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The Riemannian quantitative isoperimetric inequality

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Abstract. We study the Riemannian quantitive isoperimetric inequality. We show that a direct analogue of the Euclidean quantitative isoperimetric inequality is—in general—false on a closed Riemannian manifold. In spite of this, we show that the inequality is true generically. Moreover, we show that a modified (but sharp) version of the quantitative isoperimetric inequality holds for a real analytic metric, using the Łojasiewicz–Simon inequality. The main novelty of our work is that in all our results we do not require any a priori knowledge on the structure/shape of the minimizers.

Keywords. Quantitative stability, isoperimetric inequality, Łojasiewicz inequality

1. Introduction

The isoperimetric inequality on $\mathbb{R}^n$ states that $\mathcal{P}(\Omega) \geq \mathcal{P}(B)$ for any Caccioppoli set Ω with $|\Omega| = |B|$, with equality only for $\Omega = B$ (up to a set of measure zero). That is, the isoperimetric inequality states that balls in Euclidean space have the least perimeter for their enclosed volume. Starting with Bonnesen (cf. [40]), there has been considerable activity concerning quantitative versions of the isoperimetric inequality, finding geometric conditions on a set Ω that nearly achieves equality in the isoperimetric inequality.

Recently, a (Euclidean) quantitative isoperimetric inequality holding in all dimensions has been established by Fusco, Maggi, and Pratelli [29]. They proved that if Ω is a Caccioppoli set with $|\Omega| = |B_1(0)|$, then

$$\Big(\inf_{B=B_1(x)\subset\mathbb{R}^n} |\Omega \triangle B|\Big)^2 \leq C(n)(\mathcal{P}(\Omega) - \mathcal{P}(B)). \tag{1.1}$$

By considering C^2 perturbations of the ball, one can see thet the exponent on the left hand side is sharp [31]. Subsequently, Figalli, Maggi, and Pratelli [24], and Cicalese and

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Leonardi [13], gave substantially different proofs of (1.1), and were also able to explicitly compute the constant $C(n)$ and prove that it is polynomial in n. See also [14, 28], and [1, 7] for applications of such quantitative inequalities.

In this work, we consider the analogue of (1.1) on a closed (that is, compact without boundary) Riemannian manifold. The symmetrization and optimal transport techniques of [24, 29] are not applicable even for non-quantitative isoperimetric inequalities on a general Riemannian manifold (see Section 1.2 below for more discussion), so we follow the selection principle approach of [13]. The general idea of the selection principle (which has its roots in the work of White [50]) is that by considering a "worst case scenario" for (1.1), one can reduce to the case where $\partial\Omega$ is a small $C^{1,\alpha}$ graph over ∂B. At this point work of Fuglede [26] applies (in $\mathbb{R}^n$) to show that (1.1) holds in the worst case scenario (and thus in all situations).

In a closed Riemannian manifold (M, g), it is well known that isoperimetric regions exist for all volumes $V \in (0, |M|_g)$. However, there are surprisingly few manifolds where explicit isoperimetric regions are known (see [19, Appendix H] for a recent survey). As such, methods that rely explicitly on the geometry of $B \subset \mathbb{R}^n$ cannot be directly extended to a general manifold. Indeed, an estimate of the form (1.1) is *false* in a general Riemannian manifold, even for sets which are small graphs over isoperimetric regions!

Indeed, we construct the following example in Section 4:

Theorem 1.1. *For all $n \geq 2$ there exists a closed manifold M^n with a real analytic Riemannian metric g, a uniquely isoperimetric region $\Omega \subset M$ and sets E_k with smooth boundary such that $|\Omega \triangle E_k|_g \to 0$ but the sets E_k do not satisfy the analogue of (1.1), i.e.*

$$\frac{(|\Omega \triangle E_k|_g)^2}{\mathcal{P}^g(E_k) - \mathcal{P}^g(\Omega)} \to \infty.$$

In fact, for any $\gamma > 0$ fixed, there exists a real analytic g, depending on γ, such that

$$\frac{(|\Omega \triangle E_k|_g)^{2+\gamma}}{\mathcal{P}^g(E_k) - \mathcal{P}^g(\Omega)} \to \infty.$$

Finally, we show that there is a g smooth but not real analytic on M^n such that one cannot bound $\mathcal{P}^g(E_k) - \mathcal{P}^g(E)$ by any power of $|\Omega \triangle E_k|_g$, i.e.,

$$\frac{(|\Omega \triangle E_k|_g)^{2+\gamma}}{\mathcal{P}^g(E_k) - \mathcal{P}^g(\Omega)} \to \infty \quad \textit{for all} \quad \gamma > 0.$$

As such, the natural analogue of (1.1) cannot hold in general. Nevertheless, we prove that (1.1) holds generically in the following sense. Let Γ denote the the set of C^3 metrics on a given Riemannian manifold.

Theorem 1.2. *Let M^n be a closed manifold with $2 \leq n \leq 7$. There exists an open and dense subset $\mathcal{G} \subset \Gamma$ with the following property. If $g \in \mathcal{G}$, then there exists an open dense subset $\mathcal{V} \subset (0, |M|_g)$ such that for $V_0 \in \mathcal{V}$, there is $C = C(g, V_0) > 0$ such that*

$$\mathcal{P}^g(E) - \mathcal{I}^g(V_0) \geq C\alpha_g(E)^2 \tag{1.2}$$

for any $E \subset M$ *with* $|E|_g = V_0$. *Here,*

$$\mathcal{I}_g(V_0) := \inf\{\mathcal{P}^g(\Sigma) : |\Sigma|_g = V_0\}$$

is the isoperimetric profile and the manifold Fraenkel asymmetry is

$$\alpha_g(E) := \inf\{|E \triangle \Sigma|_g : \Sigma \in \mathcal{M}^g_{V_0}\},$$

for $\mathcal{M}^g_{V_0}$ *the set of* Σ *attaining the infimum in* $\mathcal{I}_g(V_0)$.

A key element of the proof here is a bumpiness result in the spirit of [48,49], but with a volume constraint.

Remark 1.3. We note that given a metric g, fixing $V \in (0, |M|_g)$ one can always find a nearby g such that (1.2) holds for volume V (without needing to perturb V): see Corollary 5.4.

Moreover, for any real analytic metric g, we prove an analogue of (1.1) that holds for all volumes.

Theorem 1.4. *For* $2 \le n \le 7$, *assume that* (M^n, g) *is a real analytic, closed Riemannian manifold, and let* $0 < V_0 < |M|_g$. *There exist constants* $C_0 > 0, \gamma \ge 0$, *depending only on* (M, g) *and* V_0, *such that*

$$\mathcal{P}^g(E) - \mathcal{I}^g(V_0) \ge C_0 \alpha_g(E)^{2+\gamma} \tag{1.3}$$

for any $E \subset M$ *with* $|E|_g = V_0$.

As remarked above, the main difference between Theorem 1.4 and essentially all the known quantitative inequalities is that we have no *a priori* knowledge of the structure/shape or any classification of the minimizers of (2.1). For this reason we expect this method to be applicable to a variety of other problems. On the other hand, the price that we have to pay is the exponent $\gamma > 0$ (see Section 1.2 for a more in-depth comparison). We remark that our result is optimal both in the analyticity assumption and in the fact that γ might be (arbitrarily) greater than 0 (see Section 4). The restriction $2 \le n \le 7$ is due to the fact that minimizers are smooth only in these dimensions.

1.1. Idea of the proof of Theorems 1.2 and 1.4

The key idea for Theorems 1.2 and 1.4 is that the quantitative inequality (1.3) with $\gamma = 0$ corresponds to integrability of the minimizers, that is, roughly speaking, every null direction of the second variation can be killed by choosing a nearby minimizer. More precisely:

Definition 1.5. We say that a minimizer Σ of (2.1) is *integrable* if every Jacobi field on Σ with zero average is the infinitesimal generator of a one-parameter family of minimizers.

For example, in the case of the Euclidean space, balls are known to be the unique minimizers, and the zero-average part of the kernel of the Jacobi operator is composed only of infinitesimal generators of translations, that is, balls are integrable and the second order expansion gives the inequality. Since in our case integrability is in general false (see Section 4) we have to use a stronger tool, the following infinite-dimensional version of the so-called Łojasiewicz inequality.

Lemma 1.6 (Quantitative inequality and Łojasiewicz inequality). *For any $n \geq 2$, let (M^n, g) be an analytic, compact Riemannian manifold and $\Sigma \subset M$ a smooth isoperimetric region of volume $|\Sigma|_g = V_0$. There exist constants $\delta, \gamma, C_0 > 0$, depending only on (M, g) and Σ, such that if $E \subset M$ has $|E \triangle \Sigma|_g \leq \delta$ and $|E|_g = V_0$ then*

$$\mathcal{P}^g(E) - \mathcal{P}^g(\Sigma) \geq C_0 \alpha_\delta(E, g)^{2+\gamma} \tag{1.4}$$

where

$$\alpha_\delta(E, g) := \inf\{|E \triangle \tilde{\Sigma}|_g : \tilde{\Sigma} \in \mathcal{M}^g_{V_0}, \ |\tilde{\Sigma} \triangle \Sigma|_g \leq \delta\}, \tag{1.5}$$

and we recall that $\mathcal{M}^g_{V_0}$ are the isoperimetric regions of volume V_0.

If Σ is integrable, then we can take $\gamma = 0$. If Σ is strictly stable then we can replace (1.4) with the stronger

$$\mathcal{P}^g(E) - \mathcal{P}^g(\Sigma) \geq C_0 |E \triangle \Sigma|^2. \tag{1.6}$$

This is a local version of Theorem 1.4 valid in every dimension as long as Σ is smooth, and the proof of Theorem 1.4 follows from Lemma 1.6 and a simple compactness argument. On the other hand, the proof of Lemma 1.6 is a consequence of the so-called *selection principle*, introduced in [13] for the quantitative inequality in Euclidean space, and an infinite-dimensional version of the Łojasiewicz inequality for competitors E which are graphical on Σ, which replaces the so-called Fuglede inequality.

Theorem 1.2 follows by combining (1.6) with a bumpiness type theorem that guarantees that for generic metrics and values of the enclosed volume V_0, minimizers of (2.1) are strictly stable, that is, the kernel of the second variation is empty. The only additional difficulty with respect to the results in [48, 49] is the parameter V_0, which corresponds essentially to a Lagrange multiplier.

1.2. Technical discussion of related work

As mentioned above, there has been a lot of recent work on quantitative stability not just for the isoperimetric inequality but also for many other geometric (e.g. Brunn–Minkowski [23]), spectral (e.g. Faber–Krahn [6]), and functional (e.g. Sobolev [39]) inequalities. We refer to the recent survey of Fusco [27] for a more comprehensive list. When the underlying space and the extremizers are highly symmetric these results are often proven by symmetrization or rearrangement (see e.g. [12]). In this vein we would also like to point out the works [11, 32], which do not use symmetrization techniques but do exploit the richness of the symmetry group of the underlying space.

We also note that the quantitative isoperimetric inequality in the form (1.1) for regions in space-forms does hold [4,5]. Moreover, in recent work, Cavalletti–Maggi–Mondino [8] prove a qualitative version of the Lévy–Gromov isoperimetric inequality (for manifolds with lower Ricci curvature bounds). In particular, their results imply that sets nearly saturating the Lévy–Gromov isoperimetric inequality are close (in L^1) to metric balls, which are close (in L^1) to isoperimetric regions. We emphasize that this closeness in [8] is measured relative to how close the metric saturates the Lévy–Gromov inequality, rather than how close the set comes to being isoperimetric.

In the anisotropic setting, optimal transport techniques have been used with great success (see e.g. [3]). However, usually convexity of the extremizers is required (e.g. to guarantee the necessary regularity of the transport map). Other techniques, such as the selection principle, often require understanding the spectrum of the relevant energy linearized around the extremizers (to obtain estimates like Fuglede's [26]). In the generality we consider here, there is very little one can say about the structure of the extremizers (i.e. isoperimetric regions) or symmetry of the underlying space. This lack of knowledge is our primary technical obstacle.

As alluded to above, we are able to overcome this obstacle by establishing the Łojasiewicz–Simon type inequality (1.4). Łojasiewicz's work [34] was first applied to geometric analysis by Simon [43], in order to prove the regularity of solutions to certain elliptic PDE near isolated singularities. These ideas have been further developed by a number of different authors in a number of different settings, e.g., to understand the long term behavior of some gradient flows [16, 47] or to prove results in the same vein as [43], but either in the parabolic setting (see e.g. [10, 15]), or in purely variational settings (see e.g. [20]). See the introduction of [22] and the references therein for a more comprehensive history. As far as we are aware, this is the first instance of a Łojasiewicz–Simon type inequality being used to prove a quantitative stability result.

1.3. Results for stable minimal surfaces

We briefly note that the techniques used to prove Lemma 1.6 can be used to prove the following quantitative minimality result for minimal surfaces, related to the works [18, 33, 50].

Theorem 1.7. *Consider a real analytic Riemannian manifold* (M^n, g). *Assume that* $\Gamma^{n-1} \subset M$ *is a smooth stable minimal hypersurface. Then there are* $\delta, C > 0$ *and* $\gamma \geq 0$ (*depending on* Γ, M, g) *such that for* $\mathcal{M}$ *being the set of* $\tilde{\Gamma}$ *homologous to* Γ *with the same mass and small flat norm* $\mathbb{F}(\Gamma, \tilde{\Gamma}) < \delta$,[1] *for any current* S *homologous to* Γ *with* $\mathbb{F}(\Gamma, S) < \delta$, *we have*

$$\mathbf{M}(S) - \mathbf{M}(\Gamma) \geq C\Big(\inf_{\tilde{\Gamma}\in\mathcal{M}} \mathbb{F}(S, \tilde{\Gamma})\Big)^{2+\gamma}.$$

where $\mathbf{M}(\cdot)$ *is the mass* (*area*) *of the current.*

[1] Here $\mathbb{F}(\Gamma, \Gamma') = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : A + \partial B = \Gamma - \Gamma'\}$ is the flat norm (see e.g. [33, Section 2]).

This follows in a nearly identical manner to Lemma 1.6. We note that with appropriate modifications, one can prove a similar result in higher codimensions. It would be interesting to understand an analogue of Theorem 1.7 for finite index surfaces (see [50]).

1.4. Plan of the paper

Section 2 is dedicated to fixing some notations and introducing some preliminary tools, particularly the various Banach manifolds we will use in the rest of the paper. In Section 3 we prove the infinite-dimensional version of Łojasiewicz inequality (Lemma 1.6) and Theorem 1.4, while Section 4 is dedicated to its optimality. Finally, in Section 5 we prove Theorem 1.2 and the bumpy metric result needed to do that.

2. Preliminaries and notations

We start by introducing some concepts that will be used throughout the paper.

2.1. The isoperimetric problem

Recall that the *distributional perimeter* of $E \subset M$ is defined by

$$\mathcal{P}^g(E) = \sup\left\{\int_E \operatorname{div}_g(\phi)\, d\operatorname{vol}_g \;\middle|\; \phi \in C^1(M;TM),\ \|\phi\|_{L^\infty} \le 1\right\}.$$

Sometimes, when it is clear in context, we will eliminate the dependence on g from the notation. Then, for a fixed constant $0 < V_0 < |M^n|_g$, where $|\cdot|_g$ denotes the volume on M induced by g, we study the minimization problem

$$\mathcal{I}^g(V_0) := \inf\{\mathcal{P}^g(E) : E \in \mathcal{A}^g_{V_0}\} \tag{2.1}$$

where

$$\mathcal{A}^g_{V_0} := \{E \subset M : \chi_E \in \mathrm{BV}(M),\ |E|_g = V_0\}$$

is the set of *Caccioppoli sets* with volume V_0. If $\Omega \in \mathcal{A}^g_{V_0}$ attains $\mathcal{I}^g(V_0)$, we say that Ω is *isoperimetric*. We let $\mathcal{M}^g_{V_0}$ denote the set of isoperimetric regions of volume V_0.

2.2. Graphical regions

Let $\Sigma \subset M^n$ be such that $\partial\Sigma$ is smooth, and embedded, and let ν_Σ be the normal to $\partial\Sigma$ in M^n pointing outside Σ. Let $f : \partial\Sigma \to \mathbb{R}$. Then the graph of f is defined by

$$\operatorname{graph}(f) := \{(x, \exp_x(f(x)\nu_\Sigma(x))) : x \in \partial\Sigma\},$$

and we will sometimes use the notation $\operatorname{graph}(f) = \partial\Sigma + f$. Moreover we associate to each such graph a set of finite perimeter $\Sigma + f$ in such a way that $\partial(\Sigma + f) = \partial\Sigma + f = \operatorname{graph}(f)$, with orientation chosen so that $\nu_{\operatorname{graph}(f)} \cdot \nu_\Sigma \ge 0$. When the set Σ is clear from context, we will often abuse notation and use f to refer to both the function and the submanifold $\partial\Sigma + f$ or the subset $\Sigma + f$.

If $u : N^{n-1} \to \Sigma^n$ is a smooth embedding from a compact orientable manifold N^{n-1} to M^n, we will denote by $[u]$ the set of all maps of the form $u \circ \phi$, where $\phi : N \to N$ is a smooth diffeomorphism; that is, the elements of $[u]$ are all parametrizations of the same surface $u(N)$.

2.3. *Banach manifolds*

We will denote by

$$B_r^{k,\alpha}(h_0) := \{h \in C^{k,\alpha}(M) : \|h - h_0\|_{C^{k,\alpha}} < r\}.$$

We will abuse this notation a bit, writing

$$B_r^{k,\alpha}(\Sigma) := \{f \in C^{k,\alpha}(\partial\Sigma) : \|f\|_{C^{k,\alpha}} < r\},$$

and $B_r(V_0) = \{V \in \mathbb{R} : |V - V_0| < r\}$.

Given $r, V_0 > 0$, and Σ a minimizer of (2.1) for a C^3 metric g_0 with $|\Sigma|_{g_0} = V_0$, we are interested in the following sets:

$$\mathcal{B}_r(\Sigma) := \{f \in B_r^{2,\alpha}(\Sigma) : |\Sigma + f|_{g_0} = |\Sigma|_{g_0}\}, \tag{2.2}$$

$$\mathcal{B}_r(\Sigma, g_0) := \{(f, g) \in B_r^{2,\alpha}(\Sigma) \times B_r^3(g_0) : |\Sigma + f|_g = |\Sigma|_{g_0}\}, \tag{2.3}$$

$$\mathcal{B}_r(\Sigma, g_0, V_0) := \{(f, g, V) \in B_r^{2,\alpha}(\Sigma) \times B_r^3(g_0) \times B_r(V_0) : |\Sigma + f|_g = V\}. \tag{2.4}$$

It is straightforward to see that these are Banach manifolds; we sketch the proof for the reader's convenience (recall that Γ denotes the family of C^3 metrics on M).

Lemma 2.1. *Let Σ be a smooth minimizer of the isoperimetric problem* (2.1) *for the metric g_0. There exists $\delta > 0$, depending on Σ, g_0, such that $\mathcal{B}_\delta(\Sigma)$, $\mathcal{B}_\delta(\Sigma, g_0)$ and $\mathcal{B}_\delta(\Sigma, g_0, V_0)$ are separable, codimension* 1 *Banach submanifolds of the separable Banach spaces $C^{2,\alpha}(\partial\Sigma)$, $C^{2,\alpha}(\partial\Sigma) \times \Gamma$ and $C^{2,\alpha}(\partial\Sigma) \times \Gamma \times \mathbb{R}$ respectively* (*modeled on the Banach space of functions with zero average on $\partial\Sigma$ with respect to the metric g_0*).

Proof. We sketch only the case $\mathcal{B}_r(\Sigma)$, as the other two are the same. Separability follows from the separability of $C^{2,\alpha}$, so we only need to show that the function $F(f) := |\Sigma + f|_{g_0} - |\Sigma|_{g_0}$ is a submersion near 0. To do this we observe that, by a well known computation (see for instance [49, Lemma 3.1 and Section 7])

$$DF(0)[v] = \int_{\partial\Sigma} v \, d\sigma_{g_0},$$

where $d\sigma_{g_0}$ is the volume form of $\partial\Sigma$ in the metric g_0. Choosing v as a constant, we immediately see that the differential is surjective, so that there exists $\delta > 0$ depending on Σ such that $\mathcal{B}_\delta(\Sigma, g_0)$ is a Banach submanifold of $C^{2,\alpha}(\partial\Sigma)$. Since the kernel of $DF(0)$ is the space of functions $v \in C^{2,\alpha}$ such that $\int_{\partial\Sigma} v \, d\sigma_{g_0} = 0$, the proof is complete. ∎

When the precise value of r is not so important, we will write $\mathcal{B}(\Sigma)$ to mean $\mathcal{B}_r(\Sigma)$ for some fixed r, small enough as in the previous lemma, and analogously $\mathcal{B}(\Sigma, g_0)$ and $\mathcal{B}(\Sigma, g_0, V_0)$.

We will use a chart on $\mathcal{B}(\Sigma)$ defined as follows. First, we fix a map $\Xi : T_0\mathcal{B}(\Sigma) \cap U \to \mathcal{B}(\Sigma)$, where U is a neighborhood of $0 \in T_0\mathcal{B}(\Sigma)$ by taking

$$\Xi(v) := v + \xi(v) \quad \text{where } \xi(v) \in \mathbb{R} \text{ is chosen so that } |\Sigma + \Xi(v)| = |\Sigma|.$$

We notice that by Lemma 2.1, after choosing U sufficiently small, the map $D\Xi(v) : T_0\mathcal{B}(\Sigma) \to T_{\Xi(v)}\mathcal{B}(\Sigma)$ is invertible, with $D\Xi(0) = \mathrm{Id}$ and moreover, if $u_i = \Xi(v_i)$, then

$$\|u_1 - u_2\|^2_{W^{1,2}} \simeq C\|v_1 - v_2\|^2_{W^{1,2}}. \tag{2.5}$$

Moreover, we will denote by $\mathcal{P}_\Xi : T_0\mathcal{B}(\Sigma) \cap U \to \mathbb{R}$ the perimeter functional written in the coordinates defined by Ξ, namely

$$\mathcal{P}_\Xi(v) := \mathcal{P}(\Sigma + \Xi(v)), \quad v \in T_0\mathcal{B}(\Sigma) \cap U.$$

We remark that if the metric g_0 is analytic, then so is the function Ξ. We can see this in the proof of Lemma 2.1, since the submersion, F, there is analytic.

We will denote by $D\mathcal{P}_\Xi$, $D^2\mathcal{P}_\Xi$ the first and second derivative of $\mathcal{P}_\Xi$ as a map on (a subset of) $T_0\mathcal{B}(\Sigma)$. We have the following simple lemma.

Lemma 2.2. *For every $v \in U$ we have*

$$D\mathcal{P}_\Xi(v) = D\mathcal{P}(\Sigma + \Xi(v)) \circ D\Xi(v) \tag{2.6}$$

where $D\mathcal{P}(\Sigma + \Xi(v)) : C^{2,\alpha}(\Sigma + \Xi(v)) \to \mathbb{R}$ is defined by

$$D\mathcal{P}(\Sigma + \Xi(v))[w] = \frac{d}{dt}\Big|_{t=0} \mathcal{P}(\Sigma + \Xi(v) + tw).$$

In particular,

$$D\mathcal{P}_\Xi(0) = 0, \tag{2.7}$$

$$D^2\mathcal{P}_\Xi(0)[w_1, w_2] = \int_{\partial\Sigma} w_2((J_\Sigma + H_\Sigma^2)w_1 + \mathfrak{j}(w_1)) \tag{2.8}$$

where $J_\Sigma = -\Delta_\Sigma - (|A_\Sigma|^2 + \mathrm{Ric}(\nu, \nu))$ is the Jacobi operator of $\partial\Sigma$ and $\mathfrak{j}(w_1) \in \mathbb{R}$ is the unique real number such that $(J_\Sigma + H_\Sigma^2)w_1 + \mathfrak{j}(w_1)$ has zero average on $\partial\Sigma$.

2.4. *Properties of isoperimetric regions*

The following result concerning regularity of isoperimetric regions is well known (see e.g. [35]).

Theorem 2.3. *We can choose representatives of minimizers of (2.1) so that their boundaries are compact, have constant mean curvature, and are regular away from a singular set of Hausdorff dimension at most $n - 8$.*

Finally, we recall the following

Lemma 2.4. *Let Σ be an isoperimetric region in a closed Riemannian manifold (M^n, g). There exists a number $L \in \mathbb{N}$, depending on (M, g), such that the number of compact connected components of $\partial\Sigma$ is bounded by L.*

Proof. By [37, Theorem 2.2] there is $\delta > 0$ such that if $|\Sigma|_g \in (0, \delta] \cup [|M|_g - \delta, |M|_g)$ then $\partial\Sigma$ is connected (and indeed a perturbation of a coordinate sphere). Now, by Lemma C.1 if $|\Omega|_g \in (\delta, |M|_g - \delta)$, then $\partial\Omega$ has constant mean curvature $|H| \leq C = C(M, g, \delta)$. By the boundedness of H, the monotonicity formula applied to each component of $\partial\Omega$ implies that $\mathcal{P}^g(\Omega) \geq cL$ for some constant $c = c(M, g, \delta) > 0$. However, it is easy to see that $\mathcal{I}^g(V) \leq I_0 = I_0(M, g)$ for all V, by e.g. foliating (M, g) by the level sets of a Morse function. This completes the proof. ∎

This will be used in the proof of Lemma 3.4 (to prove that the kernel of an elliptic operator over $\partial\Sigma$ has finite dimension) and again in the proof of Theorem 1.2 (to conclude that there are only countably many diffeomorphism types for minimizers Σ of (2.1)).

3. Proofs of Lemma 1.6 and Theorem 1.4

The proof is divided into two parts. First, using a modification of the argument in Simon's [43], based on the Lyapunov–Schmidt reduction and the Łojasiewicz inequality for analytic functions, we prove Lemma 1.6 for graphs close to a smooth minimizer of (2.1). This can be interpreted as a generalization of Fuglede's inequality to the non-integrable case. In the second part we combine this result with a modification of the selection principle inspired by [13] to conclude the proof of Lemma 1.6.

Throughout this section, (M, g) will be fixed, so we will not make explicit the dependence on g.

3.1. Lyapunov–Schmidt reduction, integrability and strict stability

We start by recalling the following technical result whose proof is given in Appendix A.

We denote $K := \ker(D^2\mathcal{P}_\Xi(0))$. Notice that since $\partial\Sigma$ is smooth and compact (by Theorem 2.3) and $D^2\mathcal{P}_\Xi(0)$ is the quadratic form associated to an elliptic operator by Lemma 2.2, $\dim K := l < \infty$.

We let $L^2_{\mathcal{B}}$ be the Hilbert space of L^2-functions on $\partial\Sigma$ that integrate to zero. We can thus denote by $K^\perp$ the $L^2_{\mathcal{B}}$ orthogonal complement of K in $T_0\mathcal{B}(\Sigma)$.

Lemma 3.1 (Lyapunov–Schmidt reduction). *Suppose (M, g) is a C^3 manifold and Σ is a smooth[2] minimizer of (2.1). There exists a neighborhood U of 0 in $T_0\mathcal{B}(\Sigma)$ and a*

[2]By smooth here, we mean that the singular set is empty; then $\partial\Sigma$ will be as regular as g allows.

map $\Upsilon : K \cap U \to K^{\perp}$*, as regular as* g*, where the orthogonal complement is taken with respect to the* $L^2_{\mathcal{B}}$ *inner product, such that*

$$\Upsilon(0) = 0 \quad \text{and} \quad \nabla\Upsilon(0) = 0, \tag{3.1}$$

and, in addition,

$$\begin{cases} \pi_{K^{\perp}}(\nabla\mathcal{P}_{\Xi}(\zeta + \Upsilon(\zeta))) = 0 & \forall \zeta \in K \cap U, \\ \pi_K(\nabla\mathcal{P}_{\Xi}(\zeta + \Upsilon(\zeta))) = \nabla P(\zeta) & \forall \zeta \in K \cap U, \end{cases} \tag{3.2}$$

where $P : \mathbb{R}^l \to \mathbb{R}$ *is the function defined by*

$$P(\zeta) = \mathcal{P}_{\Xi}(\zeta + \Upsilon(\zeta)) \quad \text{for every } \zeta \in K \cap U$$

and we identify ζ *with the* l*-vector given by its coordinates in an orthonormal basis of the kernel* K*. Moreover, let* $\mathcal{L}$ *be the* l*-dimensional family defined by*

$$\mathcal{L} := \{\zeta + \Upsilon(\zeta) : \zeta \in U \cap K\} \subset T_0\mathcal{B}(\Sigma).$$

Now assume that g *is analytic. Then* P *is analytic and satisfies the so-called Łojasiewicz inequality at* 0 *(see* [21, Corollary 4]*): there are constants* $C, \delta > 0$ *and* $\gamma \geq 0$*, depending on* Σ*, such that if* $|\xi| < \delta$*, then*

$$P(\xi) - P(0) \geq C\Big(\inf_{\{\xi_0 :\, \nabla P(\xi_0)=0\}} |\xi - \xi_0|\Big)^{2+\gamma}. \tag{3.3}$$

For $W = \mathcal{B}_\delta(\Sigma)$ *and* $\mathcal{M}$ *the set of critical points of* $\mathcal{P} : \mathcal{B}(\Sigma) \to \mathbb{R}$ *we have*

$$\mathcal{M} \cap W = \{\Sigma + \zeta + \Upsilon(\zeta) + \Xi(\zeta + \Upsilon(\zeta)) : \zeta \in U \cap K \text{ and } \nabla P(\zeta) = 0\}, \tag{3.4}$$

and

$$\tilde{\Sigma} \in \mathcal{M} \cap W \quad \text{implies} \quad \mathcal{P}(\tilde{\Sigma}) = \mathcal{P}(\Sigma). \tag{3.5}$$

Moreover, there is a constant $C < \infty$ *such that, for all* $\zeta, \eta \in U \cap K$*,*

$$\|\nabla\Upsilon(\zeta)[\eta]\|_{C^{2,\alpha}} \leq C\|\eta\|_{C^{0,\alpha}}. \tag{3.6}$$

Finally, there exists a constant $C > 0$ *such that, writing* $u = \Xi(v)$ *and defining* $v_{\mathcal{L}} := \pi_K v + \Upsilon(\pi_K v)$ *and* $u_{\mathcal{L}} := \Xi(v_{\mathcal{L}})$*, the following key estimate holds:*

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma + u_{\mathcal{L}}) \geq C\|u - u_{\mathcal{L}}\|^2_{W^{1,2}} \quad \forall u \in \mathcal{B}_\delta(\Sigma). \tag{3.7}$$

Definition 3.2 (Integrability and strict stability). We say that a minimizer Σ of (2.1) is *integrable* if every Jacobi field $u \in K = \ker(D^2\mathcal{P}_{\Xi}(0))$ is the infinitesimal generator of a one-parameter family of critical points of $\mathcal{P}$ in $\mathcal{B}_\delta(\Sigma)$; that is, for every element $u \in K$ there exists a one-parameter family of diffeomorphisms, $(\phi_t)_{t\in(-1,1)}$, such that $\phi_0 = \mathrm{Id}$, $\frac{d}{dt}\phi_t\big|_{t=0} = u(\phi_0)\nu_\Sigma$ on $\partial\Sigma$, and

$$(\phi_t)_\sharp(\Sigma) \in \mathcal{B}_\delta(\Sigma) \quad \text{is a critical point of } \mathcal{P} \text{ in } \mathcal{B}_\delta(\Sigma) \text{ for every } t \in (-1, 1).$$

We say that a minimizer Σ of (2.1) is *strictly stable* if there exists $C > 0$, depending on Σ, such that

$$D^2\mathcal{P}_\Xi(0)[v,v] \geq C\|v\|^2_{W^{1,2}(\Sigma)} \quad \text{for every } v \in T_0\mathcal{B}_\delta(\Sigma). \tag{3.8}$$

In this case we can refine the Lyapunov–Schmidt decomposition to obtain the following lemma.

Lemma 3.3 (Lyapunov–Schmidt and integrability). *Under the assumptions of Lemma 3.1 and using the notation introduced there, if $\delta > 0$ is small enough the following holds.*

(i) *If g is analytic, then Σ is integrable if and only if the function P of Lemma 3.1 is constant. In particular, if Σ is integrable, then*

$$\mathcal{M} \cap W = \{\Sigma + \zeta + \Upsilon(\zeta) + \Xi(\zeta + \Upsilon(\zeta)) : \zeta \in U \cap K\},$$

and moreover

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) \geq C\|u - u_{\mathcal{L}}\|^2_{W^{1,2}} \quad \forall u \in \mathcal{B}_\delta(\Sigma). \tag{3.9}$$

(ii) *If Σ is strictly stable and $g \in C^3$, then $\mathcal{L} = \{\Sigma\}$ and moreover*

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) \geq C\|u\|^2_{W^{1,2}} \quad \forall u \in \mathcal{B}_\delta(\Sigma). \tag{3.10}$$

The proof of this fact is also contained in Appendix A.

3.2. Łojasiewicz inequality as a generalization of Fuglede's inequality

In this subsection we prove the main estimate of the paper.

Lemma 3.4 (Łojasiewicz meets Fuglede). *Let Σ be a smooth embedded orientable minimizer of (2.1) on a manifold (M, g).*

If g is analytic, then there exist constants $\delta(\Sigma), C(\Sigma), \gamma(\Sigma) > 0$, all depending on Σ, M, g, such that

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) \geq C(\Sigma)\alpha_{\delta(\Sigma)}(\Sigma + u)^{2+\gamma(\Sigma)} \quad \forall u \in \mathcal{B}_\delta(\Sigma). \tag{3.11}$$

If g is analytic and Σ is integrable, then we can take $\gamma \equiv 0$ in the above estimate, that is,

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) \geq C(\Sigma)\alpha_{\delta(\Sigma)}(\Sigma + u)^2 \quad \forall u \in \mathcal{B}_\delta(\Sigma). \tag{3.12}$$

If $g \in C^3$ and Σ is strictly stable, then

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) \geq C(\Sigma)|(\Sigma + u) \triangle \Sigma|^2 \quad \forall u \in \mathcal{B}_\delta(\Sigma). \tag{3.13}$$

Proof. We start by observing that

$$\inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u - \tilde{u}\|_{W^{1,2}} \geq C \inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u - \tilde{u}\|_{L^1} \geq C\alpha_\delta(\Sigma + u), \tag{3.14}$$

where the first inequality is the Poincaré inequality and the second follows from the fact that $u, \tilde{\Sigma} \in \mathcal{B}_\delta(\Sigma)$ implies that u has small $C^{1,\alpha}$ norm when reparametrized over $\tilde{\Sigma}$.

Now we prove (3.11). Let $u \in \mathcal{B}(\Sigma)$ and let $v \in U \subset T_0\mathcal{B}(\Sigma)$ be such that $u = \Xi(v)$. Let $u_{\mathcal{L}} = \Xi(v_{\mathcal{L}})$ where $v_{\mathcal{L}} \in \mathcal{L}$ is as in Lemma 3.1 and write

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) = \underbrace{\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma + u_{\mathcal{L}})}_{=:I^\perp} + \underbrace{\mathcal{P}(\Sigma + u_{\mathcal{L}}) - \mathcal{P}(\Sigma)}_{=:I_{\mathcal{L}}}, \tag{3.15}$$

For the first term we simply use (3.7), therefore to conclude we only need to estimate $I_{\mathcal{L}}$. We distinguish three cases.

Σ is strictly stable. In this case $u_{\mathcal{L}} \equiv 0$ and (3.13) follows immediately from (3.10) and (3.14).

Σ is integrable. In this case we have $I_{\mathcal{L}} = 0$ by (3.5), therefore by (3.7) and (3.15), and the fact that $v_{\mathcal{L}} + \Xi(v_{\mathcal{L}}) = \mathcal{M} \cap W$ for all $v_{\mathcal{L}} \in \mathcal{L}$, we find that

$$\mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) \geq C\|u - u_{\mathcal{L}}\|^2_{W^{1,2}} \geq C\Big(\inf_{\bar{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u - \bar{u}\|_{W^{1,2}}\Big)^2.$$

Combined with (3.14), this proves (3.12).

Σ is not integrable. We identify $v_{\mathcal{L}}$ with $\xi \in \mathbb{R}^l$, via $v_{\mathcal{L}} = \xi + \Upsilon\xi$ where ξ is the projection of v onto the kernel of $D^2\mathcal{P}_\Xi(0)$. Using the definition of P and (3.3) we get

$$\begin{aligned} I_{\mathcal{L}} &= P(u_{\mathcal{L}}) - P(0) \\ &\geq C\Big(\inf_{\{\xi_0 : \nabla P(\xi_0) = 0\}} |\xi - \xi_0|\Big)^{2+\gamma} \\ &\geq C\Big(\inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|v_{\mathcal{L}} - \tilde{v}\|_{W^{1,2}}\Big)^{2+\gamma} \\ &\geq C\Big(\inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u_{\mathcal{L}} - \tilde{u}\|_{W^{1,2}}\Big)^{2+\gamma}, \end{aligned} \tag{3.16}$$

where in the first inequality we used standard estimates for elements of the kernel $D^2\mathcal{P}_\Xi$ and (3.6), while in the last inequality we used (2.5). We combine the inequalities (3.7) and (3.16) with the simple fact that $a^{2+\gamma} + b^{2+\gamma} \geq C(\gamma)(a+b)^{2+\gamma}$ for all $a, b > 0$ to conclude that

$$\begin{aligned} \mathcal{P}(\Sigma + u) - \mathcal{P}(\Sigma) &\geq C\|u - u_{\mathcal{L}}\|^2_{W^{1,2}} + C\Big(\inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u_{\mathcal{L}} - \tilde{u}\|_{W^{1,2}}\Big)^{2+\gamma} \\ &\geq C\Big(\|u - u_{\mathcal{L}}\|_{W^{1,2}} + \inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u_{\mathcal{L}} - \tilde{u}\|_{W^{1,2}}\Big)^{2+\gamma} \\ &\geq C\Big(\inf_{\tilde{u} \in \mathcal{M} \cap \mathcal{B}_\delta(\Sigma)} \|u - \tilde{u}\|_{W^{1,2}}\Big)^{2+\gamma}, \end{aligned}$$

which, together with (3.14), concludes the proof of the proposition. ∎

3.3. *Proof of Lemma* 1.6

Let Σ, V_0 be as in Lemma 1.6. Let $\delta(\Sigma), C(\Sigma) > 0$ and $\gamma := \gamma(\Sigma) \geq 0$ be the constants given by Lemma 3.4, depending on Σ.

Given a set $E \subset \mathcal{A}_{V_0} \cap W_\delta$ of finite perimeter, where

$$W_\delta := \{F \subset M : \chi_F \in \mathrm{BV}(M),\ |F \triangle \Sigma| \leq \delta\}, \tag{3.17}$$

we can define the associated "energy" relative to Σ,

$$\mathcal{Q}(E,\gamma) := \inf\left\{\liminf_k \frac{\delta\mathcal{P}(F_k)}{\alpha_\delta(F_k)^{2+\gamma}} \,|\, \{F_k\}_k \subset \mathcal{A}_{V_0},\ \alpha_\delta(F_k) > 0,\ |F_k \triangle E| \to 0\right\}, \tag{3.18}$$

where

$$\delta\mathcal{P}(F_k) = \mathcal{P}(F_k) - \mathcal{I}(V_0)$$

is the isoperimetric defect.

With $\gamma > 0$ fixed as above, assume that there is a sequence of "bad" sets $E_k \in \mathcal{A}_{V_0} \cap W_\delta$ such that

$$\delta\mathcal{P}(E_k) \leq \frac{1}{k}\alpha_\delta(E_k)^{2+\gamma}.$$

The trivial bound of $\alpha_\delta(E_k) \leq 2V_0$ implies that $\delta\mathcal{P}(E_k) \to 0$. Note that, by compactness in the space of functions of bounded variation and the lower semicontinuity of perimeter, passing to a subsequence we can guarantee that $E_k \to \tilde{\Sigma} \in \mathcal{M} \cap W_\delta$ in the sense of sets of finite perimeter, and therefore $\alpha_\delta(E_k) \to 0$ as well. We have just shown that the (local) quantitative isoperimetric inequality is equivalent to the statement that

$$\inf_{\tilde{\Sigma} \in \mathcal{M} \cap W_\delta} \mathcal{Q}(\tilde{\Sigma},\gamma) > 0. \tag{3.19}$$

In order to prove (3.19) we are going to use the following version of the selection principle of [13].

Proposition 3.5 (Selection principle). *Assume that* $\mathcal{Q}(\Sigma,\gamma) < \infty$. *There exists a sequence of sets* $E_k \subset M$ *of finite perimeter with the following properties:*

(i) $\alpha_\delta(E_k) > 0$ *as* $k \to \infty$;

(ii) $\mathcal{Q}(E_k,\gamma) \to \inf_{\tilde{\Sigma} \in \mathcal{M} \cap W_\delta} \mathcal{Q}(\tilde{\Sigma},\gamma)$ *as* $k \to \infty$;

(iii) *there exists a smooth* $\Sigma_0 \in \mathcal{M} \cap W_\delta$ *such that*

$$\inf_{\tilde{\Sigma} \in \mathcal{M} \cap W_\delta} \mathcal{Q}(\tilde{\Sigma},\gamma) = \mathcal{Q}(\Sigma_0,\gamma)$$

and functions $u_k \in C^{1,\alpha}(\partial\Sigma_0)$ *such that* $E_k := \Sigma_0 + u_k$ *and* $\|u_k\|_{C^{1,\alpha}} \to 0$ *as* $k \to \infty$.

The proof of Proposition 3.5 is given in Appendix B and is a modification of the one in [13] with the simplification that the ambient space is compact and the complication that once again we do not know the shape of the minimizers nor the growth of the isoperimetric profile $V \mapsto \mathcal{I}(V)$. Notice that one of the reasons for this local version is the choice of δ so that $\partial\Sigma_0$ is smooth, since it is sufficiently close to Σ.

We are now ready to conclude the proof of Lemma 1.6. If $\mathcal{Q}(\Sigma, \gamma) = \infty$, then it follows from Lemma 3.4 (and the triangle inequality) that Lemma 1.6 holds. Otherwise, we can apply Proposition 3.5: since $\Sigma_0 \in W_\delta$ is a minimizer of (2.1) and $\partial\Sigma$ is assumed to be smooth, choosing δ sufficiently small depending on $\delta(\Sigma)$, ε-regularity guarantees that $E_k = \Sigma + \tilde{u}_k$ for some $\tilde{u}_k \in \mathcal{B}_{\delta(\Sigma)}(\Sigma)$.

Then by Proposition 3.5 (ii) we have

$$\inf_{\tilde{\Sigma}\in\mathcal{M}\cap W_\delta} \mathcal{Q}(\tilde{\Sigma}, \gamma) = \lim_{k\to\infty} \mathcal{Q}(E_k, \gamma) = \lim_{k\to\infty} \frac{\delta\mathcal{P}(\Sigma + \tilde{u}_k)}{\alpha_{\delta(\Sigma)}(\Sigma + \tilde{u}_k)^{2+\gamma}} \overset{(3.11)}{\geq} C(\Sigma) > 0$$

where the second equality follows from the fact that $\alpha_\delta(E_k) > 0$ for every k (i.e., Proposition 3.5 (i)). This implies (3.19) and thus concludes the proof. ∎

3.4. *Proof of Theorem* 1.4

Let $\delta(\Sigma), C(\Sigma) > 0$ and $\gamma(\Sigma) \geq 0$ be the constants of Lemma 1.6 for each $\Sigma \in \mathcal{M}$. Consider the covering $(B_{\delta(\Sigma)/2}(\Sigma))_{\Sigma\in\mathcal{M}}$ of $\mathcal{M}$ with respect to the L^1-norm, and recall that $\mathcal{M}$ is compact, so that there exists a finite subcover $(B_{\delta(\Sigma_j)/2}(\Sigma_j))_{j=1}^J$ of $\mathcal{M}$. Set

$$\delta_0 := \min_{j=1,\ldots,J} \delta(\Sigma_j), \quad \gamma_0 := \max_{j=1,\ldots,J} \gamma(\Sigma_j), \quad C_0 := \min_{j=1,\ldots,J} C(\Sigma_j). \tag{3.20}$$

We claim that there exists $\alpha_0 > 0$ such that

$$\alpha(E) < \alpha_0 \quad \text{implies} \quad \delta\mathcal{P}(E) \geq C_0\alpha(E)^{2+\gamma_0}. \tag{3.21}$$

Indeed, suppose not; then there exists a sequence E_j of sets of finite perimeter such that $\alpha(E_j) \to 0$ and

$$\delta\mathcal{P}(E_j) < C_0\alpha(E_j)^{2+\gamma_0}.$$

By a standard compactness argument, up to a subsequence $E_j \to \bar{\Sigma} \in \mathcal{M}$, so that there exists N sufficiently large satisfying

$$|E_N \triangle \bar{\Sigma}| \leq \delta_0/2.$$

Then, by the triangle inequality and the definition of $(\Sigma_j)_{j=1}^J$, we can assume without loss of generality that

$$|E_N \triangle \Sigma_1| \leq \delta(\Sigma_1),$$

and that, by our contradiction assumption and the definitions of C_0, γ_0,

$$\delta\mathcal{P}(E_N) < C(\Sigma_1)\alpha(E_N)^{2+\gamma(\Sigma_1)} \leq C(\Sigma_1)\alpha_{\Sigma_1}(E_N, \delta)^{2+\gamma(\Sigma_1)}.$$

This contradicts Lemma 1.6.

Next suppose $\alpha_0 \le \alpha(E) \le 2|M|_g$; then we recall the following fact (whose proof is a simple contradiction argument combined with the fact that M is compact):

Lemma 3.6 ([13, Lemma 3.1]). *For every $\alpha_0 > 0$ there exists $\delta_0 > 0$ such that, for any E, if $\delta\mathcal{P}(E) < \delta_0$, then $\alpha(E) < \alpha_0$.*

Then we see that in our regime $\mathcal{P}(E) - \mathcal{P}(\Sigma) \ge \delta_0$, and so

$$\delta\mathcal{P}(E) \ge \delta_0 \ge \frac{\delta_0}{(2|M|_g)^{2+\gamma_0}}\alpha(E)^{2+\gamma_0}. \tag{3.22}$$

Choosing $C_1 := \min\{C_0, \frac{\delta_0}{(2|M|_g)^{2+\gamma_0}}\}$, Theorem 1.4 follows from (3.21) and (3.22). ∎

4. Optimality of Theorem 1.4

In this section we prove Theorem 1.1, giving an example demonstrating the sharpness of Theorems 1.2 and 1.4. Finally, we discuss the possibility of extending our results to the case of non-compact, finite volume manifolds.

We begin by proving the following relatively standard result. See e.g. [41, 42, 45] for more refined statements.

Lemma 4.1. *There is $R_0 = R_0(n)$ such that for $R \ge R_0$, if we consider the product metric g_R on $\mathbb{S}^1(R) \times \mathbb{S}^{n-1}(1)$, then every isoperimetric region $\Omega \subset M$ with volume $|\Omega| = \frac{1}{2}|\mathbb{S}^1(R) \times \mathbb{S}^{n-1}(1)|$ is of the form*

$$\Omega = (t_0, t_0 + \pi R) \times \mathbb{S}^{n-1} \quad \textit{for some } t_0 \in \mathbb{R}.$$

Proof. For $n = 2$ this can easily be proven by passing to the universal cover $\mathbb{R}^2$ and using the classification of embedded constant curvature curves. We thus consider $n \ge 3$. The proof we give below holds for $3 \le n \le 7$, but can be easily modified to accommodate for a singular set in higher dimensions.

Take $R_k \to \infty$ and consider a sequence of isoperimetric regions

$$\Omega_k \subset \mathbb{S}^1(R_k) \times \mathbb{S}^{n-1}$$

with $|\Omega_k| = \frac{1}{2}|\mathbb{S}^1(R_k) \times \mathbb{S}^{n-1}| \to \infty$. By comparison with the expected minimizer, we have

$$\mathcal{P}(\Omega_k) \le 2|\mathbb{S}^{n-1}|.$$

Moreover, because the Ricci curvature of $\mathbb{S}^1(R) \times \mathbb{S}^{n-1}$ is non-negative and vanishes only in the $\mathbb{S}^1(R)$ directions, we see[3] that the reduced boundary of Ω_k has exactly one component unless Ω_k is of the form asserted in the lemma.

[3]This is a standard argument: if there are two boundary components, take a function in the second variation that is 1 on one component and $-\lambda$ on another, where λ is chosen so that the function integrates to zero. Non-negativity of the Ricci curvature implies that $|A|^2 + \mathrm{Ric}(\nu, \nu)$ vanishes identically along each component.

We now claim that the mean curvature H_k of $\partial^*\Omega_k$ remains uniformly bounded as $k \to \infty$. This follows exactly as in Lemma C.1 since all of the metrics g_R are locally isometric. Thus, using the monotonicity formula, we find that each component of $\partial^*\Omega_k$ has (extrinsic) diameter uniformly bounded, say by T_0. From this, the conclusion easily follows, because if $\partial^*\Omega_k$ has only one component, then $\partial^*\Omega_k \subset [t_k, t_k + T_0] \times \mathbb{S}^{n-1}$ for some t_k. This implies that either $|\Omega_k| = O(1)$ or $|\Omega_k| \geq \frac{3}{4}|\mathbb{S}^1(R_k) \times \mathbb{S}^{n-1}|$ for k sufficiently large. This is a contradiction. ∎

We now prove Theorem 1.1. We begin with the non-analytic case (the third assertion in the theorem) and explain how the proof can be modified for the analytic case at the end of the section. Consider a fixed $R \geq R_0$ for R_0 from the previous lemma. Consider a sequence of smooth functions $\varphi_k : \mathbb{R} \to (1/2, 2)$ such that

(1) φ_k is $2\pi R$-periodic,

(2) $\varphi_k(r) = 1$ for $|r - 1| > 1/2$ ($r \in [0, 2\pi R)$),

(3) φ_k converges in C^∞ to 1 as $k \to \infty$,

(4) $\varphi_k(1) = 1 - 1/k$ is the unique minimum of φ_k, and

(5) φ_k is strictly decreasing on $(1/2, 1)$ and strictly increasing on $(1, 3/2)$.

We claim that for k sufficiently large, the unique isoperimetric region of half the volume in the warped product metric

$$g_k = dr^2 + \varphi_k(r)^2 g_{\mathbb{S}^{n-1}}$$

on $\mathbb{S}^1(R) \times \mathbb{S}^{n-1}$ is $\Omega_k = (1, R_k) \times \mathbb{S}^{n-1}$ (or its complement) for k large enough, where $R_k = 1 + \pi R + o(1)$ as $k \to \infty$ is chosen so that $|\Omega_k| = \frac{1}{2}|\mathbb{S}^1(R) \times \mathbb{S}^{n-1}|_{g_k}$. Indeed, let $\tilde{\Omega}_k$ be any isoperimetric region for g_k, enclosing half the volume. By Lemma 4.1, and ε-regularity, for k sufficiently large, $\partial\tilde{\Omega}_k$ is a small graph over $\{t_0, t_0 + \pi R\} \times \mathbb{S}^{n-1}$ for some $t_0 \in \mathbb{R}$. Up to replacing $\tilde{\Omega}_k$ with its complement (reversing the roles of $t_0, t_0 + \pi R$), we can assume that $\varphi_k \equiv 1$ near $t_0 + \pi R$. Hence, by the argument in Lemma 4.1, the component of $\partial\tilde{\Omega}_k$ near $\{t_0 + \pi R\}$ is an exact slice $\{r_k\} \times \mathbb{S}^{n-1}$. It remains to consider the other component $\tilde{\Sigma}_k^0$ of $\partial\tilde{\Omega}_k$. By properties (4) and (5), and because Σ_k^0 is a small graph over $t_0 \times \mathbb{S}^{n-1}$, we see that[4]

$$|\tilde{\Sigma}_k^0| \geq (1 - 1/k)^2 |\mathbb{S}^{n-1}|,$$

with equality only when $\tilde{\Sigma}_k^0 = \{1\} \times \mathbb{S}^{n-1}$. Putting this together, we find $\mathscr{P}(\tilde{\Omega}_k) \geq \mathscr{P}(\Omega_k)$ with equality only when $\tilde{\Omega}_k = \Omega_k$. This proves the claim.

We fix such a sufficiently large k for the remainder of the section and write M for $(\mathbb{S}^1(R_k) \times \mathbb{S}^{n-1}, g_k)$.

[4]Indeed, the map $(1/4, 7/4) \times \mathbb{S}^{n-1} \to \{1\} \times \mathbb{S}^{n-1}$ is $(n-1)$-area non-increasing (with respect to g_k) and is strictly $(n-1)$-area decreasing on any hypersurface except $\{1\} \times \mathbb{S}^{n-1}$.

We now consider the sets $\Gamma_\delta := (1+\delta, \rho_\delta) \times \mathbb{S}^{n-1}$ where ρ_δ is chosen so that the volume of Γ_δ is equal to $\frac{1}{2}|M|$ for all $\delta > 0$ small. Let $\Gamma_0 = (1, R_k) \times \mathbb{S}^{n-1}$. Note that

$$\rho_\delta = R_k + \int_1^{1+\delta} \varphi(r)^{n-1}\, dr.$$

Thus,

$$|\Gamma_\delta \triangle \Gamma_0| = 2|\mathbb{S}^{n-1}| \int_1^{1+\delta} \varphi(r)^{n-1}\, dr \geq c\delta.$$

On the other hand, for some $C_n > 0$,

$$\mathcal{P}(\Gamma_\delta) - \mathcal{P}(\Gamma_0) = C_n\big(\varphi(1+\delta)^{n-1} - \varphi(1)^{n-1}\big)$$

(recall that $\varphi \equiv 1$ outside of a small neighborhood of 1).

Now, suppose that $\varphi(1+r) - \varphi(1)$ vanishes faster than any polynomial. Then we see that

$$\mathcal{P}(\Gamma_\delta) - \mathcal{P}(\Gamma_0) \leq C_j \delta^j$$

for any $j > 0$. This shows that it cannot be true that

$$\mathcal{P}(\Gamma_\delta) - \mathcal{P}(\Gamma_0) \geq C|\Gamma_\delta \triangle \Gamma_0|^{2+\gamma}$$

for any $C, \gamma > 0$, independent of δ.

To show that for a general analytic metric, it is necessary to allow $\gamma > 0$ (arbitrarily large) is slightly more involved. We sketch the modifications here. Choose an analytic warping function φ that is πR-periodic, with unique minimum $\varphi(1) < 1$ at 1 (and hence $1 + \pi R$), so that is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, 2)$, and so that $\varphi > 1$ outside of $(0, 2)$. Assuming $\varphi(1+x) = \varphi(1) + x^{2m} + O(x^{2m+1})$ for small x and a large positive integer m, shows that one cannot take $\gamma = 0$ (and that $\gamma > 0$ can be arbitrarily large). ∎

4.1. Manifolds with metric of finite volume

We briefly comment on the situation for (M, g) non-compact but still with finite volume. By [36], isoperimetric regions exist for all volumes $V_0 \in (0, |M|_g)$. However, it seems possible that such an (M, g) exists where some isoperimetric region has infinitely many components (compare to Lemma 2.4). While each of these components might satisfy a Łojasiewicz inequality, it seems plausible that the associated constants, γ, are unbounded, in which case one could construct a counterexample to the finite-volume analogue of Theorem 1.2 or Theorem 1.4. It would be interesting to rigorously construct such an example.

5. Proof of Theorem 1.2

We first quickly adapt some of the results in [49] to our setting and then prove Theorem 1.2.

5.1. *More Banach manifolds*

Given Σ, a minimizer of (2.1) for the metric g_0 and with volume V_0, we introduce canonical local coordinates at (Σ, g_0, V_0), whose existence in a neighborhood U is guaranteed by Lemma 2.1, defined by

$$\Xi : U \cap T_{(0,g_0,V_0)}\mathcal{B}(\Sigma, g_0, V_0) \to \mathcal{B}_\delta(\Sigma, g_0, V_0),$$

where $T_{(0,g_0,V_0)}\mathcal{B}(\Sigma, g_0, V_0) = C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma) \times \Gamma \times \mathbb{R}$, with $C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma)$ the space of functions on $\partial\Sigma$ with zero average with respect to the metric g_0. Recall that here, and throughout this section, Γ is the family of C^3 metrics on $\mathcal{M}$. In particular, we write $(\Xi^1, \Xi^2, \Xi^3) \in \mathcal{B}_\delta(\Sigma, g_0, V_0) \subset C^{2,\alpha}(\partial\Sigma) \times \Gamma \times \mathbb{R}$ for the components of Ξ. As in the previous sections, we will write

$$\mathcal{P}_\Xi(v, g, V) := \mathcal{P}^{\Xi^2(v,g,V)}(\Sigma + \Xi^1(v, g, V)).$$

Moreover, we will denote by D_u, D_{uu} the first and second derivative of $\mathcal{P}_\Xi$ with respect to its first coordinate, and by D_g the first derivative with respect to its second coordinate. A similar computation to (2.6) shows that we can identify the space of critical points for the isoperimetric problem near $(0, g_0, V_0)$ with

$$\mathcal{M}_r(\Sigma, g_0, V_0) = \{(f, g, V) \in T_{(0,g_0,V_0)}\mathcal{B}(\Sigma, g_0, V_0) : D_u\mathcal{P}_\Xi(f, g, V) = 0\}.$$

In the following, we will work with this identification.

Proposition 5.1. *There exists $0 < \delta_1 < \delta$, depending on Σ, g_0, V_0, such that $\mathcal{M}_{\delta_1}(\Sigma, g_0, V_0)$ is a separable, smooth Banach submanifold of $T_0\mathcal{B}(\Sigma, g_0, V_0)$ such that the projection*

$$\Pi : \mathcal{M}_{\delta_1}(\Sigma, g_0, V_0) \to \Gamma \times \mathbb{R} \quad \text{is a Fredholm operator of index } 0.$$

Proof. Since the statement is local we can apply [49, Theorem 1.2] with (using the notation of that paper) $X = T_0\mathcal{B}(\Sigma, g_0, V_0)$, $Y = C^{0,\alpha}(\partial\Sigma)$, $\Gamma = \Gamma \times V$ and $H(u, g, V) = \nabla_u\mathcal{P}_\Xi(u, g, V)$ the gradient induced by $D_u\mathcal{P}_\Xi$ using the scalar product of $L^2_{\mathcal{B}}$. Since $D_{uu}\mathcal{P}_\Xi(0, g_0, V_0)$ is as in (2.8), it is a self-adjoint Fredholm map of index 0. Moreover, for every non-zero $v \in \ker(D_{uu}\mathcal{P}_\Xi(0, g_0, V_0))$, let $g(s) \in \Gamma$ be the one-parameter family of metrics defined in a neighborhood of $\partial\Sigma$ by

$$g(s)(z) := (1 + sf(z))g_0(z),$$

where $f(z) = 0$ for every $z \in \partial\Sigma$. Then, following the computation in [49, Theorem 2.1] we can find f such that

$$\frac{\partial^2}{\partial s\partial t}\Big|_{t=0=s}\mathcal{P}_\Xi(tv, g(s), V(s,t)) \neq 0.$$

Therefore condition (C) of [49, Theorem 1.2] is satisfied and Proposition 5.1 is proved. ■

Finally, by a standard procedure (see [49, Theorem 2.1]), we can patch together all the local neighborhoods $\mathcal{M}(\Sigma, g_0, V_0)$ to obtain a Banach manifold containing all the critical points for (2.1) for varying metrics and values of the volume, but fixed diffeomorphism type (as we are only working with local parametrizations).

Proposition 5.2. *Let N^{n-1} and M^n be smooth compact manifolds and let Γ be the collection of C^3 Riemannian metrics on M^n. Let $[u]$ denote the class of $C^{2,\alpha}$ embeddings $u : N \to M$ up to diffeomorphism, i.e. $v \in [u]$ if and only if $v = u \circ \phi$ with $\phi : N \to N$ a smooth diffeomorphism. Let*

$$\mathcal{M}(N) := \{([u], g, V) : u(N) \text{ is the boundary of a critical point of (2.1) with respect to } g \text{ and } V\}.$$

Then $\mathcal{M}(N)$ is a smooth separable Banach manifold and the map

$$\mathcal{M}(N) \ni ([u], g, V) \mapsto \Pi([u], g, V) := (g, v) \in \Gamma \times \{V\}$$

is a Fredholm operator of index 0 and the kernel of $D\Pi([u], g, V)$ has dimension equal to the nullity of the second variation of $u(N)$ with respect to the metric g in the linear space of functions with zero average on $u(N)$.

Proof. As observed in the preliminaries, given embeddings u, v of N in M such that $\|u - v\|_{C^{2,\alpha}} \ll 1$, we can find a function $f \in C^{2,\alpha}(u(N))$ such that $v(N) = u(N) + f$, and vice versa: if $f \in C^{2,\alpha}(\partial\Sigma = u(N))$ has small norm, then we can find $v \in C^{2,\alpha}(N, M)$ with $\|u - v\|_{C^{2,\alpha}} \ll 1$ such that $\partial\Sigma + f = v(N)$.

With this identification in mind we can use Proposition 5.1 to find local charts for $\mathcal{M}$. The rest of the proposition follows exactly as in [49, Theorem 2.1]. ∎

5.2. *Proof of Theorem 1.2*

First of all notice that for every diffeomorphism type N^{n-1} we can apply Sard–Smale [44, Theorem 1.3] to $\mathcal{M}(N)$ and Π to show that for every fixed N there is an open and dense subset $\mathcal{G}_N \subset \Gamma \times \mathbb{R}$ such that every minimizer $u(N)$ of (2.1) with $(g, v) \in \mathcal{G}_N$ is non-degenerate, that is, strictly stable. Since by Lemma 2.4, every minimizer of (2.1) has finitely many connected compact components and since there are countably many diffeomorphism types for compact manifolds $(N_i)_{i\in\mathbb{N}}$, we can consider the open dense subset $\mathcal{G} := \bigcap_{i\in\mathbb{N}} \mathcal{G}_{N_i}$ of $\Gamma \times \mathbb{R}$. Its projections $\tilde{\mathcal{G}}$ and U on Γ and $\mathbb{R}$ respectively are also open and dense, since the projection is an open map.

Now let $g \in \tilde{\mathcal{G}}$ and $V \in (0, |M|_g) \cap U$. Since every smooth minimizer of (2.1) is strictly stable, there are only finitely many minimizers with volume V in the metric g. Now the result follows using (1.6) and letting $C(g)$ be the minimum of the constants in (1.6). ∎

5.3. Some further consequences

As an outcome of the previous theorem we also have the following result, valid in every dimension.

Corollary 5.3. *For an open and dense set of metrics and volumes, smooth minimizers of* (2.1) *are strictly stable and thus satisfy* (1.6) *with a constant C depending only on (M, g) and V_0. This is the generic analogue of Lemma* 1.6.

Finally, if one uses $\mathcal{B}_r(\Sigma, g_0)$ instead of $\mathcal{B}_r(\Sigma, g_0, V_0)$ and argues as in the previous two subsections, it is easy to conclude the following.

Corollary 5.4. *Let $V \in \mathbb{R}$. There exists an open and dense set of metrics, $\mathcal{G} \subset \Gamma$, such that for every $g \in \mathcal{G}$ there exists a constant $C(g, V) > 0$ such that if $\Sigma \in \mathcal{A}_V^g$ is a minimizer of* (2.1) *in (M, g), then*

$$\delta\mathcal{P}^g(E) \geq C(g, V)|E \triangle \Sigma|_g^2 \quad \textit{for every } E \in \mathcal{A}_V^g. \tag{5.1}$$

Appendix A. Proofs of Lemmas 3.1 and 3.3

In this section we prove the Lyapunov–Schmidt reduction and its version in the integrable case. For notational simplicity, we write $C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma)$ and $C^{0,\alpha}_{\mathcal{B}}(\partial\Sigma)$ for functions that integrate to zero on $\partial\Sigma$ with the obvious Hölder regularity, and similarly for $L^2_{\mathcal{B}}(\partial\Sigma)$.

Proof of Lemma 3.1. Recall that $K := \ker(D^2\mathcal{P}_\Xi(0)) \subset T_0\mathcal{B}(\Sigma)$ and define the operator

$$\mathcal{N}(\zeta) := D\mathcal{P}_\Xi(\zeta) + \pi_K\zeta : C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma) \to C^{0,\alpha}_{\mathcal{B}}(\partial\Sigma),$$

where $\pi_K, \pi_{K^\perp}$ denote the projections on $K, K^\perp$ with respect to the inner product of $L^2_{\mathcal{B}}(\partial\Sigma)$. By Lemma 2.2, we have $\mathcal{N}(0) = 0$. Furthermore,

$$D\mathcal{N}(0)[\zeta] = \frac{d}{dt}\mathcal{N}(t\zeta)\Big|_{t=0} = D(D\mathcal{P}_\Xi)(0)[\zeta] + \pi_K\zeta.$$

Note that

$$\langle \mathcal{N}(t\zeta), w\rangle = D\mathcal{P}_\Xi(t\zeta)[w] + \langle \pi_K t\zeta, w\rangle,$$

so

$$\langle D\mathcal{N}(0)[\zeta], w\rangle = D^2\mathcal{P}_\Xi(0)[\zeta, w] + \langle \pi_K\zeta, w\rangle.$$

Thus,

$$D\mathcal{N}(0)[\zeta] = -\triangle_\Sigma\,\zeta - \big(|A_\Sigma|^2 + \mathrm{Ric}(\nu_\Sigma, \nu_\Sigma) + H^2_\Sigma\big)\zeta + \jmath(\zeta) + \pi_K\zeta,$$

where $\jmath(\zeta) \in \mathbb{R}$ is unique such that $D\mathcal{N}(0)[\zeta] \in C^{0,\alpha}_{\mathcal{B}}(\partial\Sigma)$. (Recall that $\jmath(\zeta)$ appears because we are enforcing the zero-average condition.) In particular, $\ker(D\mathcal{N}(0)) = \{0\} \subset C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma)$. Note that $|\jmath(\zeta)| \leq C\|\zeta\|_{C^0}$, so by Schauder theory, $D\mathcal{N}(0) : C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma) \to C^{0,\alpha}_{\mathcal{B}}(\partial\Sigma)$ is an isomorphism. To be more explicit, since $D\mathcal{N}(0)$ is (formally) self-adjoint and injective, we know that it is surjective onto $L^2_{\mathcal{B}}(\partial\Sigma)$. Schauder theory for elliptic operators plus the estimate on the size of $\mathfrak{j}(\zeta)$ gives the isomorphism between the Hölder spaces.

We apply the inverse function theorem to $\mathcal{N}$ in this neighborhood, producing the map $\Psi := \mathcal{N}^{-1}$ which is a bijection from a neighborhood $W \subset C^{0,\alpha}_{\mathcal{B}}(\partial\Sigma)$ of the origin to a neighborhood $U \subset C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma)$ of the origin. We claim that the desired map is given by

$$\Upsilon := \pi_{K^\perp} \circ \Psi : K \cap U \to K^\perp.$$

In particular, for $\zeta \in K \cap U$ we have $\Psi(\zeta) = \zeta + \Upsilon(\zeta)$. The first conclusion of (3.1) is trivial as $\Upsilon(0) = \Upsilon(\mathcal{N}(0)) = \pi_{K^\perp}(\Psi(\mathcal{N}(0))) = 0$.

To check (3.2), we first notice that

$$\zeta = \mathcal{N}(\Psi(\zeta)) = D\mathcal{P}_\Xi(\Psi(\zeta)) + \pi_K\Psi(\zeta). \tag{A.1}$$

Applying $\pi_{K^\perp}$ to both sides we get

$$0 = \pi_{K^\perp} D\mathcal{P}_\Xi(\Psi(\zeta))$$

for $\zeta \in K \cap U$. This proves the first line of (3.2).

To prove the second line of (3.2), we compute, for any $\eta \in K$,

$$\begin{aligned}\langle \nabla P(\zeta), \eta\rangle =& D\mathcal{P}_\Xi(\zeta + \Upsilon(\zeta))[\eta + \nabla\Upsilon(\zeta)[\eta]]\\ =& D\mathcal{P}_\Xi(\zeta + \Upsilon(\zeta))[\eta],\end{aligned}$$

which implies the second claim of (3.2) (as $\eta \in K$ is arbitrary). The second equality above follows from the fact that $\nabla\Upsilon(\zeta)[\eta] \in K^\perp$ (as the image of Υ is in $K^\perp$) and then from the first line of (3.2).

The proof of (3.3) follows from the analyticity of $P : \mathbb{R}^l \to \mathbb{R}$ and the classical Łojasiewicz inequality (see [21, Corollary 4]).

To prove (3.4) we turn to (A.1). Let $\Sigma + \eta$ be an arbitrary critical point of $\mathcal{P}$ (on $\mathcal{B}(\Sigma)$) in a neighborhood of zero, and write $\eta = \Xi(v)$ for some $v \in T_0\mathcal{B}(\Sigma)$. By (2.6) we know that $D\mathcal{P}_\Xi(v) = 0$ and so we can write $v = \Psi(\zeta)$, and (A.1) reads $\zeta = \pi_K v$. This implies

$$v = \pi_K v + \pi_{K^\perp} v = \zeta + \pi_{K^\perp}\Psi(\zeta) = \zeta + \Upsilon(\zeta),$$

as desired (the condition on ∇P follows trivially from (3.2)). The other containment follows immediately from (3.2), (2.6) and the invertibility of $D\Xi$.

To show (3.5) we recall that, by the gradient version of the Łojasiewicz inequality, there exist γ_0, C_0, δ_0, depending on Σ, such that

$$|P(\xi) - P(0)|^{1-\gamma} \le C_0|\nabla P(\xi)| \quad \text{for every } \xi \in B_{\delta_0},$$

which yields $P(\xi) = P(0)$ as long as ξ is a critical point of P, as desired.

(3.6) follows from the C^1 regularity of $\Psi : C^{0,\alpha}_{\mathcal{B}}(\partial\Sigma) \to C^{2,\alpha}_{\mathcal{B}}(\partial\Sigma)$.

Finally, to prove (3.7) we notice that since Σ is a minimizer of $\mathscr{P}$ in $\mathscr{B}(\Sigma)$, there exists a constant C, depending on Σ, such that

$$D^2\mathscr{P}_\Xi(0)[\eta,\eta] \geq C\|\eta\|^2_{W^{1,2}} \quad \forall \eta \in K^\perp. \tag{A.2}$$

Then we can use a simple Taylor expansion to deduce that if as above $u = \Xi(v)$ and $u_{\mathfrak{L}} = \Xi(v_{\mathfrak{L}})$, then with the notation $v^\perp := v - v_{\mathfrak{L}} = v - \pi_K v - \Upsilon(\pi_K v) \in K^\perp$, we have

$$\begin{aligned}
&\mathscr{P}(\Sigma + u) - \mathscr{P}(\Sigma + u_{\mathfrak{L}}) = \mathscr{P}_\Xi(v) - \mathscr{P}_\Xi(v_{\mathfrak{L}}) \\
&\quad = D\mathscr{P}_\Xi(v_{\mathfrak{L}})[v^\perp] + D^2\mathscr{P}_\Xi(v_{\mathfrak{L}})[v^\perp, v^\perp] + o(\|v^\perp\|^2_{W^{1,2}}) \\
&\quad = \underbrace{\langle \nabla\mathscr{P}_\Xi(v_{\mathfrak{L}}), v^\perp\rangle_{L^2}}_{\overset{(3.2)}{=}0} + D^2\mathscr{P}_\Xi(v_{\mathfrak{L}})[v^\perp, v^\perp] + o(\|v^\perp\|^2_{W^{1,2}}) \\
&\quad = D^2\mathscr{P}_\Xi(0)[v^\perp, v^\perp] - \big(D^2\mathscr{P}_\Xi(0)[v^\perp, v^\perp] - D^2\mathscr{P}_\Xi(v_{\mathfrak{L}})[v^\perp, v^\perp]\big) \\
&\qquad + o(\|v^\perp\|^2_{W^{1,2}}) \\
&\quad \overset{(A.2)}{\geq} C\|v^\perp\|^2_{W^{1,2}} \geq C\|u - u_{\mathfrak{L}}\|^2_{W^{1,2}},
\end{aligned}$$

where the next to last inequality follows by the continuity of $D^2\mathscr{P}_\Xi$ at 0 by choosing the norm of u, and so W, small enough, together with (3.2), and the last inequality follows from (2.5). ■

Next we prove the integrable and strictly stable versions of the Lyapunov–Schmidt reduction, which are a simple modification of the argument above, essentially already contained in [2].

Proof of Lemma 3.3. The integrability condition is equivalent to

$$\forall \phi \in K\ \exists (\Psi_s)_{s\in(-1,1)} \subset C^2(\Sigma, \Sigma^\perp) : \begin{cases} \lim_{s\to 0} \Psi_s = 0, \\ D\mathscr{P}_\Xi(\Psi_s) = 0 \quad \text{for } s \in (-1,1), \\ \left.\dfrac{d}{ds}\right|_{s=0} \Psi_s = \lim_{s\to 0} \dfrac{\Psi_s}{s} = \phi. \end{cases} \tag{A.3}$$

Assume (A.3) holds, and recall the definition $P(\mu) = \mathscr{P}_\Xi(\mu + \Upsilon(\mu))$. If $P \equiv P(0)$ in a neighborhood of zero then we are done. Otherwise we can write $P(\mu) = P_p(\mu) + P_R(\mu) + P(0)$, where $P_p \not\equiv 0$, $P_p(\lambda\mu) = \lambda^p P_p(\mu/|\mu|)$ for $\lambda > 0$ and $P_R(\mu)$ is the sum of homogeneous polynomials of degrees $\geq p+1$ (here we use the analyticity of P). Note that there exists some $\phi \in K$ such that $\nabla P_p(\phi) \neq 0$. Let Ψ_s be the one-parameter family of critical points that is generated by ϕ (as in (A.3)).

As Ψ_s is a critical point, Lemma 3.1 allows us to write $\Psi_s = \phi_s + \Upsilon(\phi_s)$ where $\phi_s \in K$ and $\phi_s/s \to \phi$ as $s \downarrow 0$. Computing

$$0 = D\mathscr{P}_\Xi(\Psi_s) = \nabla P(\phi_s) = \nabla P_p(\phi_s) + \nabla P_R(\phi_s) = s^{p-1}\nabla P_p\left(\frac{\phi}{|\phi|}\right) + o(s^{p-1}),$$

we divide the above by s^{p-1} and let $s \downarrow 0$ to obtain a contradiction to $\nabla P_p(\phi) \neq 0$.

In the other direction, assume that $P \equiv P(0)$ in a neighborhood of 0. This implies that $\nabla P \equiv 0$ in a (perhaps slightly smaller) neighborhood of 0. Therefore, for any $\mu \in K$, letting $\Psi_s = s\mu + \Upsilon(s\mu)$ and recalling (3.6) establishes (A.3).

Next, since we have proven that P is constant on $\mathcal{L}$, (3.9) follows immediately from (3.7) and the fact that $u_{\mathcal{L}} \in \mathcal{L}$.

Finally, if Σ is strictly stable, then $K = \{0\}$, which immediately implies $\mathcal{L} = \{0\}$ and so $u_{\mathcal{L}} = 0$, which gives (3.10). ∎

Appendix B. Proof of Proposition 3.5

The proof of Proposition 3.5 is obtained by combining results from [13, 33, 50], and we will recall the fundamental steps over the following subsections, leaving many standard details to the reader. The basic idea is that the E_k will be minimizers to a penalized version of energy in (3.18), where the penalization guarantees that we recover $\inf_{\tilde{\Sigma} \in \mathcal{M} \cap W_\delta} \mathcal{Q}(\tilde{\Sigma}, \gamma)$ in the limit.

The existence of the E_k and the fact that they satisfy properties (i) and (ii) of Proposition 3.5 is covered in Proposition B.2. The smooth convergence of property (iii) of Proposition 3.5 is proven in Lemma B.6.

For simplicity of notation, in this section we will denote

$$\alpha(E) := \alpha_\delta(E), \quad \mathcal{A} := \mathcal{A}_{V_0}, \quad W := W_\delta.$$

We emphasize that we are *assuming* that $\mathcal{Q}(\Sigma, \gamma) < \infty$ in this section.

Before starting the proof we observe the following simple facts.

Lemma B.1 (Properties of $\mathcal{Q}(-, \gamma)$). *The energy $\mathcal{Q}(-, \gamma)$ satisfies the following properties.*

- *If $\alpha_\delta(E) > 0$, $E \subset \mathcal{A}$, then $\mathcal{Q}(E, \gamma) = \delta \mathcal{P}(E)/\alpha(E)^{2+\gamma}$.*
- *If $E_k \subset \mathcal{A}$ and $E_k \xrightarrow{L^1} E$, then $\mathcal{Q}(E, \gamma) \leq \liminf_k \mathcal{Q}(E_k, \gamma)$.* (This follows from the lower semicontinuity of perimeter and a diagonal argument.)

B.1. The penalized minimization problem

By the definition of $\mathcal{Q}(\tilde{\Sigma}, \gamma)$ and a diagonal argument, there exists $\{W_j\}_j \subset \mathcal{A}$ such that

$$\left| \mathcal{Q}(W_j, \gamma) - \inf_{\tilde{\Sigma} \in \mathcal{M} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma) \right| < \frac{1}{j}, \quad 0 < \alpha(W_j) < 1, \quad \alpha(W_j) \to 0. \tag{B.1}$$

We want to "regularize" these W_j and so we introduce the penalized functionals

$$\mathcal{Q}_j(E, \gamma) := \mathcal{Q}(E, \gamma) + \left(\frac{\alpha(E)}{\alpha(W_j)} - 1 \right)^2, \tag{B.2}$$

where $(W_j)_j$ is as in (B.1). The content of the following proposition is that minimizers to $\mathcal{Q}_j(-, \gamma)$ exist and are also an approximating sequence for $\inf_{\tilde{\Sigma} \in \mathcal{M} \cap W_\delta} \mathcal{Q}(\tilde{\Sigma}, \gamma)$ (i.e. they satisfy (B.1)).

Proposition B.2 (Minimizers of $\mathcal{Q}_j$). *There exist sets $\{E_j\}_j \subset \mathcal{A}$ of finite perimeter such that for each j, $\mathcal{Q}_j(E_j, \gamma) \leq \mathcal{Q}_j(S, \gamma)$ for all other sets $S \in \mathcal{A}$. Furthermore,*

$$\alpha(E_j) > 0, \quad \alpha(E_j) \to 0, \quad \left| \mathcal{Q}(E_j, \gamma) - \inf_{\tilde{\Sigma} \in \mathcal{M} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma) \right| \to 0.$$

Finally, perhaps after passing to a subsequence, $E_j \xrightarrow{L^1} \Sigma_0$ where $\Sigma_0 \in \mathcal{M} \cap W$ is smooth and $\mathcal{Q}(\Sigma_0, \gamma) = \inf_{\tilde{\Sigma} \in \mathcal{M} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma)$.

Proof. The existence of a minimizer follows from BV-compactness and the lower semi-continuity of the energy $\mathcal{Q}_j(-, \gamma)$ (see Lemma B.1, second bullet point).

If $\alpha(E_j) = 0$ for any $j > 1$, then $E_j \in \mathcal{M} \cap W$ and we have

$$\begin{aligned} \inf_{\tilde{\Sigma} \in \mathcal{M} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma) &\leq \mathcal{Q}(E_j, \gamma) = \mathcal{Q}_j(E_j, \gamma) - 1 \leq \mathcal{Q}_j(W_j, \gamma) - 1 \\ &= \mathcal{Q}(W_j, \gamma) - 1 \\ &\leq \inf_{\tilde{\Sigma} \in \mathcal{C} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma) + \frac{1}{j} - 1, \end{aligned}$$

which is a contradiction as long as $j > 1$.

A similar argument shows that $\alpha(E_j) \to 0$. Indeed for any subsequence E_{j_k} we have

$$\begin{aligned} \lim_k \left(\frac{\alpha(E_{j_k})}{\alpha(W_{j_k})} - 1 \right)^2 &\leq \lim_k \mathcal{Q}_{j_k}(E_{j_k}, \gamma) \leq \lim_k \mathcal{Q}_{j_k}(W_{j_k}, \gamma) = \lim_k \mathcal{Q}(W_{j_k}, \gamma) \\ &= \inf_{\tilde{\Sigma} \in \mathcal{C} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma) < \infty. \end{aligned}$$

Since $\alpha(W_j) \to 0$ it follows that $\alpha(E_j) \to 0$.

Of course, we can similarly argue that

$$\mathcal{Q}(E_j, \gamma) \leq \mathcal{Q}_j(E_j, \gamma) \leq \mathcal{Q}_j(W_j, \gamma) = \mathcal{Q}(W_j, \gamma) \leq \inf_{\tilde{\Sigma} \in \mathcal{C} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma) + 1 < \infty,$$

where we emphasize that we have assumed that $\mathcal{Q}(\Sigma, \gamma) < \infty$.

This implies that $\delta \mathcal{P}(E_j) \to 0$ so $E_j \xrightarrow{L^1} \Sigma_0$ for some $\Sigma_0 \in \mathcal{M} \cap W$. Note that $\partial \Sigma_0$ is automatically smooth by the definition of W and the assumption that $\partial \Sigma$ is smooth.

We have proven that the E_j (perhaps after passing to a subsequence) satisfy the requirements of an approximating sequence in the definition of $\mathcal{Q}(\Sigma_0, \gamma)$. Therefore,

$$\begin{aligned} \mathcal{Q}(\Sigma_0, \gamma) &\leq \lim_j \mathcal{Q}(E_j, \gamma) \leq \lim_j \mathcal{Q}_j(E_j, \gamma) \\ &\leq \lim_j \mathcal{Q}_j(W_j, \gamma) = \lim_j \mathcal{Q}(W_j, \gamma) \\ &= \inf_{\tilde{\Sigma} \in \mathcal{C} \cap W} \mathcal{Q}(\tilde{\Sigma}, \gamma). \end{aligned}$$

This implies that

$$\lim_j \mathcal{Q}(E_j,\gamma) = \inf_{\tilde{\Sigma}\in\mathcal{C}} \mathcal{Q}(\tilde{\Sigma},\gamma) = \mathcal{Q}(\Sigma_0,\gamma),$$

and finally that

$$\lim_{j\to\infty} \frac{\alpha(E_j)}{\alpha(W_j)} = 1, \tag{B.3}$$

completing the proof. ∎

B.2. *Almost-minimizers and smoothness for the E_j*

In this subsection we will prove that the E_j satisfy the hypothesis of Proposition 3.5. Note that we only have to verify the smooth convergence property (property (iii)), as the first two properties are guaranteed by Proposition B.2.

We will prove this smooth convergence by first showing that the E_j's are *almost-minimizers* for perimeter with uniform constants. Then smooth convergence will follow from regularity theory for almost-minimizers and a standard argument in the calculus of variations (see the proof of Lemma B.6 below for more details).

Our first lemma is that E_j minimizes perimeter in the class $\mathcal{A}$ up to an error which is proportional to the area of the symmetric difference between E_j and the competitor. It is important to note that the constant of proportionality is uniform over the index.

Lemma B.3. *There exist $\Lambda > 0$ and $j_0 \in \mathbb{N}$ such that for all $F \in \mathcal{A}$ and all $j \geq j_0$ we have*

$$\mathcal{P}(E_j) \leq \mathcal{P}(F) + \Lambda|E_j \triangle F|.$$

Proof. Without loss of generality we can assume that $\mathcal{P}(F) \leq \mathcal{P}(E_j)$. We also let j_0 be large enough such that

$$\begin{aligned} \alpha(E_j) &\leq 1/2, \\ |\alpha(E_j) - \alpha(W_j)| &\leq \alpha(W_j)/2, \\ \mathcal{Q}(E_j,\gamma) &\leq \inf_{\tilde{\Sigma}\in\mathcal{M}\cap W} \mathcal{Q}(\tilde{\Sigma},\gamma) + 1. \end{aligned} \tag{B.4}$$

Such a j_0 exists by Proposition B.2 and (B.3). Next we distinguish two cases.

Case 1: $\alpha(E_j)^{2+\gamma} \leq |E_j \triangle F|$. Since $\mathcal{Q}(E_j,\gamma) \leq \inf_{\tilde{\Sigma}\in\mathcal{M}\cap W} \mathcal{Q}(\tilde{\Sigma},\gamma) + 1$ and $\alpha(E_j) > 0$, we get

$$\begin{aligned} \mathcal{P}(E_j) &\leq \mathcal{I}(V_0) + \alpha(E_j)^{2+\gamma}\Big(\inf_{\tilde{\Sigma}\in\mathcal{M}\cap W} \mathcal{Q}(\tilde{\Sigma},\gamma) + 1\Big) \\ &\leq \mathcal{P}(F) + |E_j \triangle F|\Big(\inf_{\tilde{\Sigma}\in\mathcal{M}\cap W} \mathcal{Q}(\tilde{\Sigma},\gamma) + 1\Big) \\ &\leq \mathcal{P}(F) + \Lambda|E_j \triangle F|, \end{aligned} \tag{B.5}$$

completing the proof in this case.

Case 2: $|E_j \triangle F| < \alpha(E_j)^{2+\gamma}$. We know the inequality $\mathcal{Q}_j(E_j,\gamma) \le \mathcal{Q}_j(F,\gamma)$, which implies that

$$\mathcal{P}(E_j) \le \mathcal{P}(F) + \underbrace{\delta\mathcal{P}(F)\Big(\frac{\alpha(E_j)^{2+\gamma}}{\alpha(F)^{2+\gamma}} - 1\Big)}_{I} \\ + \underbrace{\alpha(E_j)^{2+\gamma}\bigg(\Big(\frac{\alpha(F)}{\alpha(W_j)} - 1\Big)^2 - \Big(\frac{\alpha(E_j)}{\alpha(W_j)} - 1\Big)^2\bigg)}_{II}. \tag{B.6}$$

We can estimate II in (B.6) as follows:

$$II \le \Big(\frac{\alpha(E_j)}{\alpha(W_j)}\Big)^2 (\alpha(F) + \alpha(E_j) - 2\alpha(W_j))(\alpha(F) - \alpha(E_j)) \\ \le C|\alpha(F) - \alpha(E_j)| \le C|F \triangle E_j|, \tag{B.7}$$

where the second inequality follows from the estimates in (B.4) and the last inequality follows from the triangle inequality. In order to estimate I we observe that by assumption $\alpha(E_j) \le 1/2$, so

$$|E_j \triangle F| \le \alpha(E_j)^{2+\gamma} \le \tfrac{1}{2}\alpha(E_j) \Rightarrow \tfrac{1}{2}\alpha(E_j) \le \alpha(F) \le 2\alpha(E_j). \tag{B.8}$$

It follows that

$$I \le \delta\mathcal{P}(F)\frac{\alpha(E_j)^{2+\gamma} - \alpha(F)^{2+\gamma}}{\alpha(F)^{2+\gamma}} \le C\mathcal{Q}(E_j,\gamma)(\alpha(E_j)^{2+\gamma} - \alpha(F)^{2+\gamma}) \\ \le C\Big(\inf_{\tilde{\Sigma}\in\mathcal{M}} \mathcal{Q}(\tilde{\Sigma},\gamma) + 1\Big)(\alpha(E_j) - \alpha(F)) \le C|F \triangle E_j|, \tag{B.9}$$

where the second inequality follows from (B.8) and the fact that $\mathcal{P}(F) \le \mathcal{P}(E_j)$, while the third inequality comes from (B.4) and the estimate $x^r - y^r \le C(x-y)$ for $0 \le y \le x \le 1$.

Putting (B.9) and (B.7) together with (B.6) finishes the proof of Case 2 and thus concludes the proof of the lemma. ■

The following result follows by a straightforward modification of the main result in [30]. It is important to note that the constants are independent of the index j.

Proposition B.4. *There exist $C = C(M,\Sigma) > 0$, $r_0 = r_0(M,\Sigma) > 0$, $\alpha = \alpha(M) \in (0,1)$ and a $j_0 \in \mathbb{N}$ (which again depends on Σ) such that for all $j \ge j_0$, all $x \in U$ and all $r < r_0$, if $\chi_F \in \mathrm{BV}(M)$ with $\chi_F = \chi_{E_j}$ on $M \setminus B(x,r)$, then*

$$\mathcal{P}(E_j; B(x,r)) \le \mathcal{P}(F; B(x,r)) + Cr^n. \tag{B.10}$$

At this point, the desired estimates for E_j would follow from [46], except for the fact that [46] only considers almost-minimizers in Euclidean space, and not for a Riemannian metric. Although this should not be a serious issue, we describe an argument below that will instead reduce the problem to Almgren's theory of $(\mathcal{F}, ct, \delta)$ *almost-minimizers* (as in [50]).

Note that Proposition B.4 immediately implies that E_j obeys an almost-monotonicity formula and all tangent cones are area-minimizing (cf. [17, Proposition 2.1]).

We now recall that the area functional of a Riemannian metric can be regarded as a parametric elliptic integrand (cf. [50, §1]) and a current T is said to be $(\mathcal{F}, ct, \delta)$ *almost-minimizing* if for all points x,

$$\mathcal{F}_x(V) \leq (1 + cr)\mathcal{F}_x(V')$$

whenever $r < \delta$, V is a piece of T inside of $B(x, r)$ and $\partial V' = \partial V$. Here $\mathcal{F}_x$ represents the parametric elliptic functional "frozen" at x (in this situation, this just means that we are using normal coordinates at x and measuring the area of V, V' with respect to the Euclidean metric in these coordinates).

Proposition B.5. *There are $\delta = \delta(M, \Sigma) > 0$, $c = c(M, \Sigma) > 0$, and $j_0 = j_0(M, \Sigma)$ such that for all $j \geq j_0$, the reduced boundary $\partial^* E_j$ is an $(\mathcal{F}, cr, \delta)$ almost-minimizer.*

Proof. We begin by observing that (thanks to the almost-monotonicity formula and the above observation about the tangent cones) there is $\delta > 0$ independent of x and j sufficiently large such that for $r < \delta$ and $x \in \partial^* E_j$, we have $P(E_j; B(x,r)) \geq Cr^{n-1}$. In fact, if $d(x, \partial^* E_j) < r/2$ then the same inequality holds, after shrinking C.

Thus, if $\chi_F \in \mathrm{BV}(M)$ has $\chi_F = \chi_{E_j}$ on $M \setminus B(x,r)$ then this lower bound, combined with Proposition B.4, yields

$$\mathcal{P}(F; B(x,r)) \geq Cr^{n-1}$$

after again shrinking C (still assuming that $d(x, \partial^* E_j) < r/2$).

We may thus apply Proposition B.4 again to conclude that

$$\mathcal{P}(E_j; B(x,r)) \leq (1 + cr)\mathcal{P}(E_j; B(x,r))$$

for $r < \delta$ and $c > 0$ appropriately chosen. Because freezing the coefficients introduces an $O(r^2)$ error, this yields the asserted claim assuming that $d(x, \partial^* E_j) \leq r/2$. Finally, if $d(x, \partial^* E_j) \geq r/2$, then we can apply the above argument with r replaced by $2r$. This completes the proof. ∎

Proposition 3.5 will now follow from standard facts about the regularity of $(\mathcal{F}, \varepsilon, \delta)$-minimizers. We write the formal statement here and collect the salient facts in the proof.

Lemma B.6. *The E_k satisfy condition* (iii) *of Proposition 3.5, that is, if $\Sigma_0 \in \mathcal{M} \cap W$ is as in Proposition B.2, then there are functions $u_k \in C^{1,\alpha}(\partial \Sigma_0)$ such that $E_k := \Sigma_0 + u_k$ and $\|u_k\|_{C^{1,\alpha}} \to 0$ as $k \to \infty$.*

Proof. For almost-minimizers, convergence in the BV sense implies convergence in the Hausdorff sense. Smooth convergence then follows from ε-regularity for $(\mathcal{F}, ct, \delta)$ almost-minimizers, exactly as in [50, p. 207]. ∎

Appendix C. Boundedness of mean curvature for isoperimetric regions

In this appendix we recall the uniform boundedness of the mean curvature of isoperimetric regions whose volume is not very small (or close to $|M|_g$).

Lemma C.1 ([9, 38]). *For $2 \le n \le 7$, fix $\delta > 0$ and (M^n, g), a closed Riemannian manifold with C^3 metric. There is $C = C(M, g, \delta) < \infty$ such that if $\Omega \in \mathcal{A}_V^g$ is an isoperimetric region with $V = |\Omega|_g \in (\delta, |M|_g - \delta)$, then the mean curvature H of $\partial\Omega$ satisfies $|H| \le C$.*

Proof. Fix (M, g) and $\delta > 0$ and assume for contradiction that there are isoperimetric regions $\Omega_j \subset (M, g)$ with $|\Omega_j|_g \in (\delta, |M|_g - \delta)$ with mean curvature H_j satisfying $\lambda_j := |H_j| \to \infty$.

Choosing $x_j \in \partial\Omega_j$, we can rescale by λ_j around x_j to find isoperimetric regions $\tilde{\Omega}_j$ in $(\tilde{M}_j, \tilde{g}_j)$. Furthermore, $(\tilde{M}_j, \tilde{g}_j)$ converges in C^3_{loc} to $\mathbb{R}^n$ equipped with the flat metric. Passing to a subsequence, $\tilde{\Omega}_j$ converges in the local Hausdorff sense to a locally isoperimetric region $\tilde{\Omega}$ in $\mathbb{R}^n$; moreover $\partial\tilde{\Omega}_j$ converges in $C^{2,\alpha}_{\mathrm{loc}}$ to $\partial\tilde{\Omega}$. Hence, $\partial\tilde{\Omega}$ has constant mean curvature ± 1. On the other hand, $\partial\tilde{\Omega}$ is stable, in the sense that

$$\int_{\partial\tilde{\Omega}} |A|^2 \varphi^2 \, d\mathcal{H}^{n-1} \le \int_{\partial\tilde{\Omega}} |\nabla\varphi|^2 \, d\mathcal{H}^{n-1}$$

for any $\varphi \in C^1_c(\partial\tilde{\Omega})$ with $\int_{\partial\tilde{\Omega}} \varphi \, d\mathcal{H}^{n-1} = 0$. Because $|H| = 1$, we find $|A|^2 \ge 1/n$, so

$$\int_{\partial\tilde{\Omega}} \varphi^2 \, d\mathcal{H}^{n-1} \le n \int_{\partial\tilde{\Omega}} |\nabla\varphi|^2 \, d\mathcal{H}^{n-1}$$

for any $\varphi \in C^1_c(\partial\tilde{\Omega})$ with $\int_{\partial\tilde{\Omega}} \varphi \, d\mathcal{H}^{n-1} = 0$.

Suppose that $\partial\tilde{\Omega}$ were compact for all choices of $x_j \in \partial\Omega_j$. Then Ω_j would be close to a union of an increasing number of regions close to coordinate spheres. Using this, we would conclude that $\mathcal{P}_g(\Omega_j) \to \infty$, a contradiction. Therefore, we will assume that $\partial\tilde{\Omega}$ is non-compact.

Standard arguments (cf. [25]) imply that there is $R > 0$ sufficiently large such that

$$\int_{\partial\tilde{\Omega}} \varphi^2 \, d\mathcal{H}^{n-1} \le n \int_{\partial\tilde{\Omega}} |\nabla\varphi|^2 \, d\mathcal{H}^{n-1}$$

for any $\varphi \in C^1_c(\partial\tilde{\Omega} \setminus B_R)$ (i.e., $\partial\tilde{\Omega}$ is strongly stable outside of a compact set).

Taking $\varphi = \psi^{(n-1)/2}$ for $\psi \in C^1_c(\partial\tilde{\Omega} \setminus B_R)$ and using Hölder's inequality, we find

$$\int_{\partial\tilde{\Omega}} \psi^{n-1} \, d\mathcal{H}^{n-1} \le C \int_{\partial\tilde{\Omega}} |\nabla\psi|^{n-1} \, d\mathcal{H}^{n-1}.$$

Choose an ambient radial function ψ that is 0 for $|x| < R$, increases to 1 for $|x| \in [R + 1, \rho]$, and then cuts off to 0 for $|x| > 2\rho$. We can arrange that $|\nabla\psi| \le C\rho^{-1}$. Thus, we find that

$$\mathcal{H}^{n-1}(\partial\tilde{\Omega} \cap (B_\rho \setminus B_R)) \le C(1 + \rho^{1-n}\mathcal{H}^{n-1}(\partial\tilde{\Omega} \cap B_{2\rho})).$$

Letting $\rho \to \infty$, we deduce a contradiction if we can show that $\mathcal{H}^{n-1}(\partial\tilde{\Omega} \setminus B_R) = \infty$ and $\mathcal{H}^{n-1}(\partial\tilde{\Omega} \cap B_\rho) \le C\rho^{n-1}$. The first fact follows from the monotonicity formula (since $|H| = 1$) applied to small balls. The second follows since $\tilde{\Omega}$ is locally isoperimetric in the sense that $\tilde{\Omega}'$ with $\tilde{\Omega} \triangle \tilde{\Omega}' \Subset B_R$ and $|\tilde{\Omega} \cap B_R| = |\tilde{\Omega}' \cap B_R|$ has $\mathcal{P}(\partial\tilde{\Omega}'; B_R) \ge \mathcal{P}(\partial\tilde{\Omega}; B_R)$, allowing us to compare $\tilde{\Omega}$ to $(\tilde{\Omega} \setminus B_\rho) \cup B_{r(\rho)}$, where $r(\rho) \le \rho$ is chosen to preserve the enclosed volume. This is a contradiction, completing the proof. ■

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