\underline{\Omega}^*(X)$-module with a basis given by

$$[\mathbb{P}^i; \mathcal{O}(1)] := [\mathbb{P}^i_X \to X, \mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X),$$

where $i \geq 0$. Theorem 5.11 will follow from this and some elementary manipulation of algebraic expressions. Suppose $\mathcal{A}$ is a line bundle on X such that $\mathcal{A} \otimes E^\vee$ is globally generated, i.e., there exists a surjection $(\mathcal{A}^{\oplus R})^\vee \to E^\vee$ for some $R > 0$. Notice that such an $\mathcal{A}$ always exists as X is quasi-projective.

Next we are going to construct an embedding of $\underline{\Omega}^*(\mathbb{P}(E))$ into $\underline{\Omega}^{*,1}(X)$. In order to do so, we will first have to consider the diagram

$$\begin{array}{ccc} & & \operatorname{colim}_n \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n})) \\ & & \downarrow j \\ \underline{\Omega}^*(\mathbb{P}(E)) & \xrightarrow{\ i\ } & \operatorname{colim}_n \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)) \\ & & \downarrow \mathcal{F} \\ & & \underline{\Omega}^{*,1}(X) \end{array} \tag{5.5}$$

where the colimits over n are induced by the obvious inclusions $\mathcal{A}^{\oplus n} \to \mathcal{A}^{\oplus n+1}$. The maps i and j are induced by obvious inclusions of summands, and they are therefore split injections by Proposition 2.14. Finally, $\mathcal{F}$ is defined by the formula

$$\mathcal{F}([V \xrightarrow{f} \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)]) := [V \xrightarrow{\pi \circ f} X, f^*\mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X), \tag{5.6}$$

although it is not obvious yet that this gives rise to a well defined morphism. Note that if we can show that $\mathcal{F}$ is injective, then $\mathcal{F} \circ i$ provides the desired embedding. It turns out that the right way to proceed is to show that j and $\mathcal{F}$ are *isomorphisms*.

Let us start with the well definedness of $\mathcal{F}$.

Lemma 5.12. *Formula* (5.6) *gives rise to a well defined homomorphism of Abelian groups*

$$\mathcal{F} : \mathrm{colim}_n\, \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)) \to \underline{\Omega}^{*,1}(X).$$

Proof. It is enough to show that the maps $\mathcal{F}_n : \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)) \to \underline{\Omega}^{*,1}(X)$ defined by

$$[V \xrightarrow{f} \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)] \mapsto [V \xrightarrow{\pi \circ f} X, f^*\mathcal{O}(1)]$$

are well defined, as they clearly commute with the structure morphisms of the colimit. By definition it is enough to check that $\mathcal{F}_n$ respects the double point cobordism relation: Suppose we have a projective quasi-smooth morphism $W \to \mathbb{P}^1 \times \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)$ such that the fibre W_∞ over ∞ is a sum of two divisors $A + B$ in W. Moreover, let us denote by $\mathscr{L}$ the pullback of $\mathcal{O}(1)$ to W, and by W_0 the fibre over 0. We now compute that

$$\begin{aligned}
&\mathcal{F}_n([A \to \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)] + [B \to \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)] - [\mathbb{P}_{A\cap B}(\mathcal{O}(A) \oplus \mathcal{O}) \to \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)])\\
&= [A \to X, \mathscr{L}|_A] + [B \to X, \mathscr{L}|_B] - [\mathbb{P}_{A\cap B}(\mathcal{O}(A) \oplus \mathcal{O}) \to X, \mathscr{L}|_{A\cap B}]\\
&= [W_0 \to X, \mathscr{L}|_{W_0}] \quad ([3, \text{Remark 6.7}])\\
&= \mathcal{F}_n([W_0 \to \mathbb{P}(\mathcal{A}^{\oplus n} \oplus E)]) \in \underline{\Omega}^{*,1}(X),
\end{aligned}$$

proving the claim. ∎

The following lemma will do a lot of work for us.

Lemma 5.13. *Let X be a derived scheme and E a vector bundle on X. Suppose that we have a line bundle $\mathscr{L}$ on a quasi-smooth and projective derived X-scheme V and surjections $s_1, s_2 : E^\vee|_V \to \mathscr{L}$ giving rise to maps $f_1, f_2 : V \to \mathbb{P}(E)$. Then*

$$[V \xrightarrow{f_1} \mathbb{P}(E)] = [V \xrightarrow{f_2} \mathbb{P}(E)] \in \underline{\Omega}^*(\mathbb{P}(E)).$$

Proof. Let $\iota_1 : \mathbb{P}(E) \simeq \mathbb{P}(E \oplus 0) \hookrightarrow \mathbb{P}(E \oplus E)$ and $\iota_2 : \mathbb{P}(E) \simeq \mathbb{P}(0 \oplus E) \hookrightarrow \mathbb{P}(E \oplus E)$ be the natural linear embeddings. As the pushforward morphism $\iota_{1!}$ is injective by Proposition 2.14, it is enough to show that

$$\iota_{1!}[V \xrightarrow{f_1} \mathbb{P}(E)] = \iota_{1!}[V \xrightarrow{f_2} \mathbb{P}(E)] \in \underline{\Omega}^*(\mathbb{P}(E \oplus E)).$$

We can now define a surjection $x_0 s_1 + x_1 s_2 : E^\vee|_{\mathbb{P}^1_V} \oplus E^\vee|_{\mathbb{P}^1_V} \to \mathscr{L}(1)$ of vector bundles on $\mathbb{P}^1 \times V$ giving rise to an algebraic cobordism

$$\mathbb{P}^1 \times V \to \mathbb{P}^1 \times \mathbb{P}(E \oplus E),$$

showing that

$$\iota_{1!}[V \xrightarrow{f_1} \mathbb{P}(E)] = \iota_{2!}[V \xrightarrow{f_2} \mathbb{P}(E)] \in \underline{\Omega}^*(\mathbb{P}(E \oplus E)).$$

On the other hand, we can argue similarly using the surjection $x_0 s_2 + x_1 s_2$ to find that

$$\iota_{1!}[V \xrightarrow{f_2} \mathbb{P}(E)] = \iota_{2!}[V \xrightarrow{f_2} \mathbb{P}(E)] \in \underline{\Omega}^*(\mathbb{P}(E \oplus E)),$$

so we are done. ∎

The following two lemmas show that j and $\mathcal{F}$ are isomorphisms.

Lemma 5.14. *Let $\mathcal{A}$ be a line bundle on X such that $\mathcal{A} \otimes E^\vee$ is globally generated. Then the morphism $j : \operatorname{colim}_n \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n})) \to \operatorname{colim}_n \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n} \oplus E))$ from diagram (5.5) is an isomorphism.*

Proof. We already know that j is an injection, so it is enough to show that it is a surjection. Consider

$$\alpha := [V \to \mathbb{P}(\mathcal{A}^{\oplus m} \oplus E)] \in \operatorname{colim}_n \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n} \oplus E))$$

corresponding to a surjection $(\mathcal{A}^{\oplus m})^\vee|_V \oplus E^\vee|_V \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle on V. By our hypotheses there exists a surjection $(\mathcal{A}^{\oplus R})^\vee \to E^\vee$ such that we can form a composition of surjections

$$(\mathcal{A}^{\oplus R+m})^\vee|_V \to (\mathcal{A}^{\oplus m})^\vee|_V \oplus E^\vee|_V \to \mathcal{L},$$

showing by Lemma 5.13 that α is in the image of j. ∎

Lemma 5.15. *The morphism $\mathcal{F} \circ j : \operatorname{colim}_n \underline{\Omega}^*(\mathbb{P}(\mathcal{A}^{\oplus n})) \to \underline{\Omega}^{*,1}(X)$ from diagram (5.5) is an isomorphism.*

Proof. Let us denote by $[\mathcal{L}] \in \underline{\Omega}^{0,1}(X)$ the class of the cycle $[X \xrightarrow{\mathrm{Id}} X, \mathcal{L}]$. Note that $[\mathcal{L}]$ is a unit with inverse given by $[\mathcal{L}^\vee]$. Using the natural equivalences $\psi_{\mathcal{A}} : \mathbb{P}(\mathcal{A}^{\oplus n}) \cong \mathbb{P}(\mathcal{O}^{\oplus n})$, which on the level of the functor of points are given by

$$[(\mathcal{A}^{\oplus n})^\vee \to \mathcal{L}] \mapsto [\mathcal{O}^{\oplus n} \to \mathcal{A} \otimes \mathcal{L}],$$

we can form the commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_n \underline{\Omega}(\mathbb{P}(\mathcal{A}^{\oplus n})) & \xrightarrow{\psi_{\mathcal{A}}} & \operatorname{colim}_n \underline{\Omega}(\mathbb{P}(\mathcal{O}^{\oplus n})) \\ \downarrow{\scriptstyle \mathcal{F} \circ j} & & \downarrow{\scriptstyle j'} \\ \underline{\Omega}^{*,1}(X) & \xrightarrow{[\mathcal{A}]\bullet} & \underline{\Omega}^{*,1}(X) \end{array}$$

The map j' is by construction the isomorphism $\underline{\Omega}^*_{\mathbb{P}^\infty}(X) \to \underline{\Omega}^{*,1}(X)$ constructed in [3, Section 6], and since $\psi_{\mathcal{A}}$ and $[\mathcal{A}]\bullet$ are isomorphisms, also $\mathcal{F} \circ j$ must be one. ∎

Let us record the following result, which is just a combination of the preceding lemmas.

Proposition 5.16. *Let X be a quasi-projective derived scheme over a Noetherian ring A, and let E be a vector bundle of rank r on X. Then the morphism $\iota_E : \underline{\Omega}^{*+r-1}(\mathbb{P}(E)) \to \underline{\Omega}^{*,1}(X)$ defined by the formula*

$$[V \xrightarrow{f} \mathbb{P}(E)] \mapsto [V \xrightarrow{\pi \circ f} X, f^*\mathcal{O}(1)],$$

where π is the natural projection $\mathbb{P}(E) \to X$, is a split injection.

We have reduced the proof of Theorem 5.11 to understanding the image of ι_E. Recall that the differentiation operators

$$\partial_{c_1} : \underline{\Omega}^{*,1}(X) \to \underline{\Omega}^{*+1,1}(X)$$

were defined by linearly extending

$$\partial_{c_1}([V \xrightarrow{f} X, \mathscr{L}]) := f_!(c_1(\mathscr{L}) \bullet [V \to V, \mathscr{L}])$$

so that they are linear over the subring $\underline{\Omega}^*(X) \hookrightarrow \underline{\Omega}^{*,1}(X)$ (cycles with trivial line bundles) and satisfy $\partial_{c_1}([\mathbb{P}^i, \mathcal{O}(1)]) = [\mathbb{P}^{i-1}, \mathcal{O}(1)]$, where $\mathbb{P}^i$ is the empty scheme for $i < 0$. Moreover, it is clear that the embedding ι_E of Proposition 5.16 changes the action of $c_1(\mathcal{O}(1))$ on the source to the action of ∂_{c_1} on the target. We can therefore use formula (4.8) of Theorem 4.8 to obtain the following result:

Lemma 5.17. *Suppose that*

$$\sum_{i \geq 0} a_i \bullet [\mathbb{P}^i, \mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X)$$

lies in the image of ι_E. Then the a_i satisfy the linear recurrence relation

$$\sum_{j=0}^{r} (-1)^j c_{r-j}(E^\vee) \bullet a_{n+j} = 0 \in \underline{\Omega}^*(X) \tag{5.7}$$

for all $n \geq 0$, where $c_0(E^\vee) = 1_X$. ∎

Let us define $b_0, \ldots, b_{r-1} \in \underline{\Omega}^{*,1}(X)$ by setting

$$b_i := \sum_{i \geq 0} a_{i,j} \bullet [\mathbb{P}^j, \mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X),$$

where

$$a_{i,j} = \begin{cases} 1_X & \text{if } i = j, \\ 0 & \text{if } j \leq r-1 \text{ and } i \neq j, \end{cases}$$

and where $a_{i,j}$ satisfy the recursion formula (5.7) for all $0 \leq i \leq r-1$. Note that $a_{i,j} = 0$ for $j \gg 0$ because Chern classes are nilpotent, and therefore the b_i are well defined. We have shown the following.

Lemma 5.18. *The image of ι_E is contained in the free $\underline{\Omega}^*(X)$-module F_E generated by $b_0, \dots, b_{r-1} \in \underline{\Omega}^{*,1}(X)$.*

Proof. Indeed, any collection of $a_i \in \underline{\Omega}^*(X)$ satisfying (5.7) can be expressed as a unique $\Omega^*(X)$-linear combination of $b_0, \dots, b_{r-1}$. ■

We are ready to prove the projective bundle formula.

Proof of Theorem 5.11. It is enough to show that the elements $b_i \in \underline{\Omega}^{*,1}(X)$ lie in the image of ι_E, because then

$$\iota_E \circ \mathscr{P}roj : \bigoplus_{i=0}^{r-1} \underline{\Omega}^{*-i}(X) \to \underline{\Omega}^{*,1}(X),$$

where $\mathscr{P}roj$ is defined as in Lemma 4.12, will surject onto F_E, proving the surjectivity of $\mathscr{P}roj$ by the injectivity of ι_E. It follows that

$$\mathscr{P}roj : \bigoplus_{i=0}^{r-1} \underline{\Omega}^{*+r-1-i}(X) \to \underline{\Omega}^*(\mathbb{P}(E))$$

is an isomorphism, because it was already proven to be an injection in Lemma 4.12, and the claim follows from Theorem 4.8.

We note that the truthfulness of the claim does not change if we twist the vector bundle E by a line bundle $\mathscr{L}$. Indeed, the derived scheme $\mathbb{P}(\mathscr{L} \otimes E)$ is isomorphic to $\mathbb{P}(E)$, but its hyperplane bundle is equivalent to $\mathscr{L}^\vee \otimes \mathcal{O}(1)$. On the other hand, it follows from the formal group law of $\underline{\Omega}^*$ and Theorem 4.8 that for all $0 \leq l \leq r-1$,

$$\begin{aligned}\iota_E&\big(c_1(\mathscr{L}^\vee \otimes \mathcal{O}(1))^l\big) \\ &= c_{l,0} \bullet \iota_E(1_{\mathbb{P}(E)}) + c_{l,1} \bullet \iota_E(c_1(\mathcal{O}(1))) + \cdots + c_{l,r-1} \bullet \iota_E(c_1(\mathcal{O}(1))^{r-1}) \in F_E\end{aligned}$$

where $c_{l,l}$ is a unit and all the other $c_{l,i}$ are nilpotent. Hence

$$\iota_E(1_X), \iota_E(c_1(\mathcal{O}(1))), \dots, \iota_E(c_1(\mathcal{O}(1))^{r-1})$$

form an $\underline{\Omega}^*(X)$-basis for F_E if and only if

$$\iota_E(1_X), \iota_E(c_1(\mathcal{O}(1))), \dots, \iota_E(c_1(\mathcal{O}(1))^{r-1})$$

do. We can therefore assume that E embeds into a trivial vector bundle by tensoring it with a suitable line bundle.

Consider then a cofibre sequence

$$E \to \mathcal{O}^{\oplus N} \to E''$$

of vector bundles, whose left hand map gives rise to a linear embedding $i : \mathbb{P}(E) \hookrightarrow \mathbb{P}^{N-1}_X$. It is clear that ι_E factors as

$$\underline{\Omega}^{*+r-1}(\mathbb{P}(E)) \xrightarrow{i_!} \underline{\Omega}^{*+N-1}(\mathbb{P}^{N-1}_X) \xrightarrow{\iota_{\mathcal{O}^{\oplus N}_X}} \underline{\Omega}^{*,1}(X),$$

and moreover, by Proposition 2.5,

$$i_!(1_{\mathbb{P}(E)}) = c_{N-r}(E''(1)).$$

Hence, denoting by $\alpha_1, \dots, \alpha_{N-r}$ the Chern roots of E'', it follows that

$$\begin{aligned} i_!(1_{\mathbb{P}(E)}) = c_{N-r}(E''(1)) &= \prod_{j=0}^{N-r} F(\alpha_j, c_1(\mathcal{O}(1))) \\ &= \sum_{j=0}^{N-1} a_i \bullet c_1(\mathcal{O}(1))^{N-1-i} \in \underline{\Omega}^{N-r}(\mathbb{P}_X^{N-1}), \end{aligned}$$

with a_{r-1} a unit and the other a_i nilpotent. In other words,

$$\iota_E(1_{\mathbb{P}(E)}) = \sum_{i \geq 0} a_i \bullet [\mathbb{P}^i, \mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X)$$

with a_{r-1} a unit and a_i nilpotent for other i (and $a_i = 0$ for $i \geq N$).

As

$$\iota_E(c_1(\mathcal{O}(1))^l) = \partial_{c_1}^l \iota_E(1_{\mathbb{P}(E)}) = \sum_{i \geq 0} a_{l+i} \bullet [\mathbb{P}^i, \mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X),$$

where a_i are as above, it is an elementary fact from linear algebra that for any $0 \leq m \leq r-1$ we can find a unique $\underline{\Omega}^*(X)$-linear combination b'_{m} of the classes $\iota_E(c_1(\mathcal{O}(1))^l)$ for $0 \leq l \leq r-1$ such that

$$b'_{\mathrm{m}} = [\mathbb{P}^m, \mathcal{O}(1)] + \sum_{i \geq r} c_i \bullet [\mathbb{P}^i, \mathcal{O}(1)] \in \underline{\Omega}^{*,1}(X).$$

As b'_{m} is in the image of ι_E, it satisfies the recursion formula (5.7), which in turn implies that it coincides with b_{m} by the construction of the latter. Hence $b_0, \dots, b_{r-1}$ lie in the image of ι_E, proving the claim. ■

Bivariant projective bundle formula for precobordism theories

Above, we have chosen to restrict to the associated cohomology theory of the universal precobordism theory, since we did not introduce the bivariant formalism in the introduction. The purpose of this section is to show that Theorem 5.11 can be easily generalized in two directions:

(1) we can replace the universal precobordism rings $\underline{\Omega}^*(X)$ with any precobordism ring $\mathbb{B}^*(X)$ (in other words, we can add relations);

(2) instead of proving the projective bundle formula for the cohomology rings $\mathbb{B}^*(X)$, we can prove an analogous theorem for all the bivariant groups $\mathbb{B}^*(X \to Y)$.

For background on bivariant precobordism theories, the reader can consult [3, Section 6].

Theorem 5.19. *Let $\mathbb{B}^*$ be a bivariant precobordism theory in the sense of* [3], *and suppose we have a morphism $X \to Y$ between derived schemes quasi-projective over a Noetherian ring A of finite Krull dimension. Let E be a vector bundle of rank r on X. Then*

(1) *we have a natural isomorphism of rings*

$$\mathbb{B}^*(\mathbb{P}(E)) \cong \mathbb{B}^*(X)[t]/\langle f \rangle,$$

where $t = e(\mathcal{O}(1))$ and $f = \sum_{i=0}^{r}(-1)^i c_{r-i}(E^\vee)t^i$;

(2) *the morphism*

$$\mathbb{B}^*(\mathbb{P}(E)) \otimes_{\mathbb{B}^*(X)} \mathbb{B}^*(X \to Y) \to \mathbb{B}^*(\mathbb{P}(E) \to Y)$$

defined by

$$\alpha \otimes \beta \mapsto \alpha \bullet \theta(\pi) \bullet \beta$$

gives an isomorphism of $\mathbb{B}^*(\mathbb{P}(E))$-modules, where π is the structure morphism $\mathbb{P}(E) \to X$.

Proof. The proof of Theorem 5.11 goes through with essentially no changes. The first task is to show that we have a natural embedding $\mathbb{B}^*(\mathbb{P}(E) \to Y) \hookrightarrow \mathbb{B}^{*,1}(X \to Y)$ analogous to the embedding of Proposition 5.16. To do this, consider a bivariant version

$$\begin{array}{ccc} & & \operatorname{colim}_n \mathbb{B}^*(\mathbb{P}(\mathcal{A}^{\oplus n}) \to Y) \\ & & \big\downarrow j \\ \mathbb{B}^*(\mathbb{P}(E) \to Y) & \xrightarrow{\ i\ } & \operatorname{colim}_n \mathbb{B}^*(\mathbb{P}(\mathcal{A}^{\oplus n} \oplus E) \to Y) \\ & & \big\downarrow \mathcal{F} \\ & & \mathbb{B}^{*,1}(X \to Y) \end{array} \tag{5.8}$$

of the diagram (5.5), where $\mathcal{A}$ is a line bundle such that we can find a surjection $(\mathcal{A}^{\oplus R})^\vee \to E^\vee$ for R large enough. We note that

- the maps i, j induced by bivariant pushforwards are injective by an obvious bivariant analogue of Proposition 2.14;
- the map j is an isomorphism (cf. Lemma 5.14);
- the map $\mathcal{F} \circ j$ is a well defined isomorphism (this is proven exactly like Lemma 5.15, but using the more general isomorphism $\mathbb{B}^*_{\mathbb{P}^\infty}(X \to Y) \cong \mathbb{B}^{*,1}(X \to Y)$ constructed in [3]), and therefore also $\mathcal{F}$ is a well defined isomorphism.

We therefore obtain an embedding $\mathbb{B}^*(\mathbb{P}(E) \to Y) \to \mathbb{B}^{*,1}(X \to Y)$ as in Proposition 5.16. As in the proof of Theorem 5.11, we are reduced to understanding the set of solutions to an easy linear recursion. Claims (1) and (2) can be proved in a very similar way to Theorem 5.11. ∎

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