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(with an appendix by Uri Bader, Yair Hartman, and Mehrdad Kalantar)

## Stationary $C^*$ -dynamical systems

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**Abstract.** We introduce the notion of stationary actions in the context of  $C^*$ -algebras. We develop the basics of the theory, and provide applications to several ergodic-theoretical and operator-algebraic rigidity problems.

**Keywords.** Stationary actions,  $C^*$ -algebras, Poisson boundary

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### 1. Introduction

Stationary actions provide a framework that includes all measure preserving actions, as well as their opposite systems, boundary actions. This framework is general enough to not suffer from an existential problem as in the case of invariant measures for nonamenable groups, and yet enjoy having enough meaningful structural properties.

In the setup of unique stationary dynamical systems, random walk theory forms connection between topological and measurable dynamics. Study of these systems, specially in the noncommutative setting, is one of the main objectives of this work. In the case of nonamenable group actions, unique stationarity is sometimes a more suitable replacement for the notion of unique ergodicity, even in the presence of invariant measures, as our results in this paper show.

We introduce the notion of stationary  $C^*$ -dynamical systems, in order to develop new tools in the study of operator algebras associated to nonamenable groups. This includes traceless  $C^*$ - and von Neumann algebras, for which many of powerful techniques from the finite-type theories are not applicable.

Let  $\mu \in \text{Prob}(\Gamma)$  be a probability measure on a countable discrete group  $\Gamma$ ,  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $\Gamma \curvearrowright \mathcal{A}$  by  $*$ -automorphisms.

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**Definition.** A state  $\tau$  on  $\mathcal{A}$  is said to be  $\mu$ -stationary if  $\sum_{g \in \Gamma} \mu(g)g\tau = \tau$ .

We are particularly interested in inner actions. The underlying philosophy here is to view a  $C^*$ -algebra not only as a single structure, but rather as a noncommutative dynamical system via the action of its unitary group by inner automorphisms. Nontriviality of this action is an exclusive feature of noncommutative  $C^*$ -algebras. From this point of view, a trace is a noncommutative invariant probability measure, and admitting a unique trace is the noncommutative counterpart of unique ergodicity. Thus, stationary states are generalizations of traces, which, in contrast, always exist. We will see that despite this level of generality, they still reveal many meaningful structural properties, and in fact, provide a context in which techniques from measurable ergodic theory, in particular random walks, can be applied to study  $C^*$ -algebras associated to discrete groups. This is not entirely in line with the conventional expectation that topological dynamics interact with  $C^*$ -algebra theory, and measurable actions with von-Neumann-algebra theory. Indeed, we introduce new techniques to use measurable boundaries in certain  $C^*$ -algebraic rigidity problems, and we also obtain von Neumann algebraic relative superrigidity results by using topological boundaries.

Another new perspective that this theory provides is an appropriate notion of minimality for noncommutative actions. The current conventional notion of minimality of an action  $\Gamma \curvearrowright \mathcal{A}$  of a group  $\Gamma$  on a  $C^*$ -algebra  $\mathcal{A}$  is that  $\mathcal{A}$  does not contain any nontrivial closed proper  $\Gamma$ -invariant ideals. Although this definition allows generalizations to the noncommutative setting of some results on commutative minimal actions, it fails to be an appropriate counterpart of minimality in many cases. For instance, with this definition, the trivial action of any group on any simple  $C^*$ -algebra is minimal, and consequently a “factor” of a minimal action may not be minimal. Alternatively, for an action  $\Gamma \curvearrowright X$  on a compact space  $X$ , one observes that the action is minimal if and only if every  $\mu$ -stationary probability  $\nu$  on  $X$  has full support, for any generating probability  $\mu$  on  $\Gamma$ . The similar property for an action  $\Gamma \curvearrowright \mathcal{A}$  appears to be a more natural notion of minimality. For instance, the trivial action on a nontrivial  $C^*$ -algebra is never minimal in this sense, and “factors” of minimal actions are minimal. In a recent work [3] which was completed after the first version of this work was published, Amrutam and the second-named author used this perspective and the results of this paper to prove simplicity of all intermediate  $C^*$ -subalgebras of crossed products of minimal  $C^*$ -simple actions.

Topological and measurable boundary actions were introduced by Furstenberg in his seminal work [25, 26] in the context of rigidity of Lie groups. These notions have recently turned out to be particularly relevant in questions of uniqueness of the canonical trace. The latter is closely related to several rigidity problems in ergodic theory and operator algebras [16, 20, 41, 49].

For instance, the problem of classifying the groups with the unique trace property, which had been open for almost 40 years, was finally settled by Breuillard, Kennedy, Ozawa, and the second named author in [16], where a characterization of this property was proven in terms of existence of faithful topological boundary actions. The original proof in [16] used the notion of injective envelopes, but a simpler proof was provided soon

after by Haagerup [30, Theorem 3.3]. In fact, Haagerup's proof has been very inspiring for our work, as it clearly shows why boundary actions are effective in this type of problems. A very closely related problem is the classification of  $C^*$ -simple groups. A group  $\Gamma$  is called  $C^*$ -simple if  $C_\lambda^*(\Gamma)$ , the reduced  $C^*$ -algebra of  $\Gamma$ , is simple, meaning that it has no nontrivial proper closed ideals. Similarly, after numerous partial results from many works over the span of four decades the first characterization of  $C^*$ -simplicity was proven by Kennedy and the second named author [41] in terms of existence of free topological boundary actions. Therefore, in particular, the above results combined imply  $C^*$ -simplicity is stronger than the unique trace property. Finally, Le Boudec [44] proved the existence of groups with faithful topological boundary actions, but no free such actions, hence completely settled the question of whether  $C^*$ -simplicity and the unique trace property are equivalent.

As an application of our theory, we prove a new characterization of  $C^*$ -simplicity in terms of unique stationarity of the canonical trace.

**Theorem** (Theorem 5.2). *A countable discrete group  $\Gamma$  is  $C^*$ -simple if and only if there is  $\mu \in \text{Prob}(\Gamma)$  such that the canonical trace is the unique  $\mu$ -stationary state on  $C_\lambda^*(\Gamma)$ .*

In particular, this result shows that  $C^*$ -simplicity is also a uniqueness property of the canonical trace. This is indeed quite natural with our point of view that  $C_\lambda^*(\Gamma)$  is rather a  $\Gamma$ - $C^*$ -algebra via the inner action: every ideal is invariant, and therefore simplicity is a noncommutative minimality problem. Now considering the commutative picture, since stationary measures always exist, existence of a unique stationary probability with full support implies minimality.

Also, our above characterization of  $C^*$ -simplicity provides an intrinsic dynamical explanation for the difference between the unique trace property and  $C^*$ -simplicity: while the former corresponds to unique ergodicity, the latter corresponds to unique stationarity. We may even give a manifestation of this in the commutative setting: every group  $\Gamma$  admits an action on a compact metric space such that the difference between unique ergodicity and unique stationarity of the action translates into the difference between the unique trace property and  $C^*$ -simplicity, as follows.

Let  $\text{Sub}_a(\Gamma)$  denote the set of all amenable subgroups of  $\Gamma$ , which is a compact space on which  $\Gamma$  acts by conjugations. Bader, Duchesne and Lecureux [7] proved that  $\Gamma$  has the unique trace property if and only if  $\Gamma \curvearrowright \text{Sub}_a(\Gamma)$  is uniquely ergodic. Here we prove:

**Theorem** (Corollary 5.7). *A countable discrete group  $\Gamma$  is  $C^*$ -simple if and only if there is  $\mu \in \text{Prob}(\Gamma)$  such that the action  $\Gamma \curvearrowright \text{Sub}_a(\Gamma)$  is uniquely  $\mu$ -stationary.*

We would like to highlight the interesting, and somehow curious fact that  $C^*$ -simplicity, a purely  $C^*$ -algebraic property, would single out certain random walks (or measures) on  $\Gamma$  that reveal its  $C^*$ -simplicity. Moreover, it turns out that these measures possess significant ergodic-theoretical properties, connecting  $C^*$ -simplicity to random walks on groups. Thus, it suggests considering  $C^*$ -simplicity of  $\Gamma$  as rather a property of the measure(s)  $\mu \in \text{Prob}(\Gamma)$  in the above theorems.

**Definition.** We say that  $\mu \in \text{Prob}(\Gamma)$  is a  $C^*$ -simple measure if the canonical trace  $\tau_0$  is the unique  $\mu$ -stationary state on  $C_\lambda^*(\Gamma)$ .

For instance, we prove:

**Theorem** (Theorem 5.4). *Suppose  $\mu \in \text{Prob}(\Gamma)$  is  $C^*$ -simple. Then any measurable  $\mu$ -stationary action with almost surely amenable stabilizers is essentially free.*

*In particular, the action of  $\Gamma$  on the Poisson boundary of  $\mu$  is essentially free.*

Hence, we are naturally led to the problem of finding  $C^*$ -simple measures. Our proof of the existence of  $C^*$ -simple measures on  $C^*$ -simple groups does not reveal concrete measures. In fact, part of our construction of the  $C^*$ -simple measure follows a similar construction in Kaimanovich–Vershik’s proof of Furstenberg’s conjecture on the existence of measures on amenable groups with trivial Poisson boundary. The following result, on the other hand, allows verifying  $C^*$ -simplicity of many concrete measures.

**Theorem** (Theorem 4.12). *Let  $\mu \in \text{Prob}(\Gamma)$  and suppose  $\Gamma$  admits an essentially free  $\mu$ -boundary which has a compact model that is uniquely  $\mu$ -stationary. Then  $\mu$  is a  $C^*$ -simple measure.*

This result also highlights another advantage of our approach in using measurable boundaries in the above problems. In contrast to the topological case, measurable boundaries have been studied extensively, and there are several powerful methods due to the fundamental work of Kaimanovich [38], for realizing these boundaries. These methods have resulted in many deep realization results (e.g. [18, 33, 38, 39, 45]). In fact, in many examples, a concrete unique stationary model of a boundary is provided, and under some regularity assumptions on the measure the corresponding stationary measure is the Poisson measure.

To further highlight the contrast to the topological case, we remark that in the case of nonamenable discrete groups, the Furstenberg boundary is always a nonmetrizable extremally disconnected space, not concretely identifiable in any known example. We take advantage of the concreteness of the topological models of measurable boundaries in order to verify their essential freeness. For example, we prove a 0-1 law (Theorem 6.2) for a class of stationary actions that include algebraic actions, which provides an essential freeness/triviality dichotomy for such actions. Using that we prove:

**Theorem** (Theorem 6.5). *Let  $\Gamma$  be a finitely generated linear group with trivial amenable radical. Then every generating measure on  $\Gamma$  is  $C^*$ -simple.*

We obtain this result by proving the existence of essentially free mean-proximal actions for linear groups. A crucial step in the proof is an extension result for mean-proximal actions that we prove jointly with Uri Bader (see Appendix A).

Furthermore, we conclude  $C^*$ -simplicity of the measures in the following contexts.

**Theorem** (Example 6.6, Theorems 6.7 and 6.8). *Every generating measure on a mapping class group or a hyperbolic group  $\Gamma$  with trivial amenable radical is  $C^*$ -simple. The same conclusion holds for finitely supported measures on  $\text{Out}(\mathbb{F}_n)$ .*

In the last section, we study unique stationarity and unique trace property relative to subgroups, and prove several ergodic-theoretical relative rigidity results.

**Theorem** (Theorem 7.5). *Let  $\mu \in \text{Prob}(\Gamma)$  and suppose  $\Gamma$  admits an essentially free  $\mu$ -boundary which has a compact model that is uniquely  $\mu$ -stationary. Then a  $\mu$ -stationary action is essentially free if its restriction to some co-amenable subgroup  $\Lambda \leq \Gamma$  is essentially free.*

All our results mentioned up to this point are obtained by applying the techniques developed here to use measurable boundaries to deduce  $C^*$ -algebraic rigidity properties. In contrast, a von-Neumann-algebraic relative superrigidity result below is proven in the last section by using topological boundary actions.

**Theorem** (Theorems 7.8 and 7.10). *Let  $\Gamma$  be a countable discrete group that admits a faithful topological boundary, and let  $\Lambda \leq \Gamma$  be an icc co-amenable subgroup. Suppose  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  is a unitary representation such that  $\pi(\Gamma)''$  is a finite von Neumann algebra. If the restriction  $\pi|_{\Lambda}$  extends to a von Neumann algebra isomorphism  $\pi(\Lambda)'' \cong \mathcal{L}(\Lambda)$ , then  $\pi$  extends to a von Neumann algebra isomorphism  $\pi(\Gamma)'' \cong \mathcal{L}(\Gamma)$ .*

Recall that by Furman [24] a group  $\Gamma$  admits a faithful topological boundary if and only if it has a trivial amenable radical (that is, it has no nontrivial amenable normal subgroups). As a corollary we obtain the following relative version of the result of Stuck–Zimmer.

**Theorem** (Theorems 7.9 and 7.10). *Let  $\Gamma$  be a countable discrete group with trivial amenable radical, and let  $\Lambda \leq \Gamma$  be a co-amenable subgroup. Then a probability measure preserving action  $\Gamma \curvearrowright (X, m)$  is essentially free if its restriction  $\Lambda \curvearrowright (X, m)$  is essentially free.*

In terms of *Invariant Random Subgroups* (IRS), the above is equivalent to the statement that every IRS of  $\Gamma$  intersects  $\Lambda$  nontrivially with positive probability. In particular, every nontrivial normal subgroup  $N \triangleleft \Gamma$  intersects  $\Lambda$  nontrivially.

The above results are relative versions of the well studied operator-algebraic super-rigidity, originated by Connes' conjecture [37]. The first major result in this direction was obtained by Bekka [9] and then was generalized by Peterson [49] (see also [20, 50]).

Very recently Boutonnet and Houdayer [13] have strengthened Peterson's work based on the concept of stationary states developed in the present paper. This breakthrough result is another manifestation of the importance of stationarity in the noncommutative setup. Indeed, it provides the strongest known result in this context (an SRS rigidity for higher rank lattices) and answers a question of Glasner and Weiss [27].

In addition to this introduction, this paper has six other sections. In Section 2 we briefly review the requisite background material. In Section 3 we recall the definitions

of measurable and topological boundaries, and prove that a unique stationary measurable boundary is a topological boundary. In Section 4 we introduce stationary  $C^*$ -dynamical systems, prove basic properties, and provide a number of examples. In particular, we show how unique stationarity implies  $C^*$ -simplicity. In Section 5 we prove our new characterization of  $C^*$ -simplicity in terms of unique stationarity of the canonical trace. We then prove various properties of  $C^*$ -simple measures, and obtain another characterization of  $C^*$ -simplicity in terms of unique stationarity of  $\text{Sub}_a$ . Section 6 is concerned with the question of freeness of unique stationary actions, and verifying that certain measures are  $C^*$ -simple. In Section 7, we apply our techniques to prove several superrigidity results relative to co-amenable subgroups.

The paper also contains an appendix, which includes our joint result with Uri Bader, an extension theorem for mean-proximal actions that we need in the proof of  $C^*$ -simplicity of generating measures on finitely generated linear groups.

## 2. Preliminaries

Throughout the paper,  $\Gamma$  is a countable discrete group, and  $\Gamma \curvearrowright X$  denotes an action of  $\Gamma$  by homeomorphisms on a compact (Hausdorff) space  $X$ . The action  $\Gamma \curvearrowright X$  is *minimal* if  $X$  has no nonempty proper closed  $\Gamma$ -invariant subset. We denote by  $\text{Prob}(X)$  the set of all Borel probability measures on  $X$ . For  $\nu \in \text{Prob}(X)$  we denote by  $\mathcal{P}_\nu$  its corresponding *Poisson map*, i.e. the unital positive  $\Gamma$ -equivariant map  $\mathcal{P}_\nu : C(X) \rightarrow \ell^\infty(\Gamma)$  defined by

$$\mathcal{P}_\nu(f)(g) = \int_X f(gx) d\nu(x), \quad g \in \Gamma, f \in C(X). \quad (1)$$

We also consider *measurable actions*, i.e. actions  $\Gamma \curvearrowright (Y, \eta)$  of  $\Gamma$  on probability spaces  $(Y, \eta)$  by measurable automorphisms. A measurable action  $\Gamma \curvearrowright (Y, \eta)$  is a *nonsingular action* (or  $\eta$  is a nonsingular measure) if  $g\eta$  and  $\eta$  are in the same measure class for every  $g \in \Gamma$ . The Poisson map associated to a nonsingular measure is defined similarly to (1). Throughout the paper, unless otherwise stated, all measurable actions are assumed to be nonsingular.

A compact  $\Gamma$ -space  $X$  is said to be a *compact model* of a measurable  $\Gamma$ -space  $(Y, \eta)$  if there exists a Borel measure  $\nu \in \text{Prob}(X)$  such that  $(Y, \eta)$  and  $(X, \nu)$  are measurably isomorphic as  $\Gamma$ -spaces. It is a well-known fact that every measurable action on a standard Borel space has a compact model which is metrizable.

For a Hilbert space  $\mathcal{H}$  we denote by  $\mathcal{B}(\mathcal{H})$  the set of all bounded operators on  $\mathcal{H}$ . A subalgebra  $\mathcal{A} \leq \mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra if it is closed in the operator norm and under taking adjoint. In this case,  $\mathcal{A}$  is *unital* if it contains the identity operator on  $\mathcal{H}$ .

If  $X$  is a compact space, then  $C(X)$  with the sup-norm and the complex conjugate as the involution is a  $C^*$ -algebra, and conversely, by Gelfand's representation theorem, any unital commutative  $C^*$ -algebra is of this form. Hence unital  $C^*$ -algebras are viewed as algebras of continuous functions on “noncommutative compact spaces”.

An element  $a$  in a  $C^*$ -algebra  $\mathcal{A}$  is said to be *positive*, written  $a \geq 0$ , if  $a = b^*b$  for some  $b \in \mathcal{A}$ . We denote by  $1_{\mathcal{A}}$  the unit element in  $\mathcal{A}$ .

A linear map  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  between  $C^*$ -algebras is *positive* if it sends positive elements to positive elements, and it is *unital* if  $\phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ . A linear  $\phi$  is a  *$*$ -homomorphism* if it is multiplicative and  $\phi(a^*) = \phi(a)^*$  for all  $a \in \mathcal{A}$ , and it is a  *$*$ -isomorphism* if it is moreover bijective. We denote by  $\text{Aut}(\mathcal{A})$  the group of all  $*$ -automorphisms on  $\mathcal{A}$ .

The noncommutative counterpart of probability measures are states on  $C^*$ -algebras. A *state* on  $\mathcal{A}$  is a positive linear functional  $\rho : \mathcal{A} \rightarrow \mathbb{C}$  with  $\rho(1_{\mathcal{A}}) = 1$ . We denote by  $\mathcal{S}(\mathcal{A})$  the (convex, weak\* compact) space of all states on  $\mathcal{A}$ . A state  $\rho$  is *faithful* if  $\rho(a) > 0$  for any nonzero positive element  $a$ . A state  $\tau \in \mathcal{S}(\mathcal{A})$  is a *trace* if  $\tau(ab) = \tau(ba)$  for all  $a, b \in \mathcal{A}$ . Obviously every state on a commutative  $C^*$ -algebra is a trace, but on the other hand there are  $C^*$ -algebras that do not admit any trace.

Let us recall the GNS construction associated to a state  $\rho \in \mathcal{S}(\mathcal{A})$ . Define the sesquilinear form  $\langle a, b \rangle_{\rho} := \rho(b^*a)$  on  $\mathcal{A}$ , let  $L^2(\mathcal{A}, \rho)$  be the induced Hilbert space, and for every  $b \in \mathcal{A}$  denote by  $\widehat{b}$  its corresponding element in  $L^2(\mathcal{A}, \rho)$ . Then the GNS representation  $\pi_{\rho} : \mathcal{A} \rightarrow \mathcal{B}(L^2(\mathcal{A}, \rho))$  is defined by  $\pi_{\rho}(a)(\widehat{b}) = \widehat{ab}$  for  $a, b \in \mathcal{A}$ .

A  $C^*$ -algebra is said to be *simple* if it does not contain any nontrivial proper closed two-sided ideal.

A *von Neumann algebra* is a  $C^*$ -algebra  $M$  that is also a dual Banach space. In this case the predual  $M_*$  is unique. A bounded linear functional on  $M$  is called *normal* if it belongs to  $M_*$ . Since  $\mathcal{B}(\mathcal{H})$  itself is a dual Banach space (the predual being the space of trace-class operators), it follows that a unital  $C^*$ -subalgebra  $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$  is a von Neumann algebra if and only if it is closed in the weak\* topology of  $\mathcal{B}(\mathcal{H})$ , or equivalently closed in the weak or strong operator topologies. By von Neumann’s bicommutant theorem a self-adjoint unital subalgebra  $M \subset \mathcal{B}(\mathcal{H})$  is a von Neumann algebra if and only if  $M'' = M$ , where  $M' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \text{ for all } y \in M\}$  is the *commutant* of  $M$  in  $\mathcal{B}(\mathcal{H})$ , and  $M'' = (M)'$ .

If  $(X, \nu)$  is a probability space, then  $L^{\infty}(X, \nu)$  is a von Neumann algebra, and every commutative von Neumann algebra is of this form. Hence, von Neumann algebras are viewed as algebras of essentially bounded measurable functions on “noncommutative probability spaces”.

The GNS representation associated to normal states on a von Neumann algebra is defined similarly to the  $C^*$ -algebra case.

### 2.1. Random walks and stationary dynamical systems

The theory of random walks on groups and their associated boundaries was introduced by Furstenberg [25,26], and later studied extensively by various people. This theory provides a framework to apply probabilistic ideas and methods in the study of analytic properties of groups. Let us briefly recall the notion of random walks on discrete groups. We refer the reader to [8,23,26] for more details.

Let  $\mu \in \text{Prob}(\Gamma)$  be *generating*, i.e.  $\Gamma$  is the semigroup generated by  $\text{Supp}(\mu) = \{g : \mu(g) > 0\}$ . The *random walk on  $\Gamma$  with law  $\mu$*  (or just the  $(\Gamma, \mu)$ -random walk)

is the time-independent Markov chain with state space  $\Gamma$ , initial distribution  $\delta_e$  (the Dirac probability measure supported at the neutral element  $e \in \Gamma$ ), and transition probabilities  $p(g, h) = \mu(g^{-1}h)$  for  $g, h \in \Gamma$ . Thus, the probability of walking on the path  $e, g_1, g_1g_2, \dots, g_1g_2 \cdots g_k$  on the first  $k + 1$  steps is  $\mu(g_1)\mu(g_2) \cdots \mu(g_k)$ .

The *space of paths* of the random walk is the probability space  $(\Omega, \mathbb{P}_\mu)$ , where  $\Omega = \Gamma^{\mathbb{N}}$  and  $\mathbb{P}_\mu$  is the Markovian measure, that is, the unique probability on  $\Omega$  defined by

$$\begin{aligned} \mathbb{P}_\mu(\{\omega \in \Omega : \omega_1 = g_1, \omega_2 = \omega_1g_2, \dots, \omega_k = \omega_{k-1}g_k\}) \\ = \mu(g_1)\mu(g_2) \cdots \mu(g_k) \end{aligned}$$

for  $g_1, g_2, \dots, g_k \in \Gamma$  and  $k \in \mathbb{N}$ .

For a fixed  $g \in \Gamma$  it follows that  $\mathbb{P}_\mu(\{\omega \in \Omega : \omega_n = g\})$ , the probability of the random walk being in position  $g$  at the  $n$ -th step, is equal to  $\mu^n(g)$ , where  $\mu^n$  is the  $n$ -th convolution power of  $\mu$ .

Alternatively,  $\mathbb{P}_\mu$  can be described as the push-forward of the Bernoulli measure  $\mu \times \mu \times \cdots$  under the transformation  $\Gamma^{\mathbb{N}} \rightarrow \Gamma^{\mathbb{N} \cup \{0\}}$ ,  $(g_k) \mapsto (\omega_k)$ , where  $\omega_0 = e$  and  $\omega_k = g_1 \cdots g_k$  for  $k \in \mathbb{N}$ . In probabilistic terms,  $g_k$  is the increment and  $\omega_k$  is the position of the random walk at time  $k$ .

Suppose  $\Gamma \curvearrowright X$  is a continuous action on a compact space, and let  $\mu \in \text{Prob}(\Gamma)$ . A Borel probability measure  $\nu$  on  $X$  is called  $\mu$ -stationary if

$$\nu = \sum_{g \in \Gamma} \mu(g)g\nu.$$

In this case we say  $(X, \nu)$  is a  $(\Gamma, \mu)$ -space, and we write  $(\Gamma, \mu) \curvearrowright (X, \nu)$ .

The basic feature of stationary measures is their existence. Unlike invariant measures, on any compact  $\Gamma$ -space  $X$  there exists at least one  $\mu$ -stationary measure. While the existence holds for arbitrary compact space, significant part of the theory is developed in the context of metrizable compact spaces. The following is the fundamental result in this context that builds the connection between stationary systems and the theory of random walks.

**Theorem 2.1** (Furstenberg). *Let  $\nu$  be a  $\mu$ -stationary measure on a metrizable compact  $\Gamma$ -space  $X$ . Then for  $\mathbb{P}_\mu$ -almost every path  $\omega \in \Omega$  the limit  $\nu_\omega := \text{weak}^* \text{-}\lim_n \omega_n \nu$  exists. Moreover,*

$$\nu = \int_{\Omega} \nu_\omega d\mathbb{P}_\mu(\omega).$$

The measures  $\nu_\omega$  are called the *conditional measures*.

We will recall and prove other basic facts about stationary dynamical systems in the more general setting of actions on  $C^*$ -algebras in later sections.

Stationary actions are also defined in measurable setting. A nonsingular action  $\Gamma \curvearrowright (Y, \eta)$  is  $\mu$ -stationary if  $\sum_{g \in \Gamma} \mu(g)g\eta = \eta$ . Note that if  $X$  is a compact  $\Gamma$ -space, and  $\mu \in \text{Prob}(\Gamma)$  is generating, then any  $\mu$ -stationary  $\nu \in \text{Prob}(X)$  is nonsingular.



## 2.2. $C^*$ -dynamical systems

We briefly recall the notion of group actions on  $C^*$ -algebras and establish the notation and terminology that we will be using. We refer the reader to [19] for more details.

A unital  $C^*$ -algebra  $\mathcal{A}$  is called a  $\Gamma$ - $C^*$ -algebra if there is an action  $\alpha : \Gamma \curvearrowright \mathcal{A}$  of  $\Gamma$  on  $\mathcal{A}$  by  $*$ -automorphisms, that is,  $\alpha$  is a group homomorphism  $\Gamma \rightarrow \text{Aut}(\mathcal{A})$ . A class of examples of  $\Gamma$ - $C^*$ -algebras that are of main interest to our work is obtained as follows. Let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a unitary representation on the Hilbert space  $\mathcal{H}_\pi$  (where  $\mathcal{U}(\mathcal{H}_\pi)$  is the group of unitary operators on  $\mathcal{H}_\pi$ ). Then  $\Gamma$  acts on  $\mathcal{B}(\mathcal{H}_\pi)$  by inner automorphisms,  $\text{Ad}_g(x) := \pi(g)x\pi(g^{-1})$ ,  $g \in \Gamma$ ,  $x \in \mathcal{B}(\mathcal{H}_\pi)$ , as it also does on any  $C^*$ -algebra  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_\pi)$  that is invariant under this action. In fact, every  $\Gamma$ - $C^*$ -algebra is formed in the above fashion for some unitary representation of  $\Gamma$ .

In particular, for any unitary representation  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ , the group  $\Gamma$  acts on the  $C^*$ -algebra  $C_\pi^*(\Gamma) := \overline{\text{span}\{\pi(g) : g \in \Gamma\}}^{\|\cdot\|} \subset \mathcal{B}(\mathcal{H}_\pi)$  by inner automorphisms. Throughout the paper  $\Gamma \curvearrowright C_\pi^*(\Gamma)$  denotes this action unless otherwise stated.

An important example is the left regular representation  $\lambda : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma))$  defined by  $(\lambda_g \xi)(h) = \xi(g^{-1}h)$ ,  $h \in \Gamma$  and  $\xi \in \ell^2(\Gamma)$ . In this case  $C_\lambda^*(\Gamma)$  is called the *reduced  $C^*$ -algebra* of  $\Gamma$ .

The *full  $C^*$ -algebra*  $C^*(\Gamma)$  of  $\Gamma$  is the universal  $C^*$ -algebra generated by  $\Gamma$  in the sense that for any unitary representation  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  there is a canonical surjective  $*$ -homomorphism  $C^*(\Gamma) \rightarrow C_\pi^*(\Gamma)$ . We consider  $\Gamma$  as a subset of  $C^*(\Gamma)$  in the natural way.

Similarly to compact spaces and probability measures (“the commutative case”), any action  $\Gamma \curvearrowright \mathcal{A}$  induces an adjoint action  $\Gamma \curvearrowright \mathcal{S}(\mathcal{A})$ . We denote by  $\mathcal{S}_\Gamma(\mathcal{A})$  the simplex of all  $\Gamma$ -invariant states, that is, states  $\nu$  such that  $g\nu = \nu$  for all  $g \in \Gamma$ . It is obvious that  $\mathcal{S}_\Gamma(\mathcal{A})$  is compact in the weak\* topology.

In the case of  $\Gamma$  acting on  $C_\pi^*(\Gamma)$  by inner automorphisms,  $\mathcal{S}_\Gamma(C_\pi^*(\Gamma))$  coincides with the set of all traces on  $C_\pi^*(\Gamma)$ . In particular, in the case of the reduced  $C^*$ -algebra,  $\mathcal{S}_\Gamma(C_\lambda^*(\Gamma))$  is never empty as it contains the *canonical trace*  $\tau_0$  (or  $\tau_0^\Gamma$  if we need to clarify its association to  $\Gamma$ ), namely the extension of the linear functional  $\sum c_g \lambda_g \mapsto c_e$ .

Similarly to the  $C^*$ -algebra case, a source of examples of  $\Gamma$ -von Neumann algebras for us is the theory of unitary representations of groups. Let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a unitary representation on the Hilbert space  $\mathcal{H}_\pi$ . Then  $\Gamma$  acts on  $\text{VN}_\pi(\Gamma)$  by inner automorphisms, where  $\text{VN}_\pi(\Gamma) := \{\pi(g) : g \in \Gamma\}'' = \overline{\text{span}\{\pi(g) : g \in \Gamma\}}^{\text{weak}^*} \subset \mathcal{B}(\mathcal{H}_\pi)$  is the von Neumann algebra generated by the representation  $\pi$ . In the case of the left regular representation,  $\text{VN}_\lambda(\Gamma) = \overline{C_\lambda^*(\Gamma)}^{\text{weak}^*}$  is called the *group von Neumann algebra* of  $\Gamma$  and is denoted by  $\mathcal{L}(\Gamma)$ . The canonical trace  $\tau_0$  extends to a normal trace on  $\mathcal{L}(\Gamma)$ .

## 2.3. Crossed product $C^*$ -algebras

The bridge between the theories of operator algebras and (topological or measure-theoretical) dynamics is made by the *crossed product* construction. Loosely speaking, the

crossed product  $C^*$ -algebra associated to an action  $\Gamma \curvearrowright \mathcal{A}$  is a  $C^*$ -algebra that contains, and is generated by, copies of  $\mathcal{A}$  and  $\Gamma$  such that the action  $\Gamma \curvearrowright \mathcal{A}$  in this bigger algebra is by inner automorphisms.

We recall the more precise definition below and refer the reader to [19] for more details.

Let  $\Gamma \curvearrowright \mathcal{A}$ , and consider the Hilbert space  $\ell^2(\Gamma, \mathcal{H}) = \{\xi : \Gamma \rightarrow \mathcal{H} \mid \sum_{g \in \Gamma} \|\xi(g)\|^2 < \infty\}$ . For  $g \in \Gamma$  and  $a \in \mathcal{A}$  define the operators  $\tilde{\lambda}(g), \iota(a) \in \mathcal{B}(\ell^2(\Gamma, \mathcal{H}))$  by  $(\tilde{\lambda}(g)\xi)(h) = \xi(g^{-1}h)$  and  $(\iota(a)\xi)(h) = (h^{-1}a)\xi(h)$ . Then the  $C^*$ -subalgebra of  $\mathcal{B}(\ell^2(\Gamma, \mathcal{H}))$  generated by  $\tilde{\lambda}(\Gamma)$  and  $\iota(\mathcal{A})$  is called the *reduced crossed product* of the action  $\Gamma \curvearrowright \mathcal{A}$ , and is denoted by  $\Gamma \rtimes_r \mathcal{A}$ . It can also be seen that  $C^*(\{\tilde{\lambda}(g) : g \in \Gamma\})$  is canonically isomorphic to  $C_\lambda^*(\Gamma)$ .

The following simple lemma, which generalizes [4, Lemma 2], and in fact follows from its proof, is key in allowing passage between classical and noncommutative settings. We present a more elementary proof which was provided to us by Hanfeng Li. We thank him for this, as well as for pointing out to us the crucial fact that [4, Lemma 2] is also valid for actions on operator systems.

**Lemma 2.2.** *Let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a unitary representation. Suppose  $\rho$  is a state on  $\mathcal{B}(\mathcal{H}_\pi)$  and  $g \in \Gamma$ . If there is a  $a \in \mathcal{B}(\mathcal{H}_\pi)$  with  $0 \leq a \leq 1$  such that  $\rho(a) = 1$  and  $\rho(\pi(g^{-1})a\pi(g)) = 0$ , then  $\rho(\pi(g)) = 0$ .*

*Proof.* Since  $0 \leq \pi(g^{-1})a\pi(g) \leq 1$  and  $\rho(\pi(g^{-1})a\pi(g)) = 0$  we get  $\rho(\pi(g^{-1})a^2\pi(g)) = 0$ . Similarly,  $\rho((1 - a)^2) = 0$ , and the Cauchy–Schwarz inequality implies that  $|\rho((1 - a)\pi(g))| \leq \rho((1 - a)^2)\rho(1) = 0$ . So we have

$$\begin{aligned} |\rho(a\pi(g))| &= |\rho(\pi(g)\pi(g^{-1})a\pi(g))| = |\rho(\pi(g^{-1})a\pi(g)\pi(g^{-1}))| \\ &\leq \rho(\pi(g^{-1})a^2\pi(g)) = 0. \end{aligned}$$

Hence  $\rho(\pi(g)) = \rho(a\pi(g)) + \rho((1 - a)\pi(g)) = 0$ . ■

#### 2.4. Positive definite functions, invariant and stationary random subgroups

A function  $\phi : \Gamma \rightarrow \mathbb{C}$  is called a *positive definite function* (pdf) if for any  $n \in \mathbb{N}$  and  $g_1, \dots, g_n \in \Gamma$  the matrix  $[\phi(g_i g_j^{-1})]_{i,j=1}^n$  is positive.

If  $\pi$  is a unitary representation of  $\Gamma$  on a Hilbert space  $\mathcal{H}_\pi$ , then for any state  $\rho$  on  $C_\pi^*(\Gamma)$  the function  $\phi(g) = \rho(\pi(g))$  is a pdf on  $\Gamma$ . Conversely, if  $\phi$  is a pdf on  $\Gamma$  then there is a unitary representation  $\pi$  (e.g. the GNS representation associated to  $\phi$  [19]), and a vector  $\xi \in \mathcal{H}_\pi$  such that  $\phi(g) = \langle \pi(g)\xi, \xi \rangle_{\mathcal{H}_\pi}$ . This yields a canonical identification between the weak\* compact convex space  $\mathcal{S}(C^*(\Gamma))$  of all states on the full  $C^*$ -algebra of  $\Gamma$ , and the space  $P_\Gamma$  of all pdf  $\phi : \Gamma \rightarrow \mathbb{C}$  normalized by  $\phi(e) = 1$ , endowed with the pointwise convergence topology. In this correspondence,  $\rho$  is a trace if and only if  $\phi$  is a *character*, i.e. a normalized conjugation invariant pdf (i.e.  $\phi(h^{-1}gh) = \phi(g)$  for all  $g, h \in \Gamma$ ).

Let  $\text{Sub}(\Gamma)$  be the space of all subgroups of  $\Gamma$  with the topology inherited from  $2^\Gamma$  (known also as the *Chabauty topology*). It is a compact space, on which  $\Gamma$  acts by conjugation  $g.\Lambda = g^{-1}\Lambda g$ . A  $\Gamma$ -invariant Borel probability measure  $\eta$  is called an *invariant random subgroup (IRS)* [1, 2]. If  $\eta$  is only  $\mu$ -stationary for some  $\mu \in \text{Prob}(\Gamma)$ , we say that  $\eta$  is a  $\mu$ -stationary random subgroup ( $\mu$ -SRS).

We denote by  $\text{Sub}_a(\Gamma)$  the closed,  $\Gamma$ -invariant subset of  $\text{Sub}(\Gamma)$  of all amenable subgroups.

**Lemma 2.3.** *Let  $\eta$  be Borel probability measure on  $\text{Sub}(\Gamma)$ . Then the function  $\phi_\eta(g) = \eta(\{\Lambda : g \in \Lambda\})$  is positive definite. If moreover  $\eta$  is supported on  $\text{Sub}_a(\Gamma)$ , then there is a state  $\rho$  on the reduced  $C^*$ -algebra  $C_\lambda^*(\Gamma)$  such that  $\phi_\eta(g) = \rho(\lambda_g)$  for every  $g \in \Gamma$ .*

*Proof.* Given a subgroup  $\Lambda \in \text{Sub}(\Gamma)$ , let  $\mathbb{1}_\Lambda$  denote its characteristic function. It is not hard to see that  $\mathbb{1}_\Lambda \in P_\Gamma$ . The map  $\text{Sub}(\Gamma) \ni \Lambda \mapsto \mathbb{1}_\Lambda \in P_\Gamma$  is clearly continuous. Let  $\bar{\eta} \in \text{Prob}(P_\Gamma)$  be the push-forward of  $\eta$ . Then  $\phi_\eta$  is the barycenter of  $\bar{\eta}$ .

If  $\Lambda \leq \Gamma$  is an amenable subgroup, then the quasi-regular representation  $\Gamma$  on  $\ell^2(\Gamma \setminus \Lambda)$  is weakly contained in the regular representation of  $\Gamma$ , which implies  $\mathbb{1}_\Lambda$  corresponds to a state on  $C_\lambda^*(\Gamma)$ . Hence if  $\eta$  is supported on amenable subgroups then the barycenter of  $\bar{\eta}$  corresponds to a state  $\rho$  on  $C_\lambda^*(\Gamma)$ . ■

### 3. Topological, measurable, and uniquely stationary boundaries

In this section we recall the notions of topological and measurable boundary actions of discrete groups. We comment on advantages of each setting over the other, and prove that a uniquely stationary measurable boundary (USB) is also a topological boundary. Thus, in the framework of such systems we may apply both topological and measure-theoretical techniques.

#### 3.1. Topological vs. measurable boundaries

For more details on boundary actions we refer the reader to [23, 26, 29].

**Definition 3.1** (Topological boundary actions). A continuous action  $\Gamma \curvearrowright X$  on a compact space  $X$  is a *topological boundary action* if for every  $\nu \in \text{Prob}(X)$  and  $x \in X$  there is a net  $g_i$  of elements of  $\Gamma$  such that  $g_i\nu \rightarrow \delta_x$  in the weak\* topology, where  $\delta_x$  is the Dirac measure at  $x$ .

It can be shown that an action  $\Gamma \curvearrowright X$  is a topological boundary action if and only if for every  $\eta \in \text{Prob}(X)$  the Poisson map  $\mathcal{P}_\eta$  is isometric [6].

**Proposition 3.2** ([26]). *There is a unique (up to  $\Gamma$ -equivariant homeomorphism) maximal  $\Gamma$ -boundary  $\partial_F\Gamma$  in the sense that every  $\Gamma$ -boundary  $X$  is a continuous  $\Gamma$ -equivariant image of  $\partial_F\Gamma$ .*

The maximal boundary  $\partial_F\Gamma$  is called the *Furstenberg boundary* of  $\Gamma$ .

**Definition 3.3** (Measurable boundary actions). Let  $\mu \in \text{Prob}(\Gamma)$ , and suppose  $\nu$  is a  $\mu$ -stationary measure on a metrizable  $\Gamma$ -space  $X$ . The action  $(\Gamma, \mu) \curvearrowright (X, \nu)$  is a  $\mu$ -boundary action if for almost every path  $\omega = (\omega_k) \in \Omega$  of the  $(\Gamma, \mu)$ -random walk, the sequence  $\omega_k \nu$  converges to a Dirac measure  $\delta_{x_\omega}$ .

In this case the map  $\mathbf{bnd} : (\Omega, \mathbb{P}_\mu) \rightarrow (X, \nu)$  defined by  $\mathbf{bnd}(\omega) = x_\omega$  is called a boundary map.

A measurable nonsingular action  $\Gamma \curvearrowright (Y, \eta)$  is called a  $\mu$ -boundary action if it is  $\mu$ -stationary and admits a compact metrizable model  $(X, \nu)$  which is a  $\mu$ -boundary action in the above sense.

**Proposition 3.4** ([26]). *There is a unique (up to  $\Gamma$ -equivariant measurable isomorphism) maximal  $\mu$ -boundary  $(\Pi_\mu, \nu_\infty)$  in the sense that every  $\mu$ -boundary  $(X, \nu)$  is a measurable  $\Gamma$ -equivariant image of  $(\Pi_\mu, \nu_\infty)$ .*

The maximal  $\mu$ -boundary  $(\Pi_\mu, \nu_\infty)$  is called the *Poisson boundary* of the pair  $(\Gamma, \mu)$  (also known sometimes as the *Furstenberg–Poisson boundary*).

One should note that since the Poisson boundary is defined up to measurable isomorphism, it should be considered as a measurable  $\Gamma$ -space.

Alternatively, boundaries can be characterized in terms of their function algebras. This is key in allowing the use of algebraic tools in the study of boundary actions.

The operator-algebraic description of topological boundaries requires some notions from the theory of injective envelopes as developed by Hamana [31]. Since we will not use this in our work here, we only recall the main result regarding this characterization, and refer the reader to [16, 41] for more details.

**Theorem 3.5** ([41, Theorem 3.11]). *Let  $\Gamma$  be a discrete group. Then  $C(\partial_F \Gamma)$  is the smallest injective object in the category of unital  $\Gamma$ - $C^*$ -algebras.*

The  $L^\infty$ -algebras of measurable boundaries are precisely invariant von Neumann subalgebras of the algebra of bounded harmonic functions.

Recall for  $\mu \in \text{Prob}(\Gamma)$  a function  $f \in \ell^\infty(\Gamma)$  is said to be  $\mu$ -harmonic if

$$f(g) = \sum_{h \in \Gamma} \mu(h) f(gh) \quad \text{for all } g \in \Gamma. \tag{2}$$

We denote by  $H^\infty(\Gamma, \mu) \subset \ell^\infty(\Gamma)$  the space of all bounded  $\mu$ -harmonic functions. Observe that  $H^\infty(\Gamma, \mu)$  is invariant under the action of  $\Gamma$  by left translations.

The space  $H^\infty(\Gamma, \mu)$  is not a subalgebra of  $\ell^\infty(\Gamma)$  in general, but the formula

$$f_1 \cdot f_2(g) := \lim_{n \rightarrow \infty} \sum_{h \in \Gamma} f_1(h) f_2(h^{-1}g) \mu^n(h) \tag{3}$$

defines a multiplication on  $H^\infty(\Gamma, \mu)$  and turns it into a commutative von Neumann algebra.

**Proposition 3.6** (Furstenberg). *The Poisson map  $\mathcal{P}_{\nu_\infty}$  defines a von Neumann algebra isomorphism  $L^\infty(\Pi_\mu, \nu_\infty) \cong H^\infty(\Gamma, \mu)$ .*

*In particular, a measurable nonsingular action  $\Gamma \curvearrowright (Y, \eta)$  is a  $\mu$ -boundary if and only if the Poisson map  $\mathcal{P}_\nu : L^\infty(X, \nu) \rightarrow H^\infty(\Gamma, \mu)$  is a von Neumann algebra embedding.*

Abstractly, we know where to find examples of boundary actions. Measurable boundaries appear whenever one has a stationary action, and topological boundaries arise whenever one has an affine action on a compact convex space.

**Proposition 3.7** (Furstenberg). *Suppose  $(X, \nu)$  is a  $(\Gamma, \mu)$ -space. Then the weak\* closure of the set  $\{\nu_\omega : \omega \in \Omega\}$  of conditional measures, with the push-forward of  $\mathbb{P}_\mu$  under the map  $\omega \mapsto \nu_\omega$ , is a  $\mu$ -boundary. Moreover, every  $\mu$ -boundary arises in this way.*

**Proposition 3.8** ([29, Theorem III.2.3]). *Suppose  $\Gamma \curvearrowright K$  is an affine action, and suppose  $K$  has no proper  $\Gamma$ -invariant compact convex subspace. Then the closure of the extreme points of  $K$  is a topological boundary. Moreover, every topological boundary of  $\Gamma$  arises in this way.*

The main advantage of measurable boundaries over their topological counterparts is that they are much easier to concretely identify. In fact, there have been extensive work in the past few decades which has led to concrete realization of the Poisson boundary for many of the groups that arise naturally as symmetries of geometric objects. We discuss some examples below. In contrast, the Furstenberg boundary  $\partial_F \Gamma$  of a nonamenable countable group  $\Gamma$  is an extremally disconnected nonmetrizable space, and not concretely realizable in any known case.

### 3.2. USB systems

The study of these systems was initiated by Furstenberg [26], who called them  $\mu$ -proximal actions. They were further studied in [28, 46].

**Definition 3.9.** Let  $\mu \in \text{Prob}(\Gamma)$ . We say that a  $\mu$ -boundary  $(X, \nu)$  is a  $(\mu)$ -USB if it has a compact model  $(K, \bar{\nu})$  such that  $\bar{\nu}$  is the unique  $\mu$ -stationary Borel probability measure on  $K$ . If  $(X, \nu)$  is the  $(\mu)$ -Poisson boundary, we say that  $(X, \nu)$  is a  $(\mu)$ -Poisson-USB.

The following is a standard technique that allows us to assume that the topological model of a USB is metrizable. This will be important for us as we will make heavy use of the existence of conditional measures.

**Lemma 3.10.** *Every USB  $(X, \nu)$  has a compact metrizable model  $(K, \bar{\nu})$  such that  $\bar{\nu}$  is the unique  $\mu$ -stationary Borel probability measure on  $K$ .*

*Proof.* Let  $(K', \eta)$  be a compact model of  $(X, \nu)$  as in Definition 3.9. As abstract probability spaces,  $\mu$ -boundaries are always standard. Therefore, using the fact that  $\Gamma$  is countable we can choose a countable,  $\Gamma$ -invariant set of elements in  $C(K')$  that includes a non-zero constant function, and is weak\* dense in  $L^\infty(K', \eta)$ . Take  $K$  to be the spectrum of the (sup-)norm closure of the algebra generated by this set, which is compact and metrizable. Then  $K$  is a continuous  $\Gamma$ -equivariant image of  $K'$ . Let  $\bar{\nu}$  be the push-forward of  $\eta$  to  $K$ .

It follows (see Corollary 4.3) that  $\bar{\nu}$  is the unique  $\mu$ -stationary probability on  $K$ , and so  $(K, \bar{\nu})$  is the desired compact metrizable model. ■

Many natural examples of Poisson boundaries are in fact USB, namely, the Poisson boundary is being realized as a unique stationary measure on a compact space. The main tool for realizing the Poisson boundary on a compact space is the strip criterion of Kaimanovich [38], which proves in many cases that the Poisson measure is actually unique. To name some examples (by no means a complete list!), we mention linear groups acting on flag varieties [18, 38, 45], hyperbolic groups acting on the Gromov boundary [38], nonelementary subgroups of mapping class groups acting on the Thurston boundary [39], and nonelementary subgroups of  $\text{Out}(\mathbb{F}_n)$  acting on the boundary of the outer space [33]. We discuss properties of these actions further in Section 6.

**Theorem 3.11.** *Let  $\mu \in \text{Prob}(\Gamma)$ , and suppose  $(X, \nu)$  is a  $\mu$ -USB. Then for any compact model of  $(X, \nu)$ , the restriction of the action to the support of the unique  $\mu$ -stationary is a topological boundary action.*

*Proof.* To simplify notations, we assume  $X$  is already a compact model, that is,  $X$  is a compact  $\Gamma$ -space and  $\nu \in \text{Prob}(X)$  is the unique  $\mu$ -stationary measure such that  $(X, \nu)$  is a  $\mu$ -boundary. Note that by unique stationarity of  $\nu$ , its support is the unique minimal component of  $X$ . Thus, by passing to  $\text{Supp}(\nu)$  if necessary, we also assume  $\Gamma \curvearrowright X$  is minimal.

Assume first that  $X$  is metrizable. We show  $\text{Prob}(X)$  does not contain any proper nonempty compact convex  $\Gamma$ -invariant subsets. Then the theorem follows from Proposition 3.8. Suppose  $\mathcal{C} \subseteq \text{Prob}(X)$  is a nonempty closed convex  $\Gamma$ -invariant set. Let  $\eta \in \mathcal{C}$ . By  $\Gamma$ -invariance and closedness of  $\mathcal{C}$  we have  $\frac{1}{n} \sum_{k=0}^{n-1} \mu^k * \eta \in \mathcal{C}$  for all  $n \in \mathbb{N}$ . Note any weak\* cluster point of this sequence is  $\mu$ -stationary, hence by uniqueness assumption  $\nu \in \mathcal{C}$ . As  $X$  is metrizable, and by the closedness of  $\mathcal{C}$  we conclude that the conditional measures  $\nu_\omega$  are in  $\mathcal{C}$  for  $\mathbb{P}_\mu$ -a.e. path  $\omega \in \Omega$ . Since  $(X, \nu)$  is a  $\mu$ -boundary,  $\nu_\omega$  are point measures for  $\mathbb{P}_\mu$ -a.e.  $\omega$ . In particular, there is some  $x \in X$  such that  $\delta_x \in \mathcal{C}$ . By  $\Gamma$ -invariance of  $\mathcal{C}$  we get  $\delta_{gx} \in \mathcal{C}$  for every  $g \in \Gamma$ , and therefore minimality of  $X$  and closedness of  $\mathcal{C}$  yield  $\{\delta_x : x \in X\} \subset \mathcal{C}$ . Now the convexity of  $\mathcal{C}$  implies  $\text{Prob}(X) = \overline{\text{conv}} \{\delta_x : x \in X\} \subset \mathcal{C}$ .

Now for the general case (not necessarily metrizable), let  $\eta \in \text{Prob}(X)$  and  $f \in C(X)$ . Let  $\mathcal{A} \subset C(X)$  be the  $\Gamma$ -invariant  $C^*$ -subalgebra generated by  $f$ . Then  $\mathcal{A} = C(Y)$  for a metrizable  $\Gamma$ -factor of  $X$ . Moreover, the push-forward of  $\nu$  on  $Y$  is the unique  $\mu$ -stationary measure on  $Y$  (see Corollary 4.3 below), and thus by the above  $\Gamma \curvearrowright Y$  is a topological boundary action. Therefore, we have  $\|\mathcal{P}_\eta(f)\| = \|f\|$ . This shows that the Poisson map  $\mathcal{P}_\eta$  is isometric. Hence, we conclude  $X$  is a topological  $\Gamma$ -boundary. ■

Perhaps the most significant application of topological boundaries so far has been in the problems of unique trace property and  $C^*$ -simplicity: the existence of a faithful topological boundary is equivalent to the unique trace property and the existence of a free

boundary is equivalent to  $C^*$ -simplicity. A subtlety in applying these characterizations is that in general one has to pass to the maximal boundary, i.e. the Furstenberg boundary, which is too “large” to concretely realize and work with.

But we will see, for example, that in order to determine the  $C^*$ -simplicity and the unique trace property of a group, it is enough to work with its USB actions (provided that the group admits such), rather than abstract topological boundaries.

Recall that any group  $\Gamma$  admits a maximal normal amenable subgroup, called the *amenable radical* of  $\Gamma$ . We denote this subgroup by  $\text{Rad}(\Gamma)$ .

**Proposition 3.12.** *Suppose  $\Gamma \curvearrowright (X, \nu)$  is a  $\mu$ -USB for some  $\mu \in \text{Prob}(\Gamma)$ . Then  $\text{Rad}(\Gamma) \subseteq \ker(\Gamma \curvearrowright (X, \nu))$ , and equality holds if  $(X, \nu)$  is the  $\mu$ -Poisson USB.*

*Proof.* The first assertion can be proven by a straightforward modification of the proof of [24, Proposition 7]. Alternatively, by Theorem 3.11 the action  $\Gamma \curvearrowright \text{Supp}(\nu)$  is a topological boundary action, hence by the conclusion of [24, Proposition 7],  $\text{Rad}(\Gamma)$  acts trivially on  $\text{Supp}(\nu)$ , and so  $\text{Rad}(\Gamma) \subset \ker(\Gamma \curvearrowright (X, \nu))$ .

For the second part of the statement, if  $(X, \nu)$  is the Poisson boundary, then the action  $\Gamma \curvearrowright (X, \nu)$  is Zimmer-amenable [53] and in particular the stabilizers  $\text{Stab}_\Gamma(x)$  are amenable for  $\nu$ -a.e.  $x \in X$ . It follows that  $\ker(\Gamma \curvearrowright (X, \nu))$  is a normal amenable group, and so  $\ker(\Gamma \curvearrowright (X, \nu)) \subset \text{Rad}(\Gamma)$ . ■

#### 4. Stationary $C^*$ -dynamical systems

Similarly to classical ergodic theory, invariant states may not exist for actions of non-amenable groups on  $C^*$ -algebras. For instance, in the case of inner action by subgroups of the unitary group of a given  $C^*$ -algebra, which is only nontrivial in the noncommutative setting, the invariant ergodic theory is only available in the tracial case. Hence, one has to appeal to other models of dynamical systems in infinite-type cases. In this section we begin studying the concept of stationary dynamical systems in the context of  $C^*$ -algebras.

Let  $\Gamma \curvearrowright \mathcal{A}$  and let  $\mu \in \text{Prob}(\Gamma)$ . The  $\mu$ -convolution map on  $\mathcal{A}$  is defined by

$$\mu * a := \sum_{g \in \Gamma} \mu(g)g^{-1}a.$$

Its adjoint induces a  $\mu$ -convolution operator on the space of states  $\mathcal{S}(\mathcal{A})$  given by

$$\mu * \tau = \sum_{g \in \Gamma} \mu(g)g\tau.$$

**Definition 4.1.** Let  $\mathcal{A}$  be a  $\Gamma$ - $C^*$ -algebra. A state  $\tau \in \mathcal{S}(\mathcal{A})$  is said to be  $\mu$ -stationary if  $\mu * \tau = \tau$ . In this case we say the pair  $(\mathcal{A}, \tau)$  is a  $(\Gamma, \mu)$ - $C^*$ -algebra.

We denote the collection of all  $\mu$ -stationary states on  $\mathcal{A}$  by  $\mathcal{S}_\mu(\mathcal{A})$ . We say that  $\mathcal{A}$  is *uniquely stationary* if there exists  $\mu \in \text{Prob}(\Gamma)$  such that  $\mathcal{S}_\mu(\mathcal{A})$  has only one element.

4.1. Basic facts

In this section we review noncommutative versions of a few basic facts about stationary actions. First, note that invariant states are  $\mu$ -stationary for every  $\mu \in \text{Prob}(\Gamma)$ . Next, we observe that in contrast to the invariant case, stationary states always exist, and moreover they can always be extended. The corresponding statement of the latter in the commutative setting is that any stationary measure on a factor can be “pulled back” (not necessarily in a unique way) to a stationary measure on the extension.

**Proposition 4.2.** *Suppose  $\mathcal{A}$  is a  $\Gamma$ - $C^*$ -algebra and  $\mathcal{B} \subset \mathcal{A}$  is a  $\Gamma$ -invariant subalgebra. Then every  $\mu$ -stationary  $\eta \in \mathcal{S}_\mu(\mathcal{B})$  can be extended to a  $\mu$ -stationary state  $\tau \in \mathcal{S}_\mu(\mathcal{A})$ .*

*In particular, for any  $\Gamma$ - $C^*$ -algebra  $\mathcal{A}$  and any  $\mu \in \text{Prob}(\Gamma)$ , the set  $\mathcal{S}_\mu(\mathcal{A})$  is nonempty.*

*Proof.* Let  $E = \{\rho \in \mathcal{S}(\mathcal{A}) : \rho|_{\mathcal{B}} = \eta\}$ . Then  $E$  is a compact convex subset of  $\mathcal{S}(\mathcal{A})$  and the convolution map by  $\mu$  is an affine contraction on  $E$ . Hence by the Tikhonov (or Kakutani) fixed point theorem there is  $\tau \in E$  such that  $\mu * \tau = \tau$ . ■

**Corollary 4.3.** *Let  $\mathcal{A}$  be a  $\Gamma$ - $C^*$ -algebra, and let  $\mathcal{B} \leq \mathcal{A}$  be a  $\Gamma$ -invariant  $C^*$ -subalgebra. Suppose  $\mu \in \text{Prob}(\Gamma)$  and  $\tau \in \mathcal{S}(\mathcal{A})$  is unique  $\mu$ -stationary. Then  $\tau|_{\mathcal{B}} \in \mathcal{S}(\mathcal{B})$  is unique  $\mu$ -stationary for the action  $\Gamma \curvearrowright \mathcal{B}$ .*

Let  $\mathcal{A}$  be a  $\Gamma$ - $C^*$ -algebra. The Poisson map  $\mathcal{P}_\tau : \mathcal{A} \rightarrow \ell^\infty(\Gamma)$  associated to a state  $\tau$  on  $\mathcal{A}$  is defined by

$$\mathcal{P}_\tau(a)(g) = \langle g^{-1}a, \tau \rangle.$$

Poisson maps are unital, positive and  $\Gamma$ -equivariant. We observe the converse.

**Lemma 4.4.** *Suppose  $\varphi : \mathcal{A} \rightarrow \ell^\infty(\Gamma)$  is a unital positive  $\Gamma$ -equivariant map. Then there is  $\tau \in \mathcal{S}(\mathcal{A})$  such that  $\varphi = \mathcal{P}_\tau$ .*

*Proof.* Suppose  $\varphi$  is as above. Define the linear functional  $\tau$  on  $\mathcal{A}$  by  $\langle a, \tau \rangle = \varphi(a)(e)$  for all  $a \in \mathcal{A}$ . Since  $\varphi$  is positive and unital, it follows that  $\tau$  is a state on  $\mathcal{A}$ , and moreover

$$\mathcal{P}_\tau(a)(g) = \langle g^{-1}a, \tau \rangle = \varphi(g^{-1}a)(e) = g^{-1}(\varphi(a))(e) = \varphi(a)(g)$$

for all  $g \in \Gamma$  and  $a \in \mathcal{A}$ . ■

**Lemma 4.5.** *Suppose  $\mathcal{A}$  is a  $\Gamma$ - $C^*$ -algebra, and let  $\mu \in \text{Prob}(\Gamma)$ . Then a state  $\tau \in \mathcal{S}(\mathcal{A})$  is  $\mu$ -stationary if and only if  $\mathcal{P}_\tau(a) \in H^\infty(\Gamma, \mu)$  for every  $a \in \mathcal{A}$ .*

*Proof.* Suppose  $\tau$  is  $\mu$ -stationary. Then for every  $a \in \mathcal{A}$  and  $g \in \Gamma$  we have

$$\begin{aligned} \sum_{h \in \Gamma} \mu(h) \mathcal{P}_\tau(a)(gh) &= \sum_{h \in \Gamma} \mu(h) \langle h^{-1}g^{-1}a, \tau \rangle = \left\langle g^{-1}a, \sum_{h \in \Gamma} \mu(h) h\tau \right\rangle \\ &= \langle g^{-1}a, \tau \rangle = \mathcal{P}_\tau(a)(g), \end{aligned}$$

which shows  $\mathcal{P}_\tau(a) \in H^\infty(\Gamma, \mu)$ .



Conversely, suppose  $\mathcal{P}_\tau(a) \in H^\infty(\Gamma, \mu)$  for all  $a \in \mathcal{A}$ . Then

$$\left\langle a, \sum_{h \in \Gamma} \mu(h)h\tau \right\rangle = \sum_{h \in \Gamma} \mu(h)\mathcal{P}_\tau(a)(h) = \mathcal{P}_\tau(a)(e) = \langle a, \tau \rangle$$

for all  $a \in \mathcal{A}$ , which implies  $\sum_{h \in \Gamma} \mu(h)h\tau = \tau$ . ■

The following fundamental result, which is the noncommutative version of Theorem 2.1, follows from a more general result in the setting of affine pointed  $(\Gamma, \mu)$ -spaces in the sense of [8, Theorem 2.16]. We include a proof for the convenience of the reader.

**Theorem 4.6.** *Suppose  $(\mathcal{A}, \tau)$  is a separable  $(\Gamma, \mu)$ - $C^*$ -algebra. Then the weak\* limits  $\tau_\omega := \lim_n \omega_n \tau$  exists for  $\mathbb{P}_\mu$ -a.e. path  $\omega \in \Omega$ . Moreover, we have*

$$\tau = \int_\Omega \tau_\omega d\mathbb{P}_\mu(\omega) \tag{4}$$

in the weak\* sense. We call the states  $\tau_\omega$  conditional states.

*Proof.* The function  $f^a(g) := \langle a, g\tau \rangle$  is  $\mu$ -harmonic for every  $a \in \mathcal{A}$  by Lemma 4.5. Therefore, by the martingale convergence theorem, the limit  $\bar{f}^a(\omega) := \lim_n \langle a, \omega_n \tau \rangle$  exists for  $\mathbb{P}_\mu$ -a.e.  $\omega \in \Omega$ . Furthermore,  $\int_\Omega \bar{f}^a(\omega) d\mathbb{P}_\mu(\omega) = \tau(a)$  (cf. [8, Theorem 2.8]).

Now let  $\mathcal{D}$  be a countable dense subset of  $\mathcal{A}$ . By the above, there is a set  $\Omega_0$  of full measure such that  $\bar{f}^a(\omega)$  exists for every  $a \in \mathcal{D}$  and  $\omega \in \Omega_0$ . By density of  $\mathcal{D}$ , it follows that  $\bar{f}^a(\omega)$  exists for every  $a \in \mathcal{A}$  and  $\omega \in \Omega_0$ . Hence the conditional states  $\tau_\omega(a) := \lim_n \omega_n \tau(a) = \bar{f}^a(\omega)$ ,  $a \in \mathcal{A}$ , exist for every  $\omega = (\omega_n) \in \Omega_0$ , and furthermore (4) holds. ■

#### 4.2. Unique stationary actions

In classical dynamics, unique ergodicity (i.e. existence of a unique invariant measure on a compact space) is equivalent to uniform convergence of averages of continuous functions in Birkhoff’s ergodic theorem. For general nonamenable groups, instead, the appropriate notion is unique stationarity. Glasner–Weiss [28] proved that a  $(G, \mu)$ -space  $(X, \nu)$  is uniquely stationary if and only if for every  $f \in C(X)$  the averages  $\frac{1}{n} \sum_{k=0}^{n-1} \mu^k * f$  of convolutions converge uniformly to  $\int f d\nu$ .

**Proposition 4.7.** *An action  $(\Gamma, \mu) \curvearrowright (\mathcal{A}, \tau)$  is uniquely stationary if and only if*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} \mu^k * a - \tau(a)1_{\mathcal{A}} \right\| \xrightarrow{n \rightarrow \infty} 0 \tag{5}$$

for all  $a \in \mathcal{A}$ .

*Proof.* Let  $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \mu^k$ . Suppose (5) holds for all  $a \in \mathcal{A}$ , and let  $\eta \in \mathcal{S}_\mu(\mathcal{A})$ . Then

$$\langle a, \eta \rangle = \langle a, \mu_n * \eta \rangle = \langle \mu_n * a, \eta \rangle \rightarrow \langle a, \tau \rangle \langle 1_{\mathcal{A}}, \eta \rangle = \langle a, \tau \rangle$$

for all  $a \in \mathcal{A}$ . Hence  $\eta = \tau$ .

Conversely, suppose  $\tau$  is the unique  $\mu$ -stationary state on  $\mathcal{A}$ . Observe that for every  $a \in \mathcal{A}$  we have  $\|\mu_n * (\mu * a - a)\| \rightarrow 0$ . Thus, if we let  $V_0 = \overline{\text{span}}\{\mu * a - a : a \in \mathcal{A}\}$ , then  $\|\mu_n * b\| \rightarrow 0$  for all  $b \in V_0$ . Also by stationarity,  $\tau$  vanishes on  $V_0$ . Consequently, for  $t \in \mathbb{C}$  and  $b \in V_0$  we get

$$\|\mu_n * (t1_{\mathcal{A}} - b) - \langle t1_{\mathcal{A}} - b, \tau \rangle 1_{\mathcal{A}}\| = \|t1_{\mathcal{A}} - \mu_n * b - \langle t1_{\mathcal{A}}, \tau \rangle 1_{\mathcal{A}}\| = \|\mu_n * b\| \rightarrow 0,$$

which shows that (5) holds for every  $a \in V = \mathbb{C} \oplus V_0$ . Next, we show  $V = \mathcal{A}$ , which completes the proof of the theorem. For this, let  $\eta \in \mathcal{A}^*$  with  $\eta|_V = 0$ . Since  $\eta$  vanishes on  $V_0$  we see that  $\eta$  is  $\mu$ -stationary. It follows from uniqueness of the Jordan decomposition  $\eta = \eta_+ - \eta_-$  that both the positive part  $\eta_+$  and the negative part  $\eta_-$  of  $\eta$  are  $\mu$ -stationary, hence are multiples of  $\tau$  by unique stationarity. In particular,  $\eta$  is a multiple of  $\tau$ , and since  $\eta(1) = 0$  we have  $\eta = 0$ . Thus, it follows from the Hahn–Banach theorem that  $V = \mathcal{A}$ . ■

### 4.3. Inner actions: stationary states as generalizations of traces

An exclusive feature of noncommutative  $C^*$ -algebras is nontriviality of the inner action by their unitary groups. This allows one to consider a  $C^*$ -algebra  $\mathcal{A}$  as rather a  $C^*$ -dynamical system. In this point of view, traces on  $\mathcal{A}$  are nothing but invariant states, which may or may not exist in general. Thus, stationary states are generalizations of traces that do always exist, and in fact may be more appropriate objects to consider when the groups involved are nonamenable.

In this section, we prove some basic properties of stationary states in this setup, and see that they satisfy some useful properties of traces.

**Lemma 4.8.** *Suppose  $\pi$  is a unitary representation of  $\Gamma$ , and consider the action  $\Gamma \curvearrowright C_\pi^*(\Gamma)$  by inner automorphisms. Let  $\mu \in \text{Prob}(\Gamma)$  be generating, and suppose  $\tau$  is a  $\mu$ -stationary state on  $C_\pi^*(\Gamma)$ . Then the left kernel  $I_\tau = \{a \in C_\pi^*(\Gamma) : \tau(a^*a) = 0\}$  of  $\tau$  is a two-sided closed ideal of  $C_\pi^*(\Gamma)$ .*

*Proof.* The inequality  $a^*b^*ba \leq \|b^*b\|a^*a$  for operators on Hilbert spaces implies the well-known fact that the left kernel of any state is a left ideal. It is also obviously closed. We show  $I_\tau$  is also  $\Gamma$ -invariant. Let  $a \in I_\tau$ . Then

$$\begin{aligned} \sum_{g \in \Gamma} \mu(g)\tau((\pi(g^{-1})a\pi(g))^*(\pi(g^{-1})a\pi(g))) &= \sum_{g \in \Gamma} \mu(g)\tau(\pi(g^{-1})a^*a\pi(g)) \\ &= \tau(a^*a) = 0, \end{aligned}$$

which implies  $\tau((\pi(g^{-1})a\pi(g))^*(\pi(g^{-1})a\pi(g))) = 0$  for every  $g \in \text{Supp}(\mu)$ . This implies  $\pi(g^{-1})I_\tau\pi(g) \subset I_\tau$  for every  $g \in \text{Supp}(\mu)$ . Since  $\mu$  is generating the same is true for every  $g \in \Gamma$ . Thus, for every finite linear combination  $b = \sum_{i=1}^n t_i \pi(g_i) \in C_\pi^*(\Gamma)$ ,  $t_i \in \mathbb{C}$ , and every  $a \in I_\tau$ , there are  $a_1, \dots, a_n \in I_\tau$  such that

$$ab = \sum_{i=1}^n t_i a\pi(g_i) = \sum_{i=1}^n t_i \pi(g_i) a_i,$$

and the latter sum is in  $I_\tau$  since it is a left ideal. This shows  $I_\tau$  is also a right ideal. ■

Since every ideal of a  $C^*$ -algebra is invariant with respect to the inner action by the unitary group, the problem of simplicity of a  $C^*$ -algebra translates into a minimality problem for a noncommutative dynamical system. Hence connection to stationarity is expected.

**Proposition 4.9.** *Let  $\pi$  be a unitary representation of  $\Gamma$ . The  $C^*$ -algebra  $C_\pi^*(\Gamma)$  is simple if and only if there is a generating  $\mu \in \text{Prob}(\Gamma)$  such that every  $\mu$ -stationary state on  $C_\pi^*(\Gamma)$  is faithful.*

*Proof.* If  $C_\pi^*(\Gamma)$  is simple and  $\mu \in \text{Prob}(\Gamma)$  is generating, then by Lemma 4.8 every  $\mu$ -stationary state is faithful.

Conversely, suppose for some generating  $\mu \in \text{Prob}(\Gamma)$ , all  $\mu$ -stationary states are faithful. Assume for the sake of contradiction that  $C_\pi^*(\Gamma)$  has a nontrivial proper ideal  $I$ . Since every ideal is invariant under the inner action of  $\Gamma$ , the action  $\Gamma \curvearrowright C_\pi^*(\Gamma)$  induces an action  $\Gamma \curvearrowright C_\pi^*(\Gamma)/I$ . By Proposition 4.2 there exists a  $\mu$ -stationary state  $\tau$  on  $C_\pi^*(\Gamma)/I$ . Composing  $\tau$  with the canonical quotient map  $C_\pi^*(\Gamma) \rightarrow C_\pi^*(\Gamma)/I$  we obtain a  $\mu$ -stationary state on  $C_\pi^*(\Gamma)$  that vanishes on  $I$ , which contradicts the assumption. ■

We note that the condition of  $\mu$  being generating is only needed for one direction of the above. In fact, if  $\mathcal{A}$  is a unital  $C^*$ -algebra, and there is a countably supported probability measure  $\mu$  on the unitary group  $\mathcal{U}(\mathcal{A})$  such that every  $\mu$ -stationary state on  $\mathcal{A}$  is faithful, then  $\mathcal{A}$  is simple.

An important special case is the reduced  $C^*$ -algebra.

**Corollary 4.10.** *If  $\Gamma \curvearrowright C_\lambda^*(\Gamma)$  is uniquely stationary then  $\Gamma$  is  $C^*$ -simple.*

*Proof.* The canonical trace  $\tau_0$  is  $\Gamma$ -invariant, and hence  $\mu$ -stationary for any  $\mu$ . Recall also that  $\tau_0$  is faithful. The corollary now follows from Proposition 4.9. ■

In Section 5 we will prove the converse of this, which provides a new characterization of  $C^*$ -simplicity. But at this point some concrete examples are in order. In particular, we demonstrate how in general unique stationarity can be deduced.

Here and throughout the paper,  $\mathbb{F}_2$  denotes the free group on two generators  $a$  and  $b$ , and  $\partial\mathbb{F}_2$  denotes its Gromov boundary, which is a compact space naturally identified with the set of all infinite reduced words in the generators. We have the natural action  $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$  by concatenation with the subsequent cancellation of pairs of consecutive inverses.

**Example 4.11.** Let  $\mu \in \text{Prob}(\mathbb{F}_2)$  be the uniform measure on the set  $\{a, a^{-1}, b, b^{-1}\}$  of generators. We show the canonical trace  $\tau_0$  is the unique  $\mu$ -stationary state on  $C_\lambda^*(\mathbb{F}_2)$ . One can see that the “uniform measure” on  $\partial\mathbb{F}_2$ , given by  $\nu([w]) = \frac{1}{4 \cdot 3^n - 1}$  where  $w$  is a finite word of length  $n$  and  $[w]$  is the set of all infinite reduced words that start with  $w$ , is the unique  $\mu$ -stationary probability on  $\partial\mathbb{F}_2$ . Moreover,  $(\partial\mathbb{F}_2, \nu)$  is a  $\mu$ -boundary.

Now, let  $\tau$  be a  $\mu$ -stationary state on  $C_\lambda^*(\mathbb{F}_2)$ . By Proposition 4.2 we can extend  $\tau$  to a  $\mu$ -stationary state  $\tilde{\tau}$  on  $\mathbb{F}_2 \rtimes_r C(\partial\mathbb{F}_2)$ , where  $\mathbb{F}_2 \curvearrowright \mathbb{F}_2 \rtimes_r C(\partial\mathbb{F}_2)$  is also by inner automorphisms. Then  $\tilde{\tau}|_{C(\partial\mathbb{F}_2)}$  is stationary and by uniqueness, this restriction is  $\nu$ . Hence  $\tilde{\tau}_\omega|_{C(\partial\mathbb{F}_2)} = \delta_{\text{bnd}(\omega)}$  for a.e.  $\omega \in \Omega$ , where  $\text{bnd} : (\Omega, \mathbb{P}_\mu) \rightarrow (\partial\mathbb{F}_2, \nu)$  is the boundary map.

It is obvious that the action  $\mathbb{F}_2 \curvearrowright (\partial\mathbb{F}_2, \nu)$  is essentially free. Hence, it follows from Lemma 2.2 that for every nontrivial  $g \in \mathbb{F}_2$ ,  $\tau_\omega(\lambda_g) = 0$  for  $\mathbb{P}_\mu$ -a.e.  $\omega$ . Thus,  $\tau_\omega = \tau_0$  for  $\mathbb{P}_\mu$ -a.e. path  $\omega \in \Omega$ . Thus, applying Theorem 4.6 we get

$$\tau = \int_{\Omega} \tau_\omega d\mathbb{P}_\mu(\omega) = \tau_0,$$

which shows  $\tau_0$  is the unique  $\mu$ -stationary state on  $C_\lambda^*(\mathbb{F}_2)$ .

We note that in the above reasoning there is nothing particularly special about the uniform measure. The above conclusion holds for any generating measure on  $\mathbb{F}_2$ . In fact, there is nothing particularly special about the free group here, the conclusion holds for any measure  $\mu$  on any groups  $\Gamma$  that admits an essentially free  $\mu$ -USB.

**Theorem 4.12.** *Suppose  $\Gamma$  is a countable discrete group, and let  $\mu \in \text{Prob}(\Gamma)$ . If  $\Gamma$  admits an essentially free  $\mu$ -USB, then the canonical trace  $\tau_0$  on  $C_\lambda^*(\Gamma)$  is uniquely  $\mu$ -stationary.*

*Proof.* Repeat the argument given in Example 4.11 above on a metrizable model of the USB (such a model exists by Lemma 3.10). ■

In particular, any such group  $\Gamma$  is  $C^*$ -simple. Of course if  $(X, \nu)$  is an essentially free USB, by Theorem 3.11 the action  $\Gamma \curvearrowright \text{Supp}(\nu)$  is a topologically free topological boundary, hence  $\Gamma$  is  $C^*$ -simple by the results of [16]. However, for those groups with essentially free USB actions, the above theorem, besides giving a much simpler proof of  $C^*$ -simplicity, reveals more than just  $C^*$ -simplicity of  $\Gamma$ , namely, a probability  $\mu \in \text{Prob}(\Gamma)$  with respect to which the canonical trace is uniquely stationary. In Section 5 we will prove that the existence of such  $\mu$  is equivalent to  $C^*$ -simplicity of  $\Gamma$ , and we deduce various properties of boundaries and random subgroups associated to such measures.

Next we give an example of a faithful uniquely stationary state on a purely infinite  $C^*$ -algebra.

**Example 4.13.** Let  $\mathcal{A} = \mathbb{F}_2 \rtimes_r C(\partial\mathbb{F}_2)$ , and let  $\mu \in \text{Prob}(\mathbb{F}_2)$  be generating. As usual, let  $\tau_0$  denote the canonical trace on  $C_\lambda^*(\mathbb{F}_2)$ . Also, let  $\nu \in \text{Prob}(\partial\mathbb{F}_2)$  be the unique  $\mu$ -stationary probability on  $\partial\mathbb{F}_2$ . Now suppose  $\tau \in \mathcal{S}_\mu(\mathcal{A})$ . Then  $\tau|_{C(\partial\mathbb{F}_2)} = \nu$ , and by Example 4.11 above,  $\tau|_{C_\lambda^*(\mathbb{F}_2)} = \tau_0$ . Hence, we have  $\tau_\omega|_{C(\partial\mathbb{F}_2)} = \delta_{\mathbf{bnd}(\omega)}$  and  $\tau_\omega|_{C_\lambda^*(\mathbb{F}_2)} = \tau_0$  for a.e. path  $\omega \in \Omega$ . So for a linear combination  $\sum_{g \in \mathbb{F}_2} f_g \lambda_g$  where  $f_g \in C(\partial\mathbb{F}_2)$  is nonzero for at most finitely many  $g \in \Gamma$ , using the fact that  $\delta_{\mathbf{bnd}(\omega)}$  is multiplicative on  $C(\partial\mathbb{F}_2)$ , we see that

$$\left\langle \sum_{g \in \mathbb{F}_2} f_g \lambda_g, \tau_\omega \right\rangle = \sum_{g \in \mathbb{F}_2} f_g(\mathbf{bnd}(\omega)) \langle \lambda_g, \tau_\omega \rangle = \sum_{g \in \mathbb{F}_2} f_g(\mathbf{bnd}(\omega)) \tau_0(\lambda_g) = f_e(\mathbf{bnd}(\omega))$$

for a.e. path  $\omega \in \Omega$ . Thus, from (4) it follows that

$$\left\langle \sum_{g \in \mathbb{F}_2} f_g \lambda_g, \tau \right\rangle = \int_{\Omega} \left\langle \sum_{g \in \mathbb{F}_2} f_g \lambda_g, \tau_\omega \right\rangle d\mathbb{P}_\mu(\omega) = \int_{\Omega} f_e(\mathbf{bnd}(\omega)) d\mathbb{P}_\mu(\omega) = \int_{\partial\mathbb{F}_2} f_e d\nu.$$

Since the set of all finite linear combinations  $\sum_{g \in \mathbb{F}_2} f_g \lambda_g$  is dense in  $\mathcal{A}$ , this formula uniquely determines  $\tau$ . Note also  $\tau = \nu \circ \mathbb{E}$ , where  $\mathbb{E} : \mathcal{A} \rightarrow C(\partial\mathbb{F}_2)$  is the canonical conditional expectation  $\sum_{g \in \mathbb{F}_2} f_g \lambda_g \mapsto f_e$  (see e.g. [19]). Since both  $\mathbb{E}$  and  $\nu = \tau|_{C(\partial\mathbb{F}_2)}$  are faithful, so is  $\tau$ . In particular, this also implies the well-known fact that  $\mathbb{F}_2 \rtimes_r C(\partial\mathbb{F}_2)$  is simple (see e.g. [4]).

Similarly, normal stationary states with respect to the inner action by the unitary group of a von Neumann algebra can provide a suitable replacement for normal traces in the case of nonfinite von Neumann algebras.

**Proposition 4.14.** *Let  $M$  be a von Neumann algebra and  $\Gamma$  a group of unitaries in  $M$ . Suppose for the action  $\Gamma \curvearrowright M$  by inner automorphisms,  $M$  admits a faithful unique normal  $\mu$ -stationary state  $\tau$  for some  $\mu \in \text{Prob}(\Gamma)$ . Then  $M$  is a factor.*

*Proof.* Suppose that  $M$  is not a factor, and let  $p \in M$  be a nontrivial central projection. Let  $\mu \in \text{Prob}(\Gamma)$  and suppose that  $\tau \in M_*$  is a faithful normal  $\mu$ -stationary state. Set  $\tau_1 := \frac{1}{\tau(p)}\tau(\cdot p)$ . Then  $\tau_1 \in M_*$  is a normal state and using the fact that every central element is fixed by  $\Gamma$ , we get

$$\begin{aligned} \sum_{g \in \Gamma} \mu(g) \tau_1(g^{-1}a) &= \frac{1}{\tau(p)} \sum_{g \in \Gamma} \mu(g) \tau((g^{-1}a)p) = \frac{1}{\tau(p)} \sum_{g \in \Gamma} \mu(g) \tau(g^{-1}(ap)) \\ &= \frac{1}{\tau(p)} \tau(ap) = \tau_1(a) \end{aligned}$$

for all  $a \in M$ , which shows  $\tau_1$  is  $\mu$ -stationary. Similarly,  $\tau_2 := \frac{1}{\tau(1-p)}\tau(\cdot(1-p))$  is a normal  $\mu$ -stationary state, and obviously  $\tau = \tau(p)\tau_1 + \tau(1-p)\tau_2$ . But since  $\tau_1(1-p) = 0$ , we have  $\tau \neq \tau_1$ , hence  $\tau$  is not the unique normal  $\mu$ -stationary state on  $M$ . ■

**Example 4.15.** We follow the notations of Example 4.13. The von Neumann algebra crossed product  $M = \mathbb{F}_2 \rtimes L^\infty(\partial\mathbb{F}_2, \nu)$  is a type III factor. The set of all linear combinations  $\sum_{g \in \mathbb{F}_2} f_g \lambda_g$ , where  $f_g \in L^\infty(\partial\mathbb{F}_2, \nu)$  is nonzero for at most finitely many  $g \in \mathbb{F}_2$ , is weak\* dense in  $M$ . The map  $\sum_{g \in \mathbb{F}_2} f_g \lambda_g \mapsto f_e$  extends to a faithful normal conditional expectation  $\mathbb{E} : M \rightarrow L^\infty(\partial\mathbb{F}_2, \nu)$ . Thus  $\sum_{g \in \mathbb{F}_2} f_g \lambda_g \mapsto \int_{\partial\mathbb{F}_2} f_e d\nu$  defines a faithful normal state  $\tau$  on  $M$ . Note that the reduced  $C^*$ -crossed product  $\mathcal{A} = \mathbb{F}_2 \rtimes_r C(\partial\mathbb{F}_2)$  is an  $\mathbb{F}_2$ -invariant weak\* dense  $C^*$ -subalgebra. From Example 4.13 we know  $\tau|_{\mathcal{A}}$  is  $\mu$ -stationary, where  $\mu \in \text{Prob}(\mathbb{F}_2)$  is any generating probability. Since  $\tau$  is normal and  $\mathcal{A}$  is weak\* dense in  $M$ , it follows  $\tau$  is a  $\mu$ -stationary state on  $M$ . Moreover, if  $\tau'$  is another normal  $\mu$ -stationary state on  $M$ , then its restriction  $\tau'|_{\mathcal{A}}$  is again  $\mu$ -stationary, hence equal to  $\tau|_{\mathcal{A}}$  by the unique stationarity property established in Example 4.13. Since both  $\tau$  and  $\tau'$  are normal and  $\mathcal{A}$  is weak\* dense in  $M$ , it follows that  $\tau' = \tau$ , which implies unique stationarity of  $\tau$ .

**Example 4.16** (Noncommutative USB). Noncommutative Poisson boundaries were defined by Izumi [35]. This concept has found many important applications in various operator-algebraic contexts. Let us briefly recall the definition. Suppose  $M$  is a

von Neumann algebra, and  $\Phi : M \rightarrow M$  is a *Markov operator*, i.e. a unital completely positive normal map. Then the fixed point space  $\text{Fix}(\Phi) = \{x \in M : \Phi(x) = x\}$  is a unital self-adjoint weak\* closed subspace of  $M$ , and there is a positive contractive idempotent  $E : M \rightarrow \text{Fix}(\Phi)$ . Endowed with the *Choi–Effros product*,  $x \circ y := E(xy)$ , the space  $\text{Fix}(\Phi)$  becomes a von Neumann algebra, called the *Poisson boundary* of  $\Phi$ , and denoted by  $H_\infty(M, \Phi)$ .

A class of examples of Markov operators are obtained from canonical extensions of convolution operators, as follows. Let  $\mu \in \text{Prob}(\Gamma)$ , and define  $\Phi_\mu : \mathcal{B}(\ell^2(\Gamma)) \rightarrow \mathcal{B}(\ell^2(\Gamma))$  by  $\Phi_\mu(x) = \sum_{g \in \Gamma} \mu(g) \rho_g x \rho_{g^{-1}}$ , where  $\rho : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma))$  is the right regular representation. Then  $\Phi_\mu$  is a Markov map on  $\mathcal{B}(\ell^2(\Gamma))$ , and Izumi proved [36] that the Poisson boundary  $H_\infty(\mathcal{B}(\ell^2(\Gamma)), \Phi_\mu)$  is canonically isomorphic to the von Neumann crossed product  $\Gamma \ltimes H_\infty(\Gamma, \mu)$ .

Now, for instance, continuing to follow the notations of Example 4.13,  $M = \mathbb{F}_2 \ltimes L^\infty(\partial\mathbb{F}_2, \nu)$  is identified with the Poisson boundary of the Markov map  $\Phi_\mu$  where  $\mu$  is the uniform measure on the set of generators. Note also that  $M$  is the von Neumann algebra generated by the reduced crossed product  $\mathcal{A} = \mathbb{F}_2 \ltimes_r C(\partial\mathbb{F}_2)$  in the GNS representation of the unique stationary state  $\tau \in \mathcal{S}(\mathcal{A})$ . Thus, the  $C^*$ -dynamical system  $\Gamma \curvearrowright (\mathcal{A}, \tau)$  gives an example of a *noncommutative Poisson USB*.

**Example 4.17.** Suppose  $\pi : C_\lambda^*(\mathbb{F}_2) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  is an irreducible representation. Consider the inner action  $\Gamma \curvearrowright \mathcal{B}(\mathcal{H}_\pi)$  by unitaries  $\pi(\lambda_g)$ ,  $g \in \Gamma$ . Let  $\mu \in \text{Prob}(\mathbb{F}_2)$  be generating, in particular  $\text{Supp}(\mu)'' = \mathcal{B}(\mathcal{H}_\pi)$ . We show in this case  $\mathcal{B}(\mathcal{H}_\pi)$  does not admit any normal  $\mu$ -stationary state. Note first that simplicity of  $C_\lambda^*(\mathbb{F}_2)$  implies  $\pi$  is injective. Suppose  $\tau \in \mathcal{B}(\mathcal{H}_\pi)_*$  is a  $\mu$ -stationary state. Then  $\tau|_{\pi(C_\lambda^*(\mathbb{F}_2))}$  is  $\mu$ -stationary, hence equals the canonical trace  $\tau_0$  by Example 4.11. Since  $\tau$  is normal, and tracial on a weak\* dense subalgebra, it follows that  $\tau$  is a normal trace on  $\mathcal{B}(\mathcal{H}_\pi)$ , which implies  $\mathcal{H}_\pi$  is finite-dimensional. But that cannot be the case since  $\pi : C_\lambda^*(\mathbb{F}_2) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$  is injective.

### 5. A new characterization of $C^*$ -simplicity

In this section we prove a new characterization of  $C^*$ -simplicity of a group  $\Gamma$  in terms of unique stationarity of the action  $\Gamma \curvearrowright C_\lambda^*(\Gamma)$ .

The following more general statement makes the connection between these properties more explicit.

**Theorem 5.1.** *Let  $\mathcal{A}$  be a separable unital  $\Gamma$ - $C^*$ -algebra with a  $\Gamma$ -invariant state  $\tau$ . Then the following are equivalent:*

- (i) *The triple  $(\Gamma, \mathcal{A}, \tau)$  satisfies the Powers property: for every  $x \in \mathcal{A}$  and  $\varepsilon > 0$ , there exists  $\mu_0 \in \text{Prob}(\Gamma)$  such that  $\|\mu_0 * x - \tau(x)1\| < \varepsilon$ .*
- (ii) *There exists  $\mu \in \text{Prob}(\Gamma)$  such that  $\tau$  is the unique  $\mu$ -stationary state on  $\mathcal{A}$ .*

*Proof.* The implication (ii) $\Rightarrow$ (i) follows from Proposition 4.7.

The proof of (i) $\Rightarrow$ (ii) follows a similar construction to one in Kaimanovich–Vershik’s proof [40, Theorem 4.3] of Furstenberg’s conjecture.

Assume (i) holds. Choose an increasing sequence  $\{n_k\}$  of positive integers such that  $(\sum_{i=1}^k \frac{1}{2^i})^{n_k} < \frac{1}{2^k}$  for all  $k \in \mathbb{N}$ . Let  $\{a_i\}_{i \in \mathbb{N}}$  be a dense subset of the unit ball of  $\mathcal{A}$ . Using the above, for every  $l \in \mathbb{N}$  choose  $\mu_l$  inductively so that

$$\|\mu_l * \mu_{k_r} * \dots * \mu_{k_1} * a_s - \tau(a_s) \mathbb{1}_{\mathcal{A}}\| < \frac{1}{2^l}$$

for all  $1 \leq s, k_1, \dots, k_r < l$ , and  $r < n_l$ . Let  $\mu = \sum_{l=1}^{\infty} \frac{1}{2^l} \mu_l \in \ell^1(\Gamma)$ . Given any  $a$  in the unit ball of  $\mathcal{A}$  and  $\varepsilon > 0$ , let  $j \in \mathbb{N}$  be such that  $\|a - a_j\| < \varepsilon$  and  $1/2^j < \varepsilon$ . Then

$$\begin{aligned} \|\mu^{n_j} * a - \tau(a) \mathbb{1}_{\mathcal{A}}\| &\leq \|\mu^{n_j} * a - \mu^{n_j} * a_j\| + \|\mu^{n_j} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\| \\ &\quad + \|\tau(a_j) \mathbb{1}_{\mathcal{A}} - \tau(a) \mathbb{1}_{\mathcal{A}}\| \\ &< 2\varepsilon + \|\mu^{n_j} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\|. \end{aligned}$$

We expand and split the term  $\|\mu^{n_j} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\|$  as follows:

$$\begin{aligned} &\left\| \sum_{\max k_i \leq j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j + \sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j \right. \\ &\quad \left. - \left( \sum_{\max k_i \leq j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \right) \tau(a_j) \mathbb{1}_{\mathcal{A}} - \left( \sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \right) \tau(a_j) \mathbb{1}_{\mathcal{A}} \right\| \\ &\leq \left\| \sum_{\max k_i \leq j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} (\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}) \right\| \\ &\quad + \left\| \sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} (\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}) \right\| \\ &\leq 2\|a_j\| \sum_{\max k_i \leq j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} + \sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \|\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\| \\ &= 2\|a_j\| \left( \sum_{i=1}^j \frac{1}{2^i} \right)^{n_j} + \sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \|\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\| \\ &\leq 2\varepsilon + \sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \|\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\|. \end{aligned}$$

Now consider one of the terms  $\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j$  in the last sum above and let  $k_j$  be the first index such that  $k_j > j$ . Then we have

$$\|\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\| \leq \|\mu_{k_j} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\| < \varepsilon,$$

where the last inequality follows from the construction of  $\{\mu_l\}$ . This implies

$$\sum_{\max k_i > j} \frac{1}{2^{k_{n_j}} \dots 2^{k_1}} \|\mu_{k_{n_j}} * \dots * \mu_{k_1} * a_j - \tau(a_j) \mathbb{1}_{\mathcal{A}}\| < \varepsilon.$$

Hence we get  $\|\mu^{n_j} * a - \tau(a) \mathbb{1}_{\mathcal{A}}\| < 5\varepsilon$ . Since  $\|\mu\| = 1$ , this yields

$$\|\mu^n * a - \tau(a) \mathbb{1}_{\mathcal{A}}\| = \|\mu^{n-n_j} * \mu^{n_j} * a - \tau(a) \mathbb{1}_{\mathcal{A}}\| \leq \|\mu^{n_j} * a - \tau(a) \mathbb{1}_{\mathcal{A}}\| < 5\varepsilon$$

for all  $n > n_j$ . Hence

$$\|\mu^n * a - \tau(a) \mathbb{1}_{\mathcal{A}}\| \xrightarrow{n \rightarrow \infty} 0$$

for all  $a \in \mathcal{A}$ , which by Proposition 4.7 implies that  $\tau$  is the unique  $\mu$ -stationary state on  $\mathcal{A}$ . ■

Now, combining Theorem 5.1 with Haagerup’s characterization of  $C^*$ -simplicity in [30, Theorem 4.5], we get the following.

**Theorem 5.2.** *A countable discrete group  $\Gamma$  is  $C^*$ -simple if and only if there exists  $\mu \in \text{Prob}(\Gamma)$  such that the canonical trace  $\tau_0$  is the unique  $\mu$ -stationary state on  $C_\lambda^*(\Gamma)$  with respect to the  $\Gamma$ -action by inner automorphisms.*

*Proof.* Assume  $\Gamma$  is  $C^*$ -simple. Let  $f = \sum_{g \in \Gamma} f(g)\delta_g$  be a function on  $\Gamma$  with finite support, and fix  $\varepsilon_0 > 0$ . We denote by  $\lambda(f) = \sum_{g \in \Gamma} f(g)\lambda_g$  the left regular representation of  $f$ . By [30, Theorem 4.5] there are  $h_1, \dots, h_n \in \Gamma$  such that

$$\left\| \frac{1}{n} \sum_{k=1}^n \lambda_{h_k^{-1}gh_k} \right\| < \varepsilon_0$$

for all  $g \in \text{Supp } f \setminus \{e\}$ . We then have

$$\begin{aligned} \left\| \frac{1}{n} \sum_{k=1}^n \lambda_{h_k^{-1}} \lambda(f) \lambda_{h_k} - \tau_0(\lambda(f)) \mathbb{1}_{C_\lambda^*(\Gamma)} \right\| &= \left\| \sum_{g \in \Gamma} \frac{1}{n} \sum_{k=1}^n f(g) \lambda_{h_k^{-1}gh_k} - \tau_0(\lambda(f)) \lambda_e \right\| \\ &\leq \sum_{g \neq e} \left\| \frac{1}{n} \sum_{k=1}^n f(g) \lambda_{h_k^{-1}gh_k} \right\| + \frac{1}{n} \|(f(e) - \tau_0(\lambda(f))) \lambda_e\| \\ &= \sum_{g \neq e} |f(g)| \left\| \frac{1}{n} \sum_{k=1}^n \lambda_{h_k^{-1}gh_k} \right\| \\ &\leq \|f\|_\infty \sum_{g \in \text{Supp } f \setminus \{e\}} \left\| \frac{1}{n} \sum_{k=1}^n \lambda_{h_k^{-1}gh_k} \right\| < \|f\|_\infty \#\{\text{Supp } f\} \varepsilon_0. \end{aligned}$$

Now, let finitely supported functions  $f_1, \dots, f_j$  on  $\Gamma$ , and  $\varepsilon > 0$  be given. Let  $F = \bigcup_{i=1}^j \text{Supp } f_i$ , and  $c = \max_i \|f_i\|_\infty$ . Then, setting  $\varepsilon_0 = \varepsilon/(c\#F)$  in the above calculations, there are  $h_1, \dots, h_n \in \Gamma$  such that for  $\mu_0 = \frac{1}{n} \sum_{k=1}^n \delta_{h_k} \in \text{Prob}(\Gamma)$  we have

$$\|\mu_0 * \lambda(f_i) - \tau_0(\lambda(f_i)) \mathbb{1}_{C_\lambda^*(\Gamma)}\| < \varepsilon$$

for all  $i = 1, \dots, j$ . Since the set  $\{\lambda(f) : f \text{ has finite support}\}$  is norm-dense in  $C_\lambda^*(\Gamma)$ , for any given  $a_1, \dots, a_j \in C_\lambda^*(\Gamma)$  and  $\varepsilon > 0$ , we may find  $\mu_0 \in \text{Prob}(\Gamma)$  such that

$$\|\mu_0 * a_i - \tau_0(a_i) \mathbb{1}_{C_\lambda^*(\Gamma)}\| < \varepsilon$$



for all  $i = 1, \dots, j$ . Hence, by Theorem 5.1, there is  $\mu \in \text{Prob}(\Gamma)$  such that  $\tau_0$  is the unique  $\mu$ -stationary state on  $C_\lambda^*(\Gamma)$ .

The converse follows from Corollary 4.10. ■

### 5.1. $C^*$ -simple measures

In this section we prove several properties of the measures  $\mu$  that “capture”  $C^*$ -simplicity in the sense of Theorem 5.2. We see that these measures possess significant ergodic-theoretical properties. Thus, in some sense, one should consider  $C^*$ -simplicity of  $\Gamma$  as a property of the measure(s)  $\mu \in \text{Prob}(\Gamma)$  in Theorem 5.2.

**Definition 5.3.** We say that a measure  $\mu \in \text{Prob}(\Gamma)$  is  $C^*$ -simple if the canonical trace  $\tau_0$  is the unique  $\mu$ -stationary state on  $C_\lambda^*(\Gamma)$ .

**Theorem 5.4.** Suppose  $\mu \in \text{Prob}(\Gamma)$  is  $C^*$ -simple. Then any measurable  $\mu$ -stationary action with almost surely amenable stabilizers is essentially free.

In particular, any Zimmer-amenable  $\mu$ -stationary action (e.g. the Poisson boundary action  $\Gamma \curvearrowright (\Pi_\mu, \nu_\infty)$ ) is essentially free.

*Proof.* Suppose  $\mu \in \text{Prob}(\Gamma)$  is  $C^*$ -simple, and let  $\Gamma \curvearrowright (X, \nu)$  be a measurable  $\mu$ -stationary action such that  $\text{Stab}_\Gamma(x)$  is amenable for  $\nu$ -almost every  $x \in X$ .

Consider the map  $\Psi : X \rightarrow \text{Sub}(\Gamma)$  defined by  $\Psi(x) = \text{Stab}_\Gamma(x)$ . Then  $\eta = \Psi_*\nu$  is an amenable  $\mu$ -SRS of  $\Gamma$ . Therefore by Lemma 2.3 there is a state  $\tau$  on  $C_\lambda^*(\Gamma)$  such that  $\tau(\lambda_g) = \eta(\{\Lambda : g \in \Lambda\})$  for all  $g \in \Gamma$ . Since  $\eta(\{\Lambda : g \in \Lambda\}) = \nu(\{x : g \in \text{Stab}_\Gamma(x)\}) = \nu(\text{Fix}(g))$ , and since  $\nu$  is  $\mu$ -stationary, it follows that

$$\begin{aligned} \sum_{g \in \Gamma} \mu(g)\tau(\lambda_{g^{-1}}\lambda_h\lambda_g) &= \sum_{g \in \Gamma} \mu(g)\nu(\text{Fix}(g^{-1}hg)) = \sum_{g \in \Gamma} \mu(g)\nu(g^{-1}\text{Fix}(h)) \\ &= \sum_{g \in \Gamma} \mu(g)g\nu(\text{Fix}(h)) = \nu(\text{Fix}(h)) = \tau(\lambda_h), \end{aligned}$$

which shows  $\tau$  is  $\mu$ -stationary. Hence,  $\tau = \tau_0$ , which yields  $\nu(\text{Fix}(g)) = 0$  for all non-trivial  $g \in \Gamma$ , i.e.  $\Gamma \curvearrowright (X, \nu)$  is essentially free. ■

Essential freeness of the abstract Poisson boundary has some ergodic-theoretical consequences, for example it implies genericity of stationary measures (see [14]).

The following corollary is a weaker conclusion of Theorem 5.4 at the topological level.

**Corollary 5.5** (see also [16, Proposition 7.6 & Remark 7.7]). Suppose  $\Gamma$  is  $C^*$ -simple. Then any minimal action  $\Gamma \curvearrowright X$  on a compact space with amenable stabilizers is topologically free, that is, the set  $\{x \in X : \text{Stab}_\Gamma(x) \text{ is trivial}\}$  is dense in  $X$ .

*Proof.* Since by Proposition 4.2 every compact  $\Gamma$ -space admits a stationary measure, it follows from Theorem 5.4 that there is some  $x \in X$  which has trivial stabilizer, and so does every point in its orbit. Now the assertion follows from minimality. ■

Also, Theorems 5.4 and 4.12 imply the following measurable version of the main result of [41].

**Corollary 5.6.** *If  $\Gamma$  is  $C^*$ -simple then  $\Gamma$  admits an essentially free measurable boundary action, and, conversely,  $\Gamma$  is  $C^*$ -simple if it admits an essentially free  $\mu$ -USB for some  $\mu \in \text{Prob}(\Gamma)$ .*

In [7] Bader, Duchesne and Lecureux proved that every amenable IRS of a group  $\Gamma$  is supported on its amenable radical  $\text{Rad}(\Gamma)$ . Consequently,  $\text{Rad}(\Gamma)$  is trivial if and only if  $\delta_{\{e\}}$  is the unique invariant probability measure on  $\text{Sub}_a(\Gamma)$ ; or that  $\Gamma$  has the unique trace property if and only if  $\Gamma \curvearrowright \text{Sub}_a(\Gamma)$  is uniquely ergodic. Thus, the following is a more concrete evidence that the difference between the unique trace property and  $C^*$ -simplicity is indeed the difference between unique ergodicity and unique stationarity.

**Corollary 5.7.** *If  $\mu$  is a  $C^*$ -simple measure on  $\Gamma$  then  $(\Gamma, \mu) \curvearrowright (\text{Sub}_a(\Gamma), \delta_{\{e\}})$  is uniquely  $\mu$ -stationary.*

*Conversely, if  $\Gamma$  is not  $C^*$ -simple then  $(\Gamma, \mu) \curvearrowright (\text{Sub}_a(\Gamma), \delta_{\{e\}})$  is never uniquely stationary.*

*Proof.* Suppose that  $\mu$  is  $C^*$ -simple, and let  $\eta$  be an amenable  $\mu$ -SRS of  $\Gamma$ . As shown in the proof of Theorem 5.4, the function  $g \mapsto \eta(\{\Lambda : g \in \Lambda\})$  extends to a  $\mu$ -stationary state  $C_\lambda^*(\Gamma)$ . Thus, by unique stationarity of the canonical trace, we get

$$\eta(\{\Lambda : g \in \Lambda\}) = 0$$

for every nontrivial  $g \in \Gamma$ . Hence

$$\eta(\text{Sub}(\Gamma) \setminus \{e\}) = \eta\left(\bigcup_{g \neq e} \{\Lambda : g \in \Lambda\}\right) \leq \sum_{g \neq e} \eta(\{\Lambda : g \in \Lambda\}) = 0,$$

which implies  $\eta = \delta_{\{e\}}$ .

Conversely, if  $\Gamma$  is not  $C^*$ -simple, then by [42, Theorem 1.1],  $\Gamma$  has a nontrivial amenable URS (that is, a minimal subset of  $\text{Sub}_a(\Gamma)$  which is not the fixed point  $\{e\}$ ). But any URS supports a  $\mu$ -stationary probability for any  $\mu \in \text{Prob}(\Gamma)$ . Hence  $\delta_{\{e\}}$  is not unique stationary on  $\text{Sub}_a(\Gamma)$  for any  $\mu \in \text{Prob}(\Gamma)$ . ■

**Remark 5.8.** Note that since for any  $\mu \in \text{Prob}(\Gamma)$  every URS supports a  $\mu$ -SRS, it follows from the above corollary that every amenable URS of a  $C^*$ -simple group  $\Gamma$  is trivial. This is one direction of one of the main results of [42]. In the proof above, we are using the other direction of that result.

It is natural to ask whether a generalization of the result of Bader–Duchesne–Lecureux about amenable IRS, which was mentioned above, holds for stationary random subgroups. It is evident that the argument presented in [7, Theorem 1.4] cannot be extended to the stationary case, and in fact any non $C^*$ -simple group  $\Gamma$  with trivial amenable radical has a nontrivial amenable SRS. But rephrasing the above corollary, it implies  $\Gamma$  is  $C^*$ -simple if and only if there is  $\mu \in \text{Prob}(\Gamma)$  such that every amenable  $\mu$ -SRS of  $\Gamma$  is trivial.

### 6. Freeness of USB: identifying $C^*$ -simple measures

Our proof of the existence of  $C^*$ -simple measures (Theorem 5.2) is not completely constructive. But, in light of the results of Section 5.1, it is natural to ask for concrete examples of  $C^*$ -simple measures. In this section we present several approaches to proving that a given measure is  $C^*$ -simple.

#### 6.1. Noetherian actions

For a probability space  $(X, \nu)$  we denote by  $MALG(\nu)$  its *measure algebra*, that is, the Boolean algebra of equivalence classes of measurable sets modulo  $\nu$ -null sets. It is a partially ordered set with respect to inclusion, and if  $\Gamma$  acts on  $(X, \nu)$  then it clearly acts on  $MALG(\nu)$ .

**Definition 6.1.** Let  $\Gamma \curvearrowright (X, \nu)$  be a measurable action. We say that a collection  $\mathcal{F} \subset MALG(\nu)$  is a  $\Gamma$ -*Noetherian lower semilattice* (NLS) if  $\mathcal{F}$  is  $\Gamma$ -invariant, closed under intersections, and any descending chain  $Y_1 \geq Y_2 \geq \dots$  in  $\mathcal{F}$  stabilizes.

The main result of this section is the following theorem, which is a generalization of the well-known fact that (noninvariant) ergodic stationary actions are atomless, and also of [39, Lemma 2.2.2]. We are grateful to Uri Bader for suggesting this direction.

**Theorem 6.2** (0-1 law for Noetherian actions). *Let  $\Gamma \curvearrowright X$ , and let  $\mu \in \text{Prob}(\Gamma)$  be generating. Suppose  $\nu \in \text{Prob}(X)$  is an ergodic  $\mu$ -stationary measure such that  $(X, \nu)$  has no nontrivial finite factor. If  $\mathcal{F}$  is a  $\Gamma$ -NLS collection in  $MALG(\nu)$ , then  $\nu(Y) \in \{0, 1\}$  for any  $Y \in \mathcal{F}$ .*

*Proof.* We prove the theorem by Noetherian induction with respect to inclusion.

Let  $Y \in \mathcal{F}$  and assume  $\nu(Z) \in \{0, 1\}$  for any  $Z < Y$ , and for the sake of contradiction suppose  $0 < \nu(Y) < 1$ . Then  $\nu(Z) = 0$  for any  $Z < Y$ , and in particular, for any  $g \in \Gamma$ , either  $gY = Y$  or  $\nu(Y \cap gY) = 0$ . By nonsingularity of  $\nu$  it moreover follows that for any  $g, h \in \Gamma$  either  $gY = hY$  or  $\nu(gY \cap hY) = 0$ . Therefore, by ergodicity, we have  $1 = \nu(\bigcup_g gY) = \sum_{[g]} \nu(gY)$ , where  $[g]$  is the equivalence class of all  $h$  such that  $hY = gY$ . In particular, the set of numbers  $\{\nu(gY)\} \subset [0, 1]$  has a maximum. Observe that these numbers are the values of a bounded harmonic function (the Poisson map of  $\mathbb{1}_Y$ ). But a bounded harmonic function with a maximum must be constant. Hence there are finitely many  $g_1, \dots, g_n \in \Gamma$  such that  $X = \bigsqcup_{i=1}^n g_i Y$ , and  $\nu(g_i Y) = 1/n$  for every  $i = 1, \dots, n$ . Since  $\nu(Y) < 1$  we must have  $n > 1$ . But this means  $(X, \nu)$  has a nontrivial finite factor, which contradicts the assumptions. ■

It is well known that  $\mu$ -boundaries do not admit nontrivial invariant factors and hence we get the following.

**Corollary 6.3.** *Let  $(X, \nu)$  be a  $\mu$ -boundary and assume there is a  $\Gamma$ -NLS  $\mathcal{F} \subset MALG(\nu)$  such that  $\text{Fix}(g) \in \mathcal{F}$  for all  $g \in \Gamma$ . Then every  $g \in \Gamma$  acts on  $X$  either trivially or essentially freely. In particular, if  $\Gamma \curvearrowright (X, \nu)$  is faithful, it is essentially free.*

**Example 6.4** (Algebraic actions). Let  $k$  be a local field, and let  $\mathbf{G}(k) \curvearrowright \mathbf{V}(k)$  be an algebraic action of a  $k$ -algebraic group on a  $k$ -variety. Let  $\Gamma \leq \mathbf{G}(k)$  be a countable subgroup, and let  $\mathcal{F}$  denote the family of all subvarieties of  $\mathbf{V}(k)$ . Then  $\mathcal{F}$  is a  $\Gamma$ -NLS. Let  $\mu \in \text{Prob}(\Gamma)$  be generating, and suppose  $\nu \in \text{Prob}(\mathbf{V}(k))$  is such that  $(\mathbf{V}(k), \nu)$  is a  $\mu$ -boundary. Then by Theorem 6.2 every subvariety has trivial  $\nu$ -measure. Furthermore, since the action is algebraic,  $\text{Fix}(g) \in \mathcal{F}$  for all  $g \in \Gamma$  and so by Corollary 6.3 every faithful  $\mu$ -boundary algebraic action is essentially free.

Recall that a  $\Gamma$ -space  $X$  is said to be *mean-proximal* if  $X$  is a  $\mu$ -USB for all generating  $\mu \in \text{Prob}(\Gamma)$ . We say that a mean-proximal space  $X$  is *essentially free* if for every generating  $\mu \in \text{Prob}(\Gamma)$  the action  $\Gamma \curvearrowright (X, \nu_\mu)$  is essentially free, where  $\nu_\mu \in \text{Prob}(X)$  is the unique  $\mu$ -stationary measure. In the following, to conclude  $C^*$ -simplicity of the measures in question, we construct a mean-proximal space and verify the essential freeness using the 0-1 law above.

In the proof of the following theorem, we need at a certain step to extend an essentially free mean-proximal action of a finite index normal subgroup. We prove, jointly with Uri Bader, the required extension results for mean-proximal actions in Appendix A.

**Theorem 6.5.** *Let  $\Gamma$  be a finitely generated linear group with trivial amenable radical. Then any generating measure on  $\Gamma$  is  $C^*$ -simple.*

*Proof.* Let  $\mathbf{H}$  denote the Zariski closure of  $\Gamma$  and denote by  $\mathbf{H}^0 \triangleleft \mathbf{H}$  the connected component of the identity. Since  $\text{Rad}(\Gamma) = \{e\}$  we may assume that  $\mathbf{H}^0$  is also semisimple. Now let  $\Gamma^0 = \Gamma \cap \mathbf{H}^0$ . Then  $\Gamma^0 \triangleleft \Gamma$  and  $[\Gamma : \Gamma^0] < \infty$ . By Theorem A.4 in Appendix A, it is enough to find an essentially free metrizable  $\Gamma^0$ -mean-proximal space. To construct such a space, we show that for any given  $g \neq e \in \Gamma^0$ , there exists a metrizable  $\Gamma^0$ -mean-proximal space  $X_g$  on which  $g$  acts essentially freely. Then the product of all  $X_g$  is the desired essentially free  $\Gamma^0$ -mean-proximal space, by Lemma A.1.

Fix some  $g \neq e \in \Gamma^0$ . Observe that  $\text{Rad}(\Gamma^0) = \{e\}$  and hence we can find an element  $h$  in the normal closure of  $g$  with an eigenvalue  $\alpha$  which is not a root of unity. Let  $R_1$  be the finitely generated ring generated by the entries of the matrices of  $\Gamma^0$ , and let  $R_2$  be the finitely generated ring generated by the polynomials that define  $\mathbf{H}^0$ . So  $R = \langle R_1, R_2, \alpha \rangle$  is a finitely generated ring and  $I = \{\alpha^z\}_{z \in \mathbb{Z}} \subset R$  is infinite. By a result of Breuillard and Gelander [15, Lemma 2.1] (see also [52, Lemma 4.1]) we can find an embedding  $R \subset k$ , where  $k$  is a local field, such that  $I \subset k$  is unbounded. Consider  $\Gamma^0$  as a subgroup of  $\text{GL}_n(k)$ . Then  $h^z \in \Gamma^0$  has an eigenvalue whose absolute value is greater than 1 for some  $z \in \mathbb{Z}$ . In particular,  $\Gamma^0$  is not relatively compact in  $\mathbf{H}^0(k)$ . Thus,  $\Gamma^0$  is Zariski dense in the  $k$ -group  $\mathbf{H}^0$ , which is connected and semisimple. Hence, by the results of Margulis [46, Lemmas IV.4.4 and IV.4.5], there exists an irreducible representation  $\pi : \mathbf{H}^0(k) \rightarrow \text{GL}_m(k)$  such that  $\pi(h^z)$  is proximal for some  $z \in \mathbb{Z}$ . Thus we may apply another result of Margulis [46, Theorem IV.3.7] to conclude that the projective space  $\mathbb{P}(k^m)$  is  $\Gamma^0$ -mean-proximal, and as  $\pi(h^z)$  is proximal, it acts nontrivially. Since  $h^z$  is in the normal closure of  $g$ , it follows that  $g$  acts nontrivially on  $\mathbb{P}(k^m)$ . Now if  $\mu \in \text{Prob}(\Gamma^0)$  is generating, and

$\nu \in \text{Prob}(\mathbb{P}(k^m))$  is the unique  $\mu$ -stationary measure, then by the 0-1 law (Theorem 6.2) and Example 6.4,  $g$  acts  $\nu$ -essentially freely on  $\mathbb{P}(k^m)$ . ■

6.2. Groups with countably many amenable subgroups

As mentioned before, there are many results on concrete realization of the Poisson boundary on a natural USB. Taking advantage of this, in the following we use some of the main results in this context to provide further examples of  $C^*$ -simple measures.

In [14] it is shown that if a group has only countably many amenable subgroups, then the action on the (abstract) Poisson boundary is essentially free for any generating  $\mu$ . Thus any generating measure  $\mu$  on such a group, for which there exists a  $\mu$ -Poisson USB, is a  $C^*$ -simple measure.

A good source of examples of groups with countably many amenable subgroups is the class of groups satisfying a “finitely generated” version of the Tits alternative. By that we mean any subgroup that does not contain a free subgroup, is a finitely generated amenable group. Examples of such groups are mapping class groups [12, 34, 47] and  $\text{Out}(\mathbb{F}_n)$  [10, 11].

**Example 6.6.** Let  $\Gamma$  be hyperbolic group and let  $\Lambda \leq \Gamma$  be a nonelementary subgroup with trivial amenable radical. Then the Gromov boundary of  $\Gamma$  is an essentially free  $\mu$ -USB for any generating measure  $\mu$  on  $\Lambda$  [38]. Hence, any generating measure on  $\Lambda$  is  $C^*$ -simple.

**Theorem 6.7.** *Let  $\Gamma$  be a mapping class group.*

- (1) *If  $\text{Rad}(\Gamma) = \{e\}$  then any generating  $\mu \in \text{Prob}(\Gamma)$  is  $C^*$ -simple.*
- (2) *Let  $\Lambda \leq \Gamma$  be a nonelementary subgroup with  $\text{Rad}(\Lambda) = \{e\}$ . Then any generating measure on  $\Lambda$  with finite entropy and finite logarithmic moment is  $C^*$ -simple.*

*Proof.* (1) By the results of [39], the Thurston boundary is mean-proximal and the unique stationary measure (for any generating  $\mu$ ) is supported on the minimal projective foliations (in fact, on the uniquely ergodic ones). By [43, Theorem 1.4], any quasi-invariant probability which is supported on the minimal projective foliations is Zimmer-amenable. Since stationary measures are quasi-invariant, we conclude that the stabilizer of a.e. point is amenable. Since there are only countably many amenable subgroups, the Thurston boundary is indeed an essentially free mean-proximal space.

(2) In [39] it is proved that under these assumptions, the Thurston boundary is a Poisson USB of  $\Lambda$ . In particular, the stabilizers are a.e. amenable and hence, as above, the Thurston boundary is an essentially free USB. ■

Note that in (1) we do not know that the  $\mu$ -boundaries are Poisson boundaries, hence we need other tools to verify the amenability of the stabilizers.

**Theorem 6.8.** *Let  $\Gamma = \text{Out}(\mathbb{F}_n)$  and let  $\Lambda \leq \Gamma$  be a nonelementary subgroup with  $\text{Rad}(\Lambda) = \{e\}$ . Then any generating, finitely supported measure on  $\Lambda$  is  $C^*$ -simple.*

*Proof.* By [33] the boundary of the outer space is a Poisson USB for any finitely supported measure on  $\Lambda$ . Hence the result follows similarly to part (2) in Theorem 6.7. ■

In particular, all nonelementary subgroups of a mapping class group or of  $\text{Out}(\mathbb{F}_n)$  are  $C^*$ -simple (a strengthening of the results of [17]).

We conclude this discussion by pointing out that if  $\mu \in \text{Prob}(\Gamma)$  admits a Poisson  $\mu$ -USB, then any  $\mu$ -boundary which is not the Poisson boundary, is not Zimmer-amenable by [48, Theorem 9.2]. Hence there is no abstract guarantee that the stabilizers would be amenable. This, in a way, highlights the importance of the tools presented in Section 6.1 to conclude essential freeness of measurable boundary actions.

### 7. Operator-algebraic superrigidity relative to subgroups

In this section we study unique stationarity and unique trace property of groups  $\Gamma$  relative to their subgroups  $\Lambda \leq \Gamma$ , and consequently derive several superrigidity results for  $\Gamma$  relative to  $\Lambda$ .

#### 7.1. Unique stationarity relative to subgroups

Recall that we have canonical inclusions  $C_\lambda^*(\Lambda) \subseteq C_\lambda^*(\Gamma)$  and  $C^*(\Lambda) \subseteq C^*(\Gamma)$ . We denote by  $\tau_0^\Gamma$  the canonical trace on both reduced and full  $C^*$ -algebras of  $\Gamma$ .

**Definition 7.1.** We say a pair  $(\Gamma, \Lambda)$  where  $\Lambda$  is a subgroup of  $\Gamma$  is  $\mu$ -stationary rigid if the canonical trace  $\tau_0^\Gamma$  is the unique  $\mu$ -stationary state on  $C^*(\Gamma)$  that restricts to the canonical trace  $\tau_0^\Lambda$  on  $C^*(\Lambda)$ .

Stationary rigidity of  $(\Gamma, \Lambda)$  entails strong rigidity properties for  $\Gamma$  relative to  $\Lambda$ .

**Theorem 7.2.** *Suppose  $(\Gamma, \Lambda)$  is a  $\mu$ -stationary rigid pair. Then a measurable  $\mu$ -stationary action  $\Gamma \curvearrowright (X, \nu)$  is essentially free if its restriction to  $\Lambda$  is essentially free.*

*Proof.* By Lemma 2.3 the function  $\phi(g) = \nu(\text{Fix}(g))$  is a pdf on  $\Gamma$ , hence extends to a state  $\tau$  on  $C^*(\Gamma)$ . Moreover, as shown in the proof of Theorem 5.4 the state  $\tau$  is  $\mu$ -stationary. Since the restriction  $\Lambda \curvearrowright (X, \nu)$  is essentially free, we have  $\nu(\text{Fix}(h)) = 0$  for all nontrivial  $h \in \Lambda$ , which means  $\phi|_\Lambda = \delta_e$ , and equivalently  $\tau|_{C^*(\Lambda)} = \tau_0^\Lambda$ . Hence,  $\tau = \tau_0^\Gamma$  by  $\mu$ -stationary rigidity of the pair  $(\Gamma, \Lambda)$ , which implies  $\phi = \delta_e$ , i.e.  $\Gamma \curvearrowright (X, \nu)$  is essentially free. ■

**Theorem 7.3.** *Suppose  $(\Gamma, \Lambda)$  is a  $\mu$ -stationary rigid pair. Then any nontrivial  $\mu$ -SRS of  $\Gamma$  intersect nontrivially with  $\Lambda$  with positive probability.*

*Proof.* Suppose  $\eta$  is a  $\mu$ -SRS of  $\Gamma$ . Then as explained in the proof of Theorem 7.2, the pdf  $g \mapsto \eta(\{\Lambda' : g \in \Lambda'\})$  extends to a  $\mu$ -stationary state  $\tau$  on  $C^*(\Gamma)$ . If  $\Lambda' \cap \Lambda = \{e\}$  for  $\eta$ -a.e.  $\Lambda' \in \text{Sub}(\Gamma)$ , then  $\eta(\{\Lambda' : h \in \Lambda'\}) = 0$  for every nontrivial  $h \in \Lambda$ , hence  $\tau|_{C^*(\Lambda)} = \tau_0^\Lambda$ . Since the pair  $(\Gamma, \Lambda)$  is  $\mu$ -stationary rigid it follows that  $\tau = \tau_0^\Gamma$ , which

implies  $\eta(\{\Lambda' : g \in \Lambda'\}) = 0$  for all nontrivial  $g \in \Gamma$ . As shown in the proof of Corollary 5.7, this implies  $\eta = \delta_{\{e\}}$ . ■

We may state a von-Neumann-algebraic relative superrigidity for a given stationary rigid pair  $(\Gamma, \Lambda)$  in the setting of unitary representations of  $\Gamma$  whose von Neumann algebras admit normal faithful stationary states. But since at this point we do not have a characterization of those von Neumann algebras, the significance of such rigidity result for applications is not clear, although by Example 4.15 the class of von Neumann algebras admitting normal faithful stationary states is strictly larger than the class of finite von Neumann algebras.

But using the fact that stationary states on  $C^*$ -algebras always exist, we obtain the following  $C^*$ -algebraic/representation-theoretical relative rigidity.

**Proposition 7.4.** *Suppose  $(\Gamma, N)$  is a  $\mu$ -stationary rigid pair, where  $N \triangleleft \Gamma$  is a normal subgroup. Let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a unitary representation. If  $\lambda_N$  is weakly contained in the restriction of  $\pi$  to  $N$ , then  $\lambda_\Gamma$  is weakly contained in  $\pi$ .*

*Proof.* Suppose  $\lambda_N$  is weakly contained in the restriction of  $\pi$  to  $N$ , i.e. there is a canonical surjective  $*$ -homomorphism  $C_\pi^*(N) \rightarrow C_\lambda^*(N)$ . In particular, the canonical trace  $\tau_0^N$  is continuous on  $C_\pi^*(N)$ . Since  $N$  is normal,  $C_\pi^*(N)$  is an invariant subalgebra of  $C_\pi^*(\Gamma)$  for the inner action by  $\Gamma$ . Thus, by Proposition 4.2 we can extend  $\tau_0^N$  to a  $\mu$ -stationary  $\rho \in \mathcal{S}(C_\pi^*(\Gamma))$ . Considering  $\rho$  as a state on  $C^*(\Gamma)$ , it is still  $\mu$ -stationary, and thus the assumption of  $\mu$ -stationary rigidity implies  $\rho = \tau_0^\Gamma$ . Hence the canonical trace  $\tau_0^\Gamma$  is continuous on  $C_\pi^*(\Gamma)$ , which implies there is a canonical surjective  $*$ -homomorphism  $C_\pi^*(\Gamma) \rightarrow C_\lambda^*(\Gamma)$ , and equivalently  $\lambda_\Gamma$  is weakly contained in  $\pi$ . ■

The following is the main result of this section: we prove that any group that admits an essentially free USB is stationary rigid relative to its co-amenable subgroups. Recall that a subgroup  $\Lambda \leq \Gamma$  is *co-amenable* if every affine action of  $\Gamma$  with a  $\Lambda$ -fixed point has a fixed point.

**Theorem 7.5.** *Let  $\mu \in \text{Prob}(\Gamma)$ . If  $\Gamma$  admits an essentially free  $\mu$ -USB, then  $(\Gamma, \Lambda)$  is a  $\mu$ -stationary rigid pair for every co-amenable subgroup  $\Lambda \leq \Gamma$ .*

*Proof.* Suppose  $\tau$  is a  $\mu$ -stationary state on  $C^*(\Gamma)$  such that  $\tau|_{C^*(\Lambda)} = \tau_0^\Lambda$ . Let  $\pi_\tau : C^*(\Gamma) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau))$  be the GNS representation of  $\tau$ . Then the Hilbert subspace  $L^2(C^*(\Lambda), \tau)$  is canonically isomorphic to  $L^2(C^*(\Lambda), \tau_0^\Lambda) = \ell^2(\Lambda)$ . Moreover,  $L^2(C^*(\Lambda), \tau)$  is invariant under  $\pi_\tau(h)$  for every  $h \in \Lambda$ , and  $\pi_\tau|_\Lambda : \Lambda \rightarrow \mathcal{B}(L^2(C^*(\Lambda), \tau))$  is unitarily equivalent to the left regular representation of  $\Lambda$ . Thus, the map  $\pi_\tau(h) \mapsto \lambda_h, h \in \Lambda$ , extends to a surjective  $*$ -homomorphism  $C_{\pi_\tau}^*(\Lambda) \rightarrow C_\lambda^*(\Lambda)$ . But, since the canonical trace  $\tau_0^\Lambda$  coincides with  $\tau$  on  $C_{\pi_\tau}^*(\Lambda)$ , and the latter is faithful, it follows that the canonical surjective  $*$ -homomorphism above is also injective. Hence, by Arveson’s Extension Theorem [5, Theorem 1.2.3], we may extend the map  $\lambda_h \mapsto \pi_\tau(h), h \in \Lambda$ , to a unital (completely) positive map  $\mathcal{B}(\ell^2(\Lambda)) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau))$ , which is then automatically  $\Lambda$ -equivariant with respect to inner actions on both sides.

Suppose  $(X, \nu)$  is an essentially free  $\mu$ -USB. Considering  $X$  as a  $\Lambda$ -space via the restriction actions, we have a unital positive  $\Lambda$ -equivariant map  $C(X) \rightarrow \ell^\infty(\Lambda)$ . Composing with the above map, we obtain a unital positive  $\Lambda$ -equivariant map  $C(X) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau))$ .

Let  $\mathcal{E} = \{\Phi : C(X) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau)) \mid \Phi \text{ is positive and unital}\}$ , and define a  $\Gamma$ -action on  $\mathcal{E}$  by  $(g \cdot \Phi)(f) := g(\Phi(g^{-1}f))$ . Then  $\mathcal{E}$  endowed with the point-weak\* topology (i.e.  $\Phi_i \rightarrow \Phi$  iff  $\Phi_i(f) \rightarrow \Phi(f)$  weak\* for every  $f \in C(X)$ ) is a compact convex  $\Gamma$ -space.

Observe that  $\Phi \in \mathcal{E}$  is a  $\Lambda$ -fixed point if and only if it is  $\Lambda$ -equivariant. So, by the above,  $\mathcal{E}$  contains a  $\Lambda$ -fixed point. Since  $\Lambda$  is a co-amenable subgroup,  $\mathcal{E}$  also contains a  $\Gamma$ -fixed point, which is a  $\Gamma$ -equivariant positive unital map  $\iota : C(X) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau))$ .

Using Proposition 4.2 we extend  $\tau$  to a  $\mu$ -stationary state  $\tilde{\tau}$  on  $\mathcal{B}(L^2(C^*(\Gamma), \tau))$ . Then  $\tilde{\tau} \circ \iota$  gives a  $\mu$ -stationary probability on  $C(X)$ , and therefore by the uniqueness assumption we have  $\tilde{\tau}|_{\iota(C(X))} = \nu \circ \iota$ . Hence  $\tilde{\tau}_\omega \circ \iota = (\tilde{\tau} \circ \iota)_\omega = \delta_{\mathbf{bnd}(\omega)}$  for a.e. path  $\omega \in \Omega$ , where  $\mathbf{bnd} : (\Omega, \mathbb{P}_\mu) \rightarrow (X, \nu)$  is the boundary map.

Let  $g \in \Gamma$  be nontrivial. Since the action  $\Gamma \curvearrowright (X, \nu)$  is essentially free,  $g \mathbf{bnd}(\omega) \neq \mathbf{bnd}(\omega)$  for a.e. path  $\omega \in \Omega$ . Consequently, for a.e.  $\omega \in \Omega$  there is  $f_\omega \in C(X)$ ,  $0 \leq f_\omega \leq 1$ , with  $\tilde{\tau}_\omega(\iota(f_\omega)) = f_\omega(\mathbf{bnd}(\omega)) = 1$  and  $\tilde{\tau}_\omega(\pi_\tau(g^{-1})\iota(f_\omega)\pi_\tau(g)) = f_\omega(g \mathbf{bnd}(\omega)) = 0$ . Hence, Lemma 2.2 implies  $\tau_\omega(\pi_\tau(g)) = \tilde{\tau}_\omega(\pi_\tau(g)) = 0$  for a.e. path  $\omega \in \Omega$ . Thus, applying Theorem 4.6 we get

$$\tau(\pi_\tau(g)) = \int_\Omega \tau_\omega(\pi_\tau(g)) d\mathbb{P}_\mu(\omega) = 0.$$

Hence  $\tau = \tau_0^\Gamma$ . ■

Applying the results of Section 6.2 we get the following.

**Corollary 7.6.** *Let  $\Gamma$  be a hyperbolic group, a mapping class group, or a finitely generated linear group, and assume that  $\text{Rad}(\Gamma) = \{e\}$ . Then for any co-amenable subgroup  $\Lambda \leq \Gamma$  the pair  $(\Gamma, \Lambda)$  is  $\mu$ -stationary rigid for any generating  $\mu$ .*

### 7.2. Unique trace property relative to subgroups

In this section we consider a relative unique ergodicity for the canonical trace. This should be considered as a relative character rigidity property.

**Definition 7.7.** We say a pair  $(\Gamma, \Lambda)$  of a group  $\Gamma$  and a subgroup  $\Lambda$  is *tracial rigid* if the canonical trace  $\tau_0^\Gamma$  on  $C^*(\Gamma)$  is the unique tracial extension of the canonical trace  $\tau_0^\Lambda$  on  $C^*(\Lambda)$ .

Recall that a von Neumann algebra is *finite* if it has a normal faithful trace.

**Theorem 7.8.** *Suppose  $(\Gamma, \Lambda)$  is a tracial rigid pair, and  $\Lambda$  is an icc group. Suppose  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  is a unitary representation such that  $M = \pi(\Gamma)''$  is a finite von Neumann algebra. If the restriction  $\pi|_\Lambda$  extends to an isomorphism  $\mathcal{L}(\Lambda) \cong \pi(\Lambda)''$  then  $\pi$  extends to an isomorphism  $\mathcal{L}(\Gamma) \cong M$ .*



*Proof.* Since  $\Lambda$  is icc,  $\pi(\Lambda)'' \cong \mathcal{L}(\Lambda)$  is a factor, and in particular admits a unique trace, which is the canonical trace  $\tau_0^\Lambda$ . Now suppose  $\tau$  is a normal trace on  $M$ . Then its restriction to  $C_\pi^*(\Gamma)$  is the canonical trace by tracial rigidity of the pair  $(\Gamma, \Lambda)$ . Since  $C_\pi^*(\Gamma)$  is weak\* dense in  $M$ , it follows that  $\tau = \tau_0^\Gamma$  is the unique trace on  $M$ , thus also faithful.

Now let  $\iota_\tau : M \rightarrow \mathcal{B}(L^2(M, \tau))$  denote the GNS map. Then the map  $\delta_g \mapsto \iota_\tau(\pi(g))$  extends to a unitary  $U_\tau$  from  $\ell^2(\Gamma)$  onto  $L^2(M, \tau)$ , and we have

$$U_\tau^* \pi(g) U_\tau \delta_h = U_\tau^* \pi(g) \iota_\tau(\pi(h)) = U_\tau^* \iota_\tau(\pi(gh)) = \delta_{gh} = \lambda_g \delta_h$$

for all  $g, h \in \Gamma$ . Hence  $\text{Ad}(U_\tau) : \mathcal{L}(\Gamma) \rightarrow M$  is the desired isomorphism. ■

**Theorem 7.9.** *Suppose  $(\Gamma, \Lambda)$  is a tracial rigid pair. Then a probability measure preserving the action  $\Gamma \curvearrowright (X, m)$  is essentially free if its restriction  $\Lambda \curvearrowright (X, m)$  is essentially free.*

*Equivalently, any nontrivial IRS of  $\Gamma$  intersects  $\Lambda$  nontrivially, with positive probability. In particular, every nontrivial normal subgroup  $N \triangleleft \Gamma$  intersects  $\Lambda$  nontrivially.*

*Proof.* For  $g \in \Gamma$  denote  $\text{Fix}(g) = \{x \in X : gx = x\}$ . Then the function  $g \mapsto m(\text{Fix}(g))$  is positive definite on  $\Gamma$  by Lemma 2.3, hence extends to a trace  $\tau$  on  $C^*(\Gamma)$ . Moreover, by invariance of  $m$  we get

$$m(\text{Fix}(hgh^{-1})) = m(h\text{Fix}(g)) = m(\text{Fix}(g))$$

for all  $g, h \in \Gamma$ , which implies  $\tau$  is a trace. If the restriction  $\Lambda \curvearrowright (X, m)$  is essentially free, then  $m(\text{Fix}(h)) = 0$  for every nontrivial  $h \in \Lambda$ , and equivalently  $\tau|_{C^*(\Lambda)} = \tau_0^\Lambda$ . Thus by the assumption of tracial rigidity of the pair  $(\Gamma, \Lambda)$  we get  $\tau = \tau_0^\Gamma$ , hence the action  $\Gamma \curvearrowright (X, m)$  is essentially free.

For the IRS formulation, it is well known that any given IRS is the push-forward of some measure preserving action via the stabilizer map  $x \mapsto \text{Stab}_\Gamma(x)$  (see for example [2]). ■

By working with topological boundaries instead of unique stationary measurable boundaries we are able to generalize Theorem 7.5 to the case of tracial pairs.

**Theorem 7.10.** *Suppose  $\Lambda \leq \Gamma$  is co-amenable. Then every tracial extension of the canonical trace  $\tau_0^\Lambda$  to  $C^*(\Gamma)$  is supported on  $\text{Rad}(\Gamma)$ .*

*In particular, if  $\Gamma$  has trivial amenable radical, then the pair  $(\Gamma, \Lambda)$  is tracial rigid.*

*Proof.* Suppose  $\tau$  is a trace on  $C^*(\Gamma)$  such that  $\tau|_{C^*(\Lambda)} = \tau_0^\Lambda$ . Denote by  $\pi_\tau : C^*(\Gamma) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau))$  the GNS representation of  $\tau$ .

Let  $g \notin \text{Rad}(\Gamma)$ . Then it follows from [24] that  $\Gamma$  admits a topological boundary action  $\Gamma \curvearrowright X$  on which  $g$  acts nontrivially. Now, an argument similar to the proof of Theorem 7.5 yields a  $\Gamma$ -equivariant unital positive map  $\iota : C(X) \rightarrow \mathcal{B}(L^2(C^*(\Gamma), \tau))$ . Extend the trace  $\tau \in \mathcal{S}(\pi_\tau(C^*(\Gamma)))$  to a state  $\rho$  on  $\mathcal{B}(L^2(C^*(\Gamma), \tau))$ . Let  $x \in X$  be such that  $gx \neq x$ , and choose  $f \in C(X)$ ,  $0 \leq f \leq 1$ , with  $f(x) = 1$  and  $f(gx) = 0$ . Consider the restriction  $\rho|_{\iota(C(X))}$  as a probability on  $X$ . Since  $\Gamma \curvearrowright X$  is a boundary action, there

is a net  $(g_i) \subset \Gamma$  such that  $g_i \rho|_{\iota(C(X))} \rightarrow \delta_x$ . By passing to a subnet if necessary, we may assume  $g_i \rho$  converges weak\* to a state  $\eta$  on  $\mathcal{B}(L^2(C^*(\Gamma), \tau))$ . Then  $\eta|_{\iota(C(X))} = \delta_x$ , and therefore  $\eta(\iota(f)) = f(x) = 1$  and  $\eta(\pi(g^{-1})\iota(f)\pi(g)) = \eta(\iota(g^{-1}f)) = f(gx) = 0$ . Hence Lemma 2.2 yields  $\tau(\pi(g)) = \eta(\pi(g)) = 0$ . This shows  $\tau = \tau_0^\Gamma$ . ■

**Corollary 7.11.** *Let  $\Gamma$  be a group with trivial amenable radical. Suppose  $\Lambda \leq \Gamma$  is co-amenable, and suppose  $\Lambda$  is a character rigid group. Then a probability measure preserving action of  $\Gamma$  on the standard Lebesgue space is essentially free if its restriction to  $\Lambda$  is ergodic.*

*Proof.* Any ergodic measure preserving action of a character rigid group is essentially free. Thus, the assertion follows immediately from Theorems 7.9 and 7.10. ■

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**Appendix A. Extending mean-proximal actions**

In this appendix we prove an extension theorem for mean-proximal actions that is needed in our proof of  $C^*$ -simplicity of generating measures on linear groups (Theorem 6.5). Our results below can be proven in more general forms, but we state them as we need them in this work.

Throughout this section,  $\Gamma$  is a countable discrete group. By a  $\Gamma$ -space we mean a compact space  $X$  on which  $\Gamma$  acts by homeomorphisms. A  $\Gamma$ -space  $X$  is said to be  $\Gamma$ -mean-proximal if for every generating  $\mu \in \text{Prob}(\Gamma)$ , there exists a unique  $\mu$ -stationary measure  $\nu_\mu \in \text{Prob}(X)$  such that  $(X, \nu_\mu)$  is a  $(\Gamma, \mu)$ -boundary (see Definition 3.3).

**Lemma A.1.** *Let  $\{X_j\}$  be a countable collection of metrizable  $\Gamma$ -mean-proximal spaces. Then  $\prod_j X_j$  equipped with the diagonal action is a  $\Gamma$ -mean-proximal space.*

*Proof.* Fix a generating  $\mu \in \text{Prob}(\Gamma)$ , and let  $(\Omega, \mathbb{P}_\mu)$  denote the path space of the  $(\Gamma, \mu)$ -random walk (see Section 2.1). For each  $j$ , let  $\pi_{j'} : \prod_j X_j \rightarrow X_{j'}$  be the corresponding projection map. Let  $\nu$  be a  $\mu$ -stationary probability on  $\prod_j X_j$ , and for each  $j$  denote by  $\nu^j$  the push-forward of  $\nu$  under  $\pi_j$ . Then  $\nu^j \in \text{Prob}(X_j)$  is  $\mu$ -stationary, hence by the uniqueness assumption, it is also a  $(\Gamma, \mu)$ -boundary. Since the collection  $\{X_j\}$  is countable, we can find  $\Omega_0 \subset \Omega$  with  $\mathbb{P}_\mu(\Omega_0) = 1$  such that for every  $\omega \in \Omega_0$ , the conditional measures  $(\nu^j)_\omega$  are Dirac measures for all  $j$  (see Furstenberg’s Theorem 2.1 for the definition). Since the projections  $\pi_j$  are equivariant, we observe that  $(\nu_\omega)^j = (\nu^j)_\omega$  for all  $j$  and  $\omega \in \Omega_0$ . Thus, it follows that  $\nu_\omega$  is a Dirac measure for all  $\omega \in \Omega_0$  (note that the product space is also metrizable). This implies  $(\prod_j X_j, \nu)$  is a  $(\Gamma, \mu)$ -boundary. The uniqueness of  $\nu$  follows from the fact that every stationary measure is completely determined by its conditional measures (Theorem 2.1), and that the conditional measures  $\nu_\omega$  project onto the Dirac measures  $(\nu^j)_\omega$ . ■

**Theorem A.2.** *Let  $\Gamma$  be a finitely generated group, and let  $X$  be a metrizable  $\Gamma$ -mean-proximal space. Then there exists a metrizable  $\Gamma$ -mean-proximal space  $Y$  such that  $X$  is a factor of  $Y$ , and the action  $\Gamma \curvearrowright Y$  extends to an action  $\text{Aut}(\Gamma) \curvearrowright Y$ .*

*Proof.* For  $\alpha \in \text{Aut}(\Gamma)$  let  $X_\alpha$  be a copy of  $X$  equipped with the action  $g \cdot x = \alpha^{-1}(g)x$ . Obviously  $X_\alpha$  is a metrizable  $\Gamma$ -mean-proximal for every  $\alpha \in \text{Aut}(\Gamma)$ . Since  $\Gamma$  is finitely generated,  $\text{Aut}(\Gamma)$  is countable, hence Lemma A.1 shows that  $Z = \prod_\alpha X_\alpha$  equipped with the diagonal action of  $\Gamma$  is mean-proximal. In particular,  $Z$  contains a unique minimal component  $Y$ , which is also  $\Gamma$ -mean-proximal, and hence a topological boundary by Theorem 3.11.

On the other hand, the group  $\text{Aut}(\Gamma)$  also acts naturally on  $Z$  by permuting the indices, namely  $(\beta \cdot z)_{\alpha_0} = z_{\beta^{-1}\alpha_0}$  for  $\beta, \alpha_0 \in \text{Aut}(\Gamma)$  and  $z = (z_\alpha)_{\alpha \in \text{Aut}(\Gamma)} \in Z$ . One observes the relation  $g \cdot \beta \cdot z = \beta \cdot \beta^{-1}(g) \cdot z$  for  $g \in \Gamma, \beta \in \text{Aut}(\Gamma)$ , and  $z \in Z$ . Thus, if  $Z' \subset Z$  is  $\Gamma$ -invariant, then so is  $\beta(Z')$  for all  $\beta \in \text{Aut}(\Gamma)$ . In particular,  $\beta(Y) = Y$  for every  $\beta \in \text{Aut}(\Gamma)$  by uniqueness. Hence the restriction to  $Y$  defines an action  $\text{Aut}(\Gamma) \curvearrowright Y$ .

Now fix  $g \in \Gamma$ , and let  $\beta_g \in \text{Aut}(\Gamma)$  be the inner automorphism  $\beta_g(h) = ghg^{-1}$ . Considering  $g$  and  $\beta_g$  as homeomorphisms on  $Z$  via the above actions of  $\Gamma$  and  $\text{Aut}(\Gamma)$ , straightforward calculations show the composition  $g^{-1}\beta_g : Z \rightarrow Z$  is equivariant with respect to the diagonal action of  $\Gamma$ .

In particular,  $g^{-1}\beta_g(Y) = Y$  by uniqueness. But since  $\Gamma \curvearrowright Y$  is a topological boundary action, it follows that  $g^{-1}\beta_g$  is the identity map on  $Y$ , which implies the restriction of the action  $\text{Aut}(\Gamma) \curvearrowright Y$  to  $\Gamma \leq \text{Aut}(\Gamma)$  (via inner automorphisms) coincided with the diagonal action of  $\Gamma$  when restricted to  $Y$ , and this completes the proof. ■

We will use the notion of recurrent subgroups in order to relate boundary actions of a group to its finite index subgroups. We recall the definition, and refer the reader to [26] for more details.

Let  $\mu \in \text{Prob}(\Gamma)$  be generating. A subgroup  $\Lambda \leq \Gamma$  is said to be  $\mu$ -recurrent if almost every path of the  $(\Gamma, \mu)$ -random walk passes through  $\Lambda$  (infinitely many times, automatically). In this case, there exists a hitting measure  $\theta \in \text{Prob}(\Lambda)$  such that the restriction map  $\ell^\infty(\Gamma) \rightarrow \ell^\infty(\Lambda)$  yields an isometric isomorphism between the algebras of bounded harmonic functions  $H^\infty(\Gamma, \mu)$  and  $H^\infty(\Lambda, \theta)$ .

**Lemma A.3.** *Let  $X$  be a metrizable  $\Gamma$ -space and let  $\Lambda \leq \Gamma$  be  $\mu$ -recurrent for a generating  $\mu \in \text{Prob}(\Gamma)$ . Then any  $\mu$ -stationary measure  $\nu \in \text{Prob}(X)$  is  $\theta$ -stationary, where  $\theta \in \text{Prob}(\Lambda)$  is the  $\mu$ -hitting measure. Moreover, if  $\nu \in \text{Prob}(X)$  is a  $(\Lambda, \theta)$ -boundary then it is also a  $(\Gamma, \mu)$ -boundary.*

*In particular, if  $(X, \nu)$  is a  $\theta$ -USB, then it is also a  $\mu$ -USB.*

*Proof.* For a proof of the first assertion see [32, Corollary 2.14]. The second part follows directly from the definitions. ■

We say a  $\Gamma$ -mean-proximal space  $X$  is *essentially free* if for any generating measure  $\mu \in \text{Prob}(\Gamma)$ , the action  $\Gamma \curvearrowright (X, \nu_\mu)$  is essentially free.

**Theorem A.4.** *Let  $\Gamma$  be a finitely generated group with trivial amenable radical, and let  $\Lambda$  be a normal subgroup of  $\Gamma$  of finite index. If  $\Lambda$  admits an essentially free metrizable mean-proximal action, then so does  $\Gamma$ .*

*Proof.* Let  $X$  be an essentially free metrizable  $\Lambda$ -mean-proximal space, and let  $Y$  be the  $\Gamma$ -space given by Theorem A.2 (considering  $\Gamma \leq \text{Aut}(\Lambda)$ ). Since  $Y$  is a  $\Lambda$ -extension of  $X$ , it is  $\Lambda$ -essentially free.

Since  $\Lambda$  has finite index in  $\Gamma$ , it is  $\mu$ -recurrent for any generating  $\mu \in \text{Prob}(\Gamma)$ , thus Lemma A.3 implies that  $Y$  is also  $\Gamma$ -mean-proximal. To see that  $Y$  is also  $\Gamma$ -essentially free, let  $\mu \in \text{Prob}(\Gamma)$  be generating, and let  $\theta \in \text{Prob}(\Lambda)$  be the corresponding hitting measure. Denote by  $\nu = \nu_\mu = \nu_\theta \in \text{Prob}(Y)$  the unique stationary measure.

Since  $\Lambda \curvearrowright (X, \nu)$  is essentially free, the stabilizers  $\text{Stab}_\Gamma(x)$  are finite subgroups of  $\Gamma$  for  $\nu$ -almost every  $x$ . Thus, the push-forward  $\eta$  of  $\nu$  under the stabilizer map  $x \mapsto \text{Stab}_\Gamma(x)$  is a stationary measure on the compact space  $\text{Sub}(\Gamma)$  (of all subgroups of  $\Gamma$ , endowed with the conjugate action of  $\Gamma$ ) that is supported on finite subgroups. Note that  $\Gamma$  has only countably many finite subgroups, hence  $\eta$  is invariant (see e.g. the discussion in Section 6.2), i.e. an IRS. But by [7] every amenable IRS is supported on the amenable radical of  $\Gamma$ , which is trivial by assumption. Hence  $\text{Stab}_\Gamma(x)$  is trivial for  $\nu$ -almost every  $x$ , and this finishes the proof. ■

**Remark A.5.** The assumption of metrizability was not necessary in Lemmas A.1 and A.3, and was only needed in Theorem A.2 to ensure metrizability of the space  $Y$ . In all conclusions, the general case can be reduced to the metrizable case by either passing to a metrizable model, or considering metrizable factors and taking an inverse limit. Since we only use the results of this appendix in situations where all spaces under consideration are metrizable, we did not take those extra steps and just tailored the statements to our particular purposes here. But, the fact that we can deduce these results for arbitrary products and inverse limits leads to a deeper point: there is a theory of mean-proximal actions parallel to those of proximal and strongly proximal actions. One can prove the existence of a universal action, and canonical extension results. We intend to devote a followup paper to a conceptual study of a class of topological dynamical systems that include all these examples.

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