J. Eur. Math. Soc. 25, 1349–1377 (2023) DOI 10.4171/JEMS/1213

Nicolas Burq · Iván Moyano

Propagation of smallness and control for heat equations

Received December 13, 2019

Abstract. In this note we investigate propagation of smallness properties for solutions to heat equations. We consider spectral projector estimates for the Laplace operator with Dirichlet or Neumann boundary conditions on a Riemanian manifold with or without boundary. We show that using the new approach for the propagation of smallness of Logunov and Malinnikova (2018) allows one to extend the spectral projector type estimates of Jerison and Lebeau (1999) from localisation on open sets to localization on arbitrary sets of non-zero Lebesgue measure; we can actually go beyond and consider sets of non-vanishing $d - \delta$ ($\delta > 0$ small enough) Hausdorff measure. We show that these new spectral projector estimates allow one to extend Logunov–Malinnikova's propagation of smallness results to solutions to heat equations. Finally, we apply these results to the null controllability of heat equations with controls localized on sets of positive Lebesgue measure. The main novelty here is that we can drop the constant coefficient assumptions of Apraiz et al. (2013, 2014) on the Laplace operator (or the analyticity assumption of Escauriaza et al. (2017) and Lebeau and Moyano (2019)) and deal with Lipschitz coefficients. Another important novelty is that we get the first (non-one-dimensional) exact controllability results with controls supported on measure zero sets.

Keywords. Propagation of smallness, heat equations, control, uniqueness

1. Introduction

In this note we are interested in understanding the propagation of smallness and in control for solutions to heat equations and their connections with the propagation of smallness for high frequency sums of eigenfunctions of the Laplace operator on a compact Riemanian manifold (M, g) with boundary. Let

$$\Delta = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g}\, g^{ij} \partial_j)$$

be the Laplace–Beltrami operator on M and let (e_k) be a family of eigenfunctions of $-\Delta$,

Nicolas Burq: Mathématiques, Université Paris-Saclay, UMR 8628 du CNRS, Bât. 307, 91405 Orsay Cedex, France, and Institut Universitaire de France; nicolas.burq@universite-paris-saclay.fr

Iván Moyano: Laboratoire J.A. Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, UMR 7351 du CNRS, 06108 Nice Cedex 02, France; ivan.moyano@unice.fr

Mathematics Subject Classification (2020): 35A02, 35Q93, 35K05, 58J35

with eigenvalues $\lambda_k^2 \to +\infty$ forming a Hilbert basis of $L^2(M)$,

$$-\Delta e_k = \lambda_k^2 e_k, \quad e_k|_{\partial M} = 0 \text{ (Dirichlet condition) or } \partial_\nu e_k|_{\partial M} = 0 \text{ (Neumann condition)}.$$

(Note that we will also denote by Δ the more general operator defined in (1.1) below.) Now, we consider an arbitrary finite linear combination of the form

$$\phi = \sum_{\lambda_k \leq \Lambda} u_k e_k(x),$$

and given a small subset $E \subset M$ (of positive Lebesgue measure or at least not too small in a sense to be made precise later), we want to understand how L^p norms of the restrictions of ϕ to the set E dominate Sobolev norms of ϕ on M.

In the case of domains and constant coefficient Laplace operators and subsets of positive Lebesgue measure, or in the case of Lipschitz metrics and open subsets E, this is now quite well understood [2, 5]. Here we shall be interested in the two cases where M is a $W^{2,\infty}$ compact manifold of dimension d with or without boundary (endowed with a Lipschitz metric) and observation domains E of positive Lebesgue measure or even of positive $(d-\delta)$-dimensional Hausdorff content for $\delta > 0$ small enough, but depending only on the dimension of the manifold M.

Here and below by $W^{2,\infty}$ manifolds, we mean that the changes of charts are $C^1 \cap W^{2,\infty}$ maps (C^1 with second order distribution derivatives bounded a.e. or equivalently the derivatives of the change of charts are Lipschitz functions). We allow slightly more general operators than Laplace–Beltrami operators and assume that M is endowed with a Lipschitz (positive definite) metric g and a Lipschitz (positive) density κ. Let

$$\Delta = \frac{1}{\kappa(x)} \operatorname{div} g^{-1}(x)\kappa(x)\nabla_x = \operatorname{div}_\kappa \nabla_g \tag{1.1}$$

be the corresponding Laplace operator. When $\kappa(x) = \sqrt{\det g(x)}$, we recover the usual Laplace–Beltrami operator on (M, g).

In all the results below, the manifold M will be assumed to satisfy the $W^{2,\infty}$ regularity above, and unless stated explicitly otherwise, Δ stands for the operator defined by (1.1) with Dirichlet or Neumann boundary condition if $\partial M \neq \emptyset$. Recall that the *$d$-Hausdorff content* (or *measure*) of a set $E \subset \mathbb{R}^n$ is

$$\mathcal{C}^d_{\mathcal{H}}(E) = \inf\Big\{\sum_j r_j^d;\ E \subset \bigcup_j B(x_j, r_j)\Big\},$$

and the *Hausdorff dimension* of E is defined as

$$\dim_{\mathcal{H}}(E) = \inf\{d;\ \mathcal{C}^d_{\mathcal{H}}(E) = 0\}.$$

We shall denote by $|E|$ the Lebesgue measure of the set E. Let us recall that the Hausdorff content of order n is equivalent to the Lebesgue measure,

$$\exists C_d, c_d > 0\ \forall A \text{ Borel set}, \quad c_d|A| \leq \mathcal{C}^d_{\mathcal{H}}(A) \leq C_d|A|,$$

and

$$\mathcal{C}^d_{\mathcal{H}}(E) > 0 \implies \forall d' < d,\ \mathcal{C}^{d'}_{\mathcal{H}}(E) \geq \inf(1, \mathcal{C}^d_{\mathcal{H}}(E)) \tag{1.2}$$

(indeed, $\sum_j r_j^{d'} \geq 1$ if there exists i_0 such that $r_{i_0} \geq 1$, and otherwise $\sum_j r_j^{d'} \geq \sum_j r_j^d$).

The value of the Hausdorff content is not invariant by diffeomorphisms, but the Hausdorff dimension is invariant by Lipschitz diffeomorphisms, as shown by

Proposition 1.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz diffeomorphism, such that*

$$\|\nabla_x(\phi)\|_{L^\infty} \leq C. \tag{1.3}$$

Then, for any $\sigma > 0$,

$$\mathcal{C}^\sigma_{\mathcal{H}}(E) > m \implies \mathcal{C}^\sigma_{\mathcal{H}}(\phi(E)) > C^{-\sigma} m. \tag{1.4}$$

Proof. Indeed, assume that $E \subset \bigcup_j B(x_j, r_j)$. Then $\phi(E) \subset \bigcup_j \phi(B(x_j, r_j))$. But, according to (1.3), with $y_j = \phi(x_j)$, we have

$$\|\phi(x) - \phi(y_j)\| \leq C\|x - x_j\| \implies \phi(B(x_j, r_j)) \subset B(\phi(x_j), Cr_j),$$

As a consequence,

$$\begin{aligned}\mathcal{C}^\sigma_{\mathcal{H}}(E) &= \inf\Big\{\sum_j r_j^\sigma;\ E \subset \bigcup_j B(x_j, r_j)\Big\}\\ &\geq C^{-\sigma} \inf\Big\{\sum_j r_j^\sigma;\ \phi(E) \subset \bigcup_j B(y_j, Cr_j)\Big\} = C^{-\sigma}\mathcal{C}^\sigma_{\mathcal{H}}(\phi(E)). \qquad \blacksquare\end{aligned}$$

Our first result is the following generalization of Jerison–Lebeau's work [5].

Theorem 1. *There exists $\delta \in (0,1)$ (depending only on the dimension of the manifold M) such that for any $m > 0$, there exist $C, D > 0$ such that for any $E_1 \subset M$ with $|E_1| \geq m$, any $E_2 \subset M$ satisfying*

$$\mathcal{C}^{d-\delta}_{\mathcal{H}}(E_2) > m, \tag{1.5}$$

and any $\Lambda > 0$, we have

$$\phi = \sum_{\lambda_k \leq \Lambda} u_k e_k(x) \implies \|\phi\|_{L^\infty(M)} \leq Ce^{D\Lambda}\|\phi 1_{E_1}\|_{L^1(M)}, \tag{1.6}$$

$$\phi = \sum_{\lambda_k \leq \Lambda} u_k e_k(x) \implies \|\phi\|_{L^\infty(M)} \leq Ce^{D\Lambda} \sup_{x \in E_2} |\phi(x)|. \tag{1.7}$$

Remark 1.2. The assumption (1.5) is *not* invariant by change of variables. It has to be understood in a fixed local chart (and we shall prove Theorem 1 in a chart). Taking $0 < \delta' < \delta$, we could have replaced it by $\dim_{\mathcal{H}}(E) > d - \delta'$ (which implies $\mathcal{C}^{d-\delta}_{\mathcal{H}} > 0$ and is invariant by Lipschitz diffeomorphisms). Of course replacing $\delta > 0$ by any $0 < \delta' < \delta$ does not substantially change the final result (as we have no control on the actual value of the constant δ). For the sake of consistency with [8–10] we kept (1.5).

Remark 1.3. Notice that in Theorem 1 no assumption is made on the set E_2 other than the positivity of the Hausdoff content. This implies that in the presence of a boundary, the estimate (1.7) also holds when E_2 is concentrated arbitrarily closely to ∂M (with uniform constants).

As a consequence of these spectral projector estimates we deduce the following observability estimates and controllability results for the heat equation.

Theorem 2 (Null controllability from sets of positive measure). *Let $F \subset (0,T) \times M$ of positive Lebesgue measure. Then there exists $C > 0$ such that for any $u_0 \in L^2(M)$ the solution $u = e^{t\Delta}u_0$ to the heat equation*

$$\begin{aligned} &\partial_t u - \Delta u = 0, \quad u|_{t=0} = u_0, \\ &u|_{\partial M} = 0 \ (\textit{Dirichlet condition}) \ \textit{or} \\ &\partial_\nu u|_{\partial M} = 0 \ (\textit{Neumann condition}), \end{aligned}$$

satisfies (*recall that κ is defined in* (1.1))

$$\|e^{T\Delta}u_0\|_{L^2(M)} \le C \int_F |u|(t,x)\kappa(x)\,dx\,dt. \tag{1.8}$$

As a consequence, for all $u_0, v_0 \in L^2(M)$ there exists $f \in L^\infty(F)$ such that the solution to

$$\begin{aligned} &(\partial_t - \Delta)u = f 1_F(t,x), \quad u|_{t=0} = u_0, \\ &u|_{\partial M} = 0 \ \textit{or} \ \partial_\nu u|_{\partial M} = 0, \end{aligned} \tag{1.9}$$

satisfies

$$u|_{t=T} = e^{T\Delta}v_0.$$

Theorem 3 (Observability and exact controllability from measure zero sets). *There exists $\delta \in (0,1)$ (depending only on the dimension of the manifold M) which depends only on the dimension of the manifold M such that for any $E \subset M$ of positive $(d-\delta)$-dimensional Hausdorff content, and any $J \subset (0,T)$ of positive Lebesgue measure, there exists $C > 0$ such that for any $u_0 \in L^2(M)$ the solution $u = e^{t\Delta}u_0$ to the heat equation*

$$\partial_t u - \Delta u = 0, \quad u|_{\partial M} = 0 \ \textit{or} \ \partial_\nu u|_{\partial M} = 0,$$

satisfies

$$\|e^{T\Delta}u_0\|_{L^2(M)} \le C \int_J \sup_{x \in E} |u|(t,x)\,dt. \tag{1.10}$$

As a consequence, under the additional assumption that E is a closed subset of M, for all $u_0, v_0 \in L^2(M)$ there exists a Borel measure μ supported on $(0,T) \times E$ such that the solution to

$$\begin{aligned} &(\partial_t - \Delta)u = \mu(t,x) 1_{J \times E}, \quad u|_{t=0} = u_0, \\ &u|_{\partial M} = 0 \ \textit{or} \ \partial_\nu u|_{\partial M} = 0, \end{aligned} \tag{1.11}$$

satisfies

$$u|_{t \ge T} = e^{t\Delta}v_0.$$

We refer to Section 5 (see (5.4)) for the precise meaning of (1.11). Actually, we can even go a step further and show that the $d+1$-dimensional heat equation can be steered to zero by using measure-valued controls supported on a set of space-time Hausdorff measure $d-\delta$.

Theorem 4 (Observability and exact controllability using controls localized at fixed times). *Take $\delta \in (0,1)$ as in Theorem 1. Let $m > 0$, $\tau \in (0,1)$ and $D > 0$. There exists $C > 0$ such that if $E_1 \subset M$ satisfies $|E_1| \geq m$ and $E_2 \subset M$ satisfies $\mathcal{C}^{d-\delta}_{\mathcal{H}}(E_2) \geq m$, then for any sequence $(s_n)_{n\in\mathbb{N}}$,*

$$J = \{0 < \cdots < s_n < \cdots < s_0 < T\}$$

converging to 0 not too fast,

$$\exists \tau \in (0,1),\ \forall n \in \mathbb{N}, \quad s_n - s_{n+1} \geq \tau(s_{n-1} - s_n),$$

we have that for any $u_0 \in L^2(M)$, the solution $u = e^{t\Delta}u_0$ to the heat equation

$$\partial_t u - \Delta u = 0, \quad u|_{\partial M} = 0 \text{ or } \partial_\nu u|_{\partial M} = 0,$$

satisfies

$$\|e^{T\Delta}u_0\|_{L^2(M)} \leq C \sup_{n\in\mathbb{N}} e^{-\frac{D}{s_n - s_{n+1}}} \int_{E_1} |e^{s_n\Delta}u_0|(s_n, x)|\, dx \tag{1.12}$$

and

$$\|e^{T\Delta}u_0\|_{L^2(M)} \leq C \sup_{n\in\mathbb{N},\, x\in E_2} e^{-\frac{D}{s_n - s_{n+1}}} |e^{s_n\Delta}u_0|(s_n, x). \tag{1.13}$$

As a consequence, under the additional assumption that E_1 is a closed subset of M, given any sequence $(t_n)_{n\in\mathbb{N}}$,

$$J = \{0 < t_0 < \cdots < t_n < \cdots < T\},$$

converging to T not too fast,

$$\exists 0 < \tau < 1,\ \forall n \in \mathbb{N}, \quad t_{n+1} - t_n \geq \tau(t_n - t_{n-1}), \tag{1.14}$$

for all $u_0, v_0 \in L^2(M)$ there exists a sequence (f_j) of functions on E_1 such that

$$\sum_j e^{\frac{D}{t_{j+1} - t_j}} \|f_j\|_{L^\infty(E_1)} < +\infty,$$

and the solution to

$$\begin{aligned} &(\partial_t - \Delta)u = \sum_{j=1}^{+\infty} \delta_{t=t_j} \otimes f_j(x) 1_{E_1}, \quad u|_{t=0} = u_0, \\ &u|_{\partial M} = 0 \text{ or } \partial_\nu u|_{\partial M} = 0, \end{aligned} \tag{1.15}$$

satisfies

$$u|_{t>T} = e^{t\Delta}v_0.$$

Similarly under the additional assumption that E_2 *is a closed subset of* M*, given any sequence* $(t_n)_{n\in\mathbb{N}}$*,*

$$J = \{0 \le t_0 < \cdots < t_n < \cdots < T\}$$

converging not too fast to T *as in* (1.14)*, for all* $u_0, v_0 \in L^2(M)$*, there exists* (μ_j) *a sequence of Borel measure supported on* E_2 *such that*

$$\sum_j e^{\frac{D}{t_{j+1}-t_j}} |\mu_j|(E_2) < +\infty,$$

and the solution to

$$\begin{aligned} &(\partial_t - \Delta)u = \sum_{j=1}^{+\infty} \delta_{t=t_j} \otimes \mu_j(x) 1_{E_2}, \quad u|_{t=0} = u_0, \\ &u|_{\partial M} = 0 \text{ or } \partial_\nu u|_{\partial M} = 0, \end{aligned} \tag{1.16}$$

satisfies

$$u|_{t>T} = e^{t\Delta} v_0.$$

The meaning of solution to (1.15), (1.16) is also explained in Section 5.

Remark 1.4. We have $t_{j+1} - t_j \le T - t_j$. As a consequence,

$$\|f_j\|_{L^\infty(E_1)} \le Ce^{-\frac{D}{T-t_j}}, \quad |\mu_j|(E_1) \le Ce^{-\frac{D}{T-t_j}}$$

which means that our controls are exponentially small when $j \to +\infty$ $(t \to T)$.

The plan of the paper is as follows. In Section 2 we show how, for manifolds without boundary, the estimate for spectral projectors (Theorem 1) follows quite easily from Logunov–Malinnikova's results [8–10] combined with Jerison–Lebeau's method [5]. Then in Section 3, we show how to extend the results to the case of manifolds with boundary. When the manifold is smooth, this is quite standard as we can extend it by reflection across the boundary using geodesic coordinates. This allows us to define a new $W^{2,\infty}$ manifold without boundary (the double manifold), which is topologically two copies of the original manifold glued along the boundary, and into which these two copies embed isometrically. At our low regularity level, the use of geodesic coordinate systems is prohibited and a careful work is required to perform this extension. We actually provide a natural alternative for geodesic systems (see Proposition 3.4). We believe that this construction of the double manifold at this low regularity level has interest of its own. In Section 4 we prove the propagation of smallness and observation estimates for solutions to heat equation (estimates (1.8), (1.10) and (1.12) in Theorems 2, 3 and 4), by adapting a proof in Apraiz et al. [2], which in turn relied on mixing ideas from Miller [11] and Phung–Wang [12], following the pioneering work by Lebeau–Robbiano [7]. Finally, in Section 5 we prove the exact controllability results by adapting quite classical duality methods to our setting. Here we also improve on previous results by allowing control supported on a sequence of times (hence a measure zero set in time).

2. Proof of the spectral inequalities for compact manifolds

In this section we give a proof of Theorem 1 in the case of a manifold without boundary. We first show in Section 2.1 that the estimate (1.6) is actually a straightforward consequence of the results obtained by Logunov and Malinnikova [10]. In Section 2.2 we combine [10] with the spectral estimates on open sets obtained by Jerison and Lebeau [5] to get (1.7) when $\partial M = \emptyset$.

We deal with the case $\partial M \neq \emptyset$ in Section 3.

2.1. *The spectral inequality for very small sets implies the spectral inequality for sets of non-zero measure*

Here we prove that (1.7) implies (1.6). Assume that $|E_1| > m$ is given and consider

$$\phi = \sum_{\lambda_k \leq \Lambda} u_k e_k(x) \quad \text{with} \quad \|\phi\|_{L^2(M)} = 1.$$

Let $F \subset M$ with $|F| > m/2$. According to (1.2) (and the fact that the Lebesgue measure and $\mathcal{C}^d_{\mathcal{H}}$ are equivalent), we have

$$\mathcal{C}^d_{\mathcal{H}}(F) > c_d m/2, \quad \text{so} \quad \mathcal{C}^{d-\delta}_{\mathcal{H}}(F) \geq \min(1, c_d m/2).$$

Now, according to (1.7), for any such $F \subset M$ with $|F| \geq m/2$, we have

$$\|\phi\|_{L^\infty(M)} \leq C e^{D\Lambda} \sup_{x \in F} |\phi(x)|. \tag{2.1}$$

with constants uniform as long as $|F| \geq m/2$. Let

$$F = \left\{ x \in E_1;\, |\phi(x)| \leq \frac{1}{2C} e^{-D\Lambda} \|\phi\|_{L^\infty(M)} \right\}.$$

If $|F| \geq m/2$, we have

$$\|\phi\|_{L^\infty(M)} \leq C e^{D\Lambda} \sup_{x \in F} |\phi(x)| \leq \|\phi\|_{L^\infty(M)}/2,$$

which shows that F cannot satisfy (2.1) (because $\|\phi\|_{L^2(M)} = 1$, so $\phi \not\equiv 0$). Hence, $|F| < m/2$, and consequently

$$\int_{E_1} |\phi(x)|\, dx \geq \int_{E_1 \setminus F} |\phi(x)|\, dx \geq \frac{|E_1|}{4C} e^{-D\Lambda} \|\phi\|_{L^\infty(M)},$$

which implies (1.6).

2.2. *Proof of the precise estimate for compact manifolds*

Let $m > 0$ and let $E_2 \subset M$ be a given set with $\mathcal{C}^{d-\delta}_{\mathcal{H}}(E_2) > m$. Our goal is to obtain estimate (1.7) in this case.

We first localize the estimate on a coordinate patch. Since M is compact, there exists a finite covering

$$M \subset \bigcup_{j=1}^{N} U_j,$$

and $W^{2,\infty}$ diffeomorphisms $\psi_j : V_j \to \mathbb{R}^d$, with V_j a neighborhood of U_j in M. From [5, Theorem 14.6] there exist $C, D > 0$ depending only on M such that for any $j \in \{1, \dots, N\}$,

$$\|\phi\|_{L^2(M)} \leq C e^{D\Lambda} \|\phi\|_{L^2(U_j)}. \tag{2.2}$$

Let j_0 be such that

$$\mathcal{C}_{\mathcal{H}}^{d-\delta}(E_2 \cap U_{j_0}) > \frac{m}{N}.$$

We now work in a coordinate patch U_{j_0} and define the sets

$$V = \psi_{j_0}(V_{j_0}), \quad U = \psi_{j_0}(U_{j_0}), \quad F = \psi_{j_0}(E_2 \cap U_{j_0}).$$

Observe that, as ψ_{j_0} is a diffeomorphism of class $W^{2,\infty}$ by hypothesis, we must have

$$\mathcal{C}_{\mathcal{H}}^{d-\delta}(F) > C(\psi_0)\frac{m}{N}. \tag{2.3}$$

Now, denote by f_k and φ the images of e_k and ϕ by the push forward $(\psi_{j_0})_*$, which are defined on V. Consider the functions

$$u(t,x) = \sum_{\lambda_k \leq \Lambda} u_k \frac{\sinh(\lambda_k t)}{\lambda_k} f_k(x), \quad \varphi(x) := \sum_{\lambda_k \leq \Lambda} u_k f_k(x) = \partial_t u|_{t=0},$$

for $(u_k)_k$ given. Here by convention we set, for $\lambda = 0$, $\frac{\sinh(\lambda t)}{\lambda} = t$. We have

$$\left(\frac{1}{\kappa} \operatorname{div} g^{-1}\kappa \nabla_x + \partial_t^2\right) u = 0 \iff (\operatorname{div} g^{-1}\kappa\nabla_x + \partial_t \kappa \partial_t) u = 0.$$

Consider for $T_2 > T_1 > 0$ the sets

$$\mathcal{K} := [-T_1, T_1] \times \overline{U}, \quad \Omega := (-T_2, T_2) \times V, \quad E = \{0\} \times F,$$

which by construction satisfy the inclusions $E \subset \mathcal{K} \subset \Omega$. Next, thanks to (2.3), we can write

$$\mathcal{C}_{\mathcal{H}}^{n-1-\delta}(E) > m' \quad \text{for } n = d+1, \quad m' = C(\psi_0)\frac{m}{N}. \tag{2.4}$$

For sufficiently small $\delta > 0$ we can now apply [10, Theorem 5.1] and get from (2.4)

$$\sup_{\mathcal{K}} |\nabla_{t,x} u| \leq C \left(\sup_{E} |\nabla_{t,x} u|\right)^{\alpha} \left(\sup_{\Omega} |\nabla_{t,x} u|\right)^{1-\alpha}. \tag{2.5}$$

We now need a variant of Sobolev embedding, which we prove for the reader's convenience:

Proposition 2.1. *There exists $\sigma > 0$ such that for*

$$\mathcal{H}^\sigma = D((-\Delta)^{\sigma/2})$$

endowed with its natural norm

$$\|u\|_{\mathcal{H}^\sigma} = \Big(\sum_k |u_k|^2(1+\lambda_k)^{2\sigma}\Big)^{1/2},$$

we have

$$\|\nabla_x u\|_{L^\infty} + \|u\|_{L^\infty} \leq C\|u\|_{\mathcal{H}^\sigma}.$$

Remark 2.2. For smooth metrics and compact manifolds (without boundary), the space $\mathcal{H}^\sigma$ coincides with the usual Sobolev space H^σ, and Lemma 2.1 is just the usual Sobolev injection. At our level of regularity it is no more the case, because the spaces H^σ and $\mathcal{H}^\sigma$ coincide for $0 \leq \sigma \leq 2$ but no further.

Proof of Proposition 2.1. We start with a lemma about eigenfunctions of the Laplace operator.

Lemma 2.3. *For all $\sigma_1 > d/2$ there exists $C > 0$ such that for all eigenfunctions e_k of the Laplace operator we have*

$$\|\nabla_x e_k\|_{L^\infty} \leq C(1+\lambda_k)^{1+\sigma_1}\|e_k\|_{L^2}, \quad \|e_k\|_{L^\infty} \leq C\lambda_k^{1+\sigma_1}\|e_k\|_{L^2}.$$

Proof of Lemma 2.3. We start with the bound for $\|e_k\|_{L^\infty}$. For $n \leq 3$, it follows from elliptic regularity that

$$\|e_k\|_{H^2} \leq C(\|\Delta e_k\|_{L^2} + \|e_k\|_{L^2}) \leq C(1+\lambda^2),$$

which implies for $0 \leq s \leq 2$,

$$\|e_k\|_{H^s} \leq C(1+\lambda^2)^{s/2},$$

and Lemma 2.3 follows from Sobolev embeddings. For higher dimensions, we shall use the following results from [4] about weak solutions to

$$-\sum_{i,j} \partial_{y_i} a_{i,j} \partial_{y_j} w + cw = f \tag{2.6}$$

with

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \leq \Lambda|\xi|^2.$$

Theorem 5 ([4, Theorem 3.8, combined with Corollary 3.2]). *Let $w \in H^1(B(0,1))$ be a weak solution to* (2.6). *Assume that $a_{i,j} \in C^0(\overline{B(0,1)})$, $c \in L^n(B(0,1))$ and*

$f \in L^q(B(0,1))$ for some $q \in (n/2, n)$. Then $w \in C^\alpha(B(0,1))$ for $\alpha = 2 - n/q \in (0,1)$. Moreover, there exists $M = M(\lambda, \Lambda, \|c\|_{L^n}, \tau) > 0$ such that

$$\begin{aligned}\|w\|_{C^{0,\alpha}(B(0,1/2))} &= \sup_{B(0,1/2)} |w| + \sup_{x,y\in B(0,1/2),\, x\neq y} \frac{|w(x)-w(y)|}{|x-y|^\alpha} \\ &\leq M(\|f\|_{L^q(B(0,1))} + \|w\|_{H^1(B(0,1))}),\end{aligned} \tag{2.7}$$

where τ is the uniform continuity modulus of the functions $a_{i,j}$,

$$\forall y, y' \in B(0,1), \quad |a_{i,j}(y) - a_{i,j}(y')| \leq \tau(|y-y'|).$$

Using a partition of unity and applying this lemma in charts with $c = -\lambda_k^2\kappa(x)$, $f = 0$, we find that $e_k \in L^\infty$ (with an implicit bound in terms of λ_k). Then applying again the result with $c = 0$, $f(x) = \lambda_k^2\kappa(x)e_k(x)$, we get (choosing $q = n/2 + 0$, i.e. arbitrarily close to $n/2$)

$$\|e_k\|_{L^\infty} \leq M(\lambda_k^2\|e_k\|_{L^q} + \|e_k\|_{H^1}) \leq M(\lambda_k^2\|e_k\|_{L^\infty}^{1-\theta}\|e_k\|_{L^2}^\theta + \lambda_k\|e_k\|_{L^2})$$

with $\theta = \frac{2}{q} = \frac{4}{n} - 0$, and consequently

$$\|e_k\|_{L^\infty} \leq M'(\lambda_k^{2/\theta}\|e_k\|_{L^2} + \lambda_k\|e_k\|_{L^2}) \leq C(1+\lambda_k)^{n/2+0}\|e_k\|_{L^2}.$$

Now, we turn to the estimates for $\|\nabla_x e_k\|_{L^\infty}$. In this case we shall use

Theorem 6 ([4, Theorem 3.13, combined with Theorem 1.3]). *Let $w \in H^1(B(0,1))$ be a weak solution to (2.6). Assume that $a_{i,j} \in C^\alpha(\overline{B(0,1)})$ and $f \in L^q(B(0,1))$ for some $q > n$. Then $\nabla_y w \in C^\alpha(B(0,1))$ for $\alpha = 1 - n/q \in (0,1)$. Moreover, there exists $M = M(n, \lambda, \|a_{i,j}\|_{C^\alpha}) > 0$ such that*

$$\begin{aligned}\|\nabla_y w\|_{C^{0,\alpha}(B(0,1/2))} &= \sup_{B(0,1/2)} |\nabla_y w| + \sup_{y,y'\in B(0,1/2),\, y\neq y'} \frac{|\nabla_y w(y) - \nabla_y w(y')|}{|y-y'|^\alpha} \\ &\leq M(\|f\|_{L^q(B(0,1))} + \|w\|_{H^1(B(0,1))}).\end{aligned} \tag{2.8}$$

Using again a partition of unity and applying now this lemma in charts with $c = 0$, $f(x) = \lambda_k^2\kappa(x)e_k(x)$, we get (choosing $q = n + 0$, i.e. arbitrarily close to n)

$$\|\nabla_x e_k\|_{L^\infty} \leq M(\lambda_k^2\|e_k\|_{L^q} + \|e_k\|_{H^1}) \leq M(\lambda_k^2\|e_k\|_{L^\infty}^{1-\theta}\|e_k\|_{L^2}^\theta + \lambda_k\|e_k\|_{L^2})$$

with $\theta = \frac{2}{q} = \frac{2}{n} - 0$, and consequently

$$\|e_k\|_{L^\infty} \leq M(\lambda_k^{2+(n/2+0)(1-\theta)}\|e_k\|_{L^2} + \lambda_k\|e_k\|_{L^2}) \leq C(1+\lambda_k)^{n/2+1+0}\|e_k\|_{L^2}. \quad \blacksquare$$

Let us now come back to the proof of Proposition 2.1. From Weyl's formula,

$$\lambda_k \sim k^{1/d}.$$

As a consequence, we have

$$\begin{aligned}\Big\|\sum_k u_k e_k\Big\|_{L^\infty} &\le C\sum_k |u_k|(1+\lambda_k)^{\sigma_1}\\ &\le C\Big(\sum_k |u_k|^2(1+\lambda_k)^{2\sigma_1+2p}\Big)^{1/2}\Big(\sum_k (1+\lambda_k)^{-2p}\Big)^{1/2}\\ &\le C\|u\|_{\mathcal{H}^{\sigma_1+p}} \qquad (2.9)\end{aligned}$$

as long as $2p > d$. ∎

We now come back to the proof of (1.7). Using Sobolev embedding, we observe

$$\sup_\Omega |\nabla_{t,x}u| \le C\|\nabla_{t,x}u(t)\|_{\mathcal{H}^\sigma} \le Ce^{(T_2+1)\Lambda}\|\phi\|_{L^2(M)}.$$

By definition of u, we have

$$\nabla_x u|_E = 0, \quad u|_E = 0, \quad \partial_t u|_E = \varphi 1_F.$$

We deduce from (2.2) and (2.5) that

$$\begin{aligned}\|\phi\|_{L^2(M)} &\le Ce^{D\Lambda}\|\phi\|_{L^2(U_j)} \le C'e^{D\Lambda}\sup_{U_j}|\phi|\\ &\le C'e^{D\Lambda}\sup_\Omega|\nabla_{t,x}u| \le C''e^{D\Lambda}\Big(\sup_E|\nabla_{t,x}u|\Big)^\alpha\Big(\sup_{\mathcal{K}}|\nabla_{t,x}u|\Big)^{1-\alpha}\\ &\le C''e^{D'\Lambda}\Big(\sup_F|\phi|\Big)^\alpha\|\phi\|_{L^2}^{1-\alpha}, \qquad (2.10)\end{aligned}$$

which implies

$$\|\phi\|^\alpha_{L^2(M)} \le C''e^{D'\Lambda}\Big(\sup_F|\phi|\Big)^\alpha.$$

Another use of Sobolev embedding concludes the proof of (1.7).

3. The double manifold

In this section we give the proof of Theorem 1 for a manifold with boundary M and Dirichlet or Neumann boundary conditions on ∂M. The classical idea is to reduce this question to the case of a manifold without boundary by gluing two copies of M along the boundary in such a way that the new double manifold $\widetilde{M}$ inherits a Lipschitz metric, which allows one to apply the previous results (without boundary) to this double manifold. However, this procedure of gluing has to be done properly, as otherwise the resulting glued metric might not even be continuous. The main difficulty in our context comes from the fact that the usual method for this *doubling* procedure relies on the use of a reflection principle in geodesic coordinate systems. However, the existence of such coordinate systems requires at least C^2 (resp. C^3) regularity for the metric (resp. the domain), compared to our $W^{1,\infty}$ and $W^{2,\infty}$ assumptions, to get a C^1 (hence integrable) geodesic flow. To circumvent this technical difficulty, we shall define a *pseudo-geodesic system* relying on a regularization of the normal direction to the boundary, which will be $W^{2,\infty}$ and tangent at the boundary to the "geodesic coordinate system" (which actually does not exist at this low regularity level).

Let $\widetilde{M} = \overline{M} \times \{-1, 1\}/\partial M$ be the double space made up of two copies of $\overline{M}$ where we identify the points on the boundary, $(x, -1)$ and $(x, 1)$, $x \in \partial M$.

Theorem 7 (The double manifold). *Let g be given. There exists a $W^{2,\infty}$ structure on the double manifold $\widetilde{M}$, a metric $\widetilde{g}$ of class $W^{1,\infty}$ on $\widetilde{M}$, and a density $\widetilde{\kappa}$ of class $W^{1,\infty}$ on $\widetilde{M}$ such that the following hold.*

- *The maps*

$$i^{\pm} : M \ni x \mapsto (x, \pm 1) \in \widetilde{M} = M \times \{\pm 1\}/\partial M$$

are isometric embeddings.

- *The density induced on each copy of M is the density κ,*

$$\widetilde{\kappa}|_{M \times \{\pm 1\}} = \kappa.$$

- *For any eigenfunction e with eigenvalue λ^2 of the Laplace operator $-\Delta = -\frac{1}{\kappa} \operatorname{div} g^{-1} \kappa \nabla$ with Dirichlet or Neumann boundary conditions, there exists an eigenfunction $\widetilde{e}$ with the same eigenvalue λ of the Laplace operator $-\Delta = -\frac{1}{\widetilde{\kappa}} \operatorname{div} \widetilde{g}^{-1} \widetilde{\kappa} \nabla$ on $\widetilde{M}$ such that*

$$\widetilde{e}|_{M \times \{1\}} = e, \quad \widetilde{e}|_{M \times \{-1\}} = \begin{cases} -e & (\textit{Dirichlet boundary conditions}), \\ e & (\textit{Neumann boundary conditions}). \end{cases} \tag{3.1}$$

Corollary 3.1. *Estimate* (1.7) *for manifolds without boundary implies* (1.6) *for Dirichlet or Neumann boundary conditions, and in the case of Dirichlet boundary conditions, we could even add any constant to the spectral projector and replace ϕ by*

$$\Psi = u_0 + \sum_{\lambda_k \leq \Lambda} u_k e_k(x).$$

Remark 3.2. Since the vector spaces generated respectively by the Dirichlet or Neumann eigenfunctions are dense in $L^2(M)$, the vector space generated by their extensions as defined in (3.1) is dense in $L^2(\widetilde{M})$. We deduce that there exists a Hilbert basis of $L^2(\widetilde{M})$ made up of eigenfunctions of $\widetilde{\Delta}$ on $\widetilde{M}$ which are the extensions of the Dirichlet and Neumann eigenfunctions of Δ on M.

To prove Theorem 7, we are going to endow $\widetilde{M}$ with a $W^{2,\infty}$ manifold structure and a Lipschitz metric $\widetilde{g}$ which coincides with the original metric g on each copy of M. For this we just need to work near the boundary ∂M (as away from ∂M, $\widetilde{M}$ coincides with one of the copies $M \times \{\pm 1\}$). Consider a point $x_0 \in \partial M$. There exists a covering $\partial M \subset \bigcup_{j=1}^{N} U_j$ (here ∂M is seen as a subset of $\overline{M}$), where U_j are open subsets of $\overline{M}$, and there are $W^{2,\infty}$ diffeomorphisms

$$\psi_j : V_j \to \mathbb{R}^d = \mathbb{R}_y \times \mathbb{R}_x^{d-1}, \quad \overline{U_j} \subset V_j \subset \overline{M},$$

such that $\psi_j(V_j) = B(0,1)_{y,x} \cap \{y \geq 0\}$ and $\psi_j(U_j) \subset [0,\epsilon] \times B(0,\delta)_x$ for some $\delta, \epsilon > 0$ small enough. Here $W^{2,\infty}$ regularity means $W^{2,\infty} \cap C^1$ regularity of every change of charts

$$\psi_k \circ \psi_j^{-1} : \psi_j(U_k \cap U_j) \to \mathbb{R}^d.$$

Let $a = a(y,x)$ be the metric in this coordinate system, which is hence $W^{1,\infty}$ and defined for

$$\|x\| \leq \delta', \quad y \in [0,\epsilon'], \quad \delta' < \delta < 1, \quad \epsilon' < \epsilon.$$

For any $x \in \{y = 0\}$, consider the vector defined by

$$n(x) = (\lambda(x))^{-1/2} a^{-1}(0,x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \lambda(x) = (1,0) \cdot a^{-1}(0,x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.2}$$

One can check that

$$x \mapsto n(x) \in W^{1,\infty}(B(0,\delta')) \tag{3.3}$$

is the inward normal to the boundary for the metric a at the point $(0,x) \in \psi_j(\partial M \cap V_j)$. Indeed,

$${}^t n(x) a(0,x) n(x) = (\lambda(x))^{-1} (1,0) a^{-1}(0,x) a(0,x) a^{-1}(0,x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1,$$

which makes $n(x)$ unitary and if $X \in \mathbb{R}_x^{d-1}$, then

$${}^t n(x) a(0,x) \begin{pmatrix} 0 \\ X \end{pmatrix} = (\lambda(x))^{-1/2} (-1,0) a^{-1}(0,x) a(0,x) \begin{pmatrix} 0 \\ X \end{pmatrix} = 0,$$

which proves that $n(x)$ is orthogonal to the vectors tangent to the boundary. Finally, since its first component is positive, $n(x)$ points inward.

We now study the regularity of the quasi-geodesic coordinates. Let $\chi \in C_0^\infty(B(0,\delta'))$ be equal to 1 in $B(0,\delta)$, and

$$m(s,z) = e^{-\langle sD_z \rangle + 1} (\chi n)(0,z) \quad \text{with } \langle z \rangle = (1 + |z|^2)^{1/2}.$$

Lemma 3.3. *For any $q \in \mathbb{R}$ and $j = 1, \ldots, d-1$, the operators*

$$\langle sD_z \rangle^q e^{-\langle sD_z \rangle + 1} \quad \text{and} \quad sD_{z_j} \langle sD_z \rangle^q e^{-\langle sD_z \rangle + 1}$$

are uniformly bounded on $L^\infty(\mathbb{R}^{d-1})$ with respect to $s \in \mathbb{R}$.

Proof. Indeed, these are convolution operators with kernels

$$K_{1,q}(z) = \frac{1}{s^{d-1}} \mathcal{F}(\langle \xi \rangle^q e^{-\langle \xi \rangle + 1}) \left(\frac{z}{s} \right), \quad K_{2,q} = \frac{1}{s^{d-1}} \mathcal{F}(i \xi_j \langle \xi \rangle^q e^{-\langle \xi \rangle + 1}) \left(\frac{z}{s} \right),$$

where $\mathcal{F}$ stands for the usual Fourier transform on $\mathbb{R}^{d-1}$. Since the functions

$$\xi \mapsto \langle \xi \rangle^q e^{-\langle \xi \rangle}, \quad \xi \mapsto \xi_j \langle \xi \rangle^q e^{-\langle \xi \rangle} \tag{3.4}$$

are in the Schwartz class, we deduce that the kernels $K_{1,q}$ and $K_{2,q}$ are uniformly bounded (with respect to $s \geq 0$) in $L^1(\mathbb{R}_x^{d-1})$, which implies that the corresponding operators are bounded on $L^\infty(\mathbb{R}_x^{d-1})$ (uniformly with respect to $s \geq 0$). ■

According to (3.3), the map $(s, z) \mapsto m(s, z)$ is Lipschitz and therefore so is $z \mapsto m(0, z)$. Using Lemma 3.3 and the basic relations

$$\frac{d}{ds} \langle s D_z \rangle = s D_z^2 \langle s D_z \rangle^{-1}, \quad \frac{d}{ds} e^{-\langle s D_z \rangle + 1} = -s D_z^2 \langle s D_z \rangle^{-1} e^{-\langle s D_z \rangle + 1}$$

we deduce that the map

$$[-\epsilon', \epsilon'] \times B(0, \delta') \ni (s, z) \mapsto \phi_j(s, z) = z + s m(s, z) = z + s e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0})$$

is $W^{2,\infty}$. Indeed, since by assumption $\chi n \in L^\infty$ and $\nabla_z(\chi n) \in L^\infty$, a direct calculation gives, using Lemma 3.3,

$$\nabla_z \phi_j(s, z) = 1 + s e^{-\langle s D_z \rangle + 1} (\nabla_z(\chi n|_{s=0})) \in L^\infty((-\epsilon, \epsilon) \times B(0, \delta)_z). \tag{3.5}$$

$$\begin{aligned} \partial_s \phi_j(s, z) &= e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0}) - s^2 D_z^2 \langle s D_z \rangle^{-1} e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0}) \\ &= (1 - \langle s D_z \rangle + \langle s D_z \rangle^{-1}) e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0}) \\ &\in L^\infty((-\epsilon, \epsilon) \times B(0, \delta)_z), \end{aligned} \tag{3.6}$$

$$\nabla_z^2 \phi_j(s, z) = s \nabla_z e^{-\langle s D_z \rangle + 1} (\nabla_z(\chi n|_{s=0})) \in L^\infty((-\epsilon, \epsilon) \times B(0, \delta)_z). \tag{3.7}$$

$$\begin{aligned} \partial_s \nabla_z \phi_j(s, z) &= (1 - \langle s D_z \rangle + \langle s D_z \rangle^{-1}) e^{-\langle s D_z \rangle + 1} (\nabla_z(\chi n|_{s=0})) \\ &\in L^\infty((-\epsilon, \epsilon) \times B(0, \delta)_z), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \partial_s^2 \phi_j(s, z) &= -(1 - \langle s D_z \rangle + \langle s D_z \rangle^{-1}) s D_z^2 \langle s D_z \rangle^{-1} e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0}) \\ &\quad - s D_z^2 (\langle s D_z \rangle^{-1} + \langle s D_z \rangle^{-3}) e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0}) \\ &= s D_z^2 (1 - 2\langle s D_z \rangle^{-1} - \langle s D_z \rangle^{-2} - \langle s D_z \rangle^{-3}) e^{-\langle s D_z \rangle + 1} (\chi n|_{s=0}) \\ &= \sum_{p=1}^{d-1} s D_{z_p} (1 - 2\langle s D_z \rangle^{-1} - \langle s D_z \rangle^{-2} - \langle s D_z \rangle^{-3}) e^{-\langle s D_z \rangle + 1} (D_{z_p}(\chi n|_{s=0})) \\ &\in L^\infty((-\epsilon, \epsilon) \times B(0, \delta)_z). \end{aligned} \tag{3.9}$$

The differential of ϕ_j at $s = 0$ is

$$d_{s,z} \phi_j|_{s=0} = \begin{pmatrix} \chi n(0, z)_y & 0 \\ \chi n(0, z)_z & \mathrm{Id} \end{pmatrix},$$

which, according to (3.2) and the fact that a is positive definite, is invertible for $z \in B(0,\delta)$. Hence, we deduce that ϕ_j is a $W^{2,\infty}$ diffeomorphism from a neighborhood of $\{0\}_s \times B(0,\delta)_z$ to a neighborhood of $\{0\}_s \times B(0,\delta)_z$. Notice also that since ϕ_j sends the half-plane $\{s>0\}$ to itself, so does its inverse. As a consequence, shrinking U_j to a possibly smaller U_j' we get a covering

$$\partial M \subset \bigcup_{j=1}^{N} U_j'$$

and $W^{2,\infty}$ diffeomorphisms $\psi_j' = \phi_j^{-1} \circ \psi_j$ such that after this change of variable, the metric $b(s,z)$ is given for $s \geq 0$ by

$$b(s,z) = {}^t\psi_j' a(y,z) \psi_j'.$$

In particular, for $s = 0^+$ we get

$$b(0^+,z) = \begin{pmatrix} n(z)_y & n(z)_z \\ 0 & \mathrm{Id} \end{pmatrix} a(0,z) \begin{pmatrix} n(z)_y & 0 \\ n(z)_z & \mathrm{Id} \end{pmatrix}. \tag{3.10}$$

Since $n(z)$ is the normal to the boundary we have

$${}^t n(z) a(0,z) n(z) = 1, \quad \big(0,Z\big)\, a(0,z) n(z) = 0, \quad \forall Z \in \mathbb{R}^{d-1}.$$

We deduce

$$b(0^+,z) = \begin{pmatrix} 1 & 0 \\ 0 & b'(z) \end{pmatrix}, \tag{3.11}$$

with $b'(z)$ positive definite. We have just proved

Proposition 3.4. *Assume that M is a $W^{2,\infty}$ manifold of dimension d with boundary, endowed with a Lipschitz (positive definite) metric g and a Lipschitz (positive) density κ. Let*

$$\Delta = \frac{1}{\kappa} \operatorname{div} g^{-1}(x) \kappa \nabla_x = \operatorname{div}_\kappa \nabla_g. \tag{3.12}$$

Then near any point $X_0 \in \partial M$ there exists a $W^{2,\infty}$ coordinate system

$$\begin{aligned} & X_0 = (0,0) \in \mathbb{R}_y \times \mathbb{R}_z^{d-1}, \\ & \Omega = (0,+\infty) \times \mathbb{R}^{d-1}, \quad \partial\Omega = \{0\} \times \mathbb{R}^{d-1}, \\ & \Delta = \frac{1}{\kappa(y,z)} {}^t\nabla_{y,z} \kappa(y,z) b(y,z) \nabla_{y,z}, \quad b|_{\partial\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & b'(z) \end{pmatrix}. \end{aligned} \tag{3.13}$$

Remark 3.5. In a geodesic coordinate system, we would have a diagonal form for the metric as in (3.11) in a neighborhood of the boundary. Proposition 3.4 corresponds to the fact that our coordinate system is, at the boundary, "tangent to a geodesic coordinate system".

Summarizing, we have defined a covering $\bigcup_{j=1}^{N} U'_j \supset \partial M$ and $W^{2,\infty}$ diffeomorphisms

$$\psi'_j : U'_j \to \mathbb{R}^d = \mathbb{R}_s \times \mathbb{R}^{d-1}_x$$

such that $\psi'_j(V_j) \subset B(0,1)_{s,x} \cap \{s \geq 0\}$, and after the change of variables ψ'_j, the metric takes the form (3.11) on the boundary $\{s = 0\}$.

We can now perform the gluing by defining a covering of ∂M (now seen as a subset of $\widetilde{M}$),

$$\partial M \subset \bigcup_{j=1}^{N} U'_j \times \{-1, 1\} = \bigcup_{j=1}^{N} \widetilde{U}_j$$

where we identify the points in $U'_j \cap \partial M \times \{-1, 1\}$, and define the map

$$\Psi_j : U'_j \times \{\epsilon\} \ni z \mapsto \begin{cases} \psi'_j(x) & \text{if } \epsilon = 1, \\ S \circ \psi'_j(x) & \text{if } \epsilon = -1, \end{cases}$$

where

$$S(s, z) = (-s, z).$$

To conclude the proof of the first part of Theorem 7, it remains to check that

- the image of the metric on $\widetilde{M}$ induced by the metrics on the two copies of $\overline{M}$ is well defined and Lipschitz,
- the changes of charts
$$\Psi_k \circ \Psi_j^{-1} : \Psi_j(\widetilde{U}_k \cap \widetilde{U}_j) \to \mathbb{R}^d$$
are $W^{2,\infty}$,
- the density $\widetilde{\kappa}$ obtained by gluing the two copies of κ on each copy of M is $W^{1,\infty}$.

The first result follows from (3.11) because on $\Psi_j(U'_j \times \{1\})$ the metric is given by

$$b(s, x) 1_{s \geq 0},$$

while on $\Psi_j(U'_j \times \{-1\})$ it is given by

$$S' b \circ S(s, x) S' 1_{s \leq 0} = \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 1 & {}^t r(s, x) \\ r(s, x) & b'(-s, x) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} 1_{s \leq 0}, \tag{3.14}$$

where from (3.13), $r(0, x) = 0$.

As a consequence, the two metrics coincide on $\{s = 0\}$ and they define a Lipschitz metric on $(-\epsilon', \epsilon) \times B(0, \delta)$. To check the $W^{2,\infty}$ smoothness of the change of charts, we write

$$\Psi_k \circ \Psi_j^{-1} = \begin{cases} \psi'_k \circ (\psi'_j)^{-1} & \text{on } \{s \geq 0\}, \\ S \circ \psi'_k \circ (\psi'_j)^{-1} \circ S & \text{on } \{s \leq 0\}. \end{cases}$$

Taking derivatives we get

$$d_{s,z}\Psi_k \circ \Psi_j^{-1} = \begin{cases} d_{s,z}(\phi'_k \circ (\phi'_j)^{-1}) & \text{on } \{s > 0\}, \\ \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} d_{s,z}(\phi'_k \circ (\phi'_j)^{-1}) \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} & \text{on } \{s < 0\}. \end{cases} \tag{3.15}$$

We now remark that by construction the differential $d_{y,x}\phi'|_{\partial M}$ sends the normal to the boundary to the normal $\binom{1}{0}$ to the boundary and sends all vectors tangent to the boundary to tangent vectors $\binom{0}{Z'}$. As a consequence,

$$d_{s,z}(\phi'_k \circ (\phi'_j)^{-1})|_{\{s=0\}} = \begin{pmatrix} 1 & 0 \\ 0 & q(z) \end{pmatrix}. \tag{3.16}$$

We deduce from (3.16) that the two limits of the differentials as $s \to 0^+$ and as $s \to 0^-$ coincide:

$$d_{s,z}\Psi_k \circ \Psi_j^{-1}|_{s=0^+} = d_{s,z}\Psi_k \circ \Psi_j^{-1}|_{s=0^-}, \tag{3.17}$$

and consequently the differential is C^0. Let us now study the L^∞ boundedness of derivatives of order 2. The case of space derivatives $d^2_{z,z}$ or $d^2_{s,z}$ is easy because we just have to take an additional tangential derivative d_z in (3.17). Such derivatives are tangent to the boundary $\{s = 0\}$, giving

$$d^2_{z,z}\Psi_k \circ \Psi_j^{-1}|_{s=0^+} = d^2_{z,z}\Psi_k \circ \Psi_j^{-1}|_{s=0^-} = d^2_{z,z}\phi'_k|_{\partial M} \circ (\phi'_j)^{-1}|_{\{s=0\}}.$$

Finally, the case of $d^2_{s,s}$ follows from the jump formula and the use of (3.17) which shows that the first order derivatives have no jump, because

$$\begin{aligned} &\frac{\partial^2}{\partial s^2}\big(\psi'_k \circ (\psi'_j)^{-1} 1_{s>0} + S \circ \psi'_k \circ (\psi'_j)^{-1} \circ S 1_{s<0}\big) \\ &\quad = \frac{\partial^2}{\partial s^2}(\psi'_k \circ (\psi'_j)^{-1}) 1_{s>0} + \frac{\partial^2}{\partial s^2}\big(S \circ \psi'_k \circ (\psi'_j)^{-1} \circ S 1_{s<0}\big) \\ &\qquad + \left(\frac{\partial}{\partial s}(\psi'_k \circ (\psi'_j)^{-1}|_{s=0^+}) - \frac{\partial}{\partial s}(S \circ \psi'_k \circ (\psi'_j)^{-1} \circ S|_{s=0^-})\right) \otimes \delta_{s=0} \\ &\qquad + \big(\psi'_k \circ (\psi'_j)^{-1}|_{s=0^+} - S \circ \psi'_k \circ (\psi'_j)^{-1} \circ S|_{s=0^-}\big) \otimes \delta'_{s=0} \\ &\quad = \frac{\partial^2}{\partial s^2}\big(\psi'_k \circ (\psi'_j)^{-1}\big) 1_{s>0} + \frac{\partial^2}{\partial s^2}\big(S \circ \psi'_k \circ (\psi'_j)^{-1} \circ S\big) 1_{s<0}. \end{aligned} \tag{3.18}$$

The last result for the density $\widetilde{\kappa}$ follows from this $W^{2,\infty}$ regularity of the change of charts.

It remains to prove the second part in Theorem 7 (about the eigenfunctions). Let e be an eigenfunction of $-\Delta$ on M with Dirichlet boundary condition, associated to the eigenvalue λ^2. We define

$$\widetilde{e}(x, \pm 1) = \pm e(x).$$

This definition makes sense because $e(x) = 0 = -e(x)$ on the boundary. Now we check that $\widetilde{e}$ is an eigenfunction of $\widetilde{\Delta}$ on $\widetilde{M}$. Away from the boundary ∂M this is clear while near a point $x \in \partial M \subset \widetilde{M}$ we can work in a coordinate chart $(\Psi_j, \widetilde{U}_j)$. In this chart,

$$\widetilde{e}(s,z) = \begin{cases} e(s,z) & \text{if } s > 0, \\ -e(-s,z) & \text{if } s < 0. \end{cases}$$

In $\{\pm s > 0\}$, $\widetilde{e}$ satisfies $-\widetilde{\Delta}\widetilde{e} = \lambda^2 e$, and near ∂M in our coordinate systems, we have

$$\widetilde{e}(s,z) = e(s,z)1_{s>0} - e(-s,z)1_{s<0} = e(s,z)1_{s>0} + f(s,z)1_{s<0}, \quad f(s,z) = -e(-s,z), \tag{3.19}$$

and we have

$$\begin{aligned} \nabla_x(\widetilde{e})(s,x) &= (\nabla_x e)1_{s>0} + (\nabla_x f)1_{s<0}, \\ \partial_s(\widetilde{e})(s,x) &= (\partial_s e)1_{s>0} + (\partial_s f)1_{s<0} + (e(0^+,z) - f(0^-,z)) \otimes \delta_{s=0} \\ &= (\partial_s e)1_{s>0} + (\partial_s f)1_{s<0}, \end{aligned} \tag{3.20}$$

where we have used the fact that according to the Dirichlet boundary condition, $e(0^+,z) = f(0^-,z) = 0$. Now, according to (3.11), we get

$$\widetilde{b}(s,z) = \begin{pmatrix} b_{1,1}(s,z) & r(s,z) \\ {}^t r(s,z) & b'(s,z) \end{pmatrix}$$

with $b_{1,1}(0,z) = 1$, $r(0,z) = 0$, and we deduce from (3.20) and the jump formula that

$$\begin{aligned} &-\widetilde{\Delta}(\widetilde{e})(s,x) = \lambda^2 \widetilde{e}(s,x) \\ &+ \frac{1}{\kappa(0,z)} b_{1,1}(0,z)\big((\partial_s e)(0^+,z) - (\partial_s f)(0^-,z) + r(0,z)((\nabla_x e)(0^+,z) - (\nabla_x f)(0^-,z))\big) \\ &\otimes \delta_{s=0} = \lambda^2 \widetilde{e}(s,x) \end{aligned} \tag{3.21}$$

where we have used $r(0,z) = 0$ and the fact that since $f(s,z) = -e(-s,z)$ we have

$$\partial_s f(0^-,z) = \partial_s e(0^+,z).$$

This ends the proof of Theorem 7 for Dirichlet boundary conditions. The proof in the case of Neumann boundary conditions is similar by defining

$$\widetilde{e}(s,z) = \begin{cases} e(s,z) & \text{if } s > 0, \\ e(-s,z) & \text{if } s < 0. \end{cases}$$

4. Propagation of smallness for the heat equation

In this section we show how the first parts in Theorems 2 and 3 (i.e. estimates (1.8) and (1.10)) follow from Theorem 1. Here we closely follow [2, Section 2], which in turn relied on mixing ideas from [11], interpolation inequalities and the telescopic series method from [12]. Indeed, Theorem 8 is actually slightly more general than [2, Theorem 5], as the constants do not depend on the distance to the boundary but only on the Lebesgue measure of E, and the interpolation exponent ($1-\epsilon$ below) can be taken arbitrarily close to 1. The first step is to deduce interpolation inequalities from Theorem 1.

Theorem 8 (cf. [2, Theorem 6]). *Let $\epsilon \in (0,1)$ and $m > 0$. Assume that $|E_1| \geq m$ and $\mathcal{C}^{d-\delta}_{\mathcal{H}}(E_2) > m$. Then there exist $N, C > 0$ such that for all $0 \leq s < t$,*

$$\|e^{t\Delta} f\|_{L^2(M)} \leq N e^{\frac{N}{t-s}} \|e^{t\Delta} f\|^{1-\epsilon}_{L^1(E_1)} \|e^{s\Delta} f\|^{\epsilon}_{L^2(M)}, \tag{4.1}$$

$$\|e^{t\Delta} f\|_{L^2(M)} \leq N e^{\frac{N}{t-s}} \|e^{t\Delta} f\|^{1-\epsilon}_{L^\infty(E_2)} \|e^{s\Delta} f\|^{\epsilon}_{L^2(M)}. \tag{4.2}$$

Corollary 4.1. *Let $m > 0$. Assume that $|E_1| \geq m$ and $\mathcal{C}^{d-\delta}_{\mathcal{H}}(E_2) > m$. Then for any $D, B \geq 1$ there exist $A, C > 0$ such that for all $0 < t_1 < t_2 \leq T$,*

$$e^{-\frac{A}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(M)} - e^{-\frac{DA}{t_2-t_1}} \|e^{t_1\Delta} f\|_{L^2(M)} \leq C e^{-\frac{B}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^1(E_1)}, \tag{4.3}$$

$$e^{-\frac{A}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(M)} - e^{-\frac{DA}{t_2-t_1}} \|e^{t_1\Delta} f\|_{L^2(M)} \leq C e^{-\frac{B}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^\infty(E_2)}, \tag{4.4}$$

and for any J with $|J \cap (t_1, t_2)| \geq (t_2 - t_1)/3$,

$$e^{-\frac{A}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(M)} - e^{-\frac{DA}{t_2-t_1}} \|e^{t_1\Delta} f\|_{L^2(M)} \leq C \int_{t_1}^{t_2} 1_J(s) \|e^{s\Delta} f\|_{L^1(E_1)}\, ds, \tag{4.5}$$

$$e^{-\frac{A}{t_2-t_1}} \|e^{t_2\Delta} f\|_{L^2(M)} - e^{-\frac{DA}{t_2-t_1}} \|e^{t_1\Delta} f\|_{L^2(M)} \leq C \int_{t_1}^{t_2} 1_J(s) \|e^{s\Delta} f\|_{L^\infty(E_2)}\, ds. \tag{4.6}$$

Proof. Let us first deduce Corollary 4.1 from Theorem 8 adapting [2]. Let $A > 0$. From Theorem 8 we get, using the Young inequality $ab \leq (1-\epsilon)a^{1/(1-\epsilon)} + \epsilon b^{\epsilon^{-1}}$,

$$\begin{aligned}
e^{-\frac{A}{t_2-t_1}} \|e^{t_2\Delta} u\|_{L^2(M)} &\leq N e^{\frac{N-A}{t_2-t_1}} \|e^{t_2\Delta} f\|^{1-\epsilon}_{L^\infty(E_1)} \|e^{t_1\Delta} f\|^{\epsilon}_{L^2(M)} \\
&\leq N e^{-\frac{A}{2(t_2-t_1)}} \|e^{t_2\Delta} f\|^{1-\epsilon}_{L^\infty(E_1)} e^{\frac{N-A/2}{t_2-t_1}} \|e^{t_1\Delta} f\|^{\epsilon}_{L^2(M)} \\
\leq N^{(1-\epsilon)^{-1}} (1-\epsilon) e^{-\frac{A}{2(1-\epsilon)(t_2-t_1)}} &\|e^{t_2\Delta} f\|_{L^\infty(E_1)} + \epsilon e^{\frac{N-A/2}{\epsilon(t_2-t_1)}} \|e^{t_1\Delta} f\|_{L^2(M)}
\end{aligned} \tag{4.7}$$

and (4.3) follows from choosing $2\epsilon < D^{-1}$ in Theorem 8 and then

$$A \geq 2B, \quad \frac{A/2 - N}{\epsilon} > DA.$$

The proof of (4.4) is similar. Let us now turn to the proof of (4.6). From the assumption $|J \cap (t_1, t_2)| \geq (t_2 - t_1)/3$, we deduce

$$|J \cap (t_1 + (t_2 - t_1)/6, t_2)| \geq \frac{t_2 - t_1}{6}. \tag{4.8}$$

Now, from (4.2), for $t \in (t_1 + (t_2 - t_1)/6, t_2)$, we have

$$\begin{aligned}
\|e^{t_2\Delta} f\|_{L^2(M)} \leq \|e^{t\Delta} f\|_{L^2(M)} &\leq N e^{\frac{N}{t-t_1}} \|e^{t\Delta} f\|^{1-\epsilon}_{L^1(E_2)} \|e^{t_1\Delta} f\|^{\epsilon}_{L^2(M)} \\
&\leq N e^{\frac{6N}{t_2-t_1}} \|e^{t\Delta} f\|^{1-\epsilon}_{L^1(E_2)} \|e^{t_1\Delta} f\|^{\epsilon}_{L^2(M)}.
\end{aligned} \tag{4.9}$$

Integrating this inequality on $J \cap (t_1 + (t_2 - t_1)/6, t_2)$ and using the Hölder inequality gives

$$|J \cap (t_1 + (t_2 - t_1)/6, t_2)| \, \|e^{t_2 \Delta} f\|_{L^2(M)} \\ \leq N e^{\frac{6N}{t_2 - t_1}} \left(\int_{t_1 + (t_2 - t_1)/6}^{t_2} 1_J(t) \|e^{t\Delta} f\|_{L^1(E_2)} \, dt \right)^{1-\epsilon} \|e^{t_1 \Delta} f\|_{L^2(M)}^{\epsilon}, \tag{4.10}$$

which using (4.8) (and replacing $6N$ by $6N + 1$) gives

$$\|e^{t_2 \Delta} f\|_{L^2(M)} \\ \leq N e^{\frac{6N+1}{t_2 - t_1}} \left(\int_{t_1 + (t_2 - t_1)/6}^{t_2} 1_J(t) \|e^{t\Delta} f\|_{L^1(E_2)} \, dt \right)^{1-\epsilon} \|e^{t_1 \Delta} f\|_{L^2(M)}^{\epsilon}. \tag{4.11}$$

The rest of the proof of (4.6) follows now the same lines as the proof of (4.4). Finally, the proof of (4.5) is similar. ∎

Remark 4.2. The proof above shows that in (4.5) and (4.6), we can replace the sets E_1, E_2 by sets $E_1(t)$, $E_2(t)$ if we assume that $|E_1(t)| \geq m$ and $\mathcal{C}_{\mathcal{H}}^{d-\delta}(E_2(t)) \geq m$ *uniformly* with respect to $t \in I$, so that we can apply Theorem 8 with $E_1(t)$ and $E_2(t)$.

Proof of Theorem 8. Let $0 \leq s < t$ and for $f \in L^2(M)$ let

$$f = \Pi_\Lambda f + \Pi^\Lambda f,$$

where Π_Λ is the orthogonal projector on the vector space generated by $\{e_k; \lambda_k \leq \Lambda\}$. We have

$$\begin{aligned} \|e^{t\Delta} f\|_{L^2(M)} &\leq \|e^{t\Delta} \Pi_\Lambda f\|_{L^2(M)} + \|e^{t\Delta} \Pi^\Lambda f\|_{L^2(M)} \\ &\leq N e^{N\Lambda} \|e^{t\Delta} \Pi_\Lambda f\|_{L^1(E_1)} + \|e^{t\Delta} \Pi^\Lambda f\|_{L^2(M)} \\ &\leq N e^{N\Lambda} (\|e^{t\Delta} f\|_{L^1(E_1)} + \|e^{t\Delta} \Pi^\Lambda f\|_{L^2(M)}) + \|e^{t\Delta} \Pi^\Lambda f\|_{L^2(M)} \\ &\leq (N+1) e^{N\Lambda} (\|e^{t\Delta} f\|_{L^1(E_1)} + e^{-\Lambda^2 (t-s)} \|e^{s\Delta} \Pi^\Lambda f\|_{L^2(M)}) \\ &\leq (N+1) e^{N\Lambda} (\|e^{t\Delta} f\|_{L^1(E_1)} + e^{-\Lambda^2 (t-s)} \|e^{s\Delta} f\|_{L^2(M)}). \end{aligned} \tag{4.12}$$

Since

$$\sup_{\Lambda \geq 0} e^{N\Lambda - \epsilon \Lambda^2 (t-s)} = e^{\frac{N^2}{4\epsilon(t-s)}},$$

we deduce

$$\|e^{t\Delta} f\|_{L^2(M)} \\ \leq (N+1) e^{\frac{N^2}{4\epsilon(t-s)}} \left(e^{\epsilon \Lambda^2 (t-s)} \|e^{t\Delta} f\|_{L^1(E_1)} + e^{-(1-\epsilon)\Lambda^2 (t-s)} \|e^{s\Delta} f\|_{L^2(M)} \right). \tag{4.13}$$

Since Λ is a free parameter, and $t - s > 0$, we can minimize the right hand side of (4.13) with respect to the parameter $\alpha = e^{-\frac{\Lambda^2}{2}(t-s)} \in (0, 1)$, by choosing

$$e^{\Lambda^2 (t-s)} = \frac{\|e^{s\Delta} f\|_{L^2(M)}}{\|e^{t\Delta} f\|_{L^1(E_1)}},$$

which gives

$$\|e^{t\Delta} f\|_{L^2(M)} \leq 2(N+1)e^{\frac{N^2}{4\epsilon(t-s)}} (\|e^{t\Delta} f\|^2_{L^1(E_1)})^{1-\epsilon} (\|e^{s\Delta} f\|_{L^2(M)})^{\epsilon}, \tag{4.14}$$

and thus (4.1) follows.

To prove (4.2) we have to adapt the method. Using Lemma 2.1 we get

$$\begin{aligned}\|e^{t\Delta} f\|_{L^2(M)} &\leq \|e^{t\Delta}\Pi_\Lambda f\|_{L^2(M)} + \|e^{t\Delta}\Pi^\Lambda f\|_{L^2(M)} \\ &\leq Ne^{N\Lambda}\|e^{t\Delta}\Pi_\Lambda f\|^2_{L^\infty(E_2)} + \|e^{t\Delta}\Pi^\Lambda f\|_{L^2(M)} \\ &\leq Ne^{N\Lambda}(\|e^{t\Delta} f\|_{L^\infty(E_2)} + \|e^{t\Delta}\Pi^\Lambda f\|_{\mathcal{H}^\sigma(M)}) + \|e^{t\Delta}\Pi^\Lambda f\|_{L^2(M)} \\ &\leq (N+1)e^{N\Lambda}(\|e^{t\Delta} f\|_{L^\infty(E_2)} + \|e^{t\Delta}\Pi^\Lambda f\|_{\mathcal{H}^\sigma(M)}). \end{aligned} \tag{4.15}$$

Let us study the quantity

$$\|e^{t\Delta}\Pi^\Lambda f\|^2_{\mathcal{H}^\sigma(M)} = \sum_{\lambda_k > \Lambda} (e^{-2\lambda_k^2(t-s)}\lambda_k^{2\sigma})e^{-2\lambda_k^2 s}|f_k|^2.$$

Since

$$\sup_{\lambda_k \geq \Lambda} e^{-2\epsilon\lambda_k^2(t-s)}\lambda_k^{2\sigma} \leq \left(\Lambda + \frac{\sigma}{\epsilon(t-s)}\right)^{2\sigma},$$

we deduce

$$\begin{aligned}\|e^{t\Delta}\Pi^\Lambda f\|^2_{\mathcal{H}^\sigma(M)} &\leq \left(\Lambda + \frac{\sigma}{2\epsilon(t-s)}\right)^{2\sigma} \sum_{\lambda_k > \Lambda} (e^{-2(1-\epsilon)\lambda_k^2(t-s)})e^{-2\lambda_k^2 s}|f_k|^2 \\ &\leq \left(\Lambda + \frac{\sigma}{2\epsilon(t-s)}\right)^{2\sigma} e^{-2(1-\epsilon)\Lambda^2(t-s)}\|e^{s\Delta} f\|_{L^2} \\ &\leq C_{\epsilon,\sigma}e^{\Lambda}e^{-2(1-2\epsilon)\Lambda^2(t-s)}\|e^{s\Delta} f\|_{L^2}, \end{aligned} \tag{4.16}$$

and coming back to (4.15), we get

$$\|e^{t\Delta} f\|_{L^2(M)} \leq (N_{\epsilon,\sigma})e^{(N+1)\Lambda}(\|e^{t\Delta} f\|_{L^\infty(E)} + e^{-2(1-2\epsilon)\Lambda^2(t-s)}\|e^{s\Delta} f\|_{L^2}). \tag{4.17}$$

The rest of the proof of Theorem 8 follows by the same optimization argument as before. ∎

Once Corollary 4.1 is established, the rest of the proof of (1.8), (1.10), (1.12) and (1.13) closely follows [2, Section 2]. For completeness we recall the proof. Let us start with the simpler (1.12). From (4.3) with $t_1 = s_{n+1}$, $t_2 = s_n$, and $D = \tau^{-1}$ we have

$$e^{-\frac{A}{s_n - s_{n+1}}}\|e^{s_n\Delta} f\|_{L^2(M)} - e^{-\frac{DA}{s_n - s_{n+1}}}\|e^{s_{n+1}\Delta} f\|_{L^2(M)} \leq Ce^{-\frac{B}{s_n - s_{n+1}}}\|e^{s_n\Delta} f\|_{L^1(E_1)}. \tag{4.18}$$

Since $s_{n+1} - s_{n+2} \geq \frac{s_n - s_{n+1}}{D}$, we deduce

$$\begin{aligned} &e^{-\frac{A}{s_n - s_{n+1}}}\|e^{s_n\Delta} f\|_{L^2(M)} - e^{-\frac{A}{s_{n+1} - s_{n+2}}}\|e^{s_{n+1}\Delta} f\|_{L^2(M)} \\ &\quad \leq Ce^{-\frac{B}{s_n - s_{n+1}}}\|e^{s_n\Delta} f\|_{L^1(E_1)} \leq C'e^{-\frac{B-1}{(s_n - s_{n+1})}}\|(s_n - s_{n+1})e^{s_n\Delta} f\|_{L^1(E_1)}. \end{aligned} \tag{4.19}$$

Summing the telescopic series (4.19), and using

$$e^{-\frac{A}{s_n - s_{n+1}}} \|e^{s_n \Delta} f\|_{L^2(M)} \le e^{-\frac{A}{s_n - s_{n+1}}} \|f\|_{L^2(M)} \xrightarrow[n\to\infty]{} 0,$$

we get (recall that $s_0 = T$)

$$\begin{aligned} e^{-\frac{A}{T-s_1}} \|e^{T\Delta} f\|_{L^2(M)} &\le C \sum_{n=0}^{+\infty} e^{-\frac{B-1}{s_n - s_{n+1}}} (s_n - s_{n+1}) \|e^{s_n \Delta} f\|_{L^1(E_1)} \\ &\le C \sup_n e^{-\frac{B-1}{s_n - s_{n+1}}} \|e^{s_n \Delta} f\|_{L^1(E_1)}, \end{aligned} \tag{4.20}$$

which proves (1.12). The proof of (1.13) is the same.

To prove (1.10) we need the following lemma from [12] about the structure of density points of sets of positive measure on $(0, T)$.

Lemma 4.3 ([12, Proposition 2.1]). *Let J be a subset of positive measure in $(0, T)$. Let l be a density point of J. Then for any $z > 1$ there exists $l_1 \in (l, T)$ such that the sequence defined by*

$$l_{m+1} - l = z^{-m}(l_1 - l)$$

satisfies

$$|J \cap (l_{m+1}, l_m)| \ge (l_m - l_{m+1})/3.$$

Now, we apply this result with $z = 2$ and from (4.6) with $D = 2$ and $t_1 = l_{m+1}, t_2 = l_m$ we get

$$\begin{aligned} e^{-\frac{M}{l_m - l_{m+1}}} \|e^{l_m \Delta} f\|_{L^2(M)} - e^{-\frac{2M}{l_m - l_{m+1}}} \|e^{l_{m+1} \Delta} f\|_{L^2(M)} \\ \le C \int_{l_{m+1}}^{l_m} 1_J(s) \|e^{s\Delta} f\|_{L^\infty(E)}\, ds. \end{aligned} \tag{4.21}$$

Noticing that $\frac{2}{l_m - l_{m+1}} = \frac{1}{l_{m+1} - l_{m+2}}$, we get

$$\begin{aligned} e^{-\frac{M}{l_m - l_{m+1}}} \|e^{l_m \Delta} f\|_{L^2(M)} - e^{-\frac{M}{l_{m+1} - l_{m+2}}} \|e^{l_{m+1} \Delta} f\|_{L^2(M)} \\ \le C \int_{l_{m+1}}^{l_m} 1_J(s) \|e^{s\Delta} f\|_{L^\infty(E)}\, ds. \end{aligned} \tag{4.22}$$

Now summing the telescopic series (4.22), and using

$$\lim_{m\to+\infty} e^{-\frac{M}{l_{m+1} - l_{m+1}}} = 0,$$

we get

$$e^{-\frac{M}{l_1 - l_2}} \|e^{l_1 \Delta} f\|_{L^2(M)} \le C \int_{l_1}^{l} 1_J(s) \|e^{s\Delta} f\|_{L^\infty(E)}\, ds,$$

which (since $T > l_1$) implies (1.10).

To prove (1.8), we need an elementary consequence of Fubini's Theorem.

Lemma 4.4. *Let $F \subset M \times (0,T)$ be a set of positive Lebesgue measure. Working in coordinates, we can assume that $F \subset B(x_0, r_0) \times (0,T)$. For almost every $t \in (0,T)$ the sets*

$$E_t = F \cap M \times \{t\} \quad \textit{and} \quad J = \left\{ t \in (0,T);\, |E_t| \geq \frac{|F|}{2T} \right\}$$

are measurable and

$$|J| \geq \frac{|F|}{2T|B(x_0, r_0)|}.$$

Proof. Indeed, from Fubini,

$$|F| = \int_J |E_t|\,dt + \int_{(0,T)\setminus J} |E_t|\,dt \leq |J|\,|B(x_0,r_0)| + \frac{|F|}{2}. \qquad \blacksquare$$

Now, the proof of (1.8) follows exactly the same lines as the proof of (1.10) above by noticing that (4.1) will hold for $E = E_t$ with constants that are uniform with respect to $t \in I$ (because then $|E_t| \geq \frac{|F|}{2T}$); see Remark 4.2.

5. Control for heat equations on "very small sets"

Here we give the proof of the exact controllability parts in Theorems 3 and 4 (this part in Theorem 2 is very classical and we shall leave it to the reader). We start with Theorems 3. Since $J \subset (0,T)$ has positive Lebesgue measure, so does $J \cap (\epsilon, T)$ for some $0 < \epsilon < T$, and hence we can assume $J \subset (\epsilon, T)$. By subadditivity of the Hausdorff content,

$$\mathcal{C}^r_{\mathcal{H}}\Big(\bigcup_{j=1}^{+\infty} A_j\Big) \leq \sum_{j=1}^{+\infty} \mathcal{C}^r_{\mathcal{H}}(A_j)$$

with

$$A_j = E \cap \{x \in M;\, d(x, \partial M) \geq 1/j\}, \quad j \in \mathbb{N},$$

we deduce that there exists j_0 such that

$$\mathcal{C}^{d-\delta}_{\mathcal{H}}(E \cap A_{j_0}) > 0$$

because otherwise we would have

$$\mathcal{C}^{d-\delta}_{\mathcal{H}}\Big(E \cap \bigcup_{j=1}^{+\infty} A_{j_0}\Big) = \mathcal{C}^{d-\delta}_{\mathcal{H}}(E \setminus \partial M) = 0, \quad \text{so} \quad \mathcal{C}^{d-\delta}_{\mathcal{H}}(E) = 0.$$

As a consequence, replacing E by $E \cap A_{j_0}$, we can assume that

$$\exists \epsilon > 0\ \forall x \in E, \quad d(x, \partial M) > \epsilon. \tag{5.1}$$

For $w_0 \in L^2(M)$ let $w = e^{(T-t)\Delta} w_0$ be the solution to the backward heat equation

$$\begin{aligned} &(\partial_t + \Delta) w = 0, \quad u|_{t=T} = w_0, \\ &w|_{\partial M} = 0 \text{ or } \partial_\nu w|_{\partial M} = 0. \end{aligned} \tag{5.2}$$

Let σ be as in Lemma 2.1. Notice that for any $\epsilon > 0$, we have $w \in C^0([\epsilon, T]; \mathcal{H}^\sigma)$, and so

$$w \in C^0([\epsilon, T] \times M), \qquad \sup_{(t,x)\in(\epsilon,T)\times E} |w|(t,x) \leq C\|w_0\|_{L^2(M)}$$

Consider the set

$$X = \{e^{(T-t)\Delta} w_0|_{J\times E};\ w_0 \in L^2(M)\}.$$

We endow X with the norm inherited from $L^1((0,T); L^\infty(M))$ and have

$$\|w\|_X = \|w\|_{L^1(J;L^\infty(E))} \leq C \sup_{(t,x)\in(\epsilon,T)\times E} |w|(t,x) \leq C\|v_0\|_{L^2(M)}.$$

By the observation estimate (1.10), applied to $\widetilde{J} \times E$, $\widetilde{J} = T - J$, we have

$$\|w|_{t=0}\|_{L^2(M)} \leq C \int_{\widetilde{J}} \|e^{s\Delta} w_0\|_{L^\infty(E)} ds = C \int_J \|e^{(T-t)\Delta} w_0\|_{L^\infty(E)}\, dt \leq C\|w\|_X. \tag{5.3}$$

As a consequence, for any $u_0, v_0 \in L^2(M)$, the map

$$X \ni w \mapsto (w|_{t=0}, u_0 - v_0)_{L^2(M)}$$

is well defined because if $w_1 = w_2 \in X$, then from (5.3), $w_1|_{t=0} = w_2|_{t=0}$. Also from (5.3), this map is a continuous linear form on X. By the Hahn–Banach Theorem, there exists an extension as a continuous linear form to the whole space

$$L^1((0,T); C^0(E)).$$

By the Riesz Representation Theorem, there exists

$$\mu \in L^\infty((0,T); \mathcal{M}(E))$$

(here $\mathcal{M}(E)$ is the set of Borel measures on the metric space E) such that this linear form is given by

$$L^1((0,T); C^0(E)) \ni w \mapsto \int_{(0,T)\times E} w(t,x)\, d\mu.$$

We can extend μ by restriction to $L^1((0,T); C^0(M))$ in the following way:

$$L^1((0,T); C^0(M)) \ni w \mapsto \int_{(0,T)\times E} w|_{((0,T)\times E)}(t,x)\, d\mu,$$

which defines an element (still denoted by μ) of $L^\infty((0,T); \mathcal{M}(M))$, supported on $[\epsilon, T] \times E$ (here we have used the assumption that E is a closed set).

Let us now check that the solution to

$$\begin{aligned} &(\partial_t - \Delta)z = \mu(t,x) 1_{E\times(0,T)}, \quad z|_{t=0} = 0, \\ &z|_{\partial M} = 0 \text{ or } \partial_\nu z|_{\partial M} = 0, \end{aligned} \tag{5.4}$$

satisfies

$$z|_{t=T} = e^{T\Delta}(u_0 - v_0), \quad z|_{t=0} = 0$$

and consequently choosing

$$u = e^{t\Delta}u_0 - z$$

proves the second part in Theorem 3. First we have to make sense of (5.4) (and show that the right hand side $\mu(t,x)1_{E\times(0,T)}$ is an admissible source term). The first step is to prove that $\mathcal{H}^\sigma$ is dense in $C^0(M)$ subject to Dirichlet boundary conditions (or Neumann in a sense to be specified). Of course, the set $\mathcal{H}^\sigma$ being defined in terms of the eigenfunctions of the Laplace operator with Dirichlet or Neumann boundary conditions depends on this choice of boundary conditions, and in the next lemma, we make this dependence explicit.

Lemma 5.1. *For all $\sigma > 0$, the set $\mathcal{H}^\sigma_D$ is dense in the set of continuous functions on $\overline{M}$ vanishing on ∂M, while the set $\mathcal{H}^\sigma_N$ is dense in the set of continuous functions on $\overline{M}$.*

Proof. Let $u_0 \in C^0(\overline{M})$ vanish on the boundary ∂M. Then the function defined on the double manifold by

$$\widetilde{u}_0(x,\pm 1) = \pm u_0(x)$$

is clearly continuous on the double manifold $\widetilde{M}$. We shall say that $\widetilde{u}_0$ is *odd*. Clearly the set of odd C^1 functions on $\widetilde{M}$ is dense in the set of C^0 odd functions on $\widetilde{M}$. Now for any $\widetilde{v}_0$ C^1 and odd, working in the double manifold, we can apply the maximum principle for the heat semigroup $(e^{t\widetilde{\Delta}})_{t\geq 0}$, whereby the family $(e^{t\widetilde{\Delta}}\widetilde{v}_0)_{t\geq 0}$ is uniformly bounded in $L^\infty(M)$ by $\|\widetilde{v}_0\|_{L^\infty(M)}$. Then applying again the maximum principle to $(\nabla_x e^{t\widetilde{\Delta}}\widetilde{v}_0)_{t\geq 0}$ we find that $(e^{t\widetilde{\Delta}}\widetilde{v}_0)_{t\geq 0}$ is bounded in $W^{1,\infty}(\widetilde{M})$. It clearly converges to v_0 in $H^1(M) = W^{1,2}(M)$ as $t \to 0$ (by decomposition with respect to the eigenbasis of $\widetilde{\Delta}$ defined in Remark 3.2), and consequently it converges to u_0 in $W^{1,p}(M)$ for all $2 \leq p < +\infty$, which implies convergence to $\widetilde{v}_0$ in $C^0(M)$. Now the decomposition of $\widetilde{v}_0$ with respect to the set of eigenfunctions of the Laplace operator $\widetilde{\Delta}$ involves only odd eigenfunctions, hence eigenfunctions $\widetilde{e}$ which are of the form

$$\widetilde{e}(x,\pm 1) = e(x),$$

where e is an eigenfunction of the Laplace operator on M with Dirichlet boundary conditions (see Remark 3.2). As a consequence, for any $t > 0$,

$$e^{t\widetilde{\Delta}}\widetilde{v}_0|_{M\times\{1\}} = e^{t\Delta_D}v_0 \in \mathcal{H}^\sigma_D(M).$$

This implies that $\mathcal{H}^\sigma_D(M)$ is dense in the set of continuous functions on $\overline{M}$ vanishing on ∂M. To prove that $\mathcal{H}^\sigma_N(M)$ is dense in the set of continuous functions on $\overline{M}$, we proceed similarly replacing the odd extension by the even extension

$$\widetilde{u}_0(x,\pm 1) = u_0(x),$$

which sends the set of continuous functions on $\overline{M}$ to the set of continuous functions on $\widetilde{M}$ (here we do not require the vanishing of u_0 on the boundary). ∎

The density of $\mathcal{H}^\sigma_N$ in C^0 implies that the map

$$\mathcal{M}(M) \ni \nu \mapsto \widetilde{\nu} = \nu|_{\mathcal{H}^\sigma_N} \in \mathcal{H}^{-\sigma}_N,$$

is onto and consequently any measure $\nu \in \mathcal{M}(M)$ can be seen as an element of $\mathcal{H}_N^{-\sigma}$, the dual space of $\mathcal{H}_N^{\sigma}$. Respectively, since $\mathcal{H}_D^{\sigma}$ is dense in the set of functions vanishing on ∂M, any measure $\nu \in \mathcal{M}(M)$ supported away from the boundary can be seen as an element of $\mathcal{H}_D^{-\sigma}$, the dual space of $\mathcal{H}_D^{\sigma}$. As a consequence, we can solve (5.4) by using the natural spectral decomposition in $\mathcal{H}^{-\sigma}$, i.e.,

$$\mu = \sum_k \langle \mu, e_k \rangle (t) e_k,$$

with $\langle \mu, e_k \rangle (t)$ supported in (ϵ, T) and

$$\operatorname*{ess\,sup}_{t \in (0,T)} \sum_k \lambda_k^{-2\sigma} |\langle \mu, e_k \rangle|^2 (t) < +\infty.$$

Let $w_0 \in L^2(M)$ and let w_N be the solution to (5.2) with v_0 replaced by

$$w_{0,N} = \sum_{k \leq N} (w_0, e_k) e_k,$$

and z_N the solution to (5.4) with μ replaced by

$$\mu_N = \sum_{k \leq N} \langle \mu, e_k \rangle (t) e_k.$$

We have

$$\begin{aligned} 0 &= \int_0^T ((\partial_t + \Delta) w_N, z_N)_{L^2} = [(w_N, z_N)_{L^2}]_0^T - \int_0^T (w_N, (-\partial_t + \Delta) z_N)_{L^2} \\ &= (w_{0,N}, z_N|_{t=T})_{L^2} - \int_0^T (w_N, \mu_N)_{L^2}. \end{aligned} \tag{5.5}$$

We now let N tend to infinity. Then

$$\begin{aligned} & w_{0,N} \to w_0 \quad \text{in } L^2, \\ & z_N|_{t=T} \to z|_{t=T} \quad \text{in } \mathcal{H}^{-\sigma}, \quad \text{so} \quad z_N|_{t=T} \to z|_{t=T'} \quad \text{in } L^2, \\ & w_N \to w \quad \text{in } C^0([0,T]; \mathcal{H}^{\sigma}), \\ & \mu_N \to \mu \quad \text{in } L^\infty([0,T]; \mathcal{H}^{-\sigma}). \end{aligned} \tag{5.6}$$

We deduce that we can pass to the limit in (5.5) and get

$$0 = (w_0, z|_{t=T})_{L^2} - \int_0^T w(t,x) 1_{t \in (0,T)} \, d\mu.$$

From the definition of μ we have

$$\int_0^T w(t,x) d\mu = (w|_{t=0}, u_0 - v_0)_{L^2} = (e^{T\Delta} w_0, u_0 - v_0)_{L^2}.$$

We finally get

$$\forall w_0 \in L^2, (w_0, z|_{t=T})_{L^2} = (e^{T\Delta} w_0, u_0 - v_0)_{L^2}, \quad \text{so} \quad z|_{t=T} = e^{T\Delta}(u_0 - v_0).$$

Thus $u = e^{t\Delta} u_0 - z$ satisfies the second part of Theorem 3.

We now turn to the second part in Theorem 4 and highlight the modifications required in the proof above. We shall focus on the case $E = E_2$ and assume that E satisfies (5.1). Let $J = \{t_n;\ n \in \mathbb{N}\} \cup \{T\} \subset [t_0, T]$ (recall that $t_0 > 0$) and $\widetilde{J} = T - J = \{s_n\} \cup \{0\}$. Let

$$X = \{e^{(T-t)\Delta} v_0|_{J\times E};\ v_0 \in L^2(M)\} \subset C^0(J \times E),$$

endowed with the sup norm. Then according to (1.13) with $s_n = T - t_n$, the linear form

$$X \ni w \mapsto (w|_{t=0}, u_0 - v_0)_{L^2(M)}$$

is well defined and continuous, and more precisely bounded by

$$C \sup_{n\in\mathbb{N}, x\in E} e^{-\frac{B}{T-t_n}} |w(t_n, x)|. \tag{5.7}$$

Indeed (notice that $T - t_n \geq t_{n+1} - t_n$),

$$\begin{aligned}\|w|_{t=0}\|_{L^2(M)} = \|e^{T\Delta} v_0\|_{L^2(M)} &\leq C \sup_{n\in\mathbb{N}, x\in E} e^{-\frac{B}{t_{n+1}-t_n}} |w(t_n, x)| \\ &\leq C \sup_{n\in\mathbb{N}, x\in E} e^{-\frac{B}{T-t_n}} |w(t_n, x)|.\end{aligned}$$

According to the Hahn–Banach theorem [13, Theorem 3.2], we can extend this map to the whole space $C^0(J \times E)$, so that it is still bounded by (5.7). By the Riesz Representation Theorem, this continuous linear form can be represented by a measure $\mu \in \mathcal{M}(J \times E)$ which still satisfies the same bound (5.7). As previously we can extend this measure to a measure on $[0, T] \times M$ which is supported in $J \times E$. Hence this measure takes the form

$$\mu = \sum_n \delta_{t=t_n} \otimes \mu_n + \delta_{t=T} \otimes \mu_\infty,$$

with μ_j, μ_∞ measures on M supported by E. Using (5.7) we get

$$\sum_n e^{\frac{B}{T-t_n}} |\mu_n|(E) < +\infty, \quad \mu_\infty = 0.$$

Now we can easily make sense of solving

$$(\partial_t - \Delta) z = \sum_n \delta_{t=t_n} \otimes \mu_n, \quad z|_{ty=0} = 0$$

with Dirichlet or Neumann boundary conditions in $L^\infty([0, T); \mathcal{H}^{-\sigma})$, by simply noticing that the solution to this equation is the solution to the homogeneous heat equation on (t_n, t_{n+1}) which satisfies the jump condition

$$z|_{t_n+0} - z|_{t_n-0} = \mu_n \in \mathcal{H}^{-\sigma}.$$

Since

$$\sum_n \|\mu_n\|_{\mathcal{H}^{-\sigma}} \leq C \sum_n |\mu_n|(E) < +\infty,$$

we deduce that actually $\lim_{t\to T, t<T} z(t)$ exists in $\mathscr{H}^{-\sigma}$, and consequently the solution exists and is unique in $[0,+\infty)$ (defined on $[T,+\infty)$ as the solution of the homogeneous heat equation). We now write the analog of the integration by parts formula (5.5). Let z_N, w_N and $\mu_{n,N}$ be the projections of z, v and μ_n on the space spanned by the first N eigenfunctions. On (t_n, t_{n+1}), we have

$$\begin{aligned} 0 &= \int_{t_n}^{t_{n+1}} ((\partial_t + \Delta) w_N, z_N)_{L^2} = [(w_N, z_N)_{L^2}]_{t_n}^{t_{n+1}} - \int_{t_n}^{t_{n+1}} (w_N, (-\partial_t + \Delta) z_N)_{L^2} \\ &= (w_N|_{t_{n+1}}, z_N|_{t=t_{n+1}-0})_{L^2} - (w_N|_{t_n}, z_N|_{t=t_n+0})_{L^2}, \end{aligned} \tag{5.8}$$

which implies (using $z_N|_{t=0} = 0$ and $\lim_{n\to+\infty} w_N|_{t=t_n} = w_N(T)$) that

$$\begin{aligned} 0 &= \int_0^T ((\partial_t + \Delta) w_N, z_N)_{L^2} \\ &= \sum_n (w_N|_{t_{n+1}}, z_N|_{t=t_{n+1}-0})_{L^2} - (w_N|_{t_n}, z_N|_{t=t_n+0})_{L^2} \\ &= \lim_{k\to+\infty} (w_N(t_k), z_N|_{t=t_k-0})_{L^2} + \sum_{n=1}^{k-1} (w_N|_{t_n}, z_N|_{t=t_n-0} - z_N|_{t=t_n+0})_{L^2} \\ &= (w_N(T), z_N|_{t=T})_{L^2} - \sum_n (w_N|_{t_n}, \mu_{n,N})_{L^2} \\ &= (w_N, z_N|_{t=T})_{L^2} - \int_0^T w_N(t)\, d\mu_N. \end{aligned}$$

We can now let $N \to +\infty$ and get

$$(u_0 - v_0, z|_{t=T})_{L^2} = \int_0^T w(t)\, d\mu,$$

and we conclude as previously that $u = e^{t\Delta} u_0 - z$ satisfies (with Dirichlet or Neumann boundary conditions)

$$(\partial_t - \Delta) u = -\sum_n \delta_{t=t_n} \otimes \mu_n, \quad u|_{t=0} = u_0,\ u|_{t=T} = v_0.$$

This proves the second part in Theorem 4, in the case $E = E_2$.The case $E = E_1$ is proved similarly by replacing in the proof above (1.13) by (1.12).

Acknowledgments. We would like to thank a referee for pointing out a mistake in the previous proof of the construction of the double manifold (Section 3).

Funding. N.B. is supported by Institut Universitaire de France and ANR grant ISDEEC, ANR-16-CE40-0013. Part of this work was done while N.B. was in residence during the fall 2019 at the Mathematical Science Research Institute (MSRI) with support by the NSF grant DMS-1440140. I.M. was partially supported by the European Research Council (ERC) MAFRAN grant under the European's Union Horizon 2020 research and innovation programme (grant agreement No 726386) while he was a research associate in PDEs at DPMMS, University of Cambridge, U.K.

References

[1] Apraiz, J., Escauriaza, L.: Null-control and measurable sets. ESAIM Control Optim. Calc. Var. **19**, 239–254 (2013) Zbl 1262.35118 MR 3023068

[2] Apraiz, J., Escauriaza, L., Wang, G., Zhang, C.: Observability inequalities and measurable sets. J. Eur. Math. Soc. **16**, 2433–2475 (2014) Zbl 1302.93040 MR 3283402

[3] Escauriaza, L., Montaner, S., Zhang, C.: Analyticity of solutions to parabolic evolutions and applications. SIAM J. Math. Anal. **49**, 4064–4092 (2017) Zbl 1377.35048 MR 3713902

[4] Han, Q., Lin, F.: Elliptic Partial Differential Equations. 2nd ed., Courant Lecture Notes in Math. 1, Courant Inst. Math. Sci., New York, and Amer. Math. Soc., Providence, RI (2011) Zbl 1210.35031 MR 2777537

[5] Jerison, D., Lebeau, G.: Nodal sets of sums of eigenfunctions. In: Harmonic Analysis and Partial Differential Equations (Chicago, IL, 1996), Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 223–239 (1999) Zbl 0946.35055 MR 1743865

[6] Lebeau, G.. Moyano, I.: Spectral inequalities for the Schrödinger operator. arXiv:1901.03513 (2019)

[7] Lebeau, G., Robbiano, L.: Contrôle exact de l'équation de la chaleur. Comm. Partial Differential Equations **20**, 335–356 (1995) Zbl 0819.35071 MR 1312710

[8] Logunov, A.: Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. Ann. of Math. (2) **187**, 221–239 (2018) Zbl 1384.58020 MR 3739231

[9] Logunov, A.: Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture. Ann. of Math. (2) **187**, 241–262 (2018) Zbl 1384.58021 MR 3739232

[10] Logunov, A., Malinnikova, E.: Quantitative propagation of smallness for solutions of elliptic equations. In: Proc. International Congress of Mathematicians (Rio de Janeiro, 2018), Vol. III, World Sci., Hackensack, NJ, 2391–2411 (2018) Zbl 1453.35061 MR 3966855

[11] Miller, L.: A direct Lebeau–Robbiano strategy for the observability of heat-like semigroups. Discrete Contin. Dynam. Systems Ser. B **14**, 1465–1485 (2010) Zbl 1219.93017 MR 2679651

[12] Phung, K. D., Wang, G.: An observability estimate for parabolic equations from a measurable set in time and its applications. J. Eur. Math. Soc. **15**, 681–703 (2013) Zbl 1258.93037 MR 3017049

[13] Rudin, W.: Functional Analysis. 2nd ed., McGraw-Hill, New York (1991) Zbl 0867.46001 MR 1157815