J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1232

Published by EMS Press

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Nonuniqueness of minimizers for semilinear optimal control problems

Received February 11, 2020

Abstract. A counterexample to uniqueness of global minimizers of semilinear optimal control problems is given. The lack of uniqueness occurs for a special choice of the state-target in the cost functional. Our arguments also show that, for some state-targets, there exist local minimizers which are not global. When this occurs, gradient-type algorithms may be trapped by local minimizers, thus missing global ones. Furthermore, the issue of convexity of a quadratic functional in optimal control is analyzed in an abstract setting.

As a corollary of nonuniqueness of minimizers, a nonuniqueness result for a coupled elliptic system is deduced.

Numerical simulations have been performed illustrating the theoretical results.

We also discuss the possible impact of the multiplicity of minimizers on the turnpike property in long time horizons.

Keywords. Semilinear elliptic equations, nonuniqueness, global minimizer, lack of convexity, optimal control

1. Introduction

We produce a counterexample to uniqueness of optimal controls in semilinear control. Both the case of internal control and boundary control are considered. To fix ideas, we focus on the case of a quadratic functional and a semilinear governing state equation. However, our techniques are applicable to a wide range of optimal control problems governed by a nonlinear state equation.

1.1. Lack of uniqueness of minimizers

In the context of boundary control, we consider the control problem

$$\begin{gathered}\min_{u\in L^\infty(\partial B(0,R))} J(u), \quad \text{where}\\ J(u)=\frac{1}{2}\int_{\partial B(0,R)}|u|^2\,d\sigma(x)+\frac{\beta}{2}\int_{B(0,R)}|y-z|^2\,dx,\end{gathered} \tag{1.1}$$

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Mathematics Subject Classification (2020): Primary 93C20; Secondary 35J47, 35J61

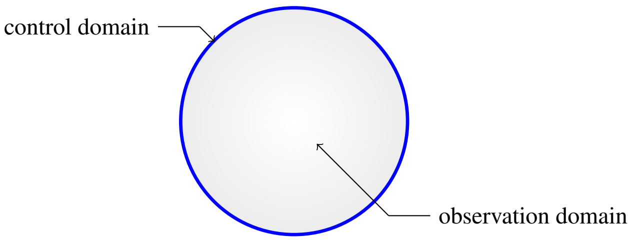


Fig. 1. Control and observation domains. The control domain is the boundary of the ball.

where $u = u(x)$ is the control and $y = y(x)$ is the associated state, a solution to the semilinear equation

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } B(0,R), \\ y = u & \text{on } \partial B(0,R). \end{cases} \tag{1.2}$$

Here $B(0,R)$ is the ball of $\mathbb{R}^n$ centered at the origin of radius R, with $n = 1,2,3$. The nonlinearity $f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ is strictly increasing and $f(0) = 0$. The target z is in $L^\infty(B(0,R))$ and $\beta > 0$ is a penalization parameter. As β increases, the distance between the optimal state and the target decreases.

In Appendix A we analyze the well-posedness of the state equation (1.2) and the existence of a global minimizer $\overline{u} \in L^\infty(\partial B(0,R))$ for the functional J defined above. As we shall see in the following result, for a special target, the global minimizer is not unique.

Theorem 1.1. *Consider the control problem* (1.1)–(1.2). *Assume, in addition,*

$$f''(y) \neq 0 \quad \forall y \neq 0. \tag{1.3}$$

There exists a target $z \in L^\infty(B(0,R))$ such that the functional J defined in (1.1) *admits* (*at least*) *two global minimizers.*

To give a first explanation of the above result, we introduce the *control-to-state map*

$$G : L^\infty(\partial B(0,R)) \to L^2(B(0,R)), \quad u \mapsto y_u, \tag{1.4}$$

with y_u the solution to (1.2) with control u. Then, for any control $u \in L^\infty(\partial B(0,R))$, the functional (1.1) reads

$$J(u) = \frac{1}{2}\int_{\partial B(0,R)} |u|^2 \, d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u) - z|^2 \, dx. \tag{1.5}$$

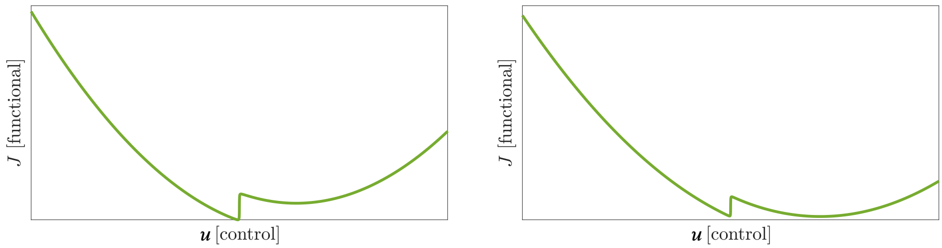
We have two summands here. The first one is convex, being a squared norm. The second one is a squared norm composed with $u \mapsto G(u) - z$. Now, under the assumption (1.3), the map $u \mapsto G(u)$ is nonlinear. Thus, the term $\int_{B(0,R)} |G(u) - z|^2 \, dx$, for a special target z, is not convex and generates the lack of uniqueness of minimizers.

The proof of Theorem 1.1 can be found in Section 3.1. The main steps of that proof are:

Step 1. **Reduction to constant controls:** by choosing radial targets and using the rotational invariance of $B(0,R)$, we reduce to the case where the control set is made up of constant controls.

Step 2. **Existence of two local minimizers:** we look for a target such that there exist two *local* minimizers ($u_1 < 0$ and $u_2 > 0$) for the functional J (see Fig. 2).

Step 3. **Existence of two global minimizers:** by the former step and a bisection argument, we prove the existence of a target such that J admits two *global* minimizers.



(a) Nonuniqueness of *local* minimizers (b) Nonuniqueness of *global* minimizers

Fig. 2. Functional versus control. This plot is obtained by drawing in MATLAB the graph of J defined in (1.1), with $R = 1$ and nonlinearity $f(y) = y^3$.

The special target yielding nonuniqueness is a step function changing sign in the observation domain, as in Fig. 3.

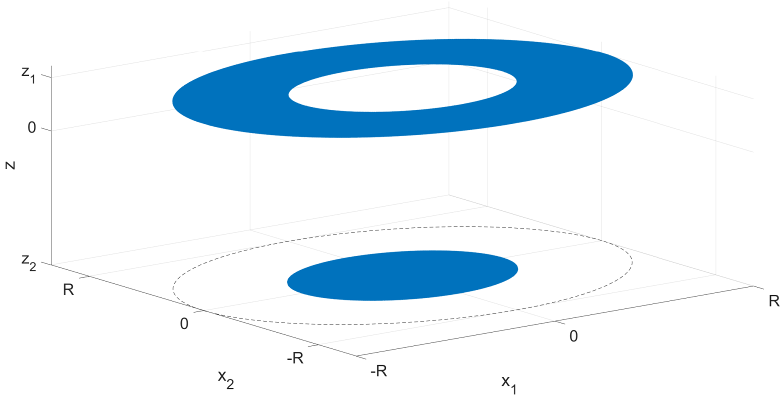


Fig. 3. Target yielding nonuniqueness in boundary control. The constructed target z (in blue) is a step function, taking values z_1 and z_2.

The above techniques can be applied, with some modifications, to the internal control problem

$$\min_{u \in L^2(B(0,r))} J(u), \quad \text{where}$$
$$J(u) = \frac{1}{2}\int_{B(0,r)} |u|^2\,dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |y - z|^2\,dx, \tag{1.6}$$

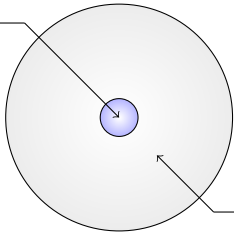


Fig. 4. Control and observation domains.

where

$$\begin{cases} -\Delta y + f(y) = u\chi_{B(0,r)} & \text{in } B(0,R), \\ y = 0 & \text{on } \partial B(0,R). \end{cases} \tag{1.7}$$

The nonlinearity $f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ is strictly increasing and $f(0) = 0$. The control acts in $B(0,r)$ with $r \in (0,R)$. The observation set is in $B(0,R) \setminus B(0,r)$ (see Fig. 4). The target z is in $L^2(B(0,R) \setminus B(0,r))$, while $\beta > 0$ is a penalization parameter.

The well-posedness of the state equation follows from [5, Theorem 4.7, p. 29], while the existence of a global minimizer in $L^2(B(0,r))$ for (1.6)–(1.7) can be shown by the direct method of the calculus of variations.

Theorem 1.2. *Consider the control problem* (1.6)–(1.7). *Assume, in addition,*

$$f''(y) \neq 0 \quad \forall y \neq 0. \tag{1.8}$$

There exists a target $z \in L^\infty(B(0,R) \setminus B(0,r))$ such that the functional J defined in (1.6) *admits* (*at least*) *two global minimizers.*

The proof can be found in Section 3.2.

A by-product of our nonuniqueness results is the lack of uniqueness of solutions $(\overline{y},\overline{q})$ to the optimality system

$$\begin{cases} -\Delta \overline{y} + f(\overline{y}) = -\overline{q}\chi_{B(0,r)} & \text{in } B(0,R), \\ \overline{y} = 0 & \text{on } \partial B(0,R), \\ -\Delta \overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{B(0,R)\setminus B(0,r)} & \text{in } B(0,R), \\ \overline{q} = 0 & \text{on } \partial B(0,R). \end{cases} \tag{1.9}$$

In the case of internal control, we can deduce the following corollary.

Corollary 1.3. *Under the assumptions of Theorem* 1.2, *there exists a target $z \in L^\infty(B(0,R) \setminus B(0,r))$ such that* (1.9) *admits* (*at least*) *two distinguished solutions* $(\overline{y}_1,\overline{q}_1)$ *and* $(\overline{y}_2,\overline{q}_2)$.

This follows from Theorem 1.2, together with the first order optimality conditions for the optimization problem (1.6)–(1.7) (see [10]).

Similarly, in the context of boundary control, nonuniqueness for (1.1) leads to nonuniqueness of solution to the optimality system

$$\begin{cases} -\Delta \overline{y} + f(\overline{y}) = 0 & \text{in } B(0,R), \\ \overline{y} = \dfrac{\partial}{\partial n}\overline{q} & \text{on } \partial B(0,R), \\ -\Delta \overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z) & \text{in } B(0,R), \\ \overline{q} = 0 & \text{on } \partial B(0,R). \end{cases} \tag{1.10}$$

To the best of our knowledge, the issue of uniqueness of minimizers has not been addressed so far for large targets z. Indeed, the uniqueness of the optimal control has been proved under smallness conditions on the target [26, Section 3.2] or on the adjoint state [1, Theorem 3.2]. In particular, in [1, Theorem 3.2] the uniqueness holds provided that the adjoint state is strictly smaller than an explicit constant [1, (3.6)].

The issue of uniqueness of minimizers for elliptic problems is of primary importance when studying the turnpike property for the corresponding time-evolution control problem (see [**?**, 26, 29, 30]). Indeed, the existence of multiple global minimizers for the steady problem generates multiple potential attractors for the time-evolution problem.

The control problems we are treating are classical. General surveys on the topic are [10] by Eduardo Casas and Mariano Mateos and [31, Chapter 4] by Fredi Tröltzsch. The interested reader is also referred to [2–4, 7–9, 12, 20, 23, 25, 27] and the references therein.

1.2. Lack of convexity

Before proving our main result on nonuniqueness of global minimizers, we observe that, for some targets, quadratic functionals of the optimal control governed by nonlinear state equations are not convex.

Theorem 1.4. *Consider the optimal control problem introduced in* (1.6)–(1.7)*. Then we have two possibilities:*

(1) *If f is linear, then J is convex for any target $z \in L^2(B(0,R) \setminus B(0,r))$.*

(2) *If f is not linear, then there exists a target $z \in L^2(B(0,R) \setminus B(0,r))$ such that J is not convex.*

In the literature, it is well known that convexity cannot be proved by standard techniques in case the state equation is nonlinear (see, for instance, [1] and [31, Section 4]). However, to the best of our knowledge, no explicit counterexamples to convexity are available. In this work, the lack of convexity can be deduced as a consequence of the lack of uniqueness (Theorem 1.1). Anyway, we prefer to prove Theorem 1.4 in Section 2 as a particular case of the following theorem, which holds in a general functional framework and basically asserts that a quadratic functional of the optimal control is convex for any target if and only if its control-to-state map is affine.

Theorem 1.5. *Let U and H be real Hilbert spaces. Let $G : U \to H$ be a function. Set*

$$J : U \to H, \quad J(u) := \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u) - z\|_H^2, \tag{1.11}$$

where $z \in H$. Then the following are equivalent:

(1) *for any target $z \in H$, J is convex;*

(2) *G is affine.*

In the application of Theorem 1.5 to optimal control, H is the observation space, U is the control space and G is the control-to-state map. The vector $z \in H$ is the given target for the state. Note that Theorem 1.5 applies both to steady and time-evolution control problems. Furthermore, the map G is not required to be smooth.

We sketch the proof of (1)⇒(2): we prove the lack of convexity if the control-to-state map G is not affine. For the time being, we assume that G is of class C^2. In the complete proof in Section 2, the smoothness of G is not required.

We start by developing the functional (1.11), for any control $u \in U$:

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u) - z\|^2 = \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u)\|_H^2 + \frac{1}{2}\|z\|_H^2 - \langle G(u), z\rangle \\ &= P(u) + \frac{1}{2}\|z\|_H^2 - \langle G(u), z\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of H and

$$P(u) := \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u)\|_H^2.$$

Now, since G is not affine, there exists a control $u_1 \in U$ and a direction $v_1 \in U$ such that the second directional derivative of G at u_1 along v_1 does not vanish:

$$D^2 G(u_1)(v_1, v_1) \neq 0. \tag{1.12}$$

Take as target $z^k := k D^2 G(u_1)(v_1, v_1)$, with $k > 0$ to be made precise later, and compute the second differential of the functional J at u_1 along direction v_1:

$$\begin{aligned} \langle D^2 J(u_1) v_1, v_1 \rangle &= \frac{d^2}{dv_1^2} P(u_1) - \langle D^2 G(u_1)(v_1, v_1), z^k \rangle \\ &= \frac{d^2}{dv_1^2} P(u_1) - k\|D^2 G(u_1)(v_1, v_1)\|_H^2 < 0 \end{aligned}$$

if we choose k sufficiently large. This shows the lack of convexity in the smooth case. The general nonsmooth case is handled in Section 2.

Theorem 1.5 can be applied to internal and boundary control, both in the elliptic and parabolic contexts.

The lack of convexity and of uniqueness of minimizers is a serious warning for numerics. Indeed, if the problem is not convex the convergence of gradient methods is not guaranteed a priori. Furthermore, by employing our techniques, one can find several counterexamples where there exist local minimizers which are not global. Thus, gradient methods may converge to local minimizers, missing global ones.

The rest of the manuscript is organized as follows. In Section 2, we prove Theorem 1.5 and we deduce Theorem 1.4. In Section 3, we provide a counterexample to uniqueness of global minimizers, in the context of boundary control (Section 3.1) and internal control (Section 3.2). In Section 4, we perform numerical simulations which explain and confirm our theoretical results. In the appendices, we prove some lemmas needed for our construction.

2. Lack of convexity: proofs of Theorems 1.5 and 1.4

In the proof of Theorem 1.5, we need the following lemma from linear algebra.

Lemma 2.1. *Let V_1 and V_2 be two real vector spaces. Then a function $G : V_1 \to V_2$ is affine if and only if, for any $\lambda \in [0, 1]$ and $(v, w) \in V_1^2$,*

$$G((1-\lambda)v + \lambda w) = (1-\lambda)G(v) + \lambda G(w). \tag{2.1}$$

Proof of Theorem 1.5. (2)⇒(1) If G is affine, by direct computations and convexity of the square of Hilbert norms, J is convex for any $z \in H$.

(1)⇒(2) Assume now G is not affine. We construct a target $z \in H$ such that J is not convex.

In what follows, we denote by $\langle \cdot, \cdot \rangle$ the scalar product of H.

Step 1: *Proof of the existence of $\tilde{\lambda} \in [0, 1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that*

$$\langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle < (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle.$$

First of all, up to changing the sign of z^0, it is sufficient to prove the existence of $\tilde{\lambda} \in [0, 1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that

$$\langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle \neq (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle. \tag{2.2}$$

Reasoning by contradiction, if (2.2) were not true, for any $z \in H$, every $(u_1, u_2) \in U^2$ and each $\lambda \in [0, 1]$,

$$\langle z, G((1-\lambda)u_1 + \lambda u_2)\rangle = (1-\lambda)\langle z, G(u_1)\rangle + \lambda\langle z, G(u_2)\rangle.$$

By the arbitrariness of z, this leads to

$$G((1-\lambda)u_1 + \lambda u_2) = (1-\lambda)G(u_1) + \lambda G(u_2)$$

for any $\lambda \in [0, 1]$ and $(u_1, u_2) \in U^2$. Then, by Lemma 2.1, G is affine, which contradicts our hypothesis. This finishes this step.

Step 2: *Conclusion.* By the first step, there exist $\tilde{\lambda} \in [0, 1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that

$$\langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle < (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle.$$

Now, arbitrarily fix $k \in \mathbb{N}^*$. Set as target

$$z^k := kz^0.$$

We develop J with target z^k, getting for any $u \in U$,

$$\begin{aligned} J(u) &= \tfrac{1}{2}\|u\|_U^2 + \tfrac{1}{2}\|G(u) - z^k\|_H^2 = \tfrac{1}{2}\|u\|_U^2 + \tfrac{1}{2}\|G(u)\|_H^2 + \tfrac{1}{2}\|z^k\|_H^2 - \langle z^k, G(u)\rangle \\ &= P(u) + \tfrac{1}{2}\|z^k\|_H^2 - \langle z^k, G(u)\rangle, \end{aligned}$$

where

$$P : U \to \mathbb{R}, \quad u \mapsto \tfrac{1}{2}\|u\|_U^2 + \tfrac{1}{2}\|G(u)\|_H^2.$$

At this point, we introduce

$$\begin{aligned} c_1 &:= (1-\tilde{\lambda})P(\tilde{u}_1) + \tilde{\lambda}P(\tilde{u}_2) - P((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2), \\ c_2 &:= (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle - \langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle. \end{aligned}$$

Then, taking z^k as target,

$$(1-\tilde{\lambda})J(\tilde{u}_1) + \tilde{\lambda}J(\tilde{u}_2) - J((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2) = c_1 - kc_2.$$

By the first step, $c_2 > 0$. Then, for k large enough,

$$(1-\tilde{\lambda})J(\tilde{u}_1) + \tilde{\lambda}J(\tilde{u}_2) - J((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2) = c_1 - kc_2 < 0,$$

which yields

$$(1-\tilde{\lambda})J(\tilde{u}_1) + \tilde{\lambda}J(\tilde{u}_2) < J((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2),$$

i.e. the desired lack of convexity of J. ∎

Theorem 1.5 applies to semilinear control, both in the elliptic case and in the parabolic case. We show how to apply Theorem 1.5 for the control problem (1.6)–(1.7), thus proving Theorem 1.4.

Proof of Theorem 1.4. Take

- $U = L^2(B(0,r))$;
- $H = L^2(B(0,R) \setminus B(0,r))$ with scalar product $\langle v_1, v_2\rangle := \beta \int_{B(0,R)\setminus B(0,r)} v_1 v_2 \, dx$;
- the map

  $$G : L^2(B(0,r)) \to L^2(B(0,R) \setminus B(0,r)), \quad u \mapsto y_u\restriction_{B(0,R)\setminus B(0,r)},$$

  where y_u fulfills (1.7) with control u.

Then, by Theorem 1.5, we have two possibilities:

(1) If G is linear, then J is convex for any target $z \in L^2(B(0,R) \setminus B(0,r))$.

(2) If G is not linear, then there exists a target $z \in L^2(B(0,R) \setminus B(0,r))$ such that J is not convex.

It remains to prove that G is linear if and only if f is. Now, if f is linear, the linearity of G follows from linear PDE theory [17, Part I]. Suppose now G is linear. We have to prove that for any $\alpha, \beta, \theta_1, \theta_2 \in \mathbb{R}$,

$$f(\alpha\theta_1 + \beta\theta_2) = \alpha f(\theta_1) + \beta f(\theta_2).$$

To this end, let us introduce a cut-off function $\zeta \in C^\infty(\mathbb{R}^n)$ such that

- $\zeta(0) = 1$;
- $\operatorname{supp}(\zeta) \subset\subset B(0, r)$.

For $i = 1, 2$, set $y_{\theta_i} := \theta_i \zeta$ and $u_{\theta_i} := [-\Delta y_{\theta_i} + f(y_{\theta_i})]\restriction_{B(0,r)}$. Then, by the linearity of G,

$$\begin{aligned} f(\alpha y_{\theta_1} + \beta y_{\theta_2}) &= f(\alpha G(u_{\theta_1}) + \beta G(u_{\theta_2})) = f(G(\alpha u_{\theta_1} + \beta u_{\theta_2})) \\ &= \Delta G(\alpha u_{\theta_1} + \beta u_{\theta_2}) + (\alpha u_{\theta_1} + \beta u_{\theta_2})\chi_{B(0,r)} \\ &= \alpha \Delta G(u_{\theta_1}) + \beta \Delta G(u_{\theta_2}) + \alpha u_{\theta_1}\chi_{B(0,r)} + \beta u_{\theta_2}\chi_{B(0,r)} \\ &= \alpha f(y_{\theta_1}) + \beta f(y_{\theta_2}), \end{aligned}$$

whence

$$\begin{aligned} f(\alpha\theta_1 + \beta\theta_2) &= f(\alpha y_{\theta_1}(0) + \beta y_{\theta_2}(0)) \\ &= \alpha f(y_{\theta_1}(0)) + \beta f(y_{\theta_2}(0)) \\ &= \alpha f(\theta_1) + \beta f(\theta_2), \end{aligned}$$

as required. ∎

3. Lack of uniqueness

In this section, we prove our nonuniqueness results. We start with boundary control (Theorem 1.1), to later deal with internal control (Theorem 1.2).

3.1. Boundary control

Hereafter, we will work with radial targets, defined below.

Definition 3.1. A function $z : B(0, R) \to \mathbb{R}$ is said to be *radial* if there exists $\phi : [0, R] \to \mathbb{R}$ such that, for any $x \in B(0, R)$, we have $z(x) = \phi(\|x\|)$.

We introduce the control-to-state map

$$G : L^\infty(\partial B(0, R)) \to L^2(B(0, R)), \quad u \mapsto y_u, \tag{3.1}$$

where y_u is the solution to (1.2) with control u. Then set

$$\begin{aligned} &I : L^\infty(\partial B(0, R)) \times L^2(B(0, R)) \to \mathbb{R}, \\ &I(u, z) := \frac{1}{2}\int_{\partial B(0,R)} |u|^2 \, d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u)|^2 \, dx - \beta \int_{B(0,R)} G(u) z \, dx, \end{aligned} \tag{3.2}$$

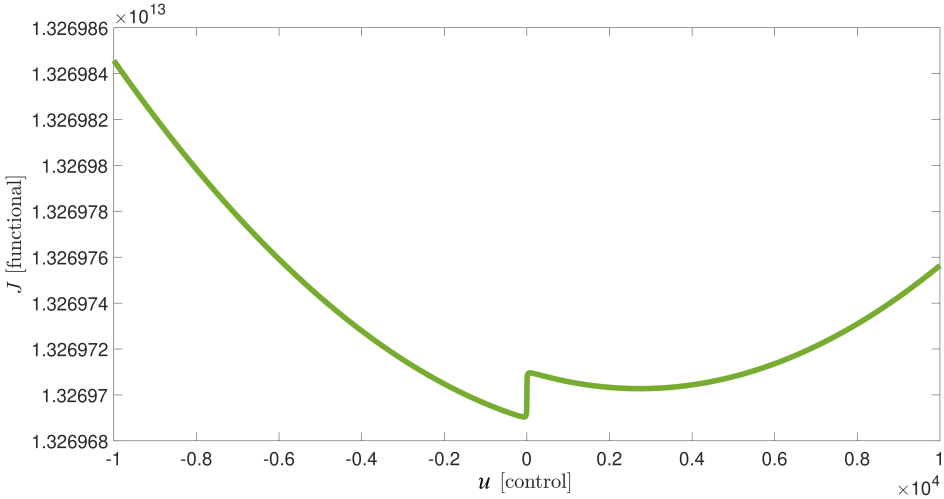


Fig. 5. Functional versus control (nonuniqueness of *local* minimizers). The graph of J defined in (1.1), with $R = 1$, $f(y) = y^3$ and $z = 260000\chi_{(0,1/4)\cup(3/4,1)} - 10300000\chi_{(1/4,3/4)}$.

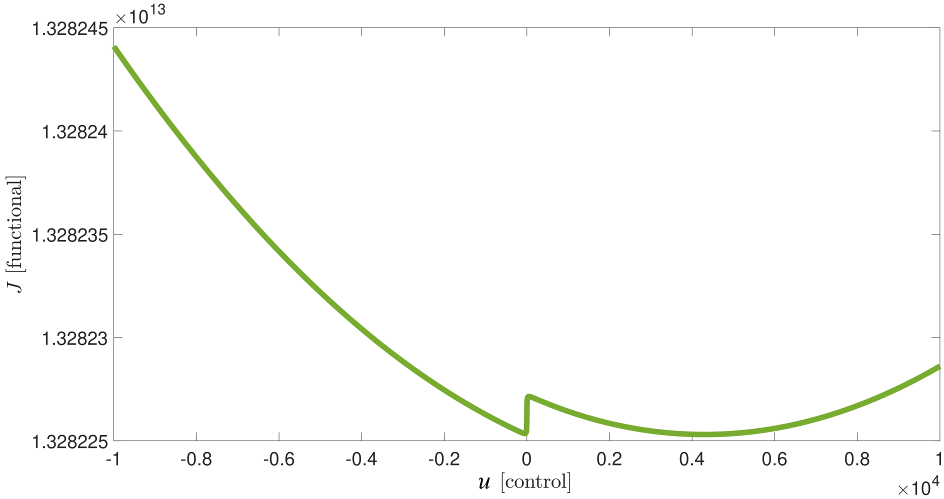


Fig. 6. Functional versus control (nonuniqueness of *global* minimizers). The graph of J defined in (1.1) with $R = 1$, $f(y) = y^3$ and $z = 410000\chi_{(0,1/4)\cup(3/4,1)} - 10300000\chi_{(1/4,3/4)}$.

where G is the control-to-state map introduced in (3.1). One recognizes that, for any target $z \in L^\infty(B(0,R))$, $I(\cdot, z) + \frac{\beta}{2}\|z\|^2_{L^2(B(0,R))}$ coincides with the functional J defined in (1.1) with target z. Then, for any $z \in L^\infty(B(0,R))$, minimizing $I(\cdot, z)$ is equivalent to minimizing J with target z. This translation is convenient, because $I(0, z) = 0$ for any target $z \in L^\infty(B(0,R))$.

We establish some important properties of solutions of the state equation (1.2):

- The unique constant solution of the equation $-\Delta y + f(y) = 0$ in any domain $\Omega \subset B(0,R)$ is $y \equiv 0$ (Lemma A.2). In particular, $G(u) = 0$ if and only if $u = 0$.
- By the comparison principle, if $u \geq 0$ on $\partial B(0,R)$ and $u \not\equiv 0$, then $G(u)(x) > 0$ in $B(0,R)$.
- By the comparison principle, if $u \leq 0$ on $\partial B(0,R)$ and $u \not\equiv 0$, then $G(u)(x) < 0$ in $B(0,R)$.

We introduce

$$h_1 : L^\infty(B(0,R)) \to \mathbb{R}, \quad h_1(z) := \inf\{I(u,z) \mid u \equiv k,\ k \in (-\infty, 0]\}, \tag{3.3}$$

$$h_2 : L^\infty(B(0,R)) \to \mathbb{R}, \quad h_2(z) := \inf\{I(u,z) \mid u \equiv k,\ k \in [0, +\infty)\}. \tag{3.4}$$

Lemma 3.2. *Let $C = (-\infty, 0]$ or $C = [0, +\infty)$.*

(1) *For any $z \in L^\infty(B(0,R))$, there exists $u_z \in C$ such that*

$$I(u_z, z) = \inf_C I(\cdot, z).$$

Furthermore, for any minimizer u_z,

$$|u_z| \leq \sqrt{\frac{\beta}{R^{n-1} n\alpha(n)}}\, \|z\|_{L^2},$$

where $n\alpha(n)$ is the surface area of $\partial B(0,1) \subset \mathbb{R}^n$, the unit sphere.

(2) *The map*

$$h : L^\infty(B(0,R)) \to \mathbb{R}, \quad h(z) := \inf_C I(\cdot, z),$$

is continuous.

We prove Lemma 3.2 in Appendix A. We now state the second lemma.

Lemma 3.3. *Assume there exists $z^0 \in L^\infty(B(0,R))$ such that*

$$h_1(z^0) < 0 \quad \textit{and} \quad h_2(z^0) < 0,$$

where h_1 and h_2 are defined in (3.3) and (3.4) resp. Then there exists a target $\tilde{z} \in L^\infty(B(0,R))$ such that

$$h_1(\tilde{z}) = h_2(\tilde{z}) < 0.$$

The proof of Lemma 3.3 can also be found in Appendix A. The following lemma is the key point for the proof of existence of two local minimizers for (1.1). At this point we employ the nonlinearity of the state equation (1.2).

Lemma 3.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^\infty$. Take $f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ strictly increasing with*

$$f''(y) \neq 0 \quad \forall y \neq 0.$$

Let $u_- < 0 < u_{+,1} < u_{+,2}$ be three constant controls. For any $u \in L^\infty(\partial\Omega)$, let $G(u)$ be the solution to

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ y = u & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Set

$$\overline{\lambda} := \frac{\int_\Omega G(u_{+,2})(x)\,dx}{\int_\Omega G(u_{+,1})(x)\,dx}, \tag{3.6}$$

$$\omega_1 := \{x \in \Omega \mid G(u_{+,2})(x) < \overline{\lambda} G(u_{+,1})(x)\}, \tag{3.7}$$

$$\omega_2 := \{x \in \Omega \mid G(u_{+,2})(x) > \overline{\lambda} G(u_{+,1})(x)\}. \tag{3.8}$$

There exists $i \in \{1, 2\}$ such that the matrix

$$\Gamma := \beta \begin{bmatrix} \int_{\omega_1} G(u_-)\,dx & \int_{\omega_2} G(u_-)\,dx \\ \int_{\omega_1} G(u_{+,i})\,dx & \int_{\omega_2} G(u_{+,i})\,dx \end{bmatrix} \tag{3.9}$$

is invertible.

Proof of Lemma 3.4. To simplify the notation, we set $y_1 := G(u_{+,1})$ and $y_2 := G(u_{+,2})$.

Step 1: *For any $\lambda \in \mathbb{R}$ the set*

$$E_\lambda := \{x \in \Omega \mid y_2(x) = \lambda y_1(x)\}$$

has Lebesgue measure zero. We start with the case $\lambda \leq 1$. By the strong maximum principle [17, Theorem 8.19 p. 198], for any $x \in \Omega$, $G(u_{+,2})(x) > y_1(x)$. Hence, for any $\lambda \leq 1$, the set E_λ is empty.

For $\lambda > 1$, suppose, for contradiction, that E_λ has strictly positive Lebesgue measure. For any $x \in \Omega$, we have

$$-\Delta y_1(x) + f(y_1(x)) = 0, \tag{3.10}$$

$$-\Delta y_2(x) + f(y_2(x)) = 0. \tag{3.11}$$

By definition, for any $x \in E_\lambda$, $y_2(x) = \lambda y_1(x)$, whence by (3.11) and Lemma A.4 applied twice we get, a.e. in E_λ,

$$-\lambda \Delta y_1(x) + f(\lambda y_1(x)) = 0. \tag{3.12}$$

Multiplying (3.10) by λ, we have

$$-\lambda \Delta y_1(x) + \lambda f(y_1(x)) = 0. \tag{3.13}$$

By subtracting (3.12) and (3.13), we obtain

$$f(\lambda y_1(x)) = \lambda f(y_1(x)) \quad \text{a.e. } x \in E_\lambda. \tag{3.14}$$

Now, we have supposed that E_λ has a positive Lebesgue measure. Hence, by Lemma A.3, there exists an accumulation point $\hat{x} \in \Omega$ and a corresponding sequence $\{x_m\}_{m\in\mathbb{N}} \subset E_\lambda$ such that

$$x_m \xrightarrow[m\to+\infty]{} \hat{x}. \tag{3.15}$$

Now, by (3.14), we have

$$f(\lambda y_1(x_m)) = \lambda f(y_1(x_m)), \quad \forall m \in \mathbb{N}. \tag{3.16}$$

Since $u_i \in C^0(\partial\Omega)$, it follows that $y_i \in H^1(\Omega) \cap C^0(\overline{\Omega})$ by Proposition 1. Then, taking the limit as $m \to +\infty$ in the above expression, we get

$$f(\lambda y_1(\hat{x})) = \lambda f(y_1(\hat{x})). \tag{3.17}$$

By (3.16) and (3.17), we have

$$\frac{f(\lambda y_1(x_m)) - f(\lambda y_1(\hat{x}))}{\lambda y_1(x_m) - \lambda y_1(\hat{x})} = \frac{\lambda f(y_1(x_m)) - \lambda f(y_1(\hat{x}))}{\lambda y_1(x_m) - \lambda y_1(\hat{x})}. \tag{3.18}$$

Taking the limit as $m \to +\infty$ on both sides and using the continuity of y_1 we get $f'(\lambda y_1(\hat{x})) = f'(y_1(\hat{x}))$. Now, by [17, Theorem 8.19, p. 198], $y_1(\hat{x}) > 0$. Hence by the Rolle Theorem applied to f', there exists $\xi > 0$ such that

$$f''(\xi) = 0, \tag{3.19}$$

contrary to assumption. This finishes Step 1.

Set now

$$\Lambda := \begin{bmatrix} \int_{\omega_1} G(u_{+,1})\,dx & \int_{\omega_2} G(u_{+,1})\,dx \\ \int_{\omega_1} G(u_{+,2})\,dx & \int_{\omega_2} G(u_{+,2})\,dx \end{bmatrix}.$$

Step 2: $\Omega \setminus [\omega_1 \cup \omega_2]$ *has Lebesgue measure zero and the matrix* Λ *is invertible.* By the above reasoning, the set $E_{\overline{\lambda}} = \Omega \setminus [\omega_1 \cup \omega_2]$ has Lebesgue measure zero. Now, by the strong maximum principle, y_1 and y_2 are strictly positive in Ω and $\overline{\lambda} \neq 0$. Hence,

$$\begin{aligned}\det(\Lambda) &= \int_{\omega_1} y_1\,dx \int_{\omega_2} y_2\,dx - \int_{\omega_1} y_2\,dx \int_{\omega_2} y_1\,dx \\ &> \overline{\lambda} \int_{\omega_1} y_1\,dx \int_{\omega_2} y_1\,dx - \overline{\lambda} \int_{\omega_1} y_1\,dx \int_{\omega_2} y_1\,dx = 0.\end{aligned}$$

Step 3: *Conclusion.* Let us assume, for contradiction, that the matrix Γ is not invertible. Then, for $i = 1, 2$, there exists $\lambda_i \in \mathbb{R}$ such that

$$\begin{bmatrix} \int_{\omega_1} G(u_{+,i})\,dx \\ \int_{\omega_2} G(u_{+,i})\,dx \end{bmatrix} = \lambda_i \begin{bmatrix} \int_{\omega_1} G(u_{-})\,dx \\ \int_{\omega_2} G(u_{-})\,dx \end{bmatrix}. \tag{3.20}$$

Since the controls are nonzero constants, by [17, Theorem 8.19, p. 198], all the above integrals are nonvanishing, whence $\lambda_i \neq 0$. Then

$$\begin{bmatrix} \int_{\omega_1} G(u_{+,2})\,dx \\ \int_{\omega_2} G(u_{+,2})\,dx \end{bmatrix} = \lambda_2 \begin{bmatrix} \int_{\omega_1} G(u_{-})\,dx \\ \int_{\omega_2} G(u_{-})\,dx \end{bmatrix} = \frac{\lambda_2}{\lambda_1} \begin{bmatrix} \int_{\omega_1} G(u_{+,1})\,dx \\ \int_{\omega_2} G(u_{+,1})\,dx \end{bmatrix}. \tag{3.21}$$

By (3.21), the matrix Λ is not invertible, contradicting Step 2. ■

Proof of Theorem 1.1. **Step 1:** *Reduction to constant controls.* Suppose that for some radial target z, the optimal control is not constant. Then, by Lemma A.5, there exists an orthogonal matrix M such that $u \circ M \neq u$. Now,

$$\begin{aligned} I(u \circ M, z) &= \frac{1}{2}\int_{\partial B(0,R)} |u \circ M|^2\,d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u \circ M)|^2\,dx \\ &\quad - \beta \int_{B(0,R)} G(u \circ M) z\,dx \\ &= \frac{1}{2}\int_{\partial B(0,R)} |u|^2\,d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u)|^2\,dx - \beta\int_{B(0,R)} G(u) z\,dx \\ &= I(u,z), \end{aligned}$$

where we have employed the change of variable $\gamma(x) = Mx$ and Lemma A.7. Thus, u and $u \circ M$ are two distinguished global minimizers for $I(\cdot, z)$, as desired. It remains to prove nonuniqueness when, for any radial targets, all the optimal controls are constants.

Step 2: *Existence of a special target $z^0 \in L^\infty(B(0,R))$ such that $I(\cdot, z^0)$ admits (at least) two local minimizers among constant controls.* By Lemma 3.4, there exist two controls $u_- < 0 < u_+$ such that (3.9) is invertible. We start by proving the existence of a special target $z^0 \in L^\infty(B(0,R))$ such that $I(u_-, z^0) < 0$ and $I(u_+, z^0) < 0$.

For any $z^0 \in L^\infty(B(0,R))$, we have $I(u_-, z^0) < 0$ and $I(u_+, z^0) < 0$ if and only if

$$\begin{cases} \beta \int_{B(0,R)} G(u_-) z^0\,dx > \frac{R^{n-1} n\alpha(n)}{2} |u_-|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(u_-)|^2\,dx, \\ \beta \int_{B(0,R)} G(u_+) z^0\,dx > \frac{R^{n-1} n\alpha(n)}{2} |u_+|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(u_+)|^2\,dx, \end{cases} \tag{3.22}$$

where G is the control-to-state map introduced in (3.1) and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. In what follows, we work with *sign-changing* targets

$$z^0 := \begin{cases} z_1^0 & \text{in } \omega_1, \\ z_2^0 & \text{in } \omega_2, \end{cases}$$

where $(z_1^0, z_2^0) \in \mathbb{R}^2$ and ω_1 and ω_2 are defined in (3.7) and (3.8) respectively. (z_1^0, z_2^0) are degrees of freedom we need in the remainder of the proof. With the above choice of target, inequalities (3.22) are satisfied if (z_1^0, z_2^0) satisfies the linear system

$$\begin{cases} z_1^0\beta\int_{\omega_1} G(u_-)\,dx + z_2^0\beta\int_{\omega_2} G(u_-)\,dx = c_1, \\ z_1^0\beta\int_{\omega_1} G(u_+)\,dx + z_2^0\beta\int_{\omega_2} G(u_+)\,dx = c_2, \end{cases} \tag{3.23}$$

with constant terms

$$c_1 := \frac{R^{n-1}n\alpha(n)}{2}|u_-|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(u_-)|^2\,dx + 1,$$
$$c_2 := \frac{R^{n-1}n\alpha(n)}{2}|u_+|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(u_+)|^2\,dx + 1.$$

The 2×2 coefficient matrix of the above linear system reads

$$\Gamma = \beta \begin{bmatrix} \int_{\omega_1} G(u_-)\,dx & \int_{\omega_2} G(u_-)\,dx \\ \int_{\omega_1} G(u_+)\,dx & \int_{\omega_2} G(u_+)\,dx \end{bmatrix}.$$

By (3.9), the matrix Γ is invertible. Therefore, by the Rouché–Capelli theorem, there exists a solution to the linear system (3.23). The solution (z_1^0, z_2^0) defines a special target

$$z^0 := \begin{cases} z_1^0 & \text{in } \omega_1, \\ z_2^0 & \text{in } \omega_2, \end{cases}$$

such that $I(u_-, z^0) < 0$ and $I(u_+, z^0) < 0$.

We now show that $I(\cdot, z^0)$ admits (at least) two local minimizers. Indeed, by Lemma 3.2 (1), there exist

$$u_1 \le 0 \quad \text{such that} \quad I(u_1, z^0) = \inf\{I(u, z) \mid u \equiv k,\ k \le 0\},$$
$$u_2 \ge 0 \quad \text{such that} \quad I(u_2, z^0) = \inf\{I(u, z) \mid u \equiv k,\ k \ge 0\}.$$

Now,

$$I(u_1, z^0) = \inf\{I(u, z) \mid u \equiv k,\ k \le 0\} \le I(u_-, z^0) < 0 = I(0, z^0),$$
$$I(u_2, z^0) = \inf\{I(u, z) \mid u \equiv k,\ k \ge 0\} \le I(u_+, z^0) < 0 = I(0, z^0).$$

Thus, the control u_1 minimizes $I(\cdot, z^0)$ on the half-line $(-\infty, 0)$, while u_2 minimizes $I(\cdot, z^0)$ on the half-line $(0, +\infty)$. We have found two distinct local minimizers u_1 and u_2 for $I(\cdot, z^0)$ in $\mathbb{R}$.

Step 3: *Conclusion.* Recall the definition of h_1 and h_2 given by (3.3) and (3.4) resp. In Step 2, we have determined $z^0 \in L^\infty(B(0, R))$ such that $h_1(z^0) < 0$ and $h_2(z^0) < 0$. To finish our proof it suffices to find $\tilde{z} \in \mathbb{R}^n$ such that $h_1(\tilde{z}) = h_2(\tilde{z}) < 0$. This follows from Lemma 3.3. ∎

3.2. Internal control

We introduce the well-known concept of radial control.

Definition 3.5. A control $u : B(0,r) \to \mathbb{R}$ is said to be *radial* if there exists $\psi : [0,r] \to \mathbb{R}$ such that, for any $x \in B(0,r)$, we have $u(x) = \psi(\|x\|)$.

Our strategy to prove Theorem 1.2 resembles the one of Theorem 1.1, except for Step 1, which consists now in reduction to radial controls instead of constant controls.

We define the control-to-state map

$$G : L^2(B(0,r)) \to L^2(B(0,R)), \quad u \mapsto y_u, \tag{3.24}$$

where y_u is the solution to (1.7) with control u. Then set

$$\begin{aligned} I &: L^2(B(0,r)) \times L^\infty(B(0,R) \setminus B(0,r)) \to \mathbb{R}, \\ I(u,z) &:= \frac{1}{2}\int_{B(0,r)} |u|^2\, dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u)|^2\, dx \\ &\quad - \beta \int_{B(0,R)\setminus B(0,r)} G(u) z\, dx, \end{aligned} \tag{3.25}$$

where G is the control-to-state map introduced in (3.24). One recognizes that, for any $z \in L^\infty(B(0,R) \setminus B(0,r))$, $I(\cdot,z) + \frac{\beta}{2}\|z\|^2_{L^2(B(0,R)\setminus B(0,r))}$ coincides with the functional J defined in (1.6) with target z. Then, for any $z \in L^\infty(B(0,R) \setminus B(0,r))$, minimizing $I(\cdot,z)$ is equivalent to minimizing J with target z. This translation is convenient, because $I(0,z) = 0$ for any target $z \in L^\infty(B(0,R) \setminus B(0,r))$.

We establish some important properties of the solutions of the state equation (1.7):

- The unique constant solution of the equation $-\Delta y + f(y) = 0$ in $B(0,R)$, with $y = 0$ on $\partial B(0,R)$ is $y \equiv 0$ (Lemma B.2). In particular, $G(u) = 0$ if and only if $u = 0$.
- By the comparison principle, if $u \geq 0$ in $B(0,r)$ and $u \not\equiv 0$, then $G(u)(x) > 0$ in $B(0,R)$; and if $u \leq 0$ in $B(0,r)$ and $u \not\equiv 0$, then $G(u)(x) < 0$ in $B(0,R)$.

We define

$$\mathscr{U}_r := \{u \in L^2(B(0,r)) \mid u \text{ is radial}\}. \tag{3.26}$$

We have

$$\mathscr{U}_r = \mathscr{U}_r^- \cup \mathscr{U}_r^- \tag{3.27}$$

with

$$\begin{aligned} \mathscr{U}_r^- &:= \{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} \leq 0\}, \\ \mathscr{U}_r^+ &:= \{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} \geq 0\}. \end{aligned} \tag{3.28}$$

We introduce

$$h_1 : L^\infty(B(0,R) \setminus B(0,r)) \to \mathbb{R}, \quad h_1(z) := \inf\{I(u,z) \mid u \in \mathscr{U}_r^-\}, \tag{3.29}$$

$$h_2 : L^\infty(B(0,R) \setminus B(0,r)) \to \mathbb{R}, \quad h_2(z) := \inf\{I(u,z) \mid u \in \mathscr{U}_r^+\}. \tag{3.30}$$

We formulate the first lemma.

Lemma 3.6. *Let $C = \mathscr{U}_r^-$ or $C = \mathscr{U}_r^+$.*

(1) *For any $z \in L^\infty(B(0,R) \setminus B(0,r))$, there exists $u_z \in C$ such that*

$$I(u_z, z) = \inf_C I(\cdot, z).$$

Furthermore, for any minimizer u_z,

$$\|u_z\|_{L^2(B(0,r))} \le \sqrt{\beta}\,\|z\|_{L^2}.$$

(2) *The map*

$$h : L^\infty(B(0,R) \setminus B(0,r)) \to \mathbb{R}, \quad z \mapsto \inf_C I(\cdot, z),$$

is continuous.

The proof of Lemma 3.6 resembles the one of Lemma 3.2, available in Appendix A. We now state the second lemma needed to prove Theorem 1.2.

Lemma 3.7. *Assume there exists $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$ such that*

$$h_1(z^0) < 0 \quad \textit{and} \quad h_2(z^0) < 0,$$

where h_1 and h_2 are defined in (3.29) and (3.30) resp. Then there exists $\tilde{z} \in L^\infty(B(0,R) \setminus B(0,r))$ such that

$$h_1(\tilde{z}) = h_2(\tilde{z}) < 0.$$

The above lemma can be proved by following the arguments for Lemma 3.3 in Appendix A. The next lemma is the foundation of the proof of the existence of two local minimizers for (1.6). The nonlinearity of the state equation (1.7) will play a key role in the proof.

Lemma 3.8. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^\infty$, and $\omega \subsetneq \Omega$ a nonempty open subset. Take $f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ strictly increasing with*

$$f''(y) \neq 0 \quad \forall y \neq 0.$$

Let $u_- < 0 < u_{+,1} < u_{+,2}$ be three constant controls. For any $u \in L^2(\omega)$, let $G(u)$ be the solution to

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.31}$$

Set

$$\overline{\lambda} := \frac{\int_\Omega G(u_{+,2})(x)\,dx}{\int_\Omega G(u_{+,1})(x)\,dx}, \tag{3.32}$$

$$\omega_1 := \{x \in \Omega \setminus \omega \mid G(u_{+,2})(x) < \overline{\lambda} G(u_{+,1})(x)\}, \tag{3.33}$$

$$\omega_2 := \{x \in \Omega \setminus \omega \mid G(u_{+,2})(x) > \overline{\lambda} G(u_{+,1})(x)\}. \tag{3.34}$$

There exists $i \in \{1,2\}$ *such that*

$$\Gamma := \beta \begin{bmatrix} \int_{\omega_1} G(u_-)\,dx & \int_{\omega_2} G(u_-)\,dx \\ \int_{\omega_1} G(u_{+,i})\,dx & \int_{\omega_2} G(u_{+,i})\,dx \end{bmatrix} \tag{3.35}$$

is invertible.

The proof of the above lemma resembles the one of Lemma 3.4. A key point is that, in the complement of the control region, for $i = 1,2$ we have

$$-\Delta G(u_{+,i}) + f(G(u_{+,i})) = 0 \quad \text{in } \Omega \setminus \omega. \tag{3.36}$$

Proof of Theorem 1.2. **Step 1:** *Reduction to radial controls.* Suppose for some radial target z, the optimal control u is not radial, that is, there exists an orthogonal matrix M such that $u \circ M \neq u$. By Lemma B.3, we have $G(u \circ M) = G(u) \circ M$. Now,

$$\begin{aligned} I(u \circ M, z) &= \frac{1}{2}\int_{B(0,r)} |u \circ M|^2\,dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u \circ M)|^2\,dx \\ &\quad - \beta \int_{B(0,R)\setminus B(0,r)} G(u \circ M) z\,dx \\ &= \frac{1}{2}\int_{B(0,r)} |u|^2\,dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u)|^2\,dx \\ &\quad - \beta \int_{B(0,R)\setminus B(0,r)} G(u) z\,dx \\ &= I(u,z), \end{aligned}$$

where we have employed the change of variable $\gamma(x) = Mx$. Thus, u and $u \circ M$ are two distinguished global minimizers for $I(\cdot, z)$, as desired. It remains to prove nonuniqueness when, for any radial target, all the optimal controls are radial. Hereafter, for a radial target z, we will consider the restriction of the functional $I(\cdot, z)$ to $\mathscr{U}_r$.

Step 2: *Existence of a special target* $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$ *such that* $I(\cdot, z^0)$ *admits (at least) two local minimizers, among radial controls.* By Lemma 3.8, there exist two controls $u_- < 0 < u_+$ such that (3.35) is invertible. Proceeding as in Step 2 of the proof of Theorem 1.1, one can prove the existence of a special target

$$z^0 := \begin{cases} z_1^0 & \text{in } \omega_1, \\ z_2^0 & \text{in } \omega_2, \end{cases}$$

such that $I(u_-, z^0) < 0$ and $I(u_+, z^0) < 0$. Note that in this case ω_1 and ω_2 are defined in (3.33) and (3.34) respectively.

We now show that $I(\cdot, z^0)$ admits (at least) two local minimizers in $\mathscr{U}_r$. Indeed, the set $\mathscr{U}_r$ (introduced in (3.26)) splits as

$$\mathscr{U}_r = \mathscr{U}_r^- \cup \mathscr{U}_r^+,$$

with

$$\mathscr{U}_r^- = \{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} \leq 0\}, \quad \mathscr{U}_r^+ = \{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} \geq 0\},$$

where we have used the fact that for any radial control u, by Lemma B.3, $G(u)$ is radial and (by elliptic regularity [13, Theorem 4, p. 334]) continuous, so that $G(u)\restriction_{\partial B(0,r)}$ is a real number.

By Lemma 3.6 (1), there exist

$$u_1 \in \mathscr{U}_r^- \quad \text{such that} \quad I(u_1, z^0) = \inf_{\mathscr{U}_r^-} I(\cdot, z^0),$$
$$u_2 \in \mathscr{U}_r^+ \quad \text{such that} \quad I(u_2, z^0) = \inf_{\mathscr{U}_r^+} I(\cdot, z^0).$$

Now, for any control $u \in \{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} = 0\}$, we have

$$I(u_1, z^0) = \inf_{\mathscr{U}_r^-} I(\cdot, z^0) \leq I(u_-, z^0) < 0 \leq I(u, z^0),$$
$$I(u_2, z^0) = \inf_{\mathscr{U}_r^+} I(\cdot, z^0) \leq I(u_+, z^0) < 0 \leq I(u, z^0).$$

Then necessarily u_1 is a local minimizer for $I(\cdot, z^0)$ in the open set $\{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} < 0\}$ and u_2 is a local minimizer for $I(\cdot, z^0)$ in the open set $\{u \in \mathscr{U}_r \mid G(u)\restriction_{\partial B(0,r)} > 0\}$. Hence, we have found two distinct local minimizers u_1 and u_2 for $I(\cdot, z^0)$ in $\mathscr{U}_r$.

Step 3: *Conclusion.* Recall the definitions of h_1 and h_2 in (3.29) and (3.30) resp. In Step 2, we have determined $z^0 \in L^\infty(B(0, R) \setminus B(0, r))$ such that $h_1(z^0) < 0$ and $h_2(z^0) < 0$. To finish our proof it suffices to find $\tilde{z} \in \mathbb{R}^n$ such that $h_1(\tilde{z}) = h_2(\tilde{z}) < 0$. This follows from Lemma 3.7. ∎

4. Numerical simulations

We have performed a numerical simulation in the context of boundary control. We illustrate in Fig. 7 an example with step target

$$z(x) := \begin{cases} 410000 & \text{for } 0 < x < 1/4 \text{ and } 3/4 < x < 1, \\ -10300000 & \text{for } 1/4 < x < 3/4. \end{cases} \tag{4.1}$$

As we have seen in the proof of Theorem 1.1, we can reduce to the case of constant controls on the boundary. In our case, the space dimension is $n = 1$. Thus, we have reduced to the case where the same control acts on both endpoints $x = 0$ and $x = 1$. Hence, we plot in Fig. 7 the restriction $J\restriction_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, the functional J being defined in (1.1).

There exist two distinguished global minimizers:

- a negative one $u_1 \cong -50$;
- a positive one $u_2 \cong 4298$.

The corresponding optimal states are depicted in Fig. 8 and Fig. 9.

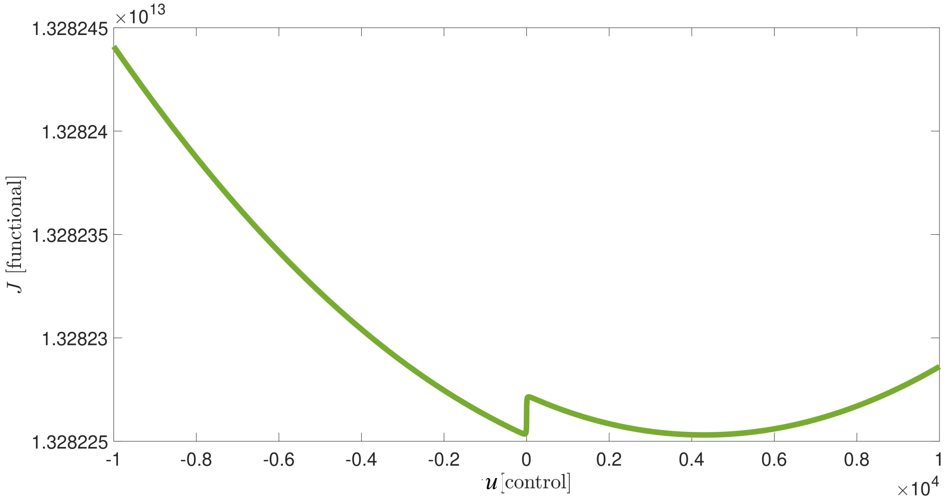


Fig. 7. Functional versus control (nonuniqueness of *global* minimizers). The graph of J defined in (1.1), with space dimension $n = 1$, $R = 1$, weighting parameter $\beta = 1$ and target (4.1).

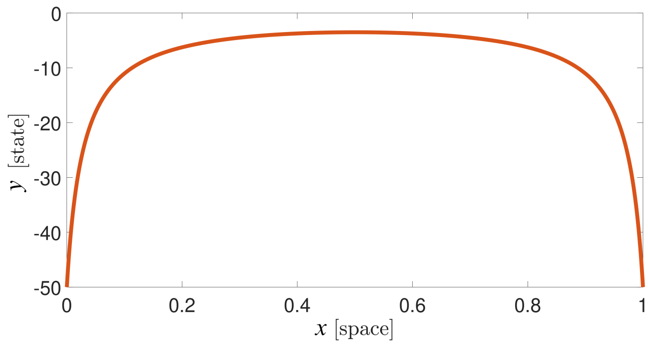


Fig. 8. State associated with control $u = -50$.

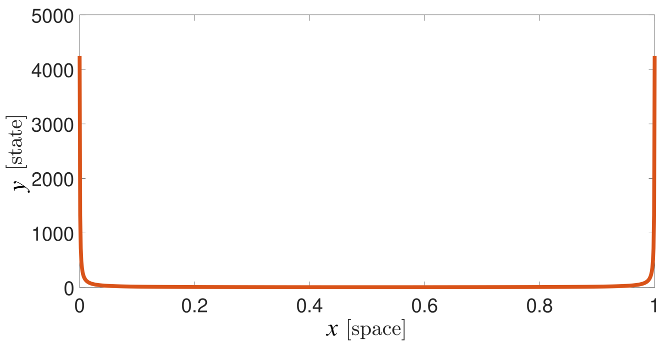


Fig. 9. State associated with control $u = 4298$.

The idea behind this example is that two optimal strategies are available:

- take a large positive control u_2 to better approximate the target in $(0, 1/4) \cup (3/4, 1)$;
- take a negative control u_1 to keep the state closer to the target in $(1/4, 3/4)$.

Note that $|u_1| < |u_2|$. Indeed, the control acts at the endpoints $x = 0$ and $x = 1$ of the space domain. Next, the effect of the control is stronger in $(0, 1/4) \cup (3/4, 1)$ than in $(1/4, 3/4)$. For this reason, it is worth to take a large positive control to better approximate the target in $(0, 1/4) \cup (3/4, 1)$. On the other hand, it is less convenient to take a very negative control to approximate the target in $(1/4, 3/4)$ (see the local estimates for semilinear equations [21] and [15, proof of Theorem 1.3]).

In Fig. 7 we observe that the functional has a different behaviour close to zero and away from zero. This can be explained by studying the behaviour of the control-to-state map (3.1):

- close to zero, (3.1) is close to its linearization around zero;
- far from zero, (3.1) is strongly influenced by the nonlinearity $f(y) = y^3$, thus producing a drastic change in the shape of the functional.

Numerical simulations have been performed in MATLAB. We now explain the numerical methods employed.

Firstly choose an interval of controls $[-M, M]$, where to study the functional J. Then our goal is to plot $J\restriction_{[-M,M]} : [-M, M] \to \mathbb{R}$.

For the interval $[-M, M]$, we choose an equi-spaced grid $v_i = -M + (i-1)\frac{2M}{N_c - 1}$ with $i = 1, \dots, N_c$ and $N_c \in \mathbb{N} \setminus \{0\}$.

Now, for each control v_i, we need to find numerically the corresponding state y_i solving the following PDE with cubic nonlinearity:

$$\begin{cases} -(y_i)_{xx} + (y_i)^3 = 0, & x \in (0, 1), \\ y_i(0) = y_i(1) = v_i. \end{cases} \tag{4.2}$$

Following [6, Section 4.3.2], we solve (4.2) by a fixed-point type algorithm with relaxation. Namely, in any iteration k, we determine the solution $y_{i,k}$ to the linear PDE

$$\begin{cases} -(y_{i,k})_{xx} + (\theta_{i,k-1})^2 y_{i,k} = 0, & x \in (0, 1), \\ y_{i,k}(0) = y_{i,k}(1) = v_i, \end{cases} \tag{4.3}$$

and we set $\theta_k := \frac{1}{2}\theta_{i,k-1} + \frac{1}{2}y_k$. The initial guess $\theta_{i,0}$ is taken to be y_{i-1}, i.e. the solution to (4.2) with control v_{i-1}.

To compute the solution to the linear PDE (4.3), we choose a finite difference scheme with uniform space grid $x_j = \frac{j-1}{\Delta x}$, where $j = 1, \dots, N_x$, $N_x \in \mathbb{N} \setminus \{0\}$ and $\Delta x := \frac{1}{N_x - 1}$. Then $y_{i,k} = (y_{i,k,j})_j$ is an N_x-dimensional discrete vector solution to

$$\begin{cases} \dfrac{-y_{i,k,j-1} + 2y_{i,k,j} - y_{i,k,j+1}}{(\Delta x)^2} + (\theta_{i,k-1,j})^2 y_{i,k,j} = 0, & j = 2, \dots, N_x - 1, \\ y_{i,k,1} = y_{i,k,N_x} = v_i. \end{cases}$$

Once we have determined the state y_i, we evaluate the functional J at the control v_i. The integral appearing in (1.1) can be computed by quadrature methods. We are now in a position to plot the functional $J\restriction_{[-M,M]} : [-M, M] \to \mathbb{R}$.

Note that, as far as we know, the actual convergence of the fixed-point method described has not been proved. However, for any control v_i, we are able to check that the state computed solves the finite difference version of the nonlinear problem (4.2) up to a small error.

An extensive literature is available on numerical approximation of solutions to (4.2) (see, for instance, [18] for a survey). Let us mention two alternative numerical methods.

The first one is a finite difference-Newton method presented in [22, Section 2.16.1]. The idea is to discretize (4.2) directly. This leads to a nonlinear equation in finite dimensions, solved by a Newton method.

Another option is to find the solution to (4.2) as the minimizer of the convex functional

$$K(y) = \frac{1}{2}\int_0^1 |y_x|^2\,dx + \frac{1}{4}\int_0^1 y^4\,dx$$

over the affine space

$$\mathcal{A} := \{y \in H^1(0,1) \mid y(0) = y(1) = v\}.$$

5. Conclusions and open problems

We have illustrated a general methodology to show lack of convexity for quadratic functionals with nonlinear state equations (Theorem 1.5). Furthermore, we have exhibited a counterexample to uniqueness of global minimizers in optimal control of semilinear elliptic equations (Theorems 1.1 and 1.2).

We list some interesting problems, which, to the best of our knowledge, have not been addressed in the literature so far.

5.1. *General space domain*

Our counterexample to uniqueness of minimizers in semilinear control relies on the rotational invariance of the space domain $B(0,R)$ to reduce to constant/radial controls. It would be interesting to enhance the techniques developed to more general space domains.

5.2. *Relations to the turnpike property*

Consider the time-evolution control problem associated to (1.6)–(1.7),

$$\min_{u\in\mathcal{U}_T} J_T(u), \quad \text{where}$$
$$J_T(u) = \frac{1}{2}\int_0^T\int_{B(0,r)} |u|^2\,dx\,dt + \frac{\beta}{2}\int_0^T\int_{B(0,R)\setminus B(0,r)} |y-z|^2\,dx\,dt, \tag{5.1}$$

where $\mathscr{U}_T := L^2((0,T)\times B(0,r))$ and the state y associated to the control u is a solution to the semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_{B(0,r)} & \text{in } (0,T)\times B(0,R), \\ y = 0 & \text{on } (0,T)\times \partial B(0,R), \\ y(0,x) = y_0(x) & \text{in } B(0,R). \end{cases} \tag{5.2}$$

The nonlinearity f is C^3 and nondecreasing, with $f(0) = 0$. The assumptions on the state equation are the same as in [26, Section 3]. An optimal control for the above problem is denoted by u^T, while the corresponding optimal state by y^T.

We rewrite (1.6)–(1.7) with an "s" subscript to stress the steady-state character of the problem:

$$\begin{aligned} &\min_{u_s\in L^2(B(0,r))} J_s(u_s), \quad \text{where} \\ &J_s(u_s) = \frac{1}{2}\int_{B(0,r)} |u_s|^2\,dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |y_s - z|^2\,dx, \end{aligned} \tag{5.3}$$

where

$$\begin{cases} -\Delta y_s + f(y_s) = u_s\chi_{B(0,r)} & \text{in } B(0,R), \\ y_s = 0 & \text{on } \partial B(0,R). \end{cases} \tag{5.4}$$

We denote by $(\overline{u},\overline{y})$ an optimal pair, where $\overline{u}$ is an optimal control and $\overline{y}$ the corresponding optimal state.

Consider a target z such that J_s has two distinguished global minimizers, as in Theorem 1.2. Choose any initial datum $y_0 \in L^\infty(B(0,R))$ for the evolution equation (5.2). Let u^T be a minimizer for (5.1). Then a question arises: if the turnpike property is satisfied, which minimizer for (5.3)–(5.4) attracts the optimal solutions to (5.1)–(5.2)? Namely, for which optimal pair $(\overline{u},\overline{y})$ for (5.3)–(5.4) do we have the estimate

$$\|u^T(t) - \overline{u}\|_{L^\infty(B(0,r))} + \|y^T(t) - \overline{y}\|_{L^\infty(B(0,R))} \le K[e^{-\mu t} + e^{-\mu(T-t)}], \quad \forall t\in[0,T],$$

where the constants K and $\mu > 0$ are independent of the time horizon T?

According to [26, Theorem 1, Section 3], this depends on the sign of the second differential of the functional J_s computed at the minima, which in turn is linked to the sign of the term $\beta\chi_{B(0,R)\setminus B(0,r)} - f''(\overline{y})\overline{q}$.

Appendix A. Preliminaries for boundary control

In this section, we present some results in boundary control. We accomplish them in a general space domain Ω.

Let Ω be a bounded open subset of $\mathbb{R}^n$, with boundary $\partial\Omega \in C^\infty$. The nonlinearity $f \in C^1(\mathbb{R})$ is increasing and $f(0) = 0$. We introduce the class of test functions

$$\mathscr{C} := \{\varphi \in C^2(\overline{\Omega}) \mid \varphi(x) = 0,\ \forall x \in \partial\Omega\}$$

and the following notion of solution.

Definition A.1. Let $u \in L^\infty(\partial\Omega)$. Then $y \in L^\infty(\Omega)$ is said to be a *solution* to the boundary value problem

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ y = u & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

if for any test function $\varphi \in \mathscr{C}$, we have

$$\int_\Omega [-y\Delta\varphi + f(y)\varphi]\,dx + \int_{\partial\Omega} u \frac{\partial\varphi}{\partial n}\,d\sigma(x) = 0,$$

where n is the outward normal to $\partial\Omega$.

We have the following existence and uniqueness result, inspired by [14, proof of Proposition 5.1].

Proposition 1. *Let $u \in L^\infty(\partial\Omega)$. There exists a unique solution $y \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$ to (A.1), with*

$$\|y\|_{L^{2^*}(\Omega)} \le K\|u\|_{L^2(\partial\Omega)}, \tag{A.2}$$

the constant $K = K(\Omega)$ being independent of the nonlinearity f and $2^* = \frac{2n}{n-1}$. If the boundary control u is in $H^{1/2}(\partial\Omega) \cap C^0(\partial\Omega)$, then in fact $y \in H^1(\Omega) \cap C^0(\overline{\Omega})$.

One of the key points of the proof will be the increasing character of the nonlinearity.

Proof of Proposition 1. **Step 1:** *Solving a nonhomogeneous linear problem.* By [24, Théorème 7.4, p. 202], there exists a unique solution $y_1 \in H^{1/2}(\Omega)$ to the nonhomogeneous boundary value problem

$$\begin{cases} -\Delta y_1 = 0 & \text{in } \Omega, \\ y_1 = u & \text{on } \partial\Omega. \end{cases} \tag{A.3}$$

The boundary value u is in $L^\infty(\partial\Omega)$. Hence, by a comparison argument, we have $y_1 \in L^\infty(\Omega)$.

Step 2: *Solving a homogeneous semilinear problem.* Since the nonlinearity f is increasing, by adapting the techniques of [5, Theorem 4.7, p. 29], there exists a unique $y_2 \in H_0^1(\Omega)$ solution to

$$\begin{cases} -\Delta y_2 + f(y_1 + y_2) = 0 & \text{in } \Omega, \\ y_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.4}$$

By a comparison argument, since $y_1 \in L^\infty(\Omega)$, we have $y_2 \in L^\infty(\Omega)$. Then $y = y_1 + y_2 \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$ is the unique solution to (A.1).

Step 3: *Proof of the estimate* (A.2). By a comparison argument, we have

$$|y| \le \hat{y} \quad \text{a.e. } \Omega, \tag{A.5}$$

with

$$\begin{cases} -\Delta \hat{y} = 0 & \text{in } \Omega, \\ \hat{y} = |u| & \text{on } \partial\Omega. \end{cases} \tag{A.6}$$

Now, by [24, Théorème 7.4, p. 202] the solution $\hat{y}$ is in $H^{1/2}(\Omega)$, with

$$\|\hat{y}\|_{H^{1/2}(\Omega)} \leq K\|u\|_{L^2(\partial\Omega)}. \tag{A.7}$$

The above inequality, together with the fractional Sobolev embedding $H^{1/2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ (see e.g. [11, Theorem 6.7]), yields

$$\|\hat{y}\|_{L^{2^*}(\Omega)} \leq \|\hat{y}\|_{H^{1/2}(\Omega)} \leq K\|u\|_{L^2(\partial\Omega)},$$

whence by (A.5), we have

$$\|y\|_{L^{2^*}(\Omega)} \leq \|\hat{y}\|_{L^{2^*}(\Omega)} \leq K\|u\|_{L^2(\partial\Omega)},$$

with $K = K(\Omega)$, as required.

Step 4: *Improved regularity.* Since $\partial\Omega \in C^\infty$, by [24, Théorème 7.4, p. 202] and [16, Proposition 1.29, p. 14] the solution y_1 to (A.3) is in $H^1(\Omega) \cap C^0(\overline{\Omega})$. Now, y_2 solves the linear problem

$$\begin{cases} -\Delta y_2 + c y_2 = -f(y_1) & \text{in } \Omega, \\ y_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{A.8}$$

with bounded coefficient

$$c(x) := \begin{cases} \dfrac{f(y_1(x) + y_2(x)) - f(y_1(x))}{y_2(x)}, & y_2(x) \neq 0, \\ f'(y_1(x)), & y_2(x) = 0. \end{cases}$$

Then, by [24, Théorème 7.4, p. 202] and [17, Theorem 8.30, p. 206] applied to (A.8), y_2 is in $H^1(\Omega) \cap C^0(\overline{\Omega})$. Hence, $y = y_1 + y_2 \in H^1(\Omega) \cap C^0(\overline{\Omega})$, as desired. ■

We now state and prove some lemmas needed in the main body of the article.

Lemma A.2. *Let $u \in L^\infty(\partial\Omega)$ be a control. Let y be the solution to (A.1) with control u. Assume the nonlinearity f is strictly increasing and y is constant. Then $y \equiv 0$ and $u \equiv 0$.*

Proof. Suppose there exists $c \in \mathbb{R}$ such that $y(x) = c$ for any $x \in \Omega$. Then, by Definition A.1, for any test function $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega [-c\Delta\varphi + f(c)\varphi]\, dx = 0,$$

where n is the outward normal to $\partial\Omega$ and $C_c^\infty(\Omega)$ denotes the class of infinitely differentiable functions with compact support in Ω. Integrating by parts, we have

$$\int_\Omega f(c)\varphi\, dx = 0$$

for any $\varphi \in C_c^\infty(\Omega)$, which leads to $f(c) = 0$. Now, $f(0) = 0$ and f is strictly increasing. Hence $f(c) = 0$ if and only if $c = 0$, whence $y \equiv 0$ and $u \equiv 0$. ■

In the next lemma, x^i and $\mathbf{e}^i$ denote respectively the ith component of the vector $x \in \mathbb{R}^n$ and the ith element of the canonical base of $\mathbb{R}^n$.

Lemma A.3. *Let Ω be an open set. Let $E \subset \Omega$ be a Lebesgue measurable set with positive Lebesgue measure. Then for a.e. $\hat{x} \in \Omega$ and for any $i = 1, \ldots, n$ there exists a sequence $\{x_m^i\}_{m\in\mathbb{N}} \subset \mathbb{R}$ (with $x_m^i \mathbf{e}^i + \hat{x} \neq \hat{x}$) such that*

$$x_m^i \mathbf{e}^i + \hat{x} \xrightarrow[m\to+\infty]{} \hat{x}. \tag{A.9}$$

Proof. Let us introduce the set of component-isolated points of E,

$$E_{\text{ci}} := \bigcup_{\substack{i=1,\ldots,n \\ r>0}} \{x \in E \mid B_{i,r} \cap E = \{x\}\}, \tag{A.10}$$

where

$$B_{i,r} := \{(x^1, \ldots, x^{i-1}, y, x^{i+1}, \ldots, x^n) \mid y \in [x^i - r, x^i + r]\}. \tag{A.11}$$

Step 1: *Reduction to a single component and radius.* By the the above definitions and the density of rational numbers in the reals, we have

$$E_{\text{ci}} = \bigcup_{\substack{i=1,\ldots,n \\ r>0 \text{ and } r\in\mathbb{Q}}} \{x \in E \mid B_{i,r} \cap E = \{x\}\}. \tag{A.12}$$

By the countable additivity of the Lebesgue measure, we reduce the task to proving that, for any $i = 1, \ldots, n$ and any $r > 0$, the set

$$E_{\text{ci},i,r} := \{x \in E \mid B_{i,r} \cap E = \{x\}\} \tag{A.13}$$

is Lebesgue measurable and has Lebesgue measure zero.

Step 2: *Conclusion.* The measurability of $E_{\text{ci},i,r}$ follows from the continuity of the distance function. Let us compute its measure. For any $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \in \mathbb{R}^{n-1}$, set

$$E_{(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n)} := \{y \in \mathbb{R} \mid (x^1, \ldots, x^{i-1}, y, x^{i+1}, \ldots, x^n) \in E_{\text{ci},i,r}\}, \tag{A.14}$$

where we have dropped the subscript ci, i, r to avoid heavy notation.

Now, by definition of $E_{\text{ci},i,r}$, for any $x \in E_{\text{ci},i,r}$, the set $E_{(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n)}$ is at most countable, whence of null Lebesgue measure. Then, by Fubini's theorem, we have

$$\mu_{\text{Leb}}(E_{\text{ci},i,r}) = \int_{\mathbb{R}^{n-1}} \mu_{\text{Leb}}(E_{(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n)})\, d(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = 0,$$

as required. ■

Lemma A.4. *Let* Ω *be an open set. Let* y_1 *and* y_2 *be two functions of class* $C^2(\Omega)$*. Then*

$$\mu_{\text{Leb}}(\{x \in E \mid y_1(x) = y_2(x)\}) = \mu_{\text{Leb}}(\{x \in E \mid y_1(x) = y_2(x) \text{ and } \nabla y_1(x) = \nabla y_2(x)\}).$$

Proof. If $\mu_{\text{Leb}}(\{x \in E \mid y_1(x) = y_2(x)\})$, the conclusion follows. Otherwise, let us apply Lemma A.3, getting for a.e. $\hat{x} \in \Omega$ and for any $i = 1, \dots, n$ a sequence $\{x^i_m\}_{m \in \mathbb{N}} \subset \mathbb{R}$ (with $x^i_m \mathbf{e}^i + \hat{x} \neq \hat{x}$) such that

$$x^i_m \mathbf{e}^i + \hat{x} \xrightarrow[m \to +\infty]{} \hat{x}.$$

Then

$$\frac{\partial y_2}{\partial x_i}(\hat{x}) = \lim_{m \to +\infty} \frac{y_2(x^i_m \mathbf{e}^i + \hat{x}) - y_2(\hat{x})}{x^i_m \mathbf{e}^i + \hat{x} - \hat{x}} = \lim_{m \to +\infty} \frac{y_1(x^i_m \mathbf{e}^i + \hat{x}) - y_1(\hat{x})}{x^i_m \mathbf{e}^i + \hat{x} - \hat{x}} = \frac{\partial y_1}{\partial x_i}(\hat{x}),$$

whence $\nabla y_2(\hat{x}) = \nabla y_1(\hat{x})$, as required. ∎

Lemma A.5. *Let* $u \in L^\infty(\partial B(0, R))$ *be nonconstant. Then there exists an orthogonal matrix* M *such that*

$$u \circ M \neq u. \tag{A.15}$$

Proof. In the present proof, we denote by $\tilde{u}$ a representative of the equivalence class $u \in L^\infty(\partial B(0, R))$. By [28, Theorem 7.7], a.e. $x \in \partial B(0, R)$ is a Lebesgue point for $\tilde{u}$, whence there exist Lebesgue points $x_1 \neq x_2$ such that $\tilde{u}(x_1) \neq \tilde{u}(x_2)$. Let M be an orthogonal matrix such that $Mx_1 = x_2$. Since x_1 and x_2 are Lebesgue points, there exists $r > 0$ such that

$$\int_{\partial B(0,R) \cap B(x_1,r)} \tilde{u}_M(x)\, dx = \int_{\partial B(0,R) \cap B(x_2,r)} \tilde{u}(y)\, dy \neq \int_{\partial B(0,R) \cap B(x_1,r)} \tilde{u}(x)\, dx,$$

where we have used the change of variable $y := Mx$ and $\tilde{u}_M(x) := \tilde{u}(Mx)$. This shows that $u \circ M \neq u$, as required. ∎

We state and prove a well-known result: the rotational invariance of the Laplacian.

Lemma A.6. *Let* $\varphi \in C^2(\Omega)$ *and let* M *be an* $n \times n$ *orthogonal matrix. Then, for any* $x \in \Omega$*,*

$$\Delta(\varphi \circ M) = \Delta(\varphi) \circ M \quad \text{in } \Omega. \tag{A.16}$$

Proof. By the chain rule and the orthogonality of M, we have

$$\text{Hess}(\varphi \circ M) = M^{-1}[\text{Hess}(\varphi) \circ M]M,$$

whence, by the similarity invariance of the trace, for any $x \in \Omega$,

$$\Delta(\varphi \circ M) = \text{Trace}(\text{Hess}(\varphi \circ M)) = \text{Trace}(M^{-1}[\text{Hess}(\varphi) \circ M]M) = \Delta(\varphi) \circ M,$$

as required. ∎

Lemma A.7. *Consider a rotational invariant domain Ω. Let $u \in L^\infty(\partial\Omega)$ be a control and let y be the solution to (A.1), with control u. Let M be an orthogonal matrix. Set $u_M(x) := u(M(x))$ and $y_M(x) := y(M(x))$. Then y_M is a solution to*

$$\begin{cases} -\Delta y_M + f(y_M) = 0 & in\ \Omega, \\ y_M = u_M & on\ \partial\Omega, \end{cases} \tag{A.17}$$

in the sense of Definition A.1. If in addition $u_M = u$ for any orthogonal matrix M, then $y_M = y$, so y is a radial solution.

Proof. As per Definition A.1, let us check that for any test function $\varphi \in \mathscr{C}$, we have

$$\int_\Omega [-y_M(x)\Delta\varphi(x) + f(y_M(x))\varphi(x)]\,dx + \int_{\partial\Omega} u_M(x)\frac{\partial\varphi(x)}{\partial n}\,d\sigma(x) = 0. \tag{A.18}$$

Set $\tilde{x} := Mx$. Since the matrix M is orthogonal, $|\det(M)| = 1$, whence by the change of variables Theorem, the definition of y_M and Lemma A.6,

$$\begin{aligned} &\int_\Omega [-y_M(x)\Delta\varphi(x) + f(y_M(x))\varphi(x)]\,dx \\ &\qquad = \int_\Omega [-y(\tilde{x})\Delta_x\varphi(M^{-1}\tilde{x}) + f(y(\tilde{x}))\varphi(M^{-1}\tilde{x})]\,d\tilde{x} \\ &\qquad = \int_\Omega [-y(\tilde{x})\Delta_{\tilde{x}}\varphi(M^{-1}\tilde{x}) + f(y(\tilde{x}))\varphi(M^{-1}\tilde{x})]\,d\tilde{x} \\ &\qquad = \int_{\partial\Omega} u(\tilde{x})\nabla_{\tilde{x}}\varphi(M^{-1}\tilde{x})\cdot n(\tilde{x})\,d\sigma(\tilde{x}), \end{aligned} \tag{A.19}$$

where in the last inequality we have used that y is a solution to (A.1) with control u. Now, we change back variable $x := M^{-1}\tilde{x}$ in (A.19), getting

$$\int_{\partial\Omega} u(\tilde{x})\nabla_{\tilde{x}}\varphi(M^{-1}\tilde{x})\cdot n(\tilde{x})\,d\sigma(\tilde{x}) = \int_{\partial\Omega} u(Mx)\nabla_x\varphi(x)M^{-1}\cdot Mn(x)\,d\sigma(\tilde{x}),$$

whence (A.18) follows. Therefore, if the control is radial, then for any orthogonal matrix M, y_M is the solution to the same boundary value problem. The uniqueness for (A.1) yields $y_M = y$. ∎

We now prove the existence of a global minimizer for the functional J defined in (1.1)–(1.2). This will be given by the coercivity in L^2 of J, enhanced by employing the regularity of the solutions to the optimality system. As we did in the former section, we are going to accomplish this task in a general space domain Ω. Consider the optimal control problem

$$\begin{aligned} &\min_{u\in L^\infty(\partial\Omega)} J(u), \quad \text{where} \\ &J(u) = \frac{1}{2}\int_{\partial\Omega} |u|^2\,d\sigma(x) + \frac{\beta}{2}\int_\Omega |y-z|^2\,dx, \end{aligned} \tag{A.20}$$

where

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ y = u & \text{on } \partial\Omega. \end{cases} \tag{A.21}$$

Here Ω is a bounded open subset of $\mathbb{R}^n$ with $n = 1, 2, 3$ and $\partial\Omega \in C^\infty$. The nonlinearity $f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ is strictly increasing and $f(0) = 0$. The target z is in $L^\infty(\Omega)$ and $\beta > 0$ is a penalization parameter.

Proposition 2. *Let $z \in L^\infty(\Omega)$ be a target for the state and let J be the corresponding functional, defined in (A.20)–(A.21). There exists a global minimizer $\overline{u} \in L^\infty(\partial\Omega)$ for J.*

Proof of Proposition 2. **Step 1:** *Existence of a minimizer for a constrained problem.* Let $a, b \in \mathbb{R}$ with $a < 0 < b$ and consider the convex set

$$\mathbb{K} := \{u \in L^\infty(\partial\Omega) \mid a \le u \le b, \text{ a.e. } \partial\Omega\}.$$

Under the same assumptions of (A.20)–(A.21), we consider the constrained optimal control problem

$$\begin{aligned} &\min_{u \in \mathbb{K}} J(u), \quad \text{where} \\ &J(u) = \frac{1}{2} \int_{\partial\Omega} |u|^2 \, d\sigma(x) + \frac{\beta}{2} \int_\Omega |y - z|^2 \, dx, \end{aligned} \tag{A.22}$$

where

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } \Omega, \\ y = u & \text{on } \partial\Omega. \end{cases} \tag{A.23}$$

By using the techniques in [10], we have the existence of an optimal control $\overline{u}_{(a,b)} \in \mathbb{K}$, and any optimal control is given by $\overline{u}_{(a,b)} = \mathbb{P}_{[a,b]}(\frac{\partial \overline{q}_{(a,b)}}{\partial n})$ with

$$\begin{cases} -\Delta \overline{y}_{(a,b)} + f(\overline{y}_{(a,b)}) = 0 & \text{in } \Omega, \\ \overline{y}_{(a,b)} = \mathbb{P}_{[a,b]}\left(\dfrac{\partial \overline{q}_{(a,b)}}{\partial n}\right) & \text{on } \partial\Omega, \\ -\Delta \overline{q}_{(a,b)} + f'(\overline{y}_{(a,b)})\overline{q}_{(a,b)} = \beta(\overline{y}_{(a,b)} - z) & \text{in } \Omega, \\ \overline{q}_{(a,b)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.24}$$

where $\mathbb{P}_{[a,b]}$ is the projector

$$\mathbb{P}_{[a,b]}(\xi) := \begin{cases} a & \text{if } \xi \le a, \\ \xi & \text{if } a < \xi < b, \\ b & \text{if } \xi \ge b. \end{cases} \tag{A.25}$$

Step 2: *L^∞ bounds for optimal controls uniform on $(a, b) \in \mathbb{R}^2$ with $a < 0 < b$.* Since $a < 0 < b$, the null control 0 is in $\mathbb{K}$. Then, for any optimal control $\overline{u}_{(a,b)}$ for (A.22)–(A.23), we have

$$\frac{1}{2} \int_{\partial\Omega} |\overline{u}_{(a,b)}|^2 \, d\sigma(x) \le J(\overline{u}_{(a,b)}) \le J(0) \le K,$$

whence

$$\|\overline{u}_{(a,b)}\|_{L^2(\partial\Omega)} \leq K, \tag{A.26}$$

where $K = K(\Omega, f, \beta, z)$ is independent of (a, b).

We now bootstrap in the optimality system (A.24), to get the desired L^∞ bound, given the above L^2 bound.

First of all, by a comparison argument, we have

$$|\overline{y}_{(a,b)}| \leq \hat{y}_{(a,b)}, \quad \text{a.e. } \Omega, \tag{A.27}$$

with

$$\begin{cases} -\Delta \hat{y}_{(a,b)} = 0 & \text{in } \Omega, \\ \hat{y}_{(a,b)} = |\overline{u}_{(a,b)}| & \text{on } \partial\Omega. \end{cases} \tag{A.28}$$

Comparison also gives

$$|\overline{q}_{(a,b)}| \leq \hat{q}_{(a,b)} \quad \text{and} \quad \left|\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right| \leq \left|\frac{\partial \hat{q}_{(a,b)}}{\partial n}\right|, \quad \text{a.e. } \Omega, \tag{A.29}$$

with

$$\begin{cases} -\Delta \hat{q}_{(a,b)} = \beta|\overline{y}_{(a,b)} - z| & \text{in } \Omega, \\ \hat{q}_{(a,b)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{A.30}$$

Now, by [24, Théorème 7.4, p. 202], the solution $\hat{y}_{(a,b)}$ is in $H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)$ and

$$\|\hat{y}_{(a,b)}\|_{L^3(\Omega)} \leq K\|\hat{y}_{(a,b)}\|_{H^{1/2}(\Omega)} \leq K\|\overline{u}_{(a,b)}\|_{L^2(\partial\Omega)} \leq K,$$

where the first inequality is given by the Sobolev embedding $H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)$ valid for space dimension $n = 1, 2, 3$ (see e.g. [11, Theorem 6.7]) and the last inequality is justified by (A.26). By (A.27),

$$\|\overline{y}_{(a,b)}\|_{L^3(\Omega)} \leq \|\hat{y}_{(a,b)}\|_{L^3(\Omega)} \leq K.$$

We now concentrate on the adjoint equation. By [19, Theorem 2.4.2.5, p. 124] applied to (A.30), we have $\hat{q}_{(a,b)} \in W^{2,3}(\Omega)$ with

$$\|\hat{q}_{(a,b)}\|_{W^{2,3}(\Omega)} \leq K\|\overline{y}_{(a,b)} - z\|_{L^3(\Omega)} \leq K[\|\overline{y}_{(a,b)}\|_{L^3(\Omega)} + \|z\|_{L^\infty(\Omega)}] \leq K.$$

By the trace theorem [19, Theorem 1.5.1.3, p. 38] applied to $\nabla \hat{q}_{(a,b)}$,

$$\left\|\frac{\partial \hat{q}_{(a,b)}}{\partial n}\right\|_{L^4(\partial\Omega)} \leq K\|\hat{q}_{(a,b)}\|_{W^{2,3}(\Omega)} \leq K.$$

By (A.29), we then have

$$\left\|\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right\|_{L^4(\partial\Omega)} \leq \left\|\frac{\partial \hat{q}_{(a,b)}}{\partial n}\right\|_{L^4(\partial\Omega)} \leq K, \tag{A.31}$$

whence

$$\|\overline{u}_{(a,b)}\|_{L^4(\partial\Omega)} = \left\|\mathbb{P}_{[a,b]}\left(\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right)\right\|_{L^4(\partial\Omega)} \leq \left\|\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right\|_{L^4(\partial\Omega)} \leq K.$$

By using the definition of solution by transposition for (A.28) and the above estimate, we get

$$\|\hat{y}_{(a,b)}\|_{L^4(\Omega)} \leq K\|\overline{u}_{(a,b)}\|_{L^4(\partial\Omega)} \leq K,$$

whence, by (A.27),

$$\|\overline{y}_{(a,b)}\|_{L^4(\Omega)} \leq \|\hat{y}_{(a,b)}\|_{L^4(\Omega)} \leq K.$$

In conclusion, we employ the elliptic regularity [19, Theorem 2.4.2.5, p. 124] in (A.30) to get

$$\|\hat{q}_{(a,b)}\|_{W^{2,4}(\Omega)} \leq K\|\overline{y}_{(a,b)} - z\|_{L^4(\Omega)} \leq K,$$

whence, by Sobolev embeddings in space dimension $n = 1, 2, 3$,

$$\|\hat{q}_{(a,b)}\|_{C^1(\overline{\Omega})} \leq \|\hat{q}_{(a,b)}\|_{W^{2,4}(\Omega)} \leq K\|y - z\|_{L^4(\Omega)} \leq K.$$

Now, (A.29) yields

$$\left\|\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right\|_{C^0(\partial\Omega)} \leq \left\|\frac{\partial \hat{q}_{(a,b)}}{\partial n}\right\|_{C^0(\partial\Omega)} \leq \|\hat{q}_{(a,b)}\|_{C^1(\overline{\Omega})} \leq K, \tag{A.32}$$

which in turn implies

$$\|\overline{u}_{(a,b)}\|_{L^\infty(\partial\Omega)} = \left\|\mathbb{P}_{[a,b]}\left(\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right)\right\|_{L^\infty(\partial\Omega)} \leq \left\|\frac{\partial \overline{q}_{(a,b)}}{\partial n}\right\|_{L^\infty(\partial\Omega)} \leq K,$$

where the last inequality follows from (A.32). We then have the estimate

$$\|\overline{u}_{(a,b)}\|_{L^\infty(\partial\Omega)} \leq K, \quad \forall a, b \in \mathbb{R} \text{ with } a < 0 < b, \tag{A.33}$$

the constant $K = K(\Omega, f, \beta, z)$ being independent of (a, b). This finishes this step.

Step 3: *Conclusion.* Let K be the upper bound appearing in (A.33). We want to show that, for any control $u \in L^\infty(\partial\Omega)$ with $\|u\|_{L^\infty(\partial\Omega)} > K$,

$$J(u) > \inf_{B^{L^\infty}(0,K)} J,$$

Indeed, for any control $u \in L^\infty(\partial\Omega)$ with $\|u\|_{L^\infty(\partial\Omega)} > K$, set $b := \|u\|_{L^\infty(\partial\Omega)} + 1$, $a := -b$, and define accordingly the control set

$$\mathbb{K} := \{u \in L^\infty(\partial\Omega) \mid a \leq u \leq b, \text{ a.e. } \partial\Omega\}.$$

By definition of a and b, the control u is in $\mathbb{K}$, and by (A.33),

$$J(u) > \inf_{B^{L^\infty}(0,K)} J, \tag{A.34}$$

as desired. Now, by Step 1, there exists $\overline{u} \in \overline{B^{L^\infty}(0, K)}$ minimizing J in $\overline{B^{L^\infty}(0, K)}$. By (A.34), $\overline{u}$ is in fact a global minimizer for J in $L^\infty(\partial\Omega)$, concluding the proof. ∎

Proof of Proposition 3.2. **Step 1:** *Proof of* (1). Fix $z \in L^\infty(B(0,R))$. The existence of a minimizer u_z is a consequence of the direct methods in the calculus of variations. Moreover, by (3.2), the definition of minimizer and $G(0) = 0$,

$$\begin{aligned}\tfrac{1}{2}R^{n-1}n\alpha(n)|u_z|^2 &\le I(u_z, z) + \frac{\beta}{2}\int_{B(0,R)} |z|^2\,dx \\ &\le I(0,z) + \frac{\beta}{2}\int_{B(0,R)} |z|^2\,dx = \frac{\beta}{2}\int_{B(0,R)} |z|^2\,dx,\end{aligned}$$

which yields $\frac{1}{2}|u_z|^2 \le \frac{\beta}{2R^{n-1}n\alpha(n)}\int_{B(0,R)} |z|^2\,dx$, as required.

Step 2: *Proof of* (2). Fix $M \in \mathbb{R}^+$. For any pair of targets $(z_1, z_2) \in L^\infty(B(0,R))^2$ such that

$$\|z_1\|_{L^2} \le M \quad \text{and} \quad \|z_2\|_{L^2} \le M,$$

and for each control $u \in C$ such that $|u| \le \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}\,M$, we have

$$\begin{aligned}I(u, z_2) - I(u_{z_1}, z_1) &= I(u, z_2) - I(u, z_1) + I(u, z_1) - I(u_{z_1}, z_1) \\ &\ge -|I(u, z_2) - I(u, z_1)| + 0 = -\beta\left|\int_{B(0,R)} G(u)(z_1 - z_2)\,dx\right| \ge -K\|z_2 - z_1\|_{L^\infty},\end{aligned}$$

where the last inequality is justified by $|u| \le \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}\,M$ and the continuity of the control-to-state map G.

Then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$I(u, z_2) - I(u_{z_1}, z_1) > -\varepsilon$$

whenever $\|z_2 - z_1\|_{L^\infty} < \delta_\varepsilon$.

Now, by the first step, any minimizer u_{z_2} for $I(\cdot, z_2)$ satisfies

$$|u_{z_2}| \le \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}\,\|z_2\|_{L^2} \le \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}\,M.$$

Thus, we have proved that

$$\inf_C I(\cdot, z_2) - \inf_C I(\cdot, z_1) = I(u_{z_2}, z_2) - I(u_{z_1}, z_1) > -\varepsilon.$$

Exchanging the roles of z_1 and z_2, one can get

$$\inf_C I(\cdot, z_1) - \inf_C I(\cdot, z_2) > -\varepsilon.$$

This yields the continuity of h. ■

Proof of Lemma 3.3. If $h_1(z^0) = h_2(z^0)$, we take $\tilde{z} := z^0$, concluding the proof. Let us now suppose $h_1(z^0) \ne h_2(z^0)$. We consider the case $h_1(z^0) < h_2(z^0)$.

Step 1: *Proof of the existence of* $\mu_0 \geq 0$ *such that*

- $\forall \mu \in [0, \mu_0]$, $h_2(z^0 + \mu) < 0$,
- $h_1(z^0 + \mu_0) = 0$.

First of all, we observe that $h_2(z^0 + \mu) < 0$ for any $\mu \geq 0$. Indeed, since $h_2(z^0) < 0$, there exists $u_2 > 0$ such that $I(u_2, z^0) < 0$. Then

$$\begin{aligned} h_2(z^0 + \mu) &\leq I(u_2, z^0 + \mu) \\ &= \frac{R^{n-1} n\alpha(n)}{2} |u_2|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(u_2)|^2 \, dx - \beta \int_{B(0,R)} (z^0 + \mu) G(u_2) \, dx \\ &= I(u_2, z^0) - \mu\beta \int_{B(0,R)} G(u_2) \, dx \leq I(u_2, z^0) < 0, \end{aligned}$$

where we have used the fact that $G(u_2) \geq 0$ a.e. in $B(0, R)$.

We now prove that $h_1(z^0 + \mu_0) = 0$ for $\mu_0 = \|z^0\|_{L^\infty}$. Indeed, for any $v \leq 0$,

$$\begin{aligned} &I(v, z^0 + \mu_0) \\ &\quad = \frac{R^{n-1} n\alpha(n)}{2} |v|^2 + \frac{\beta}{2} \int_{B(0,R)} |G(v)|^2 \, dx - \beta \int_{B(0,R)} (z^0 + \mu_0) G(v) \, dx \geq 0, \end{aligned}$$

since $z^0 + \mu_0 \geq 0$ and $G(v) \leq 0$ a.e. in $B(0, R)$. This finishes the first step.

Step 2: *Conclusion.* Set

$$g : [0, \mu_0] \to \mathbb{R}, \quad \mu \mapsto h_2(z^0 + \mu) - h_1(z^0 + \mu).$$

Since $h_1(z^0) < h_2(z^0)$, we have $g(0) > 0$, and by Step 1, $g(\mu_0) < 0$. Then, by continuity, there exists $\mu_1 \in (0, \mu_0)$ such that $g(\mu_1) = 0$. Hence,

$$\tilde{z} := z^0 + \mu_1$$

is the desired target. Indeed, by definition of g and μ_1, $h_1(\tilde{z}) = h_2(\tilde{z})$. Furthermore, since $\mu_1 \in (0, \mu_0)$, by Step 1, $h_2(\tilde{z}) < 0$. This concludes the proof for $h_1(z^0) < h_2(z^0)$. The proof for $h_1(z^0) > h_2(z^0)$ is similar. ■

Appendix B. Preliminaries for internal control

We now consider the state equation (1.7) on a general domain. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^2$ and $n = 1, 2, 3$. The nonlinearity $f \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ is strictly increasing and $f(0) = 0$. The control acts in a nonempty open subset ω of Ω.

We introduce the concept of solution, following [5, Theorem 4.7, p. 29].

Definition B.1. Let $u \in L^2(\omega)$. Then $y \in H_0^1(\Omega)$ is said to be a *solution* to

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{B.1}$$

if $f \in L^1(\Omega)$ and for any test function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega [\nabla y \cdot \nabla \varphi + f(y)\varphi]\, dx = \int_\omega u\varphi\, dx.$$

The well-posedness of (B.1) follows from [5, Theorem 4.7, p. 29].

Lemma B.2. *Let $u \in L^\infty(\omega)$ be a control. Let y be the solution to (B.1) with control u. Assume the nonlinearity f is strictly increasing and y is constant in $\Omega \setminus \omega$. Then $y \equiv 0$ and $u \equiv 0$.*

Proof. Suppose there exists $c \in \mathbb{R}$ such that $y(x) = c$ for any $x \in \Omega \setminus \omega$. Then, by Definition A.1, for any $\varphi \in C_c^\infty(\Omega \setminus \omega)$, we have

$$\int_\Omega f(c)\varphi\, dx = \int_\Omega [\nabla y \cdot \nabla \varphi + f(y)\varphi]\, dx = \int_\omega u\varphi\, dx = 0.$$

The arbitrariness of $\varphi \in C_c^\infty(\Omega \setminus \omega)$ leads to $f(c) = 0$. Now, $f(0) = 0$ and f is strictly increasing. Hence $f(c) = 0$ if and only if $c = 0$, whence $y \equiv 0$ and $u \equiv 0$. ∎

Lemma B.3. *In the notation of (B.1), consider rotational invariant domains Ω and ω. Let $u \in L^\infty(\omega)$ be a control and let y be the solution to (A.1) with control u. Let M be an orthogonal matrix. Set $u_M(x) := u(M(x))$ and $y_M(x) := y(M(x))$. Then y_M is a solution to*

$$\begin{cases} -\Delta y_M + f(y_M) = u_M \chi_\omega & \text{in } \Omega, \\ y_M = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.2}$$

in the sense of Definition B.1. If in addition $u_M = u$ for any orthogonal matrix M, then $y_M = y$, so y is a radial solution.

Proof. As per Definition B.1, let us check that for any test function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega [\nabla y_M \cdot \nabla \varphi + f(y_M)\varphi]\, dx = \int_\omega u_M \varphi\, dx. \tag{B.3}$$

Set $\tilde{x} := Mx$. Since the matrix M is orthogonal, $|\det(M)| = 1$, whence by the change of variables theorem and the definition of y_M,

$$\begin{aligned}
&\int_\Omega [\nabla y_M \cdot \nabla \varphi + f(y_M)\varphi]\, dx \\
&\qquad = \int_\Omega [(\nabla_{\tilde{x}} y(Mx) M) \cdot \nabla \varphi + f(y_M)\varphi]\, dx \\
&\qquad = \int_\Omega [\nabla_{\tilde{x}} y(Mx) \cdot (\nabla_x \varphi(x) M^{-1}) + f(y_M)\varphi]\, dx \\
&\qquad = \int_\Omega [\nabla_{\tilde{x}} y(\tilde{x}) \cdot \nabla_{\tilde{x}} \varphi(M^{-1}\tilde{x}) + f(y_M(M^{-1}\tilde{x}))\varphi(M^{-1}\tilde{x})]\, d\tilde{x} \\
&\qquad = \int_\Omega [\nabla_{\tilde{x}} y(\tilde{x}) \cdot \nabla_{\tilde{x}} \varphi(M^{-1}\tilde{x}) + f(y(\tilde{x}))\varphi(M^{-1}\tilde{x})]\, d\tilde{x} \\
&\qquad = \int_\omega u(\tilde{x})\varphi(M^{-1}\tilde{x})\, d\tilde{x},
\end{aligned} \tag{B.4}$$

where in the last inequality we have used that y is a solution to (A.1) with control u. Now, we change back variable $x := M^{-1}\tilde{x}$ in (B.4), getting

$$\int_\omega u(\tilde{x})\varphi(M^{-1}\tilde{x})\, d\tilde{x} = \int_\omega u(Mx)\varphi(x)\, dx = \int_\omega u_M(x)\varphi(x)\, dx,$$

whence (B.3) follows. Therefore, if the control is radial, then for any orthogonal matrix M, y_M is the solution to the same boundary value problem. The uniqueness for (B.1) yields $y_M = y$. ■

Acknowledgements. We are grateful to Professor Enrique Zuazua for his helpful remarks on the manuscript. We thank Professor Martin Gugat for his interesting questions. We gratefully acknowledge the referees' interesting comments.

Funding. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 694126-DYCON).

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