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Hodge–Tate decomposition for non-smooth spaces

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Abstract. In this article, we generalize the Hodge–Tate decomposition of p-adic étale cohomology to non-smooth rigid spaces. Our strategy is to study pro-étale cohomology of rigid spaces introduced by Scholze, using the resolution of singularities and the simplicial method.

Keywords. p-adic Hodge theory, étale cohomology, rigid analytic spaces, resolution of singularities

Contents

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1. Introduction

1.1. Goals and main results

Let X be a compact complex manifold. One of the most important invariants of X is its singular cohomology group $\mathrm{H}^n_{\mathrm{Sing}}(X, \mathbb{C})$, defined in a transcendental way. The de Rham Theorem allows us to compute those cohomology groups by using differential forms: there exists a natural isomorphism

$$\mathrm{H}^n_{\mathrm{Sing}}(X, \mathbb{C}) \cong \mathrm{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}),$$

where $\Omega^\bullet_{X/\mathbb{C}}$ is the analytic de Rham complex of X over $\mathbb{C}$. Moreover, induced by the Hodge filtration of the de Rham complex, there exists a (Hodge–de Rham) spectral sequence

$$E_1^{i,j} = \mathrm{H}^j(X, \Omega^i_{X/\mathbb{C}}) \Longrightarrow \mathrm{H}^{i+j}(X, \Omega^\bullet_{X/\mathbb{C}})$$

computing the de Rham cohomology. This spectral sequence often degenerates; in fact, we have a stronger result when X is a compact Kähler manifold.

Theorem 1.1.1 (Hodge decomposition). *Let X be a compact Kähler manifold. Then its singular cohomology $\mathrm{H}^n_{\mathrm{Sing}}(X, \mathbb{C})$ admits a canonical decomposition into a direct sum of the vector spaces $\mathrm{H}^{i,j}(X)$ of harmonic (i, j)-forms for $i + j = n$.*

The vector space $\mathrm{H}^{i,j}(X)$ is of the same dimension as $\mathrm{H}^j(X, \Omega^i_{X/\mathbb{C}})$. As a consequence, the Hodge–de Rham spectral sequence for X,

$$E_1^{i,j} = \mathrm{H}^j(X, \Omega^i_{X/\mathbb{C}}) \Longrightarrow \mathrm{H}^{i+j}(X, \Omega^\bullet_{X/\mathbb{C}}),$$

degenerates at its first page.

The above assumption holds in particular when X is the analytification of a smooth projective algebraic variety Y over $\mathbb{C}$. Moreover, in the aforementioned setting, by the GAGA principle, the above spectral sequence can be replaced by the algebraic Hodge–de Rham spectral sequence using the algebraic differential forms of Y over $\mathbb{C}$. The singular cohomology can thus be computed using a purely algebraic method.

Now we turn to non-archimedean geometry. Let us fix a complete and algebraic closed p-adic field extension $K/\mathbb{Q}_p$.

Proper smooth rigid spaces, introduced by Tate in the 1960s, form a natural analogue of compact complex manifolds in this setting. Here typical examples include analytifications of proper smooth algebraic varieties over K. Since the non-archimedean field K is totally disconnected, singular cohomology is not a meaningful invariant of X. Instead, the correct analogue should be the p-adic étale cohomology groups $\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}_p)$, which are defined by associating an étale site $X_{\text{ét}}$ to the rigid space X, and then taking the inverse limit of the étale cohomology groups $\mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}/p^n)$. Moreover, analogously to the complex setting, we have the following result computing étale cohomology:

Theorem 1.1.2 (Hodge–Tate filtration). *Let X be a smooth proper rigid space over K. Then we have a natural spectral sequence*

$$E_2^{i,j} = \mathrm{H}^i(X, \Omega^j_{X/K})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} K.$$

The spectral sequence degenerates at its E_2-page.

As pointed out by a referee, in contrast to Theorem 1.1.1, we do not need any "Kähler condition" in the p-adic setting.

When X is an algebraic scheme, the result was proved in many cases by Faltings. Later on Scholze constructed the spectral sequence above for rigid spaces, and proved the degeneracy for those X defined over a discretely valued field. Here we note that in the latter setting when X comes from a discretely valued subfield (with perfect residue field), the filtration above is Galois equivariant and admits a canonical splitting, which is called the *Hodge–Tate decomposition*. The case of general proper smooth rigid spaces that are not necessarily defined over a discrete subfield is handled in [3] by using a spreading-out technique of Conrad–Gabber. Similar to the complex geometry, we note that one of the biggest advantages of this decomposition is that the left side is given by a coherent cohomology, which is of more algebraic nature than p-adic étale cohomology.

The goal of our article is to generalize the Hodge–Tate decomposition to the case when X is not necessarily smooth. In scheme theory, one strategy to generalize from the smooth setting to the non-smooth setting is to use the h-topology, introduced by Voevodsky [37]. The h-topology is defined by refining the étale topology (on the category of all finite type K-schemes over X) by allowing not just étale coverings, but also proper surjective maps. In particular, thanks to the resolution of singularities in characteristic zero, any h-covering can be refined by smooth schemes, which makes the h-topology "locally smooth". Later Geisser [15] introduced the éh-topology, which is a variant of Voevodsky's theory and is defined by a smaller but more amenable collection of covering families.

Inspired by those ideas, we introduce the éh-topology $X_{\text{éh}}$ for a rigid space X, where coverings are generated by étale coverings, universal homeomorphisms, and coverings associated to blowups (see Section 2). Similarly to scheme theory, our éh-topology is locally smooth, and there exists a natural morphism of sites $\pi_X : X_{\text{éh}} \to X_{\text{ét}}$. In addition, by sheafifying the usual sheaf $\Omega^j_{/K}$ of continuous differentials in the éh-topology, we obtain the sheaf $\Omega^j_{\text{éh},/K}$ of éh-differentials on the éh-site $X_{\text{éh}}$ over K. (We will use $\Omega^j_{\text{éh}}$ to abbreviate the notation when the base is clear.)

Our main theorems are the following:

Theorem 1.1.3 (Hodge–Tate decomposition). *Let X be a proper rigid space over a complete algebraically closed non-archimedean field $K/\mathbb{Q}_p$ of characteristic zero. Then there exists a natural spectral sequence*

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K,$$

where $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is the i-th éh-cohomology group of $\Omega^j_{\text{éh}}$, and is equipped with a Galois action by the Tate twist of weight j when X is defined over a discretely valued *subfield. The spectral sequence satisfies the following:*

(i) *The cohomology group* $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ *is a finite-dimensional* K*-vector space that vanishes unless* $0 \leq i, j \leq n$.

(ii) *The spectral sequence degenerates at the* E_2*-page.*

(iii) *If* X *is a smooth rigid space,* $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ *is isomorphic to* $\mathrm{H}^i(X, \Omega^j_{X/K})(-j)$*, and the spectral sequence is the same as the Hodge–Tate spectral sequence for smooth proper rigid spaces in Theorem 1.1.2.*

The éh-cohomology group $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ above is not an exotic construction; in fact, this is an analogue of the classical Deligne–Du Bois cohomology in the rigid spaces setting, and can be computed by using cohomologies of coherent sheaves over the rigid space. To see this, we notice that the map of sites $\pi_X : X_{\text{éh}} \to X_{\text{ét}}$ provides us with a natural quasi-isomorphism

$$R\Gamma(X_{\text{ét}}, R\pi_{X*}\Omega^j_{\text{éh}}) \cong R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}).$$

As proved in Proposition 6.0.1, each higher pushforward $R^s\pi_{X*}\Omega^j_{\text{éh}}$ is a coherent sheaf over X. So the associated Leray spectral sequence above is computed by rigid cohomologies of coherent sheaves.

Moreover, éh-cohomology for a proper rigid space X coincides with its algebraic version when X comes from a proper algebraic variety. More precisely, assume $X = Y^{\text{an}}$ is the analytification of a proper variety Y over K, and the base field K is isomorphic to $\mathbb{C}$ abstractly. We can then get a functorial isomorphism

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{H}^i(Y, \underline{\Omega}^j_Y),$$

where $\underline{\Omega}^j_Y$ is the j-th graded piece of the Deligne–Du Bois complex $\underline{\Omega}^\bullet_Y$ for the complex algebraic variety Y [26, §7.3]. Furthermore, the cohomology group $\mathrm{H}^i(Y, \underline{\Omega}^j_Y)$ is isomorphic to the j-th graded factor for the Hodge filtration of the singular cohomology group $\mathrm{H}^i_{\text{Sing}}(Y(\mathbb{C}), \mathbb{C})$. In fact, there exists a filtered isomorphism between the éh-cohomology of X and the singular cohomology of Y, so that the above isomorphism is obtained by taking the j-th graded factor (see §5.2 for details).

Here we also mention that the analogous h-cohomology for algebraic varieties has already appeared in the p-adic Hodge theory, in Beilinson's work [1]. Beilinson gives a different proof of the Hodge–Tate decomposition for algebraic varieties using de Jong's alterations. In the complex settings, the h-cohomology (and the éh-cohomology) of differentials appears for example in [15, 20, 25], where the relation between h-cohomology and different types of singularities in complex algebraic geometry is studied.

Finally, we mention that in Theorem 1.1.3, the rigid space X may not be defined over a discretely valued subfield (in which case the Tate twist appearing there will not enter the picture). However, when $X = Y_K$ is defined over a discretely valued subfield, the decomposition theorem above can be obtained from a comparison between étale cohomology and an éh version of de Rham cohomology, generalizing the étale-de Rham comparison for the smooth case in [28]. More precisely, let B_{dR} be Fontaine's de Rham period ring, which is equipped with a natural filtration (see the discussion in §7.1). We then have the following result:

Theorem 1.1.4 (étale-éh de Rham comparison). *Let Y be a proper rigid space over a discretely valued subfield K_0 of $K/\mathbb{Q}_p$ that has a perfect residue field. Then there exists a* $\mathrm{Gal}(K/K_0)$*-equivariant filtered isomorphism*

$$\mathrm{H}^n(Y_{K\,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}} = \mathrm{H}^n(Y_{\text{éh}}, \Omega^{\bullet}_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}},$$

whose 0*-th graded piece is*

$$\mathrm{H}^n(Y_{K\,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \bigoplus_{i+j=n} \mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh},/K_0}) \otimes_{K_0} K(-j).$$

The isomorphism is functorial with respect to Y over K_0. In particular, the p-adic Galois representation $\mathrm{H}^n(Y_{K\,\text{ét}}, \mathbb{Q}_p)$ *is de Rham.*

Here recall that the filtration on the left side above is the tensor product filtration for the trivial filtration of étale cohomology and the natural filtration on B_{dR}; the filtration on the right side is the tensor product filtration for the Hodge filtration of the éh de Rham complex and the filtration of B_{dR}.

1.2. Ideas of proof

In this subsection, we sketch the ideas of the proof of Theorem 1.1.3.

Let $K/\mathbb{Q}_p$ be a fixed complete algebraic closed non-archimedean field extension, and let X be a rigid space over K. One way to compute the p-adic étale cohomology $\mathrm{H}^*(X_{\text{ét}}, \mathbb{Q}_p)$ is to use the pro-étale topology. Scholze introduces the pro-étale site $X_{\text{proét}}$ using the perfectoid geometry, together with a natural morphism of Grothendieck topologies

$$\nu : X_{\text{proét}} \to X_{\text{ét}}.$$

Assuming X is proper and K is complete and algebraically closed, Scholze [28,29] shows that there exists an isomorphism

$$\mathrm{H}^*(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} K \cong \mathrm{H}^*(X_{\text{proét}}, \widehat{\mathcal{O}}_X),$$

where $\widehat{\mathcal{O}}_X$ is the completed pro-étale structure sheaf on $X_{\text{proét}}$. So by the Leray spectral sequence, the study of p-adic étale cohomology can be broken into two parts: the study of $R^i\nu_*\widehat{\mathcal{O}}_X$, and the understanding of its cohomology.

When X is a smooth rigid space, we have $R^i\nu_*\widehat{\mathcal{O}}_X = \Omega^i_{X/K}(-i)$ [29, Proposition 3.23]. Here the twist "$(-i)$" means that when $X = X_0 \times_{K_0} K$ comes from a smooth rigid space X_0 over a finite extension $K_0/\mathbb{Q}_p$, the sheaf $R^i\nu_*\widehat{\mathcal{O}}_X$ is isomorphic to the $(-i)$-th Tate twist of the i-th continuous differential sheaf over K, as a sheaf of modules equipped with a Galois action.

In general when X is not necessarily smooth over K, we use the aforementioned éh-topology to extend the étale site to a locally smooth site $X_{\text{éh}}$, and denote the natural map of sites as $\pi_X : X_{\text{éh}} \to X_{\text{ét}}$. We sheafify the presheaf of K-linear continuous differential forms to get the éh-sheaf $\Omega^i_{\text{éh}}$ of differentials. Then we have the following result, connecting the higher direct image $R^i\nu_*\widehat{\mathcal{O}}_X$ of the completed pro-étale structure sheaf with the éh-sheaf of differentials:

Theorem 1.2.1 (éh-proét spectral sequence). *Let X be a rigid space over K. Then there exists an E_2 (Leray) spectral sequence of $\mathcal{O}_X$-modules*

$$E_2^{i,j} = R^i\pi_{X*}\Omega^j_{\text{éh}}(-j) \Longrightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X.$$

When X is smooth over K, the higher direct image $R^i\pi_{X*}\Omega^j_{\text{éh}}$ vanishes for $j \in \mathbb{N}$ and $i > 0$, with

$$R^0\pi_{X*}\Omega^j_{\text{éh}}(-j) = \Omega^j_{X/K}(-j)$$

being the continuous differential sheaf of X over K, together with the $(-j)$-th Tate twist when X is defined over a discretely valued subfield.

Here we note that in the scheme case, the analogous result holds for the Deligne–Du Bois complex: for a smooth algebraic variety Y over $\mathbb{C}$, the j-th graded factor $\underline{\Omega}^j_Y$ of the filtered Deligne–Du Bois complex is quasi-isomorphic to the j-th Kähler differential sheaf $\Omega^j_{Y/\mathbb{C}}[-j]$ (cf. [13,25]).

For the proof of the theorem, we first introduce the v-topology for the given rigid space X (defined in [31]) in Section 3. Here the v-topology serves as a common extension of the pro-étale topology and the éh-topology. With the help of the descent result between the pro-étale site and the v-site, we reduce the problem to the study of éh-differentials for rigid spaces in Theorem 4.0.2. The rest of the proof will then be devoted to the study of éh-differentials for smooth rigid spaces, in Section 5. Here we follow the idea by Geisser [15], showing the vanishing of a cone for the natural map

$$\Omega^i_{X/K} \to R\pi_{X*}\Omega^i_{\text{éh}}$$

by comparing éh and étale cohomologies, with the help of the covering structure in $X_{\text{éh}}$ studied in Section 2.

Remark 1.2.2. At this moment, we mention that when the rigid space X is defined over a discretely valued subfield of K, the éh-proét spectral sequence in Theorem 1.2.1 is Galois equivariant by functoriality, and hence degenerates at its E_2-page by the result of Tate. Together with the known finiteness for the p-adic étale cohomology for proper spaces, we in particular get the proof of Theorem 1.1.3 in the case when X comes from a discretely valued subfield. For the reader's convenience, we also point out that this only needs the first half of the article, namely Sections 2 to 4.[1]

[1] Here to recover the Hodge–Tate decomposition for proper smooth rigid spaces as in Theorem 1.2.1 (iii), it suffices to compare it with the known decomposition result for this special case as in [28, Corollary 1.8]. This implies that under this assumption the cohomology of éh-differentials coincides with that of continuous differentials, and in particular does not require results in Section 5. We thank the referee for pointing this out to the author.

Using Theorem 1.2.1 we can show the coherence and cohomological boundedness of $R\nu_*\widehat{\mathcal{O}}_X$ in the non-smooth case, in Section 6:

Theorem 1.2.3 (Finiteness). *Let X be a rigid space over K. Then the higher direct images $R^i\pi_{X*}\Omega^j_{\text{éh}}$ are coherent and vanish unless $0 \le i, j \le \dim(X)$.*

Next, we study the degeneracy of the éh-proét spectral sequence in the general case. As we saw in Remark 1.2.2, the Hodge–Tate decomposition follows from the Galois action when X comes from a small subfield. It is then natural to ask if the degeneracy holds for more general X, or even on the level of the derived category. Our next main result confirms the splitting of the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ into its cohomology sheaves in the derived category, under the assumption of X being strongly liftable (see Definition 7.4.1). This condition is satisfied for example when X is defined over a discretely valued subfield of K that has perfect residue field (Example 7.4.2), or when X is proper over K (Proposition 7.4.4).

Theorem 1.2.4. *Assume X is a quasi-compact and strongly liftable rigid space over K. Then there exists a non-canonical quasi-isomorphism*

$$R\nu_*\widehat{\mathcal{O}}_X \to \bigoplus_j R\pi_{X*}(\Omega^j_{\text{éh}}(-j)[-j]).$$

In particular, the éh-proét spectral sequence

$$E_2^{i,j} = R^i\pi_{X*}\Omega^j_{\text{éh}}(-j) \Longrightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X$$

degenerates at the E_2-page.

In particular, when X is smooth, the above decomposition degenerates into the following simpler form:

Corollary 1.2.5. *Assume X is quasi-compact, strongly liftable, and smooth over K. Then there exists a non-canonical quasi-isomorphism*

$$R\nu_*\widehat{\mathcal{O}}_X \to \bigoplus_{i=0}^{\dim(X)} \Omega^i_{X/K}(-i)[-i].$$

In fact, the isomorphism is functorial among strong lifts of X; for the precise statement, we refer the reader to Theorem 7.4.9.

When X is smooth, we prove the theorem above by comparing the analytic cotangent complex of X over $\mathrm{B}^+_{\mathrm{dR}}$ with the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$, which builds the bridge between liftability and degeneracy. For the non-smooth setting, we first generalize most of the previous results to smooth quasi-compact (truncated) simplicial rigid spaces, and then we use cohomological descent for smooth éh-hypercovers. The idea is to show that the above isomorphism for smooth rigid spaces is sufficiently functorial to enhance it to simplicial settings, by constructing those enhanced maps by hand. This is given in Section 7.

At the end of Section 7, we give an application of the degeneracy, on the higher direct image of éh-differential forms. By a recent work on the almost purity theorem in [4], we can show that the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ lives in cohomological degrees $[0, \dim(X)]$ for any rigid space X over K (Proposition 7.5.2). Moreover, if we assume X is proper over K, then the degeneracy of the éh-proét spectral sequence implies the vanishing of $R^i\pi_{X*}\Omega^j_{\text{éh}}$ for $i + j > \dim(X)$. Note that vanishing of cohomology sheaves is a local statement, so we can state the upshot as follows:

Theorem 1.2.6. *Let X be a locally compactifiable rigid space over K. Then*

$$R^i\pi_{X*}\Omega^j_{\text{éh}} = 0 \quad \textit{for } i + j > \dim(X).$$

We remark that the theorem gives an improvement on the cohomological boundedness in Theorem 1.2.3 for X being locally compactifiable.

It is natural to ask if the vanishing above is true for all rigid spaces. Here we make a stronger conjecture.

Conjecture 1.2.7. *Let X be a rigid space over K. Then the éh-proét spectral sequence degenerates at the E_2-page. In particular, $R^i\pi_{X*}\Omega^j_{\text{éh}} = 0$ for $i + j > \dim(X)$.*

In complex geometry, analogous vanishing results hold for graded pieces of the Deligne–Du Bois complex of a compact complex variety X over $\mathbb{C}$, as in [26, Theorem 7.29] (the statement is proved by Guillen–Navarro and Aznar–Puerta–Steenbrink independently, but it is more known as Kawamata–Viehweg type vanishing). Moreover, with the help of the rigid GAGA theorem [8, Appendix A1], our result implies the vanishing of the Deligne–Du Bois complex for proper varieties over $\mathbb{C}$.[2] We mention that the proof here essentially makes use of p-adic Hodge theory, while the proof in [26] uses mixed Hodge structures. It is thus interesting to ask if we can produce more similar results from Hodge theory in complex geometry, using tools from p-adic Hodge theory instead.

In Section 8, we provide a comparison between pro-étale cohomology and éh de Rham cohomology for proper rigid spaces over a discretely valued subfield K_0, generalizing the smooth case developed in [28]. The idea is to use the simplicial method and the cohomological descent developed in Section 7, together with the éh-descent of differentials for smooth rigid spaces (Theorem 4.0.2). Our main result in this section is the following, which generalizes the smooth case in [28]:

Theorem 1.2.8 ((pro-étale)-éh de Rham comparison). *Let Y be a proper rigid space over K_0. Then there exists a* $\mathrm{Gal}(K/K_0)$*-equivariant filtered quasi-isomorphism*

$$R\Gamma(Y_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} \to R\Gamma(Y_{K,\text{proét}}, \mathbb{B}_{\mathrm{dR}}).$$

[2] Our vanishing result for éh-differentials directly implies part (b) of [26, Theorem 7.29]. To get the global vanishing in part (a), let X be a projective variety of dimension d, $\mathcal{L}$ an ample line bundle, Y the affine cone of X, and $\mathrm{Bl}(Y)$ the blowup of Y at the origin, where the latter is also the $\mathbb{A}^1$-bundle over X defined by $\mathcal{L}^{-1}$. We can then apply the distinguished triangle of the éh-differential sheaves for the blowup square of Y. Using the projection map from $\mathrm{Bl}(Y)$ onto X together with the vanishing of the Deligne–Du Bois complexes at Y as in part (b), we can get part (a) via an induction argument starting from the top differential degree $d + 1$.

Here the left side above has the tensor product filtration, which is given by the Hodge filtration of the éh de Rham complex and the canonical filtration on B_{dR}. On the right side, the cofficient $\mathbb{B}_{\mathrm{dR}}$ is the de Rham sheaf for X over the pro-étale site (see [28, Definition 6.1], and Definition 8.0.1) with its natural filtration. As an upshot, we obtain the degeneracy for the éh version of the Hodge–de Rham spectral sequence at its E_1-page, under the assumptions on Y as above (Proposition 8.0.8): namely, the following natural spectral sequence degenerates at the first page:

$$E_1^{p,q} = \mathrm{H}^q(Y_{\text{éh}}, \Omega^p_{\text{éh},/K_0}) \Longrightarrow \mathrm{H}^{p+q}(Y_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}).$$

Finally, we deduce the generalized Hodge–Tate spectral sequence and the decomposition in Section 9, using all of our results about the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ developed in this article.

In summary, the article is structured as follows. In Section 2, we introduce the éh-topology for rigid spaces. Here we prove the local smoothness of this topology, and discuss in detail the topological structure of éh-coverings. Then in Section 3, we introduce the pro-étale topology and v-topology, and discuss the necessary comparison theorems. Next, we connect those two topologies in Section 4. Here we reduce the theorem on the éh-proét spectral sequence to the éh-descent of differentials. Section 5 is devoted to the proof of the éh-descent, with the help of which we get the comparison between éh-cohomology and singular cohomology when the rigid space comes from an algebraic variety. In Section 6, we obtain coherence and cohomological boundedness using éh-hypercovers. In Section 7, we study the degeneracy phenomenon of the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$, using the cotangent complex and the simplicial method. As an application, we improve the cohomological boundedness of $R\pi_*\Omega^j_{\text{éh}}$ for locally compactifiable spaces. In Section 8, we give a comparison between éh de Rham cohomology and pro-étale cohomology for proper rigid spaces over a discretely valued field. Finally, in Section 9, we explain the proof of the Hodge–Tate decomposition.

As in the theory of perfectoid spaces, we use the language of adic spaces throughout the article. We refer the reader to Huber's book [22] for basic results about adic spaces.

2. Éh-topology

In this section, we introduce the éh-topology and study its local structure.

2.1. *Rigid spaces*

We first give a quick review of rigid spaces, following [22].

Let K be a complete non-archimedean extension of $\mathbb{Q}_p$. Denote by Rig_K the category of *rigid spaces over* $\mathrm{Spa}(K)$; its objects are adic spaces that are locally of finite type over $\mathrm{Spa}(K, \mathcal{O}_K)$. Any $X \in \mathrm{Rig}_K$ can be covered by affinoid open subspaces, each of the form $\mathrm{Spa}(A, A^+)$ with A being a quotient of the convergent power series ring $K\langle T_1, \dots, T_n\rangle$ for some n. Here A^+ is an integrally closed open subring of A that is of topologically finite type over $\mathcal{O}_K$, and A is complete with respect to the p-adic topology on K. By the finite type condition it can be shown that any such A^+ is equal to A°, the subring of all power-bounded elements in A [21, Lemma 4.4]. To simplify the notation we abbreviate $\mathrm{Spa}(A, A^\circ)$ as $\mathrm{Spa}(A)$ in this setting. Unless otherwise mentioned, in the following discussion we always assume that X is a rigid space over $\mathrm{Spa}(K)$.

For each adic space X, we can define two presheaves: $\mathcal{O}_X$ and $\mathcal{O}_X^+$, such that when the affinoid space $U = \mathrm{Spa}(B, B^+) \subset X$ is open and B is complete, we have

$$\mathcal{O}_X(U) = B, \quad \mathcal{O}_X^+(U) = B^+.$$

It is known that for any $X \in \mathrm{Rig}_K$, both $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves. We could also define coherent sheaves over rigid spaces, so that locally the category $\mathrm{Coh}(\mathrm{Spa}(B))$ of coherent sheaves over $\mathrm{Spa}(B)$ is equivalent to the category $\mathrm{Mod}_{\mathrm{fp}}(B)$ of finitely presented B-modules [24, Theorem 2.3.3]. An important example of coherent sheaves are *continuous differentials* $\Omega^i_{X/Y}$ for a map $X \to Y$ of rigid spaces, which is a coherent sheaf of $\mathcal{O}_X$-modules over X [22, §1.6].[3] Locally for a map $A \to B$ of affinoid algebras, it could be defined by taking the p-adic completion and inverting p at the algebraic differential module of B_0 over A_0, where $A_0 \to B_0$ is a map of rings of definition over $\mathcal{O}_K$ that are topologically of finite type.

Recall that a *coherent ideal* is defined as a subsheaf $\mathcal{I}$ of ideals in $\mathcal{O}_X$ that is locally of finite presentation over $\mathcal{O}_X$. It is known that when $X = \mathrm{Spa}(A) \in \mathrm{Rig}_K$, there is a bijection between coherent ideals $\mathcal{I}$ of X and ideals of A, given by

$$\mathcal{I} \mapsto \mathcal{I}(X), \quad \widetilde{I} \leftarrow\!\shortmid I.$$

Here $\widetilde{I}$ is the sheaf of $\mathcal{O}_X$-modules associated to

$$U \mapsto I \otimes_A \mathcal{O}_X(U).$$

For a coherent ideal $\mathcal{I}$, we can define an *analytic closed*[4] *subset* of X by taking

$$Z := \{x \in X \mid \mathcal{O}_{X,x} \neq \mathcal{I}_{X,x}\} = \{x \in X \mid |f(x)| = 0, \forall f \in \mathcal{I}\}.$$

The subset Z has a canonical adic space structure such that when $X = \mathrm{Spa}(A)$ and $\mathcal{I} = \widetilde{I}$, we have $Z = \mathrm{Spa}(A/I, (A/I)^\circ) =: V(I)$.

2.2. *Blowups*

Before we introduce the éh-topology on Rig_K, we first recall the construction of blowup in rigid spaces, following [8, §4.1].

Let $X \in \mathrm{Rig}_K$ be a rigid space, $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent sheaf, and $Z = V(\mathcal{I})$ be the analytic closed subset defined by $\mathcal{I}$ as in §2.1. Following Conrad [8, §2.3, §4.1], we define the blowup of X along Z as follows:

[3] To simplify notations, we always use Ω^i to denote the sheaves of continuous i-differentials in our article, instead of algebraic ones. We will explicitly mention it when the latter comes up.

[4] It is also called a *Zariski closed subset* in the literature. Here we follow [5].

Definition 2.2.1. The *blowup* $\mathrm{Bl}_Z(X)$ of X along Z is the X-rigid space

$$\mathrm{Proj}^{\mathrm{an}}\Big(\bigoplus_{n\in\mathbb{N}} \mathcal{J}^n\Big),$$

which is the relatively analytified Proj of the graded algebra $\bigoplus_{n\in\mathbb{N}} \mathcal{J}^n$ over the rigid space X (see [8, §2.3]). It is called a *smooth blowup* if the blowup center Z is a smooth rigid space over K.

Remark 2.2.2. We warn the reader that our definition of a smooth blowup is different from some existing ones, where both X and Z are required to be smooth.

When $X = \mathrm{Spa}(A)$ is affinoid, the blowup of a rigid space is in fact the "pullback" of the schematic blowup $\mathrm{Bl}_I(\mathrm{Spec}(A))$ of $\mathrm{Spec}(A)$ at the ideal I along the map $\mathrm{Spa}(A) \to \mathrm{Spec}(A)$ of locally ringed spaces. More precisely, consider the following natural diagram of locally ringed spaces:

$$\begin{array}{ccc} \mathrm{Bl}_{V(I)}(\mathrm{Spa}(A)) & \longrightarrow & \mathrm{Bl}_I(\mathrm{Spec}(A)) \\ \downarrow & & \downarrow \\ \mathrm{Spa}(A) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Then from the universal property of the relative analytification functor [8, Theorem 2.2.5, Lemma 2.2.3], we have: for a given rigid space Y, there exists a functorial bijection between the collection of morphisms $h : Y \to \mathrm{Bl}_{V(I)}(\mathrm{Spa}(A))$ of rigid spaces over $\mathrm{Spa}(A)$ and the collection of commutative diagrams

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & \mathrm{Bl}_I(\mathrm{Spec}(A)) \\ {\scriptstyle g}\downarrow & & \downarrow \\ \mathrm{Spa}(A) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

where f is a map of locally ringed spaces and g is a morphism of rigid spaces.

As in scheme theory, $\mathrm{Bl}_Z(X)$ has the following universal property (see [8, after Definition 4.1.1]): for any $f : Y \to X$ in Rig_K such that the pullback $f^*\mathcal{J}$ is invertible, the map f factors uniquely through $\mathrm{Bl}_Z(X) \to X$. Consequently, the blowup map is an isomorphism when restricted to the open complement $X \setminus Z$. Moreover, it can be shown by the universal property that rigid blowup is compatible with flat base change and analytification of schematic blowup [8, Theorem 2.3.8]. More precisely, for a flat map $g : Y \to X$ of rigid spaces (i.e. $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$ for any $y \in Y$ over $x \in X$), we have

$$\mathrm{Bl}_{g^*\mathcal{J}}(Y) = \mathrm{Bl}_Z(X) \times_X Y.$$

When $X = X_0^{\mathrm{an}}$ is an analytification of a scheme X_0 of finite type over K, with Z being defined by an ideal sheaf $\mathcal{J}_0$ of $\mathcal{O}_{X_0}$, we have

$$\mathrm{Bl}_Z(X) = \mathrm{Bl}_{\mathcal{J}_0}(X_0)^{\mathrm{an}}.$$

We also note that the blowup map $\mathrm{Bl}_Z(X) \to X$ is proper. This is because by the coherence of $\mathcal{I}$, locally $\mathcal{I}$ can be written as a quotient of a finite free module, which (locally) produces a closed immersion of $\mathrm{Bl}_Z(X)$ into a projective space over X, thus is proper over X. Moreover, if both the center Z and the ambient space X are smooth over K, then the blowup itself $\mathrm{Bl}_Z(X)$ is also smooth.

2.3. Universal homeomorphisms

Another type of morphism that will be used later is universal homeomorphisms.

Definition 2.3.1. Let $f : X' \to X$ is a morphism of rigid spaces over K. It is called a *universal homeomorphism* if for any morphism $g : Y \to X$ of rigid spaces, the base change $X' \times_X Y \to Y$ is a homeomorphism.

The following proposition gives a characterization of universal homeomorphisms of rigid spaces:

Proposition 2.3.2. *Let $f : X \to Y$ be a morphism of rigid spaces over* $\mathrm{Spa}(K)$*. Then it is a universal homeomorphism if and only if the following two conditions hold:*

(i) *f is a finite morphism of rigid spaces.*

(ii) *For any pair of affinoid open subsets $V = \mathrm{Spa}(A) \subset Y$ and $U = f^{-1}(V) = \mathrm{Spa}(B)$, the corresponding map of schemes*

$$\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is a universal homeomorphism of schemes.

Proof. Assume f is a universal homeomorphism. Let $x \in X$ be a rigid point. Since the map f is quasi-finite, by [22, Proposition 1.5.4] there exist open neighborhoods $U \subset X$ of x and V of $f(x)$ such that $f(U) \subset V$ and $f : U \to V$ is finite. We may assume both U and V are connected. On the one hand, the finiteness of $f : U \to V$ implies the image of U is closed. On the other hand, as f is a homeomorphism, $f(U)$ is open in Y, and thus open in V. Combining these, we see V is exactly equal to $f(U)$ with $U = f^{-1}(V)$. So by the density of the rigid points $x \in X$, there exists an open covering V_i of Y such that $f^{-1}(V_i)$ is finite over V_i. Hence we get the finiteness of f.

To check the universal homeomorphism for the corresponding map of affine schemes, we recall from the Stack Project [33, Tag 04DC] that $\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism of schemes if and only if it is integral, universally injective and surjective. Since both A and B are K-algebras, where K is an extension over $\mathbb{Q}_p$, it suffices to show the following claim.

Lemma 2.3.3. *Let $f : \mathrm{Spa}(B) \to \mathrm{Spa}(A)$ be a universal homeomorphism of affinoid rigid spaces. Then the induced map $\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes is integral, bijective, and induces isomorphisms on their residue fields.*

Proof of Lemma. As we have just shown above, the map $A \to B$ of affinoid algebras is finite, thus $\widetilde{f}$ is a finite (hence integral) map of schemes.

For the rest of the claim, we first consider its restriction to closed points. Let $\widetilde{y}$ be a closed point of the scheme $\mathrm{Spec}(A)$, whose residue field $\kappa(\widetilde{y})$ is a finite extension of the p-adic field K. The defining ideal of $\widetilde{y}$ in the scheme $\mathrm{Spec}(A)$ induces a unique rigid point y of the rigid space $Y = \mathrm{Spa}(A)$, whose residue field is equal to $\kappa(\widetilde{y})$. By assumption, the base change of the universal homeomorphism f along the closed immersion $\{y\} \to Y$ induces a universal homeomorphism $X_y := \mathrm{Spa}(\kappa(\widetilde{y})) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(B) \to \mathrm{Spa}(\kappa(\widetilde{y}))$, whose natural map to $X = \mathrm{Spa}(B)$ is a closed immersion. This implies that the reduced subspace of X_y is a rigid point in $\mathrm{Spa}(B)$, and the corresponding closed subscheme inside $\mathrm{Spec}(B)$ is supported at a unique closed point. Here we also notice that the residue field of X_y is a finite separable extension of $\kappa(\widetilde{y})$. Moreover, applying the universal homeomorphism at the base change $X_y \times_{\mathrm{Spa}(\kappa(\widetilde{y}))} X_y \to X_y$, we see that the residue field of X_y is isomorphic to $k(\widetilde{y})$. As a consequence, $\widetilde{f}$ induces a bijection and isomorphisms of residue fields when restricted to their closed points.

To finish the proof, it suffices to extend the claim to non-closed points. The bijectivity of $\widetilde{f} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ follows from the density of closed points. To see this, we may assume $\mathrm{Spec}(A)$ is irreducible. Then as $\widetilde{f}$ is a finite morphism whose image contains all closed points, we get the surjectivity of $\widetilde{f}$. For the injectivity, by the homeomorphism between $\mathrm{Spa}(B)$ and $\mathrm{Spa}(A)$, the scheme $\mathrm{Spec}(B)$ admits a unique irreducible component (hence a unique generic point), and has the same dimension as $\mathrm{Spec}(A)$. Finally, as the induced map of $\widetilde{f}$ on the generic fields is finite and separable, its isomorphism follows from the bijection of points. ■

Conversely, assume f satisfies the two conditions in the statement of Proposition 2.3.2. We first notice that both are invariant under any base change of rigid spaces. We let $V = \mathrm{Spa}(A)$ and $f^{-1}(V) = \mathrm{Spa}(B)$ be two open affinoid open subsets of Y and X respectively. Note that since f is finite, for a morphism $\mathrm{Spa}(C) \to \mathrm{Spa}(A)$ of affinoid rigid spaces, the base change $\mathrm{Spa}(C) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(B)$ is exactly $\mathrm{Spa}(B \otimes_A C)$ [22, 1.4.2]. In particular, $\mathrm{Spa}(C) \times_{\mathrm{Spa}(A)} \mathrm{Spa}(B) \to \mathrm{Spa}(C)$ is a finite morphism of rigid spaces, with the underlying map of schemes being a universal homeomorphism. As a consequence, both conditions (i) and (ii) are base change invariant, and to show $f : X \to Y$ is a universal homeomorphism of rigid spaces, it suffices to show that f itself is a homeomorphism. Moreover, by finiteness, as f is both closed and continuous, it remains to show the bijectivity of f as a map of rigid spaces.

Now we pick any point $y \in Y$, and consider the completed residue field with its valuation ring $(k(y), k(y)^+)$ of y. We take an open affinoid neighborhood $V = \mathrm{Spa}(A)$ of y with $f^{-1}(V) = \mathrm{Spa}(B)$. Then the base change of the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of schemes gives

$$\mathrm{Spec}(B \otimes_A k(y)) \to \mathrm{Spec}(k(y)),$$

which is a universal homeomorphism by assumption. Here the target has exactly one point, and by finiteness we have $B \otimes_A k(y) = B \widehat{\otimes}_A k(y)$. So by assumption the reduced subscheme $\mathrm{Spec}(B \hat{\otimes} k(y))_{\mathrm{red}}$ is equal to $k(y)$ (since they are of characteristic zero). We then note that the adic spectrum

$$\mathrm{Spa}(B \hat{\otimes}_A k(y), B^\circ \otimes_{A^\circ} k(y)^+)$$

is exactly the preimage of y in the rigid space X along the morphism f. Notice that the integral closure of $k(y)^+$ in $k(y)$ is contained in the quotient ring of $B^\circ \otimes_{A^\circ} k(y)^+$ by its nilpotent elements, which has to be $k(y)^+$ itself (as the integral closure is contained in the field $k(y)$ and is finite over $k(y)^+$). In this way, the preimage $f^{-1}(y)$ has exactly one point x whose residue field has valuation equal to $(k(y), k(y)^+)$. Hence f is bijective, and thus a homeomorphism. ∎

Finally, when the target is assumed to be a smooth rigid space, there is no non-trivial universal homeomorphism:

Proposition 2.3.4. *Let X be a smooth rigid space, and X' be a reduced rigid space. Then any universal homeomorphism $f : X' \to X$ is an isomorphism.*

Proof. By Proposition 2.3.2, every universal homeomorphism $f : X' \to X$ can be covered by morphisms of affinoid spaces $\mathrm{Spa}(B) \to \mathrm{Spa}(A)$, where the underlying morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a universal homeomorphism. So it suffices to show that when $X = \mathrm{Spa}(A)$ is a smooth affinoid rigid space over $\mathrm{Spa}(K)$, A is a seminormal ring (so any universal homeomorphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ from a reduced scheme is an isomorphism). But by the smoothness of X, A is a regular ring (by [22, Corollary 1.6.10], locally X is étale over the adic spectrum of the Tate algebras $K\langle T_i \rangle$, which is regular). So A is normal, and thus seminormal. ∎

2.4. *Éh-topology and its structure*

Now we can introduce the éh-topology on Rig_K.

Definition 2.4.1. The *éh-topology* on the category Rig_K is the Grothendieck topology such that the covering families are generated by the following types of morphisms:

- étale coverings;
- universal homeomorphisms;
- coverings associated to blowups: $\mathrm{Bl}_Z(X) \amalg Z \to X$, where Z is a closed analytic subset of X.

In the sense of Grothendieck pretopology in [36, Exposé II.1], this means that a family $\{X_\alpha \to X\}$ of maps is in the set $\mathrm{Cov}(X)$ if $\{X_\alpha \to X\}$ can be refined by finite compositions of the three classes of maps above.

We denote by $\mathrm{Rig}_{K,\text{éh}}$ the big éh-site on Rig_K given by the éh-topology. For a given rigid space X over K, we define $X_{\text{éh}}$ as the localization of $\mathrm{Rig}_{K,\text{éh}}$ on X (in the sense of [33, Tag 00XZ], i.e. it is defined on the category of K-rigid spaces over X with the éh-topology.

Remark 2.4.2. (1) We notice that a covering associated to a blowup $\mathrm{Bl}_Z(X) \amalg Z \to X$ is always surjective: by the discussion in §2.2, $\mathrm{Bl}_Z(X) \to X$ is an isomorphism when restricted to $X \setminus Z$.

(2) Among the three classes of maps above, a covering associated to a blowup is not base change invariant in general. But note that for any morphism $Y \to X$, the pullback of the blowup $X' = \mathrm{Bl}_Z(X) \to X$ along $Y \to X$ can be refined by the blowup

$$\begin{array}{cccc} \mathrm{Bl}_{Y\times_X Z}(Y) \amalg Y \times_X Z & & & \\ & \searrow \quad \searrow & & \\ & Y \times_X \mathrm{Bl}_Z(X) \amalg Y \times_X Z & \longrightarrow & \mathrm{Bl}_Z(X) \amalg Z \\ & \downarrow & & \downarrow \\ & Y & \longrightarrow & X \end{array}$$

We call $\mathrm{Bl}_{Y\times_X Z}(Y) \amalg Y \times_X Z$ the *canonical refinement* for the base change of the blowup.

(3) Though denoted as $X_{\text{éh}}$, this site is still a big site. As an extreme case, when $X = \mathrm{Spa}(K)$, the site $X_{\text{éh}}$ is identical with $\mathrm{Rig}_{K,\text{éh}}$.

Remark 2.4.3. Here we note that our definition of éh-topology is different from h-topology. One of the main differences is that the éh-topology excludes ramified coverings.

For example, consider the n-fold cover map $f : \mathbb{B}^1 \to \mathbb{B}^1$ of the unit disc, which sends the coordinate T to T^n. Then f is a finite surjective map that is relatively smooth at all rigid points except $T = 0$, where it is ramified. If f is an éh-covering, then by Theorem 2.4.11 which we will prove later, f can be refined by a finite composition of coverings associated to smooth blowups and étale coverings. Notice that étale coverings are unramified maps that preserve smoothness and dimensions. Moreover, smooth blowups of a one-dimensional smooth rigid space are isomorphic to that space. In this way, such a finite composition will not produce a covering that is ramified at any rigid points, and we get a contradiction.

Example 2.4.4. Let X be a rigid space. Let $X' = X_{\mathrm{red}}$ be the reduced subspace of X. Then $X' \to X$ is a universal homeomorphism, which is then an éh-covering. So in the éh-topology, every space is locally reduced.

Proposition 2.4.5. *Let X be a quasi-compact quasi-separated rigid space over K. Assume X_i for $i = 1, \ldots, n$ are irreducible components of X (see [6]). Then the map*

$$\coprod_{i=1}^{n} X_i \to X$$

is an éh-covering.

Proof. We first claim that the canonical map $\pi : \mathrm{Bl}_{X_1}(X) \to X$ factors through $\bigcup_{i>1} X_i \to X$; in other words, the image of π is disjoint from $X_1 \setminus \bigcup_{i>1} X_i$.

Let $x \in X_1 \setminus \bigcup_{i>1} X_i$. Take any open neighborhood $U \subset X_1 \setminus \bigcup_{i>1} X_i$ of x. Then the base change of π along the open immersion $U \to X$ becomes

$$\mathrm{Bl}_{U \bigcap X_1}(U) \to U,$$

by the flatness of $U \to X$ and the discussion in §2.2. But by our choice of U, the intersection $U \cap X_1$ is exactly the whole space U, which by definition leads to the emptiness of $\mathrm{Bl}_{U \cap X_1}(U)$. Thus the intersection of $\mathrm{Bl}_{X_1}(X)$ with $p^{-1}(U)$ is empty, and the point x is not contained in the image of π.

Finally, note that the claim leads to the commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{X_1}(X) \amalg X_1 & \longrightarrow & X \\ & \searrow \qquad \nearrow & \\ & (\bigcup_{i>1} X_i) \amalg X_1 & \end{array}$$

which shows that the map $(\bigcup_{i>1} X_i) \amalg X_1 \to X$ is also an éh-covering. Thus by induction on the number of components n, we get the result. ∎

Here we define a specific type of éh-covering.

Definition 2.4.6. An éh-covering $f : X' \to X$ of rigid spaces is said to be a *proper éh-covering* if f is proper, and there exists a nowhere dense analytic closed subset $Z_{\mathrm{red}} \subset X_{\mathrm{red}}$ such that

$$f|_{f^{-1}(X \setminus Z)_{\mathrm{red}}} : f^{-1}(X \setminus Z)_{\mathrm{red}} \to X_{\mathrm{red}} \setminus Z_{\mathrm{red}}$$

is an isomorphism.

As an example, a covering associated to a blowup with the center being nowhere dense is a proper éh-covering.

The idea of allowing blowups in the definition of the éh site is to make all rigid spaces éh-locally smooth. To make this explicit, we recall Temkin's non-embedded disingularization:

Theorem 2.4.7 ([34, Theorems 1.2.1, 5.2.2]). *Let X be a generically reduced, quasi-compact rigid space over* $\mathrm{Spa}(K)$. *Then there exists a composition of finitely many smooth blowups $X_n \to X_{n-1} \to \cdots \to X_0 = X$ such that X_n is smooth.*

Corollary 2.4.8 (Local smoothness). *For any quasi-compact rigid space X, there exists a proper* éh*-covering $f : X' \to X_{\mathrm{red}}$ such that X' is a smooth rigid space over* $\mathrm{Spa}(K)$. *Moreover, f is a composition of finitely many coverings associated to smooth blowups.*

Proof. By Temkin's result, we may let $X_n \to \cdots \to X_0 = X_{\mathrm{red}}$ be the blowups in that theorem such that the center of each $p_i : X_i \to X_{i-1}$ is a smooth analytic subset Z_{i-1} of X_{i-1}. Then by taking the composition of the covering associated to the blowup associated to each p_i, the map

$$X' := X_n \amalg \Big(\coprod_{i=0}^{n-1} Z_i \Big) \to X_{\mathrm{red}}$$

is a proper éh-covering such that X' is smooth. So we get the result. ∎

Finally, we give a useful result about the structure of éh-coverings. In order to do this, we need the embedded strong desingularization by Temkin:

Theorem 2.4.9 (Embedded desingularization, Temkin [35, Theorems 1.1.9, 1.1.13]). *Let X be a quasi-compact smooth rigid space over* $\mathrm{Spa}(K)$*, and $Z \subset X$ be an analytic closed subspace. Then there exists a finite sequence $X' = X_n \to \cdots \to X_0 = X$ of smooth blowups such that the strict transform of Z is also smooth.*

Corollary 2.4.10. *Any blowup $f : Y \to X$ over a smooth quasi-compact rigid space X can be refined by a composition of finitely many smooth blowups.*

Proof. Assume Y is given by $\mathrm{Bl}_Z(X)$, where $Z \subset X$ is a closed analytic subspace. Then by embedded desingularization, we can find $g : X' \to X$ to be a composition of finitely many smooth blowups such that the strict transform Z' of Z is smooth over K. Here the total transform of Z is $g^{-1}(Z) = Z' \cup E_Z$, where E_Z is a divisor. Next we could blow up Z' in X' and get $h : X'' \to X'$. Note that h itself is a smooth blowup. In this way, $h \circ g$ is a composition of finitely many smooth blowups that factorizes through $f : Y \to X$, by the universal property of f and the observation that the preimage of Z along $h \circ g$ is the divisor $h^{-1}(Z') \cup h^{-1}(E_Z)$. ■

Theorem 2.4.11. *Let $X \in \mathrm{Rig}_K$ be a quasi-compact smooth rigid space and $f : X' \to X$ be an éh-covering. Then f can be refined by a composition of finitely many étale coverings and coverings associated to smooth blowups over X.*

Proof. By the definition of the éh-topology, a given éh-covering f can be refined by a finite composition of étale coverings, universal homeomorphisms, and coverings associated to blowups. So up to refinement we may write f as $f : X' = X_n \to X_{n-1} \to \cdots \to X_0 = X$, where each transition map $f_i : X_i \to X_{i-1}$ is of one of the above three types.

Now we produce a refinement we want, by performing the following operations on f starting from $i = 1$:

- If $X_1 \to X_0$ is an étale morphism, then we are done for $i = 1$.
- If $X_1 \to X_0$ is a universal homeomorphism, then by Proposition 2.3.4 we may take the reduced subspace of X_1, which is isomorphic to X_0 and thus is smooth.
- If $X_1 \to X_0$ is a covering associated to a blowup, then by Proposition 2.4.10, the associated blowup can be refined by a finite composition of smooth blowups. We let $X_1' \to X_1$ be the disjoint union of that refinement with all of the centers. Then we take the base change of $X_n \to \cdots \to X_1$ along $X_1' \to X_1$ and get a new covering $X_n \times_{X_1} X_1' \to \cdots \to X_1' \to X_0 = X$, i.e.

$$\begin{array}{ccccccc} X_n \times_{X_1} X_1' & \longrightarrow & \cdots & \longrightarrow & X_1' & & \\ & & & & \downarrow & \searrow & \\ X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X_0 = X \end{array}$$

Furthermore, starting at $j = 2$, we do the following operation and increase j by 1 each time: If $X_j \to X_{j-1}$ is a covering associated to a blowup, we refine the map $X_j \times_{X_1} X'_1 \to X_{j-1} \times_{X_1} X'_1$ by its canonical refinement $X'_j \to X_{j-1} \times X'_1$ (see Remark 2.4.2), and take the base change of the chain $X_n \times_{X_1} X'_1 \to \cdots \to X_j \times_{X_1} X'_1$ along $X'_j \to X_j \times_{X_1} X'_1$.[5]

After the discussion of the above three possibilities, $X_n \to \cdots \to X_0$ is refined by a finite composition $X'_n \to \cdots \to X'_1 \to X_0$ such that

- $X'_1 \to X_0$ is a composition of finitely many étale coverings and coverings associated to smooth blowups;
- $X'_n \to X'_1$ is a composition of $n-1$ éh-coverings and coverings associated to blowups, and universal homeomorphisms.

In this way, we could do the above operation for $X'_i \to X'_{i-1}$ and $i = 2, \ldots,$ each time get a new chain of coverings $X''_n \to \cdots \to X_0$ such that $X''_i \to X_0$ is a finite composition of smooth blowups and étale coverings, and $X''_n \to X''_i$ is a composition of $n - i$ coverings of three generating classes. Hence after finitely many operations, we are done. ■

Corollary 2.4.12. *Any éh-covering of a quasi-compact rigid space X can be refined by a composition*

$$X_2 \to X_1 \to X_0 = X,$$

where $X_1 = X_{\mathrm{red}}$, the map $X_2 \to X_1$ is a finite composition of étale coverings and coverings associated to smooth blowups, and X_2 is smooth over K.

Proof. Let $X' \to X$ be a given éh-covering. By Example 2.4.4, $X_1 := X_{\mathrm{red}} \to X_0$ is an éh-covering. And by the local smoothness of the éh-topology (Corollary 2.4.8), there exists a composition of finitely many coverings associated to smooth blowups $Y_1 \to X_1$ such that Y_1 is smooth. So the base change of $X' \times_X X_1 \to X_1$ along $Y_1 \to X_1$ becomes an éh-covering whose target is smooth and quasi-compact. Hence by Theorem 2.4.11, we could refine $X' \times Y_1 \to Y_1$ by $X_2 \to Y_1$, where the latter is a finite composition of étale coverings and coverings associated to smooth blowups. Finally, notice that an étale map or a smooth blowup will not change the smoothness. Hence the composition $X_2 \to X_1 \to X_0$ satisfies the requirement. ■

3. Pro-étale topology and v-topology

In this section, we recall the pro-étale topology and the v-topology over a given rigid space, in order to build a bridge between the pro-étale topology and the éh-topology. We mostly follow Scholze's foundational work [28, 31], together with the Berkeley lecture notes [32] by Scholze and Weinstein.

[5] The covering associated to a blowup is not preserved under base change, thus we need to adjust all of the maps in $X_n \times_{X_1} X'_1 \to \cdots \to X'_1$ so that they will then become exactly those three types of morphisms.

3.1. Small v-sheaves

Let Perfd be the category of perfectoid spaces. They are adic spaces that have an open affinoid covering $\{\mathrm{Spa}(A_i, A_i^+), i\}$ such that each A_i is a perfectoid algebra. Since many of our constructions are large, we need to avoid set-theoretical issues. Following [31, §4], we fix an uncountable cardinality κ with some conditions, and only consider those perfectoid spaces, morphisms, and algebras that are "κ-small". We refer to Scholze's paper for details, and will follow this convention throughout the section.

We first recall the v-topology defined on the category Perfd.

Definition 3.1.1 ([31, Definition 8.1]). The *big v-site* Perfd_v is the Grothendieck topology on the category Perfd for which a collection $\{f_i \mid X_i \to X,\ i \in I\}$ of morphisms is a covering family if for each quasi-compact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasi-compact open $V_i \subset X_i$ such that $|U| = \bigcup_{i\in J} f(|V_i|)$.

Here the index category I is assumed to be κ-small.

It is known that the v-site Perfd_v is subcanonical; namely the presheaf represented by any $X \in$ Perfd is a v-sheaf. Moreover, both integral and rational completed structure sheaves $\widehat{\mathcal{O}}^+ : X \mapsto \widehat{\mathcal{O}}_X^+(X)$ and $\widehat{\mathcal{O}} : X \mapsto \widehat{\mathcal{O}}_X(X)$ are v-sheaves on Perfd [31, Corollary 8.6, Theorem 8.7].

We then introduce a special class of v-sheaves that admits a geometric structure, generalizing perfectoid spaces. Consider the subcategory Perf of Perfd consisting of perfectoid spaces of characteristic p. We can equip Perf with the pro-étale topology and the v-topology to get two sites $\mathrm{Perf}_{\text{proét}}$ and Perf_v respectively.

Definition 3.1.2 ([31, Definition 12.1]). A *small v-sheaf* is a sheaf Y on Perf_v such that there is a surjective map $X \to Y$ of v-sheaves, where X is a representable sheaf of some κ-small perfectoid space in characteristic p.

By the definition and the subcanonicality of the v-topology, any perfectoid space X in characteristic p produces a small v-sheaf.

Here is a non-trivial example.

Example 3.1.3 ([32, §9.4]). Let K be a p-adic extension of $\mathbb{Q}_p$, i.e., K is complete with respect to a non-archimedean valuation extending that of $\mathbb{Q}_p$. Then we can produce a presheaf $\mathrm{Spd}(K)$ on Perf such that for each $Y \in$ Perf, we take

$$\mathrm{Spd}(K)(Y) := \{\text{isomorphism classes of pairs } (Y^\sharp, \iota : (Y^\sharp)^\flat \to Y)\},$$

where $Y^\sharp$ is a perfectoid space (of characteristic zero) over K, and ι is an isomorphism of perfectoid spaces identifying $Y^\sharp$ as an untilt of Y. It can be shown that $\mathrm{Spd}(K)$ is in fact a small v-sheaf.

By the tilting correspondence, it can be shown that there is an equivalence between the category Perfd_K of perfectoid spaces over K, and the category of perfectoid spaces Y in characteristic p with a structure morphism $Y \to \mathrm{Spd}(K)$ [32, Theorem 9.4.4].

One of the main reasons we introduce small v-sheaves is that this brings both perfectoid spaces and rigid spaces into a single framework. More precisely, we have the following fact:

Proposition 3.1.4 ([31, Definition 15.5], [32, Proposition 10.2.3]). *Let K be a p-adic extension of $\mathbb{Q}_p$ as in Example 3.1.3. There is a functor*

$$\{\textit{analytic adic spaces over } \mathrm{Spa}(K)\} \to \{\textit{small } v\textit{-sheaves over } \mathrm{Spd}(K)\},$$
$$X \mapsto X^{\diamond},$$

such that when X is a perfectoid space over $\mathrm{Spa}(K)$, the small v-sheaf $X^{\diamond}$ coincides with the representable sheaf for the tilt $X^{\flat}$.

Moreover, the restriction of this functor to the subcategory of seminormal rigid spaces gives a fully faithful embedding

$$\{\textit{seminormal rigid spaces over } \mathrm{Spa}(K)\} \to \{\textit{small } v\textit{-sheaves over } \mathrm{Spd}(K)\}.$$

Here we remark that every perfectoid space is seminormal [24, Theorem 3.7.4].

We can also define the "topological structure on $X^{\diamond}$": in [31, Definition 10.1], Scholze defines the concept of being open, étale and finite étale for a morphism of pro-étale sheaves over Perfd. In particular, for each small v-sheaf $X^{\diamond}$ coming from an adic space, we can define its small étale site $X^{\diamond}_{\text{ét}}$. Those morphisms between small v-sheaves are compatible with maps of adic spaces, and we have

Proposition 3.1.5 ([31, Lemma 15.6]). *For each $X \in \mathrm{Rig}_K$, the functor $Y \mapsto Y^{\diamond}$ induces an equivalence of small étale sites*

$$X_{\text{ét}} \cong X^{\diamond}_{\text{ét}},$$

where the site on the left is the small étale site of the rigid space X defined in [22].

This generalizes the tilting correspondence of perfectoid spaces between characteristic zero and characteristic p.

3.2. Pro-étale and v-topoi over X

In this subsection, we recall the small pro-étale site and the v-site associated to a given rigid space $X \in \mathrm{Rig}_K$, for K being a p-adic field. Our goal is to produce a topology over X that is large enough to include both the pro-étale topology and the éh-topology, and study the relation between their cohomologies.

We start by recalling basic concepts around the topology of small v-sheaves.

First recall that a perfectoid space X is called *quasi-compact* if every open covering admits a finite refinement, and *quasi-separated* if for any pair of quasi-compact perfectoid spaces Y, Z over X, the fiber product $Y \times_X Z$ is also quasi-compact.

The concept of quasi-compactness and quasi-separatedness can be generalized to pro-étale sheaves and small v-sheaves. A small v-sheaf $\mathcal{F}$ is called *quasi-compact* if for any family of morphisms $f_i : X_i \to \mathcal{F}$, $i \in I$, such that $\coprod_{i\in I} X_i \to \mathcal{F}$ is surjective and I is κ-small, there is a finite subcollection $J \subset I$ such that $\coprod_{j\in J} X_j \to \mathcal{F}$ is surjective. Here X_i are (pro-étale sheaves that are representable by) affinoid perfectoid spaces. The *quasi-separatedness* for small v-sheaves is defined as for perfectoid spaces.

Now we are able to define two topoi over a given rigid space X.

Definition 3.2.1. Let $X \in \mathrm{Rig}_K$ be a rigid space over the p-adic field K.

(i) The *small pro-étale site over* X, denoted by $X_{\text{proét}}$, is the Grothendieck topology on the full subcategory of pro-objects in $X_{\text{ét}}$ that are pro-étale over X, in the sense of [28, §3]. Its covering families are defined as those jointly surjective pro-étale morphisms $\{f_i : Y_i \to Y,\ i \in I\}$ such that for any quasi-compact open immersion $U \to Y$, there exists a finite subset $J \subset I$ and quasi-compact open $V_j \subset Y_j$ for $j \in J$, satisfying $|U| = \bigcup_{j\in J} f_j(|V_j|)$. We call the *pro-étale topos* over X, denoted by $\mathrm{Sh}(X_{\text{proét}})$.

(ii) The v-*site over* X is defined as the site $\mathrm{Perf}_v|_{X^\diamond}$ of perfectoid spaces in characteristic p over $X^\diamond$, with the covering structure given by the v-topology. Namely, the site is defined on the category of pairs $(Y, f : Y \to X^\diamond)$, where Y is (the representable sheaf of) a perfectoid space in characteristic p, and $f : Y \to X^\diamond$ is a map of v-sheaves over Perf_v. A collection $\{(Y_i, Y_i \to X^\diamond) \to (Y, Y \to X^\diamond)\}$ of maps is a covering in this site if $\{Y_i \to Y\}$ is a covering in the v-site. We call $\mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond})$ the v-*topos* over X.

Remark 3.2.2. The v-site $\mathrm{Perf}_v|_{X^\diamond}$ above is constructed as the localization (restriction) of the v-site Perf_v at the sheaf $X^\diamond$. The general discussion of the localization of a site at a sheaf, which generalizes the localization of a site at an object, can be found for example in [33, Tag 04GY]. Here we note that by [33, Tag 0791], the v-topos over X is isomorphic to the localization topos $\mathrm{Sh}(\mathrm{Perf}_v)|_{X^\diamond}$ of the v-topos $\mathrm{Sh}(\mathrm{Perf}_v)$ at the small v-sheaf $X^\diamond$.

Remark 3.2.3. Given a rigid space X, we can also form the characteristic zero analogue of the v-site $\mathrm{Perfd}_v|_X$ on the category of perfectoid spaces over X (cf. Definition 3.2.1 (ii)). The tilting correspondence and the definition of $X^\diamond$ induce a natural equivalence between the v-sites $\mathrm{Perf}_v|_{X^\diamond}$ and $\mathrm{Perfd}_v|_X$, sending an affinoid perfectoid space $Z \to X^\diamond$ to the associated tilt $Z^\sharp \to X$.

Let $X \in \mathrm{Rig}_K$ be a rigid space. Then there is a natural morphism of topoi $\lambda = (\lambda^{-1}, \lambda_*) : \mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}) \to \mathrm{Sh}(X_{\text{proét}})$. The inverse functor λ^{-1} is computed via the functor $(-)^\diamond$ as in Proposition 3.1.4. More precisely, when $Y \in X_{\text{proét}}$ is affinoid perfectoid with associated complete adic space is $\hat{Y}$, the inverse $\lambda^{-1}(Y)$ is the small v-sheaf $\hat{Y}^\diamond$ over $X^\diamond$, representable by the tilt $\hat{Y}^\flat$. As affinoid perfectoid objects form a basis in $X_{\text{proét}}$ [28, Proposition 4.8], this allows us to extend λ^{-1} to the whole category $X_{\text{proét}}$. In particular, by using the Galois descent as in [31, Proposition 15.4], for a rigid space X' that is finite étale over X, we have $\lambda^{-1}(X') = X'^\diamond$. Here we remark that by loc. cit. the functor λ^{-1} realizes a pro-étale presentation into an actual limit of v-sheaves: when Y is affinoid perfectoid with a pro-étale presentation $\{Y_i\}$ over X, we have $Y^\diamond \cong \varprojlim Y_i^\diamond$.

When $\{Y_i \to Y\}$ is a pro-étale covering of affinoid perfectoid objects over X, the inverse image $\{\hat{Y}_i^\diamond \to \hat{Y}^\diamond\}$ forms a v-covering of representable v-sheaves over $X^\diamond$. For a general pro-étale sheaf $\mathcal{F}$ over $X_{\text{proét}}$, the functor λ^{-1} sends $\mathcal{F}$ to the v-sheaf associated to the presheaf

$$Z \mapsto \varinjlim_{\substack{Z \to \hat{W}^\diamond \text{ in } \mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}), \\ \text{affinoid perfectoid } W \in X_{\text{proét}}}} \mathcal{F}(W).$$

Here we note that when Z is equal to the small v-sheaf $\hat{Y}^\diamond$ for $\hat{Y}$ a perfectoid space underlying a pro-étale object Y over X, the above direct limit is $\mathcal{F}(Y)$. On the other hand, the functor λ_* is the direct image functor, given by

$$\lambda_* \mathcal{G}(Y) = \mathcal{G}(\hat{Y}^\flat) \quad \text{for affinoid perfectoid } Y \in X_{\text{proét}}.$$

We define the *untilted completed structure sheaves* $\widehat{\mathcal{O}}_v$ and $\widehat{\mathcal{O}}_v^+$ on $\mathrm{Perf}_v|_{X^\diamond}$ by sending $Z \to X^\diamond$ to

$$\widehat{\mathcal{O}}_v(Z) := \widehat{\mathcal{O}}(Z^\sharp), \quad \widehat{\mathcal{O}}_v^+(Z) := \widehat{\mathcal{O}}^+(Z^\sharp),$$

where $Z^\sharp$ is the untilt of Z given by the map $Z \to X^\diamond \to \mathrm{Spd}(K)$, as in Proposition 3.1.4. By [31, Theorem 8.7], both are sheaves on $\mathrm{Perf}_v|_{X^\diamond}$. Here we notice that under the (tilting) equivalence in Remark 3.2.3, the sheaves $\widehat{\mathcal{O}}_v$ and $\widehat{\mathcal{O}}_v^+$ are sent to the completed structure sheaves $\widehat{\mathcal{O}}$ and $\widehat{\mathcal{O}}^+$ over $\mathrm{Perfd}_v|_X$ in characteristic zero.

Furthermore, we have the following comparison result on completed pro-étale structure sheaves.

Proposition 3.2.4 ((pro-étale)-v comparison). *The direct image map induces the following canonical isomorphism of sheaves on $X_{\text{proét}}$:*

$$\lambda_* \widehat{\mathcal{O}}_v^+ \to \widehat{\mathcal{O}}_X^+.$$

Moreover, for $i > 0$ the sheaf $R^i \lambda_* \widehat{\mathcal{O}}_v^+$ is almost zero.

By inverting p, similar results hold for $\lambda_* \widehat{\mathcal{O}}_v$ and $R^i \lambda_* \widehat{\mathcal{O}}_v$. In particular, the pro-étale cohomology of $\widehat{\mathcal{O}}_X$ satisfies v-hyperdescent.

Here we follow the convention of the almost mathematics as in [31, §3].

Proof. We first recall that for any quasi-compact analytic adic space Y over K, there exists a pro-étale covering of Y by perfectoid spaces [31, Lemma 15.3]. In particular, the pro-étale site $X_{\text{proét}}$ admits a basis given by affinoid perfectoid spaces that are pro-étale over X. So it suffices to check the above isomorphism and vanishing condition for $Y \in X_{\text{proét}}$ that are affinoid perfectoid.

The direct image of the untilted integral complete structure sheaf is the pro-étale sheaf associated to

$$Y \mapsto \Gamma(Y, \lambda_* \widehat{\mathcal{O}}_v^+) = \Gamma(\hat{Y}^\diamond, \widehat{\mathcal{O}}_v^+),$$

where $Y \in X_{\text{proét}}$ is affinoid perfectoid. But note that since $\hat{Y}^{\diamond} \cong \hat{Y}^{\flat}$ is the representable sheaf of an affinoid perfectoid space over $X^{\diamond}$, by construction of $\widehat{\mathcal{O}}_v^+$ we have

$$\Gamma(\hat{Y}^{\diamond}, \widehat{\mathcal{O}}_v^+) = \Gamma((\hat{Y}^{\flat})^{\sharp}, \widehat{\mathcal{O}}^+).$$

Here $\hat{Y}$ is the perfectoid space associated to the object $Y \in X_{\text{proét}}$, and $\hat{Y}^{\flat}$ is the tilt of $\hat{Y}$. So by the isomorphism of perfectoid spaces $(\hat{Y}^{\flat})^{\sharp} \cong \hat{Y}$, we see that $\lambda_* \widehat{\mathcal{O}}_v^+$ is the pro-étale sheaf associated to the presheaf

$$Y \mapsto \Gamma(Y, \widehat{\mathcal{O}}^+),$$

which is exactly the completed pro-étale structure sheaf over $X_{\text{proét}}$. Thus we get the above equality.

For the higher direct image, we first note that $R^i \lambda_* \widehat{\mathcal{O}}_v^+$ is the pro-étale sheaf on $X_{\text{proét}}$ associated to the presheaf

$$Y \mapsto \mathrm{H}_v^i(\hat{Y}^{\flat}, \widehat{\mathcal{O}}_v^+)$$

for Y being affinoid perfectoid in $X_{\text{proét}}$. By the construction of $\widehat{\mathcal{O}}_v^+$, the tilting correspondence $\mathrm{Perf}_v|_{X^{\diamond}} \cong \mathrm{Perfd}_v|_X$ in Remark 3.2.3 identifies the sheaf $\widehat{\mathcal{O}}_v^+$ over $\mathrm{Perf}_v|_{X^{\diamond}}$ with $\widehat{\mathcal{O}}^+$ over $\mathrm{Perfd}_v|_X$. In particular, we have the natural isomorphism of cohomology

$$\mathrm{H}_v^i(\hat{Y}^{\flat}, \widehat{\mathcal{O}}_v^+) \cong \mathrm{H}_v^i((\hat{Y}^{\flat})^{\sharp}, \widehat{\mathcal{O}}^+) \cong \mathrm{H}_v^i(\hat{Y}, \widehat{\mathcal{O}}^+),$$

which is almost zero by [31, Proposition 8.8] and the assumption on Y. So we are done. ∎

4. Éh-proét spectral sequence

In this section, we connect all the topologies we have defined, and consider the éh-proét spectral sequence.

Let X be a rigid space over K, for K a complete and algebraically closed p-adic field. We denote by $X_{\text{éh}}$ the localization of the big éh-site $\mathrm{Rig}_{K,\text{éh}}$ at X. Then the functor $Y \mapsto Y^{\diamond}$ induces a morphism of topoi

$$\alpha : \mathrm{Sh}(\mathrm{Perf}_v|_{X^{\diamond}}) \to \mathrm{Sh}(X_{\text{éh}}),$$

where $\alpha^{-1} Y = Y^{\diamond}$ for $Y \to X$ being the representable sheaf of an adic space. We let $X_{\text{ét}}$ be the small étale site of X, consisting of rigid spaces that are étale over X.

Consider the following commutative diagram of topoi over X:

$$\begin{array}{ccc}
\mathrm{Sh}(X_{\text{proét}}) & \xrightarrow{\nu} & \mathrm{Sh}(X_{\text{ét}}) \\
\uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\pi_X} \\
\mathrm{Sh}(\mathrm{Perf}_v|_{X^{\diamond}}) & \xrightarrow{\alpha} & \mathrm{Sh}(X_{\text{éh}})
\end{array}$$

Here we note that the diagram is functorial with respect to X. In particular, when $X = X_0 \times_{K_0} K$ is a pullback of X_0 along a non-archimedean field extension K/K_0, the diagram is equipped with a continuous action of $\mathrm{Aut}(K/K_0)$.

Now by the proét-v comparison (Proposition 3.2.4), we have

$$R\nu_*\widehat{\mathcal{O}}_X = R\nu_* R\lambda_*\widehat{\mathcal{O}}_v = R\pi_* R\alpha_*\widehat{\mathcal{O}}_v.$$

This induces a Leray spectral sequence

$$E_2^{i,j} = R^i\pi_* R^j\alpha_*\widehat{\mathcal{O}}_v \Longrightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X.$$

We then notice that by the above comparison again, the sheaf $R^j\alpha_*\widehat{\mathcal{O}}_v$ is the éh-sheafification of the presheaf

$$X_{\text{éh}} \ni Y \mapsto \mathrm{H}^j(Y_v^\diamond, \widehat{\mathcal{O}}_v) = \mathrm{H}^j(Y_{\text{proét}}, \widehat{\mathcal{O}}_Y).$$

When Y is smooth, it is in fact the j-th continuous differential:

Fact 4.0.1 ([29, Proposition 3.23]). *Let Y be a smooth affinoid rigid space over K. Then we have a canonical isomorphism*

$$\mathrm{H}^j(Y_{\text{proét}}, \widehat{\mathcal{O}}_Y) = \Omega^j_{Y/K}(Y)(-j),$$

where $\Omega^j_{Y/K}$ is the sheaf of j-th continuous differential forms. Here the "$(-j)$" means that the cohomology is equipped with an action of the Galois group $\mathrm{Gal}(K/K_0)$ by the Tate twist of weight j, when $Y = Y_0 \times_{K_0} K$ is a base change of a smooth rigid space Y_0 over a complete discretely valued field K_0 whose residue field is perfect, and K is complete and algebraically closed.

In this way, by the local smoothness of the éh-topology (Corollary 2.4.8) and functoriality, the sheaf $R^j\alpha_*\widehat{\mathcal{O}}_v$ on $X_{\text{éh}}$ is the éh-sheaf associated to

$$\text{smooth } Y \mapsto \Omega^j_{Y/K}(Y)(-j).$$

We call the éh-sheaf associated to $Y \mapsto \Omega^j_{Y/K}(Y)$ the *j-th éh-differential over K*, denoted by $\Omega^j_{\text{éh},/K}$. When the base field K is clear, we use $\Omega^j_{\text{éh}}$ to simplify the notation. Substituting this into the spectral sequence above, we get the *éh-proét spectral sequence*

$$R^i\pi_{X*}\Omega^j_{\text{éh}}(-j) \Longrightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X.$$

Our first main result is the following descent result for the éh-differential, which we will prove in the next section.

Theorem 4.0.2 (éh-descent). *Assume $X \in \mathrm{Rig}_K$ is a smooth rigid space over $\mathrm{Spa}(K)$. Then for each $j \in \mathbb{N}$, we have*

$$R\pi_{X*}\Omega^j_{\text{éh}} = R^0\pi_{X*}\Omega^j_{\text{éh}}[0] = \Omega^j_{X/K}.$$

Remark 4.0.3. When $i = j = 0$, the section $\mathcal{O}_{\text{éh}}(X)$ of the éh-structure sheaf on any rigid space X is $\mathcal{O}(X^{\text{sn}})$, where X^{sn} is the seminormalization of X_{red}. In other words, $\mathcal{O}_{\text{éh}} = \mathcal{O}^{\text{sn}}$. This follows from [32, Proposition 10.2.3].

5. Éh-descent for differentials

In this section, we prove the descent for éh-differentials of a smooth rigid space $X \in \mathrm{Rig}_K$, where K is any p-adic field (not necessarily algebraically closed). At the end of the section, we apply the éh-descent to the case when X comes from an algebraic variety, to relate the éh-cohomology to the Deligne–Du Bois complex (cf. [13], [26, §7]).

5.1. Éh-descent

We will follow the idea in [15], showing the vanishing of the cone C for $\Omega^j_{X/K} \to R\pi_*\Omega^j_{\text{éh}}$ by comparing étale cohomology and éh-cohomology.

We first show the long exact sequence of continuous differentials for coverings associated to blowups.

Proposition 5.1.1. *Let $f : X' \to X$ be a blowup of a smooth rigid space X along a smooth and nowhere dense closed analytic subset $i : Y \subset X$, with the pullback $g : Y' = X' \times_X Y \to Y$. Then the functoriality of Kähler differentials induces the following distinguished triangle in the derived category of X:*

$$\Omega^j_{X/K} \to Rf_*\Omega^j_{X'/K} \oplus i_*\Omega^j_{Y/K} \to i_*Rg_*\Omega^j_{Y'/K}. \qquad (*)$$

Proof. We first note that since the argument is local on X, it suffices to show that for any given rigid point $x \in X$, there exists a small open neighborhood of x such that the result is true over that neighborhood. So we may assume $X = \mathrm{Spa}(A)$ is affinoid, admitting an étale morphism to $\mathbb{B}^n_K = \mathrm{Spa}(K\langle x_1, \dots, x_n\rangle)$ by [22, Corollary 1.6.10], and Y is of dimension r, given by $\mathrm{Spa}(A/I)$ for an ideal I of A. Moreover, by refining X to a smaller open neighborhood of x if necessary, we could choose a collection of local parameters $f_1, \dots, f_r$ and $g_1, \dots, g_{n-r}$ at x such that $\{g_l\}$ locally generates the ideal defining Y in X. In this way, by the differential criterion for étaleness [22, Proposition 1.6.9], we could assume $Y \to X$ is an étale base change of the closed immersion

$$\mathbb{B}^r_K \to \mathbb{B}^n_K.$$

In particular, the blowup diagram for $\mathrm{Bl}_Y(X) \to X$ locally is the étale base change of $\mathrm{Bl}_{\mathbb{B}^r_K}(\mathbb{B}^n_K)$ along $X \to \mathbb{B}^n_K$.

Then we notice that the blowup of $\mathbb{B}^n$ along $\mathbb{B}^r$ is equivalent to the generic fiber of the p-adic (formal) completion of the blowup

$$\mathbb{A}^r_{\mathcal{O}_K} \to \mathbb{A}^n_{\mathcal{O}_K}.$$

Furthermore, as proved in [16, IV. Theorem 1.2.1], there exists a natural distinguished triangle

$$\Omega^j_{\mathbb{A}^n/\mathcal{O}_K} \to Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r}(\mathbb{A}^n)/\mathcal{O}_K} \oplus i_*\Omega^j_{\mathbb{A}^r/\mathcal{O}_K} \to i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r}(\mathbb{A}^n)\times_{\mathbb{A}^n}\mathbb{A}^r/\mathcal{O}_K}. \qquad (**)$$

Now we make the following claim:

Claim 5.1.2. *The sequence* $(*)$ *for* $(X, Y) = (\mathbb{B}^n_K, \mathbb{B}^r_K)$ *can be given by the generic base change of the derived* p*-adic completion of the distinguished triangle* $(**)$.

Granting the claim, since both derived completion and generic base change are exact functors, we are done.

Proof of Claim. We first notice that since $\mathbb{A}^n_{\mathcal{O}_K} = \mathrm{Spec}(\mathcal{O}_K[T_1, \ldots, T_n])$ is p-torsion free (thus flat over $\mathcal{O}_K$), by [33, Tag 0923], for a complex $C \in D(\mathbb{A}^n_{\mathcal{O}_K})$ its p-adic derived completion is given by

$$R\varprojlim(C \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/p^m\mathcal{O}_K).$$

Moreover, note that differentials of $\mathbb{A}^n_{\mathcal{O}_K}, \mathbb{A}^r_{\mathcal{O}_K}, \mathrm{Bl}_{\mathbb{A}^r_{\mathcal{O}_K}}(\mathbb{A}^n_{\mathcal{O}_K})$ and $\mathrm{Bl}_{\mathbb{A}^r_{\mathcal{O}_K}}(\mathbb{A}^n_{\mathcal{O}_K}) \times_{\mathbb{A}^n_{\mathcal{O}_K}} \mathbb{A}^r_{\mathcal{O}_K}$ over $\mathcal{O}_K$ are all flat over $\mathcal{O}_K$. We use the notations $\mathbb{A}^n_m$ and $\mathbb{A}^r_m$ to abbreviate the schemes $\mathbb{A}^n_{\mathcal{O}_K/p^m}$ and $\mathbb{A}^r_{\mathcal{O}_K/p^m}$ respectively. Then the derived base change of $(**)$ along $\mathcal{O}_K \to \mathcal{O}_K/p^m$ can be written as

$$\Omega^j_{\mathbb{A}^n_m/(\mathcal{O}_K/p^m)} \to Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)} \oplus i_*\Omega^j_{\mathbb{A}^r_m/(\mathcal{O}_K/p^m)} \to i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)\times_{\mathbb{A}^n_m}\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}. \quad (***)$$

Here we use the formula $\Omega^j_{Y/\mathcal{O}_K} \otimes^L_{\mathcal{O}_K} \mathcal{O}_K/p^m = \Omega^j_{Y_m/(\mathcal{O}_K/p^m)}$ for a smooth $\mathcal{O}_K$-scheme Y, together with the derived base change formula for a proper morphism [33, Tag 07VK]. Hence the derived p-adic completion of $(**)$ is then computed by the derived limit of $(***)$ for $m \in \mathbb{N}$.

Finally, we discuss those derived limits term by term. For $C_m = \Omega^j_{\mathbb{A}^n_m/(\mathcal{O}_K/p^m)}$ or $C_m = i_*\Omega^j_{\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}$, since their transition maps are surjective, the derived limit has no higher cohomology and we have

$$R\varprojlim \Omega^j_{\mathbb{A}^n_m/(\mathcal{O}_K/p^m)} = \Omega^j_{\widehat{\mathbb{A}}^n/\mathcal{O}_K}, \quad R\varprojlim i_*\Omega^j_{\mathbb{A}^r_m/(\mathcal{O}_K/p^m)} = \Omega^j_{\widehat{\mathbb{A}}^r/\mathcal{O}_K}.$$

For $C_m = Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)}$ or $i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)\times_{\mathbb{A}^n_m}\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}$, recall we have the formula for the derived functors

$$Rf_*R\varprojlim_m = R\varprojlim_m Rf_*.$$

Hence

$$R\varprojlim_m Rf_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)} \cong Rf_*R\varprojlim_m \Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)/(\mathcal{O}_K/p^m)} = Rf_*\Omega^j_{\mathrm{Bl}_{\widehat{\mathbb{A}}^r}(\widehat{\mathbb{A}}^n)/\mathcal{O}_K}.$$

The analogous formula holds for $C_m = i_*Rg_*\Omega^j_{\mathrm{Bl}_{\mathbb{A}^r_m}(\mathbb{A}^n_m)\times_{\mathbb{A}^n_m}\mathbb{A}^r_m/(\mathcal{O}_K/p^m)}$.

In this way, the derived limit of $(***)$ is isomorphic to

$$\Omega^j_{\widehat{\mathbb{A}}^n/\mathcal{O}_K} \to Rf_*\Omega^j_{\mathrm{Bl}_{\widehat{\mathbb{A}}^r}(\widehat{\mathbb{A}}^n)/\mathcal{O}_K} \oplus i_*\Omega^j_{\widehat{\mathbb{A}}^r/\mathcal{O}_K} \to i_*Rg_*\Omega^j_{\mathrm{Bl}_{\widehat{\mathbb{A}}^r}(\widehat{\mathbb{A}}^n)\times_{\widehat{\mathbb{A}}^n}\widehat{\mathbb{A}}^r/\mathcal{O}_K}.$$

Finally, taking the base change of this distinguished triangle along $\mathbb{Z}_p \to \mathbb{Q}_p$, we get $(*)$ for the pair of discs. ■

This completes the proof of Proposition 5.1.1. ■

Corollary 5.1.3. *Under the above notation for the smooth blowup, we get a natural long exact sequence of étale cohomology of continuous differentials:*

$$\cdots \to \mathrm{H}^j(X_{\text{ét}}, \Omega^i_{X/K}) \to \mathrm{H}^j(X'_{\text{ét}}, \Omega^i_{X'/K}) \oplus \mathrm{H}^j(Y_{\text{ét}}, \Omega^i_{Y/K}) \to \mathrm{H}^j(Y'_{\text{ét}}, \Omega^i_{Y'/K}) \to \cdots .$$

Similarly there exists a long exact sequence of the covering associated to a blowup for éh-cohomology:

Proposition 5.1.4. *Let $f : X' \to X$ be a morphism of rigid spaces over* $\mathrm{Spa}(K)$*, $Y \subset X$ be a nowhere dense analytic closed subspace, and $Y' = Y \times_X X'$ be the pullback. Let X be separated. Assume they satisfy one of the following two conditions:*

(i) *$X' \to X$ is a blowup along Y.*

(ii) *X is quasi-compact, Y is an irreducible component of X, and X' is the union of all the other irreducible components of X.*

Then the functoriality of differentials induces a natural long exact sequence of cohomology:

$$\cdots \to \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}) \to \mathrm{H}^j(X'_{\text{éh}}, \Omega^i_{\text{éh}}) \bigoplus \mathrm{H}^j(Y_{\text{éh}}, \Omega^i_{\text{éh}}) \to \mathrm{H}^j(Y'_{\text{éh}}, \Omega^i_{\text{éh}}) \to \cdots ,$$

where $\Omega^i_{\text{éh}}$ is the éh-sheafification of the i-th continuous differential forms.

Proof. For the rigid space $Z \in \mathrm{Rig}_K$, we denote by h_Z the abelianization of the éh-sheafification of the representable presheaf

$$W \mapsto \mathrm{Hom}_{\mathrm{Spa}(K)}(W, Z).$$

Then for an éh-sheaf $\mathcal{F}$ of abelian groups, we have

$$\mathrm{H}^j(Z_{\text{éh}}, \mathcal{F}) = \mathrm{Ext}^j_{\text{éh}}(h_Z, \mathcal{F}),$$

since h_Z is the final object in the category of sheaves of abelian groups over $Z_{\text{éh}}$. So back to our proof, it suffices to prove the exact sequence of éh-sheaves

$$0 \to h_{Y'} \to h_{X'} \oplus h_Y \to h_X \to 0$$

under each of the above two conditions.

Assume $\alpha : Z \to X$ is a K-morphism. Then since $X' \amalg Y \to X$ is an éh-covering (see Definition 2.4.1 and Proposition 2.4.5), the element $\alpha \in h_X(Z)$ is locally given by a map $Z \times_X (X' \amalg Y) \to X' \amalg Y$, which is an element in $h_{X'}(Z \times_X X') \oplus h_Y(Z \times_X Y)$, so we get the surjectivity.

Now assume $(\sum n_r \beta_r, \sum m_s \gamma_s)$ is an element in $h_{X'}(Z) \oplus h_Y(Z)$ whose image is 0 in $h_X(Z)$. After refining Z by an admissible covering of quasi-compact affinoid open subsets if necessary, we may assume Z is quasi-compact affinoid. By taking a further éh-covering of Z, we may also assume Z is smooth and connected, given by $Z = \mathrm{Spa}(A)$ for A integral.

Then we look at the composition of those maps with $(f, i) : X' \amalg Y \to X$.

- Assume $f \circ \beta_1 = f \circ \beta_2$ for some elements β_i. In the first setting of the proposition, since $X' \to X$ is a blowup along a nowhere dense (Zariski) closed subset, the restrictions of β_1 and β_2 to the open subset $Z \setminus f^{-1}(Y)$ coincide. So by the assumption that Z is integral (thus equal-dimensional), we see that either the closed analytic subset $f^{-1}(Y)$ is the whole Z and both β_1 and β_2 come from $Z \to Y \times_X X' = Y'$, or $f^{-1}(Y)$ is nowhere dense analytic in Z. If $f^{-1}(Y)$ is nowhere dense in Z, then β_1 and β_2 agree on a Zariski open dense subset of Z. So by looking at open affinoid subsets of X', the separatedness assumption implies $\beta_1 = \beta_2$ [19, Chap. II, Exercise 4.2]. In the second setting, note that $X' \to X$ is a closed immersion. So $f \circ \beta_1 = f \circ \beta_2$ implies $\beta_1 = \beta_2$.
- Assume $i \circ \gamma_1 = i \circ \gamma_2$ for some elements γ_i. Then we get the identity of γ_1 and γ_2 again by the injectivity of the closed immersion $i : Y \to X$.
- Assume there exists an equality $f \circ \beta_i = i \circ \gamma_j$. Since the composition $f \circ \beta_i$ is mapped into the analytic subset $Y \subset X$, the map $\beta_i : Z \to X'$ factors through $Z \to X' \times_X Y = Y'$. So β_i comes from $h_{Y'}(Z)$, and by the injectivity of $Y \to X$ again γ_j comes from $h_{Y'}(Z)$.

In this way, by combining all of those identical β_i and γ_j and canceling the coefficients, the rest of $(\sum n_r\beta_r, \sum m_s\gamma_s)$ all come from $h_{Y'}(Z)$, thus the short sequence is exact at the middle.

Finally, injectivity of $h_{Y'} \to h_{X'} \oplus h_Y$ follows from the fact that $Y' \to X'$ is a closed immersion. ■

Remark 5.1.5. Proposition 5.1.4 (ii) can be regarded as an éh-version of the Mayer–Vietoris sequence.

Proof of Theorem 4.0.2. Now we prove the descent for the éh-differential.

Let $\mathrm{Rig}_{K,\text{ét}}$ be the big étale site of rigid spaces over K. It consists of rigid spaces over K, and its topology is defined by étale coverings. Then there exists a natural map $\pi : \mathrm{Rig}_{K,\text{éh}} \to \mathrm{Rig}_{K,\text{ét}}$ of big sites that fits into the diagram

$$\begin{array}{ccc} \mathrm{Rig}_{K,\text{ét}} & \xrightarrow{\iota_X} & X_{\text{ét}} \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi_X} \\ \mathrm{Rig}_{K,\text{éh}} & \longrightarrow & X_{\text{éh}} \end{array}$$

The sheaf $\Omega^i_{\text{éh}}$ on $\mathrm{Rig}_{K,\text{éh}}$ is defined as the éh-sheafification of the continuous differential, which leads to the equality

$$\Omega^i_{\text{éh}} = \pi^{-1}\Omega^i_{/K},$$

where $\Omega^i_{/K}$ is the i-th continuous differential on $\mathrm{Rig}_{K,\text{ét}}$. Moreover, for any $Y \in \mathrm{Rig}_K$, the direct images along $\mathrm{Rig}_{K,\text{ét}} \to Y_{\text{ét}}$ and $\mathrm{Rig}_{K,\text{éh}} \to Y_{\text{éh}}$ are exact. So it is safe to use $\mathcal{F}|_{Y_{\text{ét}}}$ (resp. $\mathcal{F}|_{Y_{\text{éh}}}$) to denote the direct image of a sheaf on $\mathrm{Rig}_{K,\text{ét}}$ (resp. $\mathrm{Rig}_{K,\text{éh}}$) along those restriction maps, either in the derived or non-derived cases.

Let C be a cone of the adjunction map $\Omega^i_{/K} \to R\pi_*\pi^{-1}\Omega^i_{/K} = R\pi_*\Omega^i_{\text{éh}}$. It suffices to show the vanishing of C when restricted to a smooth X; in other words, for each X smooth over K, we want

$$\mathcal{H}^j(C)|_{X_{\text{ét}}} = 0, \quad \forall j.$$

We also note that as both $\Omega^i_{/K}$ and $R\pi_*\pi^*\Omega^i_{/K}$ have trivial cohomology in negative degrees, we have $\mathcal{H}^j(C)|_{X_{\text{ét}}} = 0$ for $j \leq -2$. In particular, C is left bounded.

Now we prove the above statement by contradiction. Assume C is not always acyclic when restricted to the small site $X_{\text{ét}}$ for some smooth rigid space X over K. By the left boundedness of C, we let j be the smallest degree such that $\mathcal{H}^j(C)|_{X_{\text{ét}}} \neq 0$ for some smooth X. Then $\mathcal{H}^{j-l}(C)|_{Y_{\text{ét}}} = 0$ for any $l > 0$ and any smooth Y over K. As this is a local statement, we let X to be a smooth, connected, quasi-compact quasi-separated rigid space of the smallest possible dimension such that $\mathcal{H}^j(C)|_{X_{\text{ét}}} \neq 0$. So by our assumption, there exists a non-zero element e in the cohomology group

$$\mathrm{H}^0(X_{\text{ét}}, \mathcal{H}^j(C)) = \mathrm{H}^j(X_{\text{ét}}, C).$$

Here the equality of those two cohomologies follows from the vanishing assumption for $\mathcal{H}^{j-l}(C)|_{X_{\text{ét}}}$ for $l > 0$.

We apply the preimage functor π^{-1} to the triangle

$$\Omega^i_{/K} \to R\pi_*\pi^{-1}\Omega^i_{/K} \to C,$$

and get a distinguished triangle in $D(\mathrm{Rig}_{K,\text{éh}})$

$$\pi^{-1}\Omega^i_{/K} \to \pi^{-1}R\pi_*\pi^{-1}\Omega^i_{/K} \to \pi^{-1}C.$$

Noting that since π^{-1} is exact and the adjoint map $\pi^{-1} \to \pi^{-1} \circ \pi_* \circ \pi^{-1}$ is an isomorphism, by taking the associated derived functors we get a canonical isomorphism

$$\pi^{-1}\Omega^i_{/K} \cong \pi^{-1}R\pi_*\pi^{-1}\Omega^i_{/K}.$$

So $\pi^{-1}C$ is quasi-isomorphic to 0, and there exists an éh-covering $X' \to X$ such that e will vanish when pulled back to X'.

Next we use the covering structure of the éh-topology (Theorem 2.4.11). By taking a refinement of $X' \to X$ if necessary, we assume $X' \to X$ is the composition

$$X' = X_m \to X_{m-1} \to \cdots \to X_0 = X,$$

where $X_l \to X_{l-1}$ is either a covering associated to a smooth blowup or an étale covering.

Now we discuss the vanishing of the non-zero element e along those pullbacks $X' = X_m \to \cdots \to X$. Assume $e|_{X_{l-1,\text{ét}}}$ is not equal to 0 (which is true when $l = 1$). If $X_l \to X_{l-1}$ is an étale covering, then since $e|_{X_{l-1,\text{ét}}} \in \mathrm{H}^0(X_{l-1,\text{ét}}, \mathcal{H}^j(C))$ is a global section of a non-zero étale sheaf $\mathcal{H}^j(C)$ on $X_{l-1,\text{ét}}$, the restriction of e to this étale covering will not be zero by the sheaf axioms. If $X_l \to X_{l-1}$ is a covering associated to a smooth blowup, we then make the following claim:

Claim 5.1.6. *Under the above assumption, the restriction* $e|_{X_{l,\text{ét}}}$ *in* $\mathrm{H}^0(X_{l,\text{ét}}, \mathcal{H}^j(C)) = \mathrm{H}^j(X_{l,\text{ét}}, C)$ *is not equal to* 0.

Granting the claim, since $X' \to X$ is a finite composition of those two types of coverings, the pullback of e to the cohomology group $\mathrm{H}^0(X'_{\text{ét}}, \mathcal{H}^j(C))$ cannot be 0, and we get a contradiction. Hence $C|_{X_{\text{ét}}}$ must vanish in the derived category $D(X_{\text{ét}})$ for a smooth quasi-compact rigid space X, and we get the natural isomorphism

$$\Omega^i_{X/K} \to R\pi_{X*}\Omega^i_{\text{éh}}, \quad \forall i.$$

Proof of Claim. By assumption, since $e|_{X_{l-1,\text{ét}}}$ is non-zero, it suffices to show that the map of cohomology groups

$$\mathrm{H}^j(X_{l-1,\text{ét}}, C) \to \mathrm{H}^j(X_{l,\text{ét}}, C)$$

is injective.

To simplify the notation, we let $X = X_{l-1}$, and $X' = X_l$ be the covering $\mathrm{Bl}_Y(X) \amalg Y \to X$ associated to the blowup at the smooth center $Y \subset X$. We let Y' be the pullback of Y along $\mathrm{Bl}_Y(X) \to X$. Since $X' \to X$ is a covering associated to a blowup along a smooth subspace Y of smaller dimension, by the two long exact sequences of cohomology for differentials (Corollary 5.1.3 and Proposition 5.1.4), we get

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & \mathrm{H}^j(X_{\text{ét}}, \Omega^i_{X/K}) & \to & \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{ét}}, \Omega^i_{\mathrm{Bl}_Y(X)/K}) \oplus \mathrm{H}^j(Y_{\text{ét}}, \Omega^i_{Y/K}) & \to & \mathrm{H}^j(Y'_{\text{ét}}, \Omega^i_{Y'/K}) & \to \cdots \\
 & \downarrow{\scriptstyle \tau_X} & & {\scriptstyle \tau_{\mathrm{Bl}_Y(X)}}\downarrow{\scriptstyle \tau_Y} & & \downarrow{\scriptstyle \tau_{Y'}} & \\
\cdots \to & \mathrm{H}^j(X_{\text{éh}}, \Omega^i_{\text{éh}}) & \longrightarrow & \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{éh}}, \Omega^i_{\text{éh}}) \oplus \mathrm{H}^j(Y_{\text{éh}}, \Omega^i_{\text{éh}}) & \longrightarrow & \mathrm{H}^j(Y'_{\text{éh}}, \Omega^i_{\text{éh}}) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{H}^j(X_{\text{ét}}, C) & \longrightarrow & \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{ét}}, C) \oplus \mathrm{H}^j(Y_{\text{ét}}, C) & \longrightarrow & \mathrm{H}^j(Y'_{\text{ét}}, C) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

By the assumption on j, since $\mathcal{H}^{j-l}(C)|_{Y',\text{ét}} = 0$ for $l > 0$, we have

$$\mathrm{H}^{j-1}(Y'_{\text{ét}}, C) = 0.$$

Moreover, since $\dim(X)$ is the smallest dimension such that $C|_{X_{\text{ét}}}$ is not quasi-isomorphic to 0, both $\mathrm{H}^j(Y_{\text{ét}}, C)$ and $\mathrm{H}^j(Y'_{\text{ét}}, C)$ are zero. In this way, the third row above becomes an isomorphism

$$\mathrm{H}^j(X_{\text{ét}}, C) \to \mathrm{H}^j(\mathrm{Bl}_Y(X)_{\text{ét}}, C) \oplus 0 = \mathrm{H}^j(X'_{\text{ét}}, C),$$

and we get an injection. ∎

This ends the proof of Theorem 4.0.2. ∎

Remark 5.1.7. In fact, the proof above works in a coarser topology, generated by the rigid topology, universal homeomorphisms and coverings associated to blowups. This is because all we need is the local smoothness and the distinguished triangles for cohomology of differentials, which is a coherent cohomology theory. Moreover, results here can be deduced from the pullback of this coarser topology to the éh-topology.

5.2. *Application to algebraic varieties*

Let K be the field $\mathbb{C}_p$ of p-adic complex numbers. We fix an abstract isomorphism of fields between $\mathbb{C}_p$ and $\mathbb{C}$. Our goal in this subsection is to relate éh-cohomology to singular cohomology when the rigid space comes from an algebraic variety.

More precisely, we have:

Theorem 5.2.1. *Let Y be a proper algebraic variety over $K = \mathbb{C}_p$, and let $X = Y^{\mathrm{an}}$ be its analytification as a rigid space over K. Then there exists a functorial isomorphism*

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{gr}^j \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C}),$$

where $\mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}))$ is the i-th singular cohomology of the complex manifold $Y(\mathbb{C})$ equipped with the Hodge filtration.

Proof. Let $\rho : Y_\bullet \to Y$ be a map from a simplicial smooth proper algebraic variety over K onto Y such that each $Y_n \to (\mathrm{cosk}_n Y_{\leq n})_{n+1}$ is a finite compositions of éh-coverings associated to smooth blowups (but with algebraic varieties instead of rigid spaces in Definition 2.4.1). Then the analytification $\rho^{\mathrm{an}} : X_\bullet \to X$ is an éh-hypercovering of X by smooth proper rigid spaces $X_n = Y_n^{\mathrm{an}}$. Moreover, the sheaf $\Omega^j_{X_n/K}$ of continuous j-differentials of X_n, which is a vector bundle over X_n, is canonically isomorphic to the sheafification of the sheaf $\Omega^j_{Y_n/K}$ of algebraic j-differentials of the algebraic variety Y_n over K.

Next we apply cohomological descent (§7.4), and get the natural quasi-isomorphism

$$R\pi_{X*}\Omega^j_{\text{éh}} \cong R\rho^{\mathrm{an}}_* R\pi_{X_\bullet *}\Omega^j_{\text{éh}}.$$

As each X_n is smooth over K, by Theorem 4.0.2 we have

$$R\pi_{X_n*}\Omega^j_{\text{éh}} \cong \Omega^j_{X_n/K}.$$

In particular, the derived pushforward $R\pi_{X*}\Omega^j_{\text{éh}}$ can be computed as

$$R\pi_{X*}\Omega^j_{\text{éh}} \cong R\rho^{\mathrm{an}}_*\Omega^j_{X_\bullet/K} \cong R\rho^{\mathrm{an}}_*(\Omega^j_{Y_\bullet/K})^{\mathrm{an}}.$$

We then take the derived global section to get

$$R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong R\Gamma(Y^{\mathrm{an}}, R\rho^{\mathrm{an}}_*(\Omega^j_{Y_\bullet/K})^{\mathrm{an}}).$$

As all of the algebraic varieties Y and Y_n are proper over K with each $\Omega^j_{Y_n/K}$ being coherent, by the rigid GAGA theorem [8, Appendix A1] we obtain a natural isomorphism

$$R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong R\Gamma(Y, R\rho_*\Omega^j_{Y_\bullet/K}).$$

Now by the construction, the map $Y_\bullet \to Y$ of simplicial varieties is a *smooth h-hypercovering* in the sense of [20]. In particular, as proved in [20, Theorem 7.12], the complex $R\rho_*\Omega^j_{Y_\bullet/K}$ is naturally isomorphic to the j-th graded piece of the Hodge filtration of the Deligne–Du Bois complex $\underline{\Omega}^\bullet_Y$. So we may replace the derived pushforward to get the isomorphism of cohomology groups

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{H}^i(Y, \underline{\Omega}^j_Y).$$

In this way, as the right side is isomorphic to the j-th graded piece $\mathrm{gr}^j \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C})$ of the Hodge filtration of the singular cohomology group [26, §7.3.1], we get

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \cong \mathrm{gr}^j \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C}). \qquad \blacksquare$$

We note that in the proof above, the comparison is compatible with the differential maps on both side. So the above leads to a comparison between éh de Rham cohomology and singular cohomology when X comes from an algebraic variety.

Corollary 5.2.2. *Let Y be a proper algebraic variety over $K = \mathbb{C}_p$, and let $X = Y^{\mathrm{an}}$ be its analytification as a proper rigid space over K. Then there exists a functorial filtered isomorphism*

$$\mathrm{H}^i(X_{\text{éh}}, \Omega^\bullet_{\text{éh}}) \cong \mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C}),$$

where $\mathrm{H}^i_{\mathrm{Sing}}(Y(\mathbb{C}), \mathbb{C})$ is the i-th singular cohomology of the complex manifold $Y(\mathbb{C})$, equipped with the Hodge filtration.

Remark 5.2.3. Let $X = Y^{\mathrm{an}}$ be the analytification of a proper algebraic variety Y over $\mathbb{C}_p$ as above. The proof of Theorem 5.2.1 in fact implies that the éh-cohomology $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ of $\Omega^j_{\text{éh}}$ is isomorphic to the h-cohomology $\mathrm{H}^i(Y_h, \Omega^j_h)$ (via [20, Corollary 6.16]), for the h-cohomology of the scheme Y introduced in [20]. So every computation for a proper algebraic variety Y in [20] can be used to compute the éh-cohomology of the rigid space Y^{an}.

6. Finiteness

In this section, we prove a finiteness result about $R\nu_*\widehat{\mathcal{O}}_X$ for X being a rigid space, namely the coherence and the cohomological boundedness of $R\nu_*\widehat{\mathcal{O}}_X$, where K is an arbitrary p-adic field.

Assume X is a rigid space over K.

Proposition 6.0.1 (Coherence). *The sheaf $R^n\nu_*\widehat{\mathcal{O}}_X$ of $\mathcal{O}_X$-modules is coherent.*

Proof. By the éh-proét spectral sequence $R^i\pi_{X*}\Omega^j_{\text{éh}} \Rightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X$, it suffices to show the coherence for each $R^i\pi_{X*}\Omega^j_{\text{éh}}(-j)$. We then assume X is reduced, since the direct image along $X_{\mathrm{red}} \to X$ preserves the coherence of modules. Then by local smoothness (Corollary 2.4.8), there exists an éh-hypercover $s : X_\bullet \to X$ such that each X_k is smooth and the map $s_k : X_k \to X$ is proper. Here we notice that each $R\pi_{X_k*}\Omega^j_{\text{éh}} = \Omega^j_{X_k/K}$ is coherent on X_k by the assumption on X_k and Theorem 4.0.2. So the properness of $s_k : X_k \to X$ implies that each $R^q s_{k*}\Omega^j_{X_k/K}$ is coherent over $\mathcal{O}_X$. On the other hand, thanks to cohomological descent (see the discussion later in §7.4), the derived direct image $Rs_* R\pi_{X_\bullet *}\Omega^j_{\text{éh}}$ along the éh-hypercover $X_\bullet \to X$ is quasi-isomorphic to $R\pi_{X*}\Omega^j_{\text{éh}}$. In this way, the spectral sequence associated to the simplicial object $s : X_\bullet \to X$ provides

$$E_1^{p,q} = R^q s_{p*}\Omega^j_{X_p/K} \Longrightarrow \mathcal{H}^{p+q}(Rs_* R\pi_{X_\bullet *}\Omega^j_{\text{éh}}) = R^{p+q}\pi_{X*}\Omega^j_{\text{éh}},$$

where each term on the left side is coherent over X. Hence the sheaf $R^{p+q}\pi_{X*}\Omega^j_{\text{éh}}$ is coherent on X. ∎

Next we consider the cohomological boundedness of derived direct images.

Theorem 6.0.2 (Cohomological boundedness). *For a quasi-compact rigid space X, the cohomology* $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ *vanishes unless* $0 \le i, j \le \dim(X)$.

Remark 6.0.3. The analogous statement about the boundedness of the Hodge numbers for varieties over $\mathbb{C}$ is proved by Deligne [10, Theorem 8.2.4].

Proof of Theorem 6.0.2. We use induction on the dimension of X. When X is of dimension 0, the reduced subspace X_{red} is a finite disjoint union of $\mathrm{Spa}(K')$ with K'/K finite, which is smooth over $\mathrm{Spa}(K)$. So by the local reducedness of the éh-topology and the vanishing of the higher direct image of $\iota : X_{\text{red}} \to X$, the case of dimension 0 is handled by Theorem 4.0.2.

We then assume the result is true for all quasi-compact rigid spaces of dimensions strictly smaller than $\dim(X)$. By local smoothness (Corollary 2.4.8) and the vanishing of the higher direct image along $X_{\text{red}} \to X$ again, we may assume X is reduced and there exists a composition of finitely many blowups at smooth centers,

$$X' = X_n \to \cdots \to X_1 \to X_0 = X,$$

such that $X' = X_n$ is smooth over $\mathrm{Spa}(K)$. Observe that by the property of the éh-differentials for smooth spaces, the sheaf $R^i\pi_{X_n*}\Omega^j_{\text{éh}}$ is zero unless $i = 0$ and $0 \le j \le \dim X_n = \dim X$. So the claim is true for X_n as $\mathrm{H}^i(X_{n,\text{éh}}, \Omega^j_{\text{éh}}) = \mathrm{H}^i(X_n, \Omega^j_{X_n/K})$. Moreover, to prove the claim for X, it suffices to show that if the result is true for X_{l+1}, then it is true for X_l, where $X_{l+1} \to X_l$ is the l-th blowup at a nowhere dense analytic subspace.

To simplify the notation, we let $X' = X_{l+1}$, $X = X_l$, $f : X' \to X$ be the blowup, $i : Y \to X$ be the inclusion map of the blowup center, and Y' be the preimage $X' \times_X Y$ under the map $g : Y' \to X'$. By the assumption, $\mathrm{H}^i(X'_{\text{éh}}, \Omega^j_{\text{éh}})$ vanishes unless $i, j \le \dim(X) = \dim(X')$. Furthermore, thanks to the induction hypothesis we have $\mathrm{H}^i(Y'_{\text{éh}}, \Omega^j_{\text{éh}}) = 0$ unless $i, j \le \dim(Y') < \dim(X)$. Now we consider the distinguished triangle of éh-cohomology (Proposition 5.1.4)

$$\cdots \to \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}}) \to \mathrm{H}^i(X'_{\text{éh}}, \Omega^j_{\text{éh}}) \oplus \mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh}}) \to \mathrm{H}^i(Y'_{\text{éh}}, \Omega^j_{\text{éh}}) \to \cdots.$$

We discuss all possible cases:

- If $j > \dim(X)$, then since the blowup center Y is nowhere dense in X, we have $j > \dim(X) > \dim(Y')$. So by induction hypothesis on dimensions, both $\mathrm{H}^i(Y_{\acute{e}h}, \Omega^j_{\acute{e}h})$ and $\mathrm{H}^{i-1}(Y'_{\acute{e}h}, \Omega^j_{\acute{e}h})$ vanish for every i. Moreover by the assumption on X', we know that $\mathrm{H}^i(X'_{\acute{e}h}, \Omega^j_{\acute{e}h})$ vanishes for $i \in \mathbb{N}$. So the long exact sequence leads to the vanishing for $\mathrm{H}^i(X_{\acute{e}h}, \Omega^j_{\acute{e}h})$ if $j > \dim(X)$.
- If $i > \dim(X)$, then since $i - 1 > \dim(X) - 1 \geq \dim(Y')$, by induction hypothesis on dimensions again, $\mathrm{H}^{i-1}(Y'_{\acute{e}h}, \Omega^j_{\acute{e}h})$ and $\mathrm{H}^i(Y_{\acute{e}h}, \Omega^j_{\acute{e}h})$ are zero. Similarly $\mathrm{H}^i(X'_{\acute{e}h}, \Omega^j_{\acute{e}h})$ by the assumption on X'. In this way, the long exact sequence implies that $\mathrm{H}^i(X_{\acute{e}h}, \Omega^j_{\acute{e}h})$ is zero for $i > \dim(X)$ and any $j \in \mathbb{N}$. ■

Corollary 6.0.4. *Let X be a rigid space over K. Then unless $0 \leq i, j \leq \dim(X)$, the higher direct image $R^i\pi_{X*}\Omega^j_{\acute{e}h}$ vanishes.*

Proof. We only need to note that the sheaf $R^i\pi_{X*}\Omega^j_{\acute{e}h}$ is the coherent sheaf on X associated to the presheaf $U \mapsto \mathrm{H}^i(U_{\acute{e}h}, \Omega^j_{\acute{e}h})$, for $U \subseteq X$ open and quasi-compact. ■

Corollary 6.0.5. *Let X be a rigid space over K. Then each cohomology sheaf $R^i\nu_*\widehat{\mathcal{O}}_X$ is coherent over $\mathcal{O}_X$, and vanishes unless $0 \leq i \leq 2\dim(X)$.*

We will improve the above two corollaries for locally compactifiable rigid spaces in Propositions 7.5.2 and 7.5.6, using the degeneracy result developed in the next section and the almost purity theorem of [4].

7. Degeneracy theorem

In this section, we show the degeneracy of the spectral sequence $R^i\pi_{X*}\Omega^j_{\acute{e}h}(-j) \Rightarrow R^{i+j}\nu_*\widehat{\mathcal{O}}_X$ under the condition that X is strongly liftable. More precisely, we use the cotangent complex to show the existence of a quasi-isomorphism

$$R\nu_*\widehat{\mathcal{O}}_X \cong \bigoplus_j R\pi_{X*}(\Omega^j_{\acute{e}h}(-j)[-j]),$$

assuming X is strongly liftable (see Definition 7.4.1). This condition is satisfied if X is proper over K (Proposition 7.4.4), or defined over a discretely valued subfield that has a perfect residue field (Example 7.4.2).

7.1. Cotangent complex for formal schemes and adic spaces

In this subsection, we first recall basics about the cotangent complex for adic spaces. A detailed discussion of the analytic cotangent complexes of formal schemes and adic spaces can be found in [14, §§7.1–7.3] (over $\mathcal{O}_K$ and K) and [18, §§5.1, 5.2] (over $\mathrm{A}_{\mathrm{inf}}/\xi^e$ and $\mathrm{B}^+_{\mathrm{dR}}/\xi^e$).

Let R_0 be a p-adically complete ring. Then there exists a continuous morphism $\mathbb{Z}_p \to R_0$ of adic rings. Recall that for a map $A \to B$ of complete R_0-algebras that are p-torsion free, we can define the complete cotangent complex $\widehat{\mathbb{L}}_{B/A}$ as the termwise p-adic completion of the usual cotangent complex $\mathbb{L}_{B/A}$. Here $\mathbb{L}_{B/A}$ is given by the corresponding complex of the simplicial B-module $\Omega^1_{P_\bullet(B)/A} \otimes_{P_\bullet(B)} B$, where $P_\bullet(B)$ is the standard A-polynomial resolution of B. The image of $\widehat{\mathbb{L}}_{B/A}$ in the derived category of B-modules is the p-adic derived completion of $\mathbb{L}_{B/A}$, which lives in cohomological degrees ≤ 0 such that

$$\mathrm{H}^0(\widehat{\mathbb{L}}_{B/A}) = \widehat{\Omega}^1_{B/A},$$

where $\widehat{\Omega}^1_{B/A}$ is the continuous differential of B over A and is defined as the p-adic completion of the algebraic Kähler differential $\Omega^1_{B/A}$. We note that the construction of the complex $\widehat{\mathbb{L}}_{B/A}$ is functorial with respect to complete R_0-algebras $A \to B$. So when $\mathfrak{X} \to \mathrm{Spf}(R_0)$ is an R_0-formal scheme that is p-torsion free, we can construct a complex of presheaves which assigns the complex $\widehat{\mathbb{L}}_{B/R_0}$ to an affinoid open subset $\mathrm{Spf}(B)$ in $\mathfrak{X}$. The *complete cotangent complex* $\widehat{\mathbb{L}}_{\mathfrak{X}/R_0}$ for a p-torsion free R_0-formal scheme $\mathfrak{X}$ is the actual complex of sheaves defined by sheafifying the above complex of presheaves termwise.

Now following the construction in [14, §7.2], for a map $(A, A^+) \to (B, B^+)$ of p-adic affinoid Huber pairs, we define its *analytic cotangent complex* $\mathbb{L}^{\mathrm{an}}_{(B,B^+)/(A,A^+)}$ as the colimit

$$\operatorname*{colim}_{\substack{A_0 \to B_0 \\ A_0, B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/A_0}[1/p],$$

where the colimit is indexed over the set of all maps $A_0 \to B_0$ of rings of definition in $A^+ \to B^+$, and $\widehat{\mathbb{L}}_{B_0/A_0}$ is the complete cotangent complex for a map of p-complete rings as above. We often use the notation $\widehat{\mathbb{L}}^{\mathrm{an}}_{B/A}$ instead of $\mathbb{L}^{\mathrm{an}}_{(B,B^+)/(A,A^+)}$ for simplicity, when the choice of the rings A^+ and B^+ is clear from the context. The construction is functorial with respect to the pair $(A, A^+) \to (B, B^+)$, and we can sheafify it to define the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X/Y}$ for a map $X \to Y$ of adic spaces. Here $\mathbb{L}^{\mathrm{an}}_{X/Y}$ is a complex of sheaves of $\mathcal{O}_X$-modules that lives in non-positive cohomological degrees, such that

$$\mathrm{H}^0(\mathbb{L}^{\mathrm{an}}_{X/Y}) = \Omega^1_{X/Y},$$

with $\Omega^1_{X/Y}$ being the continuous differential for the map $X \to Y$ of rigid spaces.

Remark 7.1.1. In many cases where the base ring is fixed, the colimit in the construction above can be simplified.

For example, let (R, R^+) be either a reduced topologically finite type algebra over a p-adic field, or $(\mathrm{A_{inf}}[1/p], \mathrm{A_{inf}})$ (the definition of $\mathrm{A_{inf}}$ will be recalled below), and let R_0 be the fixed ring of definition R^+ (this is guaranteed by the reducedness of A, and the boundedness of R° by for example [5, §6.2.4, Theorem 1]) or $\mathrm{A_{inf}}$ respectively. Then for an affinoid R-algebra (B, B^+), we have the natural quasi-isomorphism

$$\operatorname*{colim}_{\substack{R_0 \to B_0 \\ B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/R_0}[1/p] \to \widehat{\mathbb{L}}^{\mathrm{an}}_{B/R},$$

where the colimit ranges only over rings of definition of (B, B^+). This is because in both cases the ring R_0 is the largest ring of definition, so the index systems of colimits are cofinal to the one in the original definition.

Moreover, if in addition the integral subring B^+ of the Huber pair (B, B^+) is bounded, then the above colimit can be further simplified to one single complex by the quasi-isomorphism

$$\widehat{\mathbb{L}}_{B^+/R_0}[1/p] \to \widehat{\mathbb{L}}^{\mathrm{an}}_{B/R},$$

which follows for the same reason concerning the index system.

Remark 7.1.2. The construction of analytic cotangent complexes here is slightly different from the one used in [14, 18]: the colimit in the definition of $\mathbb{L}^{\mathrm{an}}_{(B,B^+)/(A,A^+)}$ above is over the set of *all* rings of definition, while the ones in loc. cit. are over the set of rings of definitions that are *topologically of finite type*. The reason we include all rings of definition is to extend the construction to perfectoid algebras, which are almost never topologically of finite type.

To see that those two constructions of analytic cotangent complexes for topologically finite type algebras A over $\mathrm{B}^+_{\mathrm{dR}}/\xi^e$ coincide, it suffices to notice that any ring of definition A_0 of A is contained in a ring of definition A_1 that is topologically of finite type over $\mathrm{A}_{\mathrm{inf}}/\xi^e$. When A is reduced (hence is topologically of finite type over K), the subring A° of power-bounded elements is the largest ring of definition which is topologically of finite type over $\mathcal{O}_K$ (apply [5, §6.4.1, Corollary 5] at a surjection $K\langle T_i\rangle \to A$).

For the general case when A is not necessarily reduced, this can be seen as follows: Let A_0 be the given ring of definition, I_0 be the nilpotent radical of A_0, and A_1 be a ring of definition that is topologically of finite type over $\mathrm{A}_{\mathrm{inf}}/\xi^e$ whose quotient by its nilpotent radical I_1 is $(A_{\mathrm{red}})^\circ$. Here we note that by the p-torsion-freeness of A_1/I_1 and [3, Lemma 13.4 (iii, b)], the ideal I_1 is finitely generated, say by g_j, $1 \le j \le m$. Moreover, the subring A_0 of A is contained in the union $\bigcup_{n\in\mathbb{N}} A_1[\frac{1}{p^n} I_1]$ of open subrings, as the latter is the preimage of $(A_{\mathrm{red}})^\circ$ in A along the surjection $A \to A_{\mathrm{red}}$, and $(A_0)_{\mathrm{red}} \subseteq (A_{\mathrm{red}})^\circ$ by the last paragraph for reduced rings. By assumption the subring A_0 is bounded, so we could choose an integer n large enough such that $A_0 \subset A_1[\frac{1}{p^n} I_1]$. Therefore the claim follows as the ring of definition $A_1[\frac{1}{p^n} I_1]$ admits a surjection from $\mathrm{A}_{\mathrm{inf}}/\xi^e\langle T_i, S_j\rangle$, where the map extends a surjection $\mathrm{A}_{\mathrm{inf}}/\xi^e\langle T_1, \ldots, T_l\rangle \to A_1$ and sends S_j to $\frac{1}{p^n} g_j$. Here we remind the reader that the construction makes sense as each $\frac{1}{p^n} g_j$ is nilpotent and in particular topologically nilpotent.

Lifting obstruction. One of the most important applications of the cotangent complex is to the deformation problem.

Let (R, R^+) be a p-adically complete Huber pair over $\mathbb{Q}_p$. Assume I is a closed ideal in R^+. We define S as the adic space $\mathrm{Spa}(R/I, \widetilde{R^+/I})$, and S' as the adic space $\mathrm{Spa}(R/I^2, \widetilde{R^+/I^2})$, where $\widetilde{R^+/I}$ and $\widetilde{R/I^2}$ are integral closures. Let X be a flat S-adic space. Then a *deformation* of X along $S \to S'$ is defined as a closed immersion $i : X \to X'$ of S'-adic spaces with X' being flat over S, such that the defining ideal is i^*I. Thus we have the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & X' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

We now focus on the case where the coefficient (R, R^+) is specified as below. Assume K is a complete and algebraically closed p-adic field, and let X be a quasi-compact rigid space over $\mathrm{Spa}(K)$. Recall that the ring $\mathrm{A}_{\mathrm{inf}}$ is defined as the ring of Witt vectors $W(\varprojlim_{x\mapsto x^p} \mathcal{O}_K)$. There is a canonical surjective continuous map $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$, with kernel being a principal ideal $\ker(\theta) = (\xi)$ for some fixed element $\xi \in \mathrm{A}_{\mathrm{inf}}$. We then recall that the de Rham period ring $\mathrm{B}_{\mathrm{dR}}^+$ is defined to be the ξ-adic completion of $\mathrm{A}_{\mathrm{inf}}[1/p]$. Here we abuse the notation and denote by $\theta : \mathrm{B}_{\mathrm{dR}}^+ \to K$ the canonical surjection induced by $\theta : \mathrm{A}_{\mathrm{inf}} \to \mathcal{O}_K$ as above. Note that for $n \geq 1$, we have $\mathrm{B}_{\mathrm{dR}}^+/(\xi)^n = \mathrm{A}_{\mathrm{inf}}[1/p]/(\xi)^n$, which is a complete Tate ring over $\mathbb{Q}_p$ [22, §1.1]. In particular the deformation of any rigid space X/K along $(\mathrm{B}_{\mathrm{dR}}^+/\xi^n, \mathrm{A}_{\mathrm{inf}}/\xi^n) \to (K, \mathcal{O}_K)$ is the same as the deformation along $(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi^n, \mathrm{A}_{\mathrm{inf}}/\xi^n) \to (K, \mathcal{O}_K)$.

We then note that the deformation theory along $(\mathrm{B}_{\mathrm{dR}}^+, \mathrm{A}_{\mathrm{inf}}) \to (K, \mathcal{O}_K)$ only depends on the p-adic topology. More precisely, we have a observation:

Lemma 7.1.3. *Let X be a topologically of finite type, p-torsion free formal scheme over $\mathrm{A}_{\mathrm{inf}}/\xi^N$ for some $N \in \mathbb{N}$. Let X_n be the base change of X along $\mathrm{A}_{\mathrm{inf}} \to \mathrm{A}_{\mathrm{inf}}/\xi^{n+1}$. Then we have a quasi-isomorphism*

$$\widehat{\mathbb{L}}_{X/\mathrm{A}_{\mathrm{inf}}} \to R\varprojlim_n \widehat{\mathbb{L}}_{X_n/(\mathrm{A}_{\mathrm{inf}}/\xi^{n+1})},$$

where $\widehat{\mathbb{L}}$ is the p-adic complete cotangent complex defined at the beginning of this section.

Proof. We may assume X is affinoid. Let T_n be the p-adic formal scheme $\mathrm{Spf}(\mathrm{A}_{\mathrm{inf}}/\xi^{n+1})$, and let T be the p-adic formal scheme $\mathrm{Spf}(\mathrm{A}_{\mathrm{inf}})$. Consider the following sequence of p-adic formal schemes:

$$X_n \to T_n \to T.$$

Then by taking the distinguished triangle of transitivity for usual cotangent complexes, we get

$$\mathbb{L}_{T_n/T} \otimes^L_{\mathcal{O}_{T_n}} \mathcal{O}_{X_n} \to \mathbb{L}_{X_n/T} \to \mathbb{L}_{X_n/T_n}. \qquad (*_n)$$

Note that the triangle remains distinguished in $D(\mathcal{O}_{X_n})$ after the derived p-adic completion.

We then take the derived inverse limit (with respect to n) of the p-adic derived completion of $(*_n)$. When $n \geq N$, we have $\widehat{\mathbb{L}}_{X_n/T} = \widehat{\mathbb{L}}_{X/T}$. Moreover, since X_n is the base change of X along $T_n \to T$, we have $\widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_{T_n}} \mathcal{O}_{X_n} = \widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_T} \mathcal{O}_X$ as complexes. So by taking the inverse limit with respect to n, we get

$$R\varprojlim_n(\widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_T} \mathcal{O}_X) \to \widehat{\mathbb{L}}_{X/T} \to R\varprojlim_n \widehat{\mathbb{L}}_{X_n/T_n}.$$

But note that the inverse system $\{\widehat{\mathbb{L}}_{T_n/T} \otimes^L_{\mathcal{O}_T} \mathcal{O}_X\}_n$ is in fact acyclic. This is because the cotangent complex $\widehat{\mathbb{L}}_{T_n/T}$ is isomorphic to $(\xi)^n/(\xi^{2n})[1]$, while the composition of transition maps

$$(\xi)^{2n}/(\xi^{4n})[1] \to (\xi)^{2n-1}/(\xi^{4n-2})[1] \to \cdots \to (\xi)^n/(\xi^{2n})[1]$$

is 0. In this way, by the vanishing of $R\varprojlim_n$, we get the quasi-isomorphism we need. ■

This lemma allows us to forget the complicated natural topology on $\mathrm{B}^+_{\mathrm{dR}}$ when we look at the deformation along $\mathrm{B}^+_{\mathrm{dR}} \to K$. So throughout the article, we will consider the adic space $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p], \mathrm{A}_{\mathrm{inf}})$ that is only equipped with the p-adic topology, and any cotangent complex that has $\mathrm{A}_{\mathrm{inf}}$ or $\mathrm{A}_{\mathrm{inf}}[1/p]$ as the base will be considered p-adically.

Let S and S' be the adic space $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi)$ and $\mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi^2)$ respectively. Here we note that S is also equal to $\mathrm{Spa}(K)$. Denote by i the map $X \to \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p], \mathrm{A}_{\mathrm{inf}})$. We let $\mathcal{O}_X(1)$ be the free $\mathcal{O}_X$-module of rank 1, defined by

$$i^*(\xi\mathcal{O}_S) = \mathcal{O}_X \otimes_{\mathrm{A}_{\mathrm{inf}}[1/p]} \xi\mathrm{A}_{\mathrm{inf}}[1/p] = \xi/\xi^2\mathcal{O}_X.$$

When X is defined over a discretely valued subfield, it admits a Galois action of Hodge–Tate weight -1.

Our first result is about the relation between the deformation of X and the splitting of the cotangent complex.

Proposition 7.1.4. *Let X be a rigid space over $S = \mathrm{Spa}(K)$. Then a flat lifting X' of X along $S \to S'$ induces a section s_X of $\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X \to \mathbb{L}^{\mathrm{an}}_{X/S'}$ in the distinguished triangle of transitivity*

$$\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X \to \mathbb{L}^{\mathrm{an}}_{X/S'} \to \mathbb{L}^{\mathrm{an}}_{X/S}.$$

Moreover, assume $X' \to Y'$ is a map of flat adic spaces over S' which lifts the map $f : X \to Y$ of rigid spaces over K. Then the induced sections above are functorial, in the sense that the following natural diagram of sections commutes:

$$\begin{array}{ccc} \mathbb{L}^{\mathrm{an}}_{Y/S'} & \xrightarrow{s_Y} & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{S'} \mathcal{O}_Y \\ \downarrow & & \downarrow \\ Rf_*\mathbb{L}^{\mathrm{an}}_{X/S'} & \xrightarrow{Rf_*(s_X)} & Rf_*(\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X) \end{array}$$

Proof. The base change diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

induces two sequences of maps

$$X \to S \to S', \quad X \to X' \to S'.$$

We take their distinguished triangles of transitivity [18, Corollary 5.2.18], and get

$$\begin{array}{ccccc}
 & & \mathbb{L}^{\mathrm{an}}_{X/S} & & \\
 & & \uparrow & & \\
\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X/S'} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X/X'} \\
 & & \uparrow & & \\
 & & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X & &
\end{array}$$

where both the vertical and the horizontal triangles are distinguished.

Following [33, Tag 09D8], we could extend the above to a bigger diagram

$$\begin{array}{ccccc}
\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X/S} & \longrightarrow & E \\
\| & & \uparrow & & \uparrow \\
\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X/S'} & \xrightarrow{\alpha_X} & \mathbb{L}^{\mathrm{an}}_{X/X'} \\
 & & \uparrow & & \uparrow \beta_X \\
 & & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X & = & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X
\end{array} \qquad (*)$$

where E is the cone of

$$(\mathbb{L}^{\mathrm{an}}_{X'/S'} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \oplus \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X) \to \mathbb{L}^{\mathrm{an}}_{X/S'},$$

which fits into the diagram such that all of the vertical and horizontal triangles are distinguished.

We then make the following claim.

Claim 7.1.5. *The cone E is isomorphic to* 0 *in the derived category.*

Proof of Claim. By construction, since the right vertical triangle above is distinguished, it suffices to show that

$$\beta_X : \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X \to \mathbb{L}^{\mathrm{an}}_{X/X'}$$

is a quasi-isomorphism.

We may assume that $X = \mathrm{Spa}(B, B^+)$ and $X' = \mathrm{Spa}(A, A^+)$ are affinoid and $A \otimes_{\mathrm{A_{inf}}/\xi^2} \mathcal{O}_K = A/\xi = B$. Then by construction, the above map can be rewritten as

$$\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_K B \to \operatorname*{colim}_{\substack{A_0 \to B_0 \\ A_0, B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/A_0}[1/p],$$

for (A_0, B_0) being all pairs of rings of definition of (A, A^+) and (B, B^+) respectively.

We then note that for a single pair (A_0, B_0) such that $B_0 \cong A_0/\xi$, the map

$$\rho : \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_K B \to \widehat{\mathbb{L}}_{A_0/B_0}[1/p]$$

is a quasi-isomorphism: by the surjectivity of $\mathrm{A_{inf}}/\xi^2 \to \mathrm{A_{inf}}/\xi = \mathcal{O}_K$ and $B_0 \to A_0$, applying [14, §7.1.31][6], we have

$$\widehat{\mathbb{L}}_{\mathcal{O}_K/(\mathrm{A_{inf}}/\xi^2)} \cong \mathbb{L}_{\mathcal{O}_K/(\mathrm{A_{inf}}/\xi^2)}, \qquad \widehat{\mathbb{L}}_{A_0/B_0} \cong \mathbb{L}_{A_0/B_0}.$$

So under the choice of A_0 and B_0 the map ρ can be rewritten as a map of the usual cotangent complexes

$$\rho : \mathbb{L}_{\mathcal{O}_K/(\mathrm{A_{inf}}/\xi^2)} \otimes_{\mathcal{O}_K} B_0[1/p] \to \mathbb{L}_{A_0/B_0}[1/p].$$

But by assumption, $A = A_0[1/p]$ is flat over $\mathrm{A_{inf}}/\xi^2[1/p]$, while $B = B_0[1/p]$ is given by $A \otimes_{\mathrm{A_{inf}}/\xi^2} \mathcal{O}_K$. Hence by the flatness of inverting p and the flat base change of the usual cotangent complexes, we see ρ is a quasi-isomorphism.

Finally, we only need to note that the collection of rings of definition $\{A_0 \to B_0\}$ such that $B_0 = A_0/\xi$ is cofinal with the collection of all $A_0 \to B_0$ (since any given B_0 is a subring of B that is of topologically finite type over $\mathcal{O}_K$, we can pick the generators and lift them to A along the surjection $A \to B$). So we get

$$\operatorname*{colim}_{\substack{A_0 \to B_0 \\ A_0, B_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{B_0/A_0}[1/p] \cong \operatorname*{colim}_{A_0 \text{ open bounded}} \widehat{\mathbb{L}}_{(A_0/\xi)/A_0}[1/p],$$

and the latter is quasi-isomorphic to $\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_K B$, so we are done. ■

In this way, since E is constructed so that the top horizontal and the right vertical triangles in $(*)$ are distinguished, we see that under the given assumption, E is quasi-isomorphic to 0. This allows us to get a section

$$s_X : \mathbb{L}^{\mathrm{an}}_{X/S'} \to \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_S \mathcal{O}_X,$$

defined as the composition of α_X and β_X^{-1} in $(*)$.

Finally, we check functoriality. Consider the map between two lifts

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ S' & = & S' \end{array}$$

Since each term in the big diagram $(*)$ is functorial with respect to $X \to X'$, the map of lifts induces a commutative diagram from $(*)$ for Y to the derived direct image of $(*)$ for X along $f : X \to Y$. In particular, this implies the commutativity of

[6] Though the statement there is for topologically finite type algebras over $\mathcal{O}_K$, the proof works for topologically finite type, p-torsion free algebras over $\mathrm{A_{inf}}/\xi^n$ as well.

$$\begin{array}{ccccc} \mathbb{L}^{\mathrm{an}}_{Y/S'} & \xrightarrow{\alpha_Y} & \mathbb{L}^{\mathrm{an}}_{Y/Y'} & \xrightarrow{\beta_Y^{-1}} & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes \mathcal{O}_Y \\ \downarrow & & \downarrow & & \downarrow \\ Rf_*\mathbb{L}^{\mathrm{an}}_{X/S'} & \xrightarrow{Rf_*(\alpha_X)} & Rf_*\mathbb{L}^{\mathrm{an}}_{X/X'} & \xrightarrow{Rf_*(\beta_X^{-1})} & Rf_*(\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes \mathcal{O}_X) \end{array}$$

By combining them, we get the desired map from s_Y to $Rf_*(s_X)$. ■

Finally, we note the following relation between $\mathbb{L}^{\mathrm{an}}_{X/S'}$ and $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}}$.

Lemma 7.1.6. *Let X be a smooth rigid space over* $\mathrm{Spa}(K)$*, and* $\mathbf{S} = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p], \mathrm{A}_{\mathrm{inf}})$ *be the p-adic complete adic space. Then the sequence of maps $X \to S' \to \mathbf{S}$ induces a quasi-isomorphism*

$$\mathbb{L}^{\mathrm{an}}_{X/\mathrm{A}_{\mathrm{inf}}[1/p]} \to \tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{X/(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi^2)} = \tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{X/S'}.$$

This is functorial with respect to X.

Proof. By taking the distinguished triangle of transitivity, we get

$$\mathbb{L}^{\mathrm{an}}_{S'/\mathbf{S}} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X \to \mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}} \to \mathbb{L}^{\mathrm{an}}_{X/S'}. \tag{$*$}$$

Since $S' = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi^2)$ is a closed subspace of $\mathbf{S} = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}})$ defined by the regular ideal (ξ^2), we have

$$\mathbb{L}^{\mathrm{an}}_{S'/\mathbf{S}} \otimes_{\mathcal{O}_{S'}} \mathcal{O}_X = (\xi^2)/(\xi^4) \otimes^L_{\mathcal{O}_{S'}} \mathcal{O}_X[1].$$

But by the distinguished triangle for $X \to S \to \mathbf{S}$, we have

$$\mathbb{L}^{\mathrm{an}}_{S/\mathbf{S}} \otimes_{\mathcal{O}_S} \mathcal{O}_X \to \mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}} \to \mathbb{L}^{\mathrm{an}}_{X/S},$$

where $\mathbb{L}^{\mathrm{an}}_{S/\mathbf{S}} \otimes_{\mathcal{O}_S} \mathcal{O}_X = (\xi)/(\xi^2) \otimes_K \mathcal{O}_X[1] = \xi/\xi^2\mathcal{O}_X[1]$, and $\mathbb{L}^{\mathrm{an}}_{X/S} = \Omega^1_{X/K}[0]$ by the smoothness assumption [14, Theorem 7.2.42]. In this way, since $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}}$ lives in cohomological degrees -1 and 0 and is killed by ξ^2, the image of $\mathbb{L}^{\mathrm{an}}_{S'/\mathbf{S}} \otimes^L_{\mathcal{O}_{S'}} \mathcal{O}_X = (\xi^2)/(\xi^4) \otimes^L_{\mathcal{O}_{S'}} \mathcal{O}_X[1]$ in $\mathbb{L}^{\mathrm{an}}_{X/\mathbf{S}}$ is 0. Hence the sequence $(*)$ induces a quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{X/\mathrm{A}_{\mathrm{inf}}[1/p]} \to \tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{X/(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi^2)},$$

which lives in degrees -1 and 0.

Finally, since those two distinguished triangles are functorial with respect to X, so is the quasi-isomorphism. ■

7.2. Degeneracy in the smooth setting

After the basics around the cotangent complex and the lifting criterion, we are going to prove the degeneracy theorem for smooth rigid spaces, assuming liftability to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. We fix a complete and algebraically closed p-adic field K as before.

We first prove a simple result about the cotangent complex over $\mathrm{A_{inf}}$.

Proposition 7.2.1. *Let A be an $\mathrm{A_{inf}}$-algebra. Then the natural map $\widehat{\mathbb{L}}_{A/\mathbb{Z}_p} \to \widehat{\mathbb{L}}_{A/\mathrm{A_{inf}}}$ of complete cotangent complexes is a quasi-isomorphism.*

Proof. Consider the sequence of maps

$$\mathbb{Z}_p \to \mathrm{A_{inf}} \to A.$$

By the basic properties of the usual cotangent complex of rings, we get a distinguished triangle in $D^-(A)$:

$$\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A \to \mathbb{L}_{A/\mathrm{A_{inf}}} \to \mathbb{L}_{A/\mathbb{Z}_p}.$$

Applying derived p-completion, we get the distinguished triangle

$$(\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A)^\wedge \to \widehat{\mathbb{L}}_{A/\mathrm{A_{inf}}} \to \widehat{\mathbb{L}}_{A/\mathbb{Z}_p}.$$

By the equality $(\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A)^\wedge = (\widehat{\mathbb{L}}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathrm{A_{inf}}} A)^\wedge$ and the derived Nakayama lemma, it suffices to prove the vanishing of $\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p$ (thus of $\widehat{\mathbb{L}}_{\mathrm{A_{inf}}/\mathbb{Z}_p}$). Note that $\mathrm{A_{inf}} = W(\mathcal{O}_K^\flat)$, where $\mathcal{O}_K^\flat$ is a perfect ring in characteristic p. As a consequence, since the cotangent complex of a perfect ring over $\mathbb{F}_p$ is quasi-isomorphic to zero [2, Chapter 4, Proposition 3.12], we get

$$\mathbb{L}_{\mathrm{A_{inf}}/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p/p = \mathbb{L}_{\mathcal{O}_K^\flat/\mathbb{F}_p} \cong 0.$$

Hence the p-adic completed cotangent complex $\widehat{\mathbb{L}}_{\mathrm{A_{inf}}/\mathbb{Z}_p}$ vanishes, and thus we obtain the quasi-isomorphism as in the statement. ∎

Corollary 7.2.2. *Let X be an adic space over* $\mathrm{Spa}(K, \mathcal{O}_K)$. *Then the sequence of maps $\mathbb{Q}_p \to \mathrm{A_{inf}}[1/p] \to \mathcal{O}_X$ induces a functorial quasi-isomorphism between analytic cotangent complexes,*

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathbb{Q}_p} \cong \mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathrm{A_{inf}}[1/p]}.$$

Cotangent complex and derived direct image. Now we are able to connect the cotangent complex with $R\nu_*\widehat{\mathcal{O}}_X$. Our first result is about the truncation of $R\nu_*\widehat{\mathcal{O}}_X$:

Theorem 7.2.3. *Let X be a smooth rigid space over* $\mathrm{Spa}(K)$. *Then there exists a functorial quasi-isomorphism in the derived category of $\mathcal{O}_X$-modules,*

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathrm{A_{inf}}[1/p]}(-1)[-1] \cong \tau^{\leq 1} R\nu_*\widehat{\mathcal{O}}_X.$$

Proof. To construct the isomorphism above, we will need the analytic cotangent complex for the complete pro-étale structure sheaf $\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R}$, where (R, R^+) is either $(\mathrm{A_{inf}}[1/p], \mathrm{A_{inf}})$ or $(\mathbb{Q}_p, \mathbb{Z}_p)$. We will first work at the presheaf level and do the construction for affinoid perfectoid rings, and then show that the cotangent complex is in fact a twist of the complete structure sheaf.

Step 1: *Calculation at affinoid perfectoid.* Denote by X_{ind} the indiscrete site on the category of affinoid perfectoid objects in $X_{\mathrm{pro\acute{e}t}}$: the category X_{ind} is the collection of affinoid perfectoid objects in $X_{\text{proét}}$, and the topology is such that every presheaf on X_{ind} is a sheaf. Then there exists a canonical map of sites $\delta : X_{\text{proét}} \to X_{\text{ind}}$. The inverse image functor δ^{-1} is an exact functor on abelian sheaves defined by sheafification, and we have $L\delta^{-1} = \delta^{-1}$.

Then we can define the completed structure sheaf $\widehat{\mathcal{O}}^+_{\text{ind}}$ such that for $U \in X_{\text{ind}}$ with underlying perfectoid space $\mathrm{Spa}(A, A^+)$, we have

$$\widehat{\mathcal{O}}^+_{\text{ind}}(U) = A^+, \quad \widehat{\mathcal{O}}_{\text{ind}}(U) = A.$$

Similarly we can define the cotangent complex

$$\mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}^+_X/R^+}(U) = \varinjlim_{\substack{A_0 \subset A^+ \\ A_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{A_0/R^+}, \quad \mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}_X/R}(U) = \mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}^+_X/R^+}[1/p].$$

Here the cotangent complexes for formal rings and adic rings are as at the beginning of the subsection. As a perfectoid algebra (A, A^+) is uniform, we know A° is bounded in A [27, Definiton 1.6]. In particular, the open subring A^+ of A° is also open bounded, and we have

$$\mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}^+_X/R^+}(U) = \widehat{\mathbb{L}}_{A^+/R^+}, \quad \mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}_X/R}(U) = \mathbb{L}^{\text{an}}_{A/R}$$

as complexes. We also note that by Proposition 7.2.1, the sequence of sheaves $\mathbb{Z}_p \to \mathrm{A}_{\text{inf}} \to \widehat{\mathcal{O}}^+_{\text{ind}}$ induces a quasi-isomorphism

$$\mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}_X/\mathrm{A}_{\text{inf}}[1/p]} \cong \mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}_X/\mathbb{Q}_p}.$$

To check this quasi-isomorphism it suffices to check sections at $U \in X_{\text{ind}}$, since X_{ind} has only trivial coverings.

Moreover, the map of rings $\mathbb{Z}_p \to \mathcal{O}_K \to \widehat{\mathcal{O}}^+_{\text{ind}}$ provides the natural distinguished triangle

$$\widehat{\mathbb{L}}_{\mathcal{O}_K/\mathbb{Z}_p} \widehat{\otimes}_{\mathcal{O}_K} \widehat{\mathcal{O}}^+_{\text{ind}} \to \widehat{\mathbb{L}}_{\text{ind},\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} \to \widehat{\mathbb{L}}_{\text{ind},\widehat{\mathcal{O}}^+_X/\mathcal{O}_K}.$$

Since the mod p reduction of $\mathcal{O}_K \to \widehat{\mathcal{O}}^+_{\text{ind}}$ is relatively perfect, and $\widehat{\mathbb{L}}_{\mathcal{O}_K/\mathbb{Z}_p}$ is isomorphic to the Breuil–Kisin twist $\mathcal{O}_K\{1\}[1]$ of weight -1, we have the quasi-isomorphism

$$\widehat{\mathbb{L}}_{\text{ind},\widehat{\mathcal{O}}^+_X/\mathbb{Z}_p} \cong \widehat{\mathcal{O}}^+_{\text{ind}}\{1\}[1].$$

So by inverting p, we have

$$\mathbb{L}^{\text{an}}_{\text{ind},\widehat{\mathcal{O}}_X/\mathbb{Q}_p} \cong \widehat{\mathcal{O}}_{\text{ind}}(1)[1]. \tag{$*$}$$

The same is true when we replace $\mathbb{Q}_p$ by $\mathrm{A}_{\text{inf}}[1/p]$.

Step 2: *Pro-étale cotangent complex.* Now we go back to the pro-étale topology. As above, let (R, R^+) be either $(\mathrm{A}_{\text{inf}}[1/p], \mathrm{A}_{\text{inf}})$ or $(\mathbb{Q}_p, \mathbb{Z}_p)$. We first observe that the definition of (integral) analytic cotangent complex can be extended to the whole pro-étale site $X_{\text{proét}}$, which is the complex of sheaves given by sheafifying the complex of presheaves that assigns to each object U with underlying perfectoid space $\mathrm{Spa}(A, A^+)$ the objects

$$\varinjlim_{\substack{A_0 \subset A^+ \\ A_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{A_0/R^+}, \qquad \varinjlim_{\substack{A_0 \subset A^+ \\ A_0 \text{ open bounded}}} \widehat{\mathbb{L}}_{A_0/R^+}[1/p].$$

We denote those two as

$$\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}^+_X/R^+}, \quad \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R}.$$

Here we note that the definition is compatible with the one for rigid spaces (see the discussion at the beginning of §7.1). In particular, by the functoriality of the construction, the canonical map of ringed sites $\nu : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ induces a natural map

$$\mathbb{L}^{\mathrm{an}}_{X/R} \to R\nu_* \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R}.$$

Moreover, as the collection of affinoid perfectoid open subsets forms a base for $X_{\mathrm{pro\acute{e}t}}$, the pro-étale cotangent complex is equal to the inverse image of the indiscrete cotangent complex along $\delta : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{ind}}$, i.e.

$$\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R} = \delta^{-1} \mathbb{L}^{\mathrm{an}}_{\mathrm{ind}, \widehat{\mathcal{O}}_X/R}.$$

Now we take the (derived) inverse image δ^{-1} for the quasi-isomorphism $(*)$ to get the quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R} \cong \delta^{-1} \widehat{\mathcal{O}}_{\mathrm{ind}}(1)[1].$$

On the other hand, for affinoid perfectoid U (with underlying perfectoid space $\mathrm{Spa}(A, A^+)$) in $X_{\mathrm{pro\acute{e}t}}$, the rings of sections of complete structure sheaves at U are known to be [28, Lemma 4.10]

$$\widehat{\mathcal{O}}^+_X(U) = A^+, \quad \widehat{\mathcal{O}}_X(U) = A.$$

In this way, the inverse images of indiscrete structure sheaves are identified with the complete structure sheaves over the pro-étale site

$$\delta^{-1} \widehat{\mathcal{O}}^+_{\mathrm{ind}} = \widehat{\mathcal{O}}^+_X, \quad \delta^{-1} \widehat{\mathcal{O}}_{\mathrm{ind}} = \widehat{\mathcal{O}}_X.$$

Thus we get the natural quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/R} \cong \widehat{\mathcal{O}}_X(1)[1]. \qquad (**)$$

Step 3: *Comparison.* Finally, we consider the statement of the theorem. The map $\nu : (X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X) \to (X, \mathcal{O}_X)$ between ringed sites induces a morphism of cotangent complexes

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathrm{A_{inf}}} \to R\nu_* \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/\mathrm{A_{inf}}}.$$

By Corollary 7.2.2 and the natural quasi-isomorphism $\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_X/\mathrm{A_{inf}}} = \widehat{\mathcal{O}}_X(1)[1]$ in Step 2, the map above is isomorphic to

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathbb{Q}_p} \to R\nu_* \widehat{\mathcal{O}}_X(1)[1].$$

So in order to show the quasi-isomorphism in Theorem 7.2.3, it suffices to show the quasi-isomorphism of

$$\mathbb{L}^{\mathrm{an}}_{\mathcal{O}_X/\mathbb{Q}_p}(-1)[-1] \to \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X. \tag{$***$}$$

Now, since the statement is local on X, we may assume X is affinoid, admitting an étale morphism to $\mathbb{T}^n$. Then both sides above are invariant under étale base change: the right side is a complex of étale coherent sheaves, while the base change of the left side is given by the vanishing of the relative cotangent complex for an étale map.[7] So it suffices to handle the case when $X = \mathbb{T}^n = \mathrm{Spa}(K\langle T_i^{\pm 1}\rangle, \mathcal{O}_K\langle T_i^{\pm 1}\rangle)$. But notice that the map $(***)$ can be given by inverting p at the sequence

$$\widehat{\mathbb{L}}_{\mathcal{O}_K\langle T_i^{\pm 1}\rangle/\mathbb{Z}_p}\{-1\}[-1] \to \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}^+_{\mathbb{T}^n},$$

where $\{-1\}$ is the Breuil–Kisin twist of weight 1. In this way, by the local computation in [3, Proposition 8.15], the map above induces a quasi-isomorphism

$$\widehat{\mathbb{L}}_{\mathcal{O}_K\langle T_i^{\pm 1}\rangle/\mathbb{Z}_p}\{-1\}[-1] \to \tau^{\leq 1} L\eta_{\zeta_p - 1} R\nu_* \widehat{\mathcal{O}}^+_{\mathbb{T}^n},$$

which after inverting p induces the quasi-isomorphism of analytic cotangent complexes

$$\mathbb{L}^{\mathrm{an}}_{\mathbb{T}^n/\mathbb{Q}_p}(-1)[-1] \to \tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_{\mathbb{T}^n}.$$

Hence we are done. ∎

Corollary 7.2.4. *Assume X is a smooth rigid space over K that admits a flat lift X' along $\mathrm{B}^+_{\mathrm{dR}}/\xi^2 \to K$. Then the lift X' induces a splitting of $\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X$ into a direct sum of its cohomology sheaves in the derived category.*

Moreover, the splitting is functorial with respect to the lift X'.

Proof. By Theorem 7.2.3, we have the functorial quasi-isomorphism

$$\tau^{\leq 1} R\nu_* \widehat{\mathcal{O}}_X = \mathbb{L}^{\mathrm{an}}_{X/\mathrm{A}_{\mathrm{inf}}[1/p]}(-1)[-1].$$

[7] This follows from the distinguished triangle $Lf^*\mathbb{L}^{\mathrm{an}}_{\mathbb{T}^n/\mathbb{Q}_p} \to \mathbb{L}^{\mathrm{an}}_{X/\mathbb{Q}_p} \to \mathbb{L}^{\mathrm{an}}_{X/\mathbb{T}^n}$ and the vanishing of $\mathbb{L}^{\mathrm{an}}_{X/\mathbb{T}^n}$ [14, Theorem 7.2.42], where $f : X \to \mathbb{T}^n$ is an étale morphism. Here we note that as neither X or $\mathbb{T}^n$ is topologically of finite type over $\mathbb{Q}_p$, we cannot apply [14] to get the triangle directly. To see the triangle, we first notice that as $X = \mathrm{Spa}(B, B^\circ)$ and $\mathbb{T} = \mathrm{Spa}(A, A^\circ)$ are reduced and topologically of finite type over K, by Remark 7.1.1 the analytic cotangent complex can be naturally computed as follows:

$$\mathbb{L}^{\mathrm{an}}_{X/\mathbb{Q}_p} = \widehat{\mathbb{L}}_{B^\circ/\mathbb{Z}_p}[1/p], \quad \mathbb{L}^{\mathrm{an}}_{\mathbb{T}^n/\mathbb{Q}_p} = \widehat{\mathbb{L}}_{A^\circ/\mathbb{Z}_p}[1/p], \quad \mathbb{L}^{\mathrm{an}}_{X/\mathbb{T}^n} = \widehat{\mathbb{L}}_{B^\circ/A^\circ}[1/p].$$

Moreover, the pullback $Lf^*\mathbb{L}^{\mathrm{an}}_{\mathbb{T}^n/\mathbb{Q}_p}$, which is equal to $B \otimes^L_A (\widehat{\mathbb{L}}_{A^\circ/\mathbb{Z}_p}[1/p])$, is naturally isomorphic to $(B^\circ \otimes^L_{A^\circ} \mathbb{L}_{A^\circ/\mathbb{Z}_p})^\wedge[1/p]$. So the distinguished triangle we want can be given by taking the derived p-completion and then inverting p at the distinguished triangle for the algebraic cotangent complex of $\mathbb{Z}_p \to A^\circ \to B^\circ$.

Moreover, Lemma 7.1.6 about truncation provides a functorial quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{X/\mathrm{A_{inf}}[1/p]}(-1)[-1] = (\tau^{\geq -1}(\mathbb{L}^{\mathrm{an}}_{X/(\mathrm{A_{inf}}[1/p]/\xi^2)}))(-1)[-1].$$

Finally, by Proposition 7.1.4, the right side splits into the direct sum of cohomology sheaves if X can be lifted to a flat adic space over $S' = \mathrm{Spa}(\mathrm{A_{inf}}[1/p]/\xi^2, \mathrm{A_{inf}}/\xi^2) = \mathrm{Spa}(\mathrm{B}^+_{\mathrm{dR}}/\xi^2, \mathrm{A_{inf}}/\xi^2)$, such that the splitting in Proposition 7.1.4 is functorial with respect to the lift. So we get the result. ■

We then notice that the splitting of the derived direct image is in fact true without the truncation.

Proposition 7.2.5. *Assume X is a smooth rigid space over K that admits a flat lift X' over $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Then the lift X' induces a splitting of the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ into $\bigoplus_{i\geq 0}\Omega^i_{X/K}(-i)[-i]$ in the derived category.*

Here the isomorphism is functorial with respect to lifts, in the sense that when $f' : X' \to Y'$ is a $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$ morphism between lifts of a map $f : X \to Y$ of two smooth rigid spaces over K, then the induced map $R\nu_*\widehat{\mathcal{O}}_Y \to Rf_*R\nu_*\widehat{\mathcal{O}}_X$ is compatible with the map between the direct sums of differentials.

Proof. By Corollary 7.2.4 above, the given lift to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$ induces an $\mathcal{O}_X$-linear quasi-isomorphism

$$\mathcal{O}_X[0] \oplus \Omega^1_{X/K}(-1)[-1] \to \tau^{\leq 1}R\nu_*\widehat{\mathcal{O}}_X.$$

It is functorial in the sense that if $f' : X' \to Y'$ is a $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$-morphism between lifts of a map $f : X \to Y$ of smooth rigid spaces over K, then the induced map

$$\tau^{\leq 1}R\nu_*\widehat{\mathcal{O}}_Y \to Rf_*\tau^{\leq 1}R\nu_*\widehat{\mathcal{O}}_X$$

is compatible with the section maps

$$\begin{array}{ccc}
\mathcal{O}_Y[0] & \longrightarrow & Rf_*\mathcal{O}_X[0] \\
\uparrow{\scriptstyle s_Y} & & \uparrow{\scriptstyle Rf_*(s_X)} \\
\tau^{\leq 1}R\nu_*\widehat{\mathcal{O}}_Y & \longrightarrow & Rf^*\tau^{\leq 1}R\nu_*\widehat{\mathcal{O}}_X
\end{array}$$

which are induced by the functoriality in Proposition 7.1.4, Lemma 7.1.6, and Theorem 7.2.3.

We compose the decomposition with $\tau^{\leq 1}R\nu_*\widehat{\mathcal{O}}_X \to R\nu_*\widehat{\mathcal{O}}_X$, and get

$$\Omega^1_{X/K}(-1)[-1] \to R\nu_*\widehat{\mathcal{O}}_X.$$

Here $R\nu_*\widehat{\mathcal{O}}_X$ is a commutative algebra object in the derived category $D(\mathcal{O}_X)$. Moreover, as in [11], the above map can be lifted to a canonical map

$$\bigoplus_{i\geq 0}\Omega^i_{X/K}(-i)[-i] \to R\nu_*\widehat{\mathcal{O}}_X \qquad (*)$$

of commutative algebra objects in the derived category. This can be constructed as follows: For each $i \geq 1$, the quotient map $(\Omega^1_{X/K})^{\otimes i} \to \Omega^i_{X/K}$ admits a canonical $\mathcal{O}_X$-linear section s_i given by

$$\omega_1 \wedge \cdots \wedge \omega_i \mapsto \frac{1}{n!} \sum_{\sigma \in S_i} \mathrm{sgn}(\sigma) \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}.$$

This allows us to give a canonical $\mathcal{O}_X$-linear map from $\Omega^i_{X/K}(-i)[-i]$ to $R\nu_*\widehat{\mathcal{O}}_X$, by the diagram

$$\begin{array}{ccc} (\Omega^1_{X/K}(-1)[-1])^{\otimes^L_{\mathcal{O}_X} i} & \longrightarrow & (R\nu_*\widehat{\mathcal{O}}_X)^{\otimes^L_{\mathcal{O}_X} i} \\ {\scriptstyle s_i}\uparrow & & \downarrow \\ \Omega^i_{X/K}(-i)[-i] & \dashrightarrow & R\nu_*\widehat{\mathcal{O}}_X \end{array}$$

Here the right vertical map is the multiplication induced from that of $\widehat{\mathcal{O}}_X$. We note that since X is smooth over K, the derived tensor product of $\Omega^1_{X/K}$ over $\mathcal{O}_X$ degenerates into the usual tensor product. Moreover, by construction the total map $\bigoplus_i s_i$ is multiplicative under wedge products.

Finally, it suffices to show that the isomorphism for the truncation $\tau^{\leq 1}$ can be extended to the map $(*)$ above. When X is of dimension 1, since $\Omega^i_{X/K}$ is zero for $i \geq 2$, we are done. In general, by taking open subsets if necessary we may assume X is affinoid and admits an étale map from a closed unit disc $\mathbb{B}^n_K$ over K. Here $\mathbb{B}^n_K$ admits a natural lift $\mathbb{B}^n_{\mathrm{B}^+_{\mathrm{dR}}/\xi^2}$, and by its smoothness over $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$ we can lift $\mathbb{B}^n_K \to X$ to an étale morphism $\mathbb{B}^n_{\mathrm{B}^+_{\mathrm{dR}}/\xi^2} \to X'$. Then as the map $(*)$ commutes with étale base change, it suffices to show this for the closed unit disc $\mathbb{B}^n_K$ with its canonical lift $\mathbb{B}^n_{\mathrm{B}^+_{\mathrm{dR}}/\xi^2}$. Finally, notice that both sides of $(*)$ admit the Künneth formula for products (the Künneth formula for pro-étale cohomology can be found in [3, Proposition 8.14]), we thus deduce the isomorphism of $(*)$ from the curve case. ■

7.3. *Simplicial generalizations*

We now generalize results in the past two subsections to simplicial cases.

Simplicial sites and cohomology. First we recall briefly the simplicial sites. The general discussion can be found in [33, Chapter 09VI].[8]

Consider a non-augmented simplicial object of sites $\{Y_n\}$: for each nondecreasing map $\phi : [n] \to [l]$ in Δ, where $[n]$ (resp. $[l]$) is the totally ordered set of $n+1$ (resp. $l+1$) elements, there exists a morphism of sites $u_\phi : Y_l \to Y_n$ satisfying the commutativity of

[8] We mention that the discussion of the simplicial sites and cohomology in our article might become simpler if we use the language of infinity categories, as the latter behave better than the ordinary derived category when we consider a diagram of derived objects.

diagrams induced from Δ. Then we can define its associated *non-augmented simplicial site* $Y_\bullet$, following the definition of $\mathcal{C}_{\text{total}}$ in [33, Tag 09WC]. An object of $Y_\bullet$ is defined as an object $U_n \in Y_n$ for some $n \in \mathbb{N}$, and a morphism $(\phi, f) : U_l \to V_n$ is given by a map $\phi : [n] \to [l]$ together with a map of objects $f : U_l \to u_\phi^{-1}(V_n)$ in Y_l. To give a covering of $U \in Y_n$ means to specify a collection of $V_i \in Y_n$ such that $\{V_i \to U\}$ is a covering in the site Y_n. It can be checked that the definition satisfies the axioms of a Grothendieck topology. Moreover, by allowing n to include the number -1, we can define the *augmented simplicial site* $Y_\bullet$. Here we remark that unless specifically mentioned, a simplicial site or a simplicial object in our article is always assumed to be non-augmented. Similarly, by replacing Δ by the finite category $\Delta_{\leq m}$ and assuming $n \leq m$, we get the definition of the *m-truncated simplicial site* $Y_\bullet$.

From the definition above, in order to give a (pre)sheaf on $Y_\bullet$, it is equivalent to give a collection of (pre)sheaves $\mathcal{F}^n$ on each Y_n, and for any map $\phi : [n] \to [l]$ in the index category, to specify a map of sheaves $\mathcal{F}^n \to u_{\phi *}\mathcal{F}^l$ over Y_n that is compatible with arrows in Δ. This allows us to define the derived category $D(Y_\bullet)$ of abelian sheaves on $Y_\bullet$.

We could also define the concept of simplicial ringed sites, which consist of pairs $(Y_\bullet, \mathcal{O}_{Y_\bullet})$, with $Y_\bullet$ being a simplicial site and $\mathcal{O}_{Y_\bullet}$ being a sheaf of rings on $Y_\bullet$, assuming $u_\phi : (Y_l, \mathcal{O}_{Y_l}) \to (Y_n, \mathcal{O}_{Y_n})$ are maps of ringed sites.

Remark 7.3.1. On the level of the derived category, the category $D(Y_\bullet)$ is not equivalent to the category where objects are given by specifying one in each $D(Y_n)$ together with natural boundary maps, unless we replace derived categories by derived infinity categories and also consider higher morphisms. This is the main reason why we need to reconstruct many objects at the simplicial level in this section, instead of using the known results for single site or space directly. The essential difference is that an object at the simplicial level has much stronger functoriality than a collection of objects over each individual space.

From the construction above, it is clear that there exists a map of sites $Y_\bullet \to Y_n$. The pushforward functor along this map is the restriction functor, sending the collection of sheaves $(\mathcal{F}_l)_l$ to its n-th component $\mathcal{F}_n$, and it is exact [33, Tag 09WG] for sheaves of abelian groups. Here is a useful vanishing criterion for objects in the derived category $D^+(Y_\bullet)$ of a simplicial site.

Lemma 7.3.2. *Let K be an object in the derived category $D^+(Y_\bullet)$ of an m-truncated simplicial site $Y_\bullet$ for $m \in \mathbb{N} \cup \{\infty\}$. Then K is acyclic if and only if for each $n \leq m$, the restriction $K|_{Y_n}$ in $D^+(Y_n)$ is acyclic.*

Proof. If K is acyclic, then since the restriction functor is exact, $K|_{Y_n}$ is also acyclic.

Conversely, assume $K|_{Y_n}$ is acyclic for each integer $n \leq m$. If K is not acyclic, then since K lives in $D^+(Y_\bullet)$, we may assume i is the least integer such that the i-th cohomology sheaf $\mathcal{H}^i(K) \in \mathrm{Ab}(Y_\bullet)$ is non-vanishing. Then by definition, there exists n such that

$\mathcal{H}^i(K)|_{Y_n}$ is non-zero. But again by the exactness of restriction,

$$\mathcal{H}^i(K)|_{Y_n} = \mathcal{H}^i(K|_{Y_n}),$$

and the latter is zero by assumption. So we get a contradiction, and hence K is acyclic. ■

As a small upshot, we have

Lemma 7.3.3. *Let $\lambda_\bullet : X_\bullet \to Y_\bullet$ be a morphism of two m-truncated simplicial sites for $m \in \mathbb{N} \cup \{\infty\}$ such that for each $n \leq m$, the map $\lambda_n : X_n \to Y_n$ is of cohomological descent, i.e. the canonical map induced by the adjunction $(\lambda_n^{-1}, \lambda_{n*})$,*

$$\mathcal{F} \to R\lambda_{n*}\lambda_n^{-1}\mathcal{F},$$

is a quasi-isomorphism for any $\mathcal{F} \in \mathrm{Ab}(Y_n)$. Then $\lambda_\bullet$ is also of cohomological descent: for any abelian sheaf $\mathcal{F}^\bullet$ on $Y_\bullet$, the counit map of this adjoint pair is a quasi-isomorphism

$$\mathcal{F}^\bullet \to R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet.$$

Proof. Let $\mathcal{C}$ be a cone of the map $\mathcal{F}^\bullet \to R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet$. It suffices to show the vanishing of the cone in the derived category $D(Y_\bullet)$. Then by the exactness of the restriction functor, for any $n \leq m$ the image $C|_{Y_n}$ in $D^+(Y_n)$ is also a cone of

$$\mathcal{F}^n \to (R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet)|_{Y_n} = R\lambda_{n*}\lambda_n^{-1}\mathcal{F}^n,$$

which vanishes by assumption. Since both $\mathcal{F}^\bullet$ and $R\lambda_{\bullet *}\lambda_\bullet^{-1}\mathcal{F}^\bullet$ are lower bounded, the cone $\mathcal{C}$ is also in $D^+(Y_\bullet)$ and we can use Lemma 7.3.2 above to get the result. ■

Derived direct image for smooth simplicial spaces. Next, we use simplicial tools above to generalize results on cotangent complexes and derived direct images to their simplicial versions.

Assume $f : X_\bullet \to Y_\bullet$ is a morphism of (m-truncated) simplicial quasi-compact adic spaces over a p-adic Huber pair. Then we can define the *simplicial analytic cotangent complex* $\mathbb{L}^{\mathrm{an}}_{X_\bullet/Y_\bullet}$ as an actual complex of sheaves on the simplicial site $X_\bullet$ such that the n-th term on the adic space X_n is the analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_n/Y_n}$, defined as in §7.1. In our applications, we will always assume $Y_\bullet$ is a constant simplicial space associated to $T = \mathrm{Spa}(R, R^+)$ for some p-adic Huber pair (R, R^+). We will use the notation $\mathbb{L}^{\mathrm{an}}_{X_\bullet/R}$ or $\mathbb{L}^{\mathrm{an}}_{X_\bullet/T}$ to indicate that the case is constant.

Here we emphasize that as in the definition of the analytic cotangent complex for X_n/Y_n above, the complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/Y_\bullet}$ is actual, i.e. it is defined in the category of complexes of abelian sheaves on $X_\bullet$, not just an object in the derived category.

Now let $X_\bullet$ be an (m-truncated) simplicial rigid space over $\mathrm{Spa}(K)$. Then we can form the cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[1/p]}$ over $\mathrm{A}_{\mathrm{inf}}[1/p]$. Moreover we can define the *simplicial differential sheaf* $\Omega^i_{X_\bullet/K}$ on $X_\bullet$ so that on each X_n, the component of the sheaf is $\Omega^i_{X_n/K}$.

We first generalize the result about the obstruction to lifting to the simplicial case:

Proposition 7.3.4. *Let $X_\bullet$ be an (m-truncated) smooth quasi-compact simplicial rigid space over* $\mathrm{Spa}(K)$. *Then a flat lift $X'_\bullet$ of $X_\bullet$ along $\mathrm{B}^+_{\mathrm{dR}}/\xi^2 \to K$ induces a splitting of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[1/p]}$ into the direct sum of its cohomological sheaves $\mathcal{O}_{X_\bullet}(1)[1] \oplus \Omega^1_{X_\bullet/K}[0]$ in the derived category. The quasi-isomorphism is functorial with respect to $X'_\bullet$.*

Proof. This is a combination of Proposition 7.1.4 and Lemma 7.1.6. We first notice that the sequence of maps

$$X_\bullet \to S'_\bullet = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p]/\xi^2)_\bullet \to \mathbf{S}_\bullet = \mathrm{Spa}(\mathrm{A}_{\mathrm{inf}}[1/p])_\bullet$$

induces a map

$$\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[1/p]} \to \mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}.$$

But by the proof of Lemma 7.1.6 and the vanishing Lemma 7.3.2, the complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A}_{\mathrm{inf}}[1/p]}$ lives only in degrees -1 and 0, and is isomorphic to the truncation of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}$ at $\tau^{\geq -1}$. So we reduce ourselves to considering the splitting of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}$.

Now assume $X_\bullet$ admits a flat lift $X'_\bullet$ over S'. The lift leads to the cartesian diagram

$$\begin{array}{ccc} X_\bullet & \longrightarrow & X'_\bullet \\ \downarrow & & \downarrow \\ S_\bullet & \longrightarrow & S'_\bullet \end{array}$$

which induces the simplicial version of the diagram $(*)$ in the proof of Proposition 7.1.4:

$$\begin{array}{ccccc} \mathbb{L}^{\mathrm{an}}_{X'_\bullet/S'} \otimes_{\mathcal{O}_{X'_\bullet}} \mathcal{O}_{X_\bullet} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X_\bullet/S} & \longrightarrow & E \\ \| & & \uparrow & & \uparrow \\ \mathbb{L}^{\mathrm{an}}_{X'_\bullet/S'} \otimes_{\mathcal{O}_{X'_\bullet}} \mathcal{O}_{X_\bullet} & \longrightarrow & \mathbb{L}^{\mathrm{an}}_{X_\bullet/S'} & \xrightarrow{\alpha_\bullet} & L^{\mathrm{an}}_{X_\bullet/X'_\bullet} \\ & & \uparrow & & \uparrow{\scriptstyle \beta_\bullet} \\ & & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet} & = & \mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet} \end{array}$$

The vanishing of E boils down to the vanishing of $E|_{X_n}$ by Lemma 7.3.2, which is true by assumption and Proposition 7.1.4. So we get a section map $\beta_\bullet^{-1} \circ \alpha_\bullet$, which splits $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S'}$ into the direct sum of $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S}$ and $\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet}$ in the derived category. Note that since $X_\bullet$ is smooth, the cotangent complex $\mathbb{L}^{\mathrm{an}}_{X_\bullet/S}$ is $\Omega^1_{X_\bullet/K}[0]$, while the truncation $\tau^{\geq -1}\mathbb{L}^{\mathrm{an}}_{S/S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{X_\bullet}$ is $\mathcal{O}_{X_\bullet}(1)[1]$. Thus we get the result.

Finally, the quasi-isomorphism is functorial with respect to $X'_\bullet$, since the big diagram above is functorial with respect to lifts, as in the proof of Proposition 7.1.4. ∎

We now try to connect the simplicial version of the cotangent complex with the derived direct image of the completed structure sheaves.

Let $X_\bullet$ be an (m-truncated) simplicial quasi-compact rigid space over K. It induces the following commutative diagram of topoi of simplicial sites, as a simplicial version of the diagram in Section 4:

$$\begin{array}{ccc} \mathrm{Sh}(X_{\bullet\mathrm{pro\acute{e}t}}) & \xrightarrow{\nu_\bullet} & \mathrm{Sh}(X_{\bullet\mathrm{\acute{e}t}}) \\ \uparrow{\scriptstyle\lambda_\bullet} & & \uparrow{\scriptstyle\pi_\bullet} \\ \mathrm{Sh}(\mathrm{Perf}_v|_{X_\bullet^\diamond}) & \xrightarrow{\alpha_\bullet} & \mathrm{Sh}(X_{\bullet\mathrm{\acute{e}h}}) \end{array}$$

We then define the complete pro-étale structure sheaf $\widehat{\mathcal{O}}_{X_\bullet}$ to the pro-étale simplicial site $X_{\bullet\mathrm{pro\acute{e}t}}$, by assigning $\widehat{\mathcal{O}}_{X_n}$ to the pro-étale site $X_{n\,\mathrm{pro\acute{e}t}}$. Similarly we define the untilted complete structure sheaf $\widehat{\mathcal{O}}_v$ on the site $\mathrm{Perf}_v|_{X_\bullet^\diamond}$. Here we notice that the sheaf $\widehat{\mathcal{O}}_v$ satisfies cohomological descent (§7.4) along the canonical map $\lambda_\bullet : \mathrm{Sh}(\mathrm{Perf}_v|_{X_\bullet^\diamond}) \to \mathrm{Sh}(X_{\bullet\mathrm{pro\acute{e}t}})$, by Lemma 7.3.3 and comparison results (Proposition 3.2.4). This leads to the equality

$$R\nu_{\bullet*}\widehat{\mathcal{O}}_{X_\bullet} = R\pi_{\bullet*}R\alpha_{\bullet*}\widehat{\mathcal{O}}_v.$$

The restriction of this equality to each X_n is the one in Section 4.

Define simplicial éh-sheaves $\Omega^i_{\mathrm{\acute{e}h}\bullet}$ of differentials on $X_{\bullet\mathrm{\acute{e}h}}$ such that on each $X_{n\,\mathrm{\acute{e}h}}$, the component of the sheaf is $\Omega^i_{\mathrm{\acute{e}h}}$. By the exactness of the restriction functor and the discussion in Section 4,

$$R^j\alpha_{\bullet*}\widehat{\mathcal{O}}_v = \Omega^j_{\bullet\mathrm{\acute{e}h}}(-j).$$

When X_n is smooth over K for each n, we have

$$R^j\nu_*\widehat{\mathcal{O}}_{X_\bullet} = \Omega^j_{X_\bullet/K}(-j)$$

with

$$R^i\pi_{\bullet*}R^j\alpha_{\bullet*} = \begin{cases} 0, & i > 0, \\ \Omega^j_{X_\bullet/K}(-j), & i = 0. \end{cases}$$

These are consequences of the éh-differentials for smooth rigid spaces (Theorem 4.0.2).

Proposition 7.3.5. *Let $X_\bullet$ be an m-truncated simplicial smooth quasi-compact rigid space over* $\mathrm{Spa}(K)$. *Then there exists a canonical quasi-isomorphism*

$$\tau^{\le 1}R\nu_{\bullet*}\widehat{\mathcal{O}}_{X_\bullet} \cong \mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A_{inf}}[1/p]}(-1)[-1].$$

The quasi-isomorphism is functorial with respect to $X_\bullet$.

Proof. The proposition is a simplicial version of Theorem 7.2.3.

We first notice that the map $\nu_\bullet : X_{\mathrm{pro\acute{e}t}} \to X_{\bullet\mathrm{\acute{e}t}}$ of simplicial sites induces a map of analytic cotangent complexes

$$\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A_{inf}}[1/p]} \to R\nu_{\bullet*}\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/\mathrm{A_{inf}}[1/p]}.$$

Meanwhile, the triple $\mathrm{A_{inf}}[1/p] \to K \to \widehat{\mathcal{O}}_{X_\bullet}$ provides a distinguished transitivity triangle

$$\mathbb{L}^{\mathrm{an}}_{K/\mathrm{A_{inf}}[1/p]} \otimes_K \widehat{\mathcal{O}}_{X_\bullet} \to \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/\mathrm{A_{inf}}[1/p]} \to \mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/K},$$

where the vanishing of $\mathbb{L}^{\mathrm{an}}_{\widehat{\mathcal{O}}_{X_\bullet}/K}$ follows from Lemma 7.3.2 and Step 3 of the proof of Theorem 7.2.3. So by combining the above two, we get the map

$$\Pi : \mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A_{inf}}[1/p]} \to R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}(1)[1].$$

Then we consider the induced map of the i-th cohomology sheaves $\mathcal{H}^i$. By the exactness of the restriction functor, the restricted map becomes

$$\mathcal{H}^i(\mathbb{L}^{\mathrm{an}}_{X_n/\mathrm{A_{inf}}[1/p]}) \to \mathcal{H}^i(R\nu_{n*}\widehat{\mathcal{O}}_{X_n}(1)[1]),$$

which is an isomorphism for $i = 0, -1$ by Theorem 7.2.3, and $\mathcal{H}^i(\mathbb{L}^{\mathrm{an}}_{X_n/\mathrm{A_{inf}}[1/p]})$ is zero except $i = 0, -1$. So by the vanishing of the cone, Π induces a quasi-isomorphism

$$\mathbb{L}^{\mathrm{an}}_{X_\bullet/\mathrm{A_{inf}}[1/p]} \to \tau^{\geq -1}(R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}(1)[1]),$$

which leads to the result by a twist. ∎

Combining Propositions 7.3.5 and 7.3.4, we get a simplicial version of the splitting for the truncated derived direct image:

Corollary 7.3.6. *Assume $X_\bullet$ is an m-truncated smooth quasi-compact simplicial rigid space over K, which admits a flat simplicial lift $X'_\bullet$ to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Then the lift $X'_\bullet$ induces a splitting of $\tau^{\leq 1}R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}$ into the direct sum of its cohomology sheaves $\mathcal{O}_{X_\bullet}[0] \oplus \Omega^1_{X_\bullet/K}(-1)[-1]$ in $D(X_\bullet)$. The splitting is functorial with respect to the lift $X'_\bullet$.*

Moreover, similar to Proposition 7.2.5, the splitting can be extended without derived truncations.

Corollary 7.3.7. *Assume $X_\bullet$ is an m-truncated smooth quasi-compact simplicial rigid space over K that admits a flat lift $X'_\bullet$ to $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Then the lift $X'_\bullet$ induces a splitting of the derived direct image $R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}$ into*

$$\bigoplus_{i\geq 0} \Omega^i_{X_\bullet/K}(-i)[-i]$$

in the derived category, which is also isomorphic to

$$\bigoplus_{i\geq 0} R\pi_{X_\bullet *}(\Omega^i_{\bullet \mathrm{\acute{e}h}}(-i)[-i]).$$

Proof. By Corollary 7.3.6, we have a quasi-isomorphism

$$\mathcal{O}_{X_\bullet}[0] \oplus \Omega^1_{X_\bullet/K}(-1)[-1] \to \tau^{\leq 1}R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}.$$

Then by composing with $\tau^{\leq 1}R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet} \to R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}$, similar to the proof of Proposition 7.2.5 we may construct the map

$$\bigoplus_{i\geq 0} \Omega^i_{X_\bullet/K}(-i)[-i] \to R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet},$$

whose restriction to each X_n is exactly the quasi-isomorphism in Proposition 7.2.5. Thus by the vanishing of the restriction of the cone, we see the above map is a quasi-isomorphism.

Finally, by the smoothness of $X_\bullet$, we get the second direct sum expression. ■

7.4. Degeneracy in general

We now generalize the splitting of the derived direct image to the general case, without assuming smoothness. Our main tools are cohomological descent and the simplicial generalizations from the last subsection.

Strong liftability. Before we prove the general degeneracy, we need to introduce a stronger version of liftability, in order to make use of cohomological descent.

We first give the definition.

Definition 7.4.1. Let X be a quasi-compact rigid space over K. We say X is *strongly liftable* if for each non-negative integer n, there exists an n-truncated augmented simplicial map $X'_{\leq n} \to X'$ of adic spaces over $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$, where X' and each X'_i are flat and topologically of finite type over $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$, such that the pullback along $\mathrm{B}_{\mathrm{dR}}^+/\xi^2 \to K$ induces an n-truncated smooth éh-hypercovering of X over K.

We call any such augmented simplicial map $X'_{\leq n} \to X'$ of rigid spaces (or $X'_{\leq n}$ for short) a *strong lift of length n*.

Example 7.4.2. Let $k = \mathcal{O}_K/\mathfrak{m}_K$ be the residue field of $\mathcal{O}_K$, and fix a section $i : k \to \mathcal{O}_K/p$ for the canonical surjection $\mathcal{O}_K/p \to k$ (whose existence is guaranteed by the formal smoothness of the perfect field k over $\mathbb{F}_p$ [33, Tag 031Z]). This induces an injection of fields from $W(k)[1/p]$ to K by the universal property of the Witt ring. Let K_0 be a subfield of K that is finite over $W(k)[1/p]$, and let X be a rigid space defined over K_0. We then claim that X is strongly liftable.

To see this, we first notice that as the resolution of singularities holds for rigid spaces over K_0, it suffices to show that any such field K_0 admits a map $K_0 \to \mathrm{B}_{\mathrm{dR}}^+/\xi^2$ compatible with the inclusion $K_0 \to K$ above. Recall that the ring $\mathrm{A}_{\mathrm{inf}}$ is defined as $W(\mathcal{O}_{K^\flat})$, where $\mathcal{O}_K^\flat$ is the inverse limit $\varprojlim_{x \mapsto x^p} \mathcal{O}_K/p$. By the construction of $\mathcal{O}_{K^\flat}$ and the functoriality of the inverse limit and of Frobenius maps, the section $i : k \to \mathcal{O}_K/p$ induces a homomorphism $k \to \mathcal{O}_{K^\flat}$, which is a section of the canonical surjection $\mathcal{O}_{K^\flat} \to k$. In this way, thanks to the functoriality of the Witt vector functor, we can lift the section map to $W(k) \to \mathrm{A}_{\mathrm{inf}} = W(\mathcal{O}_{K^\flat})$. As an upshot, we get the composition

$$W(k)[1/p] \to \mathrm{A}_{\mathrm{inf}}[1/p] \to \mathrm{A}_{\mathrm{inf}}[1/p]/\xi^2 = \mathrm{B}_{\mathrm{dR}}^+/\xi^2,$$

which lifts the map $W(k)[1/p] \to K$ that we started with. Finally, any finite field extension K_0 of $W(k)[1/p]$ is étale over $W(k)[1/p]$, while $\mathrm{B}_{\mathrm{dR}}^+/\xi^2 \to \mathcal{O}_K$ is a nilpotent extension of $W(k)[1/p]$-algebras. Hence K_0 admits a map to $\mathrm{B}_{\mathrm{dR}}^+/\xi^2$ by étaleness.

Here we also note that this implies strong liftability when X is defined over a discretely valued subfield $L_0 \subset K$ that has a perfect residue subfield $k' \subset k$, since any such L_0 is finite over $W(k')[1/p]$, while the latter is contained in $W(k)[1/p]$.

Example 7.4.3. Another example is the analytification of a finite type algebraic variety, by the spreading-out technique.

Let Y be a finitely presented scheme over K. By [17, §8.9.1], there exists a finitely generated $\mathbb{Q}$-subalgebra A in K together with a finitely presented A-scheme Y_0 such that $Y_0 \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) = Y$. As the map $A \to K$ factors through the fraction field of A, we may assume A is a finitely generated field extension of $\mathbb{Q}$ and Y_0 is defined over A. Notice that the transcendental degree of $\mathbb{Q}_p$ over $\mathbb{Q}$ is infinite. So by embedding a transcendental basis of A over $\mathbb{Q}$ into $\mathbb{Q}_p$, we may find a finite extension K_0 of $\mathbb{Q}_p$ such that A can be embedded into K_0. In this way, we reduce the case to Example 7.4.2, as Y^{an} can be defined over a discretely valued subfield K_0 of K that has a perfect residue field.

By the upcoming work on spreading-out of rigid spaces by Conrad–Gabber [9], it turns out that X is strongly liftable if it is a proper rigid space over K.

Proposition 7.4.4. *Let X be a proper rigid space over K. Then X is strongly liftable.*

Proof. We follow the spreading-out technique for rigid spaces by Bhatt–Morrow–Scholze [3] and study the structure of the deformation ring. However, instead of working on one rigid space, we need to work with a finite diagram of proper rigid spaces. Similar to Example 7.4.2, we fix a section $i : k = \mathcal{O}_K/\mathfrak{m}_K \to \mathcal{O}_K/p$ of the canonical surjection $\mathcal{O}_K/p \to k$, which induces an inclusion of p-adic fields $W(k)[1/p] \to K$.

Let n be any non-negative integer. By the resolution of singularities (Theorem 2.4.7), we can always construct an n-truncated smooth éh-hypercovering $X_{\leq n} \to X$ over K, where each X_i is proper over K [7, Section 4]. Then it suffices to show that there exists a proper éh-hypercovering $X_{\leq n} \to X$, together with a smooth rigid space $\mathcal{S}$ over a subfield $K_0 = W(k)[1/p]$ of K, such that the n-truncated simplicial diagram $X_{\leq n} \to X$ can be lifted to a diagram of proper K_0-rigid spaces $\mathcal{X}_{\leq n} \to \mathcal{X}$ over $\mathcal{S}$. This is because the nilpotent extension $\mathrm{B}_{\mathrm{dR}}^+/\xi^2 \to K$ is K_0-linear, so by the smoothness of $\mathcal{S}$, the map $\mathrm{Spa}(K) \to \mathcal{S}$ can be lifted to a map $\mathrm{Spa}(\mathrm{B}_{\mathrm{dR}}^+/\xi^2) \to \mathcal{S}$. Thus the base change of $\mathcal{X}_{\leq n}$ along this lifting does the job.

Now we prove the statement, imitating [3, proof of Proposition 13.15 and Corollary 13.16]. We first deal with the formal lifting over the integral base. Let $W = W(k)$ be the ring of Witt vectors for the residue field of K, and $\mathscr{C}_W$ be the category of artinian, complete local W-rings with the same residue field k. We first make the following claim:

Claim 7.4.5. *There exists an n-truncated smooth éh-hypercovering $X_{\leq n} \to X$ over K that admits a lift to an n-truncated simplicial diagram of p-adically complete, topologically finite type $\mathcal{O}_K$-formal schemes*

$$X_{\leq n}^+ \to X^+.$$

Proof. Fix an $\mathcal{O}_K$-integral model X^+ of X, whose existence is guaranteed by Raynaud's result on the relation between rigid spaces over K and admissible formal schemes over $\mathcal{O}_K$. We now construct inductively the required covering and the integral lift, following the idea of split hypercoverings (see [7, Section 4], or [33, Tag 094J] for discussions).

By the local smoothness of the éh-topology (Corollary 2.4.8), pick a smooth éh-covering $X_0 \to X$ that is proper over X. By Raynaud's result, there exists a morphism $X_0^+ \to X^+$ of $\mathcal{O}_K$-formal schemes that lifts $X_0 \to X$. This is the lift of the face map of the simplicial object at degree 0.

Assume we already have an n-truncated smooth proper éh-hypercovering $X_{\leq n} \to X$ together with the integral lift $X_{\leq n}^+ \to X^+$ over $\mathcal{O}_K$. Then recall from [7, Theorems 4.12, 4.14] that in order to extend $X_{\leq n} \to X$ to a smooth proper $n+1$-truncated hypercovering whose n-truncation is the same as $X_{\leq n}$, it is equivalent to find a smooth proper éh-covering of rigid spaces

$$N \to (\operatorname{cosk}_n X_{\leq n})_{n+1}.$$

Under the construction, the degree $n+1$ term of the resulting $n+1$-truncated hypercovering will be

$$X_{n+1} := N \amalg N',$$

for N' being some finite disjoint union of irreducible components of X_i $(0 \leq i \leq n)$ (which is also smooth and proper over K). Such a smooth proper éh-covering N exists by the local smoothness of $X_{\text{éh}}$. Furthermore, while we form this $n+1$-hypercovering of the rigid spaces, we also want to find the integral lift

$$N^+ \to (\operatorname{cosk}_n X_{\leq n}^+)_{n+1}$$

of the morphism $N \to (\operatorname{cosk}_n X_{\leq n})_{n+1}$. To do this, we use [7, Theorem 4.12] and do the same formal construction for N^+ and $X_{\leq n}^+ \to X^+$ as above, and extend the latter to an $n+1$-truncated simplicial formal schemes

$$X_{\leq n+1}^+ \to X^+,$$

where $X_{n+1}^+ = N^+ \amalg N'^+$ is an $\mathcal{O}_K$-model of X_{n+1}. In this way, the generic fiber of $X_{\leq n+1}^+ \to X^+$ is an $n+1$-simplicial object over X whose n-truncation is $X_{\leq n} \to X$, and whose $(n+1)$-th term is $X_{n+1} = N \amalg N'$, which is in fact a smooth proper éh-covering of $(\operatorname{cosk}_n X_{\leq n})_{n+1}$. Hence by the induction hypothesis we are done. ∎

We then fix such an éh-hypercovering $X_{\leq n} \to X$ with its integral model $X_{\leq n}^+ \to X^+$ as in the claim. Define the functor of deformations of the special fiber $X_{\leq n,k}^+ := X_{\leq n}^+ \times_W k$,

$$\mathit{Def} : \mathscr{C}_W \to \mathbf{Set},$$

which assigns to each $R \in \mathscr{C}_W$ the isomorphism classes of lifts of the digrams $X_{\leq n,k}^+ \to X_k^+$ along $R \to k$, such that each lifted rigid space is proper and flat over R. This functor is a deformation functor, and admits a versal deformation: to see this, we first note that as in [33, Tag 0E3U], the functor *Def* satisfies the Rim–Schlessinger condition [33, Tag 06J2]. Then we make the following claim:

Claim 7.4.6. *The tangent space $TDef := Def(k[\epsilon]/\epsilon^2)$ of the deformation functor is of finite dimension.*

Proof of Claim. Notice that there is a natural (forgetful) functor from Def to the deformation functor of the morphism $Def_{Y_s \to Y_t}$, where Y_s is the disjoint union of all the sources of arrows in the diagram $X^+_{\leq n,k}$, and Y_t is the disjoint union of all of those targets. This induces a map between tangent spaces

$$TDef \to TDef_{Y_s \to Y_t}.$$

By construction, both Y_s and Y_t are finite disjoint unions of proper rigid spaces, which are then proper over k. So from [33, Tag 0E3W], we know the tangent space $TDef_{Y_s \to Y_t}$ is finite-dimensional. Furthermore, assume D_1 and D_2 are two lifted diagrams over $k[\epsilon]$. Then the difference of D_1 and D_2 is the collection of k-derivations $\mathcal{O}_{X^+_{t(\alpha),k}} \to \alpha_* \mathcal{O}_{X^+_{s(\alpha),k}}$ satisfying certain k-linear relations so that those arrows in D_1 and D_2 commute. In particular, this consists of a subspace of $\mathrm{Der}_k(\mathcal{O}_{Y_t}, u_*\mathcal{O}_{Y_s}) = \mathrm{Hom}_{Y_t}(\Omega^1_{Y_t/k}, u_*\mathcal{O}_{Y_s})$, which by properness is again finite-dimensional. In this way, both the kernel and the target of the map $TDef \to TDef_{Y_s \to Y_t}$ are of finite dimension, thus so is the $TDef$. ■

By the above claim and [33, Tag 06IW], the deformation functor Def admits a versal object. In other words, there is a complete artinian local W-algebra R with residue field k, and a diagram $\mathcal{X}_{R,\leq n} \to \mathcal{X}_R$ of proper flat formal R-schemes deforming $X^+_{\leq n,k} \to X^+_k$, such that the induced classifying map

$$h_R := \mathrm{Hom}_W(R, -) \to Def$$

is formally smooth. Moreover, by [3, proof of Proposition 13.15], we can take the ind-completion of $\mathscr{C}_W$ and extend Def to a bigger category, which consists of local zero-dimensional W-algebras with residue field k (not necessarily noetherian). The category includes $\mathcal{O}_K/p^m$, and since $X^+_{\leq n} \to X^+$ is an $\mathcal{O}_K$-lifting of $X^+_{\leq n,k} \to X^+_k$, we see the diagram can be obtained by the base change of the universal family $\mathcal{X}_{R,\leq n} \to \mathcal{X}_R$ along $R \to \mathcal{O}_K = \varprojlim_m \mathcal{O}_K/p^m$.

Finally, we invert p at the diagram

$$\begin{array}{ccc} X^+_{\leq n} & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) \\ \downarrow & & \downarrow \\ \mathcal{X}_{R,\leq n} & \longrightarrow & \mathrm{Spf}(R) \end{array}$$

The diagram $X_{\leq n} \to X$ can then be obtained from a truncated simplicial diagram $\mathcal{X}_{R,\leq n}[1/p] \to \mathcal{X}[1/p]$ of proper K_0-rigid spaces that are flat over $\mathcal{S} = \mathrm{Spa}(R[1/p], R)$. By shrinking S to a suitable locally closed subset if necessary, we may assume S is smooth over K_0. So we are done. ■

Cohomological descent. Another preparation we need is hypercoverings and cohomological descent.

Assume we have a non-augmented simplicial site $Y_\bullet$ (truncated or not) and another site S. Let $\{a_n : Y_n \to S\}$ for $n \geq 0$ be a collection of morphisms to S that is compatible with face maps and degeneracy maps in $Y_\bullet$. Then we can define the *augmentation* morphism $a : \mathrm{Sh}(Y_\bullet) \to \mathrm{Sh}(S)$ between the topoi of $Y_\bullet$ and S, such that for an abelian sheaf $\mathcal{F}^\bullet$ on $Y_\bullet$, we have

$$a_*\mathcal{F}^\bullet = \ker(a_{0*}\mathcal{F}^0 \rightrightarrows a_{1*}\mathcal{F}^1).$$

It can be checked that the derived direct image Ra_* can be written as the composition

$$Ra_* = \mathbf{s} \circ Ra_{\bullet *},$$

where $a_\bullet : Y_\bullet \to S_\bullet$ is the morphism from $Y_\bullet$ to the constant simplicial site $S_\bullet$ associated to S, and $\mathbf{s}$ is the exact functor that takes a simplicial complex to its associated cochain complex of abelian groups. Here we say that the augmentation $a = \{Y_n \to S\}$ is of *cohomological descent* if the counit map induced by the adjoint pair (a^{-1}, a_*) is a quasi-isomorphism

$$\mathrm{id} \to Ra_*a^{-1}.$$

The augmentation allows us to compute the cohomology of sheaves on S by the spectral sequence associated to the simplicial site.

Lemma 7.4.7 ([33, Tag 0D7A]). *Let $Y_\bullet$ be a simplicial site, or an m-truncated simplicial site for $m \geq 0$, and let $a = \{a_n : Y_n \to S\}$ be an augmentation. Then for $K \in D^+(Y_\bullet)$, there exists a natural spectral sequence*

$$E_1^{p,q} = R^q a_{p*}(K|_{X_p}) \Longrightarrow R^{p+q}a_*K,$$

which is functorial with respect to $Y_\bullet \to S$ and K.

Moreover, if we assume that $Y_\bullet$ is non-truncated and the augmentation a is of cohomological descent, and if $L \in D^+(S)$, then by applying the spectral sequence to $K = a^{-1}L$ we get a natural spectral sequence

$$E_1^{p,q} = R^q a_{p*}a_p^{-1}L \Longrightarrow \mathcal{H}^{p+q}(L).$$

We need another variant of this lemma in order to use truncated hypercoverings to approximate the cohomology of S.

Proposition 7.4.8. *Let $\rho : Y_{\leq m} \to S$ be an m-truncated simplicial hypercovering of sites for $m \in \mathbb{N}$. Then for any $\mathcal{F} \in \mathrm{Ab}(S)$, the cone of the natural adjunction map*

$$\mathcal{F} \to R\rho_*\rho^{-1}\mathcal{F}$$

lives in cohomological degrees $\geq m - 1$.

Proof. Let $\widetilde{\rho} : \operatorname{cosk}_m Y_{\leq m} \to S$ be the m-th coskeleton of $\rho : Y_{\leq m} \to S$. We use the same symbols $\operatorname{cosk}_m Y_{\leq m}$ and $Y_{\leq m}$ to denote their associated simplicial sites. Then there exists a natural map of sites

$$\iota : \operatorname{cosk}_m Y_{\leq m} \to Y_{\leq m}.$$

Those two augmentations induce maps of topoi

$$\widetilde{\rho} : \operatorname{Sh}(\operatorname{cosk}_m Y_{\leq m}) \to \operatorname{Sh}(S), \quad \rho : \operatorname{Sh}(Y_{\leq m}) \to \operatorname{Sh}(S).$$

By construction, we have

$$\widetilde{\rho} = \iota \circ \rho$$

as maps of topoi. From this, for $\mathcal{F} \in \operatorname{Ab}(S)$, we get the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & R\widetilde{\rho}_* \widetilde{\rho}^{-1}\mathcal{F} \\ & \searrow \quad \nearrow & \\ & R\rho_*\rho^{-1}\mathcal{F} & \end{array}$$

Now let $\mathcal{C}$ be a cone of $\mathcal{F} \to R\rho_*\rho^{-1}\mathcal{F}$, and let $\widetilde{\mathcal{C}}$ be a cone of $\mathcal{F} \to R\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F}$. Then the above diagram induces a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathcal{H}^i(\mathcal{F}) & \longrightarrow & R^i\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} & \longrightarrow & \mathcal{H}^i(\widetilde{\mathcal{C}}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{H}^i(\mathcal{F}) & \longrightarrow & R^i\rho_*\rho^{-1}\mathcal{F} & \longrightarrow & \mathcal{H}^i(\mathcal{C}) & \longrightarrow & \cdots \end{array}$$

By Lemma 7.4.7 and the commutative diagram, we have a map of the first pages of spectral sequences

$$\begin{array}{ccc} R^q\widetilde{\rho}_{p*}\widetilde{\rho}_p^{-1}\mathcal{F} & \Longrightarrow & R^{p+q}\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} \\ \downarrow & & \downarrow \\ R^q\rho_{p*}\rho_p^{-1}\mathcal{F} & \Longrightarrow & R^{p+q}\rho_*\rho^{-1}\mathcal{F} \end{array}$$

But note that since $\widetilde{\rho}$ is the m-coskeleton of ρ, when $p + q \leq m$ the formation $R^q\widetilde{\rho}_{p*}\widetilde{\rho}_p^{-1}$ is the same as $R^q\rho_{p*}\rho_p^{-1}$. So we get the isomorphism

$$R^{p+q}\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F} \cong R^{p+q}\rho_*\rho^{-1}\mathcal{F}, \quad p + q \leq m.$$

Moreover, since ρ is an m-truncated hypercovering, by Deligne the augmentation $\widetilde{\rho}$ of the coskeleton satisfies cohomological descent. So the map $\mathcal{F} \to R\widetilde{\rho}_*\widetilde{\rho}^{-1}\mathcal{F}$ is a quasi-isomorphism. In this way, the map $\mathcal{H}^i(\mathcal{F}) \to R^i\rho_*\rho^{-1}\mathcal{F}$ is an isomorphism for $i \leq m$, and hence $\mathcal{C}$ lives in $D^{\geq m-1}(S)$. ∎

The degeneracy theorem. Now we are able to state and prove our main theorem about degeneracy.

Theorem 7.4.9. *Let X be a quasi-compact, strongly liftable rigid space of dimension n over K, and let the augmented truncated simplicial space $X'_{\le m}$ be a strong lift of X of length $m \ge 2n+2$. Then the strong lift $X'_{\le m}$ induces a quasi-isomorphism*

$$\Pi_{X'_{\le m}} : R\nu_* \widehat{\mathcal{O}}_X \to \bigoplus_{i=0}^{n} R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]).$$

The quasi-isomorphism $\Pi_{X'_{\le m}}$ is functorial among strong lifts $X'_{\le m}$ of rigid spaces of length $m \ge 2n+2$, in the sense that a map of m-truncated strong lifts $X'_{\le m} \to Y'_{\le m}$ of $f : X \to Y$ will induce the following commutative diagram in the derived category:

$$\begin{array}{ccc} Rf_* R\nu_* \widehat{\mathcal{O}}_X & \xrightarrow{Rf_*(\Pi_{X'_{\le m}})} & \bigoplus_{i=0}^{\dim(X)} Rf_* R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]) \\ \uparrow & & \uparrow \\ R\nu_* \widehat{\mathcal{O}}_Y & \xrightarrow{\Pi_{Y'_{\le m}}} & \bigoplus_{i=0}^{\dim(Y)} R\pi_{Y*}(\Omega^i_{\text{éh}}(-i)[-i]) \end{array}$$

where the right vertical map is induced by the functoriality of the Kähler differential.

Proof. We may assume $X_{\le m}$ is an m-truncated smooth proper éh-hypercovering of X that admits a lift $X'_{\le m}$ to a simplicial flat adic space over $\mathrm{B}^+_{\mathrm{dR}}/\xi^2$. Denote by $\rho : X_\bullet \to X$ the augmentation map. Then $X_{\le m}$ is also an m-truncated v-hypercovering, and we have a natural map

$$\widehat{\mathcal{O}}_v \to R\rho_{v*}\rho^{-1}\widehat{\mathcal{O}}_v \cong R\rho_{v*}\widehat{\mathcal{O}}_{\bullet v},$$

whose cone has cohomological degree $m-1 \ge 2n+1$ by Proposition 7.4.8.

We then apply derived direct image functors, and get a natural map

$$R\nu_*\widehat{\mathcal{O}}_X = R\pi_{X*}R\alpha_*\widehat{\mathcal{O}}_v \to R\pi_{X*}R\alpha_*R\rho_{v*}\widehat{\mathcal{O}}_{\bullet v} = R\rho_*R\pi_{\bullet *}R\alpha_{\bullet *}\widehat{\mathcal{O}}_{\bullet v} = R\rho_*R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet}.$$

Here the cone of the map lives in degree $\ge m-1 \ge 2n+1$.

Moreover, by Corollary 7.3.7, the strong lift $X'_{\le m}$ induces a functorial (among strong lifts) quasi-isomorphism

$$R\nu_{\bullet *}\widehat{\mathcal{O}}_{X_\bullet} \to \bigoplus_{i\ge 0} R\pi_{\bullet *}(\Omega^i_{\bullet\text{éh}}(-i)[-i]).$$

So we get the distinguished triangle

$$R\nu_*\widehat{\mathcal{O}}_X \to R\rho_*R\pi_{\bullet *}(\Omega^i_{\bullet\text{éh}}(-i)[-i]) \to \mathscr{C}_1, \tag{1}$$

where $\mathscr{C}_1 \in D^{\ge 2n+1}(X)$.

Moreover, by Corollary 7.3.7 and Proposition 7.4.8 again the truncated éh-hypercovering ρ induces a natural map

$$\begin{aligned}\bigoplus_{i\geq 0} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i]) \to \bigoplus_{i\geq 0} R\pi_*(R\rho_{\text{éh}*}\rho_{\text{éh}}^{-1}\Omega^i_{\text{éh}}(-i)[-i]) \\ = \bigoplus_{i\geq 0} R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet\text{éh}}(-i)[-i]),\end{aligned} \tag{2}$$

whose cone $\mathcal{C}_2$ lives in degrees $\geq m-1 \geq 2n+1$.

Finally, by combining (1) and (2), we get the following diagram that is functorial with respect to $X'_{\leq m}$, with both horizontal and vertical triangles distinguished:

$$\begin{array}{ccccc} & & \mathcal{C}_2 & & \\ & & \uparrow & & \\ R\nu_*\widehat{\mathcal{O}}_X & \longrightarrow & R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet\text{éh}}(-i)[-i]) & \longrightarrow & \mathcal{C}_1 \\ & & \uparrow & & \\ & & \bigoplus_{i\geq 0} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i]) & & \end{array}$$

But since $\dim(X) = n$, by the cohomological boundedness (Corollaries 6.0.4 and 6.0.5), both $R\nu_*\widehat{\mathcal{O}}_X$ and $\bigoplus_{i\geq 0} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i])$ live in degrees $\leq 2n$. Thus by taking the truncation $\tau^{\leq 2n}$, we get the quasi-isomorphism

$$R\nu_*\widehat{\mathcal{O}}_X \xrightarrow{\sim} \tau^{\leq 2n}(R\rho_* R\pi_{\bullet *}(\Omega^i_{\bullet\text{éh}}(-i)[-i])) \xleftarrow{\sim} \bigoplus_{i\geq 0}^{n} R\pi_*(\Omega^i_{\text{éh}}(-i)[-i]). \tag{3}$$

In this way, by taking $\Pi_{X'_{\leq m}}$ to be the quasi-isomorphism induced from (3), we are done. ■

Corollary 7.4.10. *Assume X is a quasi-compact rigid space over K that is either defined over a discretely valued subfield K_0 of a perfect residue field, or proper over K. Then we have a non-canonical decomposition*

$$R\nu_*\widehat{\mathcal{O}}_X \cong \bigoplus_{i=0}^{\dim(X)} R\pi_{X*}(\Omega^i_{\text{éh}}(-i)[-i]).$$

In particular, the éh-proét spectral sequence (Theorem 1.2.1) degenerates at the E_2-page.

7.5. Finiteness revisited

In this subsection, we use the degeneracy of the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ to improve the cohomological boundedness results of Section 6.

We first recall some recent work on perfection and almost purity in [4].

Theorem 7.5.1 ([4, Proposition 8.5, Theorem 10.9]). *Let A be a perfectoid ring, and B a finitely presented finite A-algebra such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is finite étale over an open subset. Then there exists a perfectoid ring B_{perfd} together with a map $B \to B_{\mathrm{perfd}}$ of A-algebras that is initial among all A-algebra maps $B \to B'$ for B' being perfectoid.*

Proposition 7.5.2. *Let X be a rigid space over K. Then $R^n\nu_*\widehat{\mathcal{O}}_X$ vanishes for all* $n > \dim(X)$.

We mention that the proof will not need the éh-proét spectral sequence developed above.

Proof. Since this is an étale local statement, and any étale covering of X does not change dimension, by passing from X to its open subsets if necessary, we may assume X admits a finite surjective map onto a torus of the same dimension.[9]

We introduce some notations. Denote by $X = \mathrm{Spa}(R, R^+)$ an affinoid rigid space over $\mathrm{Spa}(K)$. Assume there exists a finite surjective map $X \to \mathbb{T}_n = \mathrm{Spa}(K\langle T_i\rangle, \mathcal{O}_K\langle T_i\rangle)$ onto the torus of dimension n. Let $\mathbb{T}_n^\infty$ be the natural pro-étale cover of $\mathbb{T}_n$ by extracting all p^n-th roots of T_i, and let $\hat{\mathbb{T}}_n^\infty = \mathrm{Spa}(K\langle T_i^{1/p^\infty}\rangle, \mathcal{O}_K\langle T_i^{1/p^\infty}\rangle)$ be the underlying affinoid perfectoid space. Then the base change of $\mathbb{T}_n^\infty$ along the map $X \to \mathbb{T}_n$ produces a pro-étale cover $X^\infty \to X$ of X. Note that $\mathbb{T}_n^\infty \to \mathbb{T}_n$ is a $\mathbb{Z}_p(1)^n$-torsor, so we have

$$R\Gamma(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X) = R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^n, R\Gamma_{\mathrm{proét}}(X^\infty, \widehat{\mathcal{O}}_X)).$$

Thanks to the (pro-étale)-v comparison (Proposition 3.2.4), the above is given by

$$R\Gamma(X_{\mathrm{proét}}, \widehat{\mathcal{O}}_X) = R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^n, R\Gamma_v(\hat{X}^{\infty,\diamond}, \widehat{\mathcal{O}}_v)),$$

where $\hat{X}^{\infty,\diamond}$ is the small v-sheaf associated to the analytic adic space $\hat{X}^\infty$ as in Proposition 3.1.4. Here we note that since $\hat{\mathbb{T}}_n^\infty$, $\mathbb{T}_n$, and X are all affinoid, we can write $\hat{X}^\infty$ as $\mathrm{Spa}(B[1/p], B)$ for some p-adic complete $\mathcal{O}_K$-algebra B.

We then recall that for a perfectoid space Y of characteristic p with a structure map to the v-sheaf $\mathrm{Spd}(K)$, and any K-analytic adic space Z, we have the bijection [32, Proposition 10.2.3]

$$\mathrm{Hom}_{\mathrm{Spa}(K)}(Y^\sharp, Z) = \mathrm{Hom}_{\mathrm{Spd}(K)}(Y, Z^\diamond),$$

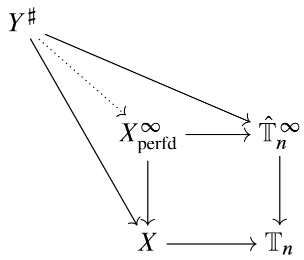
where $Y^\sharp$ is the unique untilt (as a perfectoid space over $\mathrm{Spa}(K)$) of Y associated to the structure map $Y \to \mathrm{Spd}(K)$ (Example 3.1.3). The bijection implies that as a v-sheaf over Perf_v, the small v-sheaf $\hat{X}^{\infty,\diamond}$ associated to the adic space $\hat{X}^\infty$ is the pullback of

[9] To see the existence of such surjections, we may argue as follows: as a unit disc is covered by finitely many tori, it suffices to find a finite map from an affinoid rigid space $X = \mathrm{Spa}(A, A^+)$ onto a unit disc of the same dimension. Let A_0 be a ring of definition of (A, A^+) that is of topologically finite type over $\mathcal{O}_K$. Since $A_0/\mathfrak{m}_K$ is a finite type algebra over the residue field $k = \mathcal{O}_K/\mathfrak{m}_K$, by Noether's normalization lemma we can find a subalgebra $k[x_i]$ of $A_0/\mathfrak{m}_K$ such that $A_0/\mathfrak{m}_K$ is finite over $k[x_i]$. In this way, by lifting the map to a morphism $\mathcal{O}_K\langle x_i\rangle \to A_0$, we get a finite surjective morphism from X to a disc.

the representable v-sheaf $\hat{\mathbb{T}}_n^{\infty,\flat}$ along the map $X^\diamond \to \mathbb{T}_n^\diamond$. On the other hand, given a perfectoid space Y over $\mathrm{Spd}(K)$ together with the commutative map

$$\begin{array}{ccc} Y^\sharp & \longrightarrow & \hat{\mathbb{T}}_n^\infty \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{T}_n \end{array}$$

since $X \to \mathbb{T}_n$ is finite surjective of the same dimensions, Theorem 7.5.1 implies that there exists a unique map of adic spaces $Y^\sharp \to X_{\mathrm{perfd}}^\infty = \mathrm{Spa}(B_{\mathrm{perfd}}[1/p], B_{\mathrm{perfd}})$ that fits into the commutative diagram



Comparing the pullback $\hat{X}^{\infty,\diamond}$ with the universal affinoid perfectoid space $X_{\mathrm{perfd}}^\infty$, we see that the v-sheaf $\hat{X}^{\infty,\diamond}$ is isomorphic to the representable v-sheaf $X_{\mathrm{perfd}}^{\infty,\flat}$, given by the tilt of the perfectoid space $X_{\mathrm{perfd}}^\infty$. In particular,

$$R\Gamma_v(\hat{X}^{\infty,\diamond}, \widehat{\mathcal{O}}_v) = R\Gamma_v(X_{\mathrm{perfd}}^{\infty,\flat}, \widehat{\mathcal{O}}_v).$$

Since the higher v (pro-étale) cohomology of the completed structure sheaf on an affinoid perfectoid space vanishes, by combining the equalities above we get

$$R\Gamma(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}) = R\Gamma_{\mathrm{cont}}(\mathbb{Z}_p(1)^n, B_{\mathrm{perfd}}[1/p]).$$

Finally, we note that the above object lives in cohomological degrees $[0, n]$ in the derived category of abelian groups, for the continuous group cohomology of $\mathbb{Z}_p(1)^n$ can be computed by the Koszul complex of length n [3, §7]. Thus we are done. ■

Remark 7.5.3. We mention that the cohomological bound given here is stronger than the one in Corollary 6.0.5.

Remark 7.5.4. In the proof above, the continuous group cohomology computing $R\Gamma(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X)$ can be defined concretely as

$$(R \varprojlim_m R\Gamma_{\mathrm{disc}}(\mathbb{Z}^n, R\Gamma(X_{\mathrm{pro\acute{e}t}}^\infty, \widehat{\mathcal{O}}_X^+/p^m)))[1/p],$$

where $R\Gamma_{\mathrm{disc}}(\mathbb{Z}^n, -)$ denotes the discrete group cohomology of $\mathbb{Z}^n$. Indeed, as X is affinoid (thus quasi-compact and quasi-separated), it follows that $R\Gamma(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X) = (R\varprojlim_m R\Gamma(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X^+/p^m))[1/p]$. Moreover, to compute each $R\Gamma(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}_X^+/p^m)$ we can use the Čech complex of $\widehat{\mathcal{O}}_X^+/p^m$ for the pro-étale covering $X^\infty \to X$. Finally, as the covering is a $\mathbb{Z}_p(1)^n$-torsor, the Čech complex is equivalent to the discrete group cohomology $R\Gamma_{\mathrm{disc}}(\mathbb{Z}^n, R\Gamma(X^\infty_{\text{proét}}, \widehat{\mathcal{O}}_X^+/p^m))$, by the isomorphism in [3, Lemma 7.3] for $\Gamma = \mathbb{Z}_p(1)^n$.

Definition 7.5.5. Let X be a rigid space over K. We say X *is locally compactifiable* if there exists an open covering $\{U_i \to X\}_i$ of X such that each U_i admits an open immersion into a proper rigid space Y_i over K.

By definition, any proper rigid space over K is locally compactifiable. Moreover, by Nagata's compactification in algebraic geometry, any finite type scheme over K admits an open immersion in a proper scheme over K. So the analytification of any finite type scheme over K is a locally compactifiable rigid space.

Proposition 7.5.6. *Let X be a locally compactifiable rigid space over K. Then the higher direct image $R^i\pi_{X*}\Omega^j_{\text{éh}}$ vanishes when $i + j > \dim(X)$.*

Proof. Since the vanishing of the higher direct image is a local statement, by taking an open covering, it suffices to assume X admits an open immersion $f : X \to X'$ with X' proper over K. Moreover, by dropping the irreducible components of X' that have higher dimensions, we may assume $\dim(X')$ is the same as $\dim(X)$. This is allowed as the dimension of an irreducible rigid space is not changed when we pass to its open subsets (see [7, discussion before 2.2.3]).

We then notice that the result is true for X': by Proposition 7.5.2, we know $R^n\nu_{X'*}\widehat{\mathcal{O}}_{X'}$ vanishes for $n > \dim(X')$. On the other hand, by the degeneracy in Corollary 7.4.10, each $R^i\pi_{X'*}\Omega^j_{\text{éh}}(-j)$ is a direct summand of $R^{i+j}\nu_{X'*}\widehat{\mathcal{O}}_{X'}$. This implies that when $i + j > \dim(X')$, the cohomology sheaf $R^i\pi_{X'*}\Omega^j_{\text{éh}}$ vanishes.

Finally, by the coherence proved in Section 6, since $R^i\pi_{X*}\Omega^j_{\text{éh}}$ is the sheaf associated to the presheaf $U \mapsto \mathrm{H}^i(U_{\text{éh}}, \Omega^j_{\text{éh}})$ for open subsets U inside X, the preimage of $R^i\pi_{X'*}\Omega^j_{\text{éh}}$ along f is exactly $R^i\pi_{X*}\Omega^j_{\text{éh}}$. In this way, as $\dim(X) = \dim(X')$, the vanishing of $R^i\pi_{X'*}\Omega^j_{\text{éh}}$ for $i + j > \dim(X')$ implies the vanishing of $R^i\pi_{X*}\Omega^j_{\text{éh}}$ for $i + j > \dim(X)$. So we get the result. ∎

8. (Pro-étale)-éh de Rham comparison

In this section, we give a comparison theorem between pro-étale cohomology and éh de Rham cohomology for proper rigid spaces that are defined over a discretely valued subfield. The idea is to use v-hyperdescent for the de Rham period sheaf $\mathbb{B}^+_{\mathrm{dR}}$ and the simplicial resolution.

Throughout the section, we fix a complete algebraically closed p-adic field K of rank 1, and a discretely valued subfield K_0 that has a perfect residue field, whose associated ring of Witt vectors is denoted as W.

Period sheaves for simplicial spaces. We first recall period sheaves in pro-étale topos and v-topos, following mostly [28, §6] and [30]. Here following the notations in Section 7, we extend the construction to simplicial rigid spaces directly.

Let $X_\bullet$ be a non-augmented simplicial rigid space over K_0, truncated or not. Following §7.3, we consider the commutative diagram of topoi of simplicial sites below, as a simplicial version of the diagram in Section 4:

$$\begin{array}{ccc} \mathrm{Sh}(X_{\bullet\text{proét}}) & \xrightarrow{\nu_\bullet} & \mathrm{Sh}(X_{\bullet\text{ét}}) \\ \uparrow{\scriptstyle\lambda_\bullet} & & \uparrow{\scriptstyle\pi_\bullet} \\ \mathrm{Sh}(\mathrm{Perf}_v|_{X_\bullet^\diamond}) & \xrightarrow{\alpha_\bullet} & \mathrm{Sh}(X_{\bullet\text{éh}}) \end{array}$$

We first extend the construction of various period sheaves in [12, 28] to the simplicial pro-étale site $X_{\bullet\text{proét}}$.

Definition 8.0.1. Let $X_\bullet$ be a non-augmented simplicial rigid space over K_0, truncated or not. Consider the following sheaves on $X_{\bullet\text{proét}}$:

(i) The sheaf $\mathbb{A}_{\inf} := W(\widehat{\mathcal{O}}^{\flat,+}_{\text{proét}})$, together with a canonical specializing map $\theta : \mathbb{A}_{\inf} \to \widehat{\mathcal{O}}^+_{\text{proét}}$ extending that of $\mathrm{A}_{\inf} \to \mathcal{O}_K$.

(ii) The *positive de Rham sheaf*

$$\mathbb{B}^+_{\mathrm{dR}} := \varprojlim \mathbb{A}_{\inf}[1/p]/(\ker(\theta))^n.$$

with a filtration defined by $\mathrm{Fil}^i \mathbb{B}^+_{\mathrm{dR}} = (\ker(\theta))^i \mathbb{B}^+_{\mathrm{dR}}$.

(iii) The *de Rham sheaf*

$$\mathbb{B}_{\mathrm{dR}} := \mathbb{B}^+_{\mathrm{dR}}[1/t],$$

where t is any generator of $\mathrm{Fil}^1\mathbb{B}^+_{\mathrm{dR}}$. We equip the de Rham sheaf with the filtration

$$\mathrm{Fil}^i \mathbb{B}_{\mathrm{dR}} = \sum_{t\in\mathbb{Z}} t^{-j}\mathrm{Fil}^{i+j}\mathbb{B}^+_{\mathrm{dR}}.$$

Note that when $X_\bullet$ is the truncated simplicial rigid space over $\Delta_{\leq 0}$, this recovers the non-simplicial version of the de Rham period sheaf for a rigid space $X = X_0$ as in [28]. Moreover, the positive de Rham sheaf $\mathbb{B}^+_{\mathrm{dR}}$ is filtered complete, with its i-th graded factor $\mathrm{gr}^i\mathbb{B}^+_{\mathrm{dR}}$ equal to $t^i\widehat{\mathcal{O}}_{X_{\text{proét}}} = \widehat{\mathcal{O}}_{X_{\text{proét}}}(i)$ (the non-simplicial version is in [28, Proposition 6.7]).

Next, we recall the definition of the de Rham period sheaf.

Definition 8.0.2. Let $X_\bullet$ be a simplicial smooth rigid space over K_0. Consider the following sheaves on $X_{\bullet\text{proét}}$:

(i) The *positive structure de Rham sheaf* $\mathcal{O}\mathbb{B}^+_{\mathrm{dR}}$ is defined as the sheaf associated to the presheaf sending $U \in X_{n,\text{proét}} \subset X_{\bullet\text{proét}}$ to the direct limit of the rings

$$\ker(\theta)\text{-adic completion of } \big((\mathcal{O}^+_{\text{ét}}(U_j)\,\widehat{\otimes}_W\,\mathbb{A}_{\inf}(U)\big)[1/p]),$$

where $\{U_j\}$ is a pro-étale presentation of U as in [28]. It has a filtration given by $\mathrm{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ = (\ker(\theta))^i \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$.

(ii) Let $\mathcal{F}$ be the sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+[1/t]$, with the filtration

$$\mathrm{Fil}^i \mathcal{F} = \sum_{j\in\mathbb{Z}} t^{-j} \mathrm{Fil}^{i+j} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+,$$

where t is a generator of $\mathrm{Fil}^1 \mathbb{B}_{\mathrm{dR}}^+$. The *structure de Rham sheaf* $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is then defined as the filtered completion of $\mathcal{F}$, namely

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}} := \varprojlim \mathcal{F}/\mathrm{Fil}^i \mathcal{F}.$$

Here $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ comes with a natural filtration from $\mathcal{F}$, whose i-th graded factors are isomorphic to $\mathrm{gr}^i \mathcal{F}$.

Here we note that slightly different from [30], we need to apply a filtered completion at $\mathcal{F}$ to get the structure de Rham sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. This is because the sheaf $\mathcal{F}$ is not complete under the filtration (see [12, Remark 3.11]). We also notice that as explained in [28], locally on $X_{\mathrm{pro\acute{e}t}}$ the element t exists, is a non-zero divisor, and is unique up to units. So the above notions are well-defined.

Comparisons. Let us assume $X_\bullet$ is a simplicial smooth rigid space. The sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ over $X_{\bullet\mathrm{pro\acute{e}t}}$ admits a canonical $\mathbb{B}_{\mathrm{dR}}^+$-linear connection ∇ induced from the differential map of $\mathcal{O}_{X_\bullet}$ over $X_{\bullet\mathrm{\acute{e}t}}$, with the following diagram commuting:

$$\begin{array}{ccc} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ & \xrightarrow{\nabla} & \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^1_{X_\bullet/K_0} \\ \uparrow & & \uparrow \\ \nu_\bullet^{-1}\mathcal{O}_{X_\bullet} & \xrightarrow{d} & \nu_\bullet^{-1}\Omega^1_{X_\bullet/K_0} \end{array}$$

which is functorial among simplicial smooth rigid spaces $X_\bullet$ over K_0. Moreover, the above allows us to give a natural tensor product filtration on the sequence $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}$ by taking the following subsequences:

$$\mathrm{Fil}^i(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}) = \sum_{j\in\mathbb{Z}} \mathrm{Fil}^j(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+) \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^{\geq i-j}_{X_\bullet/K_0},$$

where $\Omega^{\geq i-j}_{X_\bullet/K_0}$ is the $(i-j)$-th Hodge filtration defined by the naive truncation of the de Rham complex. This filtration is compatible with the Hodge filtration on the de Rham complex, in the sense that the connection above induces the natural map of subcomplexes

$$\nu_\bullet^{-1}\mathrm{Fil}^i \Omega^\bullet_{X_\bullet/K_0} \to \mathrm{Fil}^i(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}).$$

Furthermore, replacing the complex $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}$ by $\mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}$ above, we get the natural tensor product filtration compatible with the former:

$$\mathrm{Fil}^i(\mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}) = \sum_{j\in\mathbb{Z}} \mathrm{Fil}^j(\mathcal{O}\mathbb{B}_{\mathrm{dR}}) \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^{\geq i-j}_{X_\bullet/K_0}.$$

Now let us recall the following Poincaré Lemma on the small pro-étale site $X_{\bullet\text{proét}}$:

Lemma 8.0.3 (Poincaré Lemma, [28, Corollary 6.13], [12, Corollary 2.4.2]). *Let $X_\bullet$ be a simplicial smooth rigid space over K_0. Then the following natural sequence is an acyclic filtered complex of $\mathbb{B}_{\mathrm{dR}}^+$-linear sheaves over $X_{\bullet\text{proét}}$:*

$$0 \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^1_{X_\bullet} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^n_{X_\bullet} \to \cdots.$$

Moreover, the same holds when we replace $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ by $\mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ respectively (together with their filtrations).

Proof. The non-simplicial version of the result is proved in [28, Corollary 6.13] and [12, Corollary 2.4.2]. In general, the acyclicity of the natural sequence is checked by applying the restriction functor, as in Lemma 7.3.2. ∎

We now recall the (pro-étale)-de Rham comparison for proper smooth rigid spaces from [28].

Let $X_\bullet$ be a simplicial smooth rigid space over K_0. As discussed above, we can form the following natural map in the filtered derived category of abelian sheaves over $X_{\bullet\text{ét}}$:

$$\Omega^\bullet_{X_\bullet/K_0} \to R\nu_{\bullet*}(\mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}).$$

By taking the derived global section, we get a filtered morphism

$$R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}) \to R\Gamma(X_{\bullet K,\text{proét}}, \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}).$$

As the right side is B_{dR}-linear and the map above is compatible with the filtration on B_{dR}, the above induces a B_{dR}-linear map

$$R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} \to R\Gamma(X_{\bullet K,\text{proét}}, \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu_\bullet^{-1}\Omega^\bullet_{X_\bullet/K_0}).$$

Moreover, by endowing the complex $R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}$ with the (derived) tensor product filtration, the above is in fact a filtered map [23, Chapter 5]. The Poincaré Lemma 8.0.3 implies that the right side is filtered quasi-isomorphic to $R\nu_{\bullet*}\mathbb{B}_{\mathrm{dR}}$. So we get a filtered morphism

$$R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} \to R\Gamma(X_{\bullet K,\text{proét}}, \mathbb{B}_{\mathrm{dR}}).$$

Also note that the map is $\mathrm{Gal}(K/K_0)$-equivariant. It is in fact a quasi-isomorphism, assuming properness:

Theorem 8.0.4 ([28, Theorem 7.11]; [12, Theorem 3.2.7]). *Let $X_\bullet$ be a simplicial proper smooth rigid space over K_0. Then the following two natural maps are $\mathrm{Gal}(K/K_0)$-equivariant filtered quasi-isomorphisms:*

$$R\Gamma(X_{\bullet K,\text{proét}}, \mathbb{B}_{\mathrm{dR}}) \to R\Gamma(X_{\bullet K,\text{proét}}, \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\nu_\bullet^{-1}\mathcal{O}_{X_\bullet}} \nu^{-1}\Omega^\bullet_{X_\bullet/K_0}) \leftarrow R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}.$$

Proof. The non-simplicial version was checked in [28, Theorem 7.11] and [12, Theorem 3.2.7]. Namely, for each proper smooth rigid space X_n over K_0, the natural maps below are filtered quasi-isomorphisms:

$$\begin{aligned} &R\Gamma(X_{nK,\text{proét}}, \mathbb{B}_{\text{dR}}) \\ &\quad \to R\Gamma(X_{nK,\text{proét}}, \mathcal{O}\mathbb{B}_{\text{dR}} \otimes_{\nu_n^{-1}\mathcal{O}_{X_n}} \nu^{-1}\Omega^\bullet_{X_n/K_0}) \leftarrow R\Gamma(X_n, \Omega^\bullet_{X_n/K_0}) \otimes_{K_0} \mathrm{B}_{\text{dR}}. \end{aligned}$$

To get the simplicial version as in the statement, we take the homotopy limit over the simplicial diagram Δ. The homotopy limits of the left and the middle terms above are exactly the left and middle terms as in the statement, so it suffices to check that the following natural map is a filtered quasi-isomorphism:

$$R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}) \otimes_{K_0} \mathrm{B}_{\text{dR}} \to R \varprojlim_{[n]\in\Delta} R\Gamma(X_n, \Omega^\bullet_{X_n/K_0}) \otimes_{K_0} \mathrm{B}_{\text{dR}}.$$

This can be checked by looking at the graded pieces of B_{dR} together with the natural filtered quasi-isomorphism

$$R \varprojlim_{[n]\in\Delta} R\Gamma(X_n, \Omega^\bullet_{X_n/K_0}) \cong R\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet/K_0}).$$

So we are done for the simplicial case. ∎

Remark 8.0.5. Here we remark that the above quasi-isomorphisms between cohomology of simplicial sites also follow from Lemma 7.4.7: it suffices to show that the cone of a map vanishes, which follows from the spectral sequence in Lemma 7.4.7 and the vanishing of the cone for each individual X_n.

Moreover, we can in fact replace the de Rham complex by the éh de Rham complex to compute the cohomology:

Proposition 8.0.6. *Let $X_\bullet$ be a proper smooth rigid space over K_0. Then the map $\pi_\bullet$: $X_{\bullet\text{éh}} \to X_{\bullet\text{ét}}$ of simplicial ringed sites induces a canonical* Gal(K/K_0)*-equivariant filtered quasi-isomorphism*

$$R\Gamma(X_{\bullet\text{éh}}, \Omega^\bullet_{X_\bullet/K_0,\text{éh}}) \otimes_{K_0} \mathrm{B}_{\text{dR}} \to R\Gamma(X_{\bullet K,\text{proét}}, \mathbb{B}_{\text{dR}}).$$

Here the filtration on the complex $R\Gamma(X_{\bullet\text{éh}}, \Omega^\bullet_{X_\bullet/K_0,\text{éh}}) \otimes_{K_0} \mathrm{B}_{\text{dR}}$ is defined by the derived tensor product filtration

$$\mathrm{Fil}^i(R\Gamma(X_{\bullet\text{éh}}, \Omega^\bullet_{X_\bullet/K_0,\text{éh}}) \otimes_{K_0} \mathrm{B}_{\text{dR}}) = \varinjlim_{j\in\mathbb{Z}} R\Gamma(X_{X_\bullet\text{éh}}, \Omega^{\geq i-j}_{X_\bullet/K_0,\text{éh}}) \otimes_{K_0} \mathrm{Fil}^j \mathrm{B}_{\text{dR}}.$$

Proof. By Theorem 8.0.4 and the definition of the filtrations, to prove the above map is a filtered quasi-isomorphism, it suffices to prove this for the following map on the simplicial étale site $X_{\bullet\text{ét}}$:

$$\Omega^\bullet_{X_\bullet/K_0} \to R\pi_{X_\bullet *}\Omega^\bullet_{X_\bullet/K_0,\text{éh}}.$$

Then by the spectral sequence for the filtration given by the naive truncation of the éh de Rham complex, it suffices to show that

$$\Omega^i_{X_\bullet/K_0} \to R\pi_{X_\bullet *}\Omega^i_{X_\bullet/K_0,\text{éh}}$$

for $i \in \mathbb{N}$ and smooth $X_\bullet/K_0$ is a quasi-isomorphism, which follows from éh descent for differential forms (Theorem 4.0.2) and the vanishing criterion in terms of restrictions (Lemma 7.3.2). ■

Now we are ready to prove the (pro-étale)-de Rham comparison for non-smooth proper rigid spaces which generalizes the smooth case in Proposition 8.0.6.

Theorem 8.0.7. *Let X be a proper rigid space over K_0. Then there exists a* $\mathrm{Gal}(K/K_0)$*-equivariant filtered quasi-isomorphism*

$$R\Gamma(X_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} \to R\Gamma(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}}).$$

Proof. We first notice that the pro-étale cohomology of $\mathbb{B}_{\mathrm{dR}}$ satisfies éh-hyperdescent. More precisely, let $\rho : X_\bullet \to X$ be an éh-hypercovering such that each X_n is built from blowing-ups and is smooth and proper over K_0. Then we claim that the natural filtered map below is a filtered quasi-isomorphism

$$R\Gamma(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}}) \to R\Gamma(X_{\bullet K,\text{proét}}, \mathbb{B}_{\mathrm{dR}}).$$

To see this, let us first notice that since $\mathbb{B}_{\mathrm{dR}}$ is filtered complete, it suffices to check the quasi-isomorphism for each graded piece

$$R\Gamma(X_{K\text{proét}}, \widehat{\mathcal{O}}_X(i)) \to R\Gamma(X_{\bullet K,\text{proét}}, \widehat{\mathcal{O}}_{X_\bullet}(i)).$$

This follows from éh-hyperdescent (v-hyperdescent) of the cohomology of the pro-étale structure sheaf in Proposition 3.2.4. On the other hand, similar to the proof of Theorem 8.0.4, the natural map of sites $X_\bullet \to X$ induces a filtered quasi-isomorphism

$$R\Gamma(X_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} \to R\Gamma(X_{\bullet\text{éh}}, \Omega^\bullet_{X_\bullet/K_0,\text{éh}}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}.$$

So the rest follows from the natural filtered quasi-isomorphism as in Proposition 8.0.6,

$$R\Gamma(X_{\bullet\text{éh}}, \Omega^\bullet_{X_\bullet/K_0,\text{éh}}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}} \to R\Gamma(X_{\bullet K,\text{proét}}, \mathbb{B}_{\mathrm{dR}}).$$ ■

As the quasi-isomorphisms above are all filtered with respect to natural filtrations, by taking the 0-th graded piece, we recover the Hodge–Tate decomposition for proper rigid spaces over a discretely valued field.

Moreover, we can use the above to prove the degeneracy of the éh version of the Hodge–de Rham spectral sequence. Recall that the naive truncation of $\Omega^\bullet_{\text{éh}}$ gives a filtration on it, whose associated spectral sequence is

$$E_1^{p,q} = \mathrm{H}^q(X_{\text{éh}}, \Omega^p_{\text{éh},/K_0}) \Longrightarrow \mathrm{H}^{p+q}(X_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}).$$

This is the éh version of the Hodge–de Rham spectral sequence.

Proposition 8.0.8. *Let X be a proper rigid space over K_0. Then the éh Hodge–de Rham spectral sequence degenerates at its E_1-page.*

Proof. As each $\mathrm{H}^q(X_{\text{éh}}, \Omega^p_{\text{éh},/K_0})$ is of finite dimension over K_0 (by the properness of X and Proposition 6.0.1), it suffices to show that

$$\sum_{p+q=n} \dim_{K_0} \mathrm{H}^q(X_{\text{éh}}, \Omega^p_{\text{éh},/K_0}) = \dim_{K_0} \mathrm{H}^n(X_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0}).$$

We first take the tensor product of the K_0-vector space $\mathrm{H}^n(X_{\text{éh}}, \Omega^\bullet_{\text{éh},/K_0})$ with the field B_{dR}. Then the comparison Theorem 8.0.7 implies that the right side above is equal to

$$\dim_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}}).$$

Note that $\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}})$ is a B_{dR}-vector space of finite dimension that has a filtration compatible with that of B_{dR}, induced by the image of $\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}^+_{\mathrm{dR}})$ in $\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}})$. By the Primitive Comparison Theorem of [29], we have

$$\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}^+_{\mathrm{dR}}) \cong \mathrm{H}^n(X_{K\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}}.$$

In particular, the cohomology group $\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}^+_{\mathrm{dR}})$ is a ξ-torsion sheaf and its map to $\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}})$ is injective. Thus $\mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}})$ is a finite-dimensional B_{dR}-vector space whose dimension over B_{dR} is equal to $\mathrm{rank}_{\mathrm{B}^+_{\mathrm{dR}}} \mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}^+_{\mathrm{dR}})$. In particular, the 0-th graded piece is $\mathrm{gr}^0\mathrm{B}_{\mathrm{dR}}(= K)$-vector space $\mathrm{H}^n(X_{K\text{proét}}, \widehat{\mathcal{O}}_X)$ whose K-dimension is equal to $\dim_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}})$. In other words, we get the equality

$$\dim_{\mathrm{B}_{\mathrm{dR}}} \mathrm{H}^n(X_{K\text{proét}}, \mathbb{B}_{\mathrm{dR}}) = \dim_K \mathrm{H}^n(X_{K\text{proét}}, \widehat{\mathcal{O}}_X).$$

In this way, by the degeneracy theorem for the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ (Corollary 7.4.10), we get

$$\begin{aligned}\dim_K \mathrm{H}^n(X_{K\text{proét}}, \widehat{\mathcal{O}}_X) &= \sum_{p+q=n} \dim_K \mathrm{H}^p(X_{K\text{éh}}, \Omega^p_{\text{éh},/K}(-p)) \\ &= \sum_{p+q=n} \dim_K \mathrm{H}^p(X_{\text{éh}}, \Omega^p_{\text{éh},/K_0}),\end{aligned}$$

where the last equality follows from the coherence of $R\pi_*\Omega^p_{\text{éh},/K_0}$ on the small étale site $X_{\text{ét}}$ (Proposition 6.0.1). ■

Remark 8.0.9. As pointed out by David Hansen, the (pro-étale)-éh de Rham comparison could be extended to general de Rham local systems (in the sense of [28]) for non-smooth proper rigid spaces, not just the trivial local system. As the de Rham complex for local systems over a non-smooth rigid space does not behave very well, we need to consider locally free $\mathbb{B}_{\mathrm{dR}}$ sheaves over the v-site, instead of over the pro-étale site. Then the comparison of v-cohomology and (éh) de Rham cohomology will follow from the hyperdescent argument as in Theorem 8.0.7.

9. Hodge–Tate decomposition for non-smooth spaces

Finally, we give an application of our results to the Hodge–Tate decomposition for non-smooth spaces, as mentioned in the introduction. Throughout the section, let X be a proper rigid space over a complete algebraically closed non-archimedean field K over $\mathbb{Q}_p$.

Recall that by the Primitive Comparison Theorem [29, Theorem 3.17], we have

$$\mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \mathrm{H}^n(X_{\text{proét}}, \widehat{\mathcal{O}}_X).$$

The equality enables us to compute p-adic étale cohomology by studying pro-étale cohomology. In particular, by taking the associated derived version, the right side above can be obtained by

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = R\Gamma(X_{\text{ét}}, R\nu_*\widehat{\mathcal{O}}_X).$$

Then we recall the following diagram of topoi associated to X in Section 4:

$$\begin{array}{ccc} \mathrm{Sh}(X_{\text{proét}}) & \xrightarrow{\nu} & \mathrm{Sh}(X_{\text{ét}}) \\ \lambda \uparrow & & \uparrow \pi_X \\ \mathrm{Sh}(\mathrm{Perf}_v|_{X^\diamond}) & \xrightarrow{\alpha} & \mathrm{Sh}(X_{\text{éh}}) \end{array}$$

The (pro-étale)-v comparison (see Proposition 3.2.4) allows us to replace $R\nu_*\widehat{\mathcal{O}}_X$ by the derived direct image $R\pi_{X*}R\alpha_*\widehat{\mathcal{O}}_v$ of the untilted complete v-structure sheaf. So we have

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = R\Gamma(X_{\text{ét}}, R\pi_{X*}R\alpha_*\widehat{\mathcal{O}}_v) = R\Gamma(X_{\text{éh}}, R\alpha_*\widehat{\mathcal{O}}_v).$$

By the discussion in Section 4, we have

$$R^j\alpha_*\widehat{\mathcal{O}}_v = \Omega^j_{\text{éh}}(-j).$$

So by replacing the above equality in the Leray spectral sequence for the composition of derived functors, we get

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{proét}}, \widehat{\mathcal{O}}_X).$$

This together with the Primitive Comparison leads to the *Hodge–Tate spectral sequence* for a proper rigid space X:

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K.$$

The name is justified by the special case of the éh-differential in Theorem 4.0.2: when X is smooth, the higher direct image of the éh-differential vanishes, and the spectral sequence degenerates into

$$E_2^{i,j} = \mathrm{H}^i(X, \Omega^j_{X/K})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K,$$

with each $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})$ identified with $\mathrm{H}^i(X, \Omega^j_{X/K})$.

Now by the strong liftability of X (Proposition 7.4.4) and the Degeneracy Theorem 7.4.9, the derived direct image $R\nu_*\widehat{\mathcal{O}}_X$ is non-canonically quasi-isomorphic to the direct sum

$$\bigoplus_{i=0}^{\dim(X)} R\pi_{X*}(\Omega^j_{\text{éh}}(-j)[-j]).$$

Replacing $R\nu_*\widehat{\mathcal{O}}_X$ by this direct sum, we have

$$R\Gamma(X_{\text{proét}}, \widehat{\mathcal{O}}_X) = \bigoplus_{i=0}^{\dim(X)} R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}(-j)[-j]).$$

So after taking the n-th cohomology, we see that the Hodge–Tate spectral sequence degenerates at its E_2-page.

Theorem 9.0.1 (Hodge–Tate decomposition). *Let X be a proper rigid space over a complete algebraically closed non-archimedean field K of characteristic zero. Then there exists a natural spectral sequence to its p-adic étale cohomology*

$$E_2^{i,j} = \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j) \Longrightarrow \mathrm{H}^{i+j}(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K.$$

Here the spectral sequence degenerates at its E_2-page, and $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is a finite-dimensional K-vector space that vanishes unless $0 \le i, j \le n$.

If X is a smooth rigid space, $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is isomorphic to $\mathrm{H}^i(X, \Omega^j_{X/K})(-j)$, and the spectral sequence is the same as the Hodge–Tate spectral sequence for a smooth proper rigid space (in the sense of [29]).

Proof. The cohomological boundedness of $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ is given by Theorem 6.0.2. The finite-dimensionality follows from the properness of X, the coherence of $R^i\pi_{X*}\Omega^j_{\text{éh}}$ (Proposition 6.0.1), and the equality

$$R\Gamma(X_{\text{éh}}, \Omega^j_{\text{éh}}) = R\Gamma(X_{\text{ét}}, R\pi_{X*}\Omega^j_{\text{éh}}).$$

Moreover, when X is smooth, the isomorphism between $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ and $\mathrm{H}^i(X, \Omega^j_{X/K})(-j)$ follows from éh-descent of differential by Theorem 4.0.2. ∎

Finally, when X is defined over a discretely valued subfield K_0 of K that has a perfect residue field, the above spectral sequence is Galois equivariant. In particular, since $K(-j)^{\mathrm{Gal}(K/K_0)} = 0$ for $j \neq 0$, the boundary map from $\mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j)$ to $\mathrm{H}^{i+2}(X_{\text{éh}}, \Omega^{j-1}_{\text{éh}})(-j+1)$ is zero. In this way, the Hodge–Tate spectral sequence degenerates canonically, and the p-adic étale cohomology splits into the direct sum of distinct Hodge–Tate weights

$$\mathrm{H}^n(X_{\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \bigoplus_{i+j=n} \mathrm{H}^i(X_{\text{éh}}, \Omega^j_{\text{éh}})(-j).$$

This canonical (Galois equivariant) decomposition is functorial with respect to rigid spaces defined over K_0.

Theorem 9.0.2 (Hodge–Tate decomposition). *Let Y be a proper rigid space over a discretely valued subfield K_0 of K that has a perfect residue field. Then the spectral sequence above degenerates at its E_2-page. In fact, we have a Galois equivariant isomorphism*

$$\mathrm{H}^n(Y_{K\,\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K = \bigoplus_{i+j=n} \mathrm{H}^i(Y_{\text{éh}}, \Omega^j_{\text{éh},/K_0}) \otimes_{K_0} K(-j).$$

The isomorphism is functorial with respect to rigid spaces Y over K_0.

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