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# Does Leray's structure theorem withstand buoyancy-driven chemotaxis-fluid interaction?

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**Abstract.** In a smoothly bounded convex domain $\Omega \subset \mathbb{R}^3$, we consider the chemotaxis-Navier–Stokes model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega,\ t > 0, \\ u_t + (u \cdot \nabla) u = \Delta u + \nabla P + n \nabla \Phi, \quad \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases} \tag{$\star$}$$

proposed by Goldstein et al. to describe pattern formation in populations of aerobic bacteria interacting with their liquid environment via transport and buoyancy. Known results have asserted that under appropriate regularity assumptions on $\Phi$ and the initial data, a corresponding no-flux/no-flux/Dirichlet initial-boundary value problem is globally solvable in a framework of so-called weak energy solutions, and that any such solution eventually becomes smooth and classical.

Going beyond this, the present work focuses on the possible extent of unboundedness phenomena also on short timescales, and hence investigates in more detail the set of times in $(0, \infty)$ at which solutions may develop singularities. The main results in this direction reveal the existence of a global weak energy solution which coincides with a smooth function throughout $\overline{\Omega} \times E$, where $E$ denotes a countable union of open intervals which is such that $|(0, \infty) \setminus E| = 0$. In particular, this indicates that a similar feature of the unperturbed Navier–Stokes equations, known as Leray's structure theorem, persists even in the presence of the coupling to the attractive and hence potentially destabilizing cross-diffusive mechanism in the full system ($\star$).

**Keywords.** Chemotaxis, Navier–Stokes, singular set, partial regularity

## 1. Introduction

**Possible singularities in Navier–Stokes flows with given forces.** Questions related to regularity of weak solutions to the Navier–Stokes equations, especially due to their central role in corresponding solution theories also at levels of existence issues, have greatly stimulated substantial developments in PDE analysis even far beyond fluid-mechanical

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application areas [41]. Although deciding about the possibility of spontaneous singularity formation is still a major problem in this field, open despite remarkably comprehensive knowledge e.g. about nonexistence of self-similar blow-up or genericity of smoothness in various flavors [2, 28, 31], a contribution of great relevance in this regard, noticeably, even dates back to the first half of the last century: Namely, Jean Leray's celebrated structure theorem [24, 41] quite considerably reduces the subset of times at which a given and widely arbitrary global weak solution to the Dirichlet problem for the incompressible Navier–Stokes equations in bounded three-dimensional domains $\Omega$ may develop a singularity somewhere in space. More precisely, in its simplest form this theorem states that if $u$ is any such solution which satisfies a certain energy-type inequality naturally associated with the Navier–Stokes system, then it is possible to find $T > 0$ and an at most countable union of open subintervals of $(0, T)$ which complements a null set of times, and which is such that $u$ is smooth throughout each of these intervals, and additionally in $(T, \infty)$, as an $X$-valued mapping, with convenient choices of the function space $X$ compatible with the regularity of $\partial\Omega$, say, $X = C^2(\overline{\Omega}; \mathbb{R}^3)$ in case of smoothly bounded $\Omega$ [24, 32, 37].

Actually made already in 1934, this discovery can be viewed as a starting point for numerous substantial further developments concerning possible structures and sizes of corresponding singularity sets, e.g. including estimates for the Hausdorff dimension of the set of times at which singularities may occur [13], and even considerably detailed information about genuine spatio-temporal smoothness features in the context of studies on what is known as *partial regularity* enjoyed by certain further subclasses of so-called *suitable solutions* [2, 30].

Some natural extensions of the above structure theorem address cases in which the fluid considered is subject to a given external force, and a technique developed in [32] paved the way toward the conclusion that Leray's statement in fact remains unchanged in its essence whenever such a prescribed force is suitably regular [37, Theorem IV.5.5].

In contrast to this, regularity properties of fluid flows seem much less understood in some biologically significant situations in which the corresponding forces themselves are unknowns of the system. Such potentially self-enhancing couplings are typically present in contexts of buoyancy-driven interplay of chemotactically migrating microbial populations with a surrounding liquid environment, as experimentally found to be of relevance for pattern generation in certain bioconvection processes [7, 36]. Indeed, in the recent few years several theoretical studies have gathered considerable evidence indicating various noticeable effects of such chemotaxis-fluid interaction in frameworks of some particular models accessible to rigorous analysis, including influences of fluid flows on spatial population spreading [18, 19], and even fluid-driven suppression of bacterial aggregation in the sense of blow-up prevention [14, 20].

**Buoyancy-induced fluid forcing in bioconvection processes.** Exclusively relying on the common assumption that the respective velocity field is given, these findings predominantly focus on cases in which any gravitational feedback of microbial masses on fluid flows can be neglected; if such additional couplings are accounted for, however, much less information seems available. For instance, in the context of the particular model for oxytactic migration of swimming aerobic bacteria, as proposed in [36] according to

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, \quad \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

predominantly the presence of the source term $n\nabla\Phi$ in the Navier–Stokes subsystem seems lead to substantial challenges already at the level of basic existence theorems. Indeed, reflecting a bouyancy-induced forcing of the fluid velocity $u$ and associated pressure $P$ by fluctuations in the population density $n$ through the given gravitational potential $\Phi$, especially in light of well-known caveats from the theory of chemotaxis-driven blow-up phenomena in related fluid-free Keller–Segel systems [1, 16, 27, 34, 44] such sources seem quite far from being a priori known to fall into any class of inhomogeneities accessible to well-established theories for the Navier–Stokes equations; under general assumptions on the chemotactic sensitivity function $\chi$ and the rate $f(c)$ at which the chemical signal $c$ is consumed by cells, for instance, available regularity information on $n$ apparently reduces to bounds in $L^1(\Omega)$ obtained from mass conservation, but corresponding implications on the fluid force seem far from sufficient to ensure applicability of classical Navier–Stokes theory [31].

Accordingly, most studies on global solvability in three-dimensional domains $\Omega$ either concentrate on small-data smooth solutions [4,5,8,21], or rely on considerable restrictions on $\chi$ and $f$ [8]; a comprehensive result on global existence of weak solutions, addressing (1.1) in bounded convex domains $\Omega\subset\mathbb{R}^3$ under parameter conditions allowing for the prototypical choices $\chi\equiv 1$ and $f(c)=c$, $c\ge 0$, could be established only recently [46]. Even for simplified variants of (1.1) obtained upon suppressing the nonlinear convection term $(u\cdot\nabla)u$, clearly allowing for smooth solution components $u$ and $P$ in the decoupled case when $\nabla\Phi\equiv 0$, in the presence of chemotactic interaction only weak solutions seem available up to now [43], whereas global bounded solutions could up to now be constructed only after further system modifications, introducing appropriate additional relaxation such as diffusion enhancement at large population densities through porous medium-type operators, or including certain saturation mechanisms in the cross-diffusive term, for instance; as a selection from extensive literature in this direction, we refer to [3, 6, 9, 38–40], and also to [17, 23, 35, 48].

In line with this, the knowledge becomes quite sparse as soon as the focus is set on qualitative solution properties going beyond fundamental regularity features naturally obtained in basic existence theories. In fact, the apparently only information available in this direction to date asserts a certain long-time relaxation effect in the sense that in bounded convex three-dimensional $\Omega$, fairly arbitrary weak solutions to (1.1), if satisfying a certain quasi-energy inequality in fact enjoyed by each solution obtained through some convenient approximation procedure, eventually become smooth and classical, and that they stabilize toward a semi-trivial, and especially motion-free, equilibrium in the large time limit ([47]; cf. also [45, 49] for two-dimensional precedents partially even providing convergence rates). Widely unfathomed, however, seem possible facets of potentially destabilizing influences that well-conceivable taxis-driven cell aggregation phenomena may exert on the per se already quite delicate fluid flow regularity, and vice versa, *on short timescales*.

**Main results.** The purpose of this work is to address this issue from a perspective related to that underlying Leray's structure theorem for the unperturbed Navier–Stokes system, and we shall see that despite the evidently more complex couplings than those present in the latter, the three-dimensional version of the full chemotaxis-fluid system (1.1) in fact retains a certain generic smoothness feature in quite a similar flavor.

In order to make this more precise and most transparent, let us concentrate on (1.1) in a prototypical form, and hence throughout the sequel consider the initial-boundary value problem

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, \quad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u=0, & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

in a bounded convex domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, where accessibility to the existence theory from [46] will be provided by our standing assumptions that

$$\Phi\in W^{2,\infty}(\Omega), \tag{1.3}$$

and that

$$\begin{cases} n_0\in L\log L(\Omega) \text{ is nonnegative with } n_0\not\equiv 0,\\ c_0\in L^\infty(\Omega) \text{ is nonnegative and such that } \sqrt{c_0}\in W^{1,2}(\Omega),\\ u_0\in L^2_\sigma(\Omega), \end{cases} \tag{1.4}$$

where as usual we let $L^2_\sigma(\Omega) := \{\varphi\in L^2(\Omega)\mid \nabla\cdot\varphi=0\}$ denote the space of all solenoidal vector fields in $L^2(\Omega)$, and write $L\log L(\Omega)$ to represent the standard Orlicz space associated with the Young function $(0,\infty)\ni z\mapsto z\ln(1+z)$.

Within this framework, our main results will then reveal that at least some solutions enjoy a property of generic regularity quite in the flavor of Leray's statement:

**Theorem 1.1.** *Let $\Omega\subset\mathbb{R}^3$ be a bounded convex domain with smooth boundary, and assume (1.3) and (1.4). Then the problem (1.2) admits at least one global weak energy solution, in the sense of Definition 2.1, which has the property that there exist $T_\star>0$, a countable set $\mathcal{J}\subset\mathbb{N}$ and pairwise disjoint open intervals $I_\iota\subset(0,T_\star)$, $\iota\in\mathcal{J}$, such that*

$$\Big|(0,T_\star)\setminus\bigcup_{\iota\in\mathcal{J}} I_\iota\Big| = 0, \tag{1.5}$$

*and that after redefining* $(n, c, u)$ *on a null set in* $\Omega \times (0, \infty)$ *we have*

$$\begin{cases} n \in C^{2,1}\Big(\overline{\Omega} \times \Big(\bigcup_{\iota \in \mathcal{I}} I_\iota \cup (T_\star, \infty)\Big)\Big), \\ c \in C^{2,1}\Big(\overline{\Omega} \times \Big(\bigcup_{\iota \in \mathcal{I}} I_\iota \cup (T_\star, \infty)\Big)\Big) \quad \text{and} \\ u \in C^{2,1}\Big(\overline{\Omega} \times \Big(\bigcup_{\iota \in \mathcal{I}} I_\iota \cup (T_\star, \infty)\Big); \mathbb{R}^3\Big). \end{cases} \tag{1.6}$$

**Challenges and overall strategy.** A major difference between our analysis of (1.2) and standard approaches to the corresponding unperturbed Navier–Stokes problem, inter alia explaining the restriction in Theorem 1.1 to particular weak solutions, is rooted in the circumstance that due to its apparent sparseness, our available global a priori regularity information for (1.2) seems insufficient to warrant some essential uniqueness features in the flavor of those known from Navier–Stokes theory. In fact, unlike in initial-value problems for the latter [31] it seems unknown whether an arbitrary weak solution to (1.2), if merely known to enjoy some regularity properties inherently linked to some natural energy-type features of (1.2) (cf. Definition 2.1 and especially (2.1) and (2.2) below), must coincide with any suitably smooth solution whenever such a second solution exists.

Accordingly, besides the constitution of a local *existence* theory involving spaces $Y$ of functions $(n, c, u)$ large enough so as to be consistent with the regularity information gained from (2.1) and (2.2), deriving Theorem 1.1 will require an adequate handling of this lacking *uniqueness* property in order to make sure that a weak solution $(n, c, u)$ in question indeed is smooth near each time $t_0$ at which the size of $(n, c, u)$ in $Y$ can conveniently be controlled.

In contrast to corresponding well-established arguments from the literature on the Navier–Stokes system [32, 37], our approach will therefore predominantly operate at the level of solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ to suitably regularized variants of (1.2) (see (2.3) below), and aim at deducing estimates, ultimately in spaces of smooth functions, independent of the respective approximation parameter $\varepsilon \in (0, 1)$. Forming the origin of an additional technical complication, the temporally local character of such quantitative regularity information will suggest to finally derive smoothness near an instant $t_0$ under consideration by providing estimates throughout a *partially backward* open interval $J(t_0) \ni t_0$, instead of merely concentrating on exclusively *forward* intervals, as known to be a possible and considerably simpler procedure in the derivation of Leray's theorem for the Navier–Stokes system [32, 37].

Specifically, our approach will rest on a local theory based on an analysis of

$$y_\varepsilon(t) := \int_\Omega n_\varepsilon^p(\cdot, t) + \int_\Omega |\nabla c_\varepsilon(\cdot, t)|^{2p} + \int_\Omega |A^{\alpha/2} u_\varepsilon(\cdot, t)|^2, \quad t \geq 0,\ \varepsilon \in (0, 1), \tag{1.7}$$

for suitably chosen $p > 1$ and $\alpha > 0$, where, as throughout the sequel, we let $A = -\mathcal{P}\Delta$ denote the realization of the Stokes operator in $L^2_\sigma(\Omega)$, with its domain given by $D(A) = W^{2,2}(\Omega; \mathbb{R}^3) \cap W^{1,2}_{0,\sigma}(\Omega)$, $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega; \mathbb{R}^3) \cap L^2_\sigma(\Omega)$, and with $\mathcal{P}$ denoting the Helmholtz projection on $L^2(\Omega; \mathbb{R}^3)$, and for $\alpha \in \mathbb{R}$ we let $A^\alpha$ represent the corresponding sectorial fractional powers.

Indeed, we shall firstly see that whenever

$$p > 3/2 \quad \text{and} \quad \alpha \in (1/2, 1), \tag{1.8}$$

the short-time growth of $y_\varepsilon$ can conveniently be controlled due to the observation that $y_\varepsilon$ satisfies a superlinearly forced but autonomous ODI with $\varepsilon$-independent coefficients (Lemmas 3.7 and 3.8). The a priori information thereby gained will turn out to form a suitable starting point for a bootstrap procedure eventually providing local-in-time estimates in $C^{2+\theta,1+\theta/2}$ spaces (Lemmas 4.6 and 4.7) after each time at which $y_\varepsilon$ remains controlled by any arbitrarily large but fixed number.

In order to ensure applicability of this local regularity theory to (1.2) through an elementary observation on the sizes of certain sets containing endpoints of intervals at which a given measurable function exceeds a prescribed level (Lemma 5.1 and Corollary 5.2), in Section 6 we will thereafter complement this by making sure that the alternative hypotheses

$$p < 3 \quad \text{and} \quad \alpha < 1 \tag{1.9}$$

guarantee that the space $Y := L^p(\Omega) \times W^{1,2p}(\Omega) \times D(A^{\alpha/2})$ underlying the choice of $y_\varepsilon$ is large enough so as to contain $(n, c, u)$ throughout large sets of times due to the dissipation processes expressed in (2.1) and (2.2) (Lemma 6.5). Thanks to a suitable approximation property of $y_\varepsilon$ in the limit of vanishing $\varepsilon$ (Lemma 6.4), these arguments indeed become applicable to $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$, and the desired overall conclusion can finally be obtained due to the fortunate circumstance that the requirements in (1.8) and (1.9) can be fulfilled simultaneously.

Before going into details, let us finally remark that in line with this and our subsequent reasoning, the statement of Theorem 1.1 immediately extends to any global weak energy solution that can be gained as an accumulation point of the family of solutions to (2.3) as $\varepsilon \searrow 0$; actually, corresponding limits obtained through more general approximation procedures can be covered as well, but pursuing this in detail goes beyond the scope of this study.

## 2. Energy solutions, eventual regularity and approximation

In order to briefly specify the framework of our analysis, we firstly introduce the following solution concept which combines [46, Definition 2.1] with the essential part of [47, Definition 1.1]. For vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$, we here let $v \otimes w$ denote the matrix $(a_{ij})_{i,j\in\{1,2,3\}} \in \mathbb{R}^{3\times 3}$ defined by letting $a_{ij} := v_i w_j$ for $i, j \in \{1, 2, 3\}$.

**Definition 2.1.** Suppose that

- $n \in L^4_{loc}(\bar{\Omega} \times [0, \infty)) \cap L^2_{loc}([0, \infty); W^{1,2}(\Omega))$ is nonnegative with $n^{1/2} \in L^2_{loc}([0, \infty); W^{1,2}(\Omega))$, that
- $c \in L^\infty_{loc}(\Omega \times [0, \infty))$ is nonnegative and such that $c^{1/4} \in L^4_{loc}([0, \infty); W^{1,4}(\Omega))$, and that
- $u \in L^\infty_{loc}([0, \infty); L^2_\sigma(\Omega)) \cap L^2_{loc}([0, \infty); W^{1,2}_0(\Omega; \mathbb{R}^3))$.

Then $(n, c, u)$ will be called a *global weak energy solution* of (1.2) if

$$-\int_0^\infty \int_\Omega n\phi_t - \int_\Omega n_0\phi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla n\cdot\nabla\phi + \int_0^\infty \int_\Omega n\nabla c\cdot\nabla\phi + \int_0^\infty \int_\Omega nu\cdot\nabla\phi$$

for all $\phi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$;

$$-\int_0^\infty \int_\Omega c\phi_t - \int_\Omega c_0\phi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c\cdot\nabla\phi - \int_0^\infty \int_\Omega nc\phi + \int_0^\infty \int_\Omega cu\cdot\nabla\phi$$

for all $\phi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$;

$$-\int_0^\infty \int_\Omega u\cdot\phi_t - \int_\Omega u_0\cdot\phi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u\cdot\nabla\phi + \int_0^\infty \int_\Omega u\otimes u\cdot\nabla\phi + \int_0^\infty \int_\Omega n\nabla\Phi\cdot\phi$$

for all $\phi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3)$ satisfying $\nabla\cdot\phi \equiv 0$;

$$\frac{1}{2}\int_\Omega |u(\cdot,t)|^2 + \int_{t_0}^t \int_\Omega |\nabla u|^2 \le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \int_{t_0}^t \int_\Omega nu\cdot\nabla\Phi \tag{2.1}$$

for a.e. $t_0 > 0$ and all $t > t_0$; and there exist $\kappa > 0$ and $K > 0$ such that

$$\frac{d}{dt}\left\{\int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c} + \kappa\int_\Omega |u|^2\right\} + \frac{1}{K}\int_\Omega \left\{\frac{|\nabla n|^2}{n} + \frac{|\nabla c|^4}{c^3} + |\nabla u|^2\right\} \le K \tag{2.2}$$

in $\mathcal{D}'((0,\infty))$.

For any such solution, the main result from [47] applies so as to assert the following statement on eventual smoothness:

**Theorem A.** *Suppose that $(n, c, u)$ is any global weak energy solution of (1.2) with some initial data $n_0, c_0$ and $u_0$ satisfying (1.4). Then there exist $T_\star > 0$ and $P \in C^{1,0}(\overline{\Omega} \times [T_\star,\infty))$ such that upon redefining $(n, c, u)$ on a null set we have*

$$n \in C^{2,1}(\overline{\Omega} \times [T_\star,\infty)), \quad c \in C^{2,1}(\overline{\Omega} \times [T_\star,\infty)), \quad u \in C^{2,1}(\overline{\Omega} \times [T_\star,\infty); \mathbb{R}^3),$$

*and such that $(n, c, u, P)$ solves the boundary value problem in (1.2) classically in $\overline{\Omega} \times [T_\star,\infty)$.*

The corresponding existence theory from [46] utilizes the regularized problems

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon\cdot\nabla n_\varepsilon = \Delta n_\varepsilon - \nabla\cdot(n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon), & x\in\Omega,\ t>0,\\ c_{\varepsilon t} + u_\varepsilon\cdot\nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon, & x\in\Omega,\ t>0,\\ u_{\varepsilon t} + (Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon\nabla\Phi, \quad \nabla\cdot u_\varepsilon = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0, \quad u_\varepsilon = 0, & x\in\partial\Omega,\ t>0,\\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{2.3}$$

for $\varepsilon \in (0,1)$, where the Yosida approximation $Y_\varepsilon$ [26,31] is defined by letting

$$Y_\varepsilon v := (1+\varepsilon A)^{-1}v \quad \text{for } v \in L^2_\sigma(\Omega) \text{ and } \varepsilon \in (0,1) \tag{2.4}$$

and where setting

$$F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1+\varepsilon s) \quad \text{for } s \ge 0 \text{ and } \varepsilon \in (0,1)$$

ensures that

$$0 \le F'_\varepsilon(s) = \frac{1}{1+\varepsilon s} \le 1 \quad \text{and} \quad 0 \le F_\varepsilon(s) \le s \quad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1), \tag{2.5}$$

and that $F'_\varepsilon(s) \nearrow 1$ and $F_\varepsilon(s) \nearrow s$ as $\varepsilon \searrow 0$ for all $s > 0$. As for the initial data in (2.3), from [46] we import the requirements that

$$\begin{cases} n_{0\varepsilon} \in C_0^\infty(\Omega), \quad n_{0\varepsilon} \ge 0 \text{ in } \Omega, \quad \int_\Omega n_{0\varepsilon} = \int_\Omega n_0 \quad \text{for all } \varepsilon \in (0,1), \\ n_{0\varepsilon} \to n_0 \quad \text{in } L\log L(\Omega) \quad \text{as } \varepsilon \searrow 0, \end{cases} \tag{2.6}$$

$$\begin{cases} c_{0\varepsilon} \ge 0 \text{ in } \Omega, \quad \sqrt{c_{0\varepsilon}} \in C_0^\infty(\Omega), \quad \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } \varepsilon \in (0,1), \\ \sqrt{c_{0\varepsilon}} \to \sqrt{c_0} \quad \text{a.e. in } \Omega \text{ and in } W^{1,2}(\Omega) \quad \text{as } \varepsilon \searrow 0, \end{cases} \tag{2.7}$$

$$\begin{cases} u_{0\varepsilon} \in C_{0,\sigma}^\infty(\Omega), \quad \|u_{0\varepsilon}\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} \quad \text{for all } \varepsilon \in (0,1), \\ u_{0\varepsilon} \to u_0 \quad \text{in } L^2(\Omega) \quad \text{as } \varepsilon \searrow 0. \end{cases} \tag{2.8}$$

The following lemma summarizes some basic results concerning global existence of classical solutions and some of their elementary properties, as obtained in [46, Lemmas 2.2, 2.3, 3.9].

**Lemma 2.2.** *For each* $\varepsilon \in (0,1)$, *there exist*

$$n_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0,\infty)), \quad c_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0,\infty)), \quad u_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0,\infty); \mathbb{R}^3)$$

*such that* $n_\varepsilon > 0$ *and* $c_\varepsilon > 0$ *in* $\overline{\Omega} \times (0,\infty)$, *and such that* $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ *solves* (2.3) *classically in* $\Omega \times (0,\infty)$ *with some* $P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty))$. *Moreover,*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t > 0, \tag{2.9}$$

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } t > 0. \tag{2.10}$$

## 3. Local theory: Controlling the short-time growth of $y_\varepsilon$ when $p > 3/2$

Forming the core quantity of all our subsequent analysis, our object of investigation in this section will be the functional introduced in (1.7), with the parameters $p > 1$ and $\alpha > 0$ appearing therein still being at our disposal. Our goal will consist in making sure that the key assumptions in (1.8), and especially the requirement $p > 3/2$ therein, indeed enable us to develop a local regularity theory by deriving the autonomous ODI (3.13) for $y_\varepsilon$, and a first step toward this can be achieved by performing three quite straightforward testing procedures to (2.3):

**Lemma 3.1.** *Let $p > 1$ and $\alpha > 0$. Then there exists $C > 0$ such that with $(y_\varepsilon)_{\varepsilon\in(0,1)}$ taken from (1.7) we have*

$$y_\varepsilon'(t) + \frac{1}{C} \cdot \left\{ \int_\Omega |\nabla n_\varepsilon^{p/2}|^2 + \int_\Omega \big|\nabla|\nabla c_\varepsilon|^p\big|^2 + \int_\Omega |A^{(\alpha+1)/2} u_\varepsilon|^2 \right\}$$

$$\le C \cdot \left\{ \int_\Omega n_\varepsilon^p |\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2p-2} + \int_\Omega |\nabla c_\varepsilon|^{2p} \cdot |\nabla u_\varepsilon| \right.$$

$$\left. + \left| \int_\Omega A^\alpha u_\varepsilon \cdot \mathcal{P}\{(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon\} \right| + \left| \int_\Omega A^\alpha u_\varepsilon \cdot \mathcal{P}\{n_\varepsilon \nabla \Phi\} \right| \right\} \qquad (3.1)$$

*for all $t > 0$ and $\varepsilon \in (0,1)$.*

*Proof.* Since $\nabla \cdot u_\varepsilon = 0$, from the first equation in (2.3) and Young's inequality we obtain that for all $t > 0$,

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 = (p-1)\int_\Omega n_\varepsilon^{p-1} F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon$$

$$\le \frac{p-1}{2}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + \frac{p-1}{2}\int_\Omega n_\varepsilon^p F_\varepsilon'^2(n_\varepsilon)|\nabla c_\varepsilon|^2$$

and hence, by (2.5),

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{2(p-1)}{p}\int_\Omega |\nabla n_\varepsilon^{p/2}|^2 \le \frac{p-1}{2}\int_\Omega n_\varepsilon^p |\nabla c_\varepsilon|^2 \quad \text{for all } t > 0. \qquad (3.2)$$

Next, using that $\frac{\partial |\nabla c_\varepsilon|^2}{\partial \nu} \le 0$ on $\partial\Omega \times (0,\infty)$ by convexity of $\Omega$ [25], integrating by parts in the second equation from (2.3) we see that again due to the solenoidality of $u_\varepsilon$ and (2.5), and thanks to (2.10) and Young's inequality,

$$\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2p} = \int_\Omega |\nabla c_\varepsilon|^{2p-2}\nabla c_\varepsilon \cdot \nabla\{\Delta c_\varepsilon - F_\varepsilon(n_\varepsilon)c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon\}$$

$$= \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2p-2}\Delta|\nabla c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^{2p-2}|D^2 c_\varepsilon|^2$$

$$+ \int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon \cdot \{2(p-1)|\nabla c_\varepsilon|^{2p-4}\nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) + |\nabla c_\varepsilon|^{2p-2}\Delta c_\varepsilon\}$$

$$- \int_\Omega |\nabla c_\varepsilon|^{2p-2}\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon)$$

$$\le -\frac{2(p-1)}{p^2}\int_\Omega \big|\nabla|\nabla c_\varepsilon|^p\big|^2 - \int_\Omega |\nabla c_\varepsilon|^{2p-2}|D^2 c_\varepsilon|^2$$

$$+ (2(p-1)+\sqrt{3})\|c_0\|_{L^\infty(\Omega)}\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2p-2}|D^2 c_\varepsilon| + \int_\Omega |\nabla c_\varepsilon|^{2p} \cdot |\nabla u_\varepsilon|$$

$$\le -\frac{2(p-1)}{p^2}\int_\Omega \big|\nabla|\nabla c_\varepsilon|^p\big|^2$$

$$+ \frac{(2(p-1)+\sqrt{3})^2\|c_0\|_{L^\infty(\Omega)}^2}{4}\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2p-2} + \int_\Omega |\nabla c_\varepsilon|^{2p} \cdot |\nabla u_\varepsilon| \qquad (3.3)$$

for all $t > 0$. We finally test the third equation in (2.3), rewritten in the projected form $u_{\varepsilon t} + Au_\varepsilon = -\mathcal{P}\{(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon\} + \mathcal{P}\{n_\varepsilon \nabla\Phi\}$, by $A^\alpha u_\varepsilon$ to obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{(\alpha)/2}u_\varepsilon|^2 + \int_\Omega |A^{(\alpha+1)/2}u_\varepsilon|^2 = -\int_\Omega A^\alpha u_\varepsilon \cdot \mathcal{P}\{(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon\} + \int_\Omega A^\alpha u_\varepsilon \cdot \mathcal{P}\{n_\varepsilon \nabla\Phi\}$$

for all $t > 0$, which combined with (3.2) and (3.3) entails (3.1). ∎

Now under the announced assumption that $p > 3/2$, the first two of the five integrals on the right of (3.1) can jointly be estimated in terms of the dissipated quantity therein, and of a superlinear power of $y_\varepsilon$, by means of a Gagliardo–Nirenberg type interpolation.

**Lemma 3.2.** *Let $p > 3/2$ and $\alpha > 0$. Then for all $\eta > 0$ there exists $C(\eta) > 0$ such that whenever $\varepsilon \in (0, 1)$,*

$$\int_\Omega n_\varepsilon^p|\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2p-2} \le \eta\int_\Omega |\nabla n_\varepsilon|^2 + \eta\int_\Omega \big|\nabla|\nabla c_\varepsilon|^2\big|^2 + C(\eta)y_\varepsilon^{\frac{2p-1}{2p-3}}(t) + C(\eta) \qquad (3.4)$$

*for all $t > 0$, where $y_\varepsilon$ is as in (1.7).*

*Proof.* According to the Gagliardo–Nirenberg inequality, followed by two applications of Young's inequality which rely on the assumption $p > 3/2$ and the fact that $2(p+1)/p < \frac{2(2p-1)}{2p-3}$, we can fix $C_1 > 0$ and $C_2 = C_2(\eta) > 0$ such that

$$\begin{aligned} 2\|\varphi\|_{L^{2(p+1)/p}(\Omega)}^{2(p+1)/p} &\le C_1\|\nabla\varphi\|_{L^2(\Omega)}^{3/p}\|\varphi\|_{L^2(\Omega)}^{(2p-1)/p} + C_1\|\varphi\|_{L^2(\Omega)}^{2(p+1)/p} \\ &\le \eta\|\nabla\varphi\|_{L^2(\Omega)}^2 + C_2(\eta)\|\varphi\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} + C_1\|\varphi\|_{L^2(\Omega)}^{2(p+1)/p} \\ &\le \eta\|\nabla\varphi\|_{L^2(\Omega)}^2 + C_3(\eta)\|\varphi\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} + C_1 \quad \text{for all } \varphi \in W^{1,2}(\Omega), \end{aligned}$$

where $C_3(\eta) := C_1 + C_2(\eta)$. Twice employing this shows that again thanks to Young's inequality, with some $C_4 > 0$ we have

$$\begin{aligned} \int_\Omega n_\varepsilon^p|\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2p-2} &\le 2\int_\Omega n_\varepsilon^{p+1} + 2\int_\Omega |\nabla c_\varepsilon|^{2(p+1)} \\ &= 2\|n_\varepsilon^{p/2}\|_{L^{2(p+1)/p}(\Omega)} + 2\big\||\nabla c_\varepsilon|^p\big\|_{L^{2(p+1)/p}(\Omega)}^{2(p+1)/p} \\ &\le \eta\|\nabla n_\varepsilon^{p/2}\|_{L^2(\Omega)}^2 + C_3(\eta)\|n_\varepsilon^{p/2}\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} + C_1 \\ &\quad + \eta\big\|\nabla|\nabla c_\varepsilon|^p\big\|_{L^2(\Omega)}^2 + C_3(\eta)\big\||\nabla c_\varepsilon|^p\big\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} + C_1 \end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Since

$$\|n_\varepsilon^{p/2}\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} \le y_\varepsilon^{\frac{2p-1}{2p-3}}(t) \quad \text{and} \quad \big\||\nabla c_\varepsilon|^p\big\|_{L^2(\Omega)}^{\frac{2(2p-1)}{2p-3}} \le y_\varepsilon^{\frac{2p-1}{2p-3}}(t) \quad \text{for all } t > 0$$

by (1.7), this implies (3.4). ∎

In order to prepare our estimation of the three remaining integrals on the right-hand side of (3.1), but also one of our subsequent higher order regularity arguments in Lemma 4.4, let us explicitly recall the following well-known interpolation inequality (cf. e.g. [10, Theorem 2.14.1]).

**Lemma 3.3.** *Let $\lambda \in \mathbb{R}$, $\mu > \lambda$ and $\theta \in (\lambda, \mu)$. Then there exists $C = C(\lambda,\mu,\theta) > 0$ such that*

$$\|A^\theta \varphi\|_{L^2(\Omega)} \le C\|A^\mu \varphi\|_{L^2(\Omega)}^{\frac{\theta-\lambda}{\mu-\lambda}} \|A^\lambda \varphi\|_{L^2(\Omega)}^{\frac{\mu-\theta}{\mu-\lambda}} \quad \textit{for all } \varphi \in D(A^\mu).$$

We can thereby control the second contribution to the right-hand side of (3.1), and hence the transport-related part of the interaction in (2.3), in a flavor quite similar to that of Lemma 3.2, provided that $\alpha > 1/2$.

**Lemma 3.4.** *Let $p > 1$ and $\alpha \in (1/2, 1)$. Then for all $\eta > 0$ there exists $C(\eta) > 0$ such that for each $\varepsilon \in (0,1)$, with $y_\varepsilon$ taken from (1.7) we have*

$$\int_\Omega |\nabla c_\varepsilon|^{2p} \cdot |\nabla u_\varepsilon| \le \eta \int_\Omega \big|\nabla|\nabla c_\varepsilon|^p\big|^2 + \eta \int_\Omega |A^{(\alpha+1)/2} u_\varepsilon|^2 + C(\eta) y_\varepsilon^{\frac{2\alpha+1}{2\alpha-1}}(t) + C(\eta) \tag{3.5}$$

*for all $t > 0$.*

*Proof.* By the Cauchy–Schwarz inequality,

$$\int_\Omega |\nabla c_\varepsilon|^{2p} \cdot |\nabla u_\varepsilon| \le \big\||\nabla c_\varepsilon|^p\big\|_{L^4(\Omega)}^2 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \quad \text{for all } t > 0, \tag{3.6}$$

where due to the Gagliardo–Nirenberg inequality and (1.7), we can find $C_1 > 0$ such that

$$\begin{aligned} \big\||\nabla c_\varepsilon|^p\big\|_{L^4(\Omega)}^2 &\le C_1 \big\|\nabla|\nabla c_\varepsilon|^p\big\|_{L^2(\Omega)}^{3/2} \big\||\nabla c_\varepsilon|^p\big\|_{L^2(\Omega)}^{1/2} + C_1 \big\||\nabla c_\varepsilon|^p\big\|_{L^2(\Omega)}^2 \\ &\le C_1 \big\|\nabla|\nabla c_\varepsilon|^2\big\|_{L^2(\Omega)}^{3/2} y_\varepsilon^{1/4}(t) + C_1 y_\varepsilon(t) \end{aligned} \tag{3.7}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, and where Lemma 3.3 enables us to pick $C_2 > 0$ fulfilling

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^2(\Omega)} &= \|A^{1/2} u_\varepsilon\|_{L^2(\Omega)} \\ &\le C_2 \|A^{(\alpha+1)/2} u_\varepsilon\|_{L^2(\Omega)}^{1-\alpha} \|A^{\alpha/2} u_\varepsilon\|_{L^2(\Omega)}^{\alpha} \\ &\le C_2 \|A^{(\alpha+1)/2} u_\varepsilon\|_{L^2(\Omega)}^{1-\alpha} y_\varepsilon^{\alpha/2}(t) \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1), \end{aligned}$$

because $\|A^{\alpha/2}u_\varepsilon\|^2_{L^2(\Omega)} \le y_\varepsilon(t)$ for any such $t$ and $\varepsilon$. Since $4(1-\alpha) < 2$ according to our hypothesis that $\alpha > 1/2$, through Young's inequality a combination of this with (3.7) and (3.6) yields $C_3(\eta) > 0$ and $C_4(\eta) > 0$ such that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\int_\Omega |\nabla c_\varepsilon|^{2p} \cdot |\nabla u_\varepsilon| \\
&\quad \le C_1C_2 \big\|\nabla|\nabla c_\varepsilon|^p\big\|^{3/2}_{L^2(\Omega)} \|A^{(\alpha+1)/2}u_\varepsilon\|^{1-\alpha}_{L^2(\Omega)} y_\varepsilon^{(2\alpha+1)/4}(t) \\
&\qquad + C_1C_2\|A^{(\alpha+1)/2}u_\varepsilon\|^{1-\alpha}_{L^2(\Omega)} y_\varepsilon^{(\alpha+2)/2}(t) \\
&\quad \le \eta \big\|\nabla|\nabla c_\varepsilon|^p\big\|^2_{L^2(\Omega)} + C_3(\eta)\|A^{(\alpha+1)/2}u_\varepsilon\|^{4(1-\alpha)}_{L^2(\Omega)} y_\varepsilon^{2\alpha+1}(t) \\
&\qquad + C_1C_2\|A^{(\alpha+1)/2}u_\varepsilon\|^{1-\alpha}_{L^2(\Omega)} y_\varepsilon^{(\alpha+2)/2}(t) \\
&\quad \le \eta \big\|\nabla|\nabla c_\varepsilon|^p\big\|^2_{L^2(\Omega)} + \eta\|A^{(\alpha+1)/2}u_\varepsilon\|^2_{L^2(\Omega)} + C_4(\eta) y_\varepsilon^{\frac{2\alpha+1}{2\alpha-1}}(t) + C_4(\eta) y_\varepsilon^{\frac{\alpha+2}{\alpha+1}}(t).
\end{aligned}$$

Since $\frac{\alpha+2}{\alpha+1} < \frac{2\alpha+1}{2\alpha-1}$, a final application of Young's inequality thus yields (3.5). ∎

Likewise, through Lemma 3.3 also the third of the integrals in question can be conveniently estimated if $\alpha > 1/2$.

**Lemma 3.5.** *Let $p > 1$, $\alpha \in (1/2, 1)$ and $\rho \in (\frac{3}{4}, \frac{\alpha+1}{2})$. Then given any $\eta > 0$, one can find $C(\eta) > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\left| \int_\Omega A^\alpha u_\varepsilon \cdot \mathcal{P}\{(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon\} \right| \le \eta \int_\Omega |A^{(\alpha+1)/2}u_\varepsilon|^2 + C(\eta) y_\varepsilon^{\frac{\alpha-2\rho+2}{\alpha-2\rho+1}}(t) \quad \textit{for all } t > 0, \tag{3.8}$$

*where again $y_\varepsilon$ is as in (1.7).*

*Proof.* According to the Cauchy–Schwarz inequality and the orthogonal projection property of $\mathcal{P}$,

$$\begin{aligned}
\left| \int_\Omega A^\alpha u_\varepsilon \cdot \mathcal{P}\{(Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon\} \right| &\le \|A^\alpha u_\varepsilon\|_{L^2(\Omega)} \|(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon\|_{L^2(\Omega)} \\
&\le \|A^\alpha u_\varepsilon\|_{L^2(\Omega)} \|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}
\end{aligned} \tag{3.9}$$

for all $t > 0$. Here using that $D(A^\rho) \hookrightarrow L^\infty(\Omega; \mathbb{R}^3)$ due to our restriction $\rho > 3/4$ [12, 15], we can find $C_1 > 0$ such that since $A^\rho$ and $Y_\varepsilon$ commute on $D(A^\rho)$, and since $Y_\varepsilon$ is non-expansive on $L^2_\sigma(\Omega)$,

$$\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} \le C_1\|A^\rho Y_\varepsilon u_\varepsilon\|_{L^2(\Omega)} = C_1\|Y_\varepsilon A^\rho u_\varepsilon\|_{L^2(\Omega)} \le C_1\|A^\rho u_\varepsilon\|_{L^2(\Omega)}$$

for all $t > 0$. As furthermore $\rho < (\alpha+1)/2$ and $\alpha > 1/2$, each of the three rightmost factors in (3.9) therefore becomes accessible to Lemma 3.3, whence application of the latter, followed by Young's inequality, provides $C_2 > 0$ and $C_3(\eta) > 0$ fulfilling

$$\begin{aligned}
&\|A^\alpha u_\varepsilon\|_{L^2(\Omega)}\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)} \\
&\quad\le C_2\cdot\{\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^{\alpha}\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{1-\alpha}\}\cdot\{\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^{2\rho-\alpha}\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{\alpha-2\rho+1}\} \\
&\qquad\times\{\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^{1-\alpha}\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{\alpha}\} \\
&\quad= C_2\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^{2\rho-\alpha+1}\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{\alpha-2\rho+2} \\
&\quad\le \eta\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^2 + C_3(\eta)\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{\frac{2(\alpha-2\rho+2)}{\alpha-2\rho+1}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

because clearly $0 < 2\rho - \alpha + 1$. Again using that $\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^2 \le y_\varepsilon(t)$ for all $t > 0$ and $\varepsilon \in (0,1)$, in view of (3.9) we directly obtain (3.8) from this. ∎

The rightmost and buoyancy-induced term from Lemma 3.1 can finally be estimated in a manner sufficient for our purposes, even for arbitrary $\alpha \in (0,1)$ and any $p$ from the range $(4/3,\infty)$ larger than that determined through (1.8), by resorting to the $L^1$ bound implied by (2.9).

**Lemma 3.6.** *Let $p > 4/3$ and $\alpha \in (0,1)$. Then for each $\eta > 0$ there exists $C(\eta) > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\begin{aligned}
&\left|\int_\Omega A^\alpha u_\varepsilon\cdot\mathcal{P}\{n_\varepsilon\nabla\Phi\}\right| \\
&\quad\le \eta\int_\Omega|\nabla n_\varepsilon^{p/2}|^2 + \eta\int_\Omega|A^{(\alpha+1)/2}u_\varepsilon|^2 + C(\eta)y_\varepsilon^{\frac{(3p-1)(1-\alpha)}{(3p-1)(2-\alpha)-3}}(t) + C(\eta) \qquad (3.10)
\end{aligned}$$

*for all $t > 0$, with $y_\varepsilon$ as given by (1.7).*

*Proof.* Due to our overall assumption on boundedness of $\nabla\Phi$, we may again rely on the orthogonal projection property of $\mathcal{P}$, on Lemma 3.3 and on Young's inequality to infer that with some $C_1 > 0$ and $C_2(\eta) > 0$ we have

$$\begin{aligned}
\left|\int_\Omega A^\alpha u_\varepsilon\cdot\mathcal{P}\{n_\varepsilon\nabla\Phi\}\right| &\le \|A^\alpha u_\varepsilon\|_{L^2(\Omega)}\|n_\varepsilon\nabla\Phi\|_{L^2(\Omega)} \\
&\le C_1\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^{\alpha}\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{1-\alpha}\|n_\varepsilon\|_{L^2(\Omega)} \\
&\le \eta\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^2 + C_2(\eta)\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^{\frac{2(1-\alpha)}{2-\alpha}}\|n_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{2-\alpha}} \\
&\le \eta\|A^{(\alpha+1)/2}u_\varepsilon\|_{L^2(\Omega)}^2 + C_2(\eta)y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t)\|n_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{2-\alpha}} \qquad (3.11)
\end{aligned}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, once more because $\|A^{\alpha/2}u_\varepsilon\|_{L^2(\Omega)}^2 \le y_\varepsilon(t)$ for $t > 0$ by (1.7). Here employing the Gagliardo–Nirenberg inequality, since $\|n_\varepsilon^{p/2}\|_{L^{2/p}(\Omega)}^{2/p} = \int_\Omega n_\varepsilon =$

$\int_\Omega n_0$ for all $t > 0$ by (2.9), we see that with some $C_3(\eta) > 0$ and $C_4(\eta) > 0$ we have

$$
\begin{aligned}
C_2(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \|n_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{2-\alpha}} &= C_2(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \|n_\varepsilon^{p/2}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p(2-\alpha)}} \\
&\le C_3(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \|\nabla n_\varepsilon^{p/2}\|_{L^2(\Omega)}^{\frac{6}{(3p-1)(2-\alpha)}} \|n_\varepsilon^{p/2}\|_{L^{2/p}(\Omega)}^{\frac{2(3p-2)}{p(3p-1)(2-\alpha)}} + C_3(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \|n_\varepsilon^{p/2}\|_{L^{2/p}(\Omega)}^{\frac{4}{p(2-\alpha)}} \\
&\le C_4(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \|\nabla n_\varepsilon^{p/2}\|_{L^2(\Omega)}^{\frac{6}{(3p-1)(2-\alpha)}} + C_4(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \qquad (3.12)
\end{aligned}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Since our restrictions $p > 4/3$ and $\alpha < 1$ warrant that $\frac{6}{(3p-1)(2-\alpha)} < 2$, and since evidently $\frac{1-\alpha}{2-\alpha} < \frac{(3p-1)(1-\alpha)}{(3p-1)(2-\alpha)-3}$, two applications of Young's inequality finally show that there exists $C_5(\eta) > 0$ fulfilling

$$
\begin{aligned}
&C_4(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \|\nabla n_\varepsilon^{p/2}\|_{L^2(\Omega)}^{\frac{6}{(3p-1)(2-\alpha)}} + C_4(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \\
&\quad \le \eta \|\nabla n_\varepsilon^{p/2}\|_{L^2(\Omega)}^2 + C_5(\eta) y_\varepsilon^{\frac{(3p-1)(1-\alpha)}{(3p-1)(2-\alpha)-3}}(t) + C_4(\eta) y_\varepsilon^{\frac{1-\alpha}{2-\alpha}}(t) \\
&\quad \le \eta \|\nabla n_\varepsilon^{p/2}\|_{L^2(\Omega)}^2 + (C_4(\eta) + C_5(\eta)) y_\varepsilon^{\frac{(3p-1)(1-\alpha)}{(3p-1)(2-\alpha)-3}}(t) + C_4(\eta)
\end{aligned}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, so that (3.10) results from (3.11) and (3.12). ∎

In summary, Lemmas 3.2, 3.4 and 3.5 enable us to control the growth of $y_\varepsilon$ on the basis of Lemma 3.1 as follows.

**Lemma 3.7.** *Let $p > 3/2$ and $\alpha \in (1/2, 1)$. Then there exist $\vartheta = \vartheta(p, \alpha) > 1$ and $C = C(p, \alpha) > 0$ such that for arbitrary $\varepsilon \in (0, 1)$, the function $y_\varepsilon$ defined in (1.7) satisfies*

$$
y_\varepsilon'(t) \le C y_\varepsilon^\vartheta(t) + C \quad \textit{for all } t > 0. \qquad (3.13)
$$

*Proof.* We fix any $\rho = \rho(\alpha) \in (\frac{3}{4}, \frac{\alpha+1}{2})$ and let

$$
\vartheta = \vartheta(p, \alpha) := \max \left\{ \frac{2p-1}{2p-3}, \frac{2\alpha+1}{2\alpha-1}, \frac{\alpha - 2\rho + 2}{\alpha - 2\rho + 1}, \frac{(3p-1)(1-\alpha)}{(3p-1)(2-\alpha)-3} \right\} > 1.
$$

Then (3.13) readily results upon combining Lemma 3.1 with Lemmas 3.2 and 3.4–3.6 when applied to suitably small $\eta = \eta(p, \alpha) > 0$, and employing Young's inequality to estimate

$$
y_\varepsilon^{\frac{2p-1}{2p-3}}(t) + y_\varepsilon^{\frac{2\alpha+1}{2\alpha-1}}(t) + y_\varepsilon^{\frac{\alpha-2\rho+2}{\alpha-2\rho+1}}(t) + y_\varepsilon^{\frac{(3p-1)(1-\alpha)}{(3p-1)(2-\alpha)-3}}(t) \le 4 y_\varepsilon^\vartheta(t) + 3
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. ∎

By integration of (3.13), as the main result of this section we obtain the following quantitative information about lengths of time intervals within which the growth of $y_\varepsilon$ can conveniently be controlled.

**Lemma 3.8.** *Let $p > 3/2$ and $\alpha \in (1/2, 1)$. Then for all $k \ge 1$ there exists $T(k) = T(k; p, \alpha) \in (0, 1/k]$ with the property that whenever $\varepsilon \in (0, 1)$ and $t_0 \ge 0$ are such that with $y_\varepsilon$ taken from (1.7) we have*

$$y_\varepsilon(t_0) \le k, \tag{3.14}$$

*it follows that*

$$y_\varepsilon(t) \le 2k \quad \textit{for all } t \in (t_0, t_0 + T(k)). \tag{3.15}$$

*Proof.* By means of Lemma 3.7, we can pick $\vartheta = \vartheta(p, \alpha) > 1$ and $C_1 = C_1(p, \alpha) > 0$ such that

$$y_\varepsilon'(t) \le C_1 y_\varepsilon^\vartheta(t) + C_1 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1), \tag{3.16}$$

and given $k \ge 1$ we thereupon define

$$T(k; p, \alpha) := \min\{\overline{T}(k; p, \alpha), 1/k\} \quad \text{with} \quad \overline{T}(k; p, \alpha) := \frac{(1 - 2^{1-\vartheta})k^{1-\vartheta}}{2(\vartheta - 1)C_1}.$$

Then for fixed $t_0 \ge 0$,

$$\overline{y}(t) := \{k^{1-\vartheta} - 2(\vartheta - 1)C_1 \cdot (t - t_0)\}^{-\frac{1}{\vartheta - 1}}, \quad t \in [t_0, t_0 + \overline{T}(k; p, \alpha)],$$

defines a function $\overline{y} \in C^1([t_0, t_0 + \overline{T}(k; p, \alpha)])$ which satisfies $\overline{y}'(t) = 2C_1 \overline{y}^\vartheta(t)$ for all $t \in (t_0, t_0 + \overline{T}(k; p, \alpha))$ and $\overline{y}(t_0) = k$. In particular, $\overline{y}$ is nondecreasing and hence has the additional property that

$$1 \le k \le \overline{y}(t) \le \overline{y}(t_0 + \overline{T}(k; p, \alpha)) = 2k \quad \text{for all } t \in (t_0, t_0 + \overline{T}(k; p, \alpha)), \tag{3.17}$$

whence especially

$$\overline{y}'(t) - C_1 \overline{y}^\vartheta(t) - C_1 = C_1 \overline{y}^\vartheta(t) - C_1 \ge 0 \quad \text{for all } t \in (t_0, t_0 + \overline{T}(k; p, \alpha)).$$

Together with (3.16), through an ODE comparison this entails that whenever $\varepsilon \in (0, 1)$ and $t_0 \ge 0$ are such that (3.14) holds, we have $y_\varepsilon \le \overline{y}$ in $(t_0, t_0 + \overline{T}(k; p, \alpha))$. Therefore, (3.15) becomes a consequence of the upper estimate for $\overline{y}$ in (3.17), combined with the evident fact that $T(k; p, \alpha) \le \overline{T}(k; p, \alpha)$. ∎

## 4. Local theory for $p > 3/2$: Higher order estimates

The purpose of this section is to extend the above local regularity theory toward higher order estimates, which will be achieved on the basis of Lemma 3.8 that will form a starting point of a bootstrap procedure gradually improving our knowledge about smoothness in suitable time intervals past an instant at which (3.14) is supposed to be valid. Accordingly, throughout this section we shall rely on the assumptions $p > 3/2$ and $\alpha \in (1/2, 1)$ already made in the previous section.

In preparation for both Lemmas 4.2 and 4.3, let us first draw an essentially immediate consequence of Lemma 3.8 on the non-diffusive part of the flux appearing in the first equation from (2.3).

**Lemma 4.1.** *Let $p > 3/2$ and $\alpha \in (1/2, 1)$, and for $k \geq 1$ let $T(k) = T(k; p, \alpha)$ be as in Lemma 3.8. Then there exist $q_0 = q_0(p, \alpha) > 3$ and $C(k) = C(k; p, \alpha) > 0$ such that whenever (3.14) is satisfied for some $\varepsilon \in (0, 1)$ and $t_0 \geq 0$, we have*

$$\|F_\varepsilon'(n_\varepsilon(\cdot, t))\nabla c_\varepsilon(\cdot, t) + u_\varepsilon(\cdot, t)\|_{L^{q_0}(\Omega)} \leq C(k) \quad \text{for all } t \in (t_0, t_0 + T(k)). \tag{4.1}$$

*Proof.* As our assumptions $p > 3/2$ and $\alpha > 1/2$ warrant that $\min\{2p, \frac{6}{3-2\alpha}\} > 3$, we can fix $q_0 = q_0(p, \alpha) > 3$ such that

$$q_0 \leq 2p \quad \text{and} \quad q_0 < \frac{6}{3 - 2\alpha}.$$

Then since the latter condition herein ensures that $D(A^{\alpha/2}) \hookrightarrow L^q(\Omega; \mathbb{R}^3)$ [12, 15], we readily infer (4.1) from (2.5), Lemma 3.8 and our definition of $(y_\varepsilon)_{\varepsilon \in (0,1)}$. ■

Essentially relying on the fact that the number $q_0$ obtained above exceeds the size of the spatial dimension considered, an argument based on regularization effects of the heat semigroup yields $L^\infty$ bounds for the first solution component, involving temporal weight functions that depend on the distance to the times at which (3.14) is supposed to hold.

**Lemma 4.2.** *Let $p > 3/2$ and $\alpha \in (1/2, 1)$, and let $(T(k))_{k \geq 1} = (T(k; p, \alpha))_{k \geq 1}$ be as accordingly provided by Lemma 3.8. Then there exists $C(k) = C(k; p, \alpha) > 0$ such that if $\varepsilon \in (0, 1)$ and $t_0 \geq 0$ are such that (3.14) holds, we have*

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C(k) \cdot (t - t_0)^{-\frac{3}{2p}} \quad \text{for all } t \in (t_0, t_0 + T(k)). \tag{4.2}$$

*Proof.* With $q_0 = q_0(p, \alpha) > 3$ taken from Lemma 4.1, noting that clearly $\frac{3p}{(p-3)_+} > 3$ we fix $q = q(p, \alpha) > 3$ such that

$$q \leq q_0(p) \quad \text{and} \quad q < \frac{3p}{(p - 3)_+}, \tag{4.3}$$

whence by boundedness of $\Omega$, through Lemma 4.1 the first condition herein ensures the existence of $C_1 = C_1(k) > 0$ such that whenever (3.14) holds for some $\varepsilon \in (0, 1)$ and $t_0 \geq 0$, the function $h_\varepsilon := F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon + u_\varepsilon$ satisfies

$$\|h_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq C_1 \quad \text{for all } t \in (t_0, t_0 + T(k)). \tag{4.4}$$

In order to appropriately estimate

$$M := \sup_{t \in (t_0, t_0 + T(k))} (t - t_0)^{\frac{3}{2p}} \|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)},$$

on the basis of this, we pick any $r = r(p, \alpha) \in (3, q)$ and invoke known smoothing estimates for the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma \geq 0}$ on $\Omega$ [11, 42] to fix $C_2 = C_2(p) > 0$ and $C_3 = C_3(p, \alpha) > 0$ such that whenever $\sigma \in (0, 1)$,

$$\|e^{\sigma\Delta}\varphi\|_{L^\infty(\Omega)} \leq C_2 \sigma^{-\frac{3}{2p}} \|\varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in C^0(\overline{\Omega})$$

and

$$\|e^{\sigma\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le C_3\sigma^{-\frac{1}{2}-\frac{3}{2r}}\|\varphi\|_{L^r(\Omega)}$$

for all $\varphi \in C^1(\overline{\Omega};\mathbb{R}^3)$ such that $\varphi\cdot\nu = 0$ on $\partial\Omega$. According to a Duhamel representation associated with the first equation from (2.3), this entails that for all $t \in (t_0, t_0+T(k))$,

$$\begin{aligned}
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &= \left\| e^{(t-t_0)\Delta}n_\varepsilon(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot\{n_\varepsilon(\cdot,s)h_\varepsilon(\cdot,s)\}\,ds \right\|_{L^\infty(\Omega)} \\
&\le C_2(t-t_0)^{-\frac{3}{2p}}\|n_\varepsilon(\cdot,t_0)\|_{L^p(\Omega)} + C_3\int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2r}}\|n_\varepsilon(\cdot,s)h_\varepsilon(\cdot,s)\|_{L^r(\Omega)}\,ds,
\end{aligned}$$

because $T(k)\le 1/k\le 1$. Since furthermore

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le y_\varepsilon^{1/p}(t) \le (2k)^{1/p} \quad \text{for all } t\in[t_0,t_0+T(k))$$

thanks to (3.15), and since the second requirement in (4.3) along with the restriction $r>3$ implies that

$$\frac{qr}{q-r} - p > \frac{3q}{q-3} - p = \frac{3p-(p-3)q}{q-3} > 0$$

and hence $\frac{qr}{q-r} > p$, we may use the Hölder inequality to infer that due to (4.4), writing $a := \frac{qr-pq+pr}{qr} \in (0,1)$ we have

$$\begin{aligned}
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le C_2(t-t_0)^{-\frac{3}{2p}}\|n_\varepsilon(\cdot,t_0)\|_{L^p(\Omega)} \\
&\qquad + C_3\int_{t_0}^t (t-s)^{-1/2-\frac{3}{2r}}\|n_\varepsilon(\cdot,s)\|_{L^{\frac{qr}{q-r}}(\Omega)}\|h_\varepsilon(\cdot,s)\|_{L^q(\Omega)}\,ds \\
&\le C_2(t-t_0)^{-\frac{3}{2p}}\|n_\varepsilon(\cdot,t_0)\|_{L^p(\Omega)} \\
&\qquad + C_3\int_{t_0}^t (t-s)^{-1/2-\frac{3}{2r}}\|n_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^a\|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)}^{1-a}\|h_\varepsilon(\cdot,s)\|_{L^q(\Omega)}\,ds \\
&\le (2k)^{1/p}C_2(t-t_0)^{-\frac{3}{2p}} \\
&\qquad + (2k)^{\frac{1-a}{p}}C_1C_3M^a\int_{t_0}^t (t-s)^{-1/2-\frac{3}{2r}}(s-t_0)^{-\frac{3a}{2p}}\,ds \\
&= (2k)^{1/p}C_2(t-t_0)^{-\frac{3}{2p}} \\
&\qquad + (2k)^{\frac{1-a}{p}}C_1C_3C_4M^a(t-t_0)^{\frac{1}{2}-\frac{3}{2r}-\frac{3a}{2p}} \quad \text{for all } t\in(t_0,t_0+T(k))
\end{aligned}$$

with $C_4 := \int_0^1(1-\sigma)^{-1/2-\frac{3}{2r}}\sigma^{-\frac{3a}{2p}}\,d\sigma$ being finite thanks to the inequalities $r>3$, $a<1$ and $p>3/2$. Observing that according to the definition of $a$,

$$\frac{3}{2p} + \frac{1}{2} - \frac{3}{2r} - \frac{3a}{2p} = \frac{q-3}{2q}$$

is positive, we thus infer that

$$M \le (2k)^{1/p}C_2 + (2k)^{\frac{1-a}{p}}C_1C_3C_4M^aT^{\frac{q-3}{2q}}(k),$$

so that

$$M \le \max\{1, \{(2k)^{1/p} C_2 + (2k)^{\frac{1-a}{p}} C_1 C_3 C_4 T^{\frac{q-3}{2q}}(k)\}^{\frac{1}{1-a}}\}$$

due to the fact that $a < 1$. ∎

Now due to the latter, standard parabolic Hölder theory becomes applicable to the first equation in (2.3):

**Lemma 4.3.** *Fix $p > 3/2$ and $\alpha \in (1/2, 1)$ and let $T(k) = T(k; p, \alpha)$ be as in Lemma 3.8. Then for all $\tau \in (0, T(k))$ there exist $\gamma = \gamma(k, \tau, p, \alpha) \in (0, 1)$ and $C(k, \tau) = C(k, \tau; p, \alpha) > 0$ with the property that whenever (3.14) is valid for some $\varepsilon \in (0, 1)$ and $t_0 \ge 0$, we have*

$$\|n_\varepsilon\|_{C^{\gamma,\gamma/2}(\overline{\Omega}\times[t_0+\tau, t_0+T(k)])} \le C(k, \tau). \tag{4.5}$$

*Proof.* Again using the fact that Lemma 4.1 implies an $(\varepsilon, t_0)$-independent estimate for $(F'_\varepsilon(n_\varepsilon(\cdot, t))\nabla c_\varepsilon(\cdot, t) + u_\varepsilon(\cdot, t))_{t\in(t_0, t_0+T(k))}$ in $L^s((t_0, t_0 + T(k)); L^{q_0}(\Omega))$ with $s := \infty$ and $q_0 > 3$ as provided there, based on the bound for $n_\varepsilon$ in $L^\infty_{loc}(\overline{\Omega} \times (t_0, t_0 + T(k)])$ provided by Lemma 4.2 we may derive this from standard Hölder regularity theory for scalar parabolic equations due to the fact that these choices ensure that $\frac{1}{s} + \frac{3}{2q_0} = \frac{3}{2q_0} < \frac{1}{2}$ [29]. ∎

In order to create a temporal localization setting for our derivation of appropriate estimates for $u_\varepsilon$ from this information on $n_\varepsilon$, let us fix a function $\zeta_0 \in C^\infty([0, \infty))$ such that $0 \le \zeta_0 \le 1$ and that $\zeta_0 \equiv 0$ on $[0, 1/2]$ as well as $\zeta_0 \equiv 1$ throughout $[1, \infty)$, and let

$$\zeta^{(t_0,\tau)}(t) := \zeta_0\left(\frac{t - t_0}{\tau}\right), \quad t \ge t_0, \tag{4.6}$$

for $t_0 \ge 0$ and $\tau > 0$. Then for arbitrary $\varepsilon \in (0, 1)$ and any such $t_0$ and $\tau$,

$$v_\varepsilon(x, t) := \zeta^{(t_0,\tau)}(t) u_\varepsilon(x, t), \quad x \in \overline{\Omega},\ t \ge t_0, \tag{4.7}$$

satisfies

$$\begin{cases} v_{\varepsilon t} = \Delta v_\varepsilon - (Y_\varepsilon u_\varepsilon \cdot \nabla) v_\varepsilon + \nabla(\zeta^{(t_0,\tau)}(t) P_\varepsilon) + g_\varepsilon(x, t), & \nabla \cdot v_\varepsilon = 0, \\ & x \in \Omega,\ t > t_0, \\ v_\varepsilon = 0, & x \in \partial\Omega,\ t > t_0, \\ v_\varepsilon = 0, & x \in \Omega,\ t \in [t_0, t_0 + \tau/2], \end{cases} \tag{4.8}$$

with

$$g_\varepsilon(x, t) := \zeta^{(t_0,\tau)}(t) n_\varepsilon(x, t) \nabla\Phi(x) + \zeta_t^{(t_0,\tau)}(t) u_\varepsilon(x, t), \quad x \in \Omega,\ t > t_0. \tag{4.9}$$

A first conclusion of Lemma 4.3 then asserts local-in-time $L^\infty$ and even Hölder bounds for $A^\beta v_\varepsilon$, when considered as an $L^2(\Omega)$-valued function, thereby providing the following information about $u_\varepsilon$:

**Lemma 4.4.** *Let $p > 3/2$, $\alpha \in (1/2, 1)$, $k \geq 1$ and $T(k) = T(k;\alpha)$ be as in Lemma 3.8, and let $\beta \in (\frac{5-2\alpha}{4}, 1)$. Then for all $\tau \in (0, T(k))$ there exists $C(k,\tau) = C(k,\tau; p,\alpha,\beta) > 0$ such that if $\varepsilon \in (0,1)$ and $t_0 \geq 0$ are such that (3.14) holds,*

$$\|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C(k,\tau) \quad \text{for all } t \in (t_0+\tau, t_0+T(k)) \tag{4.10}$$

*and*

$$\|A^\beta u_\varepsilon(\cdot,t) - A^\beta u_\varepsilon(\cdot,t_\star)\|_{L^2(\Omega)} \quad \leq \quad C(k,\tau)\cdot(t-t_\star)^{1-\beta} \tag{4.11}$$

*for all $t_\star \in (t_0+\tau, t_0+T(k))$ and $t \in (t_\star, t_0+T(k))$.*

*Proof.* Once more using the fact that $\alpha > 1/2$ implies the inequality $\frac{6}{3-2\alpha} > 3$, we fix $q = q(\alpha) > 3$ such that $q < \frac{6}{3-2\alpha}$ and hence $D(A^{\alpha/2}) \hookrightarrow L^q(\Omega;\mathbb{R}^3)$ according to [12, 15]. Since $Y_\varepsilon A^{\alpha/2} = A^{\alpha/2} Y_\varepsilon$ on $D(A^{\alpha/2})$, and since $\|Y_\varepsilon\varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}$ for all $\varphi \in L^2_\sigma(\Omega)$, by means of Lemma 4.2 we thus find $C_1 = C_1(k,p,\alpha) > 0$, $C_2 = C_2(k,p,\alpha) > 0$ and $C_3 = C_3(k,\tau,p,\alpha) > 0$ such that whenever (3.14) holds for some $\varepsilon \in (0,1)$ and $t_0 \geq 0$, the functions $v_\varepsilon$, $Y_\varepsilon u_\varepsilon$ and $g_\varepsilon$ in (4.8) and (4.9) satisfy

$$\|A^{\alpha/2} v_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_1 \quad \text{for all } t \in (t_0, t_0+T(k)), \tag{4.12}$$

$$\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \leq C_2 \quad \text{for all } t \in (t_0, t_0+T(k)), \tag{4.13}$$

$$\|g_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_3 \quad \text{for all } t \in (t_0, t_0+T(k)). \tag{4.14}$$

To make appropriate use of this, we fix $\beta_0 = \beta_0(\alpha,\beta) \in (\frac{5-2\alpha}{4}, \beta)$ and note that then $D(A^{\beta_0}) \hookrightarrow W^{1,\frac{2q}{q-2}}(\Omega;\mathbb{R}^3)$ [12, 15], whence besides taking $C_4 = C_4(\beta) > 0$ and $C_5 = C_5(\beta) > 0$ such that

$$\|A^\beta e^{-\xi A}\varphi\|_{L^2(\Omega)} \leq C_4 \xi^{-\beta}\|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in L^2_\sigma(\Omega) \text{ and } \xi > 0, \tag{4.15}$$

$$\|A^{\beta+1} e^{-\xi A}\varphi\|_{L^2(\Omega)} \leq C_5 \xi^{-\beta-1}\|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in L^2_\sigma(\Omega) \text{ and } \xi > 0, \tag{4.16}$$

by using Lemma 3.3 we can choose $C_6 = C_6(\alpha,\beta) > 0$, $a = a(\alpha,\beta) \in (0,1)$ and $C_7 = C_7(\alpha,\beta) > 0$ fulfilling

$$\begin{aligned} \|\nabla\varphi\|_{L^{\frac{2q}{q-2}}(\Omega)} &\leq C_6 \|A^{\beta_0}\varphi\|_{L^2(\Omega)} \\ &\leq C_7 \|A^\beta\varphi\|^a_{L^2(\Omega)} \|A^{\alpha/2}\varphi\|^{1-a}_{L^2(\Omega)} \quad \text{for all } \varphi \in D(A^\beta). \end{aligned} \tag{4.17}$$

We now apply $A^\beta$ to a variation-of-constants representation of the accordingly defined function $v_\varepsilon$ from (4.7) to see that for arbitrary $t_\star \in [t_0, t_0 + T(k))$ and $t \in (t_\star, t_0 + T(k))$,

$$\begin{aligned}
&\|A^\beta v_\varepsilon(\cdot,t) - A^\beta v_\varepsilon(\cdot,t_\star)\|_{L^2(\Omega)} \\
&\quad = \Bigg\| - \int_{t_0}^{t_\star} A^\beta [e^{-(t-s)A} - e^{-(t_\star-s)A}] \mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla) v_\varepsilon(\cdot,s)\}\,ds \\
&\qquad - \int_{t_\star}^{t} A^\beta e^{-(t-s)A} \mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla) v_\varepsilon(\cdot,s)\}\,ds \\
&\qquad + \int_{t_0}^{t_\star} A^\beta [e^{-(t-s)A} - e^{-(t_\star-s)A}] \mathcal{P} g_\varepsilon(\cdot,s)\,ds \\
&\qquad + \int_{t_\star}^{t} A^\beta e^{-(t-s)A} \mathcal{P} g_\varepsilon(\cdot,s)\,ds \Bigg\|_{L^2(\Omega)},
\end{aligned} \tag{4.18}$$

where by (4.16), the Cauchy–Schwarz inequality, (4.13) and (4.17),

$$\begin{aligned}
&\Bigg\| - \int_{t_0}^{t_\star} A^\beta [e^{-(t-s)A} - e^{-(t_\star-s)A}] \mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla) v_\varepsilon(\cdot,s)\}\,ds \Bigg\|_{L^2(\Omega)} \\
&\quad = \Bigg\| \int_{t_0}^{t_\star} \int_{t_\star}^{t} A^{\beta+1} e^{-(\sigma-s)A} \mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla) v_\varepsilon(\cdot,s)\}\,d\sigma\,ds \Bigg\|_{L^2(\Omega)} \\
&\quad \le C_4 \int_{t_0}^{t_\star} \int_{t_\star}^{t} (\sigma-s)^{-\beta-1} \|\mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla) v_\varepsilon(\cdot,s)\}\|_{L^2(\Omega)}\,d\sigma\,ds \\
&\quad \le C_4 \int_{t_0}^{t_\star} \int_{t_\star}^{t} (\sigma-s)^{-\beta-1} \|Y_\varepsilon u_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \|\nabla v_\varepsilon(\cdot,s)\|_{L^{\frac{2q}{q-2}}(\Omega)}\,d\sigma\,ds \\
&\quad \le C_2 C_5 C_7 \int_{t_0}^{t_\star} \int_{t_\star}^{t} (\sigma-s)^{-\beta-1} \|A^\beta v_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^a \|A^{\alpha/2} v_\varepsilon(\cdot,s)\|_{L^2(\Omega)}^{1-a}\,d\sigma\,ds \\
&\quad \le C_1^{1-a} C_2 C_5 C_7 M_\varepsilon^a \int_{t_0}^{t_\star} \int_{t_\star}^{t} (s-\sigma)^{-\beta-1}\,d\sigma\,ds \\
&\quad = \frac{C_1^{1-a} C_2 C_5 C_7 M_\varepsilon^a}{\beta(1-\beta)} \cdot \{(t-t_\star)^{1-\beta} - (t-t_0)^{1-\beta} + (t_\star-t_0)^{1-\beta}\} \\
&\quad \le \frac{C_1^{1-a} C_2 C_5 C_7 M_\varepsilon^a}{\beta(1-\beta)} \cdot (t-t_\star)^{1-\beta},
\end{aligned} \tag{4.19}$$

with $M_\varepsilon := \max_{s\in[t_0,t_0+T(k)]} \|A^\beta v_\varepsilon(\cdot,s)\|_{L^2(\Omega)}$. Likewise, (4.16) and (4.14) imply that for all $t_\star \in [t_0, t_0+T(k))$ and $t \in (t_\star, t_0+T(k))$,

$$\begin{aligned}
&\Bigg\| \int_{t_0}^{t_\star} A^\beta [e^{-(t-s)A} - e^{-(t_\star-s)A}] \mathcal{P} g_\varepsilon(\cdot,s)\,ds \Bigg\|_{L^2(\Omega)} \\
&\quad = \Bigg\| - \int_{t_0}^{t_\star} \int_{t_\star}^{t} A^{\beta+1} e^{-(\sigma-s)A} \mathcal{P} g_\varepsilon(\cdot,s)\,d\sigma\,ds \Bigg\|_{L^2(\Omega)} \\
&\quad \le C_3 C_5 \int_{t_0}^{t_\star} \int_{t_\star}^{t} (\sigma-s)^{-\beta-1}\,d\sigma\,ds \le \frac{C_3 C_5}{\beta(1-\beta)} \cdot (t-t_\star)^{1-\beta},
\end{aligned} \tag{4.20}$$

and furthermore we can combine (4.15) with (4.13) and (4.17) to estimate

$$\begin{aligned}
&\left\| - \int_{t_\star}^{t} A^{\beta} e^{-(t-s)A} \mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla)v_\varepsilon(\cdot,s)\}\,ds \right\|_{L^2(\Omega)} \\
&\quad\le C_4 \int_{t_\star}^{t} (t-s)^{-\beta} \|\mathcal{P}\{(Y_\varepsilon u_\varepsilon(\cdot,s)\cdot\nabla)v_\varepsilon(\cdot,s)\}\|_{L^2(\Omega)}\,ds \\
&\quad\le C_4 \int_{t_\star}^{t} (t-s)^{-\beta} \|Y_\varepsilon u_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \|\nabla v_\varepsilon(\cdot,s)\|_{L^{\frac{2q}{q-2}}(\Omega)}\,ds \\
&\quad\le C_1^{1-a} C_2 C_4 C_7 M_\varepsilon^a \int_{t_\star}^{t} (t-s)^{-\beta}\,ds = \frac{C_1^{1-a} C_2 C_4 C_7 M_\varepsilon^a}{1-\beta}\cdot(t-t_\star)^{1-\beta} \qquad (4.21)
\end{aligned}$$

for all $t_\star \in [t_0, t_0+T(k))$ and $t \in (t_\star, t_0+T(k))$, whereas (4.15) together with (4.14) shows that

$$\begin{aligned}
\left\| \int_{t_\star}^{t} A^{\beta} e^{-(t-s)A} \mathcal{P} g_\varepsilon(\cdot,s)\,ds \right\|_{L^2(\Omega)} &\le C_4 \int_{t_\star}^{t} (t-s)^{-\beta} \|\mathcal{P} g_\varepsilon(\cdot,s)\|_{L^2(\Omega)}\,ds \\
&\le C_3 C_4 \int_{t_\star}^{t} (t-s)^{-\beta}\,ds \\
&= \frac{C_3 C_4}{1-\beta}\cdot(t-t_\star)^{1-\beta} \qquad (4.22)
\end{aligned}$$

for all $t_\star \in [t_0, t_0+T(k))$ and $t \in (t_\star, t_0+T(k))$. In view of (4.17)–(4.22), on letting $t_\star := t_0$ we firstly obtain from (4.18) that since $v_\varepsilon(\cdot,t_0)=0$, $M_\varepsilon \le C_8 + C_8 M_\varepsilon^a$ and hence $M_\varepsilon \le \max\{1, (2C_8)^{\frac{1}{1-a}}\}$ with

$$C_8 = C_8(k,\alpha,\beta) := \frac{T^{1-\beta}(k)}{1-\beta}\cdot\max\left\{\frac{C_3C_5}{\beta} + C_3C_4, \frac{C_1^{1-a}C_2C_5C_7}{\beta} + C_1^{1-a}C_2C_4C_7\right\}.$$

Having thereby proved (4.10), inserting this information into (4.19) and (4.21) we thereupon obtain (4.11) from (4.18)–(4.22) and our definition of $v_\varepsilon$. ∎

A particular consequence asserts Hölder bounds not only for $u_\varepsilon$ itself, but also for the expression $Y_\varepsilon u_\varepsilon$ forming an essential part of the nonlinear convection term in (2.3).

**Corollary 4.5.** *Let $p > 3/2$, $\alpha \in (1/2, 1)$ and $k \ge 1$, and let $T(k) = T(k; p, \alpha)$ be as given by Lemma 3.8. Then for all $\tau \in (0, T(k))$ there exist $\gamma = \gamma(k,\tau,p,\alpha) \in (0,1)$ and $C(k,\tau) = C(k,\tau;p,\alpha) > 0$ with the property that if $\varepsilon \in (0,1)$ and $t_0 \ge 0$ are such that (3.14) is satisfied, the inequality*

$$\|u_\varepsilon\|_{C^{\gamma,\gamma/2}(\overline{\Omega}\times[t_0+\tau,t_0+T(k)])} + \|Y_\varepsilon u_\varepsilon\|_{C^{\gamma,\gamma/2}(\overline{\Omega}\times[t_0+\tau,t_0+T(k)])} \le C(k,\tau) \qquad (4.23)$$

*holds.*

*Proof.* We apply Lemma 4.4 to any fixed $\beta \in (\frac{5-2\alpha}{4}, 1)$ and then infer (4.23) from (4.10) and (4.11) upon observing that, in particular, $\beta > 3/4$ and hence $D(A^\beta) \hookrightarrow C^\gamma(\overline{\Omega};\mathbb{R}^3)$ for all $\gamma \in (0, 2\beta - 3/2)$ [12, 15], and again using that $\|A^\beta Y_\varepsilon \varphi\|_{L^2(\Omega)} \le \|A^\beta \varphi\|_{L^2(\Omega)}$ for all $\varphi \in D(A^\beta)$. ∎

Once more explicitly operating on the localized problem (4.8), combining the latter with, again, Lemma 4.3 enables us to derive the following higher order estimate through standard literature on Schauder theory for the Stokes evolution equations.

**Lemma 4.6.** *Let $p > 3/2$, $\alpha \in (1/2, 1)$, $k \geq 1$ and $T(k) = T(k; p, \alpha)$ be as in Lemma 3.8. Then for each $\tau \in (0, T(k))$ one can find $\gamma = \gamma(k, \tau, p, \alpha) \in (0, 1)$ and $C(k, \tau) = C(k, \tau; p, \alpha) > 0$ with the property that whenever $\varepsilon \in (0, 1)$ and $t_0 \geq 0$ are such that (3.14) holds, we have*

$$\|u_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(\overline{\Omega}\times[t_0+\tau,t_0+T(k)])} \leq C(k, \tau). \tag{4.24}$$

*Proof.* According to Corollary 4.5 and Lemma 4.3, we can pick $\gamma_i = \gamma_i(k, \tau, p, \alpha) \in (0, 1)$ and $C_i = C_i(k, \tau, p, \alpha) > 0$, $i \in \{1, 2\}$, with the property that if (3.14) is satisfied with some $\varepsilon \in (0, 1)$ and $t_0 \geq 0$, then taking $g_\varepsilon$ as accordingly introduced in (4.9) we have

$$\|Y_\varepsilon u_\varepsilon\|_{C^{\gamma_1,\gamma_1/2}(\overline{\Omega}\times[t_0+\tau/2,t_0+T(k)])} \leq C_1, \tag{4.25}$$

$$\|g_\varepsilon\|_{C^{\gamma_2,\gamma_2/2}(\overline{\Omega}\times[t_0+\tau/2,t_0+T(k)])} \leq C_2. \tag{4.26}$$

Now due to a well-known result from Schauder theory for the Stokes evolution system [33], there exist $\gamma_3 = \gamma_3(k, \tau, p, \alpha) \in (0, 1)$ and $C_3 = C_3(k, \tau, p, \alpha) > 0$ such that if $t_0 \geq 0$, $a \in C^{\gamma_1,\gamma_1/2}(\overline{\Omega} \times [t_0 + \tau/2, t_0 + T(k)]; \mathbb{R}^{3\times3})$ and $b \in C^{\gamma_2,\gamma_2/2}(\overline{\Omega} \times [t_0 + \tau/2, t_0 + T(k)]; \mathbb{R}^3)$ are such that $b(\cdot, t_0 + \tau/2) = 0$ on $\partial\Omega$ as well as

$$\|a\|_{C^{\gamma_1,\gamma_1/2}(\overline{\Omega}\times[t_0+\tau/2,t_0+T(k)])} \leq C_1, \quad \|b\|_{C^{\gamma_2,\gamma_2/2}(\overline{\Omega}\times[t_0+\tau/2,t_0+T(k)])} \leq C_2,$$

then the problem

$$\begin{cases} w_t = \Delta w + a(x,t) \cdot \nabla w + b(x,t) + \nabla Q, \quad \nabla \cdot w = 0, & \\ & x \in \Omega,\ t \in (t_0 + \tau/2, t_0 + T(k)), \\ w = 0, & x \in \partial\Omega,\ t \in (t_0 + \tau/2, t_0 + T(k)), \\ w(x, t_0 + \tau/2) = 0, & x \in \Omega, \end{cases}$$

admits a solution $(w, Q)$ with a uniquely determined $w \in C^{2+\gamma_3,1+\gamma_3/2}(\overline{\Omega} \times [t_0 + \tau/2, t_0 + T(k)]; \mathbb{R}^3)$ fulfilling

$$\|w\|_{C^{2+\gamma_3,1+\gamma_3/2}(\overline{\Omega}\times[t_0+\tau/2,t_0+T(k)])} \leq C_3.$$

In view of (4.8), (4.25) and (4.26), an application thereof to $a := Y_\varepsilon u_\varepsilon$ and $b := g_\varepsilon$ immediately yields (4.24), because actually $g_\varepsilon(\cdot, t_0 + \tau/2) \equiv 0$ throughout $\overline{\Omega}$ by (4.9) and (4.6). ∎

According to this and to Lemma 4.3, we are now in a position to twice invoke classical Schauder theory for scalar parabolic problems to successively deduce second order estimates also for the first two solution components.

**Lemma 4.7.** *Suppose that $p > 3/2$ and $\alpha \in (1/2, 1)$, that $k \geq 1$, and that $T(k) = T(k; p, \alpha)$ is as in Lemma 3.8. Then for arbitrary $\tau \in (0, T(k))$ there exist $\gamma = \gamma(k, \tau, p, \alpha) \in (0, 1)$ and $C(k, \tau) = C(k, \tau; p, \alpha) > 0$ such that if (3.14) holds with some $\varepsilon \in (0, 1)$ and $t_0 \geq 0$, the inequalities*

$$\|n_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(\overline{\Omega}\times[t_0+\tau,t_0+T(k)])} \leq C(k, \tau), \tag{4.27}$$

$$\|c_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(\overline{\Omega}\times[t_0+\tau,t_0+T(k)])} \leq C(k, \tau) \tag{4.28}$$

*hold.*

*Proof.* Using Lemma 4.3 and Corollary 4.5 as a starting point, we can firstly derive (4.28) from (2.3) and standard parabolic Schauder theory [22] through a reasoning of quite the same flavor as that in the proof of Lemma 4.6. Thereafter, (4.28) can be seen to provide sufficient regularity information so as to warrant accessibility of (4.27) to the same token. ∎

## 5. Quantifying exceptionality of largeness: A general endpoint lemma

Our next goal will be to apply the local theory developed above, with appropriately selected parameters $p$ and $\alpha$, for suitably chosen values of $t_0$ at which solutions remain conveniently far from singular behavior, in the sense of (3.14). In Section 6 this will be achieved by means of bounds on energy dissipation rates which however, through their temporally integrated nature do not entirely rule out singular behavior, but after all provide some information about a certain exceptional character of times at which solutions may become inconveniently large.

Our quantitative exploitation of corresponding integral estimates, and hence our selection of instants $t_0$ to be used above, will be motivated by the following general observation, possibly being of independent interest, concerning endpoints of intervals of prescribed length throughout which a given function $y$ essentially exceeds some fixed value. The estimate (5.2) on the size of the set of all such points generalizes an inequality trivially valid for continuous $y$ to arbitrary measurable functions, and thereby warrants applicability to the possibly discontinuous limit object of the quantities $y_\varepsilon$ discussed before, to be precisely defined in (6.10) below.

**Lemma 5.1.** *Let $T > 0$ and $y : (0, T) \to \mathbb{R}$ be measurable. Then for each $\tau \in (0, T)$ and $k > 0$,*

$$S(k, \tau) := \{t_\star \in (\tau, T) \mid y(t) \geq k \text{ for a.e. } t \in (t_\star - \tau, t_\star)\} \tag{5.1}$$

*has the property that its outer Lebesgue measure $|S(k, \tau)|^\star$ satisfies*

$$|S(k, \tau)|^\star \leq |\{t \in (0, T) \mid y(t) \geq k\}|. \tag{5.2}$$

*Proof.* Assuming without loss of generality that $S(k, \tau)$ is not empty, we let $t_1 := \sup S(k, \tau) \in (\tau, T]$ and

$$\widehat{t}_1 := \inf\{t_\star \in (0, t_1) \mid y(t) \geq k \text{ for a.e. } t \in (t_\star, t_1)\},$$

and note that $\widehat{t}_1$ then is well-defined and nonnegative with

$$\widehat{t}_1 \le t_1 - \tau \tag{5.3}$$

according to the definitions of $t_1$ and $S(k,\tau)$. Moreover, the construction of $\widehat{t}_1$ enables us to fix a null set $N_1 \subset [0,T]$ such that

$$y(t) \ge k \quad \text{for all } t \in [\widehat{t}_1, t_1] \setminus N_1.$$

Now in the case when $\widehat{t}_1 \le \tau$, we must have $y(t) \ge k$ for all $t \in (\tau, t_1) \setminus N_1$ and hence, again by definition of $S(k,\tau)$, trivially infer that $S(k,\tau) \subset (\tau, t_1] \subset \{t \in (0,T) \mid y(t) \ge k\} \cup N_1$ and that thus (5.2) holds due to the fact that $|N_1| = 0$. Similarly, if $\widehat{t}_1 > \tau$ and

$$\Sigma_1 := \{t \in S(k,\tau) \mid t \le \widehat{t}_1\}$$

is empty, then $S(k,\tau) \subset (\widehat{t}_1, t_1] \subset \{t \in (0,T) \mid y(t) \ge k\} \cup N_1$ and hence we may conclude as before.

If $\widehat{t}_1 > \tau$ and $\Sigma_1 \ne \emptyset$, however, then

$$t_2 := \sup \Sigma_1$$

is a well-defined element of $(\tau, \widehat{t}_1]$ which, due to (5.3), in fact even satisfies $t_2 \le t_1 - \tau$. Repeating this selection process if necessary, we thus obtain an integer $j_0 \le T/\tau$ as well as finite families $(t_j)_{j=1}^{j_0}$ and $(\widehat{t}_j)_{j=1}^{0}$ such that writing $\widehat{t}_0 := T$, for all $j \in \{1,\dots,j_0\}$ we have

$$t_j = \inf\{t \in S(k,\tau) \mid t \le \widehat{t}_{j-1}\}, \quad \widehat{t}_j = \inf\{t_\star \in (0,t_j) \mid y(t) \ge k \text{ for a.e. } t \in (t_\star, t_j)\}$$

with $t_j \le \widehat{t}_{j-1} \le t_{j-1} - \tau$, and that there exist null sets $N_j \subset [0,T]$, $j \in \{1,\dots,j_0\}$, fulfilling

$$y(t) \ge k \quad \text{for all } t \in [\widehat{t}_j, t_j] \setminus N_j, \ j \in \{1,\dots,j_0\}.$$

As accordingly

$$S(k,\tau) \subset \bigcup_{j=1}^{j_0} [\widehat{t}_j, t_j] \subset \{t \in (0,T) \mid y(t) \ge k\} \cup \bigcup_{j=1}^{j_0} N_j,$$

due to an evident null set property of the rightmost union herein we again arrive at (5.2) also in this general case. ∎

An evident consequence of the latter will be of importance for our subsequent reasoning.

**Corollary 5.2.** *Let $T > 0$ and $y \in L^q((0,T))$ for some $q > 0$. Then the sets $S(k,\tau)$, $(k,\tau) \in (0,\infty) \times (0,T)$, defined in (5.1) satisfy*

$$\sup_{\tau \in (0,T)} |S(k,\tau)|^\star \to 0 \quad \text{as } k \to \infty.$$

*Proof.* This is evident from Lemma 5.1 and the fact that

$$\int_0^T |y|^q \ge k^q \cdot |\{t \in (0,T) \mid y(t) \ge k\}| \quad \text{for all } k > 0.$$ ∎

## 6. Quantifying exceptionality of largeness: Exploiting a quasi-energy structure

In accordance with Corollary 5.2, we shall next intend to identify conditions on the parameters $p$ and $\alpha$ which firstly ensure convergence of the functions from (1.7) as $\varepsilon \searrow 0$ in some appropriate sense, and which secondly warrant that the limit function $y$ thereby obtained belongs to some $L^q$ space. The following implications of a quasi-energy structure associated with (2.1) and (2.2) have been observed in [46].

**Lemma 6.1.** *For all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \int_\Omega \left\{ \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + |\nabla n_\varepsilon|^{5/4} + n_\varepsilon^{5/3} + \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + |\nabla u_\varepsilon|^2 + |u_\varepsilon|^{10/3} \right\} \le C(T) \tag{6.1}$$

*for all $\varepsilon \in (0,1)$, and*

$$\int_0^T \{ \|n_{\varepsilon t}(\cdot,t)\|^{10/9}_{(W^{1,10}(\Omega))^\star} + \|(\sqrt{c_\varepsilon})_t(\cdot,t)\|^{5/3}_{(W^{1,5/2}(\Omega))^\star} + \|u_{\varepsilon t}(\cdot,t)\|^{5/4}_{(W^{1,5}_{0,\sigma}(\Omega))^\star} \} \, dt \le C(T) \tag{6.2}$$

*for all $\varepsilon \in (0,1)$. Moreover,*

$$\sup_{\varepsilon\in(0,1)} \sup_{t>0} \left\{ \int_\Omega \frac{|\nabla c_\varepsilon(\cdot,t)|^2}{c_\varepsilon} + \int_\Omega |u_\varepsilon(\cdot,t)|^2 \right\} < \infty. \tag{6.3}$$

*Proof.* This can be obtained by simply collecting the outcomes of [46, Lemmas 3.8, 3.10, 3.11]. ∎

A straightforward interpolation between (2.9) and the first estimate implicitly contained in (6.1) yields the following further regularity information of order zero for the first solution component.

**Lemma 6.2.** *Let $p \in (1,3]$. Then for all $T > 0$ there exists $C(p,T) > 0$ such that*

$$\int_0^T \|n_\varepsilon(\cdot,t)\|^{\frac{2p}{3(p-1)}}_{L^p(\Omega)} \, dt \le C(p,T) \quad \textit{for all } \varepsilon \in (0,1). \tag{6.4}$$

*Proof.* By means of a Gagliardo–Nirenberg interpolation, we find $C_1 = C_1(p) > 0$ such that for all $t > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned} \|n_\varepsilon\|^{\frac{2p}{3(p-1)}}_{L^p(\Omega)} &= \|\sqrt{n_\varepsilon}\|^{\frac{4p}{3(p-1)}}_{L^{2p}(\Omega)} \\ &\le C_1 \|\nabla\sqrt{n_\varepsilon}\|^2_{L^2(\Omega)} \|\sqrt{n_\varepsilon}\|^{\frac{2(3-p)}{3(p-1)}}_{L^2(\Omega)} + C_1 \|\sqrt{n_\varepsilon}\|^{\frac{4p}{3(p-1)}}_{L^2(\Omega)} \\ &= \frac{C_1}{4} \cdot \left\{ \int_\Omega n_0 \right\}^{\frac{3-p}{3(p-1)}} \cdot \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_1 \cdot \left\{ \int_\Omega n_0 \right\}^{\frac{2p}{3(p-1)}}, \end{aligned} \tag{6.5}$$

because $\|\sqrt{n_\varepsilon}\|^2_{L^2(\Omega)} = \int_\Omega n_\varepsilon = \int_\Omega n_0$ for any such $t$ and $\varepsilon$ due to (2.9). In view of Lemma 6.1, an integration of (6.5) yields (6.4). ∎

In conjunction again with (6.1), the latter lemma implies a gradient estimate involving a space integrability exponent larger than that appearing in the expression $|\nabla n_\varepsilon|^{5/4}$ from (6.1), at the cost of a reduced regularity in time.

**Lemma 6.3.** *If $r \in (1, 3/2]$, then given any $T > 0$ one can find $C(r, T) > 0$ fulfilling*

$$\int_0^T \|\nabla n_\varepsilon(\cdot, t)\|_{L^r(\Omega)}^{\frac{2r}{4r-3}} dt \le C(r, T) \quad \textit{for all } \varepsilon \in (0, 1). \tag{6.6}$$

*Proof.* Two applications of the Hölder inequality show that for all $T > 0$ and each $\varepsilon \in (0, 1)$,

$$\begin{aligned}
\int_0^T \|\nabla n_\varepsilon(\cdot, t)\|_{L^r(\Omega)}^{\frac{2r}{4r-3}} dt &= \int_0^T \left\{ \int_\Omega \left\{ \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \right\}^{r/2} \cdot n_\varepsilon^{r/2} \, dx \right\}^{\frac{2}{4r-3}} dt \\
&\le \int_0^T \left\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} dx \right\}^{\frac{r}{4r-3}} \cdot \left\{ \int_\Omega n_\varepsilon^{\frac{r}{2-r}} dx \right\}^{\frac{2-r}{4r-3}} dt \\
&\le \left\{ \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} dx \, dt \right\}^{\frac{r}{4r-3}} \cdot \left\{ \int_0^T \left\{ \int_\Omega n_\varepsilon^{\frac{r}{2-r}} dx \right\}^{\frac{2-r}{3(r-1)}} dt \right\}^{\frac{3(r-1)}{4r-3}}.
\end{aligned} \tag{6.7}$$

Since our assumptions $r > 1$ and $r \le 3/2$ warrant that $p := \frac{r}{2-r}$ satisfies both $p > 1$ and $p \le 3$, and since moreover

$$\frac{2p}{3(p-1)} = \frac{2}{3 - 3 \cdot \frac{2-r}{r}} = \frac{r}{3(r-1)},$$

Lemma 6.2 applies so as to guarantee that for each $T > 0$ we can pick $C_1 = C_1(r, T) > 0$ satisfying

$$\int_0^T \left\{ \int_\Omega n_\varepsilon^{\frac{r}{2-r}} dx \right\}^{\frac{2-r}{3(r-1)}} dt = \int_0^T \|n_\varepsilon(\cdot, t)\|_{L^{\frac{r}{2-r}}(\Omega)}^{\frac{r}{3(r-1)}} dt \le C_1(r, T) \quad \text{for all } \varepsilon \in (0, 1).$$

Therefore, (6.6) results from (6.7) and Lemma 6.1. ∎

The compactness features thereby collected now prepare us for an appropriate passage to the limit $\varepsilon \searrow 0$, and especially for the definition and a convenient approximation of a function $y$ to be used when applying Corollary 5.2.

**Lemma 6.4.** *Let $p \in (1, 3)$ and $\alpha \in (0, 1)$. Then there exist a null set $N \subset (0, \infty)$ and $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that*

$$(n_\varepsilon, c_\varepsilon, u_\varepsilon) \to (n, c, u) \quad \textit{a.e. in } \Omega \times (0, \infty) \textit{ as } \varepsilon = \varepsilon_j \searrow 0 \tag{6.8}$$

*with some global weak energy solution $(n, c, u)$ of (1.2), and that furthermore the functions $y_\varepsilon$ from (1.7) satisfy*

$$y_\varepsilon(t) \to y(t) \quad \textit{for all } t \in (0, \infty) \setminus N \textit{ as } \varepsilon = \varepsilon_j \searrow 0, \tag{6.9}$$

*where*

$$y(t) \equiv y^{(p,\alpha)}(t) := \int_\Omega n^p(\cdot, t) + \int_\Omega |\nabla c(\cdot, t)|^{2p} + \int_\Omega |A^{\alpha/2}(\cdot, t)|^2, \quad t > 0. \tag{6.10}$$

*Proof.* According to the detailed derivation in [46, Lemma 4.1], a combination of the estimates from Lemma 6.1 with a straightforward extraction procedure based on an Aubin–Lions type lemma yields $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions $n, c$ and $u$ defined a.e. on $\Omega \times (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $n \geq 0$ and $c > 0$ a.e. in $\Omega \times (0,\infty)$, that (6.8) holds, and that $(n, c, u)$ forms a global weak energy solution of (1.2) in the sense of Definition 2.1.

To see that furthermore also (6.9) can be achieved for fixed $p \in (1,3)$ and $\alpha \in (0,1)$, given any such $p$ and $\alpha$ we use that $p < 3$, and that hence $\frac{3p}{p+3} < 3/2$, in choosing $r \in (1, 3/2)$ such that $r > \frac{3p}{p+3}$, and we moreover take some $q \in (3,6)$ fulfilling $q \geq 2p$. Then from Lemma 6.3, Lemma 6.1, (2.9) and (2.10) we actually know that

- $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{2r}{4r-3}}((0,T); W^{1,r}(\Omega))$,
- $(n_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^{10/9}((0,T); (W^{1,10}(\Omega))^\star)$,
- $(\sqrt{c_\varepsilon})_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T); W^{2,2}(\Omega))$,
- $((\sqrt{c_\varepsilon})_t)_{\varepsilon\in(0,1)}$ is bounded in $L^{5/3}((0,T); (W^{1,5/2}(\Omega))^\star)$,
- $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T); W^{1,2}_{0,\sigma}(\Omega))$,
- $(u_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^{5/4}((0,T); (W^{1,5}_{0,\sigma}(\Omega))^\star)$,

for all $T > 0$. Since herein $\frac{2r}{4r-3} > 1$ due to the fact that $r < 3/2$, and since the inequalities $r > \frac{3p}{p+3}$, $q < 6$ and $\alpha/2 < 1/2$ ensure that the embeddings $W^{1,r}(\Omega) \hookrightarrow L^p(\Omega)$, $W^{2,2}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ and $W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow D(A^{\alpha/2})$ are all compact in the three-dimensional setting considered, upon passing to a suitably relabeled further subsequence if necessary we may assume that again due to an Aubin–Lions lemma, with some null set $N \subset (0,\infty)$ we moreover have

$$n_\varepsilon(\cdot,t) \to n(\cdot,t) \quad \text{in } L^p(\Omega) \quad \text{for all } t \in (0,\infty)\setminus N, \tag{6.11}$$

$$\sqrt{c_\varepsilon(\cdot,t)} \to \sqrt{c(\cdot,t)} \quad \text{in } W^{1,q}(\Omega) \quad \text{for all } t \in (0,\infty)\setminus N, \tag{6.12}$$

$$u_\varepsilon(\cdot,t) \to u(\cdot,t) \quad \text{in } D(A^{\alpha/2}) \quad \text{for all } t \in (0,\infty)\setminus N \tag{6.13}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since (6.12) in particular entails that $\sqrt{c_\varepsilon(\cdot,t)} \to \sqrt{c(\cdot,t)}$ in $L^\infty(\Omega)$ for all $t \in (0,\infty)\setminus N$ as $\varepsilon = \varepsilon_j \searrow 0$ by continuity of $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, as guaranteed by our requirement that $q > 3$, it follows from (6.12) that as $\varepsilon = \varepsilon_j \searrow 0$ we also have

$$\begin{aligned} \nabla c_\varepsilon(\cdot,t) &= 2\sqrt{c_\varepsilon(\cdot,t)}\nabla\sqrt{c_\varepsilon(\cdot,t)} \\ &\to 2\sqrt{c(\cdot,t)}\nabla\sqrt{c(\cdot,t)} \\ &= \nabla c(\cdot,t) \quad \text{in } L^q(\Omega) \hookrightarrow L^{2p}(\Omega) \text{ for all } t \in (0,\infty)\setminus N \end{aligned}$$

due to the restriction that $q \geq 2p$. In conjunction with (6.11) and (6.13), this establishes (6.9). ∎

Indeed, $y$ enjoys some integrability feature in the spirit of Corollary 5.2:

**Lemma 6.5.** *Let $p \in (1,3)$ and $\alpha \in (0,1)$. Then the function $y = y^{(p,\alpha)}$ from (6.10) has the property that*

$$\int_0^T y^{q(p,\alpha)}(t)\,dt < \infty \quad \textit{for all } T > 0, \tag{6.14}$$

*where $q(p,\alpha) := \min\{\frac{2}{3(p-1)}, \frac{1}{\alpha}\}$.*

*Proof.* Given $T > 0$, from Lemma 6.1 and (2.10) we infer the existence of $C_i = C_i(T) > 0$, $i \in \{1,2,3,4\}$, such that

$$\int_0^T \int_\Omega |D^2 c_\varepsilon|^2 \le C_1 \quad \text{for all } \varepsilon \in (0,1), \tag{6.15}$$

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \le C_2 \quad \text{for all } \varepsilon \in (0,1), \tag{6.16}$$

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 \le C_3 \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1), \tag{6.17}$$

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C_4 \quad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{6.18}$$

By an application of the Gagliardo–Nirenberg inequality based on the assumption that $p < 3$, we can interpolate between (6.15) and (6.17) to see that with some $C_5 = C_5(p) > 0$ we have

$$\begin{aligned} \int_0^T \|\nabla c_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)}^{\frac{4p}{3(p-1)}}\,dt &\le C_5 \int_0^T \|D^2 c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 \|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{\frac{2(3-p)}{3(p-1)}}\,dt \\ &\le C_1 C_3^{\frac{3-p}{3(p-1)}} C_5 \quad \text{for all } \varepsilon \in (0,1), \end{aligned} \tag{6.19}$$

and utilizing Lemma 3.3 we similarly find $C_6 = C_6(\alpha) > 0$ such that

$$\begin{aligned} \int_0^T \|A^{\alpha/2} u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{2/\alpha}\,dt &\le C_6 \int_0^T \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^2 \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{2(1-\alpha)/\alpha}\,dt \\ &\le C_2 C_4^{\frac{1-\alpha}{\alpha}} C_6 \quad \text{for all } \varepsilon \in (0,1), \end{aligned} \tag{6.20}$$

because $\alpha \in (0,1)$. Since, apart from that, Lemma 6.2 provides $C_7 = C_7(p,T) > 0$ fulfilling

$$\int_0^T \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^{\frac{2p}{3(p-1)}}\,dt \le C_7 \quad \text{for all } \varepsilon \in (0,1),$$

by means of Young's inequality we can use that $pq(p,\alpha) \le \frac{2p}{3(p-1)}$ and $2q(p,\alpha) \le \frac{2}{\alpha}$ to estimate

$$\int_0^T y_\varepsilon^{q(p,\alpha)}(t)\,dt \le 3^{q(p,\alpha)} \cdot \left\{ \int_0^T \left\{ \int_\Omega n_\varepsilon^p\,dx \right\}^{q(p,\alpha)} dt + \int_0^T \left\{ \int_\Omega |\nabla c_\varepsilon|^{2p}\,dx \right\}^{q(p,\alpha)} \right.$$
$$\left. + \int_0^T \left\{ \int_\Omega |A^{\alpha/2} u_\varepsilon|^2\,dx \right\}^{q(p,\alpha)} dt \right\}$$
$$\le 3^{q(p,\alpha)} \cdot \left\{ \int_0^T \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^{\frac{2p}{3(p-1)}}\,dt + \int_0^T \|\nabla c_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)}^{\frac{4p}{3(p-1)}}\,dt \right.$$
$$\left. + \int_0^T \|A^{\alpha/2} u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}^{2/\alpha}\,dt + 2T \right\}$$
$$\le 3^{q(p,\alpha)} \cdot \{C_7 + C_1 C_3^{\frac{3-p}{3(p-1)}} C_5 + C_2 C_4^{\frac{1-\alpha}{\alpha}} C_6 + 2T\}$$

for all $\varepsilon \in (0,1)$. Since Lemma 6.4 in particular says that with $(\varepsilon_j)_{j\in\mathbb{N}}$ as provided there we have $y_\varepsilon^{q(p,\alpha)} \to y^{q(p,\alpha)}$ a.e. in $(0,T)$ as $\varepsilon = \varepsilon_j \searrow 0$, Fatou's lemma therefore implies (6.14). ∎

## 7. Genericity of smoothness: Proof of Theorem 1.1

We are now prepared to identify suitably large sets of times within which the limit $(n,c,u)$ gained in Lemma 6.4 coincides with a smooth solution to the boundary value problem in (1.2). This will be achieved in a parameter regime consistent with both (1.8) and (1.9), whence in particular both the second order local estimates from Section 4 and the approximation and integrability results from Section 6 become applicable.

A first conclusion in this direction yields open smoothness intervals around each time outside any of the sets $S(k,\tau)$ from (5.1), for arbitrarily large $k \in \mathbb{N}$ and suitably chosen $\tau = \tau(k)$:

**Lemma 7.1.** *Fix $p \in (3/2,3)$, $\alpha \in (1/2,1)$, $T > 0$ and $k_0 := 1/T$, and for integers $k > k_0$, let $S(k,T(k))$ be as correspondingly defined by (5.1), with $y = y^{(p,\alpha)}$ given by (6.10), and with $T(k) \in (0,1/k] \subset (0,T)$ taken according to Lemma 3.8. Then for each $t_\star \in (T(k),T) \setminus S(k,T(k))$ there exist an open interval $J(t_\star) \subset (0,T)$ and functions*

$$\begin{cases} \widetilde{n}^{(t_\star)} \in C^{2,1}(\overline{\Omega} \times J(t_\star)), \\ \widetilde{c}^{(t_\star)} \in C^{2,1}(\overline{\Omega} \times J(t_\star)), \\ \widetilde{u}^{(t_\star)} \in C^{2,1}(\overline{\Omega} \times J(t_\star);\mathbb{R}^3) \end{cases} \tag{7.1}$$

*such that $t_\star \in J(t_\star)$ and that the functions $n$, $c$ and $u$ from Lemma 6.4 satisfy*

$$(n,c,u) = (\widetilde{n}^{(t_\star)}, \widetilde{c}^{(t_\star)}, \widetilde{u}^{(t_\star)}) \quad \text{a.e. in } \Omega \times J(t_\star). \tag{7.2}$$

*Proof.* We take $N = N^{(\alpha)}$ as introduced in Lemma 6.4, and given $k \in \mathbb{N}$ such that $k > k_0$ we let $T(k) \in (0, 1/k]$ be as provided by Lemma 3.8. Then for fixed $t_\star \in (T(k), T) \setminus S(k, T(k))$, recalling the definition of $S(k, T(k))$ we may rely on the density of $(t_\star - T(k), t_\star) \setminus N$ in $(t_\star - T(k), t_\star)$ to find some $t_0 = t_0(t_\star) \in (t_\star - T(k), t_\star) \setminus N \subset (0, T)$ such that $y(t_0) < k$. According to Lemma 6.4, the fact that $t_0$ does not belong to $N$ ensures that with $(y_\varepsilon)_{\varepsilon\in(0,1)}$ given by (1.7) and $(\varepsilon_j)_{j\in\mathbb{N}}$ taken from Lemma 6.4 we have $y_\varepsilon(t_0) \to y(t_0)$ as $\varepsilon = \varepsilon_j \searrow 0$, whence we can pick $\varepsilon_\star = \varepsilon_\star(t_\star) \in (0, 1)$ such that

$$y_\varepsilon(t_0) \le k \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \text{ such that } \varepsilon_j < \varepsilon_\star.$$

Now in view of Lemma 4.7, the latter warrants that if we pick any $\tau = \tau(t_\star) \in (0, T(k))$ such that $t_0 + \tau < t_\star$, then there exist $\gamma = \gamma(t_\star) \in (0, 1)$ and $C_1 = C_1(t_\star) > 0$ with the property that writing $J(t_\star) := (t_0 + \tau, t_0 + T(k))$ we have

$$\|n_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(\overline{\Omega}\times\overline{J(t_\star)})} + \|c_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(\overline{\Omega}\times\overline{J(t_\star)})} + \|u_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(\overline{\Omega}\times\overline{J(t_\star)})} \le C_1$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon < \varepsilon_\star$. By means of the Arzelà–Ascoli theorem, from this we infer the existence of a subsequence $(\varepsilon_{j_i})_{i\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}} \cap (0, \varepsilon_\star)$, and of functions $\widetilde{n}^{(t_\star)}, \widetilde{c}^{(t_\star)}$ and $\widetilde{u}^{(t_\star)}$ which are such that (7.1) holds and that

$$\begin{aligned} n_{\varepsilon_{j_i}} &\to \widetilde{n}^{(t_\star)} \quad \text{in } C^{2,1}(\overline{\Omega} \times \overline{J(t_\star)}),\\ c_{\varepsilon_{j_i}} &\to \widetilde{c}^{(t_\star)} \quad \text{in } C^{2,1}(\overline{\Omega} \times \overline{J(t_\star)}),\\ u_{\varepsilon_{j_i}} &\to \widetilde{u}^{(t_\star)} \quad \text{in } C^{2,1}(\overline{\Omega} \times \overline{J(t_\star)}) \end{aligned}$$

as $i \to \infty$. In light of (6.8), this identifies $(\widetilde{n}^{(t_\star)}, \widetilde{c}^{(t_\star)}, \widetilde{u}^{(t_\star)})$ in the sense of (7.2). ∎

Now the crucial size information provided by Corollary 5.2 enables us to make sure that a suitable collection of accordingly gained time intervals from Lemma 7.1 indeed complements a null set of times.

**Lemma 7.2.** *Let $p \in (3/2, 3)$ and $\alpha \in (1/2, 1)$, and let $n, c$ and $u$ be as accordingly be obtained in Lemma 6.4. Then given any $T > 0$ one can find an open set $E \subset (0, T)$ and functions*

$$\begin{cases} \widehat{n} \in C^{2,1}(\overline{\Omega} \times E),\\ \widehat{c} \in C^{2,1}(\overline{\Omega} \times E),\\ \widehat{u} \in C^{2,1}(\overline{\Omega} \times E; \mathbb{R}^3) \end{cases} \tag{7.3}$$

*such that*

$$(n, c, u) = (\widehat{n}, \widehat{c}, \widehat{u}) \quad \textit{a.e. in } \Omega \times E, \tag{7.4}$$

$$|(0, T) \setminus E| = 0. \tag{7.5}$$

*Proof.* For $k \in \mathbb{N}$ with $k > k_0 := 1/T$ we let $S(k, T(k))$ be as defined through (5.1), with $y = y^{(\alpha)}$ taken from (6.10). Then given $t_\star \in (T(k), T) \setminus S(k, T(k))$ we let $J(t_\star) \subset (0, T)$ and $(\widetilde{n}^{(t_\star)}, \widetilde{c}^{(t_\star)}, \widetilde{u}^{(t_\star)})$ be as obtained in Lemma 7.1, and first observe that whenever $t_\star \in (T(k),T) \setminus S(k,T(k))$ and $t_{\star\star} \in (T(k),T) \setminus S(k,T(k))$, from (7.2) we clearly infer that necessarily $(\widetilde{n}^{(t_\star)}, \widetilde{c}^{(t_\star)}, \widetilde{u}^{(t_\star)}) \equiv (\widetilde{n}^{(t_{\star\star})}, \widetilde{c}^{(t_{\star\star})}, \widetilde{u}^{(t_{\star\star})})$ throughout $\overline{\Omega} \times (J(t_\star) \cap J(t_{\star\star}))$. Writing

$$E(k) := \bigcup_{t_\star \in (T(k),T)\setminus S(k,T(k))} J(t_\star), \quad k \in \mathbb{N} \cap (k_0,\infty),$$

and noting that clearly $E(k)$ is open for any such $k$, from Lemma 7.1 we thus actually infer the existence of a uniquely determined triple $(\overline{n}^{(k)}, \overline{c}^{(k)}, \overline{u}^{(k)})$ of functions

$$\overline{n}^{(k)} \in C^{2,1}(\overline{\Omega}\times E(k)), \quad \overline{c}^{(k)} \in C^{2,1}(\overline{\Omega}\times E(k)), \quad \overline{u}^{(k)} \in C^{2,1}(\overline{\Omega}\times E(k);\mathbb{R}^3) \tag{7.6}$$

such that

$$(n,c,u) = (\overline{n}^{(k)}, \overline{c}^{(k)}, \overline{u}^{(k)}) \quad \text{a.e. in } \Omega \times E(k). \tag{7.7}$$

Moreover, the trivial inclusion $(T(k),T) \setminus S(k,T(k)) \subset E(k)$ enables us to estimate

$$|(0,T) \setminus E(k)| \le |S(k,T(k))|^\star + T(k) \quad \text{for all } k \in \mathbb{N} \cap (k_0,\infty),$$

so that an application of Corollary 5.2 to $q := q(p,\alpha)$ shows that due to Lemma 6.5 we have

$$|(0,T) \setminus E(k)| \to 0 \quad \text{as } k \to \infty,$$

because $T(k) \to 0$ as $k \to \infty$ by Lemma 3.8. Therefore, letting

$$E := \bigcup_{k\in\mathbb{N},\, k>k_0} E(k)$$

defines an open set fulfilling (7.5), and similarly to the above observation noting that

$$(\overline{n}^{(k)}, \overline{c}^{(k)}, \overline{u}^{(k)}) \equiv (\overline{n}^{(l)}, \overline{c}^{(l)}, \overline{u}^{(l)}) \quad \text{in } \overline{\Omega} \times (E(k) \cap E(l)) \quad \text{for all } k,l \in \mathbb{N} \cap (k_0,\infty),$$

setting

$$(\widehat{n},\widehat{c},\widehat{u})(x,t) := (\overline{n}^{(k)}, \overline{c}^{(k)}, \overline{u}^{(k)})(x,t) \quad \text{if } (x,t) \in \overline{\Omega}\times E \text{ with } t \in E(k) \text{ for some } k > k_0,$$

we obtain functions $\widehat{n}, \widehat{c}$ and $\widehat{u}$ which are well-defined on all of $\overline{\Omega} \times E$, which satisfy (7.3) due to (7.6), and for which (7.4) holds as a consequence of (7.7). ∎

Along with the statement on eventual smoothness from Theorem A, this readily establishes our final result on generic regularity in (1.2):

*Proof of Theorem* 1.1. We apply Lemma 6.4 to any $p \in (3/2,3)$ and $\alpha \in (1/2,1)$, and employ Theorem A to fix $T_\star > 0$ such that the global weak energy solution $(n,c,u)$, as thereby obtained, upon modification on a null set in $\Omega \times (T_\star,\infty)$ satisfies

$$n \in C^{2,1}(\overline{\Omega} \times (T_\star,\infty)), \quad c \in C^{2,1}(\overline{\Omega} \times (T_\star,\infty)), \quad u \in C^{2,1}(\overline{\Omega} \times (T_\star,\infty);\mathbb{R}^3).$$

Invoking Lemma 7.2 thereafter yields an open set $E \subset (0,T_\star)$ such that $|(0,T_\star) \setminus E| = 0$, and that $(n,c,u) = (\widehat{n},\widehat{c},\widehat{u})$ a.e. in $\Omega \times E$ with some $(\widehat{n},\widehat{c},\widehat{u}) \in (C^{2,1}(\overline{\Omega} \times E))^2 \times C^{2,1}(\overline{\Omega} \times E;\mathbb{R}^3)$. Upon an evident redefinition of $(n,c,u)$ on a null set in $\Omega \times (0,T_\star)$, we readily arrive at the intended conclusion if, by suitably choosing the countable set $\mathcal{J}$, we let $(I_\iota)_{\iota\in\mathcal{J}}$ denote a family of mutually disjoint connected components of $E$. ∎

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