



Damien Gayet

Systoles and Lagrangians of random complex algebraic hypersurfaces

Received December 2, 2019

Abstract. Let $n \geq 1$ be an integer, and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth affine real hypersurface, not necessarily connected. We prove that there exist $c > 0$ and $d_0 \geq 1$ such that for any $d \geq d_0$, any smooth complex projective hypersurface Z in $\mathbb{C}P^n$ of degree d contains at least $c \dim H_*(Z, \mathbb{R})$ disjoint Lagrangian submanifolds diffeomorphic to $\mathcal{L}$, where Z is equipped with the restriction of the Fubini–Study symplectic form (Theorem 1.1). If moreover all connected components of $\mathcal{L}$ have non-vanishing Euler characteristic, which implies that n is odd, the latter Lagrangian submanifolds form an independent family in $H_{n-1}(Z, \mathbb{R})$ (Corollary 1.2). These deterministic results are consequences of a more precise probabilistic theorem (Theorem 1.23) inspired by a 2014 result by J.-Y. Welschinger and the author on random real algebraic geometry, together with quantitative Moser-type constructions (Theorem 3.4). For $n = 2$, the method provides a uniform positive lower bound for the probability that a projective complex curve in $\mathbb{C}P^2$ of given degree equipped with the restriction of the ambient metric has a systole of small size (Theorem 1.6), which is an analog of a similar bound for hyperbolic curves given by M. Mirzakhani (2013). In higher dimensions, we provide a similar result for the $(n - 1)$ -systole introduced by M. Berger (1972) (Corollary 1.14). Our results hold in the more general setting of vanishing loci of holomorphic sections of vector bundles of rank between 1 and n tensored by a large power of an ample line bundle over a projective complex n -manifold (Theorem 1.20).

Keywords. Systole, complex algebraic curve, complex projective hypersurface, Lagrangian submanifold, random geometry, Kähler geometry

1. Introduction

1.1. Disjoint Lagrangian submanifolds

On a compact orientable smooth real surface of genus $g > 1$, there exist $3g - 3$ disjoint non-contractible closed curves such that two of them are not isotopic. A natural generalization of this phenomenon in a closed symplectic manifold (X, ω) is to estimate the possible number of disjoint Lagrangian submanifolds of given diffeomorphism type in X .

Damien Gayet: Institut Fourier, Université Grenoble Alpes, CNRS UMR 5208, 38000 Grenoble, France; damien.gayet@univ-grenoble-alpes.fr

Mathematics Subject Classification (2020): 60D05, 53D12, 32Q15

The answer is easy for submanifolds which exist as compact smooth manifolds in $\mathbb{R}^{2n}$, like the torus, since by the Darboux theorem they can be implemented at any scale in X , so there exist an infinite number of them. Moreover, when the submanifold $\mathcal{L}$ possesses a smooth non-vanishing closed 1-form, which is the case for the n -torus, this form produces an infinite number of disjoint Lagrangian graphs in $T^*\mathcal{L}$, hence by Weinstein's tubular neighborhood theorem there exist an infinite number of disjoint Lagrangian submanifolds close to $\mathcal{L}$ (see Remark 1.11 below). If this is not the case or if the Euler characteristic of $\mathcal{L}$ is not zero, then it cannot be displaced by a perturbation as a disjoint submanifold (see §1.4). Furthermore, the classes of a finite family of disjoint such Lagrangian submanifolds with non-zero Euler characteristic form an independent family of the ambient homology group of degree half the dimension of X (see Lemma 2.2).

The main result. In this paper, we are interested in smooth projective complex submanifolds equipped with the restriction of the ambient Fubini–Study Kähler form. They have the same diffeomorphic type, because they can be isotoped through smooth hypersurfaces. For the latter reason, the Moser trick and the fact that the symplectic form has entire periods show that they are all also symplectomorphic (see Proposition 4.2). Moreover, they enjoy an interesting homological property: for any degree d hypersurface $Z \subset \mathbb{C}P^n$,

$$\dim H_*(Z, \mathbb{R}) \underset{d \rightarrow \infty}{\sim} \dim H_{n-1}(Z, \mathbb{R}) \underset{d \rightarrow \infty}{\sim} d^n.$$

The first asymptotic is a consequence of the Lefschetz hyperplane theorem [14] and the second one can be estimated through the Euler class of the tangent space of Z via the Euler characteristic and Chern classes. The main goal of this paper is to prove the following theorem:

Theorem 1.1. *Let $n \geq 1$ be an integer and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth real affine hypersurface, not necessarily connected. Then there exists $c > 0$ such that for any d large enough, any complex hypersurface Z of degree d in $\mathbb{C}P^n$ contains at least $c \dim H_*(Z, \mathbb{R})$ pairwise disjoint Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.*

In fact, we prove this result in the more general setting of vanishing loci of holomorphic sections of vector bundles of rank between 1 and n tensored by a large power of an ample line bundle over a projective complex n -manifold (see Theorem 1.20).

Corollary 1.2. *Under the hypotheses of Theorem 1.1,*

- (1) *if for any component $\mathcal{L}_i$ of $\mathcal{L}$, $\chi(\mathcal{L}_i) \neq 0$, then the classes in $H_{n-1}(Z, \mathbb{R})$ generated by their Lagrangian copies in Z are linearly independent;*
- (2) *if $\mathcal{L}$ is simply connected, its Lagrangian copies are not close perturbations of each other.*

Remark 1.3. (1) Note that $\chi(\mathcal{L}) \neq 0$ implies that n is odd.

(2) The real projective plane $\mathbb{R}P^2$ is a Lagrangian submanifold of $Z = \mathbb{C}P^2 \subset \mathbb{C}P^3$ but cannot be a hypersurface in $\mathbb{R}^3$ since any compact hypersurface of $\mathbb{R}^n$ is orientable.

(3) If $Z = \mathbb{C}P^2 \subset \mathbb{C}P^3$, $H_2(Z, \mathbb{Z})$ is generated by the class of a complex line $[D]$. The integral of the Fubini–Study Kähler form ω_{FS} over D is positive since ω_{FS} is positive over complex submanifolds, that is, $\langle \omega_{\text{FS}}, [D] \rangle > 0$. However, $\langle \omega_{\text{FS}}, [\mathcal{L}] \rangle = 0$ if $\mathcal{L}$ is a Lagrangian submanifold, so that $H_2(Z, \mathbb{Z})$ cannot be generated by Lagrangian classes.

In this paper, we prove Theorem 1.1, which is a special case of Theorem 1.20, through a probabilistic argument (see Theorem 1.18): if we choose at random such a projective hypersurface of given large degree, the probability that the conclusion of the theorem holds is positive. Since the hypersurfaces have the same symplectomorphism type (see Proposition 4.2), they all satisfy this property. In a parallel paper [11], we prove this theorem with a deterministic proof based on the Donaldson–Auroux method [7], [2].

Other results on disjoint Lagrangian submanifolds. As far as the author of the present work knows, essentially three types of results for disjoint Lagrangian submanifolds have been proved.

- The oldest one concerns Lagrangian spheres that naturally germ from singularities of hypersurfaces by Picard–Lefschetz theory. For instance, S. V. Chmutov [1, p. 419] proved that there exists a singular projective hypersurface of degree d with $c_n d^n + o(d^n)$ singular points, with $c_n \sim_n \sqrt{\frac{2}{\pi n}}$. When the polynomial defining this hypersurface is perturbed into a non-singular polynomial, the singularities give birth to disjoint Lagrangian spheres of the associated smooth hypersurface of the same degree.
- The second result is due to G. Mikhalkin and uses toric arguments:

Theorem 1.4 ([19, Corollary 3.1]). *For any $n \geq 2$ and $d \geq 1$, a $2h^{n-1,0}$ -dimensional subspace of $H_{n-1}(Z, \mathbb{R})$ has a basis represented by embedded Lagrangian tori and spheres, where Z is any smooth projective hypersurface of $\mathbb{C}P^n$.*

Here, $h^{n,0}$ is the geometric genus of Z , that is, the dimension of the space of global holomorphic n -forms, $H^{n,0}(X) \subset H^n(X, \mathbb{C})$. It grows like cd^n for some $c > 0$, as does the dimension of $H_{n-1}(Z, \mathbb{C})$ and $\chi(Z)$. For Lagrangian spheres, Theorem 1.4 is more precise than our Theorem 1.1, since with our method, for an even dimension $n \geq 3$, we cannot know if our Lagrangian spheres have non-trivial class in H_{n-1} , and since the constant c in our bound should be very small compared to the one given by Mikhalkin, as in Chmutov’s theorem. Moreover, our theorem does not say anything new for tori. On the other hand, Theorem 1.1 asserts that *any* real affine hypersurface is reproduced as Lagrangian submanifolds in a large quantity in projective hypersurfaces of large enough degree, and in odd dimension, with a simple topological restriction, it generates a uniform proportion of the homology of the complex hypersurface. Moreover, our result extends to any projective manifold equipped with any ample line bundle (see Theorem 1.20).

- The third type of results concerns upper bounds for the number of disjoint Lagrangian submanifolds (not necessarily spheres), and uses Floer techniques; see for instance [23] for results in open manifolds and a survey for older results of this kind.

1.2. Random complex projective hypersurfaces

The smooth projective complex hypersurfaces of a given degree d , that is, the smooth vanishing loci in $\mathbb{C}P^n$ of complex homogeneous degree d polynomials, form a very natural family of compact Kähler manifolds. Unlike real projective hypersurfaces, that is, the vanishing loci in $\mathbb{R}P^n$ of real polynomials, for fixed d all complex hypersurfaces have the same diffeomorphism type. In particular for $n = 2$, all smooth complex hypersurfaces in $\mathbb{C}P^2$ of degree d are compact connected Riemann surfaces of genus

$$g_d := \frac{1}{2}(d-1)(d-2).$$

Moreover, as said before, for any n and d , when complex hypersurfaces are equipped with the restriction of the ambient Kähler form, they all have the same symplectomorphism type (see Proposition 4.2).

In [25], the authors inaugurated the study of random vanishing loci of complex polynomials in higher dimensions (and zero sets of random holomorphic sections, see Section 1.5), studying in particular the statistics of the current of integration over these loci. In this paper, we will study the statistics of some metric and symplectic properties of these hypersurfaces equipped with the restriction of the Fubini–Study Kähler metric g_{FS} and form ω_{FS} on $\mathbb{C}P^n$. More precisely, we will be concerned with systoles and small Lagrangian submanifolds.

- $n = 2$. A source of inspiration and motivation for this paper in the case $n = 2$ was genuinely probabilistic and provided by M. Mirzakhani’s theorem on systoles of random hyperbolic curves [20] (see Theorem 1.5). One of the two main goals of the present work is in fact to find an analog of it for random complex projective curves (see Theorem 1.6).
- $n \geq 3$. With the methods we use, one finds that in higher dimensions the natural generalization of small non-contractible loops are small Lagrangian submanifolds of random hypersurfaces. Our motivation was nevertheless deterministic. The probabilistic method is partly inspired by the work of J.-Y. Welschinger and the author on random *real* algebraic manifolds [12], where we proved that any compact affine real hypersurface $\mathcal{L}$ appears a lot of times as a component of a random large degree real projective hypersurface with a uniform probability (see Theorem 1.8). Note that these components are Lagrangian submanifolds of the complexified hypersurface. In particular, this implies that any complex hypersurface of large enough degree d contains at least $c\sqrt{d}^n$ Lagrangian submanifolds diffeomorphic to $\mathcal{L}$, where $c > 0$ does not depend on d (see Remark 1.9). In this paper, we prove an analogous complex and symplectic result analogous to Theorem 1.8: any compact real hypersurface appears at least cd^n times as a small Lagrangian submanifold in a random complex projective hypersurface with a uniform positive probability (see Theorem 1.10). We emphasize that this improvement from $\sqrt{d}^n$ to d^n has an interesting topological implication: when $\chi(\mathcal{L}) \neq 0$, these disjoint submanifolds form an independent family of homology classes of cardinality comparable to the dimension of the whole homology of the complex hypersurface. As said before, the deterministic Theorem 1.6 is a direct consequence of the probabilistic Theorem 1.10.

- It can be surprising that probabilistic arguments can have deterministic consequences in this situation. The main explanation is given by Theorem 1.23 which shows that for any sequence of smaller and smaller balls B of size $1/\sqrt{d}$ in $\mathbb{C}P^n$, the Lagrangian of desired diffeomorphism type appears in the intersection of B and the random hypersurface of degree d with a uniform positive probability. This uniform localization easily implies the global Theorem 1.10, which says that with uniform probability, a uniform proportion of a packing of $\mathbb{C}P^n$ with disjoint balls of size $1/\sqrt{d}$ contain the desired Lagrangian. It happens that the order d^n of growth of $\dim H_*(Z, \mathbb{R})$ is the same as the order of the number of those packed small balls. This result itself immediately implies the deterministic consequence.
- Finally, using the universality of peak sections on Kähler manifolds equipped with ample line bundles, or the asymptotic (in the degree d) universality of the Bergman kernel, we will explain that analogous results can be proved in this general setting; see the probabilistic Theorem 1.18 and the deterministic Theorem 1.20.

Let us define the measure on the space of complex polynomials used in [25] and in this paper. Let

$$H_{d,n+1} := \mathbb{C}_{\text{hom}}^d[Z_0, \dots, Z_n]$$

be the space of complex homogeneous polynomial in $n+1$ complex variables. Its dimension equals $\binom{n+d}{n}$. For $P \in H_{d,n+1}$, denote by $Z(P) \subset \mathbb{C}P^n$ its projective vanishing locus. For P outside a codimension 1 complex subvariety of $H_{d,n+1}$, $Z(P)$ is a smooth complex hypersurface. Since for transverse polynomials P, Q , $Z(P) = Z(Q)$ is equivalent to $P = \lambda Q$ for some $\lambda \in \mathbb{C}^*$, the space of degree d hypersurfaces has the dimension of $H_{d,n+1}$ minus 1. For $n = 2$ this is $\frac{1}{2}d(d+3) \underset{g_d \rightarrow \infty}{\sim} g_d$. Note that for the hyperbolic curves, the complex moduli space has dimension $3g-3$. There exists a natural Hermitian product on $H_{d,n+1}$ given by

$$\forall P, Q \in H_{d,n+1}, \quad \langle P, Q \rangle = \int_{\mathbb{C}P^n} h_{\text{FS}}(P, Q) d\text{vol}_{g_{\text{FS}}},$$

where

$$h_{\text{FS}}(P, Q)([Z]) = \frac{P(Z)\overline{Q(Z)}}{|Z|^{2d}}$$

and g_{FS} denotes the Fubini–Study metric on $\mathbb{C}P^n$. Recall that the latter is the quotient metric induced by the projection $\mathbb{C}^{n+1} \supset \mathbb{S}^{2n+1} \rightarrow \mathbb{C}P^n$ and the standard round metric on the sphere. Then the monomials

$$\left(\sqrt{\frac{(d+n)!}{i_0! \cdots i_n!}} Z_0^{i_0} \cdots Z_n^{i_n} \right)_{\sum_{k=0}^n i_k = d} \quad (1.1)$$

form an orthonormal basis of $\mathbb{C}_{\text{hom}}^d[Z_0, \dots, Z_n]$ (see the end of the proof of Lemma 4.6). This Hermitian product induces a Gaussian probability measure on $H_{d,n+1}$. In other terms, we choose

$$P = \sum_{i_0 + \dots + i_n = d} a_{i_0 \dots i_n} \sqrt{\frac{(d+n)!}{i_0! \dots i_n!}} Z_0^{i_0} \dots Z_n^{i_n} \quad (1.2)$$

with i.i.d. Gaussian coefficients $a_I \in \mathbb{C}$ such that $\Re a_I \sim N(0, 1)$ and $\Im a_I \sim N(0, 1)$ and are independent. We denote that measure by $\mathbb{P}_d$.

1.3. Systoles of random projective curves

Let (X, h) be a compact smooth real manifold equipped with a metric h . In [20], M. Mirzakhani studied probabilistic aspects of metric parameters of (X, h) , when (X, h) is taken at random in $\mathcal{M}_g$, the moduli space of hyperbolic genus g compact Riemann surfaces. This moduli space is equipped with a natural symplectic form, the Weil–Petersson form, hence a volume form, for which $\mathcal{M}_g$ has a finite volume, and which provides a natural probability measure $\mathbb{P}_{WP, g}$ on it (see [20]). Denote by

- $\ell_{\text{sys}}(X)$ the least length of non-contractible loops in (X, h) .

M. Mirzakhani proved the following theorem:

Theorem 1.5 ([20, Theorem 4.2]). *There exist $\varepsilon_0 > 0$ and $0 < c < C$ such that for any $\varepsilon \leq \varepsilon_0$ and $g \geq 2$,*

$$c\varepsilon^2 \leq \mathbb{P}_{WP, g}[X \in \mathcal{M}_g \mid \ell_{\text{sys}}(X) < \varepsilon] \leq C\varepsilon^2.$$

We now introduce a partial analogous result for random projective curves of given degree, with a homological point of view. For any (X, h) as above, $\delta > 0$ and $c > 1$, denote by

- $N_{\text{sys}}(X, \delta, c)$ the maximal cardinality of an independent family of classes in $H_1(X, \mathbb{Z})$ such that any class in the family is represented by a circle of length between δ/c and $c\delta$.

Our first main result concerns the systoles of random complex curves in $\mathbb{C}P^2$:

Theorem 1.6. *There exist $c, d_0 \geq 1$ such that for all $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-c/\varepsilon^6) < \mathbb{P}_d[P \in H_{d,3} \mid N_{\text{sys}}(Z(P), \varepsilon/\sqrt{d}, c) \geq d^2 \exp(-c/\varepsilon^6)],$$

where $Z(P)$ is equipped with $g_{\text{FS}|Z(P)}$. In particular,

$$\exp(-c/\varepsilon^6) < \mathbb{P}_d[P \in H_{d,3} \mid \ell_{\text{sys}}(Z(P)) \leq \varepsilon/\sqrt{d}]. \quad (1.3)$$

Theorem 1.6 is a particular case of the more general Theorem 1.16, which holds for random complex curves in a projective complex manifold.

Remark 1.7. (1) Since $\dim H_1(Z(P), \mathbb{R}) = 2g_d \sim_d d^2$, the first assertion of this theorem proves that with uniform probability, there exists a basis of $H_1(Z(P), \mathbb{R})$ such that a uniform proportion of its members are represented by a loop of size less than $\varepsilon/\sqrt{d}$.

(2) If we want to compare the Fubini–Study model with the Weil–Petersson model, we would like the volumes to be equal at a given genus. This implies that the metric in

the projective setting has to be rescaled by a $\sqrt{d}$ factor. In this case the size estimates given by Theorem 1.6 become similar to the lower bound of Theorem 1.5. Note, however, that although our bound is uniform in d or g_d as in [20], the dependence on ε is very bad compared to Mirzakhani's bound.

(3) In fact, for any $x \in \mathbb{C}P^n$, with the same probability, a non-contractible loop lies in $Z(P) \cap B(x, \varepsilon/\sqrt{d})$ (see Theorem 1.23).

Other metric parameters. For the reader's convenience, we present some known results for other metric properties of projective curves. Tables 1 and 2 compare deterministic and probabilistic observables for the Weil–Petersson and Fubini–Study models.

Parameters of surfaces of genus g	Hyperbolic surfaces	Planar algebraic curves
Dimension of the moduli space	$\underset{g \rightarrow \infty}{\sim} 3g$	$\underset{d \rightarrow \infty}{\sim} g$
Curvature	-1	$\in]-\infty, 2]$ [22]
Volume	$\underset{g \rightarrow \infty}{\sim} 4\pi g$	$\underset{g \rightarrow \infty}{\sim} 4\pi g$
Diameter	$\in]0, +\infty[$	$\in [c, Cg^{5/2}]$ [9]

Tab. 1. Deterministic parameters of the two different models of real surfaces, the Weil–Petersson model with hyperbolic surfaces, and the Fubini–Study model with complex algebraic curves equipped with the induced rescaled metric $\sqrt{2\pi d} g_{\text{FS}}$ on $\mathbb{C}P^2$.

Parameters of surfaces of genus g	Hyperbolic surfaces Weil–Petersson measure	Planar algebraic curves Fubini–Study measure
Curvature	-1	$\mathbb{E}_d(K(x) \mid x \in C) \asymp -1$
Diameter	$\mathbb{P}_{\text{WP},g}(\text{Diam} \geq 40 \log g)$ $\xrightarrow[g \rightarrow \infty]{} 0$ [20]	$?$
Systole	$\mathbb{P}_{\text{WP},g}(\ell_{\text{sys}} \leq \varepsilon) \asymp \varepsilon^2$ [20]	$\mathbb{P}_d(\ell_{\text{sys}} \leq \varepsilon)$ $\geq \exp(-c/\varepsilon^6)$ [this paper]

Tab. 2. Statistics of some metric parameters. Complex algebraic curves are equipped with the induced rescaled metric $\sqrt{2\pi d} g_{\text{FS}}$ on $\mathbb{C}P^2$.

• **Volume.** By the Wirtinger theorem, any curve of degree d in $\mathbb{C}P^2$ (and any degree d hypersurface of $\mathbb{C}P^n$) has volume d (see [14]). By the Gauss–Bonnet theorem, for any hyperbolic curve of genus g , its volume equals $2\pi(2g - 2)$. Hence, for $n = 2$, for comparison with the Weil–Petersson model, we should rescale the metric g_{FS} on $\mathbb{C}P^2$ by $\sqrt{2\pi d}$, so that

$$\text{Vol}_{\sqrt{2\pi d} g_{\text{FS}|Z(P)}}(Z(P)) = 2\pi d^2 \underset{d \rightarrow \infty}{\sim} 4\pi g_d.$$

• **Curvature.** By a result by L. Ness [22, Corollary p. 60], the Gaussian curvature K of a degree d complex curve in $\mathbb{C}P^2$ equipped with the induced metric g_{FS} belongs

to $]-\infty, 2]$. Moreover, by the Gauss–Bonnet theorem, the average on $Z(P)$ of K equals

$$K_{\text{mean}} = -2\pi \frac{2gd - 2}{d} \underset{g \rightarrow \infty}{\sim} -2\pi d.$$

We can prove moreover that for all $x \in \mathbb{C}P^2$, $\mathbb{E}(K(x) \mid P(x) = 0) \asymp -d$.

• **Diameter.** Since by the maximum principle there are no compact complex curves in $\mathbb{C}^2$, no algebraic complex curve in $\mathbb{C}P^2$ exists in a ball, so that

$$\exists c > 0, \forall P \in \bigcup_{d \geq 1} H_{d,n+1}, \quad \text{Diam}(Z(P), g_{\text{FS}|Z(P)}) \geq c. \quad (1.4)$$

F. Bogomolov [6] has proved that the intrinsic diameter of planar complex curves is not bounded when the degree grows to infinity. However, S.-T. Feng and G. Schumacher [9] showed that for a given degree there exists an upper bound for the diameter:

$$\forall d \geq 1, \forall P \in H_{d,3}, \quad \text{Diam}(Z(P), g_{\text{FS}|Z(P)}) \leq 32\pi g_d^2 + o(g_d^2).$$

It should be possible, as in [20], to find a better probabilistic estimate for the diameter, and one can wonder if it is also logarithmic in d .

1.4. Small Lagrangian submanifolds of random hypersurfaces

Let (X^{2n}, ω) be a smooth symplectic manifold of dimension $2n$. Recall that ω is a closed non-degenerate 2-form. A *Lagrangian submanifold* $\mathcal{L}$ of X is an n -dimensional submanifold such that $\omega|_{T\mathcal{L}}$ vanishes. For instance, a real analytic hypersurface in $\mathbb{R}^n$ is a Lagrangian submanifold of its associated complex extension, which is a Kähler manifold for the restriction of the standard Kähler form in $\mathbb{C}^n$.

Universal real components. In [12], J.-Y. Welschinger and the present author studied random *real* projective hypersurfaces, that is, the real loci of random elements of $\mathbb{R}H_{d,n+1}$, the space of real homogeneous polynomials in $n+1$ variables and of degree d . The measure was the complex Fubini–Study (1.2) restricted to $\mathbb{R}H_{d,n+1}$. In the literature, this measure is often called the *Kostlan measure*. Let $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth real hypersurface. For any real homogeneous polynomial P , let $Z_{\mathbb{R}}(P) := Z(P) \cap \mathbb{R}P^{n+1}$, and denote by

- $N_{\mathbb{R}}(\mathcal{L}, Z_{\mathbb{R}}(P))$ the number of disjoint balls B in $\mathbb{R}P^n$ such that $B \cap Z_{\mathbb{R}}(P)$ contains a submanifold $\mathcal{L}'$ diffeomorphic to $\mathcal{L}$.

Theorem 1.8 ([12, Theorem 1.2] and [13, Theorem 2.1.1]). *Let $n \geq 1$ and $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth hypersurface, not necessarily connected. Then there exist $c > 0$ and d_0 such that for every $d \geq d_0$,*

$$c < \mathbb{P}_d[P \in \mathbb{R}H_{d,n+1} \mid N_{\mathbb{R}}(\mathcal{L}, Z_{\mathbb{R}}(P)) > c\sqrt{d}^n].$$

Remark 1.9. (1) Note that this theorem has a deterministic corollary, using the same argument as given in this paper: any compact real affine hypersurface appears at least $c\sqrt{d}^n$ times as disjoint Lagrangian submanifolds in any complex projective hypersurface of high enough degree. Indeed, the real part of a complex hypersurface defined over the reals is Lagrangian for the restriction of the Fubini–Study Kähler form, and complex projective hypersurfaces are all symplectomorphic.

(2) In [10], the author constructed real hypersurfaces with $c\sqrt{d}^n$ real spheres. The same proof, replacing a polynomial vanishing on a sphere by another polynomial gives the same corollary as the latter. Theorem 1.1 gives a cd^n lower bound, which is of the order of $\dim H_*(Z(P), \mathbb{R})$ when d grows to infinity.

(3) In fact, Theorem 1.8 holds in the more general context of Kähler compact manifolds with holomorphic line bundles equipped with real structures (see [12]).

Universal Lagrangian submanifolds. We turn now to a complex and Lagrangian analog of this theorem. As before, let $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth real hypersurface. For any compact symplectic manifold (Z, ω, h) equipped with a Riemannian metric h , any $\delta > 0$ and $c \geq 1$, denote by

- $N_{\text{Lag}}(\mathcal{L}, Z, \delta, c)$ the number of pairwise disjoint open sets containing a Lagrangian submanifold $\mathcal{L}'$ diffeomorphic to $\mathcal{L}$ and satisfying

$$\delta/c \leq \text{Diam}(\mathcal{L}', h|_{\mathcal{L}'}) \leq c\delta. \quad (1.5)$$

For polynomials, the following theorem is the main probabilistic result of this paper. It is a particular case of Theorem 1.18 below:

Theorem 1.10. *Let $n \geq 2$ and let $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth hypersurface, not necessarily connected. Then there exist $c, D, d_0 \geq 1$ such that for any $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-c/\varepsilon^D) < \mathbb{P}_d[P \in H_{d,n+1} \mid N_{\text{Lag}}(\mathcal{L}, Z(P), \varepsilon/\sqrt{d}, c) > d^n \exp(-c/\varepsilon^D)],$$

where the metric and the symplectic form on $Z(P)$ are the ones induced by the Fubini–Study metric and symplectic form on $\mathbb{C}P^n$. Moreover, if $\mathcal{L}$ is real algebraic, that is, if there exists $p \in \mathbb{R}[x_1, \dots, x_n]$ such that $\mathcal{L} = Z_{\mathbb{R}}(p)$, then D can be chosen to be $2 \deg p$.

Remark 1.11. (1) In fact, Theorems 1.1, 1.6 and 1.10 have a higher codimension generalization: instead of taking one random polynomial, one can choose $1 \leq r \leq n$ random independent polynomials $(P_1, \dots, P_r)$ of the same degree, and look at their common vanishing locus $Z(P_1, \dots, P_r) := \bigcap_{i=1}^r Z(P_i) \subset \mathbb{C}P^n$, which is now almost surely of complex codimension r . Then the same conclusions hold with the following changes: for complex curves (Theorem 1.6), we take $n \geq 2$ instead of $n = 2$, and choose $r = n - 1$. For Lagrangians (Theorem 1.10), we take $\mathcal{L} \subset \mathbb{R}^{n-r+1}$ instead of $\mathcal{L} \subset \mathbb{R}^n$. However, if $r \geq 2$, then $\mathcal{L} \subset \mathbb{R}^{n-r+1}$ must satisfy a further necessary condition: its normal bundle must be trivial. These generalizations are direct consequences of Theorem 1.16 for curves and Corollary 1.19 for higher dimensions.

(2) By the Weinstein theorem [29], a tubular neighborhood of a closed Lagrangian submanifold $\mathcal{L}$ is symplectomorphic to a tubular neighborhood of the zero section in $T^*\mathcal{L}$, so that local Lagrangian deformations of $\mathcal{L}$ can be viewed in $T^*\mathcal{L}$ as graphs of closed 1-forms on $\mathcal{L}$. If the form is exact, then it has at least two zeros and the associated graph intersects $\mathcal{L}$. In particular, if $H^1(\mathcal{L}, \mathbb{R}) = 0$, then $\mathcal{L}$ cannot be locally deformed as a disjoint Lagrangian submanifold. On the other hand, if $\mathcal{L}$ possesses a closed 1-form which does not vanish, like the torus, then there exist an infinite number of Lagrangian submanifolds diffeomorphic to $\mathcal{L}$. See [8] for topological conditions on $\mathcal{L}$ which imply non-existence of such non-vanishing forms. This remark shows that in the case of spheres, the disjoint Lagrangian spheres produced by Theorem 1.1 are not small deformations of each other.

(3) Theorem 1.10 is a consequence of the more precise Theorem 1.23, which asserts that for any sequence of balls centered at a fixed point x in $\mathbb{C}P^n$ and of size $1/\sqrt{d}$, with uniform probability $\mathcal{L}$ appears as a Lagrangian submanifold of a random vanishing locus.

Theorem 1.10 provides a simple generalization to higher dimensions of the estimate (1.3) of the systole given by Theorem 1.6. The natural object is the k -dimensional systole:

Definition 1.12 (see [3], [15]). Let (M, g) a compact smooth Riemannian manifold of dimension $n \geq 1$. For any $k \in \{1, \dots, n\}$, define

$$\ell_{\text{sys}}^k(M, g) := \frac{1}{2} \inf \{ \text{Diam}(\Sigma, g|_{\Sigma}) \mid \Sigma \subset M \text{ a } k\text{-submanifold}, H_k(M, \mathbb{Z}) \ni [\Sigma] \neq 0 \}.$$

Lemma 1.13. *Let (M, g) a compact smooth Riemannian manifold of dimension $n \geq 1$ and $k \in \{1, \dots, n-1\}$ such that $H_k(M, \mathbb{Z}) \neq 0$. Then $\ell_{\text{sys}}^k(M, g) > 0$.*

Proof. Since M is compact, there exists $r > 0$ such that any ball of size r is diffeomorphic to the unit ball in $\mathbb{R}^n$. In particular, for any closed k -submanifold $\Sigma \subset M$ with $\text{Diam}(\Sigma, g|_{\Sigma}) < r$, $H_k(M, \mathbb{Z}) \ni [\Sigma] = 0$. ■

Corollary 1.14. *Let $n \geq 2$ be an odd integer. There exist $c, D, d_0 \geq 1$ such that for any $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-c/\varepsilon^D) < \mathbb{P}_d [P \in H_{d,n+1} \mid \ell_{\text{sys}}^{n-1}(Z(P), g_{\text{FS}|Z(P)}) \leq \varepsilon/\sqrt{d}].$$

Remark 1.15. (1) Our definition of $\ell_{\text{sys}}^k(M, g)$ is different from but very close to the one given by Berger (where it is called a *carcan*), which refers to the volume of the submanifold, and not the diameter. However, our method does provide a volume estimate. In order to keep this paper not too long, we do not write it down.

(2) A similar estimate holds for holomorphic sections.

Proof of Corollary 1.14. Let $\mathcal{L} = \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then $\chi(\mathcal{L}) \neq 0$ and Corollary 1.14 is a direct consequence of Theorem 1.10 and Corollary 1.2. ■

1.5. Random sections of a holomorphic vector bundle

There are at least two natural generalizations of Theorems 1.6 and 1.10: Firstly, we can work in the setting of ample holomorphic line bundles over compact Kähler manifolds introduced by B. Schiffman and S. Zelditch [25]. Secondly, we can study the statistics of the vanishing locus of *several* random polynomials or sections, as mentioned in Remark 1.11. We present the fusion of the two generalizations, as in [13]. Let $n \geq 1$ and X be a compact complex n -dimensional manifold equipped with an ample holomorphic line bundle $L \rightarrow X$, that is, there exists a Hermitian metric h_L on L with curvature $-2i\pi\omega$ such that ω is Kähler. We denote by g_ω the associated Kähler metric. Note that by the Kodaira theorem, X can be holomorphically embedded in $\mathbb{C}P^N$ for N large enough. Let $1 \leq r \leq n$ be an integer and $E \rightarrow X$ be a holomorphic vector bundle of rank r equipped with a Hermitian metric h_E . For any degree $d \geq 1$, denote by $H^0(X, E \otimes L^d)$ the space of holomorphic sections of $E \otimes L^{\otimes d}$. By the Hirzebruch–Riemann–Roch theorem,

$$\dim H^0(X, E \otimes L^{\otimes d}) \underset{d \rightarrow \infty}{\sim} r d^n \int_X \frac{\omega^n}{n!}.$$

Let $d \text{ vol}$ be any volume form on X , and define for any $d \geq 1$ the Hermitian product on $H^0(X, E \otimes L^d)$:

$$\forall s, t \in H^0(X, E \otimes L^d), \quad \langle s, t \rangle := \int_X h_{E, L^d}(s, t) d \text{ vol}, \quad (1.6)$$

where h_{E, L^d}^d is the Hermitian metric on $E \otimes L^{\otimes d}$ associated to h_E and h_L . Then we associate to this Hermitian product the Gaussian probability measure $d\mathbb{P}_d$ on $H^0(X, E \otimes L^d)$. In other terms, for any $d \geq 1$, choosing an orthonormal basis $(S_i)_{i \in \{1, \dots, N_d\}}$ of $H^0(X, E \otimes L^d)$, where $N_d := \dim H^0(X, E \otimes L^d)$, a random section $s \in H^0(X, E \otimes L^d)$ can be written as

$$s = \sum_{i=1}^{N_d} a_i S_i,$$

where the complex coefficients a_i have $(\Re a_i)_i$ and $(\Im a_i)_i$ i.i.d. and following the same normal law $N(0, 1)$. In what follows,

- $Z(s)$ will denote the vanishing locus in X of $s \in H^0(X, E \otimes L^d)$, and
- the tuple $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d \text{ vol}, (\mathbb{P}_d)_{d \geq 1})$ will be called an *ample probabilistic model*, and an *ample model* if no probability is involved.

By Bertini's theorem, almost surely $Z(s)$ is a compact smooth codimension r complex submanifold of X .

Standard example: the Fubini–Study random polynomial mappings. For $X = \mathbb{C}P^n$, $E = \mathbb{C}P^n \times \mathbb{C}^r$, h_E the standard metric on $\mathbb{C}^r$, $L = \mathcal{O}(1)$ the hyperplane bundle, and $h_L = h_{\text{FS}}$ the Fubini–Study metric, we have

$$H^0(\mathbb{C}P^n, E \otimes L^d) = (\mathbb{C}_{\text{hom}}^d[Z_0, \dots, Z_n])^r.$$

Moreover, the monomials given by (1.1) make this identification an isometry. In other terms, a random polynomial mapping for the standard structures is an r -uple of independent random polynomials in $H_{d,n+1}$ equipped with the Gaussian measure (1.2).

Random curves. When $r = n - 1$, the vanishing locus of a section of $H^0(X, E \otimes L^d)$ is generically a smooth compact complex curve. When $n = 2$ and $r = 1$, the adjunction formula shows that its genus equals

$$g_d = \frac{1}{2}d^2 \int_X \omega^2 - \frac{1}{2}d \int_X c_1(X) \wedge \omega + 1,$$

where $c_1(X)$ denotes the first Chern class of the surface X (see [14]). Theorem 1.6 has the following natural generalization:

Theorem 1.16. *Let $n \geq 2$ be an integer. Then there exists a universal constant $c \geq 1$ such that the following holds. Let $(n, n - 1, X, L, E, h_L, \omega, g_\omega, h_E, d \text{ vol}, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. Then there exists $d_0 \geq 1$ such that for all $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-c/\varepsilon^6) < \mathbb{P}_d[s \in H^0(X, E \otimes L^d) \mid N_{\text{sys}}(Z(s), \varepsilon/\sqrt{d}, c) > d^n \text{Vol}_{g_\omega}(X) \exp(-c/\varepsilon^6)].$$

Here, the metric on $Z(s)$ is the restriction of the Kähler metric g_ω associated to ω .

Recall that N_{sys} is defined in §1.3. Note that the volume involved in Theorem 1.16 is the one associated to g_ω and not to the arbitrary volume form $d \text{ vol}$ used for the definition of the scalar product (1.6).

Theorem 1.16 means that for any degree large enough, with probability uniform in d , there exists a basis of $H_1(Z(s), \mathbb{R})$ such that a uniform proportion of its elements are represented by loops of size bounded by $\varepsilon/\sqrt{d}$.

Remark 1.17. It is classical [16, Corollary 3.6] that any compact orientable Riemann surface embeds in $\mathbb{C}P^3$. However, a degree d curve in $\mathbb{C}P^3$, that is, a holomorphic curve whose class in $H_2(\mathbb{C}P^3, \mathbb{Z})$ equals $d[D]$, where D is a line, can have different topologies, and it is not known which pairs of genus and degree exist (see [16, IV, 6]). Finally, if E is of rank 2, our model of sections of $H^0(\mathbb{C}P^3, E \otimes L^d)$ only provides strict subfamilies of the whole set of curves.

Lagrangian submanifolds. We now provide a similar Kähler generalization of Theorem 1.10, that is, for Lagrangian submanifolds. Let Σ be a complex submanifold in $\mathbb{B} \subset \mathbb{C}^n$, and $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_0|_{T\Sigma})$. For any symplectic manifold (Z, ω, h) equipped with a metric h and $\delta > 0, c > 1$, denote by

- $N(\Sigma, \mathcal{L}, Z, \delta, c)$ the maximal number of pairwise disjoint open sets $\Sigma' \subset Z$ such that Σ' contains a Lagrangian submanifold $\mathcal{L}'$ such that

$$(\mathcal{L}', \Sigma') \sim_{\text{diff}} (\mathcal{L}, \Sigma) \quad \text{and} \quad \delta/c \leq \text{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq c\delta.$$

Theorem 1.18. *Let $n \geq 2$, $1 \leq r \leq n - 1$, $\Sigma \subset \mathbb{B} \subset \mathbb{C}^n$ be a complex algebraic smooth codimension r submanifold, and $\mathcal{L} \subset \Sigma$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$. Then there exist $c, D \geq 1$ such that the following holds. Let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d \text{vol}, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. Then there exists $d_0 \geq 1$ such that for all $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-c/\varepsilon^D) < \mathbb{P}_d[s \in H^0(X, E \otimes L^{\otimes d}) \mid N(\Sigma, \mathcal{L}, Z(s), \varepsilon/\sqrt{d}, c) > d^n \text{Vol}_{g_\omega}(X) \exp(-c/\varepsilon^D)].$$

The following corollary proves that any compact smooth real affine codimension $n - r$ submanifold with trivial normal bundle appears a large number of times in the random complex codimension r submanifold, with a uniform probability:

Corollary 1.19. *Let $n \geq 2$, $1 \leq r \leq n - 1$, and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth codimension r submanifold with trivial normal bundle. Then there exist $c, D \geq 1$ such that the following holds. Let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d \text{vol}, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. Then there exists $d_0 \geq 1$ such that for all $0 < \varepsilon \leq 1$ and $d \geq d_0$,*

$$\exp(-c/\varepsilon^D) < \mathbb{P}_d[s \in H^0(X, E \otimes L^{\otimes d}) \mid N_{\text{Lag}}(\mathcal{L}, Z(s), \varepsilon/\sqrt{d}, c) > d^n \text{Vol}_{g_\omega}(X) \exp(-c/\varepsilon^D)].$$

If $\mathcal{L}$ is algebraic, one can choose D to be twice the degree of $\mathcal{L}$.

Recall that N_{Lag} is defined in §1.4. Note that when $r = 1$, that is, if $\mathcal{L}$ is a hypersurface, the condition on its normal bundle is always satisfied. Corollary 1.19 implies the following generalization of the deterministic Theorem 1.1:

Theorem 1.20. *Let $n \geq 2$, $1 \leq r \leq n$, and $\mathcal{L} \subset \mathbb{R}^n$ be a compact smooth $(n - r)$ -submanifold with trivial normal bundle. Then there exists $c > 0$ such that for any ample model $(n, r, X, L, E, h_L, \omega, g_\omega, h_E)$ and for d large enough, the zero locus of any section $s \in H^0(X, E \otimes L^d)$ vanishing transversally contains at least $cd^n \text{Vol}_{g_\omega}(X)$ disjoint Lagrangian submanifolds diffeomorphic to $\mathcal{L}$.*

Again, by the Lefschetz theorem and a computation with Chern classes, there exists $c > 0$ such that

$$\forall d \gg 1, \forall s \in H^0(X, E \otimes L^d), \quad \dim H_*(Z(s), \mathbb{R}) \underset{d \rightarrow \infty}{\sim} \dim H_{n-r}(Z(s), \mathbb{R}) \underset{d \rightarrow \infty}{\sim} cd^n;$$

see [13, Corollary 3.5.2] for a proof with an explicit constant c .

Corollary 1.21. *Under the hypotheses of Theorem 1.20,*

- (1) *if $\chi(\mathcal{L}_i) \neq 0$ for every connected component $\mathcal{L}_i$ of $\mathcal{L}$, then the classes in $H_{n-r}(Z(s), \mathbb{R})$ generated by these disjoint submanifolds are linearly independent;*
- (2) *if the $\mathcal{L}_i$'s are simply connected, no Lagrangian copy of any of them can be isotoped to another one through disjoint Lagrangian submanifolds.*

1.6. Prescribed topology in a small ball

Theorem 1.18 is a consequence of the more precise Theorem 1.23 below. This theorem is partly inspired by the work of J.-Y. Welschinger and the author. For this reason, we recall it. In [13], the following was proved:

Theorem 1.22 ([13, Proposition 2.4.2]). *Let $n \geq 2$ and $1 \leq r \leq n$. Then, for any real compact smooth $(n - r)$ -submanifold $\mathcal{L} \subset \mathbb{R}^n$ with trivial normal bundle, for any d large enough and any $x \in \mathbb{R}P^n$, with positive probability uniform in d , the zero set $Z(P)$ of a random real polynomial $P \in \mathbb{R}_{\text{hom}}^d[X_0, \dots, X_n]$ intersects $B(x, 1/\sqrt{d})$ along some components, some of which are diffeomorphic to $\mathcal{L}$.*

This theorem was in fact proved in the general setting of random sections of holomorphic real vector bundles over a projective manifold (see [13]).

We begin with an analogous version of Theorem 1.22 for smooth complex algebraic affine hypersurfaces $\Sigma \subset \mathbb{C}^n$ containing a Lagrangian submanifold $\mathcal{L}$. Note that the latter condition is not a constraint since every symplectic manifold contains a Lagrangian torus of any small enough size near every point. Note that in contrast to the real case, an affine algebraic complex hypersurface is never compact, and is connected if and only if it is the vanishing locus of an irreducible polynomial.

Let $n \geq 2$ and $1 \leq r \leq n$, Σ be a complex submanifold in $\mathbb{B} \subset \mathbb{C}^n$, $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$, and $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\text{vol}, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model (see §1.5 for the definition). For any $x \in X$, $\delta > 0$, $C > 1$, and $s \in H^0(X, E \otimes L^d)$,

- $A(\Sigma, \mathcal{L}, Z(s), x, \delta, C)$ denotes the event that there exists a smooth topological ball $B \subset X$ containing x and a Lagrangian submanifold $\mathcal{L}'$ of $(Z(s) \cap B, \omega_{|Z(s)})$ such that

$$(\mathcal{L}', Z(s) \cap B) \sim_{\text{diff}} (\mathcal{L}, \Sigma) \quad \text{and} \quad \delta/c \leq \text{Diam}(\mathcal{L}') \leq c\delta.$$

Here, the diameter is computed with respect to the induced metric on $\mathcal{L}'$. The main theorem of this paper is the following:

Theorem 1.23. *Let $n \geq 2$ and $1 \leq r \leq n - 1$ be integers, $\Sigma \subset \overline{\mathbb{B}} \subset \mathbb{C}^n$ be a smooth complex algebraic $(n - r)$ -submanifold, and $\mathcal{L} \subset \Sigma$ be a compact smooth Lagrangian submanifold of $(\Sigma, \omega_{0|T\Sigma})$. Then there exists $c \geq 1$ such that for any ample probabilistic model $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\text{vol}, (\mathbb{P}_d)_{d \geq 1})$, there exists $d_0 \geq 1$ such that for all $0 < \varepsilon \leq 1$ and $x \in X$,*

$$\forall d \geq d_0, \quad \exp(-c/\varepsilon^D) \leq \mathbb{P}_d[A(\Sigma, \mathcal{L}, Z(s), x, \varepsilon/\sqrt{d}, c)].$$

This theorem quickly implies Theorem 1.18 (see below). In fact, the same result holds for affine real hypersurfaces, not only Lagrangians, as in Corollary 1.19 and Theorem 1.22.

1.7. Ideas of the proof of the main theorems

We present the strategy of the proofs of Theorems 1.23, 1.16 and 1.18 for $r = 1$, $\varepsilon = 1$ and for polynomials. The proof relies on two main tools: the barrier method for proving uniform probability of some local topological event, and a quantitative Moser-type construction to make this event symplectic and Lagrangian. The barrier method was used for instance in a real deterministic context in [10] to construct a lot of small spheres in the real part of holomorphic or symplectic Donaldson hypersurfaces. In probabilistic contexts similar to the present work, it was used for instance in [21] to produce small components of the vanishing locus of a random function with uniform probability, and in [12] to produce small components with prescribed diffeomorphism types. The proof of the main Theorem 1.23 is roughly the following:

- Fix a point $x \in \mathbb{C}P^n$ and choose for any d a polynomial $Q_{x,d}$ vanishing along a hypersurface $Z(Q_{x,d})$ intersecting $B(x, 1/\sqrt{d})$ along a hypersurface diffeomorphic to Σ . Here, $1/\sqrt{d}$ is the natural scale for Fubini–Study or Kostlan measures. The easiest way to do this is to rescale for every d the same polynomial in an affine chart centered at x .
- Then, for small enough perturbations, the perturbed polynomial still vanishes in $B(x, 1/\sqrt{d})$ along a hypersurface isotopic to Σ . If the allowed perturbation can be quantified, typically when the two-point correlation function of the random function converges locally to a universal random function after rescaling, one can prove that with a uniform positive probability, a random polynomial of degree d vanishes in the sequence of balls $B(x, 1/\sqrt{d})$ along a hypersurface diffeomorphic to Σ . In our case, we specialize this method in two different ways, depending on the dimension n of the ambient space.
- For $n = 2$ (Theorem 1.16), we choose $\Sigma \subset \mathbb{B} \subset \mathbb{C}^2$ to be a complex curve of degree 3, hence a torus without three small disks. Then a circle whose class in $H_1(\Sigma, \mathbb{R})$ is non-trivial will still be non-trivial in $H_1(Z(P), \mathbb{R})$.
- For $n \geq 3$ (Theorem 1.18), In normal affine complex coordinates on the small ball $B(x, 1/\sqrt{d})$, the Fubini–Study form equals the standard form at x , so that the local implementation in $\mathbb{C}P^n$ of $\mathcal{L}$ is almost Lagrangian in $(Z(Q_{x,d}), \omega_{\text{FS}|TZ(Q_{x,d})})$. Since the perturbation of $Q_{x,d}$ by a random polynomial is complex and not real, there is no natural way to follow $\mathcal{L}$ as a Lagrangian perturbation in the perturbed vanishing locus Σ' . The classical way to deform objects of symplectic nature, like Lagrangians, is the Moser method. We re-prove it in our particular situation, but with a quantitative point of view (Theorem 3.4). Thanks to the latter the method keeps the perturbation of $\mathcal{L}$ inside the small ball, so that this small Lagrangian displacement happens with uniform probability. These points provide the idea of the proof of Theorem 1.23. Note that the quantitative Moser trick is needed only for dimensions $n \geq 3$ and not for our result on systoles.
- Theorems 1.16 and 1.18 are direct consequences of Theorem 1.23: if we choose in $\mathbb{C}P^n$ a maximal set of small disjoint balls, then automatically with uniform probability, at least cd^n of these balls intersect $Z(P)$ along a component diffeomorphic to Σ and contain a Lagrangian copy of $\mathcal{L}$ with a good diameter.

Organization of the paper. In Section 2, we assume Theorem 1.23 and we give the proofs of its consequences presented above. In Section 3, we give a quantitative version of the Moser trick. This part is deterministic. In Section 4, we prove Theorem 1.23.

2. Direct proofs

In this section we assume Theorem 1.23 and we give the proofs of its consequences.

2.1. From local to global

Proof of Theorem 1.18. We follow the proof given in [13, §2.5]. Let $c \geq 1$ be given by Theorem 1.23, and let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d \text{ vol}, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. Let $\Lambda_{\varepsilon, d}$ be a subset of X , maximal for the property that any two distinct points in $\Lambda_{\varepsilon, d}$ are at least $2\varepsilon/\sqrt{d}$ apart. Then the union of the balls $B(x, 2\varepsilon/\sqrt{d})$ centered at the points of $\Lambda_{\varepsilon, d}$ covers X , and the balls $B(x, \varepsilon/\sqrt{d})$ are disjoint. Denote by $N(\Lambda_{\varepsilon, d})$ the number of elements x of $\Lambda_{\varepsilon, d}$ such that $A(\Sigma, \mathcal{L}, Z(s), x, \varepsilon/\sqrt{d}, c)$ happens. Then, by Theorem 1.23,

$$\begin{aligned} |\Lambda_{\varepsilon, d}| \exp(-c/\varepsilon^D) &\leq \sum_{x \in \Lambda_{\varepsilon, d}} \mathbb{P}_d[A(\Sigma, \mathcal{L}, Z(s), x, \varepsilon/\sqrt{d}, c)] \\ &= \sum_{k=1}^{|\Lambda_{\varepsilon, d}|} k \mathbb{P}_d[N(\Lambda_{\varepsilon, d}) = k] \\ &\leq \frac{1}{2} |\Lambda_{\varepsilon, d}| e^{-c/\varepsilon^D} \mathbb{P}_d[N(\Lambda_{\varepsilon, d}) \leq \frac{1}{2} |\Lambda_{\varepsilon, d}| e^{-c/\varepsilon^D}] \\ &\quad + |\Lambda_{\varepsilon, d}| \mathbb{P}_d[N(\Lambda_{\varepsilon, d}) \geq \frac{1}{2} |\Lambda_{\varepsilon, d}| e^{-c/\varepsilon^D}]. \end{aligned}$$

Consequently, $\mathbb{P}_d[N(\Lambda_{\varepsilon, d}) \geq \frac{1}{2} |\Lambda_{\varepsilon, d}| e^{-c/\varepsilon^D}] \geq \frac{1}{2} \exp(-c/\varepsilon^D)$. Since

$$\text{Vol}_{g_\omega}(X) \leq \sum_{x \in \Lambda_{\varepsilon, d}} \text{Vol}_{g_\omega}(x, 2\varepsilon/\sqrt{d}) \underset{d \rightarrow \infty}{\sim} |\Lambda_{\varepsilon, d}| (2\varepsilon/\sqrt{d})^{2n} \text{Vol}_{g_0}(\mathbb{B}),$$

there exists a universal $c_n > 0$ and d_0 independent of $\varepsilon \leq 1$ but depending on the ample probabilistic model such that $|\Lambda_{\varepsilon, d}| \geq c_n \text{Vol}_{g_\omega}(X) d^n \varepsilon^{-2n}$ so that

$$\mathbb{P}_d[N_{\text{Lag}}(\Sigma, \mathcal{L}, Z(s), x, \varepsilon/\sqrt{d}, c) \geq c_n d^n e^{-c/\varepsilon^D} \text{Vol}_{g_\omega}(X)] \geq \frac{1}{2} e^{-c/\varepsilon^D}.$$

We can now absorb c_n into the exponential, replacing c by a smaller positive constant. ■

Proof of Theorem 1.10. This is Theorem 1.18 in the standard case and for $r = 1$. ■

2.2. From probabilistic to deterministic

Proof of Theorem 1.20. Theorem 1.20 is a direct consequence of Corollary 1.19 and the fact that the zeros of holomorphic sections of given degree d have the same diffeomorphism and symplectomorphism type, when they are equipped with the restriction of the ambient Kähler form ω (see Proposition 4.2). ■

Remark 2.1. As said before for projective hypersurfaces, in a parallel paper [11], we prove the deterministic Theorem 1.20 using the deterministic Donaldson [7] and Auroux [2] methods. In the two types of proofs, we use peak sections and a lattice of mesh of order $1/\sqrt{d}$. In both cases we prove that Lagrangian submanifolds appear in a uniform proportion of disjoint balls centered at the vertices of the lattice. An advantage of the Donaldson method is that it can be used for Donaldson hypersurfaces in a symplectic compact manifold (M, ω) equipped with an almost complex structure J . These hypersurfaces are in fact codimension 2 symplectic submanifolds which are the vanishing loci of almost holomorphic sections of high powers $L^{\otimes d}$ of a complex line bundle L over M , where L is equipped with a Hermitian metric of curvature $-2i\pi\omega$. In this general symplectic context, it is not clear which natural space of symplectic hypersurfaces can be used for probabilistic considerations. In [26], the authors replaced the holomorphic sections (which no longer exist in this general context) by the kernel of a certain elliptic operator acting on the bundle, which is the $\bar{\partial}_L$ operator if the almost complex structure is integrable and the bundle is holomorphic. However, the vanishing locus of a section in this space is a priori not symplectic. The deterministic proof is not easier, since we also need the quantitative version of the Moser method given by Theorem 3.4.

Proof of Theorem 1.1. This is Theorem 1.20 in the standard case and for $r = 1$. ■

2.3. Small non-contractible curves

We turn now to the proof of Theorem 1.16 for the systoles of random complex curves.

Proof of Theorem 1.16. Define

$$\forall (z_1, z_2) \in \mathbb{C}^2, \quad p(z_1, z_2) = z_1^3 + z_2^3 - 1.$$

By the genus formula applied to the homogenization

$$P := Z_0^3 p\left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}, \frac{Z_3}{Z_0}\right),$$

$Z(P) \subset \mathbb{C}P^2$ is a smooth torus, so that for $\rho > 0$ large enough,

$$\tilde{\Sigma} := \frac{1}{\rho}(Z(p) \cap \mathbb{B}(0, \rho)) \subset \mathbb{B} \subset \mathbb{C}^2$$

is an affine algebraic complex curve diffeomorphic to $\mathbb{T}^2 \setminus \bigcup_{i=1}^3 D_i$, where $(D_i)_{i=1}^3$ are three disjoint discs in $\mathbb{T}^2$. Embedding $\mathbb{C}^2$ into $\mathbb{C}^n$ turns $\tilde{\Sigma}$ into an affine algebraic complex curve Σ in $\mathbb{C}^n$. Let $\gamma \subset \Sigma$ be a smooth circle which is non-trivial in $H_1(\Sigma, \mathbb{Z})$ (see Figure 1). Since γ is a Lagrangian, by Theorem 1.18 there exists at least $d^n \text{Vol}_{g_\omega}(X) \exp(-c/\varepsilon^D)$ copies of (Σ, γ) in a random curve $Z(s)$ such that any copy γ_i of γ has intrinsic diameter of order $\varepsilon/\sqrt{d}$, with a uniform probability given by the theorem. The classes in $H_1(Z(s), \mathbb{R})$ generated by the copies of γ form an independent family. Indeed, if $\sum_i \lambda_i [\gamma_i] = 0$, where $(\lambda_i)_i \in \mathbb{R}^N$ and the γ_i are the distinct copies of γ , then

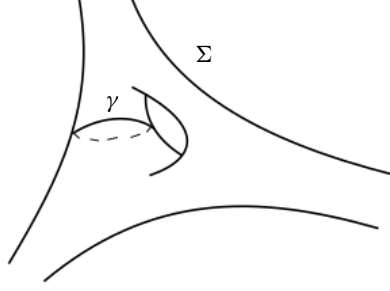


Fig. 1. A degree 3 affine complex curve Σ in $\mathbb{C}^2$ with a non-trivial loop.

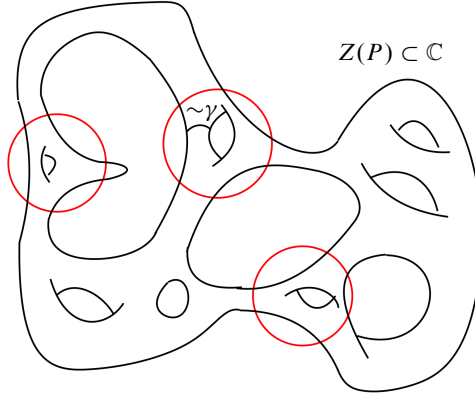


Fig. 2. A (non-realistic) degree 6 curve in $\mathbb{C}P^2$ and three small balls of size $1/\sqrt{d}$ containing the affine complex curve Σ and the non-trivial real curve γ of Figure 1.

there exist codimension 0 surfaces with boundaries $\Sigma_1, \dots, \Sigma_{N'}$ in $Z(s)$ and $(\mu_j)_j \in \mathbb{R}^{N'}$ such that $\sum_i \lambda_i \gamma_i = \sum_j \mu_j \partial \Sigma_j$. This implies that $\partial \Sigma_j$ is a sum of distinct γ_j 's. However, if γ_i is one component of the boundary of Σ_j , then the latter must contain the punctured torus $\tilde{\Sigma}$ which contains γ_i , which implies that γ_i bounds on the other side of Σ_j , which is a contradiction. ■

Proof of Theorem 1.6. Theorem 1.6 is a particular case of Theorem 1.16, with $n = 2$, $r = 1$, $X = \mathbb{C}P^2$, $E = \mathbb{C}P^2 \times \mathbb{C}$, h_E the Euclidean metric, $L = \mathcal{O}(1)$ the hyperplane bundle, h_L the Fubini–Study metric and ω the Fubini–Study Kähler form. ■

2.4. From disjoint to homologically non-trivial

Proof of Corollary 1.21. The first assertion is a direct consequence of the classical Lemma 2.2 below, remembering that Lagrangian submanifolds are totally real for any almost complex structure tamed by the symplectic form ω . The second assertion was explained in Remark 1.11. ■

Lemma 2.2. *Let $\mathcal{L} \subset (X, J)$ be any closed oriented smooth totally real dimension n submanifold in an almost complex manifold X of dimension $2n$. Then*

$$[\mathcal{L}] \cdot [\mathcal{L}] = \chi(\mathcal{L}),$$

where $[\mathcal{L}] \in H_n(X, \mathbb{Z})$ and $\chi(\mathcal{L})$ denotes the Euler characteristic of $\mathcal{L}$. If $\mathcal{L}_1, \dots, \mathcal{L}_N$ is a family of disjoint totally real submanifolds of X with non-vanishing Euler characteristic, then the family made up of their classes $[\mathcal{L}_1], \dots, [\mathcal{L}_N]$ in $H_n(X, \mathbb{R})$ is independent.

Proof. For a closed totally real $\mathcal{L} \subset X$, if h is any metric, then $JT\mathcal{L} \sim NL$, where $N\mathcal{L}$ is the normal bundle over $\mathcal{L}$. Then $\chi(\mathcal{L}) = \int_{\mathcal{L}} e(T\mathcal{L}) = \int_{\mathcal{L}} e(N\mathcal{L})$, which equals $[\mathcal{L}] \cdot [\mathcal{L}]$. For the second assertion, if $\sum_{i=k}^m a_i [\mathcal{L}_k] = 0$ in $H_n(X, \mathbb{R})$, where $\mathcal{L}_1, \dots, \mathcal{L}_m$ are pairwise disjoint totally real submanifolds, then for every j , intersecting with $[\mathcal{L}_j]$ gives $a_j [\mathcal{L}_j] \cdot [\mathcal{L}_j] = 0$ so that in our case, $a_j = 0$. ■

2.5. From smooth to algebraic

For the proof of Corollary 1.19 we will use the classical theorem of H. Seifert:

Theorem 2.3 ([24]). *Let $n \geq 2$, $1 \leq r \leq n$ and $\mathcal{L} \subset \mathbb{R}^n$ be any compact smooth $(n-r)$ -submanifold with trivial normal bundle. Then there exists a real polynomial map $p := (p_1, \dots, p_r) : \mathbb{R}^n \rightarrow \mathbb{R}^r$ and a diffeotopy of $\mathbb{R}^n$ sending $\mathcal{L}$ onto some connected components of $Z_{\mathbb{R}}(p)$. The diffeotopy can be chosen as C^1 -close to the identity map as we want.*

It is not known which hypersurfaces are diffeotopic to algebraic ones (see [5, Remark 14.1.1]).

Proof of Corollary 1.19. By Theorem 2.3, there exists a regular real polynomial map $p = (p_1, \dots, p_r) : \mathbb{R}^n \rightarrow \mathbb{R}^r$ of maximal degree $d(p) := \max_i \deg p_i$ such that $Z_{\mathbb{R}}(p) := Z(p) \cap \mathbb{R}^n$ has a compact component $\mathcal{L}'$ or a set of components diffeomorphic to $\mathcal{L}$. If $\mathcal{L}$ is algebraic, we can choose p such that $Z_{\mathbb{R}}(p) = \mathcal{L}$. By a comprehensible abuse of notation, we keep the notation $\mathcal{L}$ for $\mathcal{L}'$. After perturbation, we can assume that p , when considered as defined on $\mathbb{C}^n$, is regular, too. Then $Z_{\mathbb{R}}(p)$ is a Lagrangian submanifold of its complex vanishing locus $\Sigma := Z(p)$ equipped with the restriction of the standard Kähler form ω_0 . For a large enough $\rho > 0$, $\rho\mathbb{B}$ contains $\mathcal{L}$. We rescale the polynomial by $1/\rho$ and keep the notation p , so that $Z_{\mathbb{R}}(p) \cap \mathbb{B}$ contains $\mathcal{L}$. Then Corollary 1.19 is a consequence of Theorem 1.18 applied to the couple $(\Sigma \cap \mathbb{B}, \mathcal{L})$. ■

3. Quantitative deformations

In this section we introduce and prove deterministic lemmas and propositions which quantify how much a given specific geometrical situation can be perturbed keeping its specificity. The first part concerns the topology, the second part being Lagrangian.

3.1. Preserving the topology

The next proposition is a quantitative and deterministic version of the barrier method for functions. We first need a notation. For any linear mapping $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, where $1 \leq p \leq m$ are integers, define

$$T(A) := \inf_{|w|=1} |A^* w|, \quad (3.1)$$

where $|\cdot|$ denotes the standard Euclidean norm. We will use the following simple properties: for any $A \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$,

- $T(A) > 0$ if and only if A is onto;
- $T(A) \leq \|A\|$, where $\|A\| := \sup_{|v|=1} |Av|$;
- $\|(AA^*)^{-1}\| \leq T(A)^{-2}$;
- if $p = 1$, then $T(A) = \|A\|$;
- for any $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$, $T(A + B) \geq T(A) - \|B\|$.

The following proposition provides quantitative estimates for the perturbation of a vanishing locus on $2\mathbb{B}$. It differs from [12, Proposition 3.4] in two ways. First, it allows the vanishing locus to cross the boundary of the ball. Second, it specifies quantitatively the existence of a diffeomorphism sending the vanishing locus to its perturbation. We need indeed to understand how a Lagrangian submanifold of the locus can be moved into another Lagrangian submanifold of the perturbed locus. For this, we give quantitative estimates of the difference between the diffeomorphism and the identity. For $\eta > 0$ and C^k mappings $f, g : 2\mathbb{B} \rightarrow \mathbb{R}^p$, define

$$\forall 0 \leq j \leq k-1, \quad c_j(\eta, f, g) := \frac{1}{\eta^{2(j+1)}} \|f\|_{C^{j+1}(2\mathbb{B})}^{2j+1} \|g\|_{C^j(2\mathbb{B})}. \quad (3.2)$$

Note that c_j is a homogeneous function of degree 0; this will be crucial for probabilistic estimates (see (4.5) below).

Proposition 3.1. *Let $m \geq 1$, $1 \leq p \leq m$ and $k \geq 3$ be integers, $\eta > 0$, and $f, g : 2\overline{\mathbb{B}} \subset \mathbb{R}^m \rightarrow \mathbb{R}^p$ be C^k maps such that $\|g\|_{C^1(2\mathbb{B})} \leq \eta/8$, $c_0(\eta, f, g) \leq 1/8$ and*

$$\forall x \in 2\mathbb{B}, \quad |f(x)| < \eta \Rightarrow T(df(x)) > \eta.$$

- (1) *There exists a 1-parameter family $(\phi_t)_{t \in [0,1]}$ of diffeomorphisms with support in $2\mathbb{B}$ such that*

$$\forall t \in [0, 1], \quad (Z(f), \overline{\mathbb{B}}) \sim_{\phi_t} (Z(f + tg), \phi_t(\overline{\mathbb{B}}))$$

with $Z(f + tg) \cap \frac{1}{2}\mathbb{B} \subset \phi_t(Z(f) \cap \mathbb{B}) \subset Z(f + tg) \cap \frac{3}{2}\mathbb{B}$, $(x, t) \mapsto \phi_t(x)$ is C^{k-1} and

$$\forall t \in [0, 1], \quad \|\phi_t - \text{Id}\|_{C^0(2\mathbb{B})} \leq t c_0(\eta, f, g). \quad (3.3)$$

- (2) *Let $j = 1, 2$ and $C > 1$ be such that $c_j(\eta, f, g) \leq C$. Then there exists C' depending only on C such that*

$$\forall t \in [0, 1], \quad \|\phi_t - \text{Id}\|_{C^j(2\mathbb{B})} \leq C' t c_j(\eta, f, g). \quad (3.4)$$

In the proof of the main probabilistic Theorem 1.23 below, the different estimates for the various norms of $\phi_1 - \text{Id}$ in Proposition 3.1 will be used in different ways:

- a small C^0 norm will imply that a Lagrangian submanifold of $Z(f)$ in $\frac{1}{2}\mathbb{B}$ will be sent by $\phi_1 \in Z(f + g)$ into a submanifold of $\mathbb{B}$;
- a small C^1 norm implies that ϕ_1 is close to being symplectic, so that the image of the Lagrangian is close to being Lagrangian and can be perturbed into a genuine Lagrangian submanifold of $Z(f + g)$ (see Theorem 3.4);
- the bound for the C^2 norm will be used to estimate the intrinsic metric on the perturbation of $Z(f)$ on $Z(f + g)$, in order to obtain estimates for diameters.

Proof of Proposition 3.1. For any $t \in [0, 1]$, define $f_t := f + tg$. We first prove that

$$\forall (x, t) \in 2\mathbb{B} \times [0, 1], \quad |f_t(x)| < \eta/2 \Rightarrow T(df_t(x)) > \eta/2. \quad (3.5)$$

Indeed, $|f_t(x)| < \eta/2$ implies $|f(x)| < \eta$ since $|g(x)| < \eta/2$, so that $T(df(x)) > \eta$ by hypothesis, and since $df_t = df + tdg$ and $\|dg(x)\| < \eta/2$, we have $T(df_t(x)) > \eta/2$. In particular, for all $t \in [0, 1]$, $Z(f_t)$ is a C^{k-1} codimension p submanifold of $2\mathbb{B}$. For any $t \in [0, 1]$ and $\beta > 0$, let $V_t(\beta) := \{x \in 2\mathbb{B} \mid |f_t(x)| \leq \beta\}$. Then, by hypothesis on g ,

$$\forall t \in [0, 1], \quad Z(f_t) \subset V_0(\eta/8) \subset V_0(\eta/4) \subset V_t(\eta/2). \quad (3.6)$$

For all $(x, t) \in V_0(\eta/4) \times [0, 1]$ define $X_t(x) \in \mathbb{R}^m$ to be the projection of the origin onto the $(m - p)$ -plane

$$df_t(x)^{-1}(\{-\partial_t f_t(x)\}) \subset \mathbb{R}^m,$$

which is well defined by (3.6) and (3.5). Note that $X(x, t) = \Phi(df_t(x), g(x))$ where Φ is defined in Lemma 4.5. Lemma 4.5 shows that Φ is a smooth mapping where the first variable is onto, so that X is C^{k-1} where it is defined. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function satisfying $\chi|_{(-\infty, 1/4]} = 1$ and $\chi|_{[1/2, 1]} = 0$, and define on $2\mathbb{B}$ the family of vector fields

$$\forall (x, t) \in 2\mathbb{B} \times [0, 1], \quad \tilde{X}_t(x) := \chi\left(\frac{2}{\eta}|f(x)|\right)\chi\left(\frac{|x| - 1}{2}\right)X_t(x).$$

Then $\tilde{X}_t$ is C^{k-1} in (t, x) , for any $t \in [0, 1]$ we have $\tilde{X}_t = X_t$ over $V_0(\eta/8) \cap \frac{3}{2}\mathbb{B}$, and $\tilde{X}_t = 0$ on $(V_0(\eta/4))^c$ and on $\partial(2\mathbb{B})$. Now let $(\phi_t)_{t \in [0, 1]}$ be the family of diffeomorphisms generated by $(\tilde{X}_t)_{t \in [0, 1]}$ on $2\mathbb{B}$, that is,

$$\forall (x, t) \in 2\mathbb{B} \times [0, 1], \quad \partial_t \phi_t(x) = \tilde{X}_t(\phi_t(x)), \quad \phi_0 = \text{Id}.$$

Note that $(x, t) \mapsto \phi_t(x)$ is C^{k-1} . By construction, ϕ_t can be extended smoothly as the identity outside $2\mathbb{B}$. Since the C^0 norm of $\tilde{X}$ is bounded by the one of X , by Lemmas 4.3 and 4.5,

$$\forall t \in [0, 1], \quad \|\phi_t - \text{Id}\|_{C^0(2\mathbb{B})} \leq \frac{4t}{\eta^2} \|df\|_{C^0(2\mathbb{B})} \|g\|_{C^0(2\mathbb{B})} \leq 1/2,$$

so that $\frac{1}{2}\mathbb{B} \subset \phi_t(\mathbb{B}) \subset \frac{3}{2}\mathbb{B}$ for all $t \in [0, 1]$. Moreover, for any (x, t) such that $\phi_t(x) \in V_0(\eta/8) \cap \frac{3}{2}\mathbb{B}$,

$$\partial_t(f_t(\phi_t(x))) = g(\phi_t(x)) + df_t(\phi_t(x))(X(\phi_t(x), t)) = 0.$$

By an open-closed argument and the inclusions (3.6), this condition is satisfied if $x \in \mathbb{B}$. Consequently,

$$\forall t \in [0, 1], \quad Z(f_t) \cap \frac{1}{2}\mathbb{B} \subset \phi_t(Z(f) \cap \mathbb{B}) \subset Z(f_t) \cap \frac{3}{2}\mathbb{B}$$

and assertion (1) of the proposition is proved.

Now, since $dX_t = d\Phi(df_t(x), g(x))(d^2 f_t, dg(x))$ for all $t \in [0, 1]$, Lemma 4.5 gives

$$\begin{aligned} \max_{t \in [0, 1]} \|dX_t\|_{C^0(2\mathbb{B})} &\leq \frac{16}{\eta^4} \|g\|_{C^0(2\mathbb{B})} (2\|df\|_{C^0}^2 + \eta^2/4) \|d^2 f\|_{C^0} + \frac{4}{\eta^2} \|df\|_{C^0} \|dg\|_{C^0} \\ &\leq Kc_1(\eta, f, g), \end{aligned}$$

where K is a universal constant. Moreover, $d\tilde{X}_t = \frac{\nabla|f|}{\eta} \chi' X + \chi dX$, so that

$$\|d\tilde{X}_t\|_{C^0(2\mathbb{B})} \leq K' \left(\frac{\|df\|_{C^0}}{\eta} c_0(\eta, f, g) + c_1(\eta, f, g) \right) \leq K'' c_1(\eta, f, g),$$

where K', K'' depend only on χ . Consequently, by Lemma 4.3, there exists C' depending only on C such that

$$\|\phi_t - \text{Id}\|_{C^1(2\mathbb{B})} \leq C' t c_1(\eta, f, g).$$

This proves assertion (2) for $j = 1$.

For $j = 2$ in (2) we compute

$$d_x^2 X_t(x, t) = d^2 \Phi(df_t(x), g(x))(d^2 f_t, dg)^2 + d\Phi(df_t(x), g(x))(d^3 f_t, d^2 g(x)).$$

so that by Lemma 4.5,

$$\begin{aligned} \|d_x^2 X_t\| &\leq 14\|df\|^3 \eta^{-6} |g| \|d^2 f\|^2 + 6\eta^{-4} \|df\|^2 \|d^2 f\| \|dg\| \\ &\quad + 3\eta^{-4} \|df\|^2 |g| \|d^3 f\| + \eta^{-2} \|df\| \|d^2 g\| \\ &\leq 24c_2(\eta, f, g). \end{aligned}$$

A similar estimate for $d^2 \tilde{X}$ and Lemma 4.3 imply

$$\forall t \in [0, 1], \quad \|\phi_t - \text{Id}\|_{C^2(2\mathbb{B})} \leq t C'' c_2(\eta, f, g),$$

where C'' is a constant depending only on C . ■

3.2. Preserving Lagrangianity

The main goal of this subsection is to prove the technical Proposition 3.3 below. It asserts, in a quantitative way, that

- if some compact Lagrangian submanifold $\mathcal{L}$ lies inside a compact symplectic submanifold Σ of a symplectic manifold (M, ω) ;
- if Σ is perturbed into $\phi(\Sigma)$ by a diffeomorphism ϕ close to the identity;
- if ω is exact and perturbed by a small 2-form $d\mu$,

then there exists a perturbation $\mathcal{L}' \subset \phi(\Sigma)$ of $\mathcal{L}$ which is Lagrangian for the restriction of the perturbed form. Since we think that this quantitative proposition has its own interest, we provide a general statement and a proof for symplectic manifolds. However, in this paper we will apply it in the simple case where the ambient manifold is the unit ball of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ (see Theorem 3.4 below). Before the statement of Proposition 3.3, we need some definitions:

Definition 3.2. Let (M, h) be a smooth Riemannian manifold, possibly with boundary.

- For any continuous maps $f, g : M \rightarrow M$ define $d(f, g) := \sup_{x \in M} d(f(x), g(x))$, where d is the distance associated to h .
- For any $k \geq 0$ and any C^k vector field X on M , define $N_k(X, M) = \sup_{x \in M, 0 \leq p \leq k} \|\nabla^p X\|$ and similarly $N_k(\alpha, M)$ for any C^k form α on M . Here, ∇ denotes the Levi-Civita connection associated to h .
- For any submanifold $\mathcal{L} \subset M$, define $\text{Diam}_M(\mathcal{L}) := \max_{p, q \in \mathcal{L}} d(p, q)$.
- For any 2-form ω defined on a neighborhood of an open subset U of a manifold M equipped with a metric h , let

$$S(\omega, U) := \inf_{x \in U, X \in T_x M, |X|=1} \sup_{Y \in T_x M, |Y|=1} |\omega(X, Y)|.$$

Note that:

- $\text{Diam}_{\mathcal{L}}(\mathcal{L})$ is the intrinsic diameter of $\mathcal{L}$; note also that if $\mathcal{L}$ is a circle, then its length is bounded by its intrinsic diameter;
- if U is relatively compact, then ω is symplectic over U if and only if $S(\omega, U) > 0$;
- if ω_0 denotes the standard symplectic form on $\mathbb{R}^{2n}$, that is, $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$, then $S(\omega_0, \mathbb{R}^{2n}) = 1$;
- for any 2-forms ω and ω' ,

$$S(\omega + \omega', U) \geq S(\omega, U) - \|\omega'\|; \quad (3.7)$$

- if X is a vector field on U , $i_X \omega = \lambda$ and ω is symplectic, then

$$\|X\|_{C^0(U)} \leq \frac{1}{S(\omega, U)} \|\lambda\|_{C^0(U)}; \quad (3.8)$$

- if $f : U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ is holomorphic and vanishes transversally, then $S(\omega_0|_{TZ(f)}, U) = 1$.

Proposition 3.3. Let $1 \leq r \leq n$ be integers, (M, ω, h) a smooth symplectic $2n$ -manifold equipped with a metric, $U \subset V \subset W \subset Y$ four relatively compact open sets such that $\overline{U} \subset V$, $\overline{V} \subset W$, $\overline{W} \subset Y$, and assume that there exists a smooth 1-form λ on $\overline{Y}$ such

that $\omega|_{\bar{Y}} = d\lambda$. Let $\Sigma \subset \bar{Y}$ be a compact smooth codimension $2r$ submanifold, symplectic for $\omega|_{T\Sigma}$, $\mathcal{L}$ a compact smooth Lagrangian submanifold of $(\Sigma \cap U, \omega|_{T\Sigma})$, $\phi : Y \rightarrow Y$ a smooth diffeomorphism with support in Y , and μ a smooth 1-form on $\bar{W}$ satisfying

$$\begin{aligned} d(\phi, \text{Id}) &\leq \text{dist}(V, \partial W), \\ \|\phi^*(\lambda + \mu) - \lambda\|_{C^0(W)} &\leq \frac{1}{2} S(\omega|_{T\Sigma}, W \cap \Sigma) \text{dist}(U, \partial V), \\ \|\phi^*(d\lambda + d\mu) - d\lambda\|_{C^0(W)} &\leq \frac{1}{2} S(\omega|_{T\Sigma}, W \cap \Sigma). \end{aligned}$$

- (1) There exists a compact smooth Lagrangian submanifold $\mathcal{L}'$ of $(\phi(\Sigma) \cap W, (\omega + d\mu)|_{\phi(\Sigma)})$ such that $(\mathcal{L}, \Sigma \cap V) \sim_{\phi} (\mathcal{L}', \phi(\Sigma \cap V))$.
- (2) If furthermore $d(\phi, \text{Id}) \leq \frac{1}{8} \text{Diam}_M(\mathcal{L})$ and

$$\|\phi^*(\lambda + \mu) - \lambda\|_{C^0(W)} \leq \frac{1}{16} S(\omega|_{T\Sigma}, W \cap \Sigma) \text{Diam}_M(\mathcal{L}),$$

then $\frac{1}{2} \text{Diam}_M(\mathcal{L}) \leq \text{Diam}_{\mathcal{L}'}(\mathcal{L}')$.

- (3) Let $C > 1$. If furthermore

$$\max(S(\omega|_{T\Sigma}, W \cap \Sigma)^{-1}, N_1(\phi^*(\lambda + \mu) - \lambda, W), N_1(\omega, W), \|d\phi\|_{C^0(W)}) \leq C,$$

then there exists $C' > 0$ depending only on C , on the pair (V, W) and on the C^1 norm of h over W such that $\text{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq C' \text{Diam}_{\mathcal{L}}(\mathcal{L})$.

In the proof of Theorem 1.23, where we prove that a given affine complex hypersurface Σ with a Lagrangian submanifold $\mathcal{L}$ appears with uniform probability in a sequence of small balls, we will need Proposition 3.3 applied to the concrete context of Proposition 3.1, where Σ is the vanishing locus of a holomorphic function f , ϕ is a diffeomorphism sending $Z(f)$ onto the perturbed submanifold $Z(f + g)$, and ω is the Kähler form viewed in the chart on the standard ball. Theorem 3.4 below synthesizes these two propositions for this goal: it asserts, in a quantitative way, that if $\mathcal{L}$ is a compact Lagrangian of a vanishing locus $Z(f)$ which is symplectic for the restriction of the standard form inside the standard ball, as is the case for the real part of a complex hypersurface defined by a real polynomial, and if g is a small perturbing function, then there exists a perturbation $\mathcal{L}'$ of $\mathcal{L}$ which is a Lagrangian submanifold of $Z(f + g)$ equipped with the restriction of a perturbation $\omega_0 + d\mu$ of the standard form.

Theorem 3.4. Let $n \geq 1$ and $1 \leq r \leq n$ be integers, $\eta > 0$, and $f, g : 2\bar{\mathbb{B}} \subset \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2r}$ be smooth maps such that $\|g\|_{C^1(2\bar{\mathbb{B}})} \leq \eta/8$ and

$$\forall x \in 2\bar{\mathbb{B}}, \quad |f(x)| < \eta \Rightarrow T(df(x)) > \eta.$$

Let ω be a smooth symplectic form on $2\bar{\mathbb{B}}$ and μ be a smooth 1-form on $2\bar{\mathbb{B}}$ satisfying $\omega = \omega_0 + d\mu$ with

$$\max(c_0(\eta, f, g), c_1(\eta, f, g), \|\mu\|_{C^0(2\bar{\mathbb{B}})}, \|d\mu\|_{C^0(2\bar{\mathbb{B}})}) \leq \frac{1}{16}. \quad (3.9)$$

Let $\mathcal{L}$ be a compact smooth Lagrangian submanifold of $(Z(f) \cap \frac{1}{2}\bar{\mathbb{B}}, \omega_0|_{Z(f)})$.

(1) *There exists a smooth ball B satisfying $\frac{1}{2}\mathbb{B} \subset B \subset \frac{3}{2}\mathbb{B}$ and a compact smooth Lagrangian submanifold $\mathcal{L}'$ of $(Z(f+g) \cap B, \omega|_{Z(f+g)})$ satisfying $(\mathcal{L}, Z(f) \cap \mathbb{B}) \sim_{\text{diff}} (\mathcal{L}', Z(f+g) \cap B)$.*

(2) *If furthermore*

$$\max(c_0(\eta, f, g), c_1(\eta, f, g), \|\mu\|_{C^0(2\mathbb{B})}) \leq \frac{1}{16} \text{Diam}_{Z(f)}(\mathcal{L}), \quad (3.10)$$

then $\frac{1}{2} \text{Diam}_{Z(f)}(\mathcal{L}) \leq \text{Diam}_{\mathcal{L}'}(\mathcal{L}')$.

(3) *Let $C > 1$ be such that, furthermore,*

$$\max(c_2(\eta, f, g), N_1(\mu, 2\mathbb{B})) \leq C. \quad (3.11)$$

Then there exists $C'' > 1$ depending only on C such that $\text{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq C'' \text{Diam}_{\mathcal{L}}(\mathcal{L})$.

The various estimates for the diameters concern the restriction of the standard metric g_0 on $\mathbb{R}^{2n}$. We postpone the proof of Proposition 3.3 and prove the theorem now, which is a consequence of Propositions 3.3 and 3.1.

Proof of Theorem 3.4. By Proposition 3.1 (1, 2), there exists a family of diffeomorphisms $(\phi_t)_{t \in [0,1]} : 2\mathbb{B} \rightarrow 2\mathbb{B}$ with compact support and a universal constant $K' \geq 1$ such that, writing $\phi = \phi_1$,

$$d(\phi, \text{Id}) = \|\phi - \text{Id}\|_{C^0} \leq c_0(\eta, f, g) \leq \frac{1}{2} \quad \text{and} \quad \|d\phi - \text{Id}\|_{C^0} \leq K' c_1(\eta, f, g),$$

and $(Z(f), \mathbb{B}) \sim_{\phi} (Z(f+g), \phi(\mathbb{B}))$ with

$$Z(f+g) \cap \mathbb{B} \subset \phi(Z(f) \cap \mathbb{B}) \subset Z(f+g) \cap \frac{3}{2}\mathbb{B}.$$

Let $\lambda_0 := \sum_{i=1}^n x_i dy_i$ be the standard Liouville form, which satisfies $d\lambda_0 = \omega_0$. Note that $\|\lambda_0(x)\| \leq |x|$ for any $x \in \mathbb{R}^{2n}$. Then, using the fact that $S(\omega_0|_{TZ(f)}, 2\mathbb{B}) = 1$,

$$\begin{aligned} \|\phi^*(\lambda_0 + \mu) - \lambda_0\|_{C^0(2\mathbb{B})} &\leq \|\phi - \text{Id}\|_{C^0} + \|\lambda_0\|_{C^0} \|d\phi - \text{Id}\|_{C^0} + \|d\phi\|_{C^0} \|\mu\|_{C^0} \\ &\leq c_0(\eta, f, g) + 2c_1(\eta, f, g) + (1 + c_1) \|\mu\|_{C^0} \\ &\leq 5 \max(c_0, c_1, \|\mu\|_{C^0}) \\ &\leq \frac{1}{2} S(\omega_0|_{TZ(f)}, 2\mathbb{B} \cap Z(f)) \text{dist}(\mathbb{B}, 2\mathbb{B}) \end{aligned} \quad (3.12)$$

by (3.9). Similarly,

$$\begin{aligned} \|\phi^*(d\lambda_0 + d\mu) - d\lambda_0\|_{C^0(2\mathbb{B})} &\leq \|d\phi - \text{Id}\|_{C^0}^2 + 2\|d\phi - \text{Id}\|_{C^0} + \|d\phi\|_{C^0}^2 \|d\mu\|_{C^0} \\ &\leq c_1^2 + 2c_1 + (1 + c_1)^2 \|d\mu\|_{C^0} \\ &\leq \frac{1}{2} S(\omega_0|_{TZ(f)}, 2\mathbb{B} \cap Z(f)) \end{aligned}$$

again by (3.9). By Proposition 3.3 (1) applied to $Y = 2\mathbb{B}$, $W = \frac{3}{2}\mathbb{B}$, $V = \mathbb{B}$, $U = \frac{1}{2}\mathbb{B}$, $\Sigma = Z(f)$, and $\omega = \omega_0$, there exists a Lagrangian submanifold $\mathcal{L}'$ of $(Z(f+g) \cap \phi(\mathbb{B}), (\omega_0 + d\mu)|_{TZ(f+g)})$ such that

$$(\mathcal{L}, Z(f) \cap \mathbb{B}) \sim_{\phi} (\mathcal{L}', Z(f+g) \cap \phi(\mathbb{B})).$$

If $B := \phi(\mathbb{B})$, then $\frac{1}{2}\mathbb{B} \subset B \subset \frac{3}{2}\mathbb{B}$. Hence, the first assertion of the theorem is proved. If furthermore (3.10) is satisfied, then by (3.12), the hypotheses of Proposition 3.3 (2) are satisfied, so that $\frac{1}{2} \text{Diam}_{Z(f)}(\mathcal{L}) \leq \text{Diam}(\mathcal{L}')$. This proves the second assertion. Now, if (3.11) is satisfied, then by Proposition 3.1 (2), there exists C'' depending only on C such that $\|d^2\phi\|_{C^0(2\mathbb{B})} \leq C''$. This implies that there is a universal constant K''' and a constant C''' depending only on C such that

$$N_1[\phi^*(d\lambda_0 + d\mu) - d\lambda_0, 2\mathbb{B}] \leq K'''(\|d\phi\|_{C^0} N_1(d\mu, 2\mathbb{B}) + \|d^2\phi\|_{C^0} \|d\mu\|_{C^0}) \leq C'''.$$

Consequently,

$$\max\left(S(\omega_0|_{TZ(f)}, \frac{3}{2}\mathbb{B} \cap Z(f))^{-1}, N_1(\phi^*(d\lambda_0 + d\mu) - d\lambda_0, \frac{3}{2}\mathbb{B}), N_1(\omega_0, 2\mathbb{B})\right) \leq C^{(4)},$$

where $C^{(4)}$ depends only on C . We can now apply Proposition 3.3 (3): there exists a constant $C' > 0$ depending only on C such that $\text{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq C' \text{Diam}_{\mathcal{L}}(\mathcal{L})$. ■

The main steps for proving Proposition 3.3 are the following:

- Recall that in Proposition 3.3, a symplectic submanifold Σ is deformed by a diffeomorphism ϕ into Σ' , and the ambient symplectic form ω is deformed into $\omega + d\mu$.
- The restriction of the perturbed form to Σ' can be viewed as $\Omega' = \phi^*(\omega + d\mu)|_{T\Sigma}$ on Σ . Proposition 3.5 below constructs, in a general setting, an isotopy $(\psi_t)_t$ of local diffeomorphisms on Σ such that ψ_1 is a symplectomorphism between Ω' and a given symplectic form Ω , which is $\Omega = \omega|_{T\Sigma}$ in our case, with an explicit control of $\psi_1 - \text{Id}$ depending on $\Omega - \Omega'$ and its primitive.
- Corollary 3.6 applies this intrinsic Proposition 3.5 to the relative situation of Proposition 3.3, and transfers the latter control to controls depending on $\phi - \text{Id}$ and the perturbation of the ambient symplectic form.
- The proof of Proposition 3.3 consists in applying this corollary to the deformation of the Lagrangian submanifold.

Moser trick. The next proposition is a quantitative version of Moser's trick.

Proposition 3.5. *Let (Σ, Ω, H) be a smooth symplectic manifold, possibly with boundary, equipped with a metric H , and $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be three relatively compact open sets in Σ such that $\overline{\mathcal{U}} \subset \mathcal{V}$ and $\overline{\mathcal{V}} \subset \mathcal{W}$. Let v be a smooth 1-form on $\overline{\mathcal{W}}$ satisfying*

$$\|v\|_{C^0(\mathcal{W})} \leq \frac{1}{2} S(\Omega, \mathcal{W}) \text{dist}(\mathcal{U}, \partial\mathcal{V}) \quad \text{and} \quad \|dv\|_{C^0(\mathcal{W})} \leq \frac{1}{2} S(\Omega, \mathcal{W}).$$

- (1) *There exists a smooth family of diffeomorphisms $(\psi_t)_{t \in [0,1]} : \mathcal{W} \rightarrow \mathcal{W}$ with compact support in $\mathcal{W}$ such that*

$$\begin{aligned} \forall t \in [0, 1], \quad \psi_t^*(\Omega + t dv) &= \Omega \text{ on } \mathcal{U}, \\ \psi_t(\mathcal{U}) &\subset \mathcal{V}, \quad d(\psi_t, \text{Id}) \leq \frac{2t}{S(\Omega, \mathcal{W})} \|v\|_{C^0(\mathcal{W})}. \end{aligned}$$

(2) Let $C > 1$ and assume that

$$\max(S(\Omega, \mathcal{W})^{-1}, N_1(v), N_1(\Omega)) \leq C. \quad (3.13)$$

Then there exists $C' > 0$ depending only on $(\mathcal{U}, \mathcal{V})$, on the C^1 norm of H on $\mathcal{W}$ and on C such that $\|d\psi_t\|_{C^0(\mathcal{W})} \leq C'$.

Proof. For any $t \in [0, 1]$, let $\Omega_t := \Omega + t dv$. Then for every $t \in [0, 1]$, by hypothesis and (3.7), $S(\Omega_t, \mathcal{W}) \geq \frac{1}{2}S(\Omega, \mathcal{W})$, which is positive since $\overline{\mathcal{W}}$ is compact, so that Ω_t is symplectic on $\overline{\mathcal{W}}$. We are looking for a 1-parameter family $(\psi_t)_{t \in [0, 1]}$ of diffeomorphisms of $\mathcal{W}$ such that for all $t \in [0, 1]$, $\psi_t^*(\Omega_t) = \Omega$. Differentiating in time, and assuming that X_t is a vector field that generates ψ_t , we obtain $\partial_t \Omega_t + d(i_{X_t} \Omega_t) = 0$, or $d(v + i_{X_t} \Omega_t) = 0$. We now inverse the procedure. Let $(X_t)_{t \in [0, 1]}$ be a family of vector fields on $\mathcal{W}$ such that

$$\forall t \in [0, 1], \forall x \in \mathcal{W}, \quad i_{X_t(x)} \Omega_t(x) = -v(x). \quad (3.14)$$

Since Ω_t is non-degenerate, X_t is uniquely defined, smooth and by (3.8),

$$\forall t \in [0, 1], \quad \|X_t\|_{C^0(\mathcal{W})} \leq \frac{2}{S(\Omega, \mathcal{W})} \|v\|_{C^0(\mathcal{W})}. \quad (3.15)$$

Let $\chi : \mathcal{W} \rightarrow [0, 1]$ be a smooth cut-off function such that $\chi_{\overline{\mathcal{V}}} = 1$ and χ has compact support in $\mathcal{W}$. Let $(\psi_t)_{t \in [0, 1]}$ be the 1-parameter family of diffeomorphisms associated to χX_t . By Lemma 4.4,

$$\forall t \in [0, 1], \quad d(\psi_t, \text{Id}) \leq \frac{2t}{S(\Omega, \mathcal{W})} \|v\|_{C^0(\mathcal{W})}. \quad (3.16)$$

By hypothesis on $\|v\|$, this implies that $\psi_t(\mathcal{U}) \subset \mathcal{V}$ for all $t \in [0, 1]$. Since $\chi = 1$ over $\mathcal{V}$, we obtain $\psi_1^* \Omega_1 = \Omega$ over $\psi_1(\mathcal{U}) \subset \mathcal{V}$.

We now assume that (3.13) is satisfied and want a bound for the derivative of ψ_1 . Differentiating (3.14) gives $i_{X_t} \nabla \Omega_t + i_{\nabla X_t} \Omega_t = -\nabla v$ over $\mathcal{W}$ for all $t \in [0, 1]$, so that

$$\max_{t \in [0, 1]} \|\nabla X_t\|_{C^0(\mathcal{W})} \leq \frac{2}{S(\Omega, \mathcal{W})} (\|\nabla \Omega_t\|_{C^0(\mathcal{W})} \|X_t\|_{C^0(\mathcal{W})} + \|\nabla v\|_{C^0(\mathcal{W})})$$

and $\max_{t \in [0, 1]} \|\nabla(\chi X_t)\| \leq \max_{t \in [0, 1]} (\|\nabla X_t\|_{C^0} + \|d\chi\| \|X_t\|_{C^0})$. By Lemma 4.4 and (3.16), this implies that $\max_{t \in [0, 1]} \|d\psi_t\|_{C^0} \leq C'$, where C' depends only on the derivative of the metric on $\mathcal{W}$, on C and on χ , hence on $(\mathcal{V}, \mathcal{W})$. ■

In Corollary 3.6 below, we apply the latter proposition to the situation that is of interest for us: the construction of a symplectomorphism Ψ between a symplectic submanifold Σ in an ambient manifold (M, ω) and another submanifold $\phi(\Sigma)$ equipped with the restriction of another symplectic structure $\omega + d\mu$ which is close to ω . Then the proof of Proposition 3.3 will be a direct consequence of Corollary 3.6.

Corollary 3.6. *Assume the hypotheses of Proposition 3.3.*

(1) *There exists a smooth isotopy of embeddings $(\Psi_t)_{t \in [0,1]} : \Sigma \cap W \rightarrow \phi(\Sigma)$ satisfying*

$$\forall t \in [0, 1], \Psi_t^*((\omega + t d\mu)|_{T\phi(\Sigma)}) = \omega|_{T\Sigma} \text{ on } U \cap \Sigma \text{ with } \Psi_t(U \cap \Sigma) \subset W \cap \phi(\Sigma),$$

and

$$d(\Psi_1, \text{Id}|_{\Sigma \cap W}) \leq \frac{2}{S(\omega|_{T\Sigma}, W \cap \Sigma)} \|\phi^*(\lambda + \mu) - \lambda\|_{C^0(W)} + d(\phi, \text{Id}).$$

(2) *If furthermore the hypotheses of Proposition 3.3 (3) are satisfied, there exists $C' > 0$ depending only on U, V, Σ , the C^1 norm of h on W and on C , such that $\|d\Psi_t\|_{C^0(\Sigma \cap W)} \leq C'$ for all $t \in [0, 1]$.*

Proof. Let $j : \Sigma \rightarrow W$ be the natural injection, $\Omega := j^*\omega$ and $\nu := j^*(\phi^*(\lambda + \mu) - \lambda)$, so that $j^*(\phi^*(\omega + d\mu)) = \Omega + d\nu$. Choosing the metric H on Σ to be the one induced by the ambient metric h , the various estimates for ν are bounded by the ones for $\phi^*(\lambda + \mu) - \lambda$, so that, using the fact that the induced distance in Σ is larger than the one in M ,

$$\begin{aligned} \|\nu\|_{C^0(W \cap \Sigma)} &\leq \frac{1}{2} S(\omega|_{T\Sigma}, W \cap \Sigma) \text{dist}_\Sigma(U \cap \Sigma, \partial V \cap \Sigma), \\ \|d\nu\|_{C^0(W \cap \Sigma)} &\leq \frac{1}{2} S(\omega|_{T\Sigma}, W \cap \Sigma). \end{aligned}$$

By Proposition 3.5(1) applied to (Σ, Ω) , $\mathcal{W} = \Sigma \cap W$, $\mathcal{V} = \Sigma \cap V$, $\mathcal{U} = \Sigma \cap U$, $H = h|_{T\Sigma}$, and ν , there exists a 1-parameter family of diffeomorphisms $\psi_t : \Sigma \cap W \rightarrow \Sigma \cap W$ with compact support in $W \cap \Sigma$ such that for any $t \in [0, 1]$, $\psi_t(U \cap \Sigma) \subset V \cap \Sigma$,

$$d(\psi_t, \text{Id}) \leq \frac{2t}{S(\omega|_{T\Sigma}, W \cap \Sigma)} \|\nu\|_{C^0(W \cap \Sigma)} \leq \text{dist}(\mathcal{U}, \partial V), \quad (3.17)$$

and $\psi_t^*(j^*(\phi^*(\omega + d\mu))) = \omega|_{T\Sigma}$ on $U \cap \Sigma$. For any $t \in [0, 1]$, let $\Psi_t := \phi \circ \psi_t : W \cap \Sigma \rightarrow Y$. Then, since by hypothesis $d(\phi, \text{Id}) \leq \text{dist}(V, \partial W)$, we have $\Psi_t(U \cap \Sigma) \subset W \cap \phi(\Sigma)$ by (3.17). Moreover,

$$(\Psi_t^*(\omega + d\mu))|_{T\Sigma} = (\psi_t^*(\phi^*(\omega + t d\mu)))|_{T\Sigma} = \Omega \quad \text{on } U \cap \Sigma.$$

This proves the first assertion.

Now, assume that the hypotheses of Proposition 3.3(3) are satisfied. Then $N_1(\nu, W \cap \Sigma) \leq C$, so that by Proposition 3.5(2), there exists $C' > 0$ depending only on $(\mathcal{U}, \mathcal{V})$, the C^1 norm of H and C such that $\|d\psi_t\|_{C^0(W \cap \Sigma)} \leq C'$. Since $\|d\phi\|_{C^0(Y)} \leq C$, we have $\|d\Psi_t\|_{C^0(W \cap \Sigma)} \leq CC'$, hence the result after changing the definition of C' . ■

We can now give the proof of Proposition 3.3, which demonstrates the stability of a Lagrangian submanifold in a symplectic submanifold when the latter and the symplectic form are perturbed.

Proof of Proposition 3.3. By Corollary 3.6 there exists a smooth diffeomorphism

$$\Psi : \Sigma \cap U \rightarrow \Psi(\Sigma \cap U) \subset \phi(\Sigma) \cap W$$

such that $(\Psi^*(\omega + d\mu))|_{T\Sigma} = \omega|_{T\Sigma}$ on $U \cap \Sigma$. This implies that $\mathcal{L}' := \Psi(\mathcal{L})$ is a smooth compact Lagrangian submanifold in $(\phi(\Sigma) \cap W, (\omega + d\mu)|_{T\phi(\Sigma)})$.

Assume now that the hypotheses of Proposition 3.3 (2) are satisfied. Then by Corollary 3.6, $d(\Psi, \text{Id}) \leq \frac{1}{4} \text{Diam}_M(\mathcal{L})$. Let $p, q \in \mathcal{L}$ with $\text{Diam}_M(\mathcal{L}) = d_M(p, q)$. Then

$$\begin{aligned} \text{Diam}_{\mathcal{L}'}(\mathcal{L}') &\geq d_{\mathcal{L}'}(\Psi(p), \Psi(q)) \geq d_M(\Psi(p), \Psi(q)) \\ &\geq d_M(p, q) - 2d(\Psi, \text{Id}) \geq \frac{1}{2} \text{Diam}_M(\mathcal{L}). \end{aligned}$$

Assume now that the hypothesis of Proposition 3.3 (3) is satisfied. Again by Corollary 3.6, there exists C' such that $\|d\Psi\|_{C^0(\Sigma \cap W)} \leq C'$. This implies $\text{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq C' \text{Diam}_{\mathcal{L}}(\mathcal{L})$. Indeed, let $p', q' \in \mathcal{L}'$ and let $\gamma : [a, b] \rightarrow \mathcal{L}$ be a shortest path in $\mathcal{L}$ between $p := \Psi^{-1}(p') \in \mathcal{L}$ and $q := \Psi^{-1}(q') \in \mathcal{L}$. Then

$$d_{\mathcal{L}'}(p', q') \leq \text{Length}_{\mathcal{L}'}(\Psi(\gamma)) = \int_a^b |d\Psi(\gamma)(\gamma'(t))| dt \leq C' \text{Diam}(\mathcal{L}). \quad \blacksquare$$

4. Proof of the main local theorem

4.1. The standard setting

Proof of Theorem 1.23. We adapt the barrier method of the real context in [12] to our complex algebraic situation, and we will use the quantitative Moser method given by Theorem 3.4. For the reader's convenience, we begin with the proof in the case of standard random polynomials. Then we sketch the proof for the general setting of random holomorphic sections.

Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be regular such that $Z(p) \cap \mathbb{B} = \Sigma$. Since p is regular, there exists $\eta > 0$ such that $p : 2\mathbb{B} \rightarrow \mathbb{C}^r$ satisfies the transversality condition

$$\forall z \in 2\mathbb{B}, \quad |p(z)| < \eta \Rightarrow T(dp(z)) > \eta, \quad (4.1)$$

where T is defined by (3.1).

Since the probability measure is invariant under the symmetries of $\mathbb{C}P^n$, as also is the assertion of Theorem 1.23, it is enough to prove the theorem for $x = [1 : 0 : \dots : 0]$. Let z be the local holomorphic affine coordinates:

$$z = (z_1, \dots, z_n) := \left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0} \right) \in \mathbb{C}^n$$

defined on $\mathbb{C}P^n \setminus \{Z_0 = 0\}$. Fix $\varepsilon > 0$ and let $p_{\varepsilon, d}(z) := p(z \frac{\sqrt{d}}{\varepsilon})$. Note that $Z(p_{\varepsilon, d}) = \frac{\varepsilon}{\sqrt{d}} \Sigma$. Then for any $d \geq d(p)$, let

$$P_{\varepsilon, d}(Z) := Z_0^d p_{\varepsilon, d} \left(\frac{Z_1}{Z_0}, \dots, \frac{Z_n}{Z_0} \right) \in (\mathbb{C}_{\text{hom}}^d[Z_0, \dots, Z_n])^r. \quad (4.2)$$

By construction, $Z(P_{\varepsilon,d}) \subset \mathbb{C}P^n$ intersects the affine coordinate ball $B(0, \varepsilon/\sqrt{d})$ around $[1 : 0 : \dots : 0]$ along a small homothetical copy of Σ and contains a copy of $\mathcal{L}$. Notice that $P_{\varepsilon,d}$ is singular, since $Z(P_{\varepsilon,d})$ contains the hyperplane $\{X_0 = 0\}$ with multiplicity $d - d_0$.

In order to apply Theorem 3.4(1), we must have a bound for the perturbation of ω_0 into ω . For this, in our affine coordinates, let $\lambda_{\text{FS}} = d^c \log(1 + |z|^2)$ and $\lambda_0 = d^c |z|^2$, that is,

$$\lambda_{\text{FS}} = \frac{1}{2i} \frac{\sum_{i=1}^n (z_i d\bar{z}_i - \bar{z}_i dz_i)}{1 + |z|^2}.$$

By definition $\omega_{\text{FS}} = d\lambda_{\text{FS}}$ and $\omega_0 = d\lambda_0$, so that $\lambda_{\text{FS}} = \lambda_0 + O(\|z\|^3)$, and $d\lambda_{\text{FS}} = \omega_0 + O(\|z\|^2)$. Let ψ be the linear map $\psi(z) = z \frac{\varepsilon}{\sqrt{d}}$. Then there exists a universal constant $K > 0$ such that the pull-backs $\lambda = \frac{d}{\varepsilon^2} \psi^* \lambda_{\text{FS}}$ and $\omega = \frac{d}{\varepsilon^2} \psi^* \omega_{\text{FS}}$ satisfy

$$\begin{aligned} \forall d \geq d_{\mathcal{L}} := \frac{16K}{\min(1, \text{Diam}_{Z(p)}(\mathcal{L}))}, \quad \|\lambda - \lambda_0\|_{C^0(2\mathbb{B})} + \|\omega - \omega_0\|_{C^0(2\mathbb{B})} \\ \leq K \frac{\varepsilon^2}{d} \leq \frac{1}{16} \min(1, \text{Diam}_{Z(p)}(\mathcal{L})), \end{aligned}$$

which is the bound needed in Theorem 3.4 for the perturbation form μ and its differential (see (3.9) and (3.10)).

Now let $H_P := P_{\varepsilon,d}^\perp$ be the space orthogonal to $P_{\varepsilon,d}$ in $(\mathbb{C}_{\text{hom}}^d[Z], \langle \cdot, \cdot \rangle)$. We use a decomposition for our random polynomials adapted to $P_{\varepsilon,d}$ and H_P . Since a random polynomial can be written in any fixed orthonormal basis, we can decompose our random polynomial P as

$$P = a \frac{P_{\varepsilon,d}}{\|P_{\varepsilon,d}\|_{L^2}} + R, \quad (4.3)$$

where a is a complex Gaussian variable and $R \in H_P$ is a Gaussian random polynomial for the induced law on H_P and independent of a . The L^2 -norm of $P_{\varepsilon,d}$ is computed by Lemma 4.6 below. We want to prove that with a uniform positive lower bound, R does not perturb the first term too much, so that P still vanishes on a hypersurface diffeomorphic to Σ . Hence, we need to know when the vanishing locus of a perturbation of a function gives a diffeotopic perturbation of the vanishing locus of the function. For this, for any $d \geq d(p)$, we apply Theorem 3.4 to

$$\forall z \in 2\mathbb{B}, \quad f(z) := a P_{\varepsilon,d} \left(1, z \frac{\varepsilon}{\sqrt{d}}\right) = ap(z) \quad \text{and} \quad g(z) := \|P_{\varepsilon,d}\|_{L^2} R \left(1, z \frac{\varepsilon}{\sqrt{d}}\right).$$

By (4.1) we have

$$\forall a \in \mathbb{C}^*, \forall z \in 2\mathbb{B}, \quad |f(x)| < |a|\eta \Rightarrow T(df(x)) > |a|\eta.$$

We want now to give a uniform lower bound for the probability that the pair of random functions (f, g) on $2\mathbb{B}$ satisfies the various conditions of Theorem 3.4. In order to control the perturbation g , we decompose it as

$$g = \frac{1}{2} p_1 + \frac{1}{2} p_2 := \frac{1}{2}(g + f) + \frac{1}{2}(g - f).$$

Note that the law of $p_1 := g + f$ is the same as that of $r(z) := \|P_{\varepsilon,d}\|_{L^2} P(1, z \frac{\varepsilon}{\sqrt{d}})$, where P follows the Fubini–Study law. The same holds for $p_2 := g - f$. We use the trivial inequality

$$\mathbb{E} \sup_{2\mathbb{B}} |g| \leq \frac{1}{2} \left(\mathbb{E} \sup_{2\mathbb{B}} |p_1| + \mathbb{E} \sup_{2\mathbb{B}} |p_2| \right) \leq \mathbb{E} \sup_{2\mathbb{B}} |r|, \quad (4.4)$$

and similarly for the average of the derivative of g . Hence, it is enough to bound from above the norms of a random q .

By the Markov inequality, the independence between f and p_1, p_2 , the bound (4.4), Remark 4.7 and Lemma 4.8, there exists $K_P > 0$ depending only on P such that for all $0 < \varepsilon \leq 1, d \geq d(p), F > 0, 0 < \alpha \leq 1$, and $j \in \{0, 1, 2\}$,

$$\begin{aligned} \mathbb{P}_d[\exists a \in \mathbb{C}^*, \|g\|_{C^1} \leq |a|\eta/8, c_j(|a|\eta, f, g) \leq \alpha] \\ \geq \mathbb{P}_d \left[F \leq |a|, \|g\|_{C^1} \leq F\eta/8, \|g\|_{C^j} \leq F\alpha \frac{\eta^{2j+2}}{\|p\|_{C^3}^{2j+1}} \right] \\ \geq \frac{1}{\pi} \int_{F < |a|} e^{-|a|^2} |da| \left(1 - \frac{K_P^2}{\varepsilon^{2d(p)} \alpha^2 F^2} \right). \end{aligned} \quad (4.5)$$

Recall that c_j is defined by (3.2). For $j = 2$, let $F = F_\varepsilon := 2 \frac{K_P}{\alpha \varepsilon^{d(p)}}$ and $\alpha = \frac{1}{16} \text{Diam}_{\mathbb{R}^{2n}}(\mathcal{L})$. Then there exists a constant $C_P > 0$ depending only on P such that for all $d \geq d(p)$ and $0 < \varepsilon \leq 1$, the probability (4.5) is bounded from below by $C_P \exp(-\frac{C_P}{\varepsilon^{2d(p)} \text{Diam}_{\mathbb{R}^{2n}}^2(\mathcal{L})})$. By Theorem 3.4(1–3), there exists C'' depending only on $\text{Diam}_{\mathbb{R}^{2n}}(\mathcal{L})$ such that for $d \geq \max(d(p), d_{\mathcal{L}})$, with the same probability, there exists a topological ball B satisfying $\frac{1}{2}\mathbb{B} \subset B \subset \frac{3}{2}\mathbb{B}$, and a compact smooth Lagrangian submanifold $\mathcal{L}'$ of $(Z(f + g) \cap B, \omega|_{Z(f+g)})$ satisfying

$$(\mathcal{L}, Z(f) \cap \mathbb{B}) \sim_{\text{diff}} (\mathcal{L}', Z(f + g) \cap B)$$

with $\frac{1}{2} \text{Diam}_{Z(f)}(\mathcal{L}) \leq \text{Diam}_{\mathcal{L}'}(\mathcal{L}') \leq C'' \text{Diam}_{\mathcal{L}}(\mathcal{L})$. Here, the metrics are the various restrictions of the standard metric g_0 on the ball. However, the push-forward of the metric g_ω on the unit ball by the coordinates $z\varepsilon/\sqrt{d}$ converges uniformly in $0 \leq \varepsilon \leq 1$ to g_0 when d grows to infinity. This implies the theorem. $\blacksquare$

4.2. The general Kähler setting

The generalization of the proof of Theorem 1.23 to random holomorphic sections rests on the concept of *peak sections*, as in [12] and [13]. This object was used by Tian [28] to give estimates for the Bergman kernel, and by Donaldson [7] to prove the existence of codimension 2 symplectic submanifolds. In a way, they were already used by Hörmander to solve the Levi problem for Stein manifolds [17, Theorem 5.1.6]. They are used in the parallel paper [11] for a deterministic proof of Corollary 1.20.

Let $(n, r, X, L, E, h_L, \omega, g_\omega, h_E, d\text{vol}, (\mathbb{P}_d)_{d \geq 1})$ be an ample probabilistic model. A *peak section* of $L^{\otimes d}$ at $x \in X$ is a holomorphic section whose norm decreases exponentially fast outside x , and almost vanishes at scale $\gg 1/\sqrt{d}$, like X_0^d in the standard

projective case near the point $[1 : 0 : \dots : 0]$. One of their crucial features is that a given peak section times the monomials (1.1) in normal holomorphic coordinates form asymptotically an orthonormal family, which makes the general Kähler situation locally very similar to the standard projective one.

Proof of Theorem 1.18. Let $x \in X$, and e a local holomorphic trivialization of L near x such that $\|e\|_{h_L}$ is locally maximal at x with $\|e(x)\|_{h_L} = 1$. Then there exists a uniform (in $x \in X$) constant $c > 0$ such that for any y in a fixed neighborhood of x ,

$$\|e^{\otimes d}(y)\|_{h_L} \leq \exp(-cd\|x - y\|^2). \quad (4.6)$$

This is implied by the fact that the curvature of h_L is a Kähler form and the uniformity is implied by the compactness of X . Again, $X_0^d = e^d$ in the standard case. Let $(e_1, \dots, e_r)$ be a local holomorphic trivialization of E , orthonormal at x . Then $(e_1 \otimes e^d, \dots, e_r \otimes e^d)$ is a local holomorphic trivialization of $E \otimes L^d$ whose coordinates are called *peak sections* for x . Now, let $(p_1, \dots, p_r)$ be a polynomial map that defines the complex algebraic hypersurface Σ , and

$$s_{\varepsilon, d, p} := (p_i(\cdot \sqrt{d}/\varepsilon)e_i)_{1 \leq i \leq r} \otimes e^d,$$

which is a section of $E \otimes L^d$ defined in a fixed neighborhood of x , and is the equivalent of $P_{\varepsilon, d}$ (see (4.2) in the standard case). Now by the Hörmander L^2 -estimates (see [17] or [27] for a bundle version), $s_{\varepsilon, d, p}$ can be perturbed to a global section $\sigma_{\varepsilon, d, p} \in H^0(X, E \otimes L^d)$. Moreover, this is a classical result in Hörmander theory that the C^1 error produced by the perturbation on $B(x, (\log d)/\sqrt{d})$ is bounded by $\exp(-cd)$ [12, Lemma 3.5]. Here, the estimates (4.6) are crucial. By Lemma 3.1 this implies that $Z(s_{\varepsilon, d, p})$ is a complex $(n - r)$ -submanifold which is an isotopic perturbation of $Z(p)$.

The rest of the proof is very similar to the standard case. We decompose the random section $s \in H^0(X, E \otimes L^d)$ as

$$s = a \frac{s_{\varepsilon, d, p}}{\|s_{\varepsilon, d, p}\|_{L^2}} + \rho,$$

where $\rho \in s_{\varepsilon, d, p}^\perp$ and $s_{\varepsilon, d, p}^\perp$ is equipped with the restriction of the Gaussian measure, and a follows a complex normal law $N_{\mathbb{C}}(0, 1)$. The L^2 -norm of $s_{\varepsilon, d, p}$ has a similar equivalent to $\|P_{\varepsilon, d}\|_{L^2}$ given by Lemma 4.6. Then we look at the situation on $B(x, \varepsilon/\sqrt{d})$ which becomes a fixed $\mathbb{B} \subset \mathbb{C}^n$ after rescaling, and the sections are trivialized as functions with values in $\mathbb{C}^r$. Lemma 4.8 still holds for the trivialization q of the perturbation. In its proof of it, the only essential adaptation in the bundle case is the estimate (4.10), where the modulus of the function is compared on $B(0, \varepsilon/\sqrt{d})$ with its Fubini–Study norm. In the present situation, a similar comparison holds, since the norm of e^d varies only by a uniform positive multiplicative constant over $B(x, \varepsilon/\sqrt{d})$.

The Lagrangian part of the proof is the same, since as coordinates at a point $x \in X$ we can choose holomorphic coordinates z such that $\omega = z^*\omega_0$ at x , so that we can find a 1-form λ in the chart such that $\lambda - z^*\lambda_0 = O(|z|)$, which is the only two things we need. $\blacksquare$

Remark 4.1. Instead of peak sections, we could use the Bergman kernel, the Schwartz kernel for the projection onto the space of holomorphic sections, and the 2-point correlation function for our random model. This kernel converges at scale $1/\sqrt{d}$ to a universal kernel, the *Bargmann–Fock* kernel [4], which explains why the results on standard Fubini–Study random polynomials are similar to those for random holomorphic sections. This universality can be proved by using peak sections [28]. The kernel approach has the virtue that parts of the proof can be adapted to other random models. However, we must not overestimate this interest for some reasons. Firstly, the fact that the zeros of the sections of given degree have the same topology and symplectomorphism type is very much dependent on holomorphy, or at least asymptotic holomorphy in the Donaldson [7] and Auroux settings [2]. Secondly, the projective hypersurface inherits a natural symplectic form, which is rarely the case for other models. Thirdly, the barrier method is very much adapted to explicit local sections, like peak sections. Fourthly, the fact that this model is particularly well suited for polynomials is not directly seen from the kernel and needs some asymptotic computation. Lastly, the Bergman kernel between x and y is essentially represented by the value of the peak section associated to x evaluated at y .

We finish this section with the proof that the smooth vanishing loci all have the same symplectomorphism type:

Proposition 4.2. *Let $1 \leq r \leq n$ be integers, $E \rightarrow X$ be a holomorphic vector bundle of rank r , and $L \rightarrow X$ be a holomorphic line bundle equipped with a metric h with positive curvature $-i\omega$. For any degree $d \geq 1$, denote by $H_{\text{reg}}^0(X, E \otimes L^d)$ the space of holomorphic sections of $E \otimes L^{\otimes d}$ which vanish transversally. Then for any d large enough,*

$$\forall (s, t) \in H_{\text{reg}}^0(X, E \otimes L^d)^2, \quad (Z(s), \omega|_{Z(s)}) \sim_{\text{symp}} (Z(t), \omega|_{Z(t)}).$$

Proof. First, by Bertini’s theorem [14, p. 137], $H_{\text{sing}}^0 := H^0(X, E \otimes L^d) \setminus H_{\text{reg}}^0(X, E \otimes L^d)$ is of real codimension at least 2 in H^0 . This implies that any $s, t \in H_{\text{reg}}^0$ are joined by a path of sections in H_{reg}^0 . By the Ehresmann theorem, this implies that $Z(s)$ is diffeomorphic to $Z(t)$. Now, for a continuous family $(s_t)_{t \in [0,1]}$ of sections in H_{reg}^0 , since ω is the curvature of a line bundle, as also is its restriction to $Z(s_t)$, we have $[\omega|_{Z(s_t)}] \in H^2(Z(s_t), \mathbb{Z})$. Consequently, the pull-back in $H^2(Z(s_0), \mathbb{Z})$ of $[\omega|_{Z(s_t)}]$ by the diffeomorphism $\psi_t : Z(s_0) \rightarrow Z(s_t)$ given by the former argument is constant. In other words, $\psi_t^*[\omega|_{Z(s_t)}] = [\omega|_{Z(s_0)}]$. Then the Moser argument (see [18, Theorem 3.17]) implies that the zero sets are in fact symplectomorphic. ■

4.3. Some simple lemmas

In this subsection we give the proofs of elementary and technical lemmas that are used in the core of the proof of the quantitative Moser deformation, Proposition 3.3.

Lemmas for the deformations

Lemma 4.3. *Let $m \geq 1$ be an integer, $(X_t)_{t \in [0,1]}$ be a C^2 family of vector fields on $\mathbb{R}^n$ with compact support, and $(\phi_t)_{t \in [0,1]}$ be the associated flow.*

- (1) *For all $t \in [0, 1]$, $\|\phi_t - \text{Id}\|_{C^0(\mathbb{R}^m)} \leq t \max_{t \in [0,1]} \|X_t\|_{C^0(\mathbb{R}^m)}$.*
- (2) *Let $0 \leq j \leq 2$ and $C > 0$ be such that $\max_{t \in [0,1]} N_j(X_t, \mathbb{R}^m) \leq C$. Then there exists C' depending only on C such that*

$$\forall t \in [0, 1], \quad \|\phi_t - \text{Id}\|_{C^j(\mathbb{R}^m)} \leq C' t \max_{t \in [0,1]} N_j(X_t).$$

Proof. First, it is classical that ϕ_t is C^k in (t, x) . We have

$$\forall (x, t) \in \mathbb{R}^n \times [0, 1], \quad \phi_t(x) - x = \int_0^t X_s(\phi_s(x)) ds,$$

which implies $\|\phi_t - \text{Id}\|_{C^0(M)} \leq \max_{t \in [0,1]} \|X_t\|_{C^0(M)}$ and

$$d\phi_t - \text{Id} = \int_0^t d_x X_s(\phi_s(x)) \circ d\phi_s ds.$$

Consequently, $\|d\phi_t - \text{Id}\|_{C^0} \leq \max_t \|dX_t\|_{C^0}(t + \int_0^t \|d\phi_s - \text{Id}\| ds)$. By Gronwall, this implies

$$\|d\phi_t - \text{Id}\|_{C^0} \leq t \max_t \|dX_t\|_{C^0} \exp\left(\max_t \|dX_t\|_{C^0}\right) \leq t e^C N_1(X). \quad (4.7)$$

Now, $d^2(\phi_t - \text{Id}) = d^2\phi_t = \int_0^t d_x^2 X_s(\phi_s) d\phi_s \otimes d\phi_s + d_x X_s \circ d^2\phi_s ds$. Together with estimate (4.7), this implies

$$\|d^2\phi_t\| \leq \max_t \|d^2 X_t\|_{C^0} (1 + C e^C)^2 + \max_t \|dX_t\|_{C^0} \int_0^t \|d^2\phi_s\| ds,$$

so that by Gronwall, $\|d^2\phi_t\| \leq \max_t \|d^2 X_t\|_{C^0} (1 + C e^C)^2 \exp(C)$. ■

Unfortunately, for manifolds we need a simpler version of the latter affine lemma.

Lemma 4.4. *Let (M, h) be a smooth Riemannian manifold, $(X_t)_{t \in [0,1]}$ be a C^k family of vector fields with compact support N and $(\phi_t)_{t \in [0,1]}$ the 1-parameter group of diffeomorphism generated by $(X_t)_t$.*

- (1) *For all $t \in [0, 1]$, $d(\phi_t, \text{Id}) \leq t \max_{s \in [0,1]} \|X_s\|_{C^0(M)}$.*
- (2) *Let $C > 0$ be such that $\max_{t \in [0,1]} N_1(X_t, \mathbb{R}^m) \leq C$. Then there exists C' depending only on C and the C^1 norm of the metric on N such that $\max_{t \in [0,1]} \|d\phi_t\|_{C^0(M)} \leq C'$.*

Proof. Again, it is classical that ϕ_t is C^k in (t, x) , and

$$\forall (x, t) \in M \times [0, 1], \quad d(\phi_t(x), x) \leq \text{Length}(\{\phi_t(x)\}_{t \in [0,1]}) \leq t \max_{s \in [0,1]} \|X_s\|_{C^0(M)}.$$

Let $x \in M$ in a local chart. If t is small enough, in coordinates we have

$$\phi_t(x) - x = \int_0^t X(\phi_s(x), s) ds,$$

so that $d\phi_t - \text{Id} = \int_0^t d_x X(\phi_s(x), s) \circ d\phi_s ds$. Then there exists a constant C depending only on the compact support of X and the C^1 norm of the metric h in coordinates such that for any vector $Y \in \mathbb{R}^n$,

$$|d\phi_t(Y)|_{\phi_t(x)} \leq C|Y|_x \left(1 + \max_{s \in [0, t]} \|d_x X_s\|_{C^0(M)} \int_0^t \|d\phi_s\|_{\phi_s(x)} ds\right),$$

which implies $\|d\phi_t\|_{\phi_t(x)} \leq C(1 + \max_{s \in [0, t]} \|d_x X_s\|_{C^0(M)} \int_0^t \|d\phi_s\|_{\phi_s(x)} ds)$, and by Gronwall,

$$\|d\phi_t\|_{C^0} \leq C \exp\left(C \max_{[0, 1]} \|d_x X_s\|\right),$$

so that there exists another constant C' depending on the chart such that

$$\|d\phi_t\|_{C^0} \leq C \exp\left(C' \max_{[0, 1]} N_1(X_s)\right).$$

Since we can cover the support of X by a finite number of charts, this implies the result. $\blacksquare$

The following lemma was used in the proof of the last assertion (3.3) of Proposition 3.1.

Lemma 4.5. *Let $m \geq 1$ and $1 \leq p \leq m$ be integers, and $\Phi : M(p, m) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, where for any $(A, Y) \in M(p, m) \times \mathbb{R}^m$, $\Phi(A, Y)$ denotes the orthogonal projection of the origin onto the space $\{X \in \mathbb{R}^m \mid AX = Y\}$. Then, for any $0 \leq j \leq 2$ and for any $(A, Y) \in M(p, m) \times \mathbb{R}^m$ such that A is onto,*

$$\begin{aligned} \Phi(A, Y) &\leq T(A)^{-2} \|A\| |Y|, \\ \|d_A \Phi(A, Y)\| &\leq 3T(A)^{-4} \|A\|^2 |Y| \quad \text{and} \quad \|d_Y \Phi(A, Y)\| \leq T(A)^{-2} \|A\|, \\ \|d_{A^2}^2 \Phi(A, Y)\| &\leq 14\|A\|^3 T(A) |Y|^{-6} \quad \text{and} \quad \|d_{AY}^2 \Phi\| \leq 3T(A)^{-4} \|A\|^2, \end{aligned}$$

where T has been defined in (3.1).

Proof. Write $A = (A_1, \dots, A_p)^t$, where $A_k \in \mathbb{R}^m$ are column vectors. Since A is onto, $(\ker A)^\perp = \langle A_1, \dots, A_p \rangle$ and there exist a unique $\lambda(A, Y) = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$ such that $\Phi(A, Y) = \sum_{i=1}^p \lambda_i A_i \in A^{(-1)}(\{Y\})$, which means $AA^t \lambda = Y$, so that

$$\Phi(A, Y) = [K(A)Y, A],$$

where $[\lambda, A] := \sum_{i=1}^p \lambda_i A_i$ and $K(A) := (AA^t)^{-1}$. This implies that Φ is smooth near (A, Y) for any A onto, and linear in Y . Since

$$K(A) \leq T(A)^{-2},$$

by Cauchy–Schwarz $|\Phi(A, Y)| \leq T(A)^{-2} \|A\| |Y|$. Differentiating gives

$$d\Phi(A, Y) = [dK(A)Y + KdY, A] + [KY, dA] \quad (4.8)$$

so that $\|d_Y \Phi\| \leq T(A)^{-2} \|A\|$. If $Q(A) := AA^t$, then $\|dQ(A)\| \leq 2\|A\|$ and

$$\forall B \in M(p, m), \quad dK(A)(B) = -KdQ(A)BK,$$

so that, using $T(A) \leq \|A\|$,

$$\begin{aligned} \|dK(A)\| &\leq 2T(A)^{-4} \|A\|, \\ \|d_A \Phi\| &\leq 2T(A)^{-4} \|A\|^2 |Y| + T(A)^{-2} |Y| \leq 3T(A)^{-4} \|A\|^2 |Y|. \end{aligned}$$

Now

$$d^2 \Phi(A, Y) = [d^2 K(A)Y, A] + \text{Sym}([dK(A)Y, dA] + [K(A)dY, dA] + [dK(A)dY, A]), \quad (4.9)$$

where Sym means that the bracket is symmetrized in the two vectors in $M(p, m) \times \mathbb{R}^m$ to which $d^2 \Phi(A, Y)$ is applied. Since $d^2 K(A) = \text{Sym } KdQKdQK - Kd^2 QK$, we obtain

$$\begin{aligned} \|d^2 K(A)\| &\leq 8\|A\|^2 T(A)^{-6} + 2T(A)^{-4} \leq 10\|A\|^2 T(A)^{-6}, \\ \|d_{A^2}^2 \Phi(A, Y)\| &\leq 10\|A\|^3 T(A)^{-6} |Y| + 4T(A)^{-4} \|A\| |Y| \leq 14\|A\|^3 T(A)^{-6} |Y|, \\ \|d_{AY}^2 \Phi\| &\leq 3T(A)^{-4} \|A\|^2. \end{aligned} \quad \blacksquare$$

Lemmas for the barrier methods

Lemma 4.6. *Let $0 < \varepsilon \leq 1$, $p \in (\mathbb{C}[z_1, \dots, z_n])^r$ and $P_{\varepsilon, d} = Z_0^d p(\frac{\sqrt{d}}{\varepsilon} \cdot)$. Then, uniformly in ε ,*

$$\|P_{\varepsilon, d}\|_{L^2} \underset{d \rightarrow \infty}{\sim} \frac{1}{\pi d^{n/2}} \left\| p\left(\frac{\cdot}{\varepsilon}\right) \right\|_{\text{BF}},$$

where $\|p\|_{\text{BF}}^2 := \frac{1}{\pi^n} \int_{\mathbb{C}^n} |p(y)|^2 e^{-|y|^2} |dy|$ defines the Bargmann–Fock norm of p .

Remark 4.7. Note that for any p , there exists a constant $c > 0$ such that for any $0 < \varepsilon \leq 1$ and $d \geq 1$,

$$\|P_{\varepsilon, d}\|_{L^2} \leq \frac{c}{d^{n/2} \varepsilon^{\deg p}}.$$

Proof of Lemma 4.6. We have, by definition of the Fubini–Study measure on $\mathbb{C}P^n$,

$$\|P_{\varepsilon, d}\|_{L^2}^2 = \int_{\mathbb{S}^{2n+1}} |P_{\varepsilon, d}|^2 \frac{d\sigma}{2\pi},$$

where $d\sigma$ is the canonical measure on the sphere with volume 1 and the 2π factor corresponds to the volume of the fiber $U(1)$ of the quotient $\mathbb{S}^{2n+1} \rightarrow \mathbb{C}P^n$. Since

$$\begin{aligned} \int_{\mathbb{C}^{n+1}} |P_{\varepsilon, d}|^2 e^{-\|Z\|^2} |dZ| &= \int_0^\infty r^{2d+2n+1} e^{-r^2} dr \int_{\mathbb{S}^{2n+1}} |P_{\varepsilon, d}|^2 d\sigma \\ &= (d+n)! \int_{\mathbb{S}^{2n+1}} |P_{\varepsilon, d}|^2 d\sigma, \end{aligned}$$

where $|dZ|$ denotes the Lebesgue measure on $\mathbb{C}^{n+1}$, we have

$$\|P_{\varepsilon,d}\|_{L^2}^2 = \frac{1}{2\pi(d+n)!} \int_{\mathbb{C}^{n+1}} |Z_0|^{2d} \left| p\left(\frac{\sqrt{d}}{\varepsilon} \frac{Z'}{Z_0}\right) \right|^2 e^{-\|Z\|^2} |dZ|,$$

where $Z' = (Z_1, \dots, Z_n)$. We use the change of variable $(W_0, w) = (Z_0, Z'/Z_0)$; then $(w_0, w) = (W_0 \sqrt{1+|w|^2}, w)$, and finally $y = \sqrt{d}$ so that

$$\begin{aligned} \|P_{\varepsilon,d}\|_{L^2}^2 &= \frac{1}{2\pi(d+n)! \pi^{n+1}} \int_{\mathbb{C}^{n+1}} |W_0|^{2(d+n)} \left| p\left(\frac{\sqrt{d}}{\varepsilon} w\right) \right|^2 e^{-|W_0|^2(1+\|w\|^2)} |dW_0| |dw| \\ &= \frac{1}{2\pi(d+n)! \pi^{n+1}} \\ &\quad \times \int_{\mathbb{C}^{n+1}} |w_0|^{2(d+n)} e^{-|w_0|^2} \left| p\left(\frac{\sqrt{d}}{\varepsilon} w\right) \right|^2 \frac{1}{(1+\|w\|^2)^{d+n+1}} |dw_0| |dw| \\ &= \frac{1}{d^n} \frac{1}{2\pi(d+n)! \pi^{n+1}} \\ &\quad \times \int_{\mathbb{C}} |w_0|^{2(d+n)} e^{-|w_0|^2} |dw_0| \int_{\mathbb{C}^n} \left| p\left(\frac{y}{\varepsilon}\right) \right|^2 \frac{1}{(1+\frac{1}{d}\|y\|^2)^{d+n+1}} |dy| \\ &\underset{d \rightarrow \infty}{\sim} \frac{1}{d^n} \frac{1}{\pi^{n+1}} \int_{\mathbb{C}^n} \left| p\left(\frac{y}{\varepsilon}\right) \right|^2 e^{-|y|^2} |dy| \end{aligned}$$

uniformly in $\varepsilon \leq 1$.

Note that for any $(i_0, \dots, i_n) \in \mathbb{N}^{n+1}$ such that $\sum_k i_k = d$,

$$\begin{aligned} \|Z_0^{i_0} \dots Z_n^{i_n}\|_{L^2}^2 &= \frac{1}{2\pi(d+n)!} \int_{\mathbb{C}^{n+1}} \prod_{k=0}^{n+1} |Z_k^{i_k}|^2 e^{-\|Z\|^2} |dZ| \\ &= \frac{1}{2\pi(d+n)!} \prod_{k=0}^{n+1} \int_{\mathbb{C}} |z|^{2i_k} e^{-|z|^2} |dz| \\ &= \frac{1}{(d+n)!} \prod_{k=0}^{n+1} \int_0^\infty r^{2i_k+1} e^{-r^2} dr = \frac{i_0! \dots i_n!}{(d+n)!}. \quad \blacksquare \end{aligned}$$

The next lemma was proved in a real and general Kähler situation in [12]. We give a simple proof in the polynomial setting, in order for the article to be self-contained.

Lemma 4.8. *Let $1 \leq r \leq n$ be integers, $\varepsilon > 0$, $R \in (H_{d,n+1})^r$ be a random polynomial mapping of maximal degree d and $q(z) = R(1, z\varepsilon/\sqrt{d})$, where $z = (z_1, \dots, z_n)$. Then there exists $C > 0$ depending only on n and r such that for any $d \geq 1$, any $0 < \varepsilon \leq 1$, and any $0 \leq j \leq 2$,*

$$\mathbb{E} \left(\sup_{2\mathbb{B}} |d^j q|^2 \right) \leq C_n \frac{(d+n)!}{d!}.$$

Proof. Since q is holomorphic, we can use the mean value inequality for plurisubharmonic functions applied to $|q|^2$ (see [17]):

$$\forall z \in 2\mathbb{B}, \quad |q(z)|^2 \leq \frac{1}{\text{Vol } \mathbb{B}} \int_{z+\mathbb{B}} |q(u)|^2 du,$$

so that $\mathbb{E}(\sup_{2\mathbb{B}} |q|^2) \leq \frac{1}{\text{Vol } \mathbb{B}} \int_{3\mathbb{B}} \mathbb{E}(|q(u)|^2) du$. By (1.2) we have

$$\begin{aligned} \forall z \in 2\mathbb{B}, \quad \mathbb{E}(|q(z)|^2) &= \mathbb{E}(|R(1, z\varepsilon/\sqrt{d})|^2) \\ &= \mathbb{E}(\|R\|_{\text{FS}}^2(1, z\varepsilon/\sqrt{d}))(1 + |z|^2\varepsilon^2/d)^{2d}. \end{aligned} \quad (4.10)$$

By definition of the measure, $\mathbb{E}(\|R\|_{\text{FS}}^2)$ is constant over $\mathbb{C}P^n$. Remembering that the coordinates of R are independent random polynomials, we obtain (see decomposition (1.2))

$$\mathbb{E}(\|R\|_{\text{FS}}^2)(1, z\varepsilon/\sqrt{d}) = \mathbb{E}(\|R\|_{\text{FS}}^2(0)) = r\mathbb{E}\left(\frac{(d+n)!}{d!} |a_0|^2\right) = r\frac{(d+n)!}{d!}.$$

Moreover, $(1 + |z|^2\varepsilon/d)^{2d} \leq e^{18\varepsilon^2}$ for all $d \geq 1$ and $z \in 2\mathbb{B}$, hence the first estimate of the lemma.

For the second estimate, for any holomorphic function $f = (f_1, \dots, f_r) : \mathbb{C}^n \rightarrow \mathbb{C}^r$ define

$$\|df\|_2^2 := \sum_{i=1}^r \sum_{j=1}^n \left| \frac{\partial f_i}{\partial z_j} \right|^2.$$

Notice that $\|df\| \leq \|df\|_2$, where $\|df\|$ is the operator norm used in Proposition 3.1. Similarly to the first estimate, since the complex derivatives of q are holomorphic, we have

$$\mathbb{E}\left(\sup_{2\mathbb{B}} \|dq\|_2^2\right) \leq \frac{1}{\text{Vol } \mathbb{B}} \int_{3\mathbb{B}} \mathbb{E}(\|dq(u)\|_2^2) du$$

with $\|dq(u)\|_2^2 = \frac{\varepsilon^2}{d} \|d_{Z'} R(1, u\varepsilon/\sqrt{d})\|_2^2$, where $Z' = (Z_1, \dots, Z_n)$. As before,

$$\mathbb{E}(\|d_{Z'} P(1, u\varepsilon/\sqrt{d})\|_2^2) \leq \mathbb{E}(\|d_{Z'} P\|_{\text{FS}}^2(1, 0))e^{18\varepsilon^2}$$

with, using the linear part in Z' of the decomposition (1.2),

$$\mathbb{E}(\|d_{Z'} P(1, 0)\|_{\text{FS}}^2) = r \sum_{i=1}^n \mathbb{E}\left(\frac{(d+n)!}{(d-1)!1!} |a_{0\dots 1\dots 0}|^2\right) = rn \frac{(d+n)!}{(d-1)!},$$

which implies the second estimate of the lemma. The last estimate is similar. ■

Acknowledgments. The author would like to thank Denis Auroux, Vincent Beffara, Sylvain Courte, Thomas Delzant, Laura Monk, Alejandro Rivera, and Jean-Yves Welschinger for valuable discussions.

Funding. The research leading to these results has received funding from the French Agence nationale de la recherche, ANR-15CE40-0007-01 and from the European Research Council project ALKAGE, contract 670846 from Sept. 2015.

References

- [1] Arnol'd, V. I., Guseĭn-Zade, S. M., Varchenko, A. N.: Singularities of Differentiable Maps. Vol. II. Monogr. Math. 83, Birkhäuser Boston, Boston, MA (1988) Zbl [0659.58002](#) MR [966191](#)
- [2] Auroux, D.: Asymptotically holomorphic families of symplectic submanifolds. *Geom. Funct. Anal.* **7**, 971–995 (1997) Zbl [0912.53020](#) MR [1487750](#)
- [3] Berger, M.: À l'ombre de Loewner. *Ann. Sci. École Norm. Sup. (4)* **5**, 241–260 (1972) Zbl [0237.53035](#) MR [309009](#)
- [4] Bleher, P., Shiffman, B., Zelditch, S.: Universality and scaling of correlations between zeros on complex manifolds. *Invent. Math.* **142**, 351–395 (2000) Zbl [0964.60096](#) MR [1794066](#)
- [5] Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. *Ergeb. Math. Grenzgeb.* 36, Springer (2013) Zbl [0912.14023](#) MR [1659509](#)
- [6] Bogomolov, F. A.: On the diameter of plane algebraic curves. *Math. Res. Lett.* **1**, 95–98 (1994) Zbl [0868.53048](#) MR [1258494](#)
- [7] Donaldson, S. K.: Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.* **44**, 666–705 (1996) Zbl [0883.53032](#) MR [1438190](#)
- [8] Farber, M.: Topology of closed 1-forms and their critical points. *Topology* **40**, 235–258 (2001) Zbl [0977.58011](#) MR [1808219](#)
- [9] Feng, S.-T., Schumacher, G.: On the diameter of plane curves. *Compos. Math.* **119**, 331–334 (1999) Zbl [0956.53013](#) MR [1727135](#)
- [10] Gayet, D.: Hypersurfaces symplectiques réelles et pinceaux de Lefschetz réels. *J. Symplectic Geom.* **6**, 247–266 (2008) Zbl [1170.53069](#) MR [2448826](#)
- [11] Gayet, D.: Disjoint Lagrangians in Donaldson hypersurfaces. In preparation
- [12] Gayet, D., Welschinger, J.-Y.: Lower estimates for the expected Betti numbers of random real hypersurfaces. *J. London Math. Soc. (2)* **90**, 105–120 (2014) Zbl [1326.14139](#) MR [3245138](#)
- [13] Gayet, D., Welschinger, J.-Y.: Expected topology of random real algebraic submanifolds. *J. Inst. Math. Jussieu* **14**, 673–702 (2015) Zbl [1326.32040](#) MR [3394124](#)
- [14] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley-Interscience, New York (1978) Zbl [0408.14001](#) MR [507725](#)
- [15] Gromov, M.: Filling Riemannian manifolds. *J. Differential Geom.* **18**, 1–147 (1983) Zbl [0515.53037](#) MR [697984](#)
- [16] Hartshorne, R.: Algebraic Geometry. *Grad. Texts in Math.* 52, Springer, New York (1977) Zbl [0367.14001](#) MR [0463157](#)
- [17] Hörmander, L.: An Introduction to Complex Analysis in Several Variables. Rev. ed., North-Holland Math. Library 7, North-Holland, Amsterdam (1973) Zbl [0271.32001](#) MR [0344507](#)
- [18] McDuff, D., Salamon, D.: Introduction to Symplectic Topology. 2nd ed., Oxford Math. Monogr., Clarendon Press, Oxford Univ. Press, New York (1998) Zbl [1066.53137](#) MR [1698616](#)
- [19] Mikhalkin, G.: Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology* **43**, 1035–1065 (2004) Zbl [1065.14056](#) MR [2079993](#)
- [20] Mirzakhani, M.: Growth of Weil–Petersson volumes and random hyperbolic surfaces of large genus. *J. Differential Geom.* **94**, 267–300 (2013) Zbl [1270.30014](#) MR [3080483](#)
- [21] Nazarov, F., Sodin, M.: On the number of nodal domains of random spherical harmonics. *Amer. J. Math.* **131**, 1337–1357 (2009) Zbl [1186.60022](#) MR [2555843](#)
- [22] Ness, L.: Curvature on algebraic plane curves. I. *Compos. Math.* **35**, 57–63 (1977) Zbl [0369.32003](#) MR [453752](#)

- [23] Seidel, P.: Disjoinable Lagrangian spheres and dilations. *Invent. Math.* **197**, 299–359 (2014) Zbl [1305.53081](#) MR [3232008](#)
- [24] Seifert, H.: Algebraische Approximation von Mannigfaltigkeiten. *Math. Z.* **41**, 1–17 (1936) Zbl [62.0807.02](#) MR [1545601](#)
- [25] Shiffman, B., Zelditch, S.: Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.* **200**, 661–683 (1999) Zbl [0919.32020](#) MR [1675133](#)
- [26] Shiffman, B., Zelditch, S.: Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds. *J. Reine Angew. Math.* **544**, 181–222 (2002) Zbl [1007.53058](#) MR [1887895](#)
- [27] Skoda, H.: Morphismes surjectifs et fibrés linéaires semi-positifs. In: Séminaire Pierre Lelong-Henri Skoda (Analyse), Année 1976/77, Lecture Notes in Math. 694, Springer, Berlin, 290–324 (1978) Zbl [396.32009](#) MR [522481](#)
- [28] Tian, G.: On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.* **32**, 99–130 (1990) Zbl [0706.53036](#) MR [1064867](#)
- [29] Weinstein, A.: Symplectic manifolds and their Lagrangian submanifolds. *Adv. Math.* **6**, 329–346 (1971) Zbl [0213.48203](#) MR [286137](#)