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Trisecant flops, their associated K3 surfaces and the rationality of some cubic fourfolds

Dedicated to Antonio Lanteri on the occasion of his seventieth birthday

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Abstract. We provide a new construction of rationality for cubic fourfolds via Mori's theory and the minimal model program. As an application we present the solution of the Kuznetsov conjecture for $d = 42$ (the first open case). Our methods also show an explicit connection between the rationality of cubic fourfolds belonging to the first four admissible families $\mathcal{C}_d$ with $d = 14, 26, 38$, and 42 and some birational models of K3 surfaces of degree d contained in well known rational Fano fourfolds.

Keywords. Rationality of cubic fourfolds, flops, Mori theory

Introduction

The study of the rationality of higher-dimensional Fano manifolds is a very active area of research. Many new and interesting contributions and conjectures appeared in the last decades, mostly concerning the irrationality of very general Fano complete intersections (see for example [28, 36, 45], and also [21] and references therein). Deep recent contributions in [22] imply that the locus of geometrically rational fibres in a smooth family of projective manifolds is closed under specialization, improving substantially our understanding of the loci of rational objects in the corresponding moduli spaces (see [17] for very significant examples in dimension four). Notwithstanding, the irrationality of the very general cubic fourfold and the complete description of the rational ones remain two of the most challenging open problems.

A great amount of recent theoretical work on cubic fourfolds (see for example the surveys [16, 23]) led to the expectation that the very general ones might be irrational

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and to the specification of infinitely many irreducible divisors $\mathcal{C}_d$ of special admissible cubic fourfolds of discriminant d in the moduli space $\mathcal{C}$, whose union should be the locus of rational cubic fourfolds (*Kuznetsov Conjecture*). According to this conjecture, the rationality of cubics in $\mathcal{C}_d$ depends on the existence of an associated K3 surface in the sense of Hassett/Kuznetsov [16, 23].

The first admissible values are $d = 14, 26, 38, 42, 62, 74, 78, 86$. Our main applications of the new methods developed here will be the theoretical explanation of the role of (non-minimal) K3 surfaces in determining the rationality for $d = 14, 26, 38, 42$ together with the proof of the rationality of every cubic fourfold in $\mathcal{C}_{42}$ (the first open case of the conjecture) via the construction of a surface of degree 9 and genus 2 with five nodes admitting a congruence of 8-secant twisted cubics and contained in a general cubic fourfold in $\mathcal{C}_{42}$ (see Theorem 5.12). Let us recall that Fano showed the rationality of a general cubic fourfold in $\mathcal{C}_{14}$ (see [5, 8]), while every cubic fourfold in the irreducible divisors $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ is rational by the main results of [34] (see also [35] and Section 3 here). The proofs in [34] were achieved by constructing surfaces $S_d \subset \mathbb{P}^5$, contained in a general cubic fourfold of $\mathcal{C}_d$ and admitting a four-dimensional family of $(3e - 1)$-secant curves of degree $e \geq 2$ parametrized by a rational variety with the property that through a general point of $\mathbb{P}^5$ there passes a unique curve of the family. Then the cubics through S_d become rational sections of the universal family and hence are birational to the rational parameter space (see also Theorem 1.1 here).

This approach did not clarify the relation with the associated K3 surfaces, and even the construction of explicit birational maps from general cubics $X \subset \mathbb{P}^5$ in $\mathcal{C}_{26}$ and in $\mathcal{C}_{38}$ to $\mathbb{P}^4$ (or to other notable rational smooth fourfolds W) in [35] apparently did not provide a birational incarnation in $\mathbb{P}^4$ (or in W) of a K3 surface of genus 14, respectively of genus 20, determining the linear system of the inverse map. Indeed, the base loci of the linear systems of hypersurfaces of degree $3e - 1$ having points of multiplicity e along the corresponding S_d's giving the birational maps $\mu : X \dashrightarrow \mathbb{P}^4$ are intractable reducible schemes, while the base loci of the inverse maps $\mu^{-1} : \mathbb{P}^4 \dashrightarrow X \subset \mathbb{P}^5$ are even worse (see [35, Table 2]).

Here we study all these phenomena via Mori theory and via the Minimal Model Program to explain the birational nature of the maps $\mu : X \dashrightarrow W$ introduced above and of their inverses, which allowed us to produce a suitable birational factorization described in diagram (0.1) below. This approach provides a geometric description of the support of the base loci, and finally illustrates the relations of these explicit birational maps to the K3 surfaces associated to X (see Sections 3.4 and 5.5). Last but not least, computer-aided methods play a central role in some key points, also due to the complexity of the geometry involved.

Our method starts with the observation that many of the known examples of surfaces $S_d \subset \mathbb{P}^5$ used to describe the special divisors $\mathcal{C}_d$ for small d (not only the admissible ones) have ideal generated by cubic forms defining a map $\varphi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N$ which is birational onto its image. The restriction to a general cubic X through S_d defines a birational map $\varphi : X \dashrightarrow Y \subset \mathbb{P}^{N-1}$ with Y a general hyperplane section of Z. In many cases, the birational morphism $\tilde{\varphi} : X' = \mathrm{Bl}_{S_d} X \to Y$ is a small contraction, whose exceptional locus consists of a (union of) smooth surface(s) $T' \subset X'$ ruled by the strict transforms of trisecant lines to S_d. Since $K_{X'}$ is zero on the strict transforms of trisecant lines, the map $\tilde{\varphi}$ is a flop small contraction. In Theorem 2.6 we show that under the previous hypothesis there exists a flop $\tilde{\psi} : W' \to Y$ of the surface $T' \subset X'$ with W' a smooth projective fourfold. This *trisecant flop* $\tau : X' \dashrightarrow W'$ is constructed by analyzing the splitting of $N^*_{T'/X'}$ restricted to a general strict transform of a trisecant line to $S_d \subset \mathbb{P}^5$ (see Remark 2.4). The existence of a congruence of $(3e-1)$-secant rational curves to S_d of degree $e \geq 2$ produces an extremal ray on W' with divisorial locus, giving a birational morphism $\nu : W' \to W$ with W a $\mathbb{Q}$-factorial Fano variety having $\mathrm{Pic}(W) \simeq \mathbb{Z}$ (see Theorem 2.10). The birational morphism ν is (generically) the blow-up of an irreducible surface $U \subset W$, which for the admissible cases $d = 14, 26, 38, 42$ is a birational incarnation of the associated K3 surface to X. Moreover, the map $\mu = \nu \circ \tau \circ \lambda^{-1} : X \dashrightarrow W$ is given by a linear system of forms of degree $3e-1$ with points of multiplicity at least e along S_d, while $\mu^{-1} : W \dashrightarrow X$ is given by a linear system of divisors in $|\mathcal{O}_W(e \cdot i(W) - 1)|$ having points of multiplicity at least e along U, where $i(W)$ is the index of W. Everything is captured by the following diagram, where $R' \subset W'$ is the strict transform on W' of the locus $R \subset W$ of $i(W)$-secant lines to U:

$$\begin{tikzcd}
 & & \mathrm{Bl}_{T'} X' = \mathrm{Bl}_{R'} W' \arrow[dll, "\sigma"'] \arrow[drr, "\omega"] & & \\
X' \arrow[rrrr, dashed, "\tau"] \arrow[d, "\lambda"'] \arrow[drr, "\tilde{\varphi}"] & & & & W' \arrow[d, "\nu"] \arrow[dll, "\tilde{\psi}"'] \\
X \arrow[rr, dashed, "\varphi"] \arrow[rrrr, dashed, bend right, "\mu"'] & & Y & & W \arrow[ll, dashed, "\psi"']
\end{tikzcd} \tag{0.1}$$

In the explicit examples with $d = 14, 26, 38, 42$ considered here, diagram (0.1) together with some standard computations of the middle cohomology of fourfolds under the blow-up of smooth surfaces (see for example the discussion in [16, pp. 45–46]) shows that the (possibly singular non-minimal) K3 surface $U \subset W$ is a birational incarnation of the K3 surface *associated* to X via Hodge theory, or equivalently via derived category theory (see Section 3.4 for the complete analysis of the case $d = 38$).

The above theoretical results (and all the examples we have constructed) point out that, for low values of the discriminant d, the *birational association* between admissible cubic fourfolds and K3 surfaces passes through the construction of very special (and in many cases also singular) non-minimal birational models of these surfaces in the fourfolds W by means of peculiar linear systems of hyperplane sections (often with base points of high multiplicities) on the associated K3 surfaces, exactly as in the case $d = 14$ considered by Fano. In particular, we are able to prove the rationality of every cubic fourfold in $\mathcal{C}_d$ for $d = 14, 26, 38, 42$ via trisecant flops and to show that their inverses are determined by linear systems having a prescribed multiplicity along (non-minimal) birational models of the associated K3 surfaces.

Some of these examples of non-minimal K3 surfaces have also been studied by Voisin [48, §4] to prove vanishing results related to Lehn's Conjecture and have been later considered by Fontanari and Sernesi [10, Theorem 10]. Our constructions of K3 surfaces are based on a different geometric method developed in Section 4 and on computations, which among other things also provide explicit equations for the general K3 surfaces of genera 8, 14, 20, and 22.

A description of $\mathcal{C}_d$ via surfaces S_d with at least two cubics containing them as those considered here (and also in [29, 34, 35]) is related to the uniruledness of $\mathcal{C}_d$ (see the argument in the proof of Theorem 5.9). Since the Kodaira dimension of $\mathcal{C}_d$ is non-negative for $d \geq 86$ admissible (see [43] and also [29, Proposition 1.3]), the admissible values $d = 62, 74, 78$ might be the last ones for which the elements in $\mathcal{C}_d$ can be arranged into (homaloidal) linear systems of cubics through an irreducible surface S_d, simplifying the construction of U and W via a trisecant flop. For $d \geq 86$ admissible one might be forced to consider suitable generalizations of this approach (see Remark 2.8), increasing substantially the difficulty of the solution of other cases of the conjecture.

The techniques introduced here open the way to further applications of this circle of ideas to prove the rationality of other classes of Fano fourfolds (see [35, Section 4] and [18]).

1. Preliminaries

1.1. *Congruences of* $(3e-1)$*-secant curves of degree e to surfaces in* $\mathbb{P}^5$

The following definitions have been introduced in [34, Section 1]. Let $\mathcal{H}$ be an irreducible proper family of (rational or of fixed arithmetic genus) curves of degree e in $\mathbb{P}^5$ whose general element is irreducible. We have a diagram

$$\begin{tikzcd} \mathcal{D} \arrow[d, "\pi"'] \arrow[dr, "p"] & \\ \mathcal{H} & \mathbb{P}^5 \end{tikzcd} \tag{1.1}$$

where $\pi : \mathcal{D} \to \mathcal{H}$ is the universal family over $\mathcal{H}$ and where $p : \mathcal{D} \to \mathbb{P}^5$ is the tautological morphism. Suppose moreover that p is birational and that a general member $[C] \in \mathcal{H}$ is $(re-1)$-secant to an irreducible surface $S \subset \mathbb{P}^5$, that is, $C \cap S$ is a length $re-1$ scheme, $r \in \mathbb{N}$. We shall call such a family $\mathcal{H}$ (or $\mathcal{D}$ or $\pi : \mathcal{D} \to \mathcal{H}$) a *congruence of* $(re-1)$*-secant curves of degree e to S*. Let us remark that necessarily $\dim(\mathcal{H}) = 4$.

An irreducible hypersurface $X \in |H^0(\mathcal{I}_S(r))|$ is said to be *transversal to the congruence* $\mathcal{H}$ if the unique curve of the congruence passing through a general point $p \in X$ is not contained in X. A crucial result is the following.

Theorem 1.1 ([34, Theorem 1]). *Let* $S \subset \mathbb{P}^5$ *be a surface admitting a congruence of* $(re-1)$*-secant curves of degree e parametrized by* $\mathcal{H}$*. If* $X \in |H^0(\mathcal{I}_S(r))|$ *is an irreducible hypersurface transversal to* $\mathcal{H}$*, then* X *is birational to* $\mathcal{H}$*.*

If the map $\Phi = \Phi_{|H^0(\mathcal{I}_S(r))|} : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(\mathcal{I}_S(r)))$ *is birational onto its image, then a general hypersurface* $X \in |H^0(\mathcal{I}_S(r))|$ *is birational to* $\mathcal{H}$.

Moreover, under the previous hypothesis on Φ, *if a general element in* $|H^0(\mathcal{I}_S(r))|$ *is smooth, then every* $X \in |H^0(\mathcal{I}_S(r))|$ *with at worst rational singularities is birational to* $\mathcal{H}$.

Since $p : \mathcal{D} \to \mathbb{P}^5$ is birational, we also have a rational map

$$\varphi = \pi \circ p^{-1} : \mathbb{P}^5 \dashrightarrow \mathcal{H},$$

whose fibre through a general $p \in \mathbb{P}^5$, $F = \overline{\varphi^{-1}(\varphi(p))}$, is the unique curve of the congruence passing through p.

It is natural to ask what linear systems on $\mathbb{P}^5$ give the abstract birational maps $\varphi : \mathbb{P}^5 \dashrightarrow \mathcal{H}$ as above or their restrictions to a general $X \in |H^0(\mathcal{I}_S(r))|$. The linear system $|H^0(\mathcal{I}_S^e(re-1))|$, when not empty, contracts the fibres of φ and in [35] we showed that, quite surprisingly, in many cases this kind of linear systems can provide a birational geometric realization of φ for $r = 3$, yielding birational maps from cubic hypersurfaces through S to $\mathcal{H}$ with $\mathcal{H} = \mathbb{P}^4$ or with $\mathcal{H}$ a Fano fourfold. We shall develop a theoretical framework for these phenomena in order to be able to understand also the birational maps defined by the previous linear systems.

1.2. Divisorial contractions, small contractions and flops

We introduce some general definitions of the Minimal Model Program (MMP for short), adapting them to our setting.

Let X be a smooth projective irreducible fourfold defined over the complex field with $\rho(X) = 1$ (here $\rho(X)$ denotes the Picard number of X) and let $\varphi : X \dashrightarrow W$ be a birational map onto a smooth (or at least $\mathbb{Q}$-factorial) irreducible projective fourfold, whose base locus scheme contains a surface S with at most a finite number of nodes.

Let $\lambda : X' = \mathrm{Bl}_S X \to X$ be the blow-up of S and consider the diagram

$$\begin{tikzcd} \mathrm{Bl}_S X \arrow[d, "\lambda"'] \arrow[dr, dashed, "\tilde{\varphi}"] & \\ X \arrow[r, dashed, "\varphi"] & W \end{tikzcd} \tag{1.2}$$

When W is smooth, the complexity of the birational map $\varphi : X \dashrightarrow W$ depends on the base locus scheme of $\tilde{\varphi} : \mathrm{Bl}_S X \dashrightarrow W$. Surely the easiest case to consider is when $\tilde{\varphi} : \mathrm{Bl}_S X \to W$ is a morphism, that is, φ is a *special birational map* in the sense of Semple and Tyrrell [38] (solved by a single blow-up along a smooth irreducible variety).

If $X \subset \mathbb{P}^5$ is a cubic fourfold and if $S \subset X$ is smooth, few examples of special birational maps of the above type exist. Two examples of maps of this kind were first considered by Fano [8], revisited in modern terms in [1,5], and played a fundamental role in the formulation of the Kuznetsov Conjecture.

Example 1.2. Letting $\varphi : X \dashrightarrow W$ be a special birational map with $X \subset \mathbb{P}^5$ a cubic fourfold, letting $B \subset W$ be the base locus scheme of φ^{-1} and letting $U = B_{\mathrm{red}}$, Fano's examples are the following:

(i) $S \subset \mathbb{P}^5$ is a smooth quintic del Pezzo surface, $W = \mathbb{P}^4$, φ is given by $|H^0(\mathcal{I}_S(2))|$ and $U \subset \mathbb{P}^4$ is a surface of degree 9 and sectional genus 8 having at most a finite number of singular points corresponding to planes in X spanned by conics in S. If non-singular, the surface U is the projection from a 5-secant $\mathbb{P}^3 \subset \mathbb{P}^8$ of a smooth K3 surface of degree 14 and genus 8, and φ^{-1} is given by $|H^0(\mathcal{I}_U(4))|$.

(ii) $S \subset \mathbb{P}^5$ is a smooth quartic rational normal scroll, $W = Q \subset \mathbb{P}^5$ is a smooth quadric hypersurface, φ is given by $|H^0(\mathcal{I}_S(2))|$, and $U \subset \mathbb{P}^5$ is a surface of degree 10 and sectional genus 8 having at most a finite number of singular points corresponding to planes in X spanned by conics in S. If non-singular, the surface U is the projection from the tangent plane of a smooth K3 surface of degree 14 and genus 8, and φ^{-1} is given by $|H^0(\mathcal{I}_U(3))|_{|Q}$.

Remark 1.3. The two surfaces $S \subset \mathbb{P}^5$ appearing in Example 1.2 are the only smooth surfaces in $\mathbb{P}^5$ admitting a congruence of secant lines ($r = 3$ and $e = 1$ in the definition) (see for example [32]). The lines of the congruence contained in X describe the exceptional locus $\overline{E}$ of $\tilde{\varphi}$ (or equivalently the exceptional locus $\lambda(\overline{E})$ of $\varphi : X \dashrightarrow W$) and are birationally parametrized by the surfaces $U \subset W$.

The general MMP philosophy suggests that meaningful birational properties of (rational) cubic fourfolds might be related to small contractions from X'. So one can start to investigate birational properties of cubic fourfolds from the point of view of the MMP and to consider the most elementary links in the Sarkisov Program associated to small contractions, i.e. flops and flips (one may consult [15] for results about this program in arbitrary dimension).

Definition 1.4. Let X be a smooth irreducible projective variety (from now on, a *projective manifold*) and let $\tilde{\varphi} : X \to Y$ be a *small contraction*, i.e. $\tilde{\varphi}$ is a birational morphism onto a normal variety Y inducing an isomorphism in codimension 1 and such that $\rho(X/Y) = 1$.

If $K_X \cdot C = 0$ for every irreducible curve contracted by $\tilde{\varphi}$, then $\tilde{\varphi} : X \to Y$ is called a *small flop contraction*. A small flop contraction $\tilde{\psi} : W \to Y$ with W a projective manifold is called a *flop of* $\tilde{\varphi}$.

The resulting birational map $\tau = \tilde{\psi}^{-1} \circ \tilde{\varphi} : X \dashrightarrow W$ is usually called a *flop* if it is not an isomorphism. Since we assume $\rho(X/Y) = 1 = \rho(W/Y)$, given $\tilde{\varphi}$ one can prove that the morphism $\tilde{\psi}$, if it exists, is unique as long as τ is not an isomorphism.

One can *flop* the small contraction $\tilde{\varphi} : X \to Y$ by constructing a projective manifold V and two birational morphisms $\sigma : V \to X$ and $\omega : V \to W$ such that $\sigma^*(K_X) = \omega^*(K_W)$. This means that the exceptional locus of σ, which is divisorial by the smoothness of X, is contracted by ω and that we have a commutative diagram

$$\begin{tikzcd}
 & V \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X \arrow[rr, dashed, "\tau"'] \arrow[dr, "\tilde{\varphi}"'] & & W \arrow[dl, "\tilde{\psi}"] \\
 & Y &
\end{tikzcd} \tag{1.3}$$

First of all one may ask if there exist flops of this kind on the fourfolds $X' = \mathrm{Bl}_S X$ obtained from cubic fourfolds $X \subset \mathbb{P}^5$ by blowing up a mildly singular surface $S \subset X$. As we shall see, this is the case under some hypothesis and this occurrence is deeply related to the rationality of some *special* cubic fourfolds (or of other *special* fourfolds).

1.3. Condition $\mathcal{K}_3$ and examples of small contractions on cubic fourfolds

Let us recall that, given homogeneous forms f_i of degree $d_i \geq 1$, $i = 0, \dots, M$, a vector $(g_0, \dots, g_M)$ of homogeneous forms is a *syzygy* if $\sum_{i=0}^M f_i g_i = 0$. If $d_1 = \cdots = d_M = d$ and if $\deg(g_i) = h$ for every $i = 0, \dots, M$, then we say that $(g_0, \dots, g_M)$ is a syzygy *of degree* h and for $h = 1$ we shall say that the syzygy is *linear*. For $i < j$ the syzygies $(0, \dots, 0, f_j, 0, \dots, 0, -f_i, 0, \dots, 0)$, corresponding to the trivial identity $f_i f_j + f_j(-f_i) = 0$, are called *Koszul syzygies*. We say that the Koszul syzygies are generated by the linear ones if they belong to the submodule generated by the linear syzygies. This is *condition* $\mathcal{K}_d$ introduced by Vermeire [46].

The next result provides a wide class of examples of rational maps with linear fibres (hence birational under mildly natural geometrical assumptions on their base locus scheme).

Proposition 1.5 ([46, Proposition 2.8][1]). *Let $f_0, \dots, f_M$ be homogeneous forms in $N+1$ variables of degree $d \geq 2$ satisfying condition $\mathcal{K}_d$. Then the closure of each fibre of the rational map*

$$\varphi = (f_0 : \cdots : f_M) : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$$

is a linear space $\mathbb{P}^s$. For $s > 0$ the closure of the fibre intersects scheme-theoretically the base locus scheme of φ along a hypersurface of degree d.

Remark 1.6. Suppose that an irreducible surface $S \subset \mathbb{P}^5$ is scheme-theoretically defined by cubic equations satisfying condition $\mathcal{K}_3$. Then, by Proposition 1.5, every positive-dimensional fibre of $\varphi : \mathbb{P}^5 \dashrightarrow Z$ is a linear space $\mathbb{P}^s$ cutting S in a cubic hypersurface $S \cap \mathbb{P}^s$ if $s > 0$. In particular, $0 \leq s \leq 2$ (except some trivial cases) and $s = 2$ occurs only for planes spanned by cubic curves contained in S, which are mapped to a point by φ. Hence if condition $\mathcal{K}_3$ for $S \subset \mathbb{P}^5$ holds and if a general cubic $X \subset \mathbb{P}^5$ through S does

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[1] The hypothesis on the absence of lines in the base locus scheme of φ appearing in the original version is not necessary (see the first arXiv version of [5] for a proof).

not contain any plane spanned by cubic curves on S, the exceptional locus $T \subset X$ of the restriction of φ to X is ruled by proper trisecant lines. As we shall see in Section 2.1, the expected dimension of T is 2 so that surfaces in $\mathbb{P}^5$ defined by cubic equations satisfying condition $\mathcal{K}_3$ may naturally produce examples of small contractions on $X' = \mathrm{Bl}_S X$.

2. The trisecant flop and the extremal congruence contraction

We first introduce and study the behaviour of trisecant lines to a *general* non-degenerate irreducible projective surface $S \subset \mathbb{P}^5$.

2.1. The Hilbert scheme of trisecant lines to $S \subset \mathbb{P}^5$

For the generalities we shall follow the treatment in [4]. Let $\mathrm{Hilb}^r\, \mathbb{P}^5$ (respectively $\mathrm{Hilb}^r\, S$) be the Hilbert scheme of zero-dimensional length $r \geq 2$ subschemes of $\mathbb{P}^5$ (respectively of $S \subset \mathbb{P}^5$) and let $\mathrm{Hilb}^r_c\, \mathbb{P}^5 \subset \mathrm{Hilb}^r\, \mathbb{P}^5$ be the open non-singular subscheme consisting of *curvilinear* length r subschemes, that is, length r subschemes which, locally around every point of their support, are contained in a smooth curve of $\mathbb{P}^5$. We can define $\mathrm{Hilb}^r_c\, S$ as the scheme-theoretic intersection between $\mathrm{Hilb}^r\, S$ and $\mathrm{Hilb}^r_c\, \mathbb{P}^5$ inside $\mathrm{Hilb}^r\, \mathbb{P}^5$.

Let $\mathrm{Al}^r\, \mathbb{P}^5 \subset \mathrm{Hilb}^r_c\, \mathbb{P}^5$ denote the subscheme consisting of *aligned subschemes of length* r, that is, subschemes of length r contained in a line. Finally, the Hilbert scheme of length r aligned subschemes of S, denoted by $\mathrm{Al}^r\, S$, is the scheme-theoretic intersection of $\mathrm{Al}^r\, \mathbb{P}^5$ with $\mathrm{Hilb}^r_c\, S$.

The schemes $\mathrm{Hilb}^r_c\, \mathbb{P}^5$ and $\mathrm{Al}^r\, \mathbb{P}^5$ are smooth of dimension $5r$ and $8 + r$, respectively. Moreover, if $S \subset \mathbb{P}^5$ is smooth, then $\mathrm{Hilb}^r_c\, S$ is smooth of dimension $2r$. In particular, either $\mathrm{Al}^3\, S$ is empty or every irreducible component of $\mathrm{Al}^3\, S$ has dimension at least

$$\dim(\mathrm{Al}^3\, \mathbb{P}^5) + \dim(\mathrm{Hilb}^3_c\, S) - \dim(\mathrm{Hilb}^3_c\, \mathbb{P}^5) = 2,$$

which is therefore the *expected dimension* of $\mathrm{Al}^3\, S$. So, for an irreducible projective surface $S \subset \mathbb{P}^5$, one might expect that, with few exceptions, the Hilbert scheme $\mathrm{Al}^3\, S$ of trisecant lines is of pure dimension 2.

There exists a natural morphism of schemes

$$\mathrm{axis} : \mathrm{Al}^r\, S \to \mathbb{G}(1, 5),$$

sending each length $r \geq 2$ aligned subscheme of S to the unique line containing its support, that is, to the multisecant line to S determined by the subscheme of points (counted with multiplicity). Let $q : \mathcal{L} \to \mathbb{G}(1, 5)$ be the universal family and let $p : \mathcal{L} \to \mathbb{P}^5$ be the tautological morphism. Then

$$\mathrm{Trisec}(S) := p(q^{-1}(\mathrm{axis}(\mathrm{Al}^3\, S))) \subset \mathbb{P}^5$$

is called the *trisecant locus of* $S \subset \mathbb{P}^5$. The previous count of parameters and analysis show that the expected dimension of $\mathrm{Trisec}(S)$ is 3; that every irreducible component of

$\mathrm{Trisec}(S)$ has dimension at least 2; and that the irreducible components of dimension 2 of $\mathrm{Trisec}(S)$ are either S (in this case S is ruled by lines) or planes cutting S along a plane curve of degree at least 3 (see [4]).

By the Trisecant Lemma (see [33, Proposition 1.4.3]), a general secant line to an irreducible non-degenerate surface $S \subset \mathbb{P}^5$ is not a trisecant line. So $\dim(\mathrm{Al}^3 S) \leq 3$ and $\dim(\mathrm{Trisec}(S)) \leq 4$. Very few examples of irreducible non-degenerate surfaces $S \subset \mathbb{P}^5$ having $\dim(\mathrm{Trisec}(S)) = 4$ are known, most of them are very singular (see [31] for a description) but a complete classification is still lacking. The smooth irreducible non-degenerate surfaces $S \subset \mathbb{P}^5$ with $\dim(\mathrm{Trisec}(S)) \leq 2$ are classified in [4].

In our analysis we shall always consider the most general case $\dim(\mathrm{Trisec}(S)) = 3$. While the condition on the dimension is expected by the above parameter count, the generic smoothness of an irreducible component of $\mathrm{Al}^3 S$ is related to the dimension of the corresponding locus and to the tangential behaviour of $S \subset \mathbb{P}^5$ at the points of intersection of a general trisecant line by [14, Proposition 4.3] (see also [6, Section 1] and [30] for spectacular generalizations). We shall specialize this general result to our setting.

Proposition 2.1. *Let* $S \subset \mathbb{P}^5$ *be an irreducible projective surface and let* $L \subset \mathbb{P}^5$ *be a proper trisecant line to* S *such that* $L \cap S = \{p_1, p_2, p_3\}$, *with* p_1, p_2, p_3 *distinct smooth points of* S, *and with* $[L]$ *belonging to an irreducible component* A *of* $\mathrm{Al}^3 S$ *of dimension* 2. *Then* $\mathrm{Al}^3 S$ *is smooth at* $[L]$ *if and only if the tangent planes to* S *at the points* p_i *are in general linear position, that is,* $T_{p_j} S \cap T_{p_k} S = \emptyset$ *for any distinct* $p_j, p_k \in L \cap S$. *In particular, if this condition holds at the point* $[L] \in A$, *then* A *is generically smooth and for a general* $[L] \in A$ *the tangent planes at the points* p_i *are in general linear position.*

Moreover, in this case the irreducible component of $\mathrm{Trisec}(S)$ *corresponding to* A *has dimension* 3 *and through a general point* q *of this irreducible component there pass a finite number of trisecant lines to* S, *which are smooth points of the zero-dimensional Hilbert scheme of trisecant lines to* S *passing through* q.

This result and the previous analysis motivate the next definition.

Definition 2.2 (Expected trisecant behaviour). Let $S \subset \mathbb{P}^5$ be an irreducible non-degenerate projective surface. If $\dim(\mathrm{Trisec}(S)) = 3$ and if every irreducible component of $\mathrm{Al}^3 S$ of dimension 2, whose trisecant lines describe an irreducible component of $\mathrm{Trisec}(S)$ of dimension 3, is generically reduced (and hence generically smooth), then $S \subset \mathbb{P}^5$ is said to have the *expected trisecant behaviour.*

A trisecant line to an irreducible surface $S \subset \mathbb{P}^5$ having the expected trisecant behaviour is said to be *general* if it is the general element of an irreducible component of $\mathrm{Al}^3 S$ whose locus has dimension 3.

We now start to study the consequences of this natural condition.

Lemma 2.3. *Let* $S \subset \mathbb{P}^5$ *be an irreducible non-degenerate projective surface with the expected trisecant behaviour and with at most a finite number of singular points, let*

$L \subset \mathbb{P}^5$ be a general trisecant line and let $L' \subset \mathrm{Bl}_S \mathbb{P}^5$ denote the strict transform of L. Then

$$N_{L'/\mathrm{Bl}_S \mathbb{P}^5} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

If $\tilde{T} \subset \mathbb{P}^5$ denotes the unique irreducible component of $\mathrm{Trisec}(S)$ *containing L and if $\tilde{T}'$ denotes the strict transform of $\tilde{T}$ on $\mathrm{Bl}_S \mathbb{P}^5$, then*

$$N_{L'/\tilde{T}'} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}.$$

Furthermore, if $\tilde{T}'$ is smooth along L', then

$$N_{\tilde{T}'/\mathrm{Bl}_S \mathbb{P}^5_{|L'}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Let $S_{\mathrm{reg}} = S \setminus \mathrm{Sing}(S)$ be the locus of smooth points of an irreducible nondegenerate surface $S \subset \mathbb{P}^5$ and let $\pi : \mathrm{Bl}_S \mathbb{P}^5 \to \mathbb{P}^5$ be the blow-up of $\mathbb{P}^5$ along S. Then $\pi^{-1}(\mathbb{P}^5 \setminus \mathrm{Sing}(S))$ is a smooth variety. Since $\mathrm{Sing}(S)$ is zero-dimensional, a general $[L] \in \mathrm{Al}^3\, S$ will cut S in three smooth distinct points and L' will be contained in the smooth locus of $\mathrm{Bl}_S \mathbb{P}^5$. In particular, the normal bundle $N_{L'/\mathrm{Bl}_S \mathbb{P}^5}$ is locally free of rank 4.

The strict transforms of general trisecant lines to S determine a proper family of dimension 2 of smooth rational curves on $\mathrm{Bl}_S \mathbb{P}^5$ and the curve L' represents a smooth point of this family by hypothesis, yielding $N_{L'/\mathrm{Bl}_S \mathbb{P}^5} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Indeed, $\deg(N_{L'/\mathrm{Bl}_S \mathbb{P}^5}) = -2$ by the adjunction formula, $h^0(N_{L'/\mathrm{Bl}_S \mathbb{P}^5}) = 2$ by the generic smoothness hypothesis of the irreducible component to which L' belongs, and $h^0(N_{L'/\mathrm{Bl}_S \mathbb{P}^5}(-1)) = 0$ since through a general point of $\tilde{T}'$ there pass a finite number of curves of the family by the last part of Proposition 2.1. Then the exact sequence

$$0 \to N_{L'/\tilde{T}'} \to N_{L'/\mathrm{Bl}_S \mathbb{P}^5} \to N_{\tilde{T}'/\mathrm{Bl}_S \mathbb{P}^5_{|L'}} \tag{2.1}$$

ensures that $N_{L'/\tilde{T}'}$ is torsion free and hence locally free of rank 2. Moreover, letting $N_{L'/\tilde{T}'} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$, we deduce $a_i \leq 0$ for $i = 1, 2$. Since the curve L' moves in a family of dimension 2 inside $\tilde{T}'$, we have $h^0(N_{L'/\tilde{T}'}) \geq 2$ and hence $a_1 = a_2 = 0$. Finally, if $\tilde{T}'$ is smooth along L', then the exact sequence (2.1) is also exact on the right and $N_{\tilde{T}'/\mathrm{Bl}_S \mathbb{P}^5_{|L'}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. ■

Remark 2.4. We are interested in studying the birational properties of smooth cubic hypersurfaces $X \subset \mathbb{P}^5$ passing through an irreducible projective surface $S \subset \mathbb{P}^5$ having the expected trisecant behaviour and with at most a finite number of singular points. Retain the notation of Lemma 2.3 and suppose $L \subset X \subset \mathbb{P}^5$ is a general proper trisecant line to S contained in X. Since $N_{\mathrm{Bl}_S X/\mathrm{Bl}_S \mathbb{P}^5_{|L'}} \simeq \mathcal{O}_{\mathbb{P}^1}$, since $N_{L'/\mathrm{Bl}_S \mathbb{P}^5} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and since we have the exact sequence

$$0 \to N_{L'/\mathrm{Bl}_S X} \to N_{L'/\mathrm{Bl}_S \mathbb{P}^5} \to N_{\mathrm{Bl}_S X/\mathrm{Bl}_S \mathbb{P}^5_{|L'}} \to 0,$$

we deduce that either $N_{L'/\mathrm{Bl}_S X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ or $N_{L'/\mathrm{Bl}_S X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. If the last splitting holds, then either the family of strict transforms of trisecant lines to S contained in X to which L' belongs is two-dimensional (that is, $\mathrm{Trisec}(S) \subseteq X$) and is generically smooth, or this family is one-dimensional but not generically reduced. If $X \subset \mathbb{P}^5$ does not contain $\mathrm{Trisec}(S)$ and if S has the expected trisecant behaviour, then the family of trisecant lines to S contained in X is one-dimensional and the corresponding locus has dimension 2.

Thus when X is *sufficiently general*, when S has the expected trisecant behaviour and at most a finite number of singular points, the locus of trisecant lines to S contained in X is of pure dimension 2 and one expects that the one-dimensional families of trisecant lines to S contained in X are generically smooth as subschemes of the corresponding parameter space.

The previous natural expectation/hypothesis translates into the following conditions, letting notation be as above:

$$N_{L'/\mathrm{Bl}_S X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

If $T \subset X$ denotes the unique two-dimensional irreducible component of the locus of trisecant lines to S contained in X to which L belongs, then

$$N_{L'/T'} \simeq \mathcal{O}_{\mathbb{P}^1}.$$

Furthermore, if T' is smooth along L', then

$$N_{T'/\mathrm{Bl}_S X_{|L'}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \tag{2.2}$$

Condition (2.2) is crucial. Indeed, as we shall see in the next section, it essentially says that T' can be flopped producing another four-dimensional variety birational to $\mathrm{Bl}_S X$ and hence to X in a very natural way.

If an irreducible projective surface $S \subset \mathbb{P}^5$ has the expected trisecant behaviour and if S satisfies condition $\mathcal{K}_3$, then, for a general cubic through S, the expected splittings listed above hold for a general proper trisecant line to S contained in the cubic (see the proof of Theorem 2.6). There are also many other examples of different flavour for which the above conditions naturally hold and which naturally lead to flops of the trisecant locus contained in the cubic fourfold.

2.2. *Assumptions and main definitions*

Assumption 1. Suppose we have a smooth irreducible projective surface (the treatment can be extended to surfaces with a finite number of singular points) $S \subset \mathbb{P}^5$, scheme-theoretically defined by cubic hypersurfaces and such that the associated rational map

$$\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(\mathcal{I}_S(3))) = \mathbb{P}^N$$

is birational onto the closure of its image $Z = \overline{\varphi(\mathbb{P}^5)} \subset \mathbb{P}^N$.

Then the restriction of φ to a general $X \in |H^0(\mathcal{I}_S(3))|$ induces a birational map

$$\varphi : X \dashrightarrow Y \subset \mathbb{P}^{N-1}$$

with Y the corresponding hyperplane section of $Z \subset \mathbb{P}^N$. On $X' = \mathrm{Bl}_S\, X$ we have

$$-K_{X'} = \lambda^*(-K_X) - E = 3H' - E$$

and our hypothesis on the defining equations of S and on the birational map $\varphi : \mathbb{P}^5 \dashrightarrow Z$ can be reformulated by saying that $-K_{X'}$ is a big divisor generated by its global sections. In particular, $-K_{X'}$ is nef and big.

The induced morphism

$$\tilde{\varphi} : \mathrm{Bl}_S\, X \to Y$$

is a small contraction (with very few exceptions). Indeed, the base locus scheme of φ is the surface S and φ contracts any irreducible (rational) curve $C \subset X$ of degree $e \geq 1$ which is $3e$-secant to S, i.e. such that $\mathrm{length}(C \cap S) = 3e$ (*proper* $3e$-secant curve to S). Let us indicate by $T \subset X$ the closure of the locus of proper $3e$-secant curves to S contained in X. If $L' \subset X'$ is the strict transform of a proper trisecant line to S contained in X, let $[L']$ denote its numerical class in $N_1(X')$.

The strict transform $C' \subset X'$ of a proper $3e$-secant curve $C \subset X$ to S of degree $e \geq 1$ satisfies $[C'] = [eL']$. Therefore on $X' = \mathrm{Bl}_S\, X$ we have

$$K_{X'} \cdot C' = (E - 3H') \cdot C' = 3e - 3e = 0$$

for curves $C' \subset X'$ as above.

Definition 2.5 (Trisecant flop). Let notation and assumptions be as above. If $\tilde{\varphi} : X' \to Y$ is a small contraction of curves in $\mathbb{R}[L']$, then it is called a *trisecant flop contraction*. If $\tilde{\varphi} : X' \to Y$ is a trisecant flop contraction and if there exists a flop $\tilde{\psi} : W' \to Y$ of $\tilde{\varphi}$ with W' a projective fourfold, then the resulting birational map $\tau : X' \dashrightarrow W'$ will be called a *trisecant flop* (of $\tilde{\varphi} : X' \to Y$).

Let us remark that, by definition, if $\tilde{\varphi} : X' \to Y$ is a trisecant flop contraction, then its exceptional locus has dimension at most 2 and the irreducible components of dimension 2 are covered by proper $3e$-secant (rational) curves (in most cases they are ruled by these curves). By Zariski's Main Theorem, a positive-dimensional fibre is connected so that a general positive-dimensional fibre is smooth and irreducible.

During our study of birational maps $\varphi : \mathbb{P}^5 \dashrightarrow Z$ of the type described above, we constructed many surfaces $S \subset \mathbb{P}^5$ inducing trisecant flop contractions on a general cubic fourfold X through S; for example, surfaces satisfying condition $\mathcal{K}_3$ but not only such (see Table 1).

2.3. *Existence of a trisecant flop*

For simplicity we shall now assume as above that S is smooth. As always, let $\lambda : X' = \mathrm{Bl}_S\, X \to X$ be the blow-up of X along S, let $E \subset X'$ be the exceptional divisor and let $H' = \lambda^*(H)$, where $H \subset X$ is a hyperplane section.

The results in Section 2.1 and in Remarks 1.6 and 2.4 suggest that, under some mild assumptions, trisecant flop contractions might exist.

We shall now construct explicitly a flop of the two-dimensional irreducible components of T ruled by trisecant lines to S via φ as long as $S \subset \mathbb{P}^5$ has the expected trisecant behaviour and $\tilde{\varphi}$ is a small contraction. When these loci exhaust the exceptional locus of a trisecant flop contraction we shall obtain a *trisecant flop* of $\tilde{\varphi} : X' \to Y$. Flops of this kind have also been considered in [25] in arbitrary dimension under the stronger assumption that the splitting (2.2) holds for every line of the ruling of T'.

Theorem 2.6 (Trisecant flop). *Let notation be as above, suppose that $S \subset \mathbb{P}^5$ satisfies Assumption 1 and that it has the expected trisecant behaviour. If $T' \subset X'$ denotes the exceptional locus of the associated small contraction*

$$\tilde{\varphi} : X' = \mathrm{Bl}_S\, X \to Y,$$

then any irreducible smooth surface $\overline{T} \subseteq T'$, which is ruled via $\tilde{\varphi}$ by the strict transforms of trisecant lines to S (that is, through a general point of $\overline{T}'$ there passes a unique curve of this kind), can be flopped to produce a small contraction $\tilde{\psi} : W' \to Y$ with W' a smooth projective fourfold.

In particular, under the previous assumptions, if $\tilde{\varphi} : X' \to Y$ is a trisecant flop contraction and if $T' \subset X'$ is a smooth irreducible surface ruled via $\tilde{\varphi}$ by trisecant lines, then a trisecant flop $\tau : X' \dashrightarrow W'$ exists.

Proof. First we shall prove the second part, that is, suppose that $\overline{T} = T'$ is a smooth irreducible surface ruled via $\tilde{\varphi}$ by trisecant lines and such that $\tilde{\varphi}(T') = \overline{C} \subset Y$ is a curve. At the end we shall consider the general case in which T' is a finite union of such surfaces. The general fibre of $\tilde{\varphi} : T' \to \overline{C}$ is smooth and irreducible so that $\overline{C}$ generically coincides, as a scheme, with the parameter space of trisecant lines to S contained in X. In particular, this parameter space is generically smooth of dimension 1. Let $L' \subset T'$ be a general fibre of the restriction of $\tilde{\varphi}$ to T'. By Lemma 2.3 (see also Remark 2.4),

$$N^*_{T'/X'_{|L'}} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1). \tag{2.3}$$

Let $\sigma : X'' = \mathrm{Bl}_{T'}\, X' \to X'$ be the blow-up of X' along T', let $E' \subset X''$ be the exceptional divisor and let $C_1 \simeq \mathbb{P}^1 \subset E'$ be a positive-dimensional fibre of σ. By (2.3) we deduce that $\Sigma_{L'} = \sigma^{-1}(L') \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and we can suppose that the restriction of σ to $\Sigma_{L'}$ is identified with the projection onto the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$. Since $\Sigma_{L'}$ is also the general fibre of $\tilde{\varphi} \circ \sigma_{|E'} : E' \to \overline{C}$, we have $N_{\Sigma_{L'}/E'} \simeq \mathcal{O}_{\Sigma_{L'}}$. Let $C_2 \simeq \mathbb{P}^1 \subset E'$ be a fibre of the projection onto the second factor of $\Sigma_{L'}$. Since $N_{C_2/\Sigma_{L'}} \simeq \mathcal{O}_{\mathbb{P}^1}$ and since $N^*_{E'/X''_{|C_2}} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$ by (2.3), the exact sequence

$$0 \to N_{C_2/\Sigma_{L'}} \to N_{C_2/E'} \to N_{\Sigma_{L'}/E'_{|L'}} \simeq \mathcal{O}_{\mathbb{P}^1} \to 0$$

yields

$$N_{C_2/E'} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \tag{2.4}$$

and hence $-K_{E'} \cdot C_2 = 2$ by the adjunction formula. From $\sigma(C_2) = L'$ and from the projection formula we get $\sigma^*(H') \cdot C_2 = 1$. The Hilbert scheme of curves contained in the smooth projective variety E' is smooth and of dimension 2 at the point $[C_2]$ corresponding to $C_2 \subset E'$ by (2.4). So $[C_2]$ belongs to a unique irreducible component $\mathcal{C}$ of the Hilbert scheme with $\dim(\mathcal{C}) = 2$. Since $\sigma^*(H') \cdot C_2 = 1$ and since $-K_{E'} \cdot C_2 = 2$, the possible deformations of C_2 inside E' are either irreducible and isomorphic to $\mathbb{P}^1$, or consist of two distinct smooth irreducible rational curves $F_1, F_2 \subset E'$ intersecting in one point and such that $N_{F_i/E'} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (see [26, Sections 3.24, 3.25, 3.25.2]). Since $1 = \sigma^*(H') \cdot (F_1 + F_2)$, we can suppose $\sigma^*(H') \cdot F_1 = 1$ (yielding $E' \cdot F_1 = -1$) and $\sigma^*(H') \cdot F_2 = 0$. The last condition means that either F_2 is contracted to a point by σ and hence $E' \cdot F_2 = -1$ by (2.3), or $\sigma(F_2)$ is a curve contracted to a point by λ, that is, a positive-dimensional fibre of the blow-up λ. The first case is excluded because it would imply $E' \cdot (F_1 + F_2) = -2$, contradicting $E' \cdot C_2 = -1$. If $\sigma(F_2) \simeq \mathbb{P}^1 \subset T'$, then it is a (-1) curve on T'. From $(3H' - E) \cdot \sigma(F_2) = -E \cdot \sigma(F_2) = 1$ we would deduce that $\overline{C} = \tilde{\varphi}(\sigma(F_2))$ is a line and the projections via λ of all the fibres of $T' \to \overline{C}$ pass through the smooth point $\lambda(\sigma(F_2)) = p \in T \cap S$. Then $T \subset X$ would be a plane because it would coincide with its tangent plane at p. The intersection $T \cap S$ would contain a cubic curve since S is scheme-theoretically defined by cubics and every line contained in T would be trisecant to S. The plane T would be contracted to a point by φ (and a fortiori by $\tilde{\varphi}$) so that T would not be ruled by trisecant lines to S. In conclusion, the deformations of C_2 inside E' are all smooth, irreducible and isomorphic to $\mathbb{P}^1$, parametrized by a smooth projective surface (the splitting (2.4) necessarily holds for all deformations of C_2) and the locus of these curves is E'. The extremal ray $\mathbb{R}_+[C_2]$ determines a contraction $\omega' : E' \to R'$ with R' a smooth surface and such that every fibre of ω' is isomorphic to $\mathbb{P}^1$ [26, Theorem 3.5.1].

By the above analysis the surface R' is ruled by the curves $L'' = \omega'(\Sigma_{L'})$. There exists a morphism $\omega : X'' \to W'$ with W' a smooth irreducible projective fourfold, which is the blow-up of W' along the smooth surface R' with exceptional divisor E' and whose restriction to E' is ω' (see for example [11,27] and also [3]).

The smooth rational curves $L'' \subset R'$ are disjoint and contracted to $\overline{C}$ by the nef, big and base point free linear system $|-K_{W'}|$, yielding a morphism $\tilde{\psi} : W' \to Y$ such that $\overline{C} = \tilde{\psi}(R')$ and the surface R' is ruled by $\tilde{\psi} : R' \to \overline{C}$.

Suppose now that $T' = T_1 \cup \cdots \cup T_r$, $r \geq 2$, with T_i a smooth irreducible projective surface ruled via $\tilde{\varphi}$ by trisecant lines to S. After applying the previous construction to T_1 we produce W_1 and we change T_1 to R'_1. The strict transforms $T'_j \subset W_1$ of T_j, $j = 2, \ldots, r$, are smooth irreducible surfaces which are ruled by the strict transforms of trisecant lines to S and such that the restriction of $N^*_{T'_j/W_1}$ to a general fibre of $\tilde{\psi} : T'_j \to C_j$ satisfies (2.3). Then we can flop T'_2 and produce a smooth fourfold W_2. After r steps we get a smooth fourfold W_r birational to X' in which the smooth irreducible ruled surfaces T_j have been changed to the corresponding R'_j. The birational map $X' \dashrightarrow W_r$ is an isomorphism in codimension 1 but not an isomorphism and it has been factorized into a sequence of elementary flops (see also [15] for the general program of factorization of birational maps into elementary links according to Sarkisov). ■

We now state a useful corollary, helpful for our applications and showing that the phenomenon described above really occurs.

Corollary 2.7. *Let $S \subset \mathbb{P}^5$ be a smooth surface satisfying Assumption 1, condition $\mathcal{K}_3$ and having the expected trisecant behaviour. If $X \subset \mathbb{P}^5$ is a cubic hypersurface through S not containing any plane spanned by cubic curves on S and if $T' \subset X'$ denotes the exceptional locus of the associated small contraction $\tilde{\varphi} : X' = \mathrm{Bl}_S X \to Y$, then there exists a trisecant flop $\tilde{\psi} : W' \to Y$ of any irreducible component of T'.*

Remark 2.8. Obviously, one might only assume that $|-K_{X'}| = |3H' - E|$ is generated by global sections and big (or only nef and big but not generated by global sections) without requiring that the trisecant flop contraction is necessarily given by $|-K_{X'}|$. It is not difficult to see that in any case, for some $m \geq 1$, the linear system $|-mK_{X'}|$ gives a trisecant flop contraction. We avoided this more general approach to simplify the exposition but there are examples of trisecant flops appearing also in more general settings; see Section 3.6 of the first arXiv version of this paper and also example (xv) of Table 1.

2.4. *Trisecant flop and congruences of* $(3e - 1)$*-secant rational curves of degree* $e \geq 2$

The aim of this section is to relate the trisecant flop to (congruences of) $(3e - 1)$-secant curves to S. We start by an easy but very useful result.

Proposition 2.9 (Extremal ray generated by $(3e - 1)$-secant curves). *Let notation be as above. Suppose that $S \subset \mathbb{P}^5$ satisfies Assumption 1 and that there exists a trisecant flop $\tilde{\psi} : W' \to Y$ of a trisecant flop contraction $\tilde{\varphi} : X' = \mathrm{Bl}_S X \to Y$ with X a general cubic fourfold through S.*

If $C' \subset X'$ is the strict transform on X' of a $(3e - 1)$-secant curve to S of degree e contained in X, then the strict transform $\overline{C}'$ of C' on W' generates an extremal ray on W'.

Proof. By hypothesis there exists a trisecant flop of the trisecant flop contraction $\tilde{\varphi} : X' = \mathrm{Bl}_S X \to Y$ and hence a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_T X' = \mathrm{Bl}_R W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"] \arrow[d, "\lambda"'] \arrow[dr, "\tilde{\varphi}"'] & & W' \arrow[dl, "\tilde{\psi}"] \\
X & Y &
\end{tikzcd} \tag{2.5}$$

By definition C' is the strict transform of a curve of degree e, which is $(3e - 1)$-secant to S. Thus

$$K_{X'} \cdot C' = (E - 3H') \cdot C' = 3e - 1 - 3e = -1.$$

Consider the possible degenerations C'' of $C' \subset X'$ as sums of effective 1-cycles inside X':

$$C'' = C_1 + C_2.$$

The cycles C_1 and C_2 have degree $e_i = H' \cdot C_i$ and are $\beta_i = E \cdot C_i$-secant to S. In particular, $e_1 + e_2 = e$ and $\beta_1 + \beta_2 = 3e - 1$. Since $-K_{X'}$ is nef and since

$$1 = -K_{X'} \cdot C' = -K_{X'} \cdot (C_1 + C_2),$$

either $K_{X'} \cdot C_1 = 0$ or $K_{X'} \cdot C_2 = 0$. In conclusion, either $[C_2] = [e_2 L']$ or $[C_1] = [e_1 L']$ with L' the strict transform of a trisecant line to S contained in X. Let $\overline{C}' \subset W'$ be the strict transform of $C' \subset X'$. Then $[\overline{C}']$ generates an extremal ray because $\tilde{\varphi}$ has contracted all the rational curves in $\mathbb{R}_+[L']$. ■

The locus of the extremal ray $\mathbb{R}_+[\overline{C}']$ will determine the type of the associated elementary Mori contraction from W' onto a $\mathbb{Q}$-factorial Fano variety. Here we shall consider only the most relevant case for our applications. Examples of fibre type contractions can be constructed as long as $|(3e-1)H' - eE| \neq \emptyset$ and $S \subset \mathbb{P}^5$ has a finite number $\rho \geq 2$ of $(3e-1)$-secant curves of degree $e \geq 2$ passing through a general point of $\mathbb{P}^5$. The dimension of the general fibre of the contraction will be $\rho - 1 = 4 - \dim(\mu(X))$, where μ is the rational map on X defined by the linear system of hypersurfaces of degree $3e-1$ having points of multiplicity at least e on S.

Theorem 2.10 (Extremal contraction of the congruence). *Let notation be as above. Suppose that $S \subset \mathbb{P}^5$ satisfies Assumption 1, that it has the expected trisecant behaviour and that there exists a trisecant flop $\tilde{\psi} : W' \to Y$ of a trisecant flop contraction $\tilde{\varphi} : X' = \mathrm{Bl}_S X \to Y$ with X a general cubic fourfold through S and with T' irreducible.*

If $S \subset \mathbb{P}^5$ admits a congruence $\pi : \mathfrak{D} \to \mathcal{H}$ of $(3e-1)$-secant rational curves of degree $e \geq 2$, then the locus of curves of the congruence contained in $X \subset \mathbb{P}^5$ is an irreducible divisor $D \subset X$ and the following hold:

(1) *There exists a divisorial contraction $\nu : W' \to W$, with W a locally $\mathbb{Q}$-factorial projective Fano variety, whose exceptional locus $\overline{E}$ is the strict transform of D on W' and such that $\nu(D) = U$ is an irreducible surface supporting the base locus scheme B of ν^{-1}. The base locus scheme B is generically smooth, irreducible and ν is generically the blow-up of the surface U.*

(2) *The induced birational map $\mu' : X' \dashrightarrow W$ (or $\mu : X \dashrightarrow W$) is given by a linear system in $|(3e-1)H' - eE|$.*

(3) *Let $\overline{H}' = \nu^*(\overline{H})$ with $\overline{H} \subset W$ a generator of* Pic(W) *and let $-K_W = i(W)\overline{H}$. The induced birational morphism $\tilde{\psi} : W' \to Y$ is given by a linear system in $|i(W)\overline{H}' - \overline{E}|$, while the birational map $W' \dashrightarrow X$ is given by a linear system in $|(i(W) \cdot e - 1)\overline{H}' - e\overline{E}|$. The strict transform $D' \subset W$ of E via μ' is a locus of $(i(W) \cdot e - 1)$-secant curves to U of degree e such that through a general point of D' there passes a unique curve of the family.*

(4) *The irreducible components of T are contained in the base locus scheme of μ and their flopped images on W are contained in the base locus scheme of μ^{-1}. The flopped images of the scrolls in T are scrolls $R \subset W$ ruled by lines in W, which are $i(W)$-secant to U.*

Proof. Let notation be as above, let $p : \mathcal{D} \to \mathbb{P}^5$ be the tautological morphism of the congruence and let $\mathcal{H}$ be its parameter space. Since p is birational, the locus $\mathcal{E} \subset \mathbb{P}^5$ of points through which there pass more than one curve of the congruence has codimension at least 2 in $\mathbb{P}^5$ by Zariski's Main Theorem. Since the curves of the congruence are $(3e-1)$-secant to S and hence to X, for them to be contained in X imposes two conditions in $\mathcal{H}$ by the Bézout Theorem. Putting these two facts together we deduce that D is a divisor inside X, whose irreducibility will be proved below. By hypothesis there exists a trisecant flop of the trisecant flop contraction $\tilde{\varphi} : X' = \mathrm{Bl}_S X \to Y$ and hence the commutative diagram (2.5). Let $D' \subset X'$ be the strict transform of D on X' and let $C' \subset D'$ be the strict transform of a general curve C of the congruence $\pi : \mathcal{D} \to \mathcal{H}$ contained in X. By definition C' is the strict transform of a smooth rational curve of degree $e \geq 2$ which is $(3e-1)$-secant to S.

Let $\overline{C}' \subset W'$ be the strict transform of a general curve of the congruence $C' \subset D'$ and let $\overline{E} \subset W'$ be the strict transform of D. Then $K_{W'} \cdot \overline{C}' = -1$ and $[\overline{C}']$ generates an extremal ray by Proposition 2.9. By construction the locus of the extremal ray $\mathbb{R}_+[\overline{C}']$ is the divisor $\overline{E}$ and $N_{\overline{C}'/W'} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (see for example [2, Lemma 2.5]). By [2, Theorem 2.1] there exists a birational divisorial contraction $\nu : W' \to W$ of the extremal ray $\mathbb{R}_+[\overline{C}']$ with W a locally $\mathbb{Q}$-factorial projective variety of dimension 4 with $\mathrm{Pic}(W) = \mathbb{Z}\langle\overline{H}\rangle$ and with $\overline{H}$ ample. If $i(W)$ is defined by $-K_W = i(W)\overline{H}$, then $i(W) > 0$ since W is birational to X. Moreover, the divisor $\overline{E}$ (and hence the divisor D) is irreducible, being the exceptional locus of ν (recall that $\mathrm{Pic}(W') \simeq \mathbb{Z} \oplus \mathbb{Z}$).

Let $U = \nu(\overline{E}) \subset W$. Since the $(3e-1)$-secant curves to S belong to a congruence, through a general point of D there passes a unique curve of the congruence, as recalled at the beginning of the proof. So the same holds for $\overline{E}$, and the restriction of ν to $\overline{E}$ has general fibre isomorphic to a curve $\overline{C}'$, giving $\dim(U) = 2$. In particular, $N_{\overline{C}'/\overline{E}} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for a general $\overline{C}'$ so that the previous splitting of the normal bundle $N_{\overline{C}'/W'}$ yields $\overline{E} \cdot \overline{C}' = -1$. Therefore, there exists an open subset $U_0 \subset U$ consisting of smooth points of U and such that the fibre of $\nu : \overline{E} \to U$ over a point of U_0 is isomorphic to $\mathbb{P}^1$. Then $\overline{E}_0 = \nu^{-1}(U_0) \subseteq \overline{E}$ is smooth and, letting $W_0 \subset W$ be an open subset such that $W_0 \cap U = U_0$, $\nu^{-1}(W_0) \to W_0$ is the blow-up of W_0 along U_0 (see for example [11,27]). In particular, the base locus scheme of ν^{-1}, which is supported on U, coincides generically with U and hence it is generically smooth. All the assertions in (1) are now proved.

The birational map $\mu' = \nu \circ \tau : X' \dashrightarrow W$ is given by a linear system in $|a[(3e-1)H' - eE]|$, $a \geq 1$. Indeed, $\mathrm{Pic}(X') \simeq \mathbb{Z}\langle H'\rangle \oplus \mathbb{Z}\langle E\rangle$ and such a divisor is of the form $\alpha H' - \beta E$ with $\alpha > 0$ and $\beta > 0$. From

$$0 = (\alpha H' - \beta E) \cdot C' = \alpha e - \beta(3e-1)$$

and from $e \geq 2$, we get $\alpha = a(3e-1)$ and $\beta = ae$ with $a \geq 1$. The irreducible components of T are contained in the base locus scheme of this linear system because

$$a[(3e-1)H' - eE] \cdot L' = -a < 0$$

for every strict transform of a general 3-secant line L to S contained in X. Since the map μ' is compatible with the trisecant flop, necessarily $a = 1$. Indeed, after the blow-up of T the birational map μ' becomes a morphism so that

$$[N^*_{T'/X'} \otimes \mathcal{O}_{X'}(a[(3e-1)H' - eE])]_{|L'} \simeq \mathcal{O}_{\mathbb{P}^1}(1-a) \oplus \mathcal{O}_{\mathbb{P}^1}(1-a) \tag{2.6}$$

is generated by global sections, yielding $a \leq 1$ and hence $a = 1$, concluding the proof of (2).

We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_{T'} X' = \mathrm{Bl}_{R'} W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"'] \arrow[d, "\lambda"'] \arrow[drr, dashed, "\mu'"'] & & W' \arrow[d, "\nu"] \\
X \arrow[rr, dashed, "\mu"'] & & W
\end{tikzcd} \tag{2.7}$$

Let $\overline{H} \subset W$ be as above, let $\overline{H}'$ be its strict transform on W' and, keeping notation as in the proof of Theorem 2.6, let $L'' = \omega(\Sigma_{L'}) \subset W'$ be a general fibre of the ruling of the smooth surface $R' = \omega(E') \subset W$. Since $\nu : W' \to W$ is generically the blow-up of U, we have $-K_{W'} = i(W)\overline{H}' - \overline{E}$. The morphism $\nu \circ \omega : \mathrm{Bl}_R W' \to W$ is given by a linear system in $|(3e-1)\sigma^*(H') - e\tilde{E} - E'|$ with $\tilde{E}$ the strict transform of E on $\mathrm{Bl}_T X'$. Then

$$\nu(L'') \cdot \overline{H} = F_1 \cdot ((3e-1)\sigma^*(H') - e\tilde{E} - E') = -(F_1 \cdot E') = 1,$$

that is, $\nu(L'') \subset W$ is a *line* with respect to $\overline{H}$. By the projection formula we deduce $L'' \cdot \overline{H}' = 1$ and $0 = -K_{W'} \cdot L'' = i(W) - \overline{E} \cdot F'$ yields $i(W) = \overline{E} \cdot L''$, that is the *lines* $\nu(L'')$ are $i(W)$-secant to $U \subset W$. We also have $\overline{C}' \cdot \overline{E} = 1$ and $\overline{H}' \cdot \overline{C}' = 0$. Since the birational morphism $\tilde{\psi} : W' \to Y$ is given by a linear system in $|i(W)\overline{H}' - \overline{E}|$ as shown above, the birational map $\psi = \tilde{\psi} \circ \nu^{-1} : W \dashrightarrow Y$ is given by a linear system of divisors in $|O_W(i(W))|$ vanishing on U.

We have $\mathrm{Pic}(W') \simeq \mathbb{Z}\langle\overline{H}'\rangle \oplus \mathbb{Z}\langle\overline{E}\rangle$ so that the map $\eta' = \mu^{-1} \circ \nu : W' \dashrightarrow X$ is given by a linear system in $|\alpha'\overline{H}' - \beta'\overline{E}|$ with $\alpha' > 0$ and $\beta' > 0$. Since the general fibre $\overline{C}'$ of $\nu : \overline{E} \to U$ is sent to a curve of the congruence $\mathfrak{D}$, which by definition has degree $e \geq 2$, we deduce

$$e = (\alpha\overline{H}' - \beta\overline{E}) \cdot \overline{C}' = \beta.$$

Moreover, reasoning as above, the compatibility with the trisecant flop yields

$$-1 = (\alpha\overline{H}' - e\overline{E}) \cdot L'' = \alpha - e \cdot i(W),$$

that is, $\alpha = i(W) \cdot e - 1$. In conclusion, the birational map η' is given by a linear system in $|(i(W) \cdot e - 1)\overline{H}' - e\overline{E}|$ of dimension 5, and μ^{-1} is given by a linear system of dimension 5 of divisors in $|H^0(\mathcal{O}_W((i(W) \cdot e - 1)))|$ having points of multiplicity at least e along $U \subset W$. The previous analysis shows that the base locus of μ^{-1} contains U and $\nu(R)$, which is a locus of $i(W)$-secant *lines* to U contained in W, concluding the proof of (4).

Let $D' \subset W$ be the strict transform of $E \subset X'$ via μ, which is an irreducible divisor. Let $F \subset E$ be a positive-dimensional fibre of $\lambda : X' \to X$. Then D' is ruled by the strict transforms of the curves F and through a general point of D' there passes a unique curve of this family. Moreover, $F \cdot [(3e-1)H' - eE] = e$ and, letting $\overline{F} = \tau(F) \subset W$ and recalling that $\eta'(\overline{F})$ is a point, we have

$$0 = \overline{F} \cdot [(i(W) \cdot e - 1)\overline{H}' - e\overline{E}] = e[(i(W) \cdot e - 1) - \overline{F} \cdot \overline{E}],$$

yielding $\overline{F} \cdot \overline{E} = i(W) \cdot e - 1$. In conclusion, $\mu'(F) \subset W$ is a curve of degree e with respect to $\overline{H}$ which is $(i(W) \cdot e - 1)$-secant to $U \subset W$. This proves the last assertion in (3). ■

Remark 2.11. Obviously, one can also reverse the construction in Theorem 2.10 starting from suitable $U \subset W$ and then producing the congruences of $(3e-1)$-secant curves of degree e to a surface $S \subset X \subset \mathbb{P}^5$ by taking the image of $\overline{E}$ in X and by taking $S \subset X \subset \mathbb{P}^5$ as the surface describing the linear system defining the inverse map $\mu : X \dashrightarrow W$. In practice, as long as the trisecant flop exists, the existence of a congruence of $(3e-1)$-secant lines to S is equivalent to the existence of the surface $U \subset W$, which should be an incarnation of the associated K3 surface (see the end of Section 3.1 for the analysis of the case $d = 38$ to see one such explicit incarnation). From this point of view one associates to the pair (X, S) a pair (W, U), where W is the image of X and the surface U is naturally the parameter space of the curves of the congruence $\mathfrak{D}$ contained in X.

3. Associated K3 surfaces to cubic fourfolds in $\mathcal{C}_{38}$ via the trisecant flop

In this section, as an application of previous theoretical results, we describe birational incarnations of the K3 surfaces *associated* to the cubic fourfolds in $\mathcal{C}_{38}$ via Hodge theory or via derived category theory. For the sake of brevity, we omit a similar analysis of the cases of cubic fourfolds in $\mathcal{C}_{14}$ and $\mathcal{C}_{26}$, which has been outlined in the first arXiv version of this paper. The case of cubic fourfolds in $\mathcal{C}_{42}$ will be considered in Section 5.5.

3.1. General properties of degree 10 smooth surfaces $S_{38} \subset \mathbb{P}^5$ of sectional genus 6

Let us consider the smooth surfaces $S_{38} \subset \mathbb{P}^5$ obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having ten fixed triple points in general position. These surfaces are contained in a general cubic fourfold in the admissible divisor $\mathcal{C}_{38}$, as shown in [29]. They were also studied in [34, 35] and it was proved that every cubic fourfold in $\mathcal{C}_{38}$ is rational.

The Hilbert scheme $\mathcal{S}_{38}$ parametrizing such surfaces is explicitly unirational, that is, we can write out equations for the general member $[S_{38}] \in \mathcal{S}_{38}$ over a pure transcendental extension of the base field. From this, one can deduce that the homogeneous ideal of S_{38} is generated by ten cubic forms, whose first syzygies are generated by the linear ones. In particular, the general $S_{38} \subset \mathbb{P}^5$ satisfies condition $\mathcal{K}_3$. By Proposition 1.5 the linear system $|H^0(\mathcal{I}_{S_{38}}(3))|$ defines a birational map $\varphi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9$ onto its image Z. Through a general point $\varphi(p) \in Z$ there pass eight lines contained in Z. The pullbacks of these lines are seven secant lines to S_{38} passing through p and a 5-secant conic to S_{38}. In particular, a general $S_{38} \subset \mathbb{P}^5$ admits a congruence of 5-secant conics (see [34, Section 5] for the details of this computation, and also Section 6.1).

Moreover, we have $|H^0(\mathcal{I}^2_{S_{38}}(5))| = \mathbb{P}^4$ for a general $S_{38} \subset \mathbb{P}^5$ and the coefficients of the multidegree of the graph of the associated rational map $\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ are $(1, 5, 19, 13, 2)$ (see Section 6.1). From this one deduces that μ is dominant since the last entry is equal to 2 (this means that the closure of a general fibre of μ, $F = \overline{\mu^{-1}(\mu(p))}$ with $p \in \mathbb{P}^5$ general, has degree 2 and dimension 1); its base locus scheme $B \subset \mathbb{P}^5$ has degree $6 = 5^2 - 19$ and dimension 3. Since the unique 5-secant conic C_p to S_{38} passing through p is contracted by μ to the point $\mu(p)$ $(5 \cdot 2 - 2 \cdot 5 = 0)$, the fibre F coincides with C_p and it is irreducible (otherwise one can argue as in Section 6.1, prove that F is an irreducible conic and verify that it is 5-secant to S_{38}, which yields a different proof of the existence of the congruence of 5-secant conics).

The rationality of a general X through a general S_{38} follows by restricting μ to X. Indeed, through a general $q \in X$ there passes a unique conic C_q of the congruence, which is not contained in X by the generality of q. The conic C_q cuts X in q and in five points on S_{38}, implying that the general fibre of map $\mu = \mu_{|X} : X \dashrightarrow \mathbb{P}^4$ is a point by the Bézout Theorem.

3.2. *Small contraction defined by cubics through a general $S_{38} \subset \mathbb{P}^5$*

The (closures of the) fibres of $\varphi : \mathbb{P}^5 \dashrightarrow Z$ are linear spaces of dimension s with $0 \leq s \leq 2$. The two-dimensional fibres of

$$\tilde{\varphi} : \mathrm{Bl}_{S_{38}} \mathbb{P}^5 \to Z \subset \mathbb{P}^9$$

are the strict transforms of planes in $\mathbb{P}^5$ cutting S_{38} along plane cubic curves by Proposition 1.5. Let $C \subset S_{38} \simeq \mathrm{Bl}_{\{p_1,\dots,p_{10}\}} \mathbb{P}^2 \subset \mathbb{P}^5$ be such a cubic and recall that the embedding is given by $|10H - \sum_{i=1}^{10} 3E_i|$, using the standard notation. Since the curve C is contained in a plane, it cuts each line $E_i \subset S_{38}$ in at most one point so $C \equiv \alpha H - \sum_{i=1}^{10} a_i E_i$ with $0 \leq a_i \leq 1$ and $\alpha \geq 1$. From

$$3 = C \cdot \Big(10H - \sum_{i=1}^{10} 3E_i\Big) = 10\alpha - 3 \sum_{i=1}^{10} a_i,$$

we deduce $\alpha = 3$ and $a_i = 1$ for nine of the ten indices. Hence C is the image of a plane cubic curve passing through nine of the ten general base points on $\mathbb{P}^2$. In conclusion, there are ten two-dimensional fibres of $\tilde{\varphi}$. The other positive-dimensional fibres of $\tilde{\varphi}$ are the strict transforms of trisecant lines to S_{38}. From the equations of the base locus scheme B of μ, we deduce that B has a unique irreducible component $\tilde{B}$ of dimension 3 and degree 6, which contains S_{38}. In particular, B is generically reduced along $\tilde{B}$. An explicit computation shows that the variety $\tilde{B}$ is mapped by φ onto an irreducible surface

$V \subset Z \subset \mathbb{P}^9$, which is a Veronese surface generating a $\mathbb{P}^5 \subset \mathbb{P}^9$. The general fibre of the restriction of φ to $\tilde{B}$ has dimension 1 and hence it is a trisecant line to S_{38}. A trisecant line to S_{38} is contained in the base locus scheme B of μ because it intersects a quintic with double points along S_{38} in at least six points counted with multiplicity. Hence $\tilde{B}$ is the unique irreducible component of $\mathrm{Trisec}(S_{38})$ of dimension 3, and S_{38} has the expected trisecant behaviour because the irreducible component of $\mathrm{Al}^3\, S_{38}$ corresponding to $\tilde{B}$ is birational to the smooth irreducible surface V. From this analysis and from the previous computations, we deduce that $\mathrm{Trisec}(S_{38})$ consists of $\tilde{B}$ and of the ten planes cutting S_{38} along cubic curves.

Since the ten planes in $\mathrm{Trisec}(S_{38})$ are mapped to ten points in Z, a general hyperplane section of Z does not pass through these ten points. Hence a general cubic hypersurface $X \subset \mathbb{P}^5$ through S_{38} does not contain any of the ten planes in $\mathrm{Trisec}(S_{38})$. The restriction of $\tilde{\varphi}$ to $X' = \mathrm{Bl}_{S_{38}} X$ induces a small contraction

$$\tilde{\varphi} : X' = \mathrm{Bl}_{S_{38}} X \to Y \subset \mathbb{P}^8,$$

with $Y = Z \cap H$ the corresponding hyperplane section of Z. From the previous description we deduce that $\tilde{B} \cap X = T \cup S_{38}$ with $T \subset X$ an irreducible surface by the generality of X. Moreover, $\deg(T) = 3 \times 6 - 10 = 8$ and $\varphi(T) = C = V \cap H \subset Y = Z \cap H \subset \mathbb{P}^8$ is a smooth rational normal quartic curve (a general hyperplane section of the Veronese surface $V \subset Z$). Hence T is ruled by the trisecant lines to S_{38} contained in X via the restriction of φ. The double point formula implies that the singular locus of the rational scroll T, projection of a smooth rational normal scroll of degree 8 in $\mathbb{P}^9$, consists of six singular points. Its strict transform $T' \subset X'$ is smooth and $\tilde{\varphi}_{|T'} : T' \to C$ is a $\mathbb{P}^1$-bundle. The birational morphism $\tilde{\varphi}$ is an isomorphism between $X' \setminus T'$ and $Y \setminus C$ and hence it is a trisecant flop contraction.

3.3. The trisecant flop determined by $S_{38} \subset \mathbb{P}^5$

Theorems 2.6 and 2.10 ensure the existence of a trisecant flop $\tilde{\psi} : W' \to Y$ and of a divisorial contraction $\nu : W' \to \mathbb{P}^4$ giving a factorization of the birational map $\mu : X \dashrightarrow \mathbb{P}^4$ (see the commutative diagram (2.7) also for recalling the notation).

The scroll $\overline{R} = \nu(R') \subset \mathbb{P}^4$ has degree 6 (recall that we tensor with $\mathcal{O}_{\mathbb{P}^1}(-1)$ performing the flop, see (2.6)) and $\mu^{-1} : \mathbb{P}^4 \dashrightarrow X$ is given by a linear system in $|H^0(\mathcal{J}_U^2(9))|$ by Theorem 2.10 (3), where $U \subset \mathbb{P}^4$ is the support of the base locus $\overline{B}$ of ν^{-1} (recall that $\overline{B}$ is generically reduced so that it coincides with U generically). By Theorem 2.10 (4), the lines of the scroll $\overline{R}$ are 5-secant to U, the map ψ is given by a linear system in $|H^0(\mathcal{J}_U(5))|$ and $\psi(\overline{R}) = \tilde{\psi}(R') = C$.

The cubics through S_{38} restricted to X are mapped by μ onto quintics defining $\overline{B}$ as a scheme by Theorem 2.10 (3). Taking a basis of cubics X_i through S_{38} restricted to X, $i = 1, \ldots, 9$, their images $V_i = \mu(X_i) \subset \mathbb{P}^4$ determine the ideal of $\overline{B}$. In Section 6.1 and in the ancillary file `code_section_6.m2` (see the arXiv version of the paper) we verified that $\overline{B}$ is a smooth surface of degree 12 and sectional genus 14 so that it coincides with U as scheme. Hence $U \subset \mathbb{P}^4$ is a smooth surface of degree 12 and sectional genus 14, whose ideal is generated by nine forms of degree 5. Since $\nu^{-1}(\overline{B}) = \overline{E}$, the universal property of the blow-up yields a birational projective morphism $\delta : W' \to \mathrm{Bl}_U\,\mathbb{P}^4$ between smooth projective fourfolds. Since $\mathrm{rk}(\mathrm{Pic}(W')) = 2 = \mathrm{rk}(\mathrm{Pic}(\mathrm{Bl}_U\,\mathbb{P}^4))$, we have $W' \simeq \mathrm{Bl}_U\,\mathbb{P}^4$ by Zariski's Main Theorem. Moreover,

$$\mu^{-1} : \mathbb{P}^4 \dashrightarrow X \subset \mathbb{P}^9$$

is given by the linear system $|H^0(\mathcal{J}_U^2(9))|$ by Theorem 2.10 (3).

Let $A = \mathbb{C}[t_0, \dots, t_4]$. Using the explicit equations constructed above, we can compute the resolution of the homogeneous ideal of U:

$$0 \leftarrow I_U \leftarrow A(-5)^{\oplus 9} \leftarrow A(-6)^{\oplus 11} \leftarrow A(-7)^{\oplus 3} \leftarrow 0$$

and also verify that $p_g(U) = 1$ and $q(U) = 0$ (see Section 6.1). To conclude the analysis of $U \subset \mathbb{P}^4$ we shall follow the arguments in [7, Section 2.7], where the authors gave a different construction of the above surface via Beilinson spectral sequence methods.

The intersection matrix for the sublattice $\langle H, K_U \rangle$ of $\mathrm{Num}(U)$ is

$$\begin{pmatrix} H^2 & H \cdot K_U \\ H \cdot K_U & K_U^2 \end{pmatrix} = \begin{pmatrix} 12 & 14 \\ 14 & -11 \end{pmatrix},$$

where $K_U^2 = -11$ is deduced from the double point formula for a smooth surface in $\mathbb{P}^4$. The number N_6 of proper 6-secant lines to U (if finite) plus the number of exceptional lines, that is, (-1)-curves $L_i \subset U$ such that $H \cdot L_i = 1$ and $L_i^2 = -1$, is equal to 10 by the Le Barz formulas (see *loc. cit.*). There are no proper 6-secant lines since I_U is generated by quintic forms. Let $U' \subset \mathbb{P}^{14}$ be the first adjunction surface of U, that is, the image of the birational morphism $\gamma : U \to U'$ defined by the base point free linear system $|K_U + H|$ (see [7, Introduction] for a summary of the basic results of adjunction theory on surfaces). Since the above morphism contracts the exceptional lines on U, the morphism γ realizes U as the blow-up of U' in ten distinct points and $K_{U'}^2 = K_U^2 + 10 = -1$. Since $H = \gamma^*(H') - \sum_{i=1}^{10} L_i$, we also have $H' \cdot K_{U'} = H \cdot K_U - 10 = 4$.

Since $p_g(U') = p_g(U) = 1$ and since $K_{U'}^2 = -1$, the canonical divisor is effective but not nef and the surface U' is non-minimal. Let $E \subset U'$ be a (-1)-curve, let $K_{U'} = D + E$ $(D \geq 0)$ and let $\tilde{E} \subset U$ be an irreducible curve such that $\gamma(\tilde{E}) = E$. Then $\tilde{E}^2 \leq E^2 = -1$ and

$$0 < (K_U + H) \cdot \tilde{E} = (K_{U'} + H') \cdot E = H' \cdot E - 1$$

yields $H' \cdot E \geq 2$. From $4 = H' \cdot (D + E)$ we deduce $H' \cdot D \leq 2$ and that E is not contained in the support of D (otherwise $K_{U'} = 2E$ and $K_{U'}^2 = -4$). Then $D \cdot E = 0$, $D^2 = 0$ and $K_{U'} \cdot D = 0$. If $H' \cdot E = 2$, then $H' \cdot D = 2$. From $(K_{U'} + D) \cdot D = 0$, we deduce that D cannot be an irreducible conic. If $D = 2 \cdot \tilde{L}$, then $H' \cdot \tilde{L} = 1$ would be a line such $(K_{U'} + \tilde{L}) \cdot \tilde{L} = 0$, which is impossible. In the same way we can exclude that $H' \cdot E = 3$, conclude that $H' \cdot E = 4$ and finally that $K_{U'} = E$. Let $\epsilon : U' \to U''$ be the contraction of E to a point of the smooth surface U''. Since $K_{U''} = 0$ and since $q(U'') = q(U) = 0$, the surface U'' is a K3. The curve $\tilde{E} \subset U$ is a smooth rational normal curve of degree 4, proving that the distinct points $p_i = \gamma(L_i)$ do not belong to E. In the next section we shall develop an alternative geometric method to obtain the whole configuration of the (-1)-curves on U in explicit examples.

Let $\pi : U \to U''$ be the blow-up of U'' at the eleven distinct points described above and let $H'' = \pi_*(H)$. Then H'' is an ample divisor on U'' with a point of multiplicity 4 at p_{11} and passing simply through $p_1, \ldots, p_{10}$ and the linear system H on U is thus given by $|\pi^*(H'') - \sum_{i=1}^{10} L_i - 4E|$. Hence $(H'')^2 = (H')^2 + 10 + 16 = 12 + 10 + 16 = 38$ and $p_a(H'') = p_a(H') + 6 = 20$. The generality of X through S_{38} ensures that the K3 surface has general moduli and that $\operatorname{Pic}(U'') \simeq \mathbb{Z}$ (a direct proof of this fact will be given below). Hence the divisor H'' is very ample and gives an embedding $U'' \subset \mathbb{P}^{20}$.

3.4. The associated K3 surface to a general cubic in $\mathcal{C}_{38}$ via the trisecant flop

Via the trisecant flop and via the contraction of the curves of the congruence contained in X, we proved that the associated surface $U \subset \mathbb{P}^4$ to a general pair (X, S_{38}) is a birational incarnation of a general smooth K3 surface $U'' \subset \mathbb{P}^{20}$ of degree 38 and genus 20. We now want to show that the surface U'' (or U) is associated to X in the sense of Hodge theory (or, equivalently, of derived category theory) following the treatment by Hassett [16, Section 3].

Let us recall that for an arbitrary cubic fourfold $X \subset \mathbb{P}^5$, letting $h^{p,q}$ denote the Hodge numbers of X, we have $h^{0,4} = h^{4,0} = 0$, $h^{1,3} = h^{3,1} = 1$, $h^{2,2} = 21$. Let h denote the class of a hyperplane section of X and let

$$H^4_{\text{prim}}(X, \mathbb{Z}) \simeq \langle h^2 \rangle^{\perp} \subset H^4(X, \mathbb{Z})$$

be the primitive cohomology of X. This cohomology reminds the H^2 cohomology of a K3 surface S, modulo a Tate twist by -1, since S has Hodge numbers $h^{1,0} = h^{0,1} = 1$, $h^{1,1} = 20$. The intersection forms have signatures $(20, 2)$ for $H^4_{\text{prim}}(X, \mathbb{Z})$ by the Hodge–Riemann bilinear relations and $(19, 3)$ for $H^2(S, \mathbb{Z})(-1)$. They become compatible as long as one can find a common codimension 1 Hodge substructure of signature $(19, 2)$.

The definition of $\mathcal{C}_{38}$ (see [16, Section 2.3] for the general theory) and the fact that a general $[X] \in \mathcal{C}_{38}$ contains a surface $S_{38} \subset \mathbb{P}^5$ (see [29]) yield

$$H^{2,2}(X, \mathbb{Z}) = H^4(X, \mathbb{Z}) \cap H^1(\Omega^1_X) \simeq \mathbb{Z}\langle h^2, [S_{38}] \rangle$$

for such an X. Let $\mathcal{T}_X \subset H^4(X, \mathbb{Z})$ denote the transcendental part of the cohomology of a cubic fourfold $X \subset \mathbb{P}^5$. Then in our setting

$$\mathcal{T}_X = \langle h^2, [S_{38}] \rangle^{\perp} \subset H^4(X, \mathbb{Z})$$

has rank 21 and signature $(19, 2)$. Clearly $H^2(S_{38}, \mathbb{Z}) \simeq \mathbb{Z}^{11}$, $H^2(T', \mathbb{Z}) \simeq \mathbb{Z}^2 \simeq H^2(R', \mathbb{Z})$ and $H^2(U, \mathbb{Z}) \simeq H^2(U'', \mathbb{Z}) \oplus \mathbb{Z}^{11}$ because we blow up eleven points on U''.

Let M be a smooth projective fourfold and let $S \subset M$ be a smooth projective surface. Then

$$H^4(\mathrm{Bl}_S\, M, \mathbb{Z}) \simeq H^4(M, \mathbb{Z}) \oplus_{\perp} H^2(S, \mathbb{Z})(-1). \tag{3.1}$$

The homomorphism $H^4(M,\mathbb{Z}) \to H^4(\mathrm{Bl}_S\, M,\mathbb{Z})$ is induced by the pull-back of $\pi : \mathrm{Bl}_S\, M \to M$. Letting $E = \mathbb{P}(N^*_{S/M}) \to S$ (Grothendieck notation), the homomorphism $H^2(S,\mathbb{Z})(-1) \to H^4(\mathrm{Bl}_S\, M,\mathbb{Z})$ is the pull-back from S to $E \subset \mathrm{Bl}_S\, M$ followed by push-forward via the inclusion of E.

We have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Bl}_{T'}\, X' = \mathrm{Bl}_{R'}\, W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' = \mathrm{Bl}_{S_{38}}\, X \arrow[rr, dashed, "\tau"] \arrow[d, "\lambda"'] & & W' = \mathrm{Bl}_U\, \mathbb{P}^4 \arrow[d, "\nu"] \\
X \arrow[rr, dashed, "\mu"] & & \mathbb{P}^4
\end{tikzcd} \tag{3.2}$$

inducing an isomorphism between

$$[H^4(X,\mathbb{Z}) \oplus_\perp H^2(S_{38},\mathbb{Z})(-1)] \oplus_\perp H^2(T',\mathbb{Z})(-1)$$

and

$$[H^4(\mathbb{P}^4,\mathbb{Z}) \oplus_\perp H^2(U,\mathbb{Z})(-1)] \oplus_\perp H^2(R',\mathbb{Z})(-1)$$

respecting the Hodge structures.

For a smooth projective surface S let $\mathcal{T}_S \subset H^2(S,\mathbb{Z})$ denote the trascendental part of the cohomology of S. Then $\mathcal{T}_{R'} = \mathcal{T}_{T'} = \mathcal{T}_{S_{38}} = 0$, while $\mathcal{T}_U \simeq \mathcal{T}_{U''}$ and $\mathrm{rk}(\mathcal{T}_{U''}) = 22 - \mathrm{rk}(\mathrm{Pic}(U''))$. On the one hand,

$$\mathcal{T}_{\mathrm{Bl}_{T'}\, X'} \simeq \mathcal{T}_{\mathrm{Bl}_{S_{38}}\, X} \simeq \mathcal{T}_X = \langle h^2, [S_{38}]\rangle^\perp \subset H^4(X,\mathbb{Z})$$

implies $\mathrm{rk}(\mathcal{T}_{\mathrm{Bl}_{T'}\, X'}) = 21$. On the other hand

$$\mathcal{T}_{\mathrm{Bl}_{R'}\, W'} \simeq \mathcal{T}_{\mathrm{Bl}_U\, \mathbb{P}^4} \simeq \mathcal{T}_U \simeq \mathcal{T}_{U''}$$

yields

$$\mathrm{rk}(\mathrm{Pic}(U'')) = 22 - \mathrm{rk}(\mathcal{T}_{U''}) = 22 - \mathrm{rk}(\mathcal{T}_{\mathrm{Bl}_{R'}\, W'}) = 22 - 21 = 1.$$

If $H'' \subset U''$, let $h'' = [H''] \in H^2(U'',\mathbb{Z})$. Then $h''^\perp = \mathcal{T}_{U''} \subset H^2(U'',\mathbb{Z})(-1)$ and the previous isomorphisms defined via (3.2) induce an isomorphism

$$H^4(X,\mathbb{Z}) \supset \langle h^2, [S_{38}]\rangle^\perp \xrightarrow{\simeq} \langle h''\rangle^\perp \subset H^4(U'',\mathbb{Z})(-1)$$

respecting Hodge structures. Therefore the K3 surface U'' is associated to X in the sense of Hodge theory according to Hassett [16, Section 3.2].

The isomorphism between $H^4(\mathrm{Bl}_{T'}\, X',\mathbb{Z})$ and $H^4(\mathrm{Bl}_{R'}\, W',\mathbb{Z})$ sends the classes corresponding to the ten exceptional lines on S_{38} to the classes corresponding to the ten exceptional lines on $U \subset \mathbb{P}^4$, while the rational normal curve of degree 4 on U correspond to the class of $H^2(T',\mathbb{Z})(-1)$, $T' \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(4)) \to \mathbb{P}^1$, not contracted by ω. This gives an interpretation of the fact that $\tilde{\varphi}(T') = C = \tilde{\psi}(R')$ is a rational normal curve of degree 4 (or equivalently that T' admits a section which is a rational normal curve of degree 4) and shows that the exceptional curve $\tilde{E} \subset U$ of degree 4 is exactly $\nu(R') \cap U$.

4. A geometric method for detecting the exceptional (-1)-curves on some non-minimal K3 surfaces

We shall now consider the problem of finding the non-minimal K3 surface $U \subset W$ in the base locus scheme of the birational map $\mu : X \dashrightarrow W$ defined in the previous sections for a general cubic $[X] \in \mathcal{C}_d$ with $d = 14, 38$ in order to develop a method to be applied later in the more difficult case $d = 42$. The detection of the (-1)-curves on the (smooth non-minimal K3) surface U (or on its linear normalization if U has nodes) is a delicate and intriguing problem, which in some cases can lead to the explicit construction of the general K3 surface of degree $2g - 2$ and genus g. When this is possible, one usually gets a direct proof that the corresponding moduli space of polarized K3 surfaces is unirational. In the previous section to analyse the case $d = 38$, following the traditional approach of [7], we used the smoothness of $U \subset \mathbb{P}^4 = W$ and appealed to the Le Barz formulas together with adjunction theory in order to find the ten exceptional lines on U. Then, after the contraction of these ten (-1)-curves, there appeared a last exceptional curve, which is a quartic rational normal curve already contained in U.

It is very difficult to try to adapt these arguments to surfaces $U \subset W$ with W a Fano fourfold lying in spaces of higher dimension. So we elaborated an entirely new geometric method, which works efficiently in the cases $d = 14, 26, 38, 42$ to prove that U is the blow-up of a K3 surface of degree d and genus $g = (d + 2)/2$ *associated* to $X \in \mathcal{C}_d$. Our analysis is based on the existence of the congruence, of the map $\mu : \mathbb{P}^5 \dashrightarrow W$, and on a careful study of the known examples studied by Fano [8] (see also [5]).

Recall that by definition of congruence of $(3e - 1)$-secant curves of degree $e \geq 1$ to $S \subset \mathbb{P}^5$, we have a diagram (1.1), where $\pi : \mathcal{D} \to \mathcal{H}$ is the universal family over the parameter space $\mathcal{H}$ and where $p : \mathcal{D} \to \mathbb{P}^5$ is the tautological morphism, which by definition is birational (see Section 1.1). The *fundamental locus of the congruence*,

$$E = \overline{\{q \in \mathbb{P}^5 : \dim(p^{-1}(q)) > 0\}} \subset \mathbb{P}^5,$$

is the base locus of p^{-1} by Zariski's Main Theorem, which also implies that E is the image of the ramification locus of p. The locus E has codimension at least 2. Let $E_1, \dots, E_r$, $r \geq 1$, be the irreducible components of dimension 3 of E, if any. Suppose there exists an associated rational map $\mu : \mathbb{P}^5 \dashrightarrow W$ defined by the linear system $|H^0(\mathcal{J}_S^e(3e - 1))|$ such that a general fibre of μ is a curve of the congruence and let $C_i = \mu(E_i) \subset W$, $i = 1, \dots, r$. These are special curves in W, defined via the congruence but without any apparent relation to cubic fourfolds through S.

Let us start by describing the surfaces considered in Example 1.2 from this perspective.

Example 4.1. Let $S = \mathrm{Bl}_{\{p_1,\dots,p_4\}} \mathbb{P}^2 \subset \mathbb{P}^5$ be a smooth quintic del Pezzo surface. The embedding is given by the linear system of cubics passing through the four points $p_1, \dots, p_4$, denoted as usual by $|3L - E_1 - E_2 - E_3 - E_4| = |-K_S|$. The pencil of lines through one of the points p_i, $i = 1, \dots, 4$, and the pencil of conics through $p_1, \dots, p_4$ produce five base point free pencils of conics on S. Moreover, by considering a Cremona transformation centred at three of the four base points we can always find a representation of S as the blow-up at four points in which any of the five pencil of conics is represented by the pencil of conics through the four base points.

Let $C \subset S$ be an irreducible conic and $\Pi = \langle C \rangle = \mathbb{P}^2$ be its linear span. Since S is defined by quadratic equations, $\Pi \cap E_i$ is either empty or consists of a point (otherwise $\Pi \cap S$ would contain C and the line E_i). Let $\pi : S \to \mathbb{P}^2$ be the blow-up morphism. Then $C = dL - \sum_{i=1}^4 a_i E_i$ with $0 \leq a_i \leq 1$. From $2 = 3d - \sum_{i=1}^4 a_i$, we get $3d = 2 + \sum_{i=1}^4 a_i \leq 6$, yielding either $d = 1$ and $a_i = 1$ for only one i, or $d = 2$ and $a_i = 1$ for every i. In conclusion, C belongs to one of the five pencils described above.

Let $\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ be the map associated to the congruence of secant lines to S. By Proposition 1.5 the closures of the fibres of μ are either secant lines to S or planes cutting S along a conic. Hence the fundamental locus E of the congruence of secant lines to S is the locus of planes spanned by conics through S (through a point of the plane there pass infinitely many secant lines to S and if through a point of a secant line there passes another secant line, these lines span a plane by the analysis of the fibres of μ).

Let $\pi_1 : S \to \mathbb{P}^1$ be a morphism induced by one of the five pencils of conics and let $\pi : S \to \mathbb{P}^2$ be a blow-up morphism in which the pencil is represented by conics through the four base points. Then $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathcal{O}_S(2L - E_1 - E_2 - E_3 - E_4)$ and $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}(L)$ so that $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)) = \mathcal{O}_S(3L - E_1 - E_2 - E_3 - E_4)$. By the universal property of the product, $\alpha = \pi_1 \times \pi : S \to E_1 = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ is such that $\alpha^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_S(1)$. Hence α is an embedding and $S \subset E_1$ is a divisor of type $(1, 2)$. In conclusion, we have five Segre threefolds $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^2$, $i = 1, \dots, 5$, containing S as a divisor of type $(1, 2)$ and $E = E_1 \cup \dots \cup E_5$. Then $C_i = \mu(E_i) \subset \mathbb{P}^4$ are five lines because μ is defined by the linear system of quadrics through S (note that by restricting μ to E_i we have $(2, 2) - (1, 2) = (1, 0)$ on E_i).

Example 4.2. If $S \subset \mathbb{P}^5$ is a general quartic rational normal scroll, the fundamental locus E of the congruence of secant lines to S consists of the unique Segre threefold $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^2$ containing S as a divisor of type $(0, 2)$. Indeed, $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 2)$ so that, reasoning as above, $S \simeq \mathbb{P}^1 \times C \subset \mathbb{P}^1 \times \mathbb{P}^2 = \Sigma \subset \mathbb{P}^5$ with $C \subset \mathbb{P}^2$ a conic. The conics on S are precisely the fibres of the first projection $\mathbb{P}^1 \times C \to \mathbb{P}^1$. The fibres of $\mu : \mathbb{P}^5 \dashrightarrow W$ with $W \subset \mathbb{P}^5$ a smooth quadric hypersurface are all linear by Proposition 1.5. Let $E \subset \mathbb{P}^5$ be the fundamental locus of the congruence of secant lines to S. If $q \in E$, then through E there pass at least two secant lines to S. So $\Pi_q = \overline{\mu^{-1}(\mu(q))}$ is a plane cutting S along a conic, yielding $E \subset \Sigma$ and hence $E = \Sigma$. Then $C = \mu(E) \subset W$ is a smooth conic because on E we have $(2, 2) - (0, 2) = (2, 0)$.

In general it is not easy to determine the E_i's and then calculate their images (although possible in all the examples treated here). We shall now present a key remark, which allows us to determine the C_i's without necessarily computing the E_i's in all the known examples we have studied until now.

Geometric method. Suppose there exists a congruence of $(3e - 1)$-secant curves of degree $e \geq 1$ to $S \subset \mathbb{P}^5$ with fundamental locus $E = E_1 \cup \dots \cup E_r$ with $E_i \subset E$ the irreducible components of dimension 3 of E, if any. Suppose there exists an associated rational map $\mu : \mathbb{P}^5 \dashrightarrow W$ defined by the linear system $|H^0(\mathcal{J}_S^e(3e-1))|$ such that a general fibre of μ is a curve of the congruence and let $C_i = \mu(E_i) \subset W$, $i = 1, \dots, r$. Since through a general point of E_i the fibre of μ has dimension at least 2, we expect that $C_i \subset W$ is a curve (see Examples 4.1 and 4.2).

Let $X_j \subset \mathbb{P}^5$, $j = 1, 2$, be a general cubic through S and let $U_j \subset W$ be the associated surface contained in the base locus of the inverse of the restriction of μ to X_j. Then $C_i = \mu(X_j \cap E_i) \subset U_j$ for every $i = 1, \dots, r$ and for $j = 1, 2$, so that

$$C_1 \cup \cdots \cup C_r \subset U_1 \cap U_2.$$

Since U_1 and U_2 are moving surfaces in the fourfold W, one expects that $C_1 \cup \cdots \cup C_r$ is exactly the one-dimensional component C of $U_1 \cap U_2$.

From the equations of U_1 and U_2 we immediately derive those defining C. In the cases under consideration this allows us to verify the smoothness of C, that $C = C_1 \cup \cdots \cup C_r$ and that all the disjoint irreducible components C_i are rational (many components are lines) and that $h^1(N_{C/U}) = 0$. The knowledge of the equations of the U_j's yields $p_g(U_j) = 1$ and $q(U_j) = 0$ (at least in the smooth cases) by direct computation. Collecting all the information one proves that U is the blow-up of a K3 surface as shown by the next crucial remark.

Lemma 4.3. *Let U be a smooth projective surface with $p_g(U) = 1$ and with $q(U) = 0$. Let $C = C_1 \cup \cdots \cup C_r$, $r \geq 1$, be a smooth curve on U with C_i a rational curve for every $i = 1, \dots, r$. Then:*

(1) *$C_i^2 < 0$ for every $i = 1, \dots, r$.*

(2) *Every C_i is a (-1)-curve on U if and only if $h^1(N_{C/U}) = 0$.*

(3) *If $h^1(N_{C/U}) = 0$ and if there exists an ample divisor H on U such that $H \cdot K_U = H \cdot C$, then $K_U = C_1 + \cdots + C_r$ and U is the blow-up at r distinct points of a K3 surface U'.*

(4) *Under the hypotheses in (3), if $\pi : U \to U'$ is the blow-up and $H' = \pi(H)$, then*

$$H = \pi^*(H') - \sum_{i=1}^{r} (H \cdot C_i) C_i.$$

Proof. From $q(U) = h^1(\mathcal{O}_U) = 0$ we deduce $h^0(\mathcal{O}_U(C_i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(C_i^2)) + 1$. If $C_i^2 \geq 0$, then the rational curves in $|H^0(\mathcal{O}_U(C_i))|$ would cover U and U would be uniruled, contradicting $p_g(U) = 1$. So $C_i^2 < 0$ for every $i = 1, \dots, r$.

The C_i's are disjoint smooth rational curves, $\mathcal{O}_C(C)_{|C_i} \simeq \mathcal{O}_{\mathbb{P}^1}(C_i^2)$ and

$$h^1(N_{C/U_j}) = \sum_{i=1}^{r} h^0(\mathcal{O}_{\mathbb{P}^1}(-2 - C_i^2))$$

is equal to 0 if and only if $C_i^2 = -1$ for each $i = 1, \dots, r$, proving (2).

Since $p_g(U) = 1$, the canonical class is effective. From $K_U \cdot C_j = -1$ we deduce $K_U = C + D$ with $D \geq 0$. Then $H \cdot C = H \cdot K_U = H \cdot C + H \cdot D$ yields $H \cdot D = 0$ and hence $D = 0$. Since the C_i's are disjoint (-1)-curves, the contraction of the curves in C produces a smooth surface U' and a morphism $\pi : U \to U'$ which is the blow-up of $r \geq 1$ distinct points on U. Then $K_{U'} = 0$ and $q(U') = q(U) = 0$ imply that U' is a K3 surface. The last claim about the expression of H via pull-back of H' is now obvious. ■

Example 4.4 (Application to smooth quintic del Pezzo surfaces in $\mathbb{P}^5$). Suppose $S \subset \mathbb{P}^5$ is a smooth del Pezzo surface of degree 5. Then $U_j \subset \mathbb{P}^4$, $j = 1, 2$, are smooth surfaces of degree 9 and sectional genus 8 with $p_g(U_j) = 1$ and $q(U_j) = 0$ (the invariants are determined via the explicit equations obtained via the restriction of $\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$). The one-dimensional component C of $U_1 \cap U_2$ is a smooth curve of degree 5 and arithmetic genus -4, from which it immediately follows that C has five irreducible components and hence is the union of five distinct lines $C_1, \ldots, C_5$, as we already know.

Let $U = U_1$. After verifying that $h^1(N_{C/U}) = 0$, we deduce that every C_i is a (-1)-curve on U. Since $H \cdot K_U = -H^2 + 2g(H) - 2 = -9 + 16 - 2 = 5 = H \cdot C$ we conclude by Lemma 4.3 that $\pi : U \to U'$ is the blow-up at five distinct points $p_1, \ldots, p_5$ of the K3 surface U' and that $H' = \pi_*(H) \subset U'$ is a very ample divisor on U' such that $(H')^2 = 9 + 5 = 14$, $g(H') = 8$. Indeed, H' is ample and the generality of X_1 and of $U = U_1$ implies that $\mathrm{rk}(\mathrm{Pic}(U')) = 1$ (see the argument used at the end of Section 3.1 or compute the moduli of the K3's). Hence $U' \subset \mathbb{P}^8$ is a K3 surface of degree 14 and genus 8 and the linear system $|H|$ on U corresponds to the hyperplane sections of U' passing through $p_1, \ldots, p_5$. In particular, we see that these points impose only four independent conditions to hyperplane sections.

Example 4.5 (Application to general quartic rational normal scrolls in $\mathbb{P}^5$). For $S \subset \mathbb{P}^5$ a general rational normal scroll, the corresponding surfaces $U_j \subset W$ are smooth, have degree 10, sectional genus 7, $p_g(U_j) = 1$ and $q(U_j) = 0$. The one-dimensional component C of $U_1 \cap U_2$ is a smooth conic, which is a (-1)-curve on U_j because $h^1(N_{C/U_j}) = 0$. Since $H \cdot K_{U_j} = 2 = H \cdot C$, we deduce $K_{U_j} = C$ and we can apply Lemma 4.3. We conclude that $U_j \subset \mathbb{P}^5$ is obtained by blowing up a K3 surface $U' \subset \mathbb{P}^8$ of degree 14 and genus 8 at one point $p \in U'$ and that $|H|$ corresponds to the linear system of hyperplane sections having a point of multiplicity at least 2 at p.

Example 4.6 (Application to degree 10 surfaces $S_{38} \subset \mathbb{P}^5$). The surfaces $U_j \subset \mathbb{P}^4$, $j = 1, 2$, are smooth, have degree 12, sectional genus 14, $p_g(U_j) = 1$ and $q(U_j) = 0$ (see Section 3.1). The one-dimensional component C of $U_1 \cap U_2$ is a smooth curve of degree 14, of arithmetic genus -10 with eleven irreducible components. By projecting C generically to $\mathbb{P}^3$ (to speed computations) and to $\mathbb{P}^2$ (to read more efficiently the decomposition), we get a smooth curve of degree 14, respectively a plane curve $\tilde{C}$ of degree 14. The lines in $\tilde{C}$ disappear by taking duality, that is, the image of $\tilde{C}$ via the Gauss map of $\tilde{C}$. Then, by reflexivity, the bidual curves of $\tilde{C}$ are precisely the irreducible curves in $\tilde{C}$ different from lines. It turns out that the bidual curve of $\tilde{C}$ is a quartic curve with three nodes from which it follows that C is the disjoint union of ten lines and of a quartic rational normal curve (see also the ancillary file `code_section_6.m2`). In conclusion, C is the union of eleven smooth rational curves.

The eleven curves C_i are (-1)-curves on U_j because $h^1(N_{C/U_j}) = 0$ (see `code_section_6.m2` for the computation and then apply Lemma 4.3). Since $H \cdot K_U = -H^2 + 2g(H) - 2 = -12 + 28 - 2 = 14 = H \cdot C$, we deduce from Lemma 4.3 that there exists a contraction $\pi : U \to U'$ of the eleven (-1)-curves C_i to the eleven distinct points $p_1, \dots, p_{11}$ on the smooth K3 surface U'. Then $H' = \pi_*(H) \subset U'$ is an ample divisor on U' such that $(H')^2 = 12 + 16 + 10 = 38$ and $g(H') = 14 + 6 = 20$. For a general X through S_{38} we have that U' is a K3 surface of degree 38 and genus 20 with $\mathrm{rk}(\mathrm{Pic}(U')) = 1$ (see the argument at the end of Section 3.1 or compute the moduli of the K3's) so that $|H'|$ gives an embedding $U' \subset \mathbb{P}^{20}$. The linear system $|H|$ on U corresponds to the hyperplane section of U' passing through $p_1, \dots, p_{10}$ and having a point of multiplicity at least 4 at p_{11}. In particular, we see that these points do not impose independent conditions on hyperplane sections of U'.

5. Rationality of cubics in $\mathcal{C}_{42}$ via congruences of 8-secant twisted cubics

In this section, we first construct a family of surfaces in $\mathbb{P}^5$ to describe the divisor $\mathcal{C}_{42}$ as the locus of cubic fourfolds containing these surfaces. This family is related to an example of a surface of degree 9 and sectional genus 2 contained in a del Pezzo fivefold, which has been discovered (computationally) for the first time in [18]. Here we give an explicit and geometric description of the complete family of these surfaces inside a del Pezzo fivefold and then use them to construct surfaces in $\mathbb{P}^5$ of degree 9 and sectional genus 2 with five nodes. Finally, we use this new description of $\mathcal{C}_{42}$ to show that $\mathcal{C}_{42}$ is unirational (see Corollary 5.10) and that every cubic fourfold in $\mathcal{C}_{42}$ is rational (see Theorem 5.12).

5.1. Birational representations of del Pezzo fivefolds

A del Pezzo fivefold $V \subset \mathbb{P}^8$ is a smooth hyperplane section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$. The following result is well known in classical algebraic geometry (see e.g. [44] and [37, Section 10]).

Proposition 5.1. *Let* $V = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ *be a del Pezzo fivefold and let* $\Pi \subset V$ *be a plane with class* $\sigma_{2,2}$ *in* $\mathbb{G}(1,4)$. *Then:*

(i) *The projection from* Π *restricted to* V *induces a birational map* $V \dashrightarrow \mathbb{P}^5$ *whose base locus scheme is* Π.

(ii) *The inverse map* $\mathbb{P}^5 \dashrightarrow V \subset \mathbb{P}^8$ *is given by the linear system of quadric hypersurfaces through a rational normal cubic scroll contained in a hyperplane in* $\mathbb{P}^5$.

Remark 5.2. A del Pezzo fivefold $V = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ contains two distinct families of planes, $\mathcal{F}_1$ and $\mathcal{F}_2$, which in the Chow ring of $\mathbb{G}(1,4)$ have Schubert classes given by $\sigma_{2,2}$ and $\sigma_{3,1}$. The family $\mathcal{F}_1$ has dimension 3 and through a point of V there passes a unique plane of the family. The family $\mathcal{F}_2$ has dimension 4 and through a general point of V there passes a one-dimensional family of these planes.

If $\alpha : \mathbb{P}^5 \dashrightarrow V \subset \mathbb{P}^8$ denotes the birational map defined by the quadrics through a rational normal cubic scroll $\Delta \subset H \simeq \mathbb{P}^4 \subset \mathbb{P}^5$ as in Proposition 5.1, then the planes through a point $q = \alpha(p)$, $p \in \mathbb{P}^5 \setminus H$, correspond, respectively, to the unique plane generated by p and the directrix line of Δ and to the planes generated by p and by a line of the ruling of Δ.

Since $N_{C/V} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}$ (see for example [20, p. 403]) for an irreducible conic $C \subset V \subset \mathbb{P}^8$, the Hilbert scheme $\mathcal{C}on$ of conics contained in V has dimension $10 = h^0(N_{C/V})$, a fact that can also be deduced from a simple parameter count using the previous birational representation of V.

The next result is also classical and well-known. It can also be easily verified via an explicit computation.

Proposition 5.3. *Let $V = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be a del Pezzo fivefold and let $C \subset V$ be an irreducible conic whose linear span $\Pi = \langle C \rangle$ is not contained in V. Then:*

(i) *The projection from Π restricted to V induces a birational map $V \dashrightarrow \mathbb{P}^5$ whose base locus scheme is C.*

(ii) *The inverse map $\mathbb{P}^5 \dashrightarrow V \subset \mathbb{P}^8$ is given by the linear system of cubic hypersurfaces vanishing on a rational scroll in $\mathbb{P}^5$ of dimension 3 and degree 4, which is a projection of a smooth quartic rational normal scroll in $\mathbb{P}^6$.*

5.2. *Curves of degree 8 in $\mathbb{P}^5$ with a node and with geometric genus 2 contained in a projected rational scroll of degree 4*

Now we prove some geometrical properties about irreducible curves of degree 8 and arithmetic genus 3 in $\mathbb{P}^5$. We shall restrict ourselves to the case of curves with a node, used in what follows, although the same proof works also for smooth curves (this case has been considered in [38]).

Proposition 5.4. *Let $C \subset \mathbb{P}^5$ be a non-degenerate curve of degree 8 and arithmetic genus 3 with a node. Then:*

(i) *The curve $C \subset \mathbb{P}^5$ is the complete intersection of a pencil of quintic del Pezzo surfaces on a Segre threefold $\Sigma = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ such that the general element of the pencil is smooth.*

(ii) *The curve $C \subset \mathbb{P}^5$ has ideal generated by seven quadratic forms, defining a birational map $\psi : \mathbb{P}^5 \dashrightarrow W \subset \mathbb{P}^6$ onto a quartic hypersurface and such that $\psi(\Sigma) = Q \subset W \subset \mathbb{P}^6$ is a smooth quadric surface contained in the base locus scheme of ψ^{-1}.*

(iii) *The preimages on Σ of the lines of the two rulings of Q are, respectively, the planes of the ruling of Σ and the pencil of del Pezzo surfaces through C.*

(iv) *The map* ψ *is an isomorphism outside* $\Sigma \cup \mathrm{Sec}(C)$ *and* $\mathrm{Sec}(C)$ *is mapped onto a degree* 24 *surface* T. *The quartic hypersurface* W *has double points along* Q *and* T, *and* $\psi^{-1} : W \dashrightarrow \mathbb{P}^5$ *is given by the restriction of a linear system of quartic hypersurfaces in* $\mathbb{P}^6$ *passing simply through* T *and having double points along* Q.

Proof. Let $C \subset \mathbb{P}^5$ be an irreducible curve of degree 8 and geometric genus 2 with a node. Let $\nu : C' \to C$ be its normalization and let $\mathcal{L} = \nu^*(\mathcal{O}_C(1))$. The linear system $|H^0(\mathcal{L})|$ has dimension 6 by Riemann–Roch, it is very ample and it embeds C' in $\mathbb{P}^6$ as a smooth curve of degree 8 and genus 2. The curve $C \subset \mathbb{P}^5$ is the projection of C' from a point q on the secant variety $\mathrm{Sec}(C') \subset \mathbb{P}^6$ of C' but not belonging to the tangential surface $\mathrm{Tan}(C')$.

Let $P' = \langle p_1', \dots, p_4' \rangle$, $p_i' \in C'$, be a 4-secant $\mathbb{P}^3$ passing through q and let $P = \langle p_1, \dots, p_4 \rangle$, with $p_i \in C$ the projection from q of p_i'. Then P is a 4-secant plane to C and, letting $D = p_1 + \cdots + p_4$ be the corresponding Cartier divisor on C, a general hyperplane through P cuts C in D and in other four points $p_5, \dots, p_8$ such that $\Pi = \langle p_5, \dots, p_8 \rangle = \mathbb{P}^3$. Letting $E = p_5 + \cdots + p_8$, from the previous definitions and from Riemann–Roch we deduce $h^0(\mathcal{O}_C(D)) = 2$ and $h^0(\mathcal{O}_C(E)) = 3$. The linear system $|H^0(\mathcal{O}_C(D))|$ defines a morphism $\xi : C \to \mathbb{P}^1$ of degree 4, while $|H^0(\mathcal{O}_C(E))|$ defines a morphism $\eta : C \to \mathbb{P}^2$ birational onto its image. By the universal property of the product we have a morphism $\xi \times \eta : C \to \mathbb{P}^1 \times \mathbb{P}^2$ which composed with the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ gives a morphism $\epsilon : C \to \mathbb{P}^5$ such that $\epsilon^*(\mathcal{O}_{\mathbb{P}^5}(1)) = \mathcal{O}_C(D) \otimes \mathcal{O}_C(E) = \mathcal{O}_C(1)$. This means that the embedding $C \subset \mathbb{P}^5$ factors through $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ in such a way that ξ is the composition of ϵ with the projection onto the first factor and that η is the composition of ϵ with projection onto the second factor. In particular, $P = p \times \mathbb{P}^2 \subset \Sigma$ for some $p \in \mathbb{P}^1$. By Riemann–Roch we deduce $h^0(\mathcal{J}_C(2)) \geq 7$ so that $h^0(\mathcal{J}_{C \cup P}(2)) \geq 5$ (from $P \cap C = \{p_1, \dots, p_4\}$ we deduce that containing P imposes only two conditions on quadrics vanishing on C). Since $h^0(\mathcal{J}_\Sigma(2)) = 3$, there exists at least a pencil $\{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$ of quadrics vanishing on $C \cup P$ but not on Σ. This pencil of quadrics cuts Σ along P and along a residual pencil $\{S_\lambda\}_{\lambda \in \mathbb{P}^1}$ of divisors of type $(1, 2)$ containing C. The projection from P maps Σ onto a plane Π, and C onto a plane quartic curve. Hence C is not contained in any divisor of type $(0, 2)$ on Σ (they project from P onto a conic) and being non-degenerate in $\mathbb{P}^5$ is not contained in any divisor of type $(1, 1)$ or $(1, 0)$ or $(0, 1)$, proving that every S_λ is irreducible. Since the complete intersection of two distinct divisors of type $(1, 2)$ on Σ without common irreducible components is a curve of degree 8 and arithmetic genus 3, we conclude that C is the complete intersection of two surfaces in the pencil $\{S_\lambda\}_{\lambda \in \mathbb{P}^1}$. The projection from P restricted to S_λ resolves to a birational morphism $\sigma_\lambda : S_\lambda \to \Pi$. The planes in Σ cuts on S_λ a pencil of conics, whose image by σ_λ is a pencil of conics having a base locus scheme $Z_\lambda \subset \Pi$ of length 4. The inverse map $\sigma_\lambda^{-1} : \Pi \dashrightarrow S_\lambda$ is given by the linear system of cubics vanishing at Z_λ. Let $\tilde{C} \subset \Pi$ denote the projection of C from P. Clearly $Z_\lambda \subset \tilde{C}$ and the divisors $\{Z_\lambda\}_{\lambda \in \mathbb{P}^1}$ vary in a g_4^1 on $\tilde{C}$. Moreover, the pencil of conics through a fixed Z_λ cuts on $\tilde{C}$ the $g_4^1 = |H^0(\mathcal{O}_{\tilde{C}}((\sigma_\lambda)_*(D)))|$. On the other hand, for a fixed general divisor $\tilde{D}_\mu$ in the last g_4^1, the pencil of conics through $\tilde{D}_\mu$ cuts on $\tilde{C}$ the $g_4^1 = \{Z_\lambda\}_{\lambda \in \mathbb{P}^1}$. Hence for general

$\lambda \in \mathbb{P}^1$ the scheme Z_λ is smooth by the Bertini Theorem and the corresponding $S_\lambda \subset \mathbb{P}^5$ is a smooth quintic del Pezzo surface. Thus, part (i) is proved.

Let $C = S \cap S' \subset \Sigma$ be a curve of degree 8 and arithmetic genus 3, complete intersection of a pencil $\{S_\lambda\}_{\lambda \in \mathbb{P}^1}$ of divisors of type $(1, 2)$ on Σ, whose general member is smooth. Arguing as in [19, pp. 435–436], an iterated use of the mapping cone and the knowledge of the resolution of a $S_\lambda \subset \mathbb{P}^5$ show that $C \subset \mathbb{P}^5$ has ideal generated by seven quadratic equations (more precisely, we can obtain the complete free resolution of the ideal of $C \subset \mathbb{P}^5$, see *loc. cit.*). Let $\psi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^6$ be the map defined by $|H^0(\mathcal{I}_C(2))|$. The image $W = \overline{\psi(\mathbb{P}^5)} \subset \mathbb{P}^6$ is a quartic hypersurface (see [38] and [19]). The restriction of ψ to Σ is a linear system of dimension 3 because Σ is defined by three quadratic equations vanishing on C. Let $S \subset \Sigma$ be a quintic del Pezzo surface through C and let $\pi : S \simeq \mathrm{Bl}_{\{q_1,\dots,q_4\}} \mathbb{P}^2 \to \mathbb{P}^2$ be a birational representation such that the pencil of conics on S given by the restriction of the projection onto the first factor of Σ is mapped on $\mathbb{P}^2$ to the pencil of conics through $q_1, \dots, q_4$. Then $\pi(C)$ is a quartic curve passing through $q_1, \dots, q_4$. Hence the free part of the restriction of $|H^0(\mathcal{I}_C(2))|$ to S is given by the pencil of strict transforms of conics passing through $q_1, \dots, q_4$, and $\psi(S)$ is a line contained in $Q = \overline{\psi(\Sigma)} \subset W$. In particular, $Q \subset \mathbb{P}^3 \subset \mathbb{P}^6$ is a surface. Since each plane $\Pi \subset \Sigma$ cuts C in four points, ψ maps Π onto a line in Q cutting $\psi(S)$ in the point corresponding to $\psi(\Pi \cap S)$. In conclusion, $Q \subset \mathbb{P}^3$ is a smooth quadric surface and all the claims in (ii) and (iii) are proved. A finer analysis of the map ψ, as in [38, Section 4] or in [19, Section 3, Lemmas 3.3, 3.4], leads to the description of all the positive-dimensional fibres of ψ and of all the properties listed in part (iv). We refer to [38, p. 208] for the details of these computations. ■

The following result will play a crucial role in our geometric constructions together with the description of the map ψ defined above.

Proposition 5.5. *Let $B \subset \mathbb{P}^5$ be a rational scroll of dimension 3 and degree 4 which is a general projection of a smooth quartic rational normal scroll $B' \subset \mathbb{P}^6$. Then:*

- (i) *There exists an irreducible family $\mathcal{F}$ of dimension 15 of curves of degree 8 and geometric genus 2 on B whose general member is nodal.*
- (ii) *There exists an irreducible family $\mathcal{D}$ of dimension 16 of quintic del Pezzo surfaces in $\mathbb{P}^5$ whose general member is smooth and cuts B along a general curve of the family $\mathcal{F}$.*

Proof. Let $C' \subset \mathbb{P}^6$ be a smooth curve of degree 8 and genus 2. The projection of $C' \subset \mathbb{P}^6$ from four general points $p'_1, \dots, p'_4$ on it is a quartic plane curve $\tilde{C} \subset \mathbb{P}^2$ with a node $\tilde{p}_5$. Let $\tilde{p}_1, \dots, \tilde{p}_4 \in \tilde{C}$ be the images of $p'_1, \dots, p'_4$. The linear system of quartic curves through $\tilde{p}_1, \dots, \tilde{p}_4$ and having double points at $\tilde{p}_5$ embeds the blow-up of $\mathbb{P}^2$ at $\tilde{p}_1, \dots, \tilde{p}_5$ as a smooth surface $S \subset \mathbb{P}^7$ of degree 8 and genus 2 having $C' \subset \mathbb{P}^6$ as a hyperplane section. The linear system $|H^0(\mathcal{I}_{\{\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_5\}}(2))|$ yields a morphism $\varphi_1 : S \to \mathbb{P}^1$, while $|H^0(\mathcal{I}_{\{\tilde{p}_4,\tilde{p}_5\}}(2))|$ yields a morphism $\varphi_2 : S \to \mathbb{P}^3$. The composition of the morphism $\varphi_1 \times \varphi_2 : S \to \mathbb{P}^1 \times \mathbb{P}^3$ with the Segre embedding is given by the linear system embedding S in $\mathbb{P}^7$. Hence $S \subset \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and $C' \subset \mathbb{P}^6$ is contained in a smooth quartic rational normal scroll $B' \subset \mathbb{P}^6$, which is a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^3$.

Let $C' \subset B'$ be a curve of degree 8 and genus 2. Then $-K_{B'} \cdot C' = 18$, $\deg(N_{C'/B'}) = 20$ by the adjunction formula and $\chi(N_{C'/B'}) = 20 + 2(1 - g(C')) = 18$. Since in an explicit example we verified that $h^0(N_{C'/B'}) = 18$ and that $h^1(N_{C'/B'}) = 0$, the general curve of degree 8 and genus 2 contained in B' belongs to a unique irreducible and generically smooth component of dimension 18 of the corresponding Hilbert scheme. The family of the secant varieties $\mathrm{Sec}(C')$ with $C' \subset B'$ has dimension $21 = 18 + 3$, so through a general point $q \in \mathbb{P}^6$ there passes a family of dimension 15 of such secant varieties. The projection from q of the corresponding curves produces a 15-dimensional family of nodal curves of degree 8 and geometric genus 2 on the scroll B, projection of B' from q. Since one verifies that $h^0(N_{C/B}) = 15$ (a fact which can be easily computed with *Macaulay2*), the family $\mathcal{F}$ of such curves is irreducible, generically smooth and of dimension 15.

By Proposition 5.4, a nodal curve $C \subset B \subset \mathbb{P}^5$ of degree 8 and geometric genus 2 with a node is the complete intersection of a pencil of del Pezzo surfaces on a Segre threefold $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Let $C \subset S = S_\lambda \subset \Sigma$ with $\lambda \in \mathbb{P}^1$ general and let $\pi : S = \mathrm{Bl}_{\{q_1,\ldots,q_4\}} \mathbb{P}^2 \to \mathbb{P}^2$ be the blow-up morphism. Then we can assume that $\pi(C) = \tilde{C} \subset \mathbb{P}^2$ is a plane quartic curve with a node passing simply through $q_1, \ldots, q_4$ (see the proof of Proposition 5.4). The irreducible threefold B has ideal generated by one quadratic form and by three cubic forms so that it is scheme-theoretically defined by nine cubic forms. To study the scheme-theoretic intersection $B \cap S$ we restrict to S the linear system $|H^0(\mathcal{I}_B(3))|$. Each cubic in this linear system cuts S along a curve D containing C and such that $\pi(D) = \tilde{D}$ is a curve of degree nine having triple points at $q_1, \ldots, q_4$. Hence $\tilde{D} = \tilde{C} + \tilde{A}$ with $\tilde{A}$ a quintic curve with double points at $q_1, \ldots, q_4$. The linear system $|\tilde{A}|$ has dimension 8, it is base point free and very ample, proving that $B \cap S = C$ as schemes for $[C] \in \mathcal{F}$ general. The linear system $|\tilde{A}|$ is equivalent via a quadratic standard Cremona transformation centred at q_2, q_3, q_4 to the linear system of quartic curves having a double point at q_1 and simple base points at q_2, q_3, q_4.

By varying C and recalling that any C is a complete intersection of a pencil of quintic del Pezzo surfaces inside Σ, we get a family $\mathcal{D}$ of dimension 16 of quintic del Pezzo surfaces cutting B scheme-theoretically along a nodal curve of degree 8 and genus 2 as above. ■

5.3. *A rational surface of degree 9 and sectional genus 2 with five nodes contained in a general cubic fourfold of $\mathcal{C}_{42}$*

We can now construct a 25-dimensional family of smooth surfaces of degree 9 and sectional genus 2 on a del Pezzo fivefold $V \subset \mathbb{P}^8$. This was also achieved in [18, Lemma 3.1] via a different construction of an explicit example that corresponds to a smooth point in Hilb_V and which is related to a K3 surface of genus 11 in $\mathbb{P}^{11}$.

Proposition 5.6. *Let $V \subset \mathbb{P}^8$ be a del Pezzo fivefold.*

(i) *There exists an irreducible and generically smooth family $\mathcal{S} \subset \mathrm{Hilb}_V$ of dimension 25 of surfaces $S \subset V$ of degree 9 and sectional genus 2 whose general member is smooth.*

(ii) *There exists an irreducible and generically smooth family $\mathcal{S}_{42} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ of dimension 48 of surfaces in $\mathbb{P}^5$ of degree 9 and sectional genus 2 with five nodes whose general member is obtained as the projection from a general plane of the family $\mathcal{F}_1 \subset \mathrm{Hilb}_V$ of a general surface of the family $\mathcal{S} \subset \mathrm{Hilb}_V$.*

Proof. Let $C \subset V$ be an irreducible conic whose linear span $\Pi = \langle C \rangle$ is not contained in V, and let

$$\alpha : \mathbb{P}^5 \dashrightarrow V \subset \mathbb{P}^8 \tag{5.1}$$

be the inverse of the projection from Π (see Proposition 5.3). Let $\mathcal{D}$ be the family of quintic del Pezzo surfaces described in Proposition 5.5, and let $D \in \mathcal{D}$ be a general member. The smooth surfaces $S = \alpha(D) \subset V \subset \mathbb{P}^8$ have degree 9 and sectional genus 2 and are obtained from $\mathbb{P}^2$ via the linear system of quintics having four double points (or equivalently via the linear system of quartics with a double point and three simple base points) as shown in the proof of Proposition 5.5. Since the Hilbert scheme $\mathcal{C}on$ of conics in V has dimension 10 (see Remark 5.2) and since the conics on such a surface $S \subset \mathbb{P}^8$ belong to the unique pencil of conics on S (represented on the plane by the conics through the four base points of the linear system), we deduce that the above surfaces describe a family $\mathcal{S} \subset \mathrm{Hilb}_V$ of dimension at least $\dim(\mathcal{C}on) - 1 + \dim(\mathcal{D}) = 10 - 1 + 16 = 25$. Since we have verified in a specific example of surface $[S] \in \mathcal{S}$ that $h^0(N_{S/V}) = 25$, we deduce that the family $\mathcal{S}$ is generically smooth of dimension 25.

Let $\beta : V \dashrightarrow \mathbb{P}^5$ be the birational map induced by the projection from a plane $P \subset V$ of the family $\mathcal{F}_1$ (see Proposition 5.3). The image via β of a general $S \in \mathcal{S}$ is a surface of degree 9, sectional genus 2, cut out by nine cubics and having five nodes produced by the intersection of P with the secant variety to S, a fact which can be verified by a direct computation via *Macaulay2* by following the first steps of the algorithm described in Remark 5.8 below (see also Section 6.2).

A surface in $\mathbb{P}^5$ of degree 9, sectional genus 2 and with five nodes of the type constructed above will be denoted by S_{42}. Since the cubic rational normal scrolls in $\mathbb{P}^4$ depend on 18 parameters, we deduce that the cubic rational normal scrolls in $\mathbb{P}^5$ depend on 23 parameters. Each rational normal scroll in $\mathbb{P}^5$ determines a map β and hence a plane $P \subset V \subset \mathbb{P}^8$. The projection from P of the 25-dimensional family of surfaces of degree 9 and sectional genus 2 contained in V gives a 25-dimensional family of S_{42}. By varying the scroll we deduce that the surfaces $S_{42} \subset \mathbb{P}^5$ describe a family of dimension at least 48=23+25. Since we have verified in a specific example of surface $S_{42} \subset \mathbb{P}^5$ that $h^0(N_{S_{42}/\mathbb{P}^5}) = 48$, we can deduce that $h^0(N_{S_{42}/\mathbb{P}^5}) = 48$ for a general $S_{42} \subset \mathbb{P}^5$ as above. Since we previously proved that these surfaces depend on at least 48 parameters, we conclude that there exists a unique irreducible component $\mathcal{S}_{42}$ of the corresponding

Hilbert scheme containing the surfaces $S_{42} \subset \mathbb{P}^5$, which is generically smooth of dimension 48 and such that the general element of $\mathcal{S}_{42}$ is of the kind described above. ■

Remark 5.7. Let $V = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be a del Pezzo fivefold, let $\mathcal{F}_1 \subset \mathrm{Hilb}_V$ be the three-dimensional family of planes in V with class $\sigma_{2,2}$, and let $\mathcal{S} \subset \mathrm{Hilb}_V$ and $\mathcal{S}_{42} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ be, respectively, the 25-dimensional family of smooth surfaces in V of degree 9 and sectional genus 2, and the 48-dimensional family of 5-nodal surfaces in $\mathbb{P}^5$ of degree 9 and sectional genus 2 constructed in Proposition 5.6. To a general pair $([S],[P]) \in \mathcal{S} \times \mathcal{F}_1$, we associate a general $[S_{42}] \in \mathcal{S}_{42}$ defined by $S_{42} = \overline{\beta_P(S)}$, where $\beta_P : V \dashrightarrow \mathbb{P}^5$ denotes the projection from P. The inverse map $\alpha_\Delta = \beta_P^{-1} : \mathbb{P}^5 \dashrightarrow V$ is defined by the quadrics through a rational normal cubic scroll $\Delta \subset \mathbb{P}^4 \subset \mathbb{P}^5$, which intersects $S_{42} \subset \mathbb{P}^5$ in a curve $C \subset \mathbb{P}^5$ of degree 9 and arithmetic genus 7. We now illustrate how one can determine this scroll Δ from the surface S_{42}, and thus determine the pair (S, P) via $S = \overline{\alpha_\Delta(S_{42})}$ and $P = \mathrm{Bs}(\alpha_\Delta^{-1})$. Let $\varphi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^8$ be the rational map defined by the linear system $|H^0(\mathcal{I}_{S_{42}}(3))|$ of cubic hypersurfaces through the surface $S_{42} \subset \mathbb{P}^5$. Then φ is a birational map onto its image $Z \subset \mathbb{P}^8$, and the base locus of the inverse map $\varphi^{-1} : Z \dashrightarrow \mathbb{P}^8$ is an irreducible surface T with an immersed point $q \in T$. Then $\overline{\varphi^{-1}(T)}$ is a threefold ruled by trisecant lines to S_{42} while $\overline{\varphi^{-1}(q)}$ coincides with the scroll Δ intersecting S_{42} along a curve of degree 9 and arithmetic genus 7.[2] The previous description of φ ensures that a general $S_{42} \subset \mathbb{P}^5$ satisfies Assumption 1 in Section 2.2 and that it has the expected trisecant behaviour.

Remark 5.8. Our construction of the surface S_{42} can be easily implemented and executed in a computer program such as *Macaulay2*. For the convenience of the reader, we now summarize the algorithm for the construction of the general surface in the family $\mathcal{S}_{42}$ (see also Section 6.2).

- Let $p_1, \ldots, p_5 \in \mathbb{P}^2$ be general points. The image of the rational map defined by the linear system $|H^0(\mathcal{I}_{\{p_1^2, p_2, p_3, p_4, p_5\}}(4))|$ of plane quartic curves having a double point at p_1 and simple base points at p_2, p_3, p_4, p_5 is a smooth surface $T \subset \mathbb{P}^7$ of degree 8 and sectional genus 2. The surface T can be embedded into $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ via the product of linear systems $|H^0(\mathcal{I}_{\{p_1,p_3,p_4,p_5\}}(2))| \times |H^0(\mathcal{I}_{\{p_1,p_2\}}(2))|$.
- Let L be a general secant line to T, $H \supset L$ a general hyperplane through L, and $p \in L$ a general point. Then the projection of $C' = H \cap T \subset H = \mathbb{P}^6$ to $\mathbb{P}^5$ from p yields a one-nodal curve $C \subset \mathbb{P}^5$ of degree 8 and arithmetic genus 3 contained in a singular quartic scroll threefold $B \subset \mathbb{P}^5$, projection from p of $B' = \mathbb{P}^1 \times \mathbb{P}^3 \cap H$.
- The quadrics through C define a birational map ψ from $\mathbb{P}^5$ into a quartic hypersurface $W \subset \mathbb{P}^6$. The exceptional locus of ψ contains a Segre threefold $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, which is sent into a smooth quadric surface $Q \subset \mathbb{P}^6$. The quadric $Q \subset W$ can be

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[2] The idea of gluing a cubic scroll $\Delta \subset \mathbb{P}^5$ with another surface $B \subset \mathbb{P}^5$ along some curve $\Delta \cap B$ and then of considering the image $\overline{\alpha_\Delta(B)} \subset V$ has been systematically applied in [41] to construct other types of surfaces in a del Pezzo fivefold.

detected as the unique irreducible component of the base locus scheme of ψ^{-1} along which the base locus scheme is not generically reduced.

- The inverse images via ψ of the two lines of Q passing through a general point of Q are a plane of the ruling of Z and a smooth quintic del Pezzo surface $D \subset \mathbb{P}^5$, which intersects B along C. (Note that Σ and the quadric $Q = \overline{\psi(\Sigma)}$ are rational over their field of definition. Indeed, the syzygy matrix of the three quadrics defining a Segre threefold $\Sigma \subset \mathbb{P}^5_K$ is a 2×3 matrix of linearly independent linear forms on $\mathbb{P}^5_K$. These six linear forms can be used to define an automorphism of $\mathbb{P}^5_K$ that sends Σ into the Segre embedding of $\mathbb{P}^1_K \times \mathbb{P}^2_K$ in $\mathbb{P}^5_K$.)
- Finally, the map $\alpha : \mathbb{P}^5 \dashrightarrow V$ defined in (5.1) induces an isomorphism between D and a smooth surface S of degree 9 and sectional genus 2. The map $\beta : V \dashrightarrow \mathbb{P}^5$ defined in Proposition 5.1 induces a birational morphism from S to our surface S_{42}.

We are now ready to prove that a general cubic in $\mathcal{C}_{42}$ contains a general surface $S_{42} \subset \mathbb{P}^5$ of degree 9 and genus 2 with five nodes in the irreducible family $\mathcal{S}_{42}$ described above.

Theorem 5.9. *The irreducible divisor $\mathcal{C}_{42}$ parametrizing cubic fourfolds of discriminant* 42 *coincides with the closure of the locus of cubic fourfolds containing a rational surface S_{42} of degree* 9 *and sectional genus* 2 *with five nodes of the irreducible family $\mathcal{S}_{42}$ constructed above.*

Proof. Let h be the class of a hyperplane section of X, let h^2 be the class of 2-cycles $h \cdot h$ and remark that $h^2 \cdot h^2 = h^4 = 3$ and $h^2 \cdot S_{42} = 9$. The double point formula for $S_{42} \subset X$ (see [12, Theorem 9.3]) yields $S_{42}^2 = 41$ and shows that the restriction of the intersection form to $\langle h^2, S_{42}\rangle$ has discriminant $3 \cdot 41 - 81 = 42$. Let $\mathcal{V} \subset |H^0(\mathcal{O}_{\mathbb{P}^5}(3))| = \mathbb{P}^{55}$ be the open set corresponding to smooth cubic hypersurfaces. We verified that $h^0(\mathcal{I}_{S_{42}}(3)) = 9$ for a general $[S_{42}] \in \mathcal{S}_{42}$ and that there exists a smooth cubic hypersurface through S_{42}. Therefore the locus

$$\mathbf{C}_{42} = \{([S],[X]) : S \subset X\} \subset \mathcal{S}_{42} \times \mathcal{V}$$

has dimension $48 + 8 = 56$. The image of $\pi_2 : \mathbf{C}_{42} \to \mathcal{V}$ has dimension at most 54 because the general cubic fourfold does not contain any surface belonging to $\mathcal{S}_{42}$. For every $[X] \in \mathcal{Y}_2(\mathbf{C}_{42})$ we have

$$\dim(\pi_2^{-1}([X])) \geq \dim(\mathbf{C}_{42}) - \dim(\pi_2(\mathbf{C}_{42})) = 56 - \dim(\pi_2(\mathbf{C}_{42})) \geq 56 - 54 = 2.$$

Since $h^0(N_{S/X}) \geq \dim_{[S]}(\pi_2^{-1}([X]))$ for every $[S] \in \pi_2^{-1}([X])$, where $\dim_{[S]}(\pi_2^{-1}([X]))$ denotes the dimension of $\pi_2^{-1}([X])$ at the point $[S]$, to show that a general $[X] \in \mathcal{C}_{42}$ contains a surface S_{42} it is sufficient to verify that $h^0(N_{S_{42}/X}) = 2$ for a fixed S_{42} and for a smooth $X \in |H^0(\mathcal{I}_{S_{42}}(3))|$ (see also [29, pp. 284–285] for a similar argument). We verified this via *Macaulay2* in an explicit example and we can conclude that a general $[X] \in \mathcal{C}_{42}$ contains a surface S_{42} as above. ■

Since at each step of the algorithm summarized in Remark 5.8 we need to introduce only new independent variables, we deduce that the family $\mathcal{S}_{42}$ is unirational. In other words, our construction yields an explicit dominant rational map $\mathbb{P}^N \dashrightarrow \mathcal{S}_{42}$. As an immediate consequence of this and of Theorem 5.9, we have the following:

Corollary 5.10. *The irreducible divisor $\mathcal{C}_{42}$ is unirational.*

Remark 5.11. A different description of the divisor $\mathcal{C}_{42}$ was given in [24] as the locus of cubic fourfolds containing a rational scroll of degree 9 with eight nodes. From this Lai deduces that $\mathcal{C}_{42}$ is uniruled. This has been substantially refined in [9, Theorem 1.1], where the authors prove that the universal K3 surface of genus 22 is unirational. This implies the unirationality of the 19-dimensional moduli space $\mathcal{F}_{22}$ of polarized K3 surfaces of genus 22 and hence the unirationality of $\mathcal{C}_{42}$ by a result of Hassett (see for example [16, Corollary 25]).

5.4. *Rationality of cubics fourfolds in $\mathcal{C}_{42}$*

The new description of the general cubic fourfolds in $\mathcal{C}_{42}$ as those containing a general $[S_{42}] \in \mathcal{S}_{42}$ allows us to deduce the rationality of a general element of $\mathcal{C}_{42}$ and then, by applying [22, Theorem 1], of all cubics in $\mathcal{C}_{42}$. In the next subsection we shall also put in evidence, via the trisecant flop, the birational connection between a general pair (X, S_{42}) with $S_{42} \subset X$ and its associated K3 surface of degree 42 and of genus 22.

Theorem 5.12. *Every cubic fourfold in $\mathcal{C}_{42}$ is rational.*

Proof. Since we have given an algorithm for computing the general surface $S_{42} \subset \mathbb{P}^5$ of the family $\mathcal{S}_{42}$ (see Remark 5.8), we can explicitly construct it using *Macaulay2* and study its geometrical properties. So let $\varphi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^8$ be the rational map defined by the linear system $|H^0(\mathcal{I}_{S_{42}}(3))|$ of cubic hypersurfaces through a general surface $S_{42} \subset \mathbb{P}^5$ of the family $\mathcal{S}_{42}$. One can calculate that the map φ is birational onto its image $Z \subset \mathbb{P}^8$, which has degree 14, sectional genus 15 and ideal generated by seven cubic forms. Through a general point $q = \varphi(p)$ there pass 17 lines contained in Z, whose preimages via φ provide nine secant lines to S_{42} through p, seven 5-secant conics to S_{42} through p and one 8-secant twisted cubic to S_{42} through p. Hence S_{42} admits a congruence of 8-secant twisted cubics. Once we have determined the congruence we also have another way of detecting it. Indeed, the linear system $|H^0(\mathcal{I}^3_{S_{42}}(8))|$ of octic hypersurfaces with triple points along S_{42} defines a dominant rational map $\mu : \mathbb{P}^5 \dashrightarrow W \subset \mathbb{P}^7$ onto a smooth linear section W of $\mathbb{G}(1,4) \subset \mathbb{P}^9$. The general fibre of μ is a twisted cubic 8-secant to S_{42} so that the restriction of μ to a general cubic fourfold X through S_{42} induces a birational map $\mu_{|X} : X \dashrightarrow W$. Since a general cubic fourfold in $|H^0(\mathcal{I}_{S_{42}}(3))|$ through a general surface in $\mathcal{S}_{42}$ is rational, we conclude that every cubic fourfold of discriminant 42 is rational by the main result in [22]. We refer to Section 6.2 for more details on the above calculations. ∎

5.5. *Birational model of the associated K3 surface of degree 42 and genus 22 via the trisecant flop*

A general surface $S_{42} \subset \mathbb{P}^5$ satisfies Assumption 1 in Section 2.2 and it has the expected trisecant behaviour (see the end of Remark 5.7). By Theorem 2.6 the map μ restricted to a general cubic $X \subset \mathbb{P}^5$ through S_{42} determines a trisecant flop $\tau : X' = \mathrm{Bl}_{S_{42}} X \dashrightarrow W'$ with W' a smooth fourfold. By Theorems 5.12 and 2.10, the congruence of 8-secant twisted cubics to S_{42} induces a birational morphism $\nu : W' \to W$, which is the blow-up of a surface $\overline{B} \subset W \subset \mathbb{P}^7$.

By studying the birational map $\mu : X \dashrightarrow W$ (see Section 6.2), we find that $\overline{B} \subset W \subset \mathbb{P}^7$ is a smooth surface of degree 21 and sectional genus 18. Then $\overline{B}$ coincides with its support U, which is thus a smooth surface of degree 21 and sectional genus 18 with $p_g(U) = 1$ and $q(U) = 0$. We now apply the geometric method developed in Section 4 to deduce that U is the blow-up of a K3 surface at nine distinct points and to describe the linear system giving the embedding.

Consider two surfaces $U_j \subset W$, $j = 1, 2$, associated via μ to two general X_j's through S_{42}. The one-dimensional component C of $U_1 \cap U_2$ is a smooth curve of degree 13, of arithmetic genus -8 with nine irreducible components (a general projection of C to $\mathbb{P}^3$ allows a quick verification of all these properties), implying that $C = C_1 \cup \cdots \cup C_9$ is the disjoint union of nine smooth rational curves. More precisely, the curve $C \subset \mathbb{P}^7$ consists of five distinct lines, say $C_1, \ldots, C_5$, and of four conics $C_6, \ldots, C_9$ (see Section 6.2 and `code_section_6.m2`). Since $h^1(N_{C/U_j}) = 0$, the C_i's are (-1)-curves on U_j for every $i = 1, \ldots, 9$ and for $j = 1, 2$ by Lemma 4.3. Let $U = U_1$ and let $H \subset U$ be a hyperplane section. Since $H \cdot K_U = -H^2 + 2g(H) - 2 = -21 + 36 - 2 = 13 = H \cdot C$ we deduce from Lemma 4.3 the existence of the contraction $\pi : U \to U'$ of the C_i's to nine distinct points $p_1, \ldots, p_9$ on a smooth K3 surface U'. The divisor $H' = \pi_*(H) \subset U'$ is ample and such that $(H')^2 = 21 + 16 + 5 = 42$, $g(H') = 18 + 4 = 22$, proving that U' is a K3 surface of degree 42 and genus 22. For a general X through S_{42} we find that U' is a K3 surface of degree 42 and genus 22 with $\mathrm{rk}(\mathrm{Pic}(U')) = 1$ (one can argue as at the end of Section 3.1 or remark that U' necessarily has 19 moduli equal to the moduli of a general $X \in \mathcal{C}_{42}$) so that $|H'|$ gives an embedding $U' \subset \mathbb{P}^{22}$. The linear system $|H|$ on U corresponds to the hyperplane sections of U' passing through $p_1, \ldots, p_5$ and having a point of multiplicity at least 2 at $p_6, \ldots, p_9$. In particular, we see that these points do not impose independent conditions on hyperplane sections of $U' \subset \mathbb{P}^{22}$.

By considering the map associated to the linear system $|H + C_1 + \cdots + C_5 + 2(C_6 + \cdots + C_9)|$ on U, we construct its image $U' \subset \mathbb{P}^{22}$, which is thus a general K3 surface of degree 42 and genus 22 (see `code_section_6.m2`). From this one deduces an alternative proof of one of the main results in [9], according to which the moduli space of polarized K3 surfaces of degree 42 and genus 22 is unirational; see also [42].

6. Explicit examples of trisecant flops in *Macaulay2*

We mostly used the computer algebra system *Macaulay2* [13] with the packages *Cremona* [40] and *SpecialFanoFourfolds* [39] to study surfaces in $\mathbb{P}^5$ admitting congruences of $(3e-1)$-secant curves of degree e, the rational maps given by hypersurfaces of degree $3e-1$ having points of multiplicity e along these surfaces and also the lines contained in the images of $\mathbb{P}^5$ via the linear system of cubics through these surfaces. We refer to the documentations of these two packages for technical computational details. In particular, the first one provides tools for working with rational maps, such as determining the inverse of a birational map and computing fibres

The validity of these computations relies on the fact that the irreducible components $\mathcal{S}_d$ of the Hilbert schemes considered here are explicitly unirational. Therefore, by introducing a finite number of free parameters, one can explicitly construct the generic surface in $\mathcal{S}_d$ as a function of the specified parameters and adding more parameters one can also take the generic point of $\mathbb{P}^5$. Then one can, for instance, compute the generic fibre of the map defined by the cubics through the generic $[S_d] \in \mathcal{S}_d$, which will depend on all these parameters. In practice this is far beyond what computers can do today. Anyway, the answer we get is equivalent to the one obtained on the original field via a generic specialization of the parameters and, above all, the generic specialization commutes with this type of computation. So, using a common computer one can get an experimental proof that a certain property holds or not for the generic $[S_d]$. In the affirmative case, one can try to apply some semicontinuity arguments to get a rigorous proof.

6.1. *Rationality of cubic fourfolds in $\mathcal{C}_{38}$*

Here, we consider a specific example related to Section 3.1 (row (iii) of Table 1).

In the following code, we produce a surface $S = S_{38} \subset \mathbb{P}^5$ obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having ten randomly chosen triple points. We work over the finite field $K = \mathbb{F}_{10000019}$ for speed reasons.

```
Macaulay2, version 1.19
i1 : needsPackage "SpecialFanoFourfolds"; -- v2.5
i2 : K = ZZ/10000019;
i3 : S = surface({10,0,0,10},K);
o3 : ProjectiveVariety, surface in PP^5
```

We now compute the rational map μ defined by the linear system of quintic hypersurfaces of $\mathbb{P}^5$ which are singular along S.[3] From the information obtained by its projective degrees we deduce that μ is a dominant rational map onto $\mathbb{P}^4$ with generic fibre of dimension 1 and degree 2 and with base locus of dimension 3 and degree $5^2 - 19 = 6$.

---

[3] More generally, the command `rationalMap(S,d,e)` returns the rational map defined by a basis of the linear system $|H^0(\mathcal{I}_S^e(d))|$ of hypersurfaces of degree d having points of multiplicity e along S.

```
i4 : mu = rationalMap(S,5,2);
o4 : RationalMap (rational map from PP^5 to PP^4)
i5 : projectiveDegrees mu
o5 = {1, 5, 19, 13, 2, 0}
```

Next we compute a special random fibre F of the map μ.

```
i6 : p = point source mu;  -- a random point on P^5
i7 : F = mu^* mu p;
```

It is easy to verify directly that F is an irreducible 5-secant conic to S passing through p. Letting $\varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^9$ be the rational map defined by the linear system of cubics through S, one can also see that F coincides with the pull-back $\overline{\varphi^{-1}(L)}$ of the unique line $L \subset \overline{\varphi(\mathbb{P}^5)} \subset \mathbb{P}^9$ passing through $\varphi(p)$ that is not the image of a secant line to S passing through p (see [34, Section 5] for the details of this computation). Finally, the following lines of code tell us that the restriction μ' of μ to a randomly chosen cubic fourfold X containing S is a birational map whose inverse map is defined by forms of degree 9 and whose base locus scheme has dimension 2 and degree $9^2 - 27 = 54$.

```
i8 : X = random(3,S);
o8 : ProjectiveVariety, hypersurface in PP^5
i9 : mu' = mu|X;
o9 : RationalMap (rational map from X to PP^4)
i10 : projectiveDegrees mu'
o10 = {3, 15, 27, 9, 1}
```

The smooth surface $U \subset \mathbb{P}^4$ of degree 12 and sectional genus 14 determining the inverse map of $\mu' : X \dashrightarrow \mathbb{P}^4$ can be calculated as the non-reduced part of the base locus scheme of μ'^{-1}. Alternatively, one can use the method described in the next subsection. The full code to determine U, the exceptional curves on U, and the map $U \dashrightarrow U' \subset \mathbb{P}^{20}$ onto the K3 surface $U' \subset \mathbb{P}^{20}$ of degree 38 and genus 20 is included in `code_section_6.m2`.

6.2. *Rationality of cubic fourfolds in $\mathcal{C}_{42}$*

Here, we perform similar calculations as above, but considering a specific example related to Section 5.3 (row (0) of Table 1).

Using the algorithm given in Remark 5.8, one can calculate the homogeneous ideal of a randomly chosen surface $S = S_{42}$ in the 48-dimensional family $\mathcal{S}_{42}$ constructed in Proposition 5.6. This has been implemented in the *Macaulay2* package *SpecialFanoFourfolds*. So, to get the ideal of such a surface S and of a randomly chosen (smooth) cubic fourfold X containing it, it is enough to run the following code:

```
i11 : X = specialCubicFourfold("general cubic 4-fold of discriminant 42",K);
o11 : ProjectiveVariety, cubic fourfold containing a surface
      of degree 9 and sectional genus 2
i12 : S = ideal surface X;
```

The following is one of the ways to compute relatively quickly the rational map μ defined by the linear system of octic hypersurfaces of $\mathbb{P}^5$ having triple points along S. This calculation takes about one minute.

```
i13 : mu = rationalMap(S^3 : first gens ring S,8);
o13 : RationalMap (rational map from PP^5 to PP^7)
```

Now, with the same code, we compute a special random fibre F of the map μ.

```
i14 : p = point source mu;
i15 : F = mu^* mu p;
```

Here is a practical way to infer that F is a twisted cubic curve which is 8-secant to S.

```
i16 : ? F
o16 = smooth cubic curve of genus 0 in PP^5 cut out by 5 hypersurfaces
      of degrees (1,1,2,2,2)
i17 : ? (F + S)
o17 = 0-dimensional subscheme of degree 8 in PP^5
```

The code above also tells us that the (closure of the) image of μ is a subvariety $W \subset \mathbb{P}^7$ of dimension 4. To get the equations of W, one can use the command `image mu`, but this takes a while. A faster way is to calculate the scheme of quadrics containing W, as shown below. Then, since one verifies easily that it is a smooth connected fourfold, we deduce that W coincides with this fourfold.

```
i18 : W = image(2,mu);
```

The surface $U \subset W \subset \mathbb{P}^7$ determining the inverse map of the restriction of μ to the cubic fourfold X can be found without computing the inverse map. Indeed, the intersection of W with a general cubic hypersurface through U is given by the image $\overline{\mu(X \cap X')} \subset W$, where X' is a general cubic hypersurface through S. For efficiency, we suggest calculating $\overline{\mu(X \cap X')}$ by interpolating the images via μ of several points on $X \cap X'$. The following is a possible implementation. It gives the homogeneous ideal of $U \subset \mathbb{P}^7$ and takes about 5 minutes.

```
i19 : U = trim sum(8,j->(X' = random(3,surface X);
         ideal take((intersect apply(90,i->mu ideal point(X * X')))_*,6)));
```

The exceptional curves in the surface U can be determined by taking another randomly chosen cubic fourfold $\tilde{X}$ through S (for instance, as in line `i8`), and then by calculating the corresponding surface $\tilde{U}$ (for instance, by re-executing the line `i19`). One expects that the exceptional curves in $\tilde{U}$ are the same as those in U (see Section 4), so one tries to determine them as the top-dimensional components of the intersection $U \cap \tilde{U}$ (this can be done quickly after taking generic projections in $\mathbb{P}^3$ and $\mathbb{P}^2$). In the cases under consideration, the one-dimensional components of this intersection are nine disjoint curves: five lines $C_1, \ldots, C_5$ and four conics $C_6, \ldots, C_9$, which are all the (-1)-curves on U (see also Section 5.5). Having determined these curves, one can also calculate the map $f : U \to U' \subset \mathbb{P}^{22}$ onto the K3 surface $U' \subset \mathbb{P}^{22}$ of degree 42 and genus 22, given by the linear system $|H + \sum_{i=1}^{5} C_i + 2\sum_{i=1}^{4} C_{5+i}|$, where H denotes the hyperplane section of U. For the sake of brevity, the full code to get the map f and the equations of its image is omitted here and it is included in `code_section_6.m2` together with all the lines of code given in this section. Note also that this kind of calculations can be automated using the function `associatedK3surface` provided by the package *SpecialFanoFourfolds*.

	d	e	$S \subset X \subset \mathbb{P}^5$	W	$U \subset W$	Y
0	42	3	Rational surface of degree 9 and sectional genus 2 with 5 nodes, which is a special projection of the image of $\mathbb{P}^2$ in $\mathbb{P}^8$ via the linear system of quartic curves with 3 simple points and one double point	$\mathbb{G}(1,4) \cap \mathbb{P}^7 \subset \mathbb{P}^7$	Non-minimal K3 surface of degree 21 and sectional genus 18, cut out in $\mathbb{P}^7$ by 5 quadrics and 8 cubics	4-fold of degree 14 in $\mathbb{P}^7$ cut out by 7 cubics
i	14	2	Isomorphic projection of a smooth surface in $\mathbb{P}^6$ of degree 8 and sectional genus 3, obtained as the image of $\mathbb{P}^2$ via the linear system of quartic curves with 8 general base points	$\mathbb{P}^4$	Singular K3 surface of degree 10 and sectional genus 7, cut out by 12 quintics and having 8 singular points	4-fold of degree 28 in $\mathbb{P}^{11}$ cut out by 16 quadrics
ii	26	2	Rational scroll of degree 7 with 3 nodes	$\mathbb{P}^4$	Singular K3 surface of degree 10 and sectional genus 8, cut out by 12 quintics and one sextic, and having 3 singular points	4-fold of degree 29 in $\mathbb{P}^{11}$ cut out by 15 quadrics
iii	38	2	Smooth surface of degree 10 and sectional genus 6, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 10 with 10 general triple points	$\mathbb{P}^4$	Smooth non-minimal K3 surface of degree 12 and sectional genus 14 cut out by 9 quintics	4-fold of degree 20 in $\mathbb{P}^8$ cut out by 16 cubics
iv	26	2	Projection of a smooth del Pezzo surface of degree 7 in $\mathbb{P}^7$ from a line intersecting the secant variety in one general point	$\mathbb{G}(1,4) \cap \mathbb{P}^7 \subset \mathbb{P}^7$	Non-minimal K3 surface of degree 17 and sectional genus 11, cut out in $\mathbb{P}^7$ by 5 quadrics and 13 cubics	4-fold of degree 34 in $\mathbb{P}^{12}$ cut out by 20 quadrics
v	38	3	Rational scroll of degree 8 with 6 nodes	$\mathbb{G}(1,5) \cap \mathbb{P}^{10} \subset \mathbb{P}^{10}$	Smooth non-minimal K3 surface of degree 22 and sectional genus 14, cut out in $\mathbb{P}^{10}$ by 24 quadrics	4-fold of degree 17 in $\mathbb{P}^8$ cut out by 3 quadrics and 4 cubics
vi	14	3	Projection from 3 general internal points of a minimal K3 surface of degree 14 and sectional genus 8	Cubic fourfold	Projection from 3 general internal points of a minimal K3 surface of degree 14 and sectional genus 8	Complete intersection in $\mathbb{P}^7$ of 2 quadrics and one cubic
vii	14	3	Projection of a K3 surface of degree 10 and sectional genus 6 in $\mathbb{P}^6$ from a general point on its secant variety	Gushel–Mukai fourfold in $\mathbb{P}^8$	Smooth minimal K3 surface of degree 14 and sectional genus 8	Hypercubic section of a hyperplane section of $\mathbb{G}(1,4)$
viii	14	5	General hyperplane section of a conic bundle in $\mathbb{P}^6$ of degree 13 and sectional genus 12	Complete intersection of three quadrics in $\mathbb{P}^7$	Smooth non-minimal K3 surface of degree 13 and sectional genus 8, cut out by 9 quadrics	Hypersurface of degree 5 in $\mathbb{P}^5$
ix	14	5	General hyperplane section of a pfaffian threefold in $\mathbb{P}^6$ of degree 14 and sectional genus 15	$\mathbb{G}(1,6) \cap \mathbb{P}^{14} \subset \mathbb{P}^{14}$	Smooth minimal K3 surface of degree 14 and sectional genus 8 embedded in $\mathbb{P}^8 \subset \mathbb{P}^{14}$	Hypersurface of degree 5 in $\mathbb{P}^5$

	d	e	$S \subset X \subset \mathbb{P}^5$	W	$U \subset W$	Y
x	38	5	Smooth surface of degree 11 and sectional genus 7, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 12 with one general simple point, 4 general triple points, and 6 general quadruple points	$\mathbb{G}(1,5) \cap \mathbb{P}^{10} \subset \mathbb{P}^{10}$	Smooth non-minimal K3 surface of degree 25 and sectional genus 17, cut out in $\mathbb{P}^{10}$ by 21 quadrics	Hypersurface of degree 7 in $\mathbb{P}^5$
xi	38	3	Projection of an octic del Pezzo surface isomorphic to $\mathbb{F}_1$ from a plane intersecting the secant variety in 3 general points	$\mathbb{G}(1,3) \subset \mathbb{P}^5$	Non-minimal K3 surface of degree 13 and sectional genus 10, cut out in $\mathbb{P}^5$ by one quadric, 9 quartics, and 3 quintics	4-fold of degree 17 in $\mathbb{P}^8$ cut out by 3 quadrics and 4 cubics
xii	38	3	Projection of an octic del Pezzo surface isomorphic to $\mathbb{F}_0$ from a plane intersecting the secant variety in 3 general points (cut out by 10 cubics and one quartic)	$\mathrm{LG}_3(\mathbb{C}^6) \cap \mathbb{P}^{11} \subset \mathbb{P}^{11}$	Non-minimal K3 surface of degree 26 and sectional genus 17, cut out in $\mathbb{P}^{11}$ by 30 quadrics	4-fold of degree 18 in $\mathbb{P}^8$ cut out by 2 quadrics and 8 cubics
xiii	14	3	Isomorphic projection of a smooth surface in $\mathbb{P}^7$ of degree 8 and sectional genus 2, obtained as the image of $\mathbb{P}^2$ via the linear system of quartic curves with 4 simple base points and one double point (cut out by 10 cubics and 3 quartics)	Complete intersection of 2 quadrics in $\mathbb{P}^6$	Singular K3 surface of degree 14 and sectional genus 8, cut out in $\mathbb{P}^6$ by 2 quadrics and 9 cubics, and having one singular point	Complete intersection of 4 quadrics in $\mathbb{P}^8$
xiv	26	5	Rational scroll of degree 8 with 4 nodes (cut out by 8 cubics and 3 quartics)	$\mathbb{G}(1,3) \subset \mathbb{P}^5$	Non-minimal K3 surface of degree 14 and sectional genus 11, cut out in $\mathbb{P}^5$ by one quadric, 7 quartics, and 2 quintics	Complete intersection in $\mathbb{P}^6$ of a quadric and a quartic
xv	26	5	Surface of degree 13 and sectional genus 11 cut out by 6 cubics and with an ordinary node, which is obtained as a special projection of a minimal K3 surface of degree 26 of genus 14	Cubic fourfold	A surface of the same kind as S	–
xvi	26	6	Surface of degree 11 and sectional genus 6 cut out by 7 cubics and with 3 non-normal nodes, which is obtained as a special projection of a smooth surface of degree 11 and sec. genus 6 in $\mathbb{P}^6$	$\mathbb{S}^{10} \cap \mathbb{P}^9 \subset \mathbb{P}^9$, where $\mathbb{S}^{10} \subset \mathbb{P}^{15}$ is the spinorial variety	Non-minimal K3 surface of degree 21 and sectional genus 14, cut out in $\mathbb{P}^9$ by 16 quadrics and one cubic	Hypersurface of degree 5 in $\mathbb{P}^5$
xvii	26	5	Smooth surface of degree 11 and sectional genus 7, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 8 with 3 simple base points, 8 general double points, and 2 general triple points (cut out by 7 cubics and one quartic)	$\mathbb{G}(1,3) \subset \mathbb{P}^5$	Singular K3 surface of degree 15 and sectional genus 12, cut out in $\mathbb{P}^5$ by one quadric and 6 quartics, and having 9 singular points	Hypersurface of degree 6 in $\mathbb{P}^5$

Tab. 1. Examples of maps $\mu : X \dashrightarrow W$ as in diagram (0.1), where $[X] \in \mathcal{C}_d$ and $S \subset X$ admits a congruence of $(3e-1)$-secant rational curves of degree e.

6.3. Other worked out examples of trisecant flops

We provide a *Macaulay2* package named *TrisecantFlops*,[4] which produces explicit examples of trisecant flops in accordance to Table 1 in the next section. This package can be downloaded automatically from *SpecialFanoFourfolds*. So, by typing `trisecantFlop i` (where i is an integer between 0 and 17) will yield a birational map $\mu : X \dashrightarrow W$ as in the i-th row of Table 1. For instance, we now consider the third example:

```
i20 : mu = trisecantFlop 3;
o20 : RationalMap (birational map from cubic fourfold containing
      a surface of degree 10 and sectional genus 6 to PP^4)
i21 : projectiveDegrees inverse mu
o21 = {1, 9, 27, 15, 3}
```

We can obtain the smooth surface $S \subset \mathbb{P}^5$ of degree 10 and sectional genus 6 by giving the following command:

```
i22 : S = surface source mu;
o22 : ProjectiveVariety, surface in PP^5
i23 : (degree S, sectionalGenus S)
o23 = (10, 6)
```

Analogously, the non-minimal K3 surface $U \subset \mathbb{P}^4$ is obtained as follows:

```
i24 : U = surface target mu;
o24 : ProjectiveVariety, surface in PP^4
i25 : (degree U, sectionalGenus U)
o25 = (12, 14)
```

Finally, the following command yields an extension to $\mathbb{P}^5$ of the map $\mu : X \dashrightarrow W = \mathbb{P}^4$ whose general fibre is a 5-secant conic to the surface S:

```
i26 : extend mu;
o26 : RationalMap (dominant rational map from PP^5 to PP^4)
```

7. Summary table of examples of trisecant flops

We provide in Table 1 a list of 18 examples of maps $\mu : X \dashrightarrow W$ as in diagram (0.1), where X is a cubic fourfold in $\mathcal{C}_d$ which contains a surface $S \subset \mathbb{P}^5$ admitting a congruence of $(3e-1)$-secant rational curves of degree e. There are some different behaviours:

- $[X] \in \mathcal{C}_d$ is general except in (xi) and (xii);
- $S \subset \mathbb{P}^5$ is cut out by cubics except in (xii), (xiii), (xiv), and (xvii);
- the map $X \dashrightarrow Y$ defined by the cubics through S is birational except in (xv);
- the cubics through S satisfy condition $\mathcal{K}_3$ except in (0), (v), (viii), (x)–(xvii).

---

[4] It is available at https://github.com/giovannistagliano/TrisecantFlops.

	2-secant lines to S passing through p	5-secant conics to S passing through p	8-secant cubics to S passing through p	11-secant quartics to S passing through p	14-secant quintics to S passing through p	$h^0(\mathcal{I}_{S/\mathbb{P}^5}(3))$	$h^0(N_{S/\mathbb{P}^5})$	$h^0(N_{S/X})$
0	9	7	1	0	0	9	48	2
i	7	1	0	0	0	13	49	7
ii	7	1	0	0	0	13	44	2
iii	7	1	0	0	0	10	47	2
iv	5	1	0	0	0	14	42	1
v	9	4	1	0	0	10	47	2
vi	13	10	1	0	0	9	60	14
vii	11	6	1	0	0	10	59	14
viii	19	44	48	8	1	7	68	20
ix	21	56	42	0	1	7	77	29
x	11	16	16	4	1	7	49	1
xi	7	6	1	0	0	10	44	0
xii	7	4	1	0	0	10	44	0
xiii	9	6	1	0	0	10	49	4
xiv	11	12	16	8	1	8	49	2
xv	19	–	–	–	1	6	58	9
xvi	15	31	44	24	5	7	56	8
xvii	13	22	26	10	1	7	53	5

Tab. 2. Surfaces S contained in a cubic fourfold $X \subset \mathbb{P}^5$ as in Table 1; p is a general point of $\mathbb{P}^5$.

In Table 2, we give some additional information on the examples of surfaces $S \subset X \subset \mathbb{P}^5$ considered in Table 1. Most of this was achieved using the functions `detectCongruence` and `parameterCount` from the package *SpecialFanoFourfolds* (see also Section 6.3).

When the corresponding surface $U \subset W$ is smooth, the proof that it is a (non-minimal) K3 surfaces follows the paths used in Section 4 (see also Section 5.5) by determining explicitly the exceptional curves on U. Once these exceptional curves are determined, one finds the description of the linear system on U in terms of the hyperplane sections of the K3 surface U'. When $U \subset W$ is singular, one first determines the exceptional curves as above; then one takes a linear normalization to obtain a smooth surface and finally, if necessary, one follows the previous path.

Acknowledgments. This project started from an intuition of János Kollár that the maps $\mu : X \dashrightarrow W \subseteq \mathbb{P}^N$ might be useful to describe the congruences (see [35, Section 1.1], Section 1.1 here, and [21, Section 5, §29]). We are very indebted to him for the suggestion. We wish to thank Michele Bolognesi for his continuous support along the years and for many useful conversations on these topics. We are also very grateful to Sandro Verra for several discussions on the subject and to Brendan Hassett for his remarks on a preliminary version of the manuscript. We thank the referees for their careful reading and for their useful comments.

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