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Brown measures of free circular and multiplicative Brownian motions with self-adjoint and unitary initial conditions

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Abstract. Let Z_N be a Ginibre ensemble and let A_N be a Hermitian random matrix independent of Z_N such that A_N converges in distribution to a self-adjoint random variable x_0 in a W^* -probability space (\mathcal{A}, τ) . For each $t > 0$, the random matrix $A_N + \sqrt{t} Z_N$ converges in $*$ -distribution to $x_0 + c_t$, where c_t is a circular variable of variance t , freely independent of x_0 . We use the Hamilton–Jacobi method to compute the Brown measure ρ_t of $x_0 + c_t$. The Brown measure has a density that is *constant along the vertical direction* inside the support. The support of the Brown measure of $x_0 + c_t$ is related to the subordination function of the free additive convolution of $x_0 + s_t$, where s_t is a semicircular variable of variance t , freely independent of x_0 . Furthermore, the push-forward of ρ_t by a natural map is the law of $x_0 + s_t$.

Let $G_N(t)$ be the Brownian motion on the general linear group and let U_N be a unitary random matrix independent of $G_N(t)$ such that U_N converges in distribution to a unitary random variable u in (\mathcal{A}, τ) . The random matrix $U_N G_N(t)$ converges in $*$ -distribution to ub_t where b_t is the free multiplicative Brownian motion, freely independent of u . We compute the Brown measure μ_t of ub_t , extending the recent work by Driver–Hall–Kemp, which corresponds to the case $u = I$. The measure has a density of the special form

$$\frac{1}{r^2} w_t(\theta)$$

in polar coordinates in its support. The support of μ_t is related to the subordination function of the free multiplicative convolution of uu_t where u_t is the free unitary Brownian motion, freely independent of u . The push-forward of μ_t by a natural map is the law of uu_t .

In the special case that u is Haar unitary, the Brown measure μ_t follows the *annulus law*. The support of the Brown measure of ub_t is an annulus with inner radius $e^{-t/2}$ and outer radius $e^{t/2}$. In its support, the density in polar coordinates is given by

$$\frac{1}{2\pi t} \frac{1}{r^2}.$$

Keywords. Free probability, Brown measure, random matrices, free Brownian motion, circular law

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1. Introduction

It is a classical theorem by Wigner [43] that the eigenvalue distribution of a Gaussian unitary ensemble (GUE) G_N converges to the semicircle law. An operator $a \in \mathcal{A}$, where \mathcal{A} is a tracial von Neumann algebra, is said to be a limit in $*$ -distribution of a sequence of $N \times N$ self-adjoint random matrices A_N if, for any polynomial in two noncommuting variables,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E} \operatorname{Tr}[p(A_N, A_N^*)] = \tau[p(a, a^*)]. \quad (1.1)$$

In other words, the GUE has the limit in $*$ -distribution as an operator having the semicircle law as its spectral distribution. The operators in \mathcal{A} are called random variables.

Voiculescu [39] discovered that free probability can be used to study the large- N limit of eigenvalue distributions of $X_N + G_N$, where X_N is a sequence of self-adjoint random matrices independent of G_N , or a sequence of deterministic matrices that has a limit in distribution.

Biane proved that the limit in $*$ -distribution of the unitary Brownian motion $U_N(t)$ on the unitary group $U(N)$ is the free unitary Brownian motion in a tracial von Neumann algebra [9]. If V_N is a sequence of unitary random matrices independent of U_N or a sequence of deterministic unitary matrices that has a limit in $*$ -distribution, then free probability can also be used to study the limit of the eigenvalue distribution of $V_N U_N(t)$.

Given a self-adjoint random variable $a \in \mathcal{A}$, the spectral distribution, or the law, of a is a probability measure on \mathbb{R} defined to be the trace of the projection-valued spectral measure, whose existence is guaranteed by the spectral theorem. The law of a can be identified and computed by the Cauchy transform

$$G_a(z) = \tau((z - a)^{-1}), \quad z \in \mathbb{C}^+. \quad (1.2)$$

The spectral distribution of a unitary operator in \mathcal{A} is a probability measure on the unit circle \mathbb{T} . When we consider nonnormal random variables, the spectral theorem is no longer valid. Instead, we look at the Brown measure [16], which has been called the spectral distribution measure of a not-necessarily-normal random variable. The Brown measure of free random variables provides a natural candidate for the limit of the eigenvalue distribution of nonnormal random matrices.

In this article, we calculate the density formulas for the Brown measure of the free circular Brownian motion with self-adjoint initial condition, as well as the free multiplicative Brownian motion with unitary initial condition. The latter extends the recent work of Driver, Hall and Kemp [20] concerning the case of the free multiplicative Brownian motion starting at the identity operator. (See also the expository paper [27], which provides a gentle introduction to the PDE methods used in this paper.) Our result indicates that the Brown measure of the free *circular* Brownian motion with self-adjoint initial condition is closely related to the free *semicircular* Brownian motion with the same self-adjoint initial condition. Similarly, the free *multiplicative* Brownian motion with unitary initial condition is closely related to the free *unitary* Brownian motion with the same unitary initial condition.

After the first version of this paper was posted on arXiv, there appeared subsequent works using a similar strategy to compute the Brown measure of free Brownian motions with nontrivial initial conditions. Demni and Hamdi [19] analyze the free unitary Brownian motion with projection initial condition. Hall and the first author compute the Brown measure of the imaginary multiple of free semicircular Brownian motion with bounded self-adjoint initial condition in [28] and the Brown measure of the three-parameter free multiplicative Brownian motion with unitary initial condition in [29]. The second author [46] extends the main results for additive Brownian motions to all elliptic operators using a different approach based on free probability techniques.

1.1. Additive case

One of the fundamental nonnormal random matrix models is the *Ginibre ensemble* Z_N , which is a sequence of $N \times N$ random matrices with i.i.d. complex Gaussian entries, with variance $1/N$. The limiting empirical eigenvalue distribution, which is a normalized counting measure $\frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j}$ of the eigenvalues $\{\lambda_j\}$ of Z_N , converges to the uniform probability measure on the unit disk [23]. The reader is referred to [15] for a survey on the circular law. The limit random variable, in the sense of $*$ -distribution, is called the circular operator [38]. If we consider the *process* $Z_N(t)$ of random matrices with i.i.d. entries of complex Brownian motion at time t/N , the limiting empirical eigenvalue distribution

at time t is the uniform probability measure on the disk of radius \sqrt{t} . The limit in $*$ -distribution of $Z_N(t)$ is a “free stochastic process”, a one-parameter family of random variables in \mathcal{A} , which is called the *free circular Brownian motion* c_t . At each $t > 0$, $Z_N(t)$ has the same distribution as $\sqrt{t} Z_N$, and c_t has the same distribution as $\sqrt{t} c_1$.

The standard free circular Brownian motion starts with the condition $c_0 = 0$. We consider a more general free stochastic process: one that starts at an arbitrary random variable $x_0 \in \mathcal{A}$ and has the same increments as the standard free circular Brownian motion. Such a process has the form $x_0 + c_t$ where x_0 and c_t are freely independent.

The circular operator c_t is an R -diagonal operator [36, Lecture 15]. Given a variable $x_0 \in \mathcal{A}$ which may not be normal and $r \in \mathcal{A}$ which is R -diagonal and freely independent of x_0 , Biane and Lehner [13, Section 3] studied the Brown measure of $x_0 + r$. They obtained an explicit density formula for particular x_0 and r . In Section 5 of the same paper, they studied, more specifically, the Brown measure of $x_0 + c_t$ where $x_0 \in \mathcal{A}$ may not be normal; the density at $\lambda \notin \sigma(x_0)$ is given by

$$\frac{1}{\pi} \partial_{\bar{\lambda}} \left(\int_{t_\lambda}^t \frac{\partial_\lambda v_s(\lambda)}{s^2} ds - \frac{v_{t_\lambda}(\lambda)^2}{t_\lambda^2} \partial_\lambda t_\lambda \right)$$

where

$$v_s(\lambda) = \inf \left\{ v \geq 0 : \int_{\mathbb{R}} \frac{d\mu_{|\lambda-x_0|}(x)}{x^2 + v^2} \leq \frac{1}{s} \right\}$$

and $t_\lambda = \inf \{t : v_t(\lambda) > 0\}$. It is mentioned in their paper that $v_s(\lambda)$ and t_λ are related to the subordination function of $\tilde{x}_0 + s_t$ with respect to \tilde{x}_0 , where \tilde{x}_0 is the symmetrization of $|\lambda - x_0|$, and s_t is the free semicircular Brownian motion, the real part of $\sqrt{2} c_t$. The quantities are not very explicit; there is an integration in the time variable. Furthermore, the formula is only valid outside the spectrum of x_0 .

Our method shows that the density formula for the special case when x_0 is self-adjoint can be computed more explicitly. Our results also illustrate unexpected connections between the Brown measure of $x_0 + c_t$ and the subordination function for the free convolution of x_0 with s_t . In the rest of the paper we suppose x_0 is self-adjoint. Denote by μ the spectral distribution of x_0 . In this case, the following function v_t defined on \mathbb{R} is fundamental in our analysis:

$$v_t(a) = \inf \left\{ b \geq 0 : \int_{\mathbb{R}} \frac{d\mu(x)}{(x-a)^2 + b^2} \leq \frac{1}{t} \right\}. \tag{1.3}$$

This definition of v_t also coincides with the definition of Biane and Lehner; the measure $\mu_{|\lambda-x_0|}$ is the push-forward of μ_{x_0} by the map $x \mapsto |\lambda - x|$, for each λ , because x_0 is self-adjoint.

Our main result in Section 3 gives an explicit description of the Brown measure of $x_0 + c_t$, using the Hamilton–Jacobi method used in the recent paper [20]. In this paper, the Brown measure of $x_0 + c_t$ is computed as an absolutely continuous measure on \mathbb{R}^2 . The density is computed explicitly; the density is not given as an integral over the time parameter.

The operator $x_0 + c_t$ is the limit in $*$ -distribution (in the sense of (1.1)) of the random matrix model $X_N + Z_N(t)$, by [39, Theorem 2.2], where X_N is an $N \times N$ self-adjoint

deterministic matrix or random matrix classically independent of Z_N , with x_0 in the limit of X_N in $*$ -distribution. On the level of Brown measure convergence, Śniady [37] showed that the empirical eigenvalue distribution of $X_N + Z_N(t)$ almost surely converges weakly to the Brown measure of $x_0 + c_t$, without computing the Brown measure of $x_0 + c_t$ explicitly. (Apply [37, Theorem 6] by taking the not-necessarily-normal random matrix $A^{(N)}$ as the X_N here.) Consequently, the result in this paper gives a formula for the weak limit of the empirical eigenvalue distribution of $X_N + Z_N(t)$.

In physics literature, Burda et al. [17, 18] studied the limiting eigenvalue distribution of the sum of the Ginibre ensemble Z_N and a deterministic matrix A_N (which is not assumed to be normal) using PDE methods. The formal large- N limit of the PDE obtained in their work [18, (31)] is the same as the PDE we obtain for the additive case. They transformed the PDE into one that can be solved using the method of characteristics, whereas we use the Hamilton–Jacobi method to solve the PDE in this paper. In [18, Section 6.2], they computed explicitly the limiting eigenvalue distribution of $A_N + Z_N$ when the eigenvalue distribution of A_N is the Bernoulli distribution. It can be checked that the domain where the limiting eigenvalue distribution of $A_N + Z_N$ is nonzero agrees with our result. The density of the limiting eigenvalue distribution computed in their paper also agrees with the Brown measure density computed in our paper, after correcting minor algebraic errors. We note, however, that [17, 18] did not compute the limiting eigenvalue distribution of $A_N + Z_N$ for a general self-adjoint matrix A_N , which is the main purpose of Section 3 of our paper.

Let $G_{x_0}(z)$ be the Cauchy transform of x_0 as defined in (1.2) and let

$$H_t(z) = z + tG_{x_0}(z), \quad z \in \mathbb{C} \setminus \sigma(x_0).$$

It is known [10] that H_t maps the curves $\{(a, \pm v_t(a)) : a \in \mathbb{R}\}$, which are analytic at the points where $v_t(a) > 0$, to \mathbb{R} . The restriction of H_t to the set $\{a + ib \in \mathbb{C}^+ : b > v_t(a)\}$ is the inverse of an analytic self-map $F_t : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ on the upper half-plane such that

$$G_{x_0+s_t}(z) = G_{x_0}(F_t(z))$$

where s_t is a semicircular variable with variance t [10]. The map F_t is the *subordination function with respect to x_0* and allows one to compute $G_{x_0+s_t}$ from G_{x_0} . The following theorem summarizes the results proved in Theorems 3.13, 3.14 and Proposition 3.16.

Theorem 1.1. *The Brown measure ρ_t of $x_0 + c_t$ has the following properties.*

- (1) *The support of ρ_t is the closure of an open set. More precisely, $\text{supp } \rho_t = \overline{\Lambda_t}$ where*

$$\Lambda_t = \{a + ib \in \mathbb{C} : |b| < v_t(a)\},$$

where v_t is defined in (1.3).

- (2) *The measure ρ_t is absolutely continuous and its density is constant in the vertical direction inside the support. That is, the density w_t has the property $w_t(a + ib) = w_t(a)$ in the support of ρ_t . Moreover,*

$$w_t(a) \leq \frac{1}{\pi t},$$

and the inequality is strict unless x_0 is a scalar.

- (3) Define $\Psi_t(a + ib) = H_t(a + iv_t(a))$ on $\text{supp } \rho_t$ agreeing with H_t on the boundary of $\text{supp } \rho_t$ and constant along the vertical segments (the image of Ψ only depends on a). Then the push-forward of ρ_t by Ψ is the law of $x_0 + s_t$.
- (4) The preceding properties uniquely define ρ_t . That is, ρ_t is the unique measure whose support agrees with $\text{supp } \rho_t$, the density is constant along the vertical segments and the push-forward under Ψ is the distribution of $x_0 + s_t$. This is because Ψ restricted to \mathbb{R} is a homeomorphism [10] and the density is constant along the vertical direction.

The support of the Brown measure is symmetric about the real line. Figure 1 shows a random matrix simulation to the Brown measure of $x_0 + c_t$ where x_0 has law $\frac{1}{4}\delta_{-0.8} + \frac{3}{4}\delta_{0.8}$ and $t = 1$.

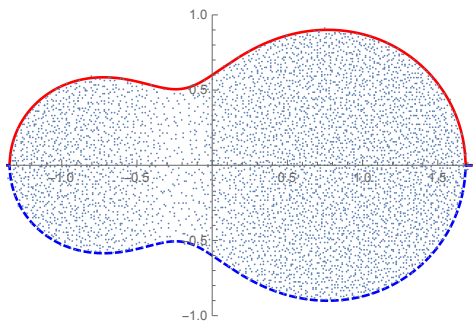


Fig. 1. Matrix simulation of eigenvalues for $x_0 + c_1$, where x_0 has distribution $\frac{1}{4}\delta_{-0.8} + \frac{3}{4}\delta_{0.8}$. The graphs of $v_t(a)$ (blue dashed) and $-v_t(a)$ (red) are superimposed.

1.2. Multiplicative case

The process $Z_N(t)$ can be viewed as a Brownian motion on the Lie algebra $\mathfrak{gl}(N, \mathbb{C})$ of the general linear group $GL(N, \mathbb{C})$, under the Hilbert–Schmidt inner product such that the real and imaginary parts are orthogonal. The Brownian motion on $GL(N, \mathbb{C})$ is defined to be the solution of the matrix-valued stochastic differential equation

$$dG_N(t) = G_N(t) dZ_N(t), \quad G_N(0) = I_N.$$

Kemp [32] proved that the limit of $G_N(t)$ in $*$ -distribution is the free stochastic process b_t that can be obtained by solving the free stochastic differential equation (see, for example, [14, 33])

$$db_t = b_t dc_t, \quad b_0 = I.$$

The process b_t , starting at the identity I , is called the *free multiplicative Brownian motion*. Hall and Kemp [30] showed that the Brown measure of b_t is supported in a certain compact set in \mathbb{C} . Later the Brown measure of b_t was computed by Driver, Hall and Kemp [20].

In this paper, we consider the free multiplicative Brownian motion starting at a unitary random variable u ; such a process has the form ub_t , where u is freely independent of b_t .

Before we state the results on the Brown measure of ub_t , we need to introduce the free unitary Brownian motion u_t , the multiplicative analogue of the semicircular Brownian motion. The free unitary Brownian motion is the solution of the free stochastic differential equation

$$du_t = iu_t ds_t - \frac{1}{2}u_t dt, \quad u_0 = I.$$

The law ν_t of u_t was computed by Biane [9]. The law of the free unitary Brownian motion with an arbitrary unitary initial condition was computed by the second author [45].

Suppose that $a \in \mathcal{A}$ is a unitary random variable with spectral distribution, or law, μ . Then μ can be identified by the moment generating function

$$\psi_a(z) = \int_{\mathbb{T}} \frac{\xi z}{1 - \xi z} d\mu(\xi), \quad z \in \mathbb{D}.$$

Now, we consider the random variable ub_t where u is any unitary random variable freely independent of b_t . Let $\bar{\mu}$ be the spectral distribution of u^* and set

$$\Phi_t(z) = z \exp\left(\frac{t}{2} \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\bar{\mu}(\xi)\right), \quad z \in \mathbb{C} \setminus \sigma(u).$$

Consider the function $r_t : (-\pi, \pi] \rightarrow (0, 1]$ defined by

$$r_t(\theta) = \sup \left\{ 0 < r < 1 : \frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) < \frac{1}{t} \right\}. \quad (1.4)$$

Because $\lim_{r \rightarrow 0} \frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) = 0$, it follows that the set over which the supremum is taken is nonempty. It is known [45] that the map Φ_t maps the curve $\{r_t(\theta)e^{i\theta} : \theta \in (-\pi, \pi]\}$, which is analytic whenever $r_t(\theta) < 1$, to the unit circle. The restriction of Φ_t to the set $\{z \in \mathbb{D} : |z| < r_t(\theta)\}$ is the inverse of the analytic self-map $\omega_t : \mathbb{D} \rightarrow \mathbb{D}$ such that

$$\psi_{u^*u_t}(z) = \psi_{u^*}(\omega_t(z)).$$

Analogous to the additive case, the map ω_t is the *subordination function with respect to u^** and is of significance in the study of free probability. The regularity results of r_t are summarized in Proposition 2.9.

We establish the following results in Theorem 4.28 and Proposition 4.31.

Theorem 1.2. *The Brown measure μ_t of ub_t satisfies the following properties.*

(1) *The support of μ_t is the closure of an open set. More precisely, $\text{supp } \mu_t = \overline{\Delta_t}$, where*

$$\Delta_t = \{r e^{i\theta} : r_t(\theta) < r < 1/r_t(\theta)\}$$

where r_t is defined in (1.4).

(2) *The measure μ_t is absolutely continuous and, inside the support, its density in polar coordinates has the form*

$$W_t(r, \theta) = \frac{1}{r^2} w_t(\theta)$$

for some function w_t depending on the argument only. That is, the density of μ_t is inversely proportional to r^2 along the radial direction, and the proportionality constant depends only on the argument θ . The Brown measure μ_t is invariant under $z \mapsto 1/\bar{z}$. Moreover,

$$w_t(\theta) < \frac{1}{\pi t}.$$

- (3) For each $\lambda \in \text{supp } \mu_t$, there exists a unique $\lambda_\theta \in \overline{\mathbb{D}} \cap \partial(\text{supp } \mu_t)$ such that λ_θ and λ have the same argument mod 2π . Define $\Gamma_t(\lambda) = \Phi_t(\lambda_\theta)$ agreeing with Φ_t on $\overline{\mathbb{D}} \cap \partial(\text{supp } \mu_t)$, constant along the radial segments. Then the push-forward of μ_t by Γ_t is the distribution of uu_t .
- (4) The preceding properties uniquely define μ_t . That is, μ_t is the unique measure whose support agrees with $\text{supp } \mu_t$, the density has the form $(1/r^2)\phi(\theta)$ in polar coordinates and the push-forward under Γ_t is the distribution of uu_t . This is because Γ_t restricted to $\overline{\mathbb{D}} \cap \partial(\text{supp } \mu_t)$ is a homeomorphism [45] and the density of μ_t is inversely proportional to r^2 along the radial direction.

Denote a Haar unitary random variable by h ; which means h is a unitary operator whose spectral distribution is the Haar measure on the unit circle. In the special case when $u = h$, the Brown measure of hb_t is the *annulus law*. It is absolutely continuous, supported in the annulus $A_t = \{e^{-t/2} \leq |z| \leq e^{t/2}\}$, rotationally invariant, and the density $W_t(r, \theta)$ is given by

$$W_t(r, \theta) = \frac{1}{2\pi t} \frac{1}{r^2}$$

on A_t . The density W_t is independent of θ because the Brown measure is rotationally invariant. We can also calculate the Brown measure of hb_t using Haagerup–Larsen’s formula for the Brown measures of R -diagonal operators [24]. Indeed, the random variable hb_t is R -diagonal [36, Proposition 15.8] and the Brown measure of hb_t can be calculated from the distribution of $(b_t^*b_t)^{1/2}$ (see Appendix for details).

Figure 2 shows a random matrix simulation of the eigenvalue distribution of $ub_{0.8}$ where u has distribution $\frac{1}{3}\delta_{e^{i\pi/3}} + \frac{2}{3}\delta_{e^{4\pi i/5}}$. It should be noted that, in this multiplicative case, it is an open problem to give a mathematical proof that even when the initial condition is the identity, the empirical eigenvalue distribution of $G_N(t)$ converges to the Brown measure of b_t .

The paper is organized as follows. Section 2 consists of some background and preliminaries of free probability theory and the definition of the Brown measure. The distributions of the sum of two self-adjoint free random variables and the product of two unitary free random variables will be described using the subordination functions. In Section 3, we compute the Brown measure of the random variable $x_0 + c_t$. The Brown measure is closely related to the subordination function and the distribution of $x_0 + s_t$. Section 4 is concerned with the Brown measure of the random variable ub_t , for both cases when u is and is not a Haar unitary, using the same Hamilton–Jacobi analysis but different initial conditions from [20]. The support and the density of the Brown measure are again related

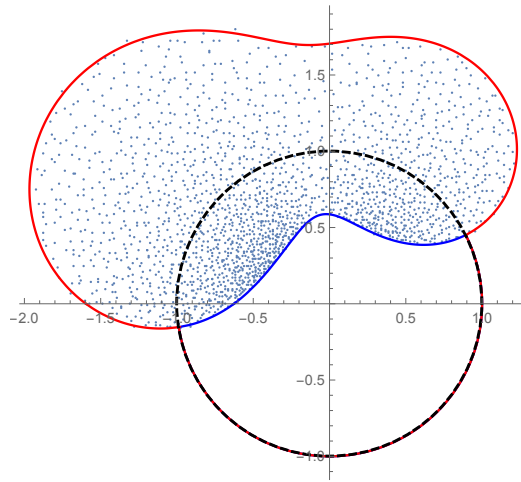


Fig. 2. Matrix simulations of eigenvalues for $ub_{0,8}$, where u has distribution $\frac{1}{3}\delta_{e^{i\pi/3}} + \frac{2}{3}\delta_{e^{4\pi i/5}}$. The curves $r_t(\theta)e^{i\theta}$ (blue), $\frac{1}{r_t(\theta)}e^{i\theta}$ (red), and the unit circle (black dashed) are superimposed.

to a subordination function; but the subordination function is the one for u^*u_t rather than uu_t . The Brown measure of ub_t is connected to the distribution of uu_t by a natural map.

2. Preliminaries

2.1. Free probability

A W^* -probability space is a pair (\mathcal{A}, τ) where \mathcal{A} is a finite von Neumann algebra and τ is a normal, faithful tracial state on \mathcal{A} . The elements in \mathcal{A} are called (noncommutative) random variables.

Unital $*$ -subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n \subseteq \mathcal{A}$ are said to be free or *freely independent* in the sense of Voiculescu if, given any $i_1, \dots, i_m \in \{1, \dots, n\}$ with $i_k \neq i_{k+1}$, and $a_{i_j} \in \mathcal{A}_{i_j}$ satisfying $\tau(a_{i_k}) = 0$ for all $1 \leq k \leq m$, we have $\tau(a_{i_1} \cdots a_{i_m}) = 0$. The random variables a_1, \dots, a_m are free or freely independent if the unital $*$ -subalgebras generated by them are free.

For any self-adjoint (resp. unitary) element $a \in \mathcal{A}$, the law or the *distribution* μ of a is a probability measure on \mathbb{R} (resp. \mathbb{T}) such that whenever f is a bounded continuous function on \mathbb{R} (resp. \mathbb{T}), we have

$$\int f d\mu = \tau(f(a)).$$

For a measure μ on the real line, the *Cauchy transform* of μ is given by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x) = \tau((z-a)^{-1}), \quad z \in \mathbb{C}^+.$$

The Cauchy transform G_μ maps the upper half-plane \mathbb{C}^+ into the lower half-plane \mathbb{C}^- . It satisfies the asymptotic property $\lim_{y \uparrow +\infty} iy G_\mu(iy) = \mu(\mathbb{R})$. The reader is referred to [1] for results about the Cauchy transform. The measure μ can be recovered from its Cauchy transform G_μ using the Stieltjes inversion formula, which expresses μ as a weak limit:

$$d\mu(x) = \lim_{y \downarrow 0} -\frac{1}{\pi} \text{Im } G_\mu(x + iy) dx. \tag{2.1}$$

The R -transform of μ is defined by

$$\mathcal{R}_\mu(z) = G_\mu^{(-1)}(z) - \frac{1}{z} \tag{2.2}$$

where $G_\mu^{(-1)}$ means the inverse function to G_μ in a truncated Stolz angle $\{z \in \mathbb{C} : \text{Im } z > \beta, |\text{Re } z| < \alpha \text{Im } z\}$ for some $\alpha, \beta > 0$.

For a measure μ on the unit circle \mathbb{T} , we consider the moment generating function on the open unit disk \mathbb{D} :

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{\xi z}{1 - \xi z} d\mu(\xi) \quad z \in \mathbb{D}.$$

The η -transform of μ is defined as

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}.$$

Then the measure μ can be recovered using the Herglotz representation theorem, as a weak limit

$$d\mu(e^{-i\theta}) = \lim_{r \uparrow 1} \frac{1}{2\pi} \text{Re} \left(\frac{1 + \eta_\mu(re^{i\theta})}{1 - \eta_\mu(re^{i\theta})} \right) d\theta. \tag{2.3}$$

When $\eta'_\mu(0) \neq 0$ (which is equivalent to the condition that μ has nonzero first moment), the Σ -transform and S -transform of μ are defined

$$\Sigma_\mu(z) = \frac{\eta_\mu^{(-1)}(z)}{z} \quad \text{and} \quad S_\mu(z) = \Sigma_\mu \left(\frac{z}{1+z} \right), \tag{2.4}$$

where these functions are defined in a neighborhood of zero.

2.2. Free Brownian motions

In free probability, the semicircle law plays a similar role to the Gaussian distribution in classical probability. The semicircle law σ_t with variance t is compactly supported in the interval $[-2\sqrt{t}, 2\sqrt{t}]$ with density

$$d\sigma_t(x) = \frac{1}{2\pi t} \sqrt{4t - x^2}.$$

Definition 2.1. (1) A free semicircular Brownian motion s_t in a W^* -probability space (\mathcal{A}, τ) is a weakly continuous free stochastic process $(s_t)_{t \geq 0}$ with free and stationary semicircular increments.

- (2) A free circular Brownian motion c_t has the form $\frac{1}{\sqrt{2}}(s_t + is'_t)$ where s_t and s'_t are two freely independent free semicircular Brownian motions.

In the unitary group $U(N)$, we can consider a Brownian motion X_t on the Lie algebra $\mathfrak{gl}(N)$, after fixing an Ad-invariant inner product. Taking the exponential map gives us a unitary Brownian motion. More precisely, the unitary Brownian motion $U_t = U_t(N)$ can be obtained by solving the Itô differential equation

$$dU_t = iU_t dX_t - \frac{1}{2}U_t dt, \quad U_0 = I.$$

Definition 2.2. In free probability, the free unitary Brownian motion can be obtained by solving the free Itô differential equation

$$du_t = iu_t ds_t - \frac{1}{2}u_t dt, \quad u_0 = I, \quad (2.5)$$

where s_t is a free semicircular Brownian motion. The free multiplicative Brownian motion b_t is the solution of the free Itô stochastic differential equation

$$db_t = b_t dc_t, \quad b_0 = I. \quad (2.6)$$

We note that the right increments of the free unitary Brownian motion u_t are free. In other words, for every $0 < t_1 < t_2 < \dots < t_n$ in \mathbb{R} , the elements

$$u_{t_1}, u_{t_1}^{-1}u_{t_2}, \dots, u_{t_{n-1}}^{-1}u_{t_n}$$

form a free family. Similarly, one can show that the process b_t has free right increments. These stochastic processes were introduced by Biane [9]. He proved that the large- N limit in $*$ -distribution of the unitary Brownian motion $U_t = U_t(N)$ is the free unitary Brownian motion, and conjectured that the large- N limit of the Brownian motion on $\mathrm{GL}(N, \mathbb{C})$ is the free multiplicative Brownian motion b_t . Kemp [32] proved that b_t is the limit in $*$ -distribution of the Brownian motion on $\mathrm{GL}(N, \mathbb{C})$.

The connection between the Brownian motions on the Lie groups $U(N)$ and $\mathrm{GL}(N, \mathbb{C})$ is natural. The heat kernel function on $\mathrm{GL}(N, \mathbb{C})$ is the analytic continuation of that on $U(N)$ (see [26, 34] for instance). Consider now the free unitary Brownian motion with initial condition uu_t where u is a unitary random variable freely independent of $(u_t)_{t>0}$. The process uu_t is the solution of the free stochastic differential equation in (2.5) with initial condition u . Similarly, the solution g_t of the free stochastic differential equation

$$dg_t = g_t dc_t, \quad g_0 = u, \quad (2.7)$$

has the form $g_t = ub_t$.

2.3. Free additive convolution

Our main results show that the Brown measure of the free circular Brownian motion with self-adjoint initial condition has direct connections with the spectral distribution of

the free additive Brownian motion with the same self-adjoint initial condition and some related analytic functions. In this section, we review some relevant facts about the free additive convolution to set up notation.

Suppose that self-adjoint random variables $x, y \in \mathcal{A}$ are freely independent. It is known that the distribution of $x + y$ is determined by the distributions of x and y . The free additive convolution of x and y is then defined to be the distribution of $x + y$. The subordination relation in free convolution was first established by Voiculescu [40] for free additive convolutions under some generic conditions, and was further extended by Biane [12] to free multiplicative convolutions and again by Voiculescu [41] to a very general setting (see also [5]). There exists a unique pair of analytic maps $\omega_1, \omega_2 : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ such that

- (1) $\text{Im } \omega_j(z) \geq \text{Im } z$ for all $z \in \mathbb{C}^+, j = 1, 2$;
- (2) $G_x(\omega_1(z)) = G_y(\omega_2(z)) = (\omega_1(z) + \omega_2(z) - z)^{-1}$ for all $z \in \mathbb{C}^+$;
- (3) $G_{x+y}(z) = G_x(\omega_1(z)) = G_y(\omega_2(z))$ for all $z \in \mathbb{C}^+$.

Point (3) tells us if we could compute (one of) the subordination functions ω_1 and ω_2 , we could compute the Cauchy transform of $x + y$ in terms of the Cauchy transform of x or y . The Cauchy transform of $x + y$ then determines the law of $x + y$ by (2.1). Although the subordination functions, in general, cannot be computed explicitly, a lot of regularity results can be deduced from the subordination relation (see [3, 8] or the survey [42, Chapter 6]). Denote by μ and ν the spectral distributions of x and y respectively. The free additive convolution of μ and ν is defined to be the spectral distribution of $x + y$ and is denoted as $\mu \boxplus \nu$. The \mathcal{R} -transform (2.2) linearizes the free additive convolution in the sense that $\mathcal{R}_{\mu \boxplus \nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z)$ in the domain where all the three \mathcal{R} -transforms are defined.

In the special case when x is an arbitrary self-adjoint variable x_0 with law μ , and y is the semicircular variable s_t , we denote ω_1 by F_t , the subordination function such that

$$G_{x_0+s_t}(z) = G_{x_0}(F_t(z)).$$

Biane [10] computed that

$$H_t(z) = z + tG_{x_0}(z), \quad z \in \mathbb{C}^+,$$

is the left inverse of F_t , that is, $H_t(F_t(z)) = z$ for $z \in \mathbb{C}^+$.

Define the function

$$v_t(a) = \inf \left\{ b > 0 : \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2} \leq \frac{1}{t} \right\}, \quad a \in \mathbb{R}. \tag{2.8}$$

So, whenever $v_t(a) > 0$, $v_t(a)$ is the unique positive b with

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2} = \frac{1}{t}. \tag{2.9}$$

Therefore, v_t satisfies the equality given in the following lemma.

Lemma 2.3 ([10, Lemma 2]). *If $v_t(a) > 0$, then*

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + v_t(a)^2} = \frac{1}{t}. \quad (2.10)$$

The function v_t is analytic at a whenever $v_t(a) > 0$. And H_t takes $\{a + iv_t(a) : a \in \mathbb{R}\}$ to the real line, since, for each $a \in \mathbb{R}$, if there exists a $b > 0$ such that

$$\operatorname{Im} H_t(a + ib) = b \left(1 - t \int_{\mathbb{R}} \frac{1}{(a-x)^2 + b^2} d\mu(x) \right) = 0,$$

then it is the unique b which satisfies (2.9).

We summarize Biane's result as follows.

Proposition 2.4 ([10, Proposition 1]). *The subordination function F_t satisfying*

$$G_{x_0+s_t}(z) = G_{x_0}(F_t(z))$$

defined on \mathbb{C}^+ is a one-to-one conformal mapping into \mathbb{C}^+ . The inverse of F_t can be computed explicitly as

$$H_t(z) = z + tG_{x_0}(z),$$

which is conformal from $\{a + ib \in \mathbb{C}^+ : b > v_t(a)\}$ to \mathbb{C}^+ . The function H_t extends to a homeomorphism from $\{a + ib \in \mathbb{C}^+ : b > v_t(a)\}$ to $\overline{\mathbb{C}^+}$ since the domain of H_t is a Jordan domain.

When $v_t(a) > 0$, the point $a + iv_t(a) \in \mathbb{C}^+$ is mapped to the real line by H_t . In fact, the law of $x_0 + s_t$ at the point $H_t(a + iv_t(a))$ is computed and expressed in terms of v_t .

Proposition 2.5 ([10, Corollary 3 and Lemma 5]). *Let*

$$\psi_t(a) = H_t(a + iv_t(a)) = a + t \int_{\mathbb{R}} \frac{(a-x) d\mu(x)}{(a-x)^2 + v_t(a)^2}.$$

Then $\psi_t : \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism and at the point $\psi_t(a)$ the law μ_t of $x_0 + s_t$ has the density given by

$$p_t(\psi_t(a)) = \frac{v_t(a)}{\pi t}.$$

Moreover, the function ψ_t satisfies

$$\psi_t'(a) \geq \frac{2}{t} v_t(a)^2 (1 + v_t'(a)^2) > 0 \quad \text{for any } a \in U_t.$$

Proposition 2.6 ([10, Proposition 3]). *The support of the law μ_t of $x_0 + s_t$ is the closure of its interior, and the number of connected components of U_t is a nonincreasing function of t .*

2.4. Free multiplicative convolution

We will show that the Brown measure for the free multiplicative Brownian motion with unitary initial condition can be described by certain analytic functions and their geometric

properties related to the free unitary Brownian motion with the same unitary initial condition. We review some basic facts about free multiplicative convolution in this section.

Let $u, v \in \mathcal{A}$ be two freely independent unitary random variables, with spectral distributions μ and ν respectively. The distribution of uv is determined by μ and ν and is denoted by $\mu \boxtimes \nu$; it is called the *free multiplicative convolution* of μ and ν . The Σ -transform (2.4) has the property that $\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_{\mu}(z)\Sigma_{\nu}(z)$ in the domain where all these Σ -transforms are defined. We refer the readers to [4, 7] for more details on free multiplicative convolution on \mathbb{T} . The subordination relation for free additive convolution was extended by Biane [12] to the multiplicative case.

Theorem 2.7 ([5, 12]). *Let (\mathcal{A}, τ) be a W^* -probability space, and $u, v \in \mathcal{A}$ two unitary random variables that are freely independent of each other with distributions μ and ν , respectively. If μ is not the Haar measure on \mathbb{T} and ν has nonzero first moment, then there exists a unique pair of analytic self-maps $\omega_1, \omega_2 : \mathbb{D} \rightarrow \mathbb{D}$ such that*

- (1) for $i = 1, 2$, $|\omega_i(z)| \leq |z|$ for $z \in \mathbb{D}$; in particular, $\omega_i(0) = 0$;
- (2) $\omega_1(z)\omega_2(z) = z\eta_{\mu \boxtimes \nu}(z)$ for all $z \in \mathbb{D}$;
- (3) $\eta_{\mu}(\omega_1(z)) = \eta_{\nu}(\omega_2(z)) = \eta_{\mu \boxtimes \nu}(z)$;

The functions η_{μ}, η_{ν} are defined in Section 2.1.

As in the additive case, (3) tells us that if we could compute (one of) the subordination functions ω_1 and ω_2 , we could compute the function $\eta_{\mu \boxtimes \nu}$ of uv in terms of η_{μ} or η_{ν} . The function η_{uv} determines the law of uv by (2.3). The subordination functions are in general impossible to compute explicitly in the multiplicative case.

When any one of unitaries u, v is a Haar unitary, one can check by the definition of free independence that all moments of uv vanish and hence the distribution of uv is always the Haar measure (uniform measure) on \mathbb{T} . We denote by h a Haar unitary and also by h the Haar measure on \mathbb{T} . Note that $\eta_h(z) \equiv 0$. When $u = h$ is a Haar unitary, the subordination function of $\eta_{h \boxtimes \nu} = \eta_h$ with respect to $\eta_{\mu} = \eta_h$ is not unique. However, we shall see that there is a canonical choice in our study.

From now on, we fix a unitary operator u that is freely independent of the free unitary Brownian motion u_t ; we do not restrict u to be not a Haar unitary variable. The spectral distribution of uu_t has been studied by the second author [44, 45]. This is the multiplicative analogue of Biane’s work presented in Section 2.3. We shall now briefly review these results. Let μ be the distribution of u , and λ_t the distribution of u_t . It is convenient for us to use the subordination function with respect to u^* but not u to describe our main results (essentially due to how the measure is recovered by the η -transform as shown in (2.3)). Then we define $\bar{\mu}$ by

$$d\bar{\mu}(e^{ix}) = d\mu(e^{-ix}),$$

which is the distribution of u^* . When u is not a Haar unitary, denote by ω_t the subordination function of $\eta_{\bar{\mu} \boxtimes \lambda_t}$ with respect to $\eta_{\bar{\mu}}$ as in Theorem 2.7. That is,

$$\eta_{\bar{\mu} \boxtimes \lambda_t}(z) = \eta_{\bar{\mu}}(\omega_t(z)). \tag{2.11}$$

When u is a Haar unitary, the subordination function is chosen to be $\omega_t(z) = e^{-t/2}z$. The subordination relation (2.11) also holds (since both sides are zero).

We now describe the left inverse function of ω_t (see [44, Lemma 3.4] and [45, Proposition 2.3]). Set

$$\Sigma_{t,\bar{\mu}}(z) = \exp\left(\frac{t}{2} \int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z} d\bar{\mu}(\xi)\right) = \exp\left(\frac{t}{2} \int_{\mathbb{T}} \frac{\xi+z}{\xi-z} d\mu(\xi)\right), \quad z \in \mathbb{C} \setminus \sigma(u). \quad (2.12)$$

It is known that $\Sigma_{t,\bar{\mu}}(z)$ is the Lévy–Khinchin representation of a free infinitely divisible distribution on the unit circle [7]. When u is a Haar unitary, (2.12) is reduced to

$$\Sigma_{t,\bar{h}}(z) = \Sigma_{t,h}(z) \equiv e^{t/2}.$$

By [45], the left inverse of ω_t is the function

$$\Phi_{t,\bar{\mu}}(z) = z \Sigma_{t,\bar{\mu}}(z).$$

That is, $\Phi_{t,\bar{\mu}}(\omega_t(z)) = z$ for all $z \in \mathbb{D}$.

Denote

$$\Omega_{t,\bar{\mu}} = \{\omega_t(z) : z \in \mathbb{D}\}. \quad (2.13)$$

The subordination function ω_t is a one-to-one conformal mapping from \mathbb{D} onto $\Omega_{t,\bar{\mu}}$ and can be extended to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega_{t,\bar{\mu}}}$. It is known [44, Lemma 3.2] that

$$\Omega_{t,\bar{\mu}} = \{z \in \mathbb{D} : |\Phi_{t,\bar{\mu}}(z)| < 1\}. \quad (2.14)$$

We now describe the boundary of $\Omega_{t,\bar{\mu}}$ and the density formula of $\mu \boxtimes \lambda_t$. More details can be found in Section 4.4. Following [45, p. 1361], we set

$$U_{t,\bar{\mu}} = \left\{ e^{i\theta} \in \mathbb{T} : \int_{-\pi}^{\pi} \frac{1}{|1 - e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) > \frac{1}{t} \right\} \quad (2.15)$$

and $U_{t,\bar{\mu}}^c = [-\pi, \pi] \setminus U_{t,\bar{\mu}}$. We also define a function $r_t : [-\pi, \pi] \rightarrow (0, 1]$ by

$$r_t(\theta) = \sup \left\{ 0 < r < 1 : \frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) < \frac{1}{t} \right\}. \quad (2.16)$$

Indeed, whenever $r_t(\theta) < 1$, $r_t(\theta)$ is the unique $r \in (0, 1)$ such that

$$\frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) = \frac{1}{t}.$$

Remark 2.8. We can prove that the function r_t is analytic when $0 < r_t(\theta) < 1$ by the implicit function theorem applied to the function $F(r, \theta) = \log |\Phi_{t,\bar{\mu}}(r e^{i\theta})|$. For each fixed θ such that $0 < r_t(\theta) < 1$, $r_t(\theta)$ is the unique solution in the unit disk such that $F(r_t(\theta), \theta) = 0$. By [45, p. 1360], we can write

$$\log |\Phi_{t,\bar{\mu}}(r e^{i\theta})| = (\log r) h_t(r, \theta)$$

for some function h_t such that $h_t(r_t(\theta), 0) = 0$ and $\partial h_t / \partial r < 0$ for all $r < 1$. (We will revisit this factorization of $\log |\Phi_{t,\bar{\mu}}(r e^{i\theta})|$ in Section 4.4.) Then we can compute

$$\frac{\partial}{\partial r} \Big|_{r=r_t(\theta)} \log |\Phi_{t,\bar{\mu}}(r e^{i\theta})| = \frac{h_t(r_t(\theta), \theta)}{r_t(\theta)} + (\log r_t(\theta)) \frac{\partial h_t}{\partial r} \Big|_{r=r_t(\theta)} > 0$$

since $h_t(r_t(\theta), \theta) = 0$ and $\partial h_t / \partial r < 0$. Now, it follows from the implicit function theorem that r_t is an analytic function in θ .

The following theorem summarizes the regularity of r_t .

Proposition 2.9 ([45]). *The function r_t defined in (2.16) is continuous everywhere and analytic at θ whenever $r_t(\theta) < 1$. The sets $\Omega_{t,\bar{\mu}}$ and $\partial\Omega_{t,\bar{\mu}}$ can be characterized by the function $r_t(\theta)$ as follows:*

- (1) $\Omega_{t,\bar{\mu}} = \{r e^{i\theta} : 0 \leq r < r_t(\theta), \theta \in [-\pi, \pi]\}$.
- (2) $\partial\Omega_{t,\bar{\mu}} = \{r_t(\theta) e^{i\theta} : \theta \in [-\pi, \pi]\}$ and $\partial\Omega_{t,\bar{\mu}}$ is a continuous closed curve which encloses the origin. For $\theta \in U_{t,\bar{\mu}}$, the value $r_t(\theta)$ is the unique solution $r \in (0, 1)$ of the equation

$$\frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) = \frac{1}{t}.$$

- (3) The map $\Phi_{t,\bar{\mu}}(r_t(\theta) e^{i\theta}) \mapsto r_t(\theta) e^{i\theta}$ is a homeomorphism from \mathbb{T} onto $\partial\Omega_{t,\bar{\mu}}$.

Proof. That r_t is continuous follows from [45, Proposition 3.7]. The analyticity of r_t when $r_t(\theta) < 1$ follows from Remark 2.8.

Point (1) is [45, Theorem 3.2(1)]. The displayed equation in (2) is just Remark 2.8. That $\partial\Omega_{t,\bar{\mu}} = \{r_t(\theta) e^{i\theta} : \theta \in [-\pi, \pi]\}$ follows from [45, Corollary 3.3] since $\Phi_{t,\bar{\mu}}$ is the left inverse of ω_t (the function ω_t is called η_t in [45]). The boundary curve $\partial\Omega_{t,\bar{\mu}}$ encloses the origin because we always have $r_t(\theta) > 0$. Point (3) also follows from [45, Proposition 3.7]; by [45, Proposition 3.7] we know that the maps $\theta \mapsto r_t(\theta) e^{i\theta}$ and $e^{i\theta} \mapsto \Phi_{t,\bar{\mu}}(r_t(\theta) e^{i\theta})$ are homeomorphisms. ■

Remark 2.10. The paper [45] did not include the Haar unitary case. However, when u is a Haar unitary, the above description for the boundary set also holds. Indeed, in this case $\Omega_{t,\bar{h}}$ is the disk centered at the origin with radius $e^{-t/2}$. In fact, one can verify that, for any $0 < r < 1$,

$$\frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{i(\theta+x)}|^2} dh(e^{ix}) = \frac{1}{2\pi} \frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - r e^{ix}|^2} dx = -\frac{1}{2 \log r}.$$

Hence $r_t(\theta) = e^{-t/2}$ for all θ in this case by the definition (2.16), and the results in Proposition 2.9 are also valid.

Being the inverse of ω_t , the restriction of the map $\Phi_{t,\bar{\mu}}$ to $\Omega_{t,\bar{\mu}}$ is a conformal map and can be extended to a homeomorphism of $\overline{\Omega_{t,\bar{\mu}}}$ onto $\overline{\mathbb{D}}$. We can write

$$\arg(\Phi_{t,\bar{\mu}}(r_t(\theta) e^{i\theta})) = \theta + t \int_{-\pi}^{\pi} \frac{r_t(\theta) \sin(\theta + x)}{|1 - r_t(\theta) e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}).$$

We can now obtain the following result, which is a slight modification of [45, Theorem 3.8].

Theorem 2.11. *Let μ_t be the free multiplicative convolution of μ and λ_t , the spectral measure of uu_t . Then μ_t has a density p_t with respect to the Haar measure given by*

$$p_t(\Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})) = -\frac{\log r_t(\theta)}{\pi t}. \quad (2.17)$$

Proof. When u is a Haar unitary, $r_t(\theta) = e^{-t/2}$ for all θ . The formula (2.17) is reduced to the Haar measure on the unit circle \mathbb{T} .

Next, consider u that is not a Haar unitary. Let q_t be the density of u^*u_t . Directly applying [45, Theorem 3.8] (in which u played the role of u^* here) gives

$$\overline{q_t(\Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta}))} = -\frac{\log r_t(\theta)}{\pi t}.$$

The spectral distribution λ_t of u_t is symmetric about the x -axis; hence $(uu_t)^* = u_t^*u^*$ has the same distribution as $u^*u_t^*$ and u^*u_t in the tracial W^* -probability space (\mathcal{A}, τ) . We then have

$$p_t(\Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})) = q_t(\overline{\Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})}) = -\frac{\log r_t(\theta)}{\pi t},$$

giving the desired result. ■

Proposition 2.12 ([45, Corollary 3.9]). *The support of the law μ_t of uu_t is the closure of its interior, and the number of connected components of $U_{t,\mu}$ is a nonincreasing function of t .*

2.5. The Brown measure

If $x \in \mathcal{A}$ is a normal random variable, then there exists a spectral measure E_x such that

$$x = \int \lambda dE_x(\lambda),$$

by the spectral theorem. The law μ_x of x is then computed as

$$\mu_x(A) = \tau(E_x(A))$$

for each Borel set A .

However, if x is not normal, then the spectral theorem does not apply. The Brown measure was introduced by Brown [16] and is a natural candidate of the spectral distribution of a nonnormal operator.

Given $x \in \mathcal{A}$, the *Fuglede–Kadison determinant* [22] $\mathcal{D}(x)$ of x is defined as

$$\mathcal{D}(x) = \exp[\tau(\log |x|)] \in [0, \infty).$$

Define a function L_x on \mathbb{C} by

$$L_x(\lambda) = \log \mathcal{D}(x) = \tau[\log |a - \lambda|].$$

This function is subharmonic. For example, if $A \in M_n(\mathbb{C})$, then

$$L_A(\lambda) = \log |\det(A - \lambda I)|^{1/n}.$$

The *Brown measure* [16] of x is then defined to be the distributional Laplacian of L_x ,

$$\rho_x = \frac{1}{2\pi} \Delta L_x.$$

In this paper, we compute the Brown measures using the following strategy. We first regularize the function L_x by looking at

$$\lambda \mapsto \tau[\log((\lambda - x)^*(\lambda - x) + \varepsilon)]$$

for $\varepsilon > 0$. For any $\varepsilon > 0$, the above quantity is always well-defined as a real number. We then take the limit as $\varepsilon \downarrow 0$ and attempt to take the Laplacian. In the cases we consider, the function L_x is indeed analytic on an open set to which the Brown measure of x gives full measure. The Brown measure of x is then the $1/(2\pi)$ multiple of the ordinary Laplacian of L_x ; that is,

$$d\rho_x(\lambda) = \frac{1}{4\pi} \Delta_\lambda \lim_{\varepsilon \downarrow 0} \tau[\log((\lambda - x)^*(\lambda - x) + \varepsilon)] d^2\lambda \tag{2.18}$$

where $d^2\lambda$ denotes the Lebesgue measure on \mathbb{R}^2 . See [35, Chapter 11] for more details and [25] for a discussion of unbounded operators.

In this paper, we compute the Brown measures of a sum or a product of two freely independent random variables. The paper [6] studied the Brown measure of polynomials of several free random variables using operator-valued free probability and linearization [2]; it does not appear to be easy to apply their framework to get analytic results in our case.

3. Free circular Brownian motion

Let $(c_t)_{t \geq 0}$ be a free circular Brownian motion. We write $x_t = x_0 + c_t$ and $x_{t,\lambda} = \lambda - x_t$, where $\lambda \in \mathbb{C}$. Define

$$S(t, \lambda, \varepsilon) = \tau[\log(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)] \tag{3.1}$$

for $\varepsilon > 0$. When $t = 0$, since x_0 is self-adjoint,

$$S(0, \lambda, \varepsilon) = \tau[\log((x_0 - \lambda)^*(x_0 - \lambda) + \varepsilon)]$$

can be written as an integral, instead of a trace,

$$S(0, \lambda, \varepsilon) = \int_{\mathbb{R}} \log(|x - \lambda|^2 + \varepsilon) d\mu(x)$$

where μ is the spectral distribution of x_0 .

In order to compute the Brown measure of $x_0 + c_t$, we need to compute the Laplacian $\Delta_\lambda S(t, \lambda, 0)$ of $S(t, \lambda, 0)$ with respect to λ , where

$$S(t, \lambda, 0) = \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon). \quad (3.2)$$

In Section 3.1, we compute a first-order, nonlinear partial differential equation of Hamilton–Jacobi type satisfied by S . We then solve a system of ODEs that depends on λ and the initial condition ε_0 . We try to choose, for each λ , an initial condition ε_0 such that

- (1) the lifetime of the solution is t , and
- (2) $\lim_{s \uparrow t} \varepsilon(s) = 0$.

Then we use the solution of the Hamilton–Jacobi equation to compute $S(t, \lambda, 0)$, which is in terms of λ and the initial condition ε_0 , and compute the Laplacian $\Delta_\lambda S(t, \lambda, 0)$.

3.1. The Hamilton–Jacobi equation

In this section, we find a first-order, nonlinear partial differential equation of Hamilton–Jacobi type satisfied by the function S defined in (3.1):

$$\frac{\partial S}{\partial t} = \varepsilon \left(\frac{\partial S}{\partial \varepsilon} \right)^2.$$

The variable ε is positive.

We remark that this PDE corresponds to the formal large- N limit of the PDE computed in [18, (27)] by Nowak et al. after identifying ε with $|w|^2$.

3.1.1. The PDE of S . We first compute the time-derivative of S , using free Itô calculus. Free stochastic calculus was developed in the 1990s by Biane, Kümmerer, Speicher, and many others; see for example [9, 14, 33].

Suppose that f_t and h_t are processes adapted to c_t . The “stochastic differentials” below involving these processes can be computed and simplified as follows (see [14, Theorem 4.1.2]):

$$\begin{aligned} dc_t f_t dc_t^* &= dc_t^* f_t dc_t = \tau(f_t) dt, \\ dc_t f_t dc_t &= dc_t^* f_t dc_t^* = 0, \\ dc_t dt &= dc_t^* dt = 0, \\ \tau(f_t dc_t h_t) &= \tau(f_t dc_t^* h_t) = 0. \end{aligned} \quad (3.3)$$

We can use the free Itô product rule for processes $a_t^{(1)}, \dots, a_t^{(n)}$ adapted to c_t :

$$\begin{aligned} d(a_t^{(1)} \dots a_t^{(n)}) &= \sum_{j=1}^n (a_t^{(1)} \dots a_t^{(j-1)}) da_t^{(j)} (a_t^{(j+1)} \dots a_t^{(n)}) \\ &+ \sum_{1 \leq j < k \leq n} (a_t^{(1)} \dots a_t^{(j-1)}) da_t^{(j)} (a_t^{(j+1)} \dots a_t^{(k-1)}) da_t^{(k)} (a_t^{(k+1)} \dots a_t^{(n)}). \end{aligned} \quad (3.4)$$

Lemma 3.1. *The time-derivative of S satisfies*

$$\frac{\partial S}{\partial t}(t, \lambda, \varepsilon) = \varepsilon \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}] \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}]. \tag{3.5}$$

Proof. Fix $t \geq 0$ and $\lambda \in \mathbb{C}$. For any ε with $\text{Re}(\varepsilon) > 0$, the operator $x_{t,\lambda}^* x_{t,\lambda} + \varepsilon$ is invertible. We can express the function $S(t, \lambda, \cdot)$ defined in (3.1) as a power series of ε and hence it can be analytically continued to the right half-plane.

For each $|\varepsilon| > \|x_{t,\lambda}^* x_{t,\lambda}\|$, we can expand $S(t, \lambda, \varepsilon) = \tau(\log(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon))$ into power series

$$\tau(\log(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)) = \log \varepsilon + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \varepsilon^n} \tau[(x_{t,\lambda}^* x_{t,\lambda})^n]. \tag{3.6}$$

If we apply (3.4) to $d((x_{t,\lambda}^* x_{t,\lambda})^n)$, we get

$$\frac{\partial}{\partial t} \tau[(x_{t,\lambda}^* x_{t,\lambda})^n] = n \sum_{j=1}^n \tau[(x_{t,\lambda}^* x_{t,\lambda})^j] \tau[(x_{t,\lambda}^* x_{t,\lambda})^{n-j-1}].$$

Thus, using (3.6), we have

$$\begin{aligned} \frac{\partial S}{\partial t}(t, \lambda, \varepsilon) &= \varepsilon \left(\frac{1}{\varepsilon} \sum_{k=0}^{\infty} \frac{(-1)^k}{\varepsilon^k} \tau[(x_{t,\lambda}^* x_{t,\lambda})^k] \right) \left(\frac{1}{\varepsilon} \sum_{l=0}^{\infty} \frac{(-1)^l}{\varepsilon^l} \tau[(x_{t,\lambda}^* x_{t,\lambda})^l] \right) \\ &= \varepsilon \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}] \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}]. \end{aligned}$$

Since all the quantities in (3.5) are analytic in ε , it indeed holds for all ε in the right half-plane; in particular, it holds for all $\varepsilon > 0$. ■

Proposition 3.2. *For each $\lambda \in \mathbb{C}$, the function $S(\cdot, \lambda, \cdot)$ satisfies the first-order nonlinear partial differential equation*

$$\frac{\partial S}{\partial t} = \varepsilon \left(\frac{\partial S}{\partial \varepsilon} \right)^2$$

with initial condition

$$S(0, \lambda, \varepsilon) = \tau(\log((x_0 - \lambda)^*(x_0 - \lambda) + \varepsilon)).$$

The PDE does not depend on the derivative of the real or imaginary parts of λ . The variables for the PDE are only t and ε .

Proof. Applying the fact [16, Lemma 1.1] that

$$\frac{\partial S}{\partial \varepsilon} = \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}]$$

to (3.5) concludes the proof. ■

The equation in Proposition 3.2 is a first-order, nonlinear PDE of Hamilton–Jacobi type (see for example, [21, Section 3.3]). We will use the Hamilton–Jacobi method to

analyze the function S . (The solution, which is the function S defined in (3.1), already exists.) The Hamiltonian function

$$H(\varepsilon, p_\varepsilon) = -\varepsilon p_\varepsilon^2 \quad (3.7)$$

satisfying

$$\frac{\partial S}{\partial t} = -H\left(\varepsilon, \frac{\partial S}{\partial \varepsilon}\right)$$

is obtained by replacing the partial derivative in Proposition 3.2 by the momentum variable p_ε and adding a minus sign.

3.1.2. Solving the differential equations. We consider the Hamilton equations for the Hamiltonian (3.7), which is the following system of two coupled ODEs:

$$\frac{d\varepsilon}{dt} = \frac{\partial H}{\partial p_\varepsilon}, \quad \frac{dp_\varepsilon}{dt} = -\frac{\partial H}{\partial \varepsilon}. \quad (3.8)$$

To apply the Hamilton–Jacobi method, we take an arbitrary initial condition $\varepsilon_0 > 0$ for ε but choose an initial condition p_0 for the momentum variable p_ε as

$$p_0 = p_\varepsilon(0) = \tau((|\lambda - x_0|^2 + \varepsilon_0)^{-1}).$$

The initial momentum p_0 can be written as an integral:

$$p_0 = \int \frac{1}{|\lambda - x|^2 + \varepsilon_0} d\mu(x), \quad (3.9)$$

where μ is the spectral distribution of x_0 . The momentum p_0 cannot be chosen arbitrarily and it depends on the initial condition ε_0 , as seen in the above formula. The Hamiltonian is a constant of motion and can be expressed as

$$H(\varepsilon, p_\varepsilon) = H(\varepsilon_0, p_0) = -\varepsilon_0 p_0^2. \quad (3.10)$$

We first state the result from [20, Proposition 6.3].

Proposition 3.3. Fix a function $H(\mathbf{x}, \mathbf{p})$ defined for \mathbf{x} in an open set $U \subseteq \mathbb{R}^n$ and \mathbf{p} in \mathbb{R}^n . Consider a smooth function $S(t, \mathbf{x})$ on $[0, \infty) \times U$ satisfying

$$\frac{\partial S}{\partial t} = -H(\mathbf{x}, \nabla_{\mathbf{x}} S).$$

Suppose the pair $(\mathbf{x}(t), \mathbf{p}(t))$ with values in $U \times \mathbb{R}^n$ satisfies the Hamilton equations

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}(\mathbf{x}(t), \mathbf{p}(t)), \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}(\mathbf{x}(t), \mathbf{p}(t))$$

with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{p}(0) = (\nabla_{\mathbf{x}} S)(0, \mathbf{x}_0)$. Then

$$S(t, \mathbf{x}(t)) = S(0, \mathbf{x}_0) - H(\mathbf{x}_0, \mathbf{p}_0)t + \int_0^t \mathbf{p}(s) \cdot \frac{d\mathbf{x}}{ds} ds \quad (3.11)$$

and

$$(\nabla_{\mathbf{x}}S)(t, \mathbf{x}(t)) = \mathbf{p}(t). \tag{3.12}$$

These two formulas are valid only as long as the solution curve $(\mathbf{x}(t), \mathbf{p}(t))$ exists in $U \times \mathbb{R}^n$.

We apply this result with $n = 1$ and $U = (0, \infty)$ in the system (3.8).

Proposition 3.4. *In the system of coupled ODEs (3.8) with $\varepsilon(0) = \varepsilon_0$ and $p_\varepsilon(0) = p_0$ given in (3.9), for any λ , the first Hamilton–Jacobi formula (3.11) reads*

$$S(t, \lambda, \varepsilon(t)) = \tau(\log(|\lambda - x_0|^2 + \varepsilon_0)) - \varepsilon_0\tau((|\lambda - x_0|^2 + \varepsilon_0)^{-1})^2t.$$

This formula is valid only as long as the solution $(\varepsilon(t), p_\varepsilon(t))$ exists in $(0, \infty) \times \mathbb{R}$; the lifetime of the solution depends on the initial condition ε_0 of ε .

Proof. Since the Hamiltonian H is given by (3.10), by (3.8) we have

$$\begin{aligned} p_\varepsilon(s) \cdot \frac{d}{ds}\varepsilon(s) &= -2\varepsilon(s)p_\varepsilon(s)^2 = 2H(\varepsilon, p_\varepsilon(s)) \\ &= 2H(\varepsilon_0, p_0) = -2\varepsilon_0\tau((|\lambda - x_0|^2 + \varepsilon_0)^{-1})^2. \end{aligned}$$

The expression in the conclusion then follows from (3.11). ■

Proposition 3.5. *Let $p_0 = \tau((|\lambda - x_0|^2 + \varepsilon_0)^{-1})$ be the initial value of p_ε . If the solution to the Hamiltonian system exists up to time t_* , then for all $t \in (0, t_*)$, the solution of p_ε is*

$$p_\varepsilon(t) = \frac{1}{1/p_0 - t}. \tag{3.13}$$

Proof. This follows directly from solving one equation in the system of ODEs (3.8)

$$\frac{dp_\varepsilon}{dt} = -\frac{\partial H}{\partial \varepsilon} = p_\varepsilon^2$$

with initial value $p_0 = \tau((|\lambda - x_0|^2 + \varepsilon_0)^{-1})$. ■

Since $\varepsilon(t) = -H(t)/p_\varepsilon(t)^2 = -H_0/p_\varepsilon(t)^2$, we have the following corollary.

Corollary 3.6. *Under the hypothesis of Proposition 3.5,*

$$\varepsilon(t) = -\left(\frac{1}{p_0} - t\right)^2 H_0 = \varepsilon_0(1 - tp_0)^2.$$

The pair $(\varepsilon(t), p_\varepsilon(t))$ given by Proposition 3.5 and Corollary 3.6 gives a solution to the Hamiltonian system (3.8) up to time

$$t_*(\lambda, \varepsilon_0) = 1/p_0,$$

when the denominator in (3.13) blows up. The value of p_0 given in (3.9) and hence the lifetime t_* of the solution depends on λ and the initial condition ε_0 .

3.1.3. *Two regimes and the connection to the subordination map.* To compute the Brown measure ρ_t of $x_0 + c_t$, we want to compute $\Delta S(t, \lambda, 0)$. Thus, we want to take $\varepsilon(t) = 0$ for the formula of $S(t, \lambda, \varepsilon(t))$ in Proposition 3.4. By Corollary 3.6, the condition $\varepsilon(t) = 0$ can be achieved by either letting $\varepsilon_0 \downarrow 0$, or considering $(1 - tp_0) \downarrow 0$ by making a suitable choice of the initial condition ε_0 .

Write $\lambda = a + ib$. As ε_0 decreases, the lifetime of the path

$$t_*(\lambda, \varepsilon_0) = \frac{1}{p_0} = \frac{1}{\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2 + \varepsilon_0}}$$

decreases. Let

$$T(\lambda) = \lim_{\varepsilon_0 \downarrow 0} t_*(\lambda, \varepsilon_0) = \frac{1}{\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2}} \quad (3.14)$$

be the lifetime of the path in the limit as $\varepsilon_0 \downarrow 0$, where the second equality comes from an application of the monotone convergence theorem. The first regime, $\varepsilon_0 \downarrow 0$, only works if the lifetime of the path is at least t when ε_0 is very small. Hence we study the set where the lifetime of the path is at least t :

$$\{\lambda \in \mathbb{C} : T(\lambda) \geq t\} = \left\{ \lambda = a + ib : \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2} \leq \frac{1}{t} \right\}.$$

The second regime is the open set

$$\Lambda_t = \{\lambda \in \mathbb{C} : T(\lambda) < t\} = \left\{ \lambda = a + ib : \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2} > \frac{1}{t} \right\}, \quad (3.15)$$

where we can use letting $(1 - tp_0) \downarrow 0$, by taking a proper limit of the initial condition ε_0 in Section 3.2.2.

We note that the first regime, letting $\varepsilon_0 \downarrow 0$, will be used in Section 3.2.1; it can be described using the subordination function F_t in Proposition 2.4 and the function v_t defined in (2.8). We note that T may not be continuous, depending on the choice of μ . For example, if $\mu = \mathbb{1}_{[0,1]}(x) \cdot 3x^2 dx$, then $T(0) = \frac{1}{3}$ but $T(a) = 0$ for all $0 < a < 1$.

We now identify Λ_t and draw a connection between the definition of Λ_t and the free additive convolution discussed in Section 2.3. Recall from (2.8) that

$$v_t(a) = \inf \left\{ b > 0 : \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2} \leq \frac{1}{t} \right\}, \quad a \in \mathbb{R}. \quad (3.16)$$

Using (3.14), by (3.15) we see that

$$\Lambda_t = \{a + ib \in \mathbb{C} : |b| < v_t(a)\}. \quad (3.17)$$

Thus, Λ_t agrees with the region defined in point (1) of Theorem 1.1. We will prove that the Brown measure ρ_t has support inside $\overline{\Lambda}_t$.

We recall from Section 2.3 that the subordination function F_t satisfying

$$G_{x_0}(F_t(z)) = G_{x_0+s_t}(z), \quad z \in \mathbb{C}^+,$$

has left inverse

$$H_t(z) = z + tG_{x_0}(z).$$

From Proposition 2.4, the function F_t can be continuously extended to $\overline{\mathbb{C}^+}$ [10, Lemmas 3 and 4] and

$$F_t(\mathbb{C}^+ \cup \mathbb{R}) = \{a + ib \in \mathbb{C}^+ : b \geq v_t(a)\}.$$

Its inverse H_t maps

$$\{a + ib \in \mathbb{C}^+ : b \geq v_t(a)\}$$

onto $\mathbb{C}^+ \cup \mathbb{R}$. Due to (3.17), the set Λ_t can be written in terms of F_t as

$$\Lambda_t = \mathbb{C} \setminus \{F_t(z), \overline{F_t(z)} : z \in \mathbb{C}^+ \cup \mathbb{R}\}.$$

The boundary of Λ_t is given by the closure of the set

$$\{z, \bar{z} : z \in F_t(\mathbb{R}) \cap \mathbb{C}^+\} = \{z, \bar{z} : z \in \mathbb{C}^+ \text{ and } H_t(z) \in \mathbb{R}\}.$$

It is notable that the subordination function F_t for $x_0 + s_t$ also plays a role in the analysis of $x_0 + c_t$, where c_t is the free circular Brownian motion.

3.2. Computation of the Laplacian of S

We now outline the strategy for the computation of the Brown measure in different regimes as follows. (i) If $T(\lambda) > t$, then the lifetime of the solution to the Hamiltonian system remains greater than t when ε_0 tends to zero. Hence, we could let $\varepsilon_0 = 0$ in (3.4). (ii) If $T(\lambda) < t$, the value of ε_0 is chosen such that $p_0 = 1/t$. This condition is equivalent to $b^2 + \varepsilon_0 = v_t(a)^2$ as in Lemma 3.9, which allows us to calculate the Brown measure in this regime. (iii) Finally, we need to eliminate any mass on the set $\{\lambda \in \mathbb{C} : T(\lambda) = t\}$. We do this by showing that the restriction of the Brown measure ρ_t to Λ_t is a probability measure; therefore, using the fact that the Brown measure is a probability measure, we conclude that there is no mass on the boundary $\{\lambda \in \mathbb{C} : T(\lambda) = t\}$. Our approach for the boundary $\{\lambda \in \mathbb{C} : T(\lambda) = t\}$ is different from the approach in [20], where the authors showed directly that the Brown measure of the free multiplicative Brownian motion puts no mass on the boundary by showing that the partial derivatives of the corresponding S are continuous.

3.2.1. Outside $\overline{\Lambda_t}$. By Corollary 3.6,

$$\varepsilon(t) = \varepsilon_0(1 - tp_0)^2$$

depends on the initial condition $\varepsilon_0 > 0$. In the case when $\lambda \notin \overline{\Lambda_t}$, we take $\varepsilon_0 \downarrow 0$ to make $\varepsilon(t) \downarrow 0$. When $\lambda \notin \overline{\Lambda_t}$ is fixed, by (3.17), we have

$$p_0 = \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2 + \varepsilon_0} \leq \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2} < \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + v_t(a)^2} \leq \frac{1}{t},$$

where the definition (2.8) of v_t was used. In this case, by (3.14), the lifetime of the solution is

$$t_*(\lambda, \varepsilon) > t \quad \text{for all } \varepsilon_0 > 0.$$

The initial momentum p_0 can then be extended continuously at $\varepsilon_0 = 0$ by the monotone convergence theorem, and $1 - tp_0 \geq 0$ for all $\varepsilon_0 \geq 0$. It is clear that for a fixed $\lambda = a + ib$ such that $|b| > v_t(a)$, p_0 is a decreasing function of $\varepsilon_0 \geq 0$ and hence $1 - tp_0$ is increasing. This proves the following lemma.

Lemma 3.7. *For a fixed $\lambda \notin \overline{\Lambda}_t$, $\varepsilon = \varepsilon_0(1 - tp_0)^2$ is an increasing continuous function of $\varepsilon_0 > 0$; ε can be extended continuously as $\varepsilon_0 \downarrow 0$. We have*

$$\lim_{\varepsilon_0 \downarrow 0} \varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon_0 \rightarrow \infty} \varepsilon = \infty.$$

Thus, for every $\varepsilon \geq 0$, there is a unique ε_0 such that $\varepsilon = \varepsilon_0(1 - tp_0)^2$.

Fix t and λ . As ε is an increasing function of ε_0 , we can express ε_0 as a function $\varepsilon_0(\varepsilon)$ of ε such that with initial condition $\varepsilon_0(\varepsilon)$, we have $\varepsilon(t) = \varepsilon$. Hence, by Proposition 3.4,

$$\begin{aligned} S(t, \lambda, \varepsilon(t)) &= \tau(\log(|\lambda - x_0|^2 + \varepsilon_0)) - \varepsilon_0 \tau((|\lambda - x_0|^2 + \varepsilon_0)^{-1})^2 t \\ &= \tau(\log(|\lambda - x_0|^2 + \varepsilon_0(\varepsilon))) - \varepsilon_0(\varepsilon) \tau((|\lambda - x_0|^2 + \varepsilon_0(\varepsilon))^{-1})^2 t. \end{aligned} \quad (3.18)$$

Using also the fact that $\varepsilon_0(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ gives the main result of this section.

Theorem 3.8. *For a fixed $\lambda \notin \overline{\Lambda}_t$,*

$$S(t, \lambda, 0) = \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon) = \int_{\mathbb{R}} \log((a - x)^2 + b^2) d\mu(x) \quad (3.19)$$

is real-valued. Thus, for all $\lambda \notin \overline{\Lambda}_t$, $S(t, \lambda, 0)$ is analytic and

$$\Delta_\lambda S(t, \lambda, 0) = 0.$$

In particular, the support of the Brown measure ρ_t of $x_0 + c_t$ is inside $\overline{\Lambda}_t$.

Proof. We first prove that the right hand side of (3.19) exists as a (finite) real number. It suffices to consider $\lambda \in \overline{\Lambda}_t^c \cap \mathbb{R}$. Let $I \subset \mathbb{R}$ be an interval that is outside the closed set $\overline{\Lambda}_t$. Then, for any $a \in I$, we have

$$0 \leq -\frac{1}{\pi} \operatorname{Im} G_{x_0}(a + i\varepsilon) = \frac{1}{\pi} \int \frac{\varepsilon}{(a - x)^2 + \varepsilon^2} d\mu(x) \leq \frac{\varepsilon}{\pi t}.$$

Letting $\varepsilon \downarrow 0$ in the above equation shows that μ is 0 in $\overline{\Lambda}_t^c \cap \mathbb{R}$. Indeed, as $\varepsilon \downarrow 0$, the measure $-\mathbb{1}_I \frac{1}{\pi} \operatorname{Im} G_{x_0}(a + i\varepsilon) da$ converges weakly to $\mu|_I$ for any interval $I \subset \overline{\Lambda}_t^c \cap \mathbb{R}$ by the Stieltjes inversion formula (2.1).

By letting $\varepsilon \downarrow 0$ in (3.18), the fact that $\varepsilon_0(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ gives (3.19). Now, since, for all $\lambda \notin \overline{\Lambda}_t$, $(a - x)^2 + b^2 > 0$, we can move the Laplacian into the integrand:

$$\Delta_\lambda S(t, \lambda, 0) = \int_{\mathbb{R}} \Delta_\lambda \log((a - x)^2 + b^2) d\mu(x) = 0.$$

Recall that the Brown measure ρ_t is defined to be the *distributional* Laplacian of the function $S(t, \lambda, 0)$ with respect to λ . The last assertion about the support of the Brown measure ρ_t follows from the fact that

$$\Delta_\lambda S(t, \lambda, 0) = 0$$

outside $\overline{\Lambda_t}$ implies that the support of the *distribution* $\Delta_\lambda S(t, \lambda, 0)$ is inside $\overline{\Lambda_t}$. ■

3.2.2. *Inside Λ_t .* By (3.15), if $\lambda \in \Lambda_t$, we have $T(\lambda) < t$ (see the definition (3.14) of $T(\lambda)$), showing that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{|\lambda - x|^2} = \int_{\mathbb{R}} \frac{d\mu(x)}{(a - x)^2 + b^2} > \frac{1}{t}, \quad \lambda \in \Lambda_t.$$

By the fact that

$$\lim_{\varepsilon_0 \downarrow 0} t_*(\lambda, \varepsilon_0) = T(\lambda)$$

for each $\lambda \in \Lambda_t$, if the initial condition $\varepsilon_0 > 0$ is small enough, the lifetime of the solution path satisfies $t_*(\lambda, \varepsilon_0) < t$. That means the solution does not exist up to time t . Since, given any λ , the function $\varepsilon_0 \mapsto t_*(\lambda, \varepsilon_0)$ is strictly increasing, and $\lim_{\varepsilon_0 \uparrow \infty} t_*(\lambda, \varepsilon_0) = \infty$, there exists a unique $\varepsilon_0(\lambda)$ such that $t_*(\lambda, \varepsilon_0(\lambda)) = t$. In other words, with this choice of $\varepsilon_0 = \varepsilon_0(\lambda)$, for $\lambda = a + ib$, we have, since $t_*(\lambda, \varepsilon_0(\lambda)) = t$,

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(a - x)^2 + b^2 + \varepsilon_0(\lambda)} = \frac{1}{t_*(\lambda, \varepsilon_0(\lambda))} = \frac{1}{t}. \tag{3.20}$$

Lemma 3.9. *With the choice of $\varepsilon_0 = \varepsilon_0(\lambda)$ satisfying (3.20), ε_0 is a function of (a, b) and is determined by*

$$b^2 + \varepsilon_0(\lambda) = v_t(a)^2. \tag{3.21}$$

Proof. This follows directly from the fact that such an $\varepsilon_0(\lambda)$ is unique. The $\varepsilon_0(\lambda)$ such that $b^2 + \varepsilon_0(\lambda) = v_t(a)^2$ satisfies (3.20) by (2.10). ■

If $t_*(\lambda, \varepsilon_0(\lambda)) = t$, then by (3.21) and Corollary 3.6, $\varepsilon(t) = 0$. This means that the solution of the system (3.8) of ODEs does not exist at time t . For all $\varepsilon_0 > \varepsilon_0(\lambda)$, the lifetime of the system (3.8) is greater than t ; hence it makes sense to look at $S(t, \lambda, \varepsilon(t))$ with any initial condition $\varepsilon_0 > \varepsilon_0(\lambda)$. Our strategy is to consider the solution of S by solving the system of ODEs (3.8) at time t as the limit as $\varepsilon_0 \downarrow \varepsilon_0(\lambda)$ so that $\varepsilon(t) \downarrow 0$.

Recall that

$$S(t, \lambda, 0) = \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon)$$

is defined as in (3.2). We now compute $\Delta_\lambda S(t, \lambda, 0)$.

Theorem 3.10. *For $\lambda = a + ib \in \Lambda_t$ with $b^2 + \varepsilon_0(\lambda) = v_t(a)^2$, using Proposition 3.4, we have*

$$S(t, \lambda, 0) = \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon) = \int_{\mathbb{R}} \log((a - x)^2 + v_t(a)^2) d\mu(x) - \frac{v_t(a)^2 - b^2}{t}.$$

In particular, $S(t, \lambda, 0)$ is analytic inside Λ_t .

The Laplacian of $S(t, \lambda, 0)$ with respect to λ is computed as

$$\Delta_\lambda S(t, \lambda, 0) = \frac{4}{t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right).$$

In particular, for $\lambda \in \Lambda_t$, $\Delta_\lambda S(t, \lambda, 0)$ is independent of $b = \text{Im } \lambda$.

Proof. If $\lambda \in \Lambda_t$ and $\varepsilon > \varepsilon_0(\lambda)$, the lifetime $t_*(\lambda, \varepsilon_0)$ is greater than t . Using Proposition 3.4, we can compute $S(t, \lambda, \varepsilon(t))$, for each initial condition $\varepsilon_0 > \varepsilon_0(\lambda)$, in terms of the initial condition ε_0 explicitly. By (3.18), at time t ,

$$\begin{aligned} S(t, \lambda, \varepsilon(t)) &= \tau(\log((a-x_0)^2 + b^2 + \varepsilon_0)) - \varepsilon_0 \tau(((a-x_0)^2 + b^2 + \varepsilon_0)^{-1})^2 t \\ &= \int_{\mathbb{R}} \log((a-x)^2 + b^2 + \varepsilon_0) d\mu(x) - \varepsilon_0 t \left(\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2 + \varepsilon_0} \right)^2. \end{aligned} \quad (3.22)$$

Let $\varepsilon_0 \downarrow \varepsilon_0(\lambda) = v_t(a)^2 - b^2$ as in (3.21). Then, in the limit, we have

$$\lim_{\varepsilon_0 \downarrow \varepsilon_0(\lambda)} \varepsilon(t) = \lim_{\varepsilon_0 \downarrow \varepsilon_0(\lambda)} \varepsilon_0 (1 - t p_0)^2 = 0,$$

because

$$\lim_{\varepsilon_0 \downarrow \varepsilon_0(\lambda)} t p_0 = t \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + v_t(a)^2} = 1.$$

Therefore, using (3.22), we have

$$\begin{aligned} S(t, \lambda, 0) &= \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon) \\ &= \lim_{\varepsilon_0 \downarrow \varepsilon_0(\lambda)} \left[\int_{\mathbb{R}} \log((a-x)^2 + b^2 + \varepsilon_0) d\mu(x) - \varepsilon_0 t \left(\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + b^2 + \varepsilon_0} \right)^2 \right] \\ &= \int_{\mathbb{R}} \log((a-x)^2 + v_t(a)^2) d\mu(x) - (v_t(a)^2 - b^2) t \left(\int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + v_t(a)^2} \right)^2 \\ &= \int_{\mathbb{R}} \log((a-x)^2 + v_t(a)^2) d\mu(x) + \frac{b^2 - v_t(a)^2}{t}, \end{aligned} \quad (3.23)$$

where the last equality comes from Lemma 2.3.

Now we compute the derivatives of $S(t, \lambda, 0)$ using (3.23). The partial derivative with respect to b is

$$\frac{\partial S}{\partial b}(t, \lambda, 0) = \frac{2b}{t}. \quad (3.24)$$

We note that, by [10, Lemma 2], v_t is analytic at any point a such that $v_t(a) > 0$. If $\lambda \in \Lambda_t$, then $v_t(a) > 0$ and the partial derivative with respect to a is

$$\begin{aligned} \frac{\partial S}{\partial a} &= \int_{\mathbb{R}} \frac{1}{(a-x)^2 + v_t(a)^2} \left(2(a-x) + \frac{\partial}{\partial a} v_t(a)^2 \right) d\mu(x) - \frac{1}{t} \frac{\partial}{\partial a} v_t(a)^2 \\ &= \left(2a + \frac{\partial}{\partial a} v_t(a)^2 \right) \int_{\mathbb{R}} \frac{d\mu(x)}{(a-x)^2 + v_t(a)^2} - \int_{\mathbb{R}} \frac{2x d\mu(x)}{(a-x)^2 + v_t(a)^2} - \frac{1}{t} \frac{\partial}{\partial a} v_t(a)^2 \\ &= \frac{2a}{t} - \int_{\mathbb{R}} \frac{2x d\mu(x)}{(a-x)^2 + v_t(a)^2}, \end{aligned} \quad (3.25)$$

where the identity in Lemma 2.3 was used. Using (3.24) and (3.25), we can compute the Laplacian

$$\begin{aligned} \Delta_\lambda S(t, \lambda, 0) &= \left(\frac{\partial^2 S}{\partial a^2} + \frac{\partial^2 S}{\partial b^2} \right) (t, \lambda, 0) = \frac{2}{t} + \frac{2}{t} - 2 \frac{d}{da} \int_{\mathbb{R}} \frac{x d\mu(x)}{(a-x)^2 + v_t(a)^2} \\ &= \frac{4}{t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x d\mu(x)}{(a-x)^2 + v_t(a)^2} \right). \quad \blacksquare \end{aligned}$$

3.3. The Brown measure of $x_0 + c_t$

Lemma 3.11. *We have*

$$\frac{1}{4\pi} \int_{\Lambda_t} \Delta_\lambda S(t, \lambda, 0) da db = 1;$$

hence, $\frac{1}{4\pi} \Delta_\lambda S(t, \lambda, 0) da db$ defines a probability measure on Λ_t .

Proof. Recall that, by Lemma 2.5,

$$\psi_t(a) = a + t \int_{\mathbb{R}} \frac{(a-x) d\mu(x)}{(a-x)^2 + v_t(a)^2}. \tag{3.26}$$

This is a computation using Fubini’s theorem. Define

$$V_t = \{a \in \mathbb{R} : v_t(a) > 0\}.$$

By Theorem 3.10,

$$\begin{aligned} \frac{1}{4\pi} \int_{\Lambda_t} \Delta_\lambda S(t, \lambda, 0) da db &= \iint_{\Lambda_t} \frac{1}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) db da \\ &= \int_{V_t} \frac{2v_t(a)}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) da \\ &= \int_{V_t} \frac{v_t(a)}{\pi t} \frac{d}{da} \left(2a - t \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) da \\ &= \int_{V_t} \frac{v_t(a)}{\pi t} \frac{d}{da} \left(a + t \int_{\mathbb{R}} \frac{a-x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) da \\ &= \int_{V_t} \frac{v_t(a)}{\pi t} d\psi_t(a) = \int_{\mathbb{R}} d(\mu \boxplus \sigma_t) = 1, \end{aligned}$$

where we used the definition of Λ_t to deduce the second identity and Lemma 2.3 to deduce the fourth identity. The last equality follows from Proposition 2.5. \blacksquare

Since the Brown measure is a probability measure, using the definition of the Brown measure, the above lemma shows that the Brown measure ρ_t of $x_0 + c_t$ is supported on $\overline{\Lambda}_t$ and is absolutely continuous on the support. Lemma 3.11 yields the following:

Corollary 3.12. *We have*

$$\int_{\Lambda_t} d\rho_t = 1.$$

Hence, the Brown measure ρ_t has the form

$$d\rho_t(\lambda) = \frac{1}{4\pi} \mathbb{1}_{\Lambda_t} \Delta_\lambda S(t, \lambda, 0) da db.$$

Theorem 3.13. *The Brown measure ρ_t of $x_0 + c_t$ has support $\overline{\Lambda_t}$ and has the form*

$$d\rho_t(a + ib) = \mathbb{1}_{\Lambda_t}(a + ib) w_t(a) db da \quad (3.27)$$

where

$$w_t(a) = \frac{1}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right).$$

The function w_t is strictly positive inside Λ_t .

Let $\Psi(\lambda) = H_t(a + iv_t(a))$, where $\lambda = a + ib \in \Lambda_t$ (see Proposition 2.4 for the definition of H_t). Then the push-forward of the Brown measure to the real line by Ψ is $\mu \boxplus \sigma_t$, where μ is the distribution of x_0 .

When $\mu = \delta_0$, one can verify that $v_t(a) = \mathbb{1}_{a^2 \leq t} \sqrt{t - a^2}$ and $w_t(a) = 1/(\pi t)$. Then the Brown measure is, as expected, the circular law.

Proof of Theorem 3.13. First note that (3.27) comes from Corollary 3.12 and Theorem 3.10. Define

$$V_t = \{a \in \mathbb{R} : v_t(a) > 0\}.$$

For the assertion about the push-forward, given any smooth function g , we have

$$\begin{aligned} & \iint_{\Lambda_t} g \circ \Psi(a, b) \frac{1}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x d\mu(x)}{(a-x)^2 + v_t(a)^2} \right) db da \\ &= \int_{V_t} g(\psi(a)) \frac{2v_t(a)}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x d\mu(x)}{(a-x)^2 + v_t(a)^2} \right) da \\ &= \int_{V_t} g(\psi(a)) \frac{v_t(a)}{\pi t} \frac{d}{da} \left(2a - t \int_{\mathbb{R}} \frac{x d\mu(x)}{(a-x)^2 + v_t(a)^2} \right) da \\ &= \int_{V_t} g(\psi(a)) \frac{v_t(a)}{\pi t} \frac{d}{da} \left(a + t \int_{\mathbb{R}} \frac{(a-x) d\mu(x)}{(a-x)^2 + v_t(a)^2} \right) da \quad (\text{by Lemma 2.3}) \\ &= \int_{V_t} g(\psi(a)) \frac{v_t(a)}{\pi t} d\psi_t(a) \quad (\text{by (3.26)}) \\ &= \int_{\mathbb{R}} g d(\mu \boxplus \sigma_t). \end{aligned}$$

The last equality follows from Proposition 2.5. The above computation also shows that when $(a, b) \in \Lambda_t$, we have

$$\frac{1}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) = \frac{1}{\pi t} \psi'_t(a) \geq \frac{2}{t^2} v_t(a)^2 (1 + v'_t(a)^2) > 0$$

by Proposition 2.5. This shows that the density of ρ_t is strictly positive inside Λ_t . \blacksquare

Theorem 3.14. *The density of the Brown measure of $x_0 + c_t$ may be expressed as*

$$w_t(a) = \frac{1}{2\pi t} \frac{d\psi_t(a)}{da}, \tag{3.28}$$

and is bounded:

$$w_t(a) \leq \frac{1}{\pi t}. \tag{3.29}$$

The inequality is strict unless μ is a Dirac mass.

Proof. Formula (3.28) was already obtained in the proof of Theorem 3.13; Theorem 4.6 in [4] can be used to prove (3.29) but we give a direct proof here.

Recall that $\psi_t(a) = H_t(a + i v_t(a))$ and $H_t(z) = z + t G_{x_0}(z)$. Let $z = a + i v_t(a)$. If $v_t(a) > 0$, we have

$$\begin{aligned} |1 - H'_t(z)| &= \left| t \frac{d}{dz} \int_{\mathbb{R}} \frac{d\mu(x)}{z - x} \right| = \left| t \int_{\mathbb{R}} \frac{d\mu(x)}{(z - x)^2} \right| \\ &\leq t \int_{\mathbb{R}} \frac{d\mu(x)}{|z - x|^2} = 1, \end{aligned} \tag{3.30}$$

where we used the definition of $v_t(a)$ in (2.8). If μ is not a Dirac mass at 0, the inequality in (3.30) must be strict, since $1/(z - x)^2$ does not have the same phase for μ -almost every x .

Since $\psi_t(a) = H_t(a + i v_t(a))$ is real-valued, by (3.30) we must have

$$0 \leq \frac{\psi_t(a)}{da} \leq 2$$

and both inequalities are strict unless μ is a Dirac mass. ■

Corollary 3.15. *The support of the Brown measure ρ_t of $x_0 + c_t$ is the closure of the open set Λ_t . The number of connected components of Λ_t is a nonincreasing function of t .*

Proof. That the support of ρ_t is the closure of Λ_t follows directly from the facts that ρ_t is strictly positive on Λ_t and it has full measure on Λ_t .

The number of connected components of the open set $V_t = \{a \in \mathbb{R} : v_t(a) > 0\}$ is non-increasing, by Proposition 2.6. Since Λ_t has the same number of components as V_t by the definition of Λ_t , the number of connected components of Λ_t is also nonincreasing. ■

Proposition 3.16. *The Brown measure ρ_t of $x_t = x_0 + c_t$ is the unique measure σ on $\overline{\Lambda_t}$ with the following two properties: (1) the push-forward of σ by the map $\lambda \mapsto \Psi_t(\lambda) = H_t(a + i v_t(a))$ is the distribution of $x_0 + s_t$, and (2) σ is absolutely continuous with respect to the Lebesgue measure and its density is given by*

$$W(a, b) = h(a)$$

for some continuous function h .

Proof. Define

$$V_t = \{a \in \mathbb{R} : v_t(a) > 0\}.$$

Let $y = \psi_t(a)$. By Theorem 2.5, the law of $x_0 + s_t$ has the density given by

$$p_t(y) = \frac{v_t(a)}{\pi t}.$$

We also recall, from Theorem 2.5, that $\psi'_t(a) > 0$ for all $a \in V_t$. The push-forward by Ψ of ρ_t can be computed, for any smooth function g , as

$$\begin{aligned} \int_{\Lambda_t} (g \circ \Psi)(a, b) h(a) da db &= \int_{V_t} (g \circ \psi_t)(a) \cdot 2v_t(a) h(a) da \\ &= \int_{V_t} (g \circ \psi_t)(a) \frac{2v_t(a) h(a)}{\psi'_t(a)} dy. \end{aligned}$$

By assumption, this means the measure $\frac{2v_t(a)h(a)}{\psi'_t(a)} dy$ is the law of $x_0 + s_t$. It follows that

$$\frac{2v_t(a)h(a)}{\psi'_t(a)} = \frac{v_t(a)}{\pi t}, \quad a \in V_t.$$

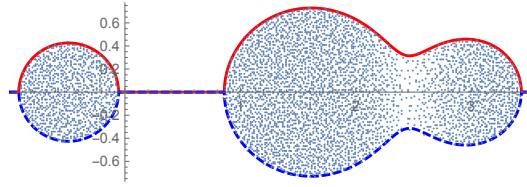
Then h can be solved explicitly, using the definition of ψ from Theorem 2.5:

$$\begin{aligned} h(a) &= \frac{1}{2\pi t} \psi'_t(a) = \frac{1}{2\pi t} \frac{d}{da} \left(a + t \int_{\mathbb{R}} \frac{a-x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) \\ &= \frac{1}{2\pi t} \frac{d}{da} \left(2a - t \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right) \quad (\text{using Lemma 2.3}) \\ &= \frac{1}{\pi t} \left(1 - \frac{t}{2} \frac{d}{da} \int_{\mathbb{R}} \frac{x}{(a-x)^2 + v_t(a)^2} d\mu(x) \right), \end{aligned}$$

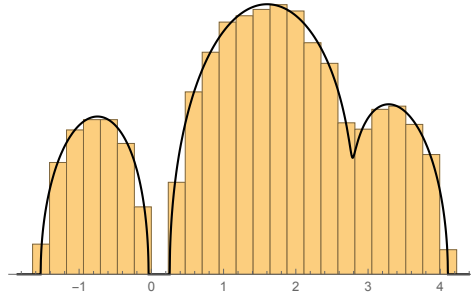
which is $w_t(a)$ in (3.27). This establishes uniqueness. ■

Before we move on to the multiplicative case, we show some computer simulations. As indicated in Section 1.1, in this additive case, we can apply the result by Śniady [37, Theorem 6]: if X_N is a sequence of $N \times N$ self-adjoint deterministic matrices or random matrices classically independent of the Ginibre ensemble $Z_N(t)$ at time t , and if x_0 is the limit of X_N in $*$ -distribution, then the empirical eigenvalue distribution of $X_N + Z_N(t)$ converges weakly to the Brown measure of $x_0 + c_t$.

The two figures below use computer simulations to compare the push-forward of the Brown measure of $x_0 + c_t$ under Ψ and the spectral distribution of $x_0 + s_t$. The latter distribution is stated in Proposition 2.5. In each of the figures, part (a) is the eigenvalue plot of $x_0 + c_t$ at $t = 0.8$, and $t = 2$; part (b) shows the histogram of the image of the eigenvalues in part (a) pushed forward under Ψ , with the density of $x_0 + s_t$ superimposed. We can see that both histograms in part (b) follow the theoretical distribution – the spectral distribution of $x_0 + s_t$.

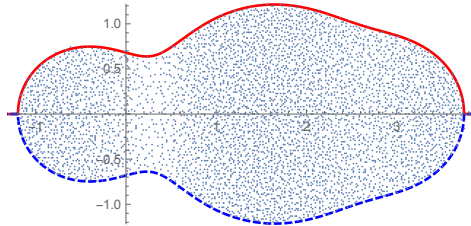


(a) Eigenvalue plot for $x_0 + c_{0.8}$ and the graphs $v_{0.8}(a)$ (red) and $-v_{0.8}(a)$ (blue dashed)

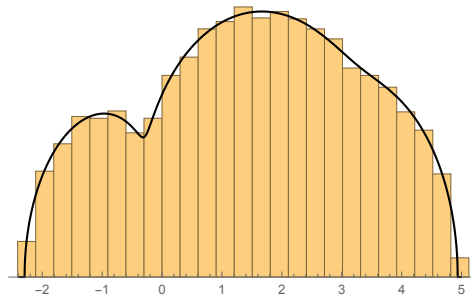


(b) Push-forward of eigenvalues of $x_0 + c_{0.8}$ by $\Psi_{0.8}$ and the density of $x_0 + s_{0.8}$ superimposed

Fig. 3. 2000×2000 matrix simulations at $t = 0.8$ for x_0 distributed as $\frac{1}{5}\delta_{-0.5} + \frac{3}{5}\delta_{1.6} + \frac{1}{5}\delta_3$.



(a) Eigenvalue plot for $x_0 + c_2$ and the graphs $v_2(a)$ (red) and $-v_2(a)$ (blue dashed)



(b) Push-forward of eigenvalues of $x_0 + c_2$ by Ψ_2 and the density of $x_0 + s_2$ superimposed

Fig. 4. 2000×2000 matrix simulations at $t = 2$ for x_0 distributed as $\frac{1}{5}\delta_{-0.5} + \frac{3}{5}\delta_{1.6} + \frac{1}{5}\delta_3$.

4. Free multiplicative Brownian motion

In this section, we compute the Brown measure of the random variable ub_t where u is a unitary random variable freely independent of the free multiplicative Brownian motion b_t . The strategy in the computation in the multiplicative case is similar to the additive case. We first compute the PDE of the corresponding S which can be solved using the Hamilton–Jacobi method. We then find all the constants of motion of the Hamilton equation and solve a system of ODEs. Because the PDE of S is the same as the one computed in [20], the analysis is similar to the arguments there, with initial conditions for our case.

The analysis in this case is much more technical than in the additive case. Nevertheless, the idea is that given a point λ on the complex plane, we want to find initial conditions λ_0 and ε_0 such that

- (1) the solution of the Hamiltonian system exists up to time t ;
- (2) $\lim_{s \uparrow t} \varepsilon(s) = 0$ and $\lim_{s \uparrow t} \lambda(s) = \lambda$.

In the process, we will see that for $\lambda \notin \overline{\Delta_{t,\mu}}$, where $\Delta_{t,\mu}$ is a certain open set described in Section 4.4, the initial condition of ε_0 can be taken arbitrarily small and the above properties still hold. Thus, when $\lambda \notin \overline{\Delta_{t,\mu}}$, we can make $\varepsilon(t)$ arbitrarily close to 0 by making ε_0 sufficiently close to 0. For $\lambda \in \overline{\Delta_{t,\mu}}$, one can show that $\Delta_\lambda S(t, \lambda, 0) = 0$ and hence the Brown measure is supported in $\overline{\Delta_{t,\mu}}$. In this paper, we use a different approach; we show that the Brown measure is 0 outside $\overline{\Delta_{t,\mu}}$ by showing that the Brown measure puts mass 1 on $\Delta_{t,\mu}$.

For $\lambda \in \Delta_{t,\mu}$, we take a proper limit of the initial condition ε_0 such that the lifetime of the solution is at least t . Then we use the Hamilton–Jacobi formulas to compute the Laplacian of S .

4.1. The differential equations

Let b_t be the free multiplicative Brownian motion and u a unitary operator that is freely independent of b_t . Denote $g_t = ub_t$. We consider the function S defined by

$$S(t, \lambda, \varepsilon) = \tau[\log((g_t - \lambda)^*(g_t - \lambda) + \varepsilon)] \tag{4.1}$$

and set

$$s_t(\lambda) = \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon).$$

Then (2.18) shows that the density $W(t, \lambda)$ of the Brown measure of g_t is computed in terms of $s_t(\lambda)$ as

$$W(t, \lambda) = \frac{1}{4\pi} \Delta_\lambda s_t(\lambda). \tag{4.2}$$

By applying the free Itô formula, one can prove the following result. The PDE in the following theorem is the same as the one in [20], except that it is written in logarithmic polar coordinates, and has a different initial condition. We refer the interested reader

to [20] for the proof for $u = I$ and the same argument works for every unitary operator u , because g_t satisfies the same free SDE, but a different initial condition than b_t .

Theorem 4.1. *The function S in (4.1) satisfies the following PDE in logarithmic polar coordinates:*

$$\frac{\partial S}{\partial t} = \varepsilon \frac{\partial S}{\partial \varepsilon} \left(1 + (|\lambda|^2 - \varepsilon) \frac{\partial S}{\partial \varepsilon} - \frac{\partial S}{\partial \rho} \right), \quad \lambda = e^\rho e^{i\theta}, \tag{4.3}$$

with initial condition

$$S(0, \lambda, \varepsilon) = \tau [\log(u - \lambda)^*(u - \lambda) + \varepsilon] = \int_{\mathbb{T}} \log(|\xi - \lambda|^2 + \varepsilon) d\mu(\xi), \tag{4.4}$$

where μ is the spectral distribution of u .

In [20], Driver, Hall and Kemp studied the properties of solutions of (4.3) with the unitary initial condition $u = I$. Since we are solving the same PDE with a different initial condition, the properties of the solution for (4.3) with the initial condition (4.4) are similar to those obtained in their work.

The equation (4.3) is a first-order, nonlinear PDE of Hamilton–Jacobi type (see for example [21, Section 3.3]). We will use polar coordinates and logarithmic polar coordinates for λ ; we write $\lambda = e^\rho e^{i\theta} = r e^{i\theta}$. We define the Hamiltonian corresponding to (4.3) by

$$H(\rho, \theta, \varepsilon, p_\rho, p_\theta, p_\varepsilon) = -\varepsilon p_\varepsilon (1 + r^2 p_\varepsilon - \varepsilon p_\varepsilon - p_\rho). \tag{4.5}$$

Although the right hand side of (4.5) is independent of θ and p_θ , the function S does depend on θ . We consider θ and p_θ as variables of H so that we can apply the second Hamilton–Jacobi formula (3.12) to compute $\frac{\partial S}{\partial \theta}$. We consider Hamilton’s equations for this Hamiltonian; that is, we consider this system of six coupled ODEs:

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial H}{\partial p_\rho}, & \frac{d\theta}{dt} &= \frac{\partial H}{\partial p_\theta}, & \frac{d\varepsilon}{dt} &= \frac{\partial H}{\partial p_\varepsilon}, \\ \frac{dp_\rho}{dt} &= -\frac{\partial H}{\partial \rho}, & \frac{dp_\theta}{dt} &= -\frac{\partial H}{\partial \theta}, & \frac{dp_\varepsilon}{dt} &= -\frac{\partial H}{\partial \varepsilon}. \end{aligned} \tag{4.6}$$

Since the right hand side of (4.5) is independent of θ and p_θ , it is clear that $d\theta/dt = dp_\theta/dt = 0$. Hence, θ and p_θ are independent of t .

Here we require that $\varepsilon(t)$ be positive, while all other quantities are real-valued. For convenience, we write

$$\lambda(t) = r(t)e^{i\theta} = e^{\rho(t)} e^{i\theta}.$$

Note that we have used, in the above equation, the fact that the Hamiltonian is independent of p_θ . The initial conditions for ρ and $\varepsilon > 0$ are arbitrary. Given

$$\rho(0) = \rho_0, \quad \varepsilon(0) = \varepsilon_0, \tag{4.7}$$

we write $r(0) = r_0 = e^{\rho_0}$, and the initial condition (4.4) of S as

$$\begin{aligned}
S(0, \lambda, \varepsilon) &= \int_{\mathbb{T}} \log(|\xi - \lambda|^2 + \varepsilon) d\mu(\xi) \\
&= \int_{-\pi}^{\pi} \log(|e^{i\alpha} - \lambda|^2 + \varepsilon) d\mu(e^{i\alpha}) \\
&= \int_{-\pi}^{\pi} \log(1 + r_0^2 - 2r_0 \cos(\theta - \alpha) + \varepsilon) d\mu(e^{i\alpha}). \tag{4.8}
\end{aligned}$$

The initial momenta $p_{\rho,0} = p_{\rho}(0)$, p_{θ} and $p_0 = p_{\varepsilon}(0)$ are chosen as

$$p_{\rho,0} = p_{\rho}(0) = \frac{\partial S(0, \lambda_0, \varepsilon_0)}{\partial \rho_0}, \quad p_{\theta} = \frac{\partial S(0, \lambda_0, \varepsilon_0)}{\partial \theta}, \quad p_0 = p_{\varepsilon}(0) = \frac{\partial S(0, \lambda_0, \varepsilon_0)}{\partial \varepsilon_0}.$$

With these choices of momenta, we will apply Proposition 3.3 to see that the momenta correspond to the partial derivatives of S along the curve $(\rho(t), \theta, \varepsilon(t))$.

Recalling that μ is the spectral distribution of u , we can write the above definitions explicitly:

$$\begin{aligned}
p_{\rho,0} = p_{\rho}(0) &= \frac{\partial S(0, \lambda_0, \varepsilon_0)}{\partial \rho_0} \\
&= \int_{-\pi}^{\pi} \frac{2r_0(r_0 - \cos(\theta - \alpha))}{1 + r_0^2 - 2r_0 \cos(\theta - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}); \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
p_{\theta} &= \frac{\partial S(0, \lambda_0, \varepsilon_0)}{\partial \theta} \\
&= \int_{-\pi}^{\pi} \frac{2r_0 \sin(\theta - \alpha)}{1 + r_0^2 - 2r_0 \cos(\theta - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}); \tag{4.10}
\end{aligned}$$

$$\begin{aligned}
p_0 = p_{\varepsilon}(0) &= \frac{\partial S(0, \lambda_0, \varepsilon_0)}{\partial \varepsilon_0} \\
&= \int_{-\pi}^{\pi} \frac{1}{1 + r_0^2 - 2r_0 \cos(\theta - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}) \\
&= \int_{\mathbb{T}} \frac{1}{|\xi - \lambda_0|^2 + \varepsilon_0} d\mu(\xi), \tag{4.11}
\end{aligned}$$

where $\lambda_0 = r_0 e^{i\theta}$.

After a change of variable to rectangular coordinates, the Hamiltonian system (4.5) is the same as the one studied in [20] because the PDE of S is the same. The solution of the system of coupled ODEs (4.6), with given initial conditions, is very similar to that in [20, Section 6]; if we do not write the initial momenta explicitly but leave as symbols $p_{\rho,0}$ and p_0 , the solutions look pretty much the same. With the new initial conditions, we will see how the Laplacian of S changes and we will be able to analyze and identify the Brown measure of ub_t .

Lemma 4.2. *The value of the Hamiltonian at $t = 0$ is*

$$H_0 = -\varepsilon_0 p_0^2. \tag{4.12}$$

Proof. We calculate

$$\begin{aligned} r_0^2 p_0 - \varepsilon_0 p_0 - p_{\rho,0} &= \int_{-\pi}^{\pi} \frac{-r_0^2 + 2r_0 \cos(\theta_0 - \alpha) - \varepsilon_0}{1 + r_0^2 - 2r_0 \cos(\theta_0 - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}) \\ &= -1 + p_0. \end{aligned}$$

Hence, $H_0 = -\varepsilon_0 p_0(1 + r_0^2 p_0 - \varepsilon_0 p_0 - p_{\rho,0}) = -\varepsilon_0 p_0^2$. ■

We record the following result which is modified from [20, Theorem 6.2] for our choice of initial conditions.

Theorem 4.3. *Assume $\lambda_0 \neq 0$ and $\varepsilon_0 > 0$. Suppose a solution to the system (4.6) with initial conditions (4.7), and (4.9)–(4.11) exists with $\varepsilon(t) > 0$ for $0 \leq t < T$. Then*

$$\begin{aligned} S(t, \lambda(t), \varepsilon(t)) &= \int_{\mathbb{T}} \log(|\xi - \lambda_0|^2 + \varepsilon_0) d\mu(\xi) - \varepsilon_0 t \left(\int_{\mathbb{T}} \frac{1}{|\xi - \lambda_0|^2 + \varepsilon_0} d\mu(\xi) \right)^2 \\ &\quad + \log |\lambda(t)| - \log |\lambda_0| \end{aligned} \tag{4.13}$$

for all $t \in [0, T)$. Furthermore, the derivatives of S with respect to ρ and ε satisfy

$$\begin{aligned} \frac{\partial S}{\partial \varepsilon}(t, \lambda(t), \varepsilon(t)) &= p_\varepsilon(t), \\ \frac{\partial S}{\partial \rho}(t, \lambda(t), \varepsilon(t)) &= p_\rho(t). \end{aligned} \tag{4.14}$$

Proof. We compute

$$\begin{aligned} p_\rho \frac{d\rho}{dt} + p_\varepsilon \frac{d\varepsilon}{dt} &= p_\rho \frac{\partial H}{\partial p_\rho} + p_\varepsilon \frac{\partial H}{\partial p_\varepsilon} \\ &= \varepsilon p_\varepsilon - 2\varepsilon p_\varepsilon(1 + r^2 p_\varepsilon - \varepsilon p_\varepsilon - p_\rho) \\ &= \varepsilon p_\varepsilon + 2H = \varepsilon p_\varepsilon + 2H_0. \end{aligned}$$

By Proposition 3.3, we have

$$S(t, \lambda(t), \varepsilon(t)) = S(0, \lambda_0, \varepsilon_0) + tH_0 + \int_0^t \varepsilon(s)p_\varepsilon(s) ds.$$

We also have

$$\frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \varepsilon p_\varepsilon,$$

which yields

$$\int_0^t \varepsilon(s)p_\varepsilon(s) ds = \log |\lambda(t)| - \log |\lambda_0|. \tag{4.15}$$

If we plug in the values of $S(0, \lambda_0, \varepsilon_0)$ in (4.4), and H_0 and p_0 in (4.12), we obtain (4.13). ■

It is very important to understand the constants of motion of a Hamiltonian system.

Proposition 4.4. *The following quantities remain constant along any solution of (4.6):*

- (1) *the Hamiltonian H ,*
- (2) *the argument θ of λ if $\lambda_0 \neq 0$,*
- (3) *the momentum p_θ with respect to θ ,*
- (4) *the function $\Psi := \varepsilon p_\varepsilon + \frac{1}{2} p_\rho$.*

Proof. These are the same constants of motion in [20, Propositions 6.4, 6.5]. It can also be checked directly using (4.6). ■

4.2. Solving the ODEs

We will need the following values to solve the coupled ODEs. Using the initial conditions (4.9)–(4.11) and the fact that Ψ is a constant of motion, we have

$$\Psi = \varepsilon p_\varepsilon + \frac{1}{2} p_\rho = \int_{-\pi}^{\pi} \frac{|\lambda_0|^2 - |\lambda_0| \cos(\theta_0 - \alpha) + \varepsilon_0}{1 + |\lambda_0|^2 - 2|\lambda_0| \cos(\theta_0 - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}). \quad (4.16)$$

We set

$$C = 2\Psi - 1 = \int_{-\pi}^{\pi} \frac{|\lambda_0|^2 - 1 + \varepsilon_0}{1 + |\lambda_0|^2 - 2|\lambda_0| \cos(\theta_0 - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}) \quad (4.17)$$

$$= p_0(r_0^2 - 1 + \varepsilon_0) \quad (4.18)$$

and

$$\delta = \frac{r_0^2 + 1 + \varepsilon_0}{r_0}. \quad (4.19)$$

Proposition 4.5. *For all t , we have*

$$\varepsilon(t) p_\varepsilon(t)^2 = \varepsilon_0 p_0^2 e^{-Ct}, \quad (4.20)$$

where C is given by (4.18).

Proof. This is established by solving the ODE that $\varepsilon(t) p_\varepsilon(t)^2$ satisfies. See [20, Proposition 6.6] for details. The difference in our initial conditions does not play a role in the proof. ■

We now adapt the proof of [20, Proposition 6.9] (which is by solving a system of coupled ODEs), under the initial conditions (4.7) and (4.9)–(4.11), to analyze the function S defined in (4.1). We calculate, by (4.5) and (4.6),

$$\frac{dp_\varepsilon}{dt} = -\frac{\partial H}{\partial \varepsilon} = -\frac{H}{\varepsilon(t)} - \varepsilon(t) p_\varepsilon(t)^2.$$

The Hamiltonian H is a constant of motion. By Lemma 4.2 and Proposition 4.5, the above equation is expressed as

$$\frac{dp_\varepsilon}{dt} = -\frac{\varepsilon_0 p_0^2}{\varepsilon(t)} - \varepsilon_0 p_0^2 e^{-Ct} = -p_\varepsilon(t)^2 e^{Ct} - \varepsilon_0 p_0^2 e^{-Ct}. \tag{4.21}$$

The ODE (4.21) can be solved explicitly as the constants C , ε_0 , and p_0 have been given. Following [20, Proposition 6.9], the solution of (4.21) is the p_ε -component of the solution of the system (4.6), which is

$$p_\varepsilon(t) = p_0 \frac{\cosh(kt) + \frac{2|\lambda_0|-\delta}{\sqrt{\delta^2-4}} \sinh(kt)}{\cosh(kt) - \frac{\delta}{\sqrt{\delta^2-4}} \sinh(kt)} e^{-Ct} \tag{4.22}$$

where

$$k = \frac{1}{2} p_0 \cdot r_0 \cdot \sqrt{\delta^2 - 4} \tag{4.23}$$

for as long as the solution to the system (4.6) exists, where we use the same choice of $\sqrt{\delta^2 - 4}$ as in the definition of k in (4.23). If $\delta = 2$, we interpret $\sinh(kt)/\sqrt{\delta^2 - 4}$ as $\frac{1}{2} p_0 r_0 t$. In addition, if $\varepsilon_0 \geq 0$, the numerator $\cosh(kt) - \frac{\delta}{\sqrt{\delta^2-4}} \sinh(kt)$ is positive for all t . Hence, the function $p_\varepsilon(t)$ is positive as long as the solution exists and its reciprocal $1/p_\varepsilon(t)$ is a real analytic function of t defined for all $t \in \mathbb{R}$. Moreover, the first time the expression (4.22) blows up is when the denominator is zero, which is

$$t_*(\lambda_0, \varepsilon_0) = \frac{2}{p_0 r_0} \frac{1}{\sqrt{\delta^2 - 4}} \tanh^{-1} \left(\frac{\sqrt{\delta^2 - 4}}{\delta} \right) \tag{4.24}$$

$$= \frac{1}{p_0 r_0} \frac{1}{\sqrt{\delta^2 - 4}} \log \left(\frac{\delta + \sqrt{\delta^2 - 4}}{\delta - \sqrt{\delta^2 - 4}} \right). \tag{4.25}$$

Here, the principal branch of the inverse hyperbolic tangent should be used in (4.24), with branch cuts $(-\infty, -1]$ and $[1, \infty)$ on the real axis, which corresponds to using the principal branch of the logarithm in (4.25). When $\delta = 2$, we interpret $t_*(\lambda_0, \varepsilon_0)$ as having its limiting value as δ approaches 2.

We now describe the limit behaviors of $\varepsilon(t)$ and $\lambda(t)$ by adapting the arguments in [20, Section 6.3]. By (4.20), we have

$$\varepsilon(t) = \frac{\varepsilon_0 p_0^2 e^{-Ct}}{p_\varepsilon(t)^2}. \tag{4.26}$$

As t approaches $t_*(\lambda_0, \varepsilon_0)$ from the left, $p_\varepsilon(t)$ remains positive until it blows up and $\varepsilon(t)$ approaches zero. That is, for any ε_0 , we have

$$\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} \varepsilon(t) = 0. \tag{4.27}$$

If $\varepsilon_0 = 0$, from (4.26) we see that the solution has $\varepsilon(t) \equiv 0$; and we deduce that $\lambda(t) \equiv \lambda_0$ from (4.15). By [20, Theorem 6.7] (our initial conditions are different from

the system in [20], but it only uses (4.26) and the fact that H is a constant of motion), if $\varepsilon(t) \rightarrow 0$ as $t \rightarrow t_*(\lambda_0, \varepsilon_0)$ with $\varepsilon_0 > 0$, then

$$\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} \rho(t) = C t_*(\lambda_0, \varepsilon_0)/2. \quad (4.28)$$

Proposition 4.6. *We have*

$$\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} p_\rho = 2\Psi = C + 1 = \lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} 2\rho(t)/t + 1. \quad (4.29)$$

Proof. Note that $\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} \varepsilon(t) p_\varepsilon(t) = \lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} \sqrt{\varepsilon(t)} \sqrt{\varepsilon(t) p_\varepsilon(t)^2} = 0$ by Proposition 4.5 and (4.27). We then use (4.16) to obtain

$$\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} p_\rho = 2\Psi.$$

Then (4.18) and (4.28) yield the result. ■

4.3. Monotonicity of the lifetime

We will show first that the lifetime of the Hamiltonian system is an increasing function of ε_0 . To this end, let us recall the following elementary lemma from [20, proof of Proposition 6.16].

Lemma 4.7. *Given $\theta \in (-\pi, \pi]$, the function g_θ defined by*

$$g_\theta(x) := \frac{x - 2 \cos \theta}{\sqrt{x^2 - 4}} \log \left(\frac{x + \sqrt{x^2 - 4}}{x - \sqrt{x^2 - 4}} \right) \quad (4.30)$$

is a strictly increasing, nonnegative, continuous function of x for $x \geq 2$ and tends to ∞ as x tends to infinity.

Proposition 4.8. *For each λ_0 , the function $t_*(\lambda_0, \varepsilon_0)$ is a strictly increasing function of ε_0 for $\varepsilon_0 \geq 0$ and*

$$\lim_{\varepsilon_0 \rightarrow +\infty} t_*(\lambda_0, \varepsilon_0) = +\infty.$$

Proof. For λ_0 fixed, recalling the definition of δ in (4.19) and $t_*(\lambda, \varepsilon)$ in (4.25), we define

$$f_{\lambda_0}(\delta) = \frac{1}{t_*(\lambda_0, \varepsilon_0)} = p_0 r_0 \sqrt{\delta^2 - 4} \log \left(\frac{\delta - \sqrt{\delta^2 - 4}}{\delta + \sqrt{\delta^2 - 4}} \right). \quad (4.31)$$

By the expression for p_0 in (4.11), we obtain

$$p_0 r_0 = \int_{-\pi}^{\pi} \frac{1}{\delta - 2 \cos(\theta_0 - \alpha)} d\mu(e^{i\alpha}).$$

We can then rewrite $f_{\lambda_0}(\delta)$ as

$$f_{\lambda_0}(\delta) = \int_{-\pi}^{\pi} \frac{1}{g_{\theta_0 - \alpha}(\delta)} d\mu(e^{i\alpha}),$$

where $g_{\theta_0-\alpha}(\delta)$ is defined in (4.30). Hence, as $t_*(\lambda_0, \varepsilon_0)$ is the reciprocal of $f_{\lambda_0}(\delta)$ as in (4.31), it then follows from Lemma 4.7 that $t_*(\lambda_0, \varepsilon_0)$ is a strictly increasing, nonnegative, continuous function of δ for $\delta \geq 2$ that tends to ∞ as $\delta \rightarrow \infty$. This finishes the proof. ■

Similar to the additive case, the complex plane \mathbb{C} is naturally divided into two sets when we calculate the Brown measure of the free multiplicative Brownian motion. The following function T determines how we divide \mathbb{C} into two sets. The analysis to achieve $\varepsilon(t) = 0$ is different in these two sets. We will see that one of these two sets has full Brown measure, and we omit the analysis of the other set in this paper. The interested readers can find that analysis in [20] (for $u = I$).

Define the function $T : \mathbb{C} \rightarrow [0, \infty)$ by

$$T(\lambda_0) = \begin{cases} \frac{1}{p_0} \frac{\log r_0^2}{r_0^2 - 1} & \text{for } \lambda_0 = r_0 e^{i\theta_0} \text{ and } r_0 \neq 1, \\ \frac{1}{p_0} & \text{for } \lambda_0 = r_0 e^{i\theta_0} \text{ and } r_0 = 1. \end{cases} \tag{4.32}$$

When $\mu = \delta_1$, the function T reduces to the function in [20]. The following result is analogous to results in [20, Section 6.4]. Roughly speaking, it says that T is the lifetime of the solution “when $\varepsilon_0 = 0$ ”.

Proposition 4.9. *Recall that $t_*(\lambda_0, \varepsilon_0)$ is defined by (4.25). Then for all nonzero λ_0 ,*

$$t_*(\lambda_0, 0) := \lim_{\varepsilon_0 \downarrow 0} t_*(\lambda_0, \varepsilon_0) = T(\lambda_0), \tag{4.33}$$

where the function T is defined in (4.32). Furthermore, when $\varepsilon_0 = 0$, we have

$$\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} \rho(t) = \rho_0. \tag{4.34}$$

Proof. We first consider the case when $|\lambda_0| = 1$. In this case, we have $\lim_{\varepsilon_0 \rightarrow 0} \delta = 2$ and $\lim_{\varepsilon_0 \rightarrow 0} \sqrt{\delta^2 - 4} = \pm \frac{|\lambda_0|^2 - 1}{|\lambda_0|}$. We can then compute the limit

$$\lim_{\delta \downarrow 2} \frac{1}{\sqrt{\delta^2 - 4}} \log \left(\frac{\delta + \sqrt{\delta^2 - 4}}{\delta - \sqrt{\delta^2 - 4}} \right) = 1 \tag{4.35}$$

and obtain

$$\lim_{\varepsilon_0 \downarrow 0} t_*(\lambda_0, \varepsilon_0) = 1/p_0,$$

where p_0 , given by (4.11), can also be written as

$$p_0 = \int_{\mathbb{T}} \frac{1}{|\xi - \lambda_0|^2} d\mu(\xi) \quad \text{with } \theta = \arg \lambda_0.$$

Hence $\lim_{\varepsilon_0 \rightarrow 0} t_*(\lambda_0, \varepsilon_0) = T(\lambda_0)$ by (4.32).

For $|\lambda_0| \neq 1$, we note that

$$\lim_{\varepsilon_0 \downarrow 0} \frac{\delta + \sqrt{\delta^2 - 4}}{\delta - \sqrt{\delta^2 - 4}} = r_0^2 \quad \text{and} \quad \lim_{\varepsilon_0 \downarrow 0} r_0 \sqrt{\delta^2 - 4} = r_0^2 - 1.$$

Hence, from (4.25), we deduce that

$$\lim_{\varepsilon_0 \downarrow 0} t_*(\lambda_0, \varepsilon_0) = \frac{1}{p_0} \frac{\log r_0^2}{r_0^2 - 1} = T(\lambda_0), \quad \text{where } \theta = \arg \lambda_0,$$

due to the expressions of $p_\varepsilon(0)$ as in (4.11).

The proof of (4.34) follows from a similar calculation using (4.28), and $\sqrt{\delta^2 - 4} = \pm \frac{|\lambda_0|^2 - 1}{|\lambda_0|}$. ■

4.4. The domains $\Delta_{t,\mu}$ and $\Omega_{t,\bar{\mu}}$, and their relations to the lifetime

Recall from Section 2.4 that the set $\Omega_{t,\bar{\mu}} = \{\omega_t(z) : z \in \mathbb{D}\}$ defined in (2.13) is star-like with respect to the origin, by Proposition 2.9. Moreover, the graph of the function r_t defined in (2.16) is exactly the boundary set $\partial\Omega_{t,\bar{\mu}}$. That is, by Proposition 2.9 (1),

$$\Omega_{t,\bar{\mu}} = \{re^{i\theta} : -\pi \leq \theta \leq \pi, 0 \leq r < r_t(\theta)\}.$$

See Figure 5 for an example. Define the open set

$$\Delta_{t,\mu} := \{z, 1/\bar{z} : z \in \overline{\mathbb{D}} \setminus \overline{\Omega_{t,\bar{\mu}}}\}.$$

We will prove in Theorem 4.28 that the closure $\overline{\Delta_{t,\mu}}$ of $\Delta_{t,\mu}$ is the support of the Brown measure of ub_t ; thus, we call it $\Delta_{t,\mu}$ instead of $\overline{\Delta_{t,\mu}}$. All other notations are related to the subordination function of u^*u_t with respect to u^* so we use $\bar{\mu}$ as subscript in those notations.

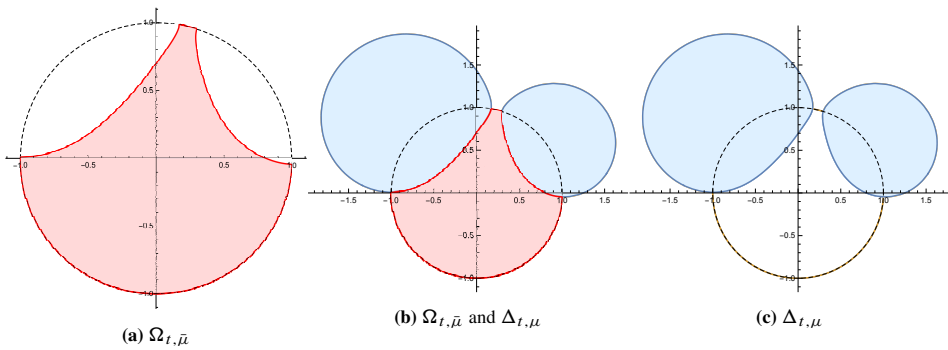


Fig. 5. The sets $\Omega_{t,\bar{\mu}}$ and $\Delta_{t,\mu}$ when u is distributed as $\frac{1}{3}\delta_{e^{\pi i/6}} + \frac{2}{3}\delta_{e^{3\pi i/4}}$ and $t = 0.8$, plotted with the unit circle (black-dashed).

In this section, we first establish the following theorem which gives the relation between the domain $\Delta_{t,\mu}$ and the function T (which is the lifetime of the solution when $\varepsilon_0 = 0$).

Theorem 4.10. For any $t > 0$, the region $\Delta_{t,\mu}$ is invariant under $\lambda \mapsto 1/\bar{\lambda}$ and we have

$$\Delta_{t,\mu} = \{re^{i\theta} : r_t(\theta) < r < 1/r_t(\theta), \theta \in U_{t,\bar{\mu}}\}, \tag{4.36}$$

where r_t is the function defined in (2.16). Moreover, $\Delta_{t,\mu}$ may be expressed as

$$\Delta_{t,\mu} = \{\lambda \in \mathbb{C} : T(\lambda) < t\}.$$

For any $\lambda \notin \overline{\Delta_{t,\mu}}$, we have $T(\lambda) > t$, that is,

$$\mathbb{C} \setminus \overline{\Delta_{t,\mu}} \subset \{\lambda \in \mathbb{C} : T(\lambda) > t\}.$$

For any $\lambda \in \partial\Delta_{t,\mu} \cap \mathbb{T}$, we have $T(\lambda) \geq t$; and for any $\lambda \in \partial\Delta_{t,\mu} \setminus \mathbb{T}$, we have $T(\lambda) = t$.

Theorem 4.10 characterizes the complement of $\overline{\Omega_{t,\mu}}$ using the function T . With this characterization, the strategy of letting $\varepsilon_0 \rightarrow 0^+$ could work on $\mathbb{C} \setminus \overline{\Omega_{t,\mu}}$, using an argument parallel to that in [20], where the case $\mu = \delta_1$ was considered. We do not analyze $\mathbb{C} \setminus \overline{\Omega_{t,\mu}}$ in this paper because we can show that $\Omega_{t,\mu}$ has full Brown measure (see Lemma 4.25).

We will prove Theorem 4.10 later. In the following, we establish a corollary of this theorem.

Corollary 4.11. *For $\lambda_0 \in \Delta_{t,\mu}$, we have $t_*(\lambda_0, 0) < t$, and for $\lambda_0 \notin \overline{\Delta_{t,\mu}}$, we have $t_*(\lambda_0, 0) > t$. For $\lambda_0 \in \partial\Delta_{t,\mu} \setminus \mathbb{T}$, we have $t_*(\lambda_0, 0) = t$, and for $\lambda_0 \in \partial\Delta_{t,\mu} \cap \mathbb{T}$, we have $t_*(\lambda_0, 0) \geq t$.*

Proof. Apply (4.33) and Proposition 4.9 and Theorem 4.10. ■

Remark 4.12. We can only conclude $t_*(\lambda, 0) \geq t$ for $\lambda \in \partial\Delta_{t,\mu} \cap \mathbb{T}$ but *not* $t_*(\lambda, 0) = t$, because $t_*(\lambda, 0) = T(\lambda)$ may not be continuous at these points and $T > t$ can occur on $\partial\Delta_{t,\mu} \cap \mathbb{T}$. For example, if $d\mu(e^{ix}) = \mathbb{1}_{[0,1]}3x^2 d(e^{ix})$, then, for all $t < 1/3$, $T(e^i) = 1/3 > t$ and $e^i \in \partial\Delta_{t,\mu} \cap \mathbb{T}$.

We now describe some important properties of the set $\Omega_{t,\bar{\mu}}$ through the subordination function. Recall that

$$\Phi_{t,\bar{\mu}}(z) = z\Sigma_{t,\bar{\mu}}(z) = z \exp\left(\frac{t}{2} \int_{\mathbb{T}} \frac{1 + \xi z}{1 - \xi z} d\bar{\mu}(\xi)\right). \tag{4.37}$$

We define the function

$$\phi = \phi(\theta) = \theta + t \int_{-\pi}^{\pi} \frac{r_t(\theta) \sin(\theta - x)}{|1 - r_t(\theta)e^{i(\theta-x)}|^2} d\mu(e^{ix}), \quad \theta \in \mathbb{R}. \tag{4.38}$$

Recall that the function ω_t extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\Omega_{t,\bar{\mu}}$; the function ϕ is a strictly increasing continuous function from $[a, a + 2\pi)$ onto some interval $[b, b + 2\pi)$. Then, given any $a \in \mathbb{R}$, ϕ is a continuous version of $\arg(\Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta}))$ for any $\theta \in [a, a + 2\pi)$; that is, $e^{i\phi} = \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$.

We write

$$\log |\Phi_{t,\bar{\mu}}(re^{i\theta})| = \log r + \operatorname{Re}\left(\frac{t}{2} \int_{\mathbb{T}} \frac{1 + \xi re^{i\theta}}{1 - \xi re^{i\theta}} d\bar{\mu}(\xi)\right) = (\log r)h_t(r, \theta), \tag{4.39}$$

where

$$h_t(r, \theta) = 1 - \frac{t}{2} \frac{r^2 - 1}{\log r} \int_{\mathbb{T}} \frac{1}{|1 - r e^{i\theta} \xi|^2} d\bar{\mu}(\xi).$$

For $\lambda \in \mathbb{C}$ with $|\lambda| \neq 0, 1$, we let

$$\begin{aligned} f(r, \theta) &= \frac{1 - h_t(r, \theta)}{t} = \frac{1}{2} \frac{r^2 - 1}{\log r} \int_{\mathbb{T}} \frac{1}{|1 - r e^{i\theta} \xi|^2} d\bar{\mu}(\xi) \\ &= \frac{1}{2} \frac{r^2 - 1}{\log r} \int_{\mathbb{T}} \frac{1}{|1 - r e^{i(\theta-x)}|^2} d\mu(e^{ix}) \\ &= \frac{1}{2} \frac{r^2 - 1}{\log r} \int_{-\pi}^{\pi} \frac{1}{1 - 2r \cos(\theta - x) + r^2} d\mu(e^{ix}). \end{aligned} \quad (4.40)$$

We will need the following elementary fact.

Lemma 4.13 ([45, Lemma 3.1]). *Given $-1 \leq y \leq 1$, define a function of r by*

$$R_y(r) = \frac{r^2 - 1}{\log r} \frac{1}{1 - 2ry + r^2}$$

on the interval $(0, 1)$. Then $R'_y(r) > 0$ for all $r \in (0, 1)$.

Corollary 4.14. *For fixed θ , the function $r \mapsto f(r, \theta)$ is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. We have*

$$\lim_{r \rightarrow 0} f(r, \theta) = \lim_{r \rightarrow \infty} f(r, \theta) = 0.$$

Moreover, $f(r, \theta) = f(1/r, \theta)$.

Proof. By Lemma 4.13, we see directly that $\frac{\partial f(r, \theta)}{\partial r} > 0$ for $r \in (0, 1)$, so $r \mapsto f(r, \theta)$ is increasing for $r \in (0, 1)$. It is straightforward to check that $f(r, \theta) = f(1/r, \theta)$; thus, $r \mapsto f(r, \theta)$ is decreasing on $(1, \infty)$. The limit assertion can be checked directly from the definition of f . ■

Hence, it makes sense to define $f(1^-, \cdot) : [-\pi, \pi] \rightarrow (0, \infty]$ as follows:

$$f(1^-, \theta) = \lim_{r \rightarrow 1^-} f(r, \theta) = \int_{-\pi}^{\pi} \frac{1}{|1 - e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) = \int_{-\pi}^{\pi} \frac{1}{|1 - e^{i(\theta-x)}|^2} d\mu(e^{ix}). \quad (4.41)$$

Then the function $T : \mathbb{C} \rightarrow [0, \infty)$ defined in (4.32) is expressed by f as follows:

$$T(\lambda) = \begin{cases} 1/f(r, \theta) & \text{for } \lambda = r e^{i\theta} \text{ and } r \neq 1, \\ 1/f(1^-, \theta) & \text{for } \lambda = r e^{i\theta} \text{ and } r = 1. \end{cases} \quad (4.42)$$

If $f(1^-, \theta) = \infty$, we understand that, for this θ , $T(e^{i\theta}) = 0$. Analogous to the additive case (see (3.14)), the function T may not be continuous on \mathbb{T} , as mentioned in Remark 4.12. If $\mu = \delta_1$, one can use (4.42) to check that T reduces to the function defined in [20].

Lemma 4.15. For each $\theta \in [-\pi, \pi]$, the function $r \mapsto T(re^{i\theta})$ is strictly decreasing for $0 < r < 1$ and strictly increasing for $r > 1$. For each θ , the minimum value of $T(re^{i\theta})$ is achieved at $r = 1$, which is $1/f(1^-, \theta)$, and we have

$$\lim_{r \rightarrow 0} T(re^{i\theta}) = \lim_{r \rightarrow \infty} T(re^{i\theta}) = \infty.$$

Moreover, $T(re^{i\theta}) = T((1/r)e^{i\theta})$.

Proof. The claim follows from Corollary 4.14 and the reciprocal relation (4.42). ■

We now can use the functions r_t defined in (2.16) and T to describe the set $\Omega_{t, \bar{\mu}}$ and its boundary.

Proposition 4.16. Let $\theta \in [-\pi, \pi]$.

- (1) When $r_t(\theta) < 1$, we have $T(r_t(\theta)e^{i\theta}) = t$. Moreover,
 - (a) $T(re^{i\theta}) > t$ for $r < r_t$;
 - (b) $T(re^{i\theta}) < t$ for $r_t < r \leq 1$.
- (2) When $r_t(\theta) = 1$, we have $T(r_t(\theta)e^{i\theta}) = T(e^{i\theta}) \geq t$, and $T(re^{i\theta}) > t$ for $r < 1$. If, in addition, $\theta \in [-\pi, \pi] \setminus \overline{U_{t, \bar{\mu}}}$, then $T(re^{i\theta}) > t$ for all $r > 0$.

Proof. When $r_t(\theta) < 1$, by Proposition 2.9 (2) we have

$$\frac{r_t(\theta)^2 - 1}{2 \log r_t(\theta)} \int_{-\pi}^{\pi} \frac{1}{|1 - r_t(\theta)e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) = \frac{1}{t},$$

which implies that $T(r_t(\theta)e^{i\theta}) = t$ by the definition (4.32) of T . For each θ , the function $r \mapsto T(re^{i\theta})$ is strictly decreasing for $0 < r < 1$ by Lemma 4.15. Items (1a) and (1b) follow.

By the definition of r_t in (2.16), when $r_t(\theta) = 1$, we have

$$\frac{r^2 - 1}{2 \log r} \int_{-\pi}^{\pi} \frac{1}{|1 - re^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) \leq \frac{1}{t}$$

for all $0 \leq r \leq r_t(\theta) = 1$. Hence $T(r_t(\theta)e^{i\theta}) \geq t$, and $T(\lambda) > t$ when $|\lambda| < 1$. If, in addition, $\theta \in [-\pi, \pi] \setminus \overline{U_{t, \bar{\mu}}}$, then by (2.15),

$$\int_{-\pi}^{\pi} \frac{1}{|1 - e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) < \frac{1}{t},$$

which shows that $T(e^{i\theta}) > t$ due to (4.32). Therefore, $T(re^{i\theta}) > t$ for all $0 < r \leq 1$ and $\theta \in [-\pi, \pi] \setminus \overline{U_{t, \bar{\mu}}}$. ■

Proposition 4.17. The open set $U_{t, \bar{\mu}}$ defined in (2.15) may be characterized by

$$U_{t, \bar{\mu}} = \{-\pi \leq \theta \leq \pi : r_t(\theta) < 1\} \tag{4.43}$$

$$= \{-\pi \leq \theta \leq \pi : T(e^{i\theta}) < t\}. \tag{4.44}$$

Moreover, $\partial\Omega_{t, \bar{\mu}} \cap \mathbb{T} = \mathbb{T} \setminus \{e^{i\theta} : \theta \in U_{t, \bar{\mu}}\}$, where $\Omega_{t, \bar{\mu}}$ is defined in (2.14).

Proof. We rewrite the definition of $U_{t,\bar{\mu}}$ in (2.15) in terms of the function f as

$$U_{t,\bar{\mu}} = \{-\pi \leq \theta \leq \pi : f(1^-, \theta) > 1/t\}.$$

Hence, by Proposition 4.14, $\theta \in U_{t,\bar{\mu}}$ if and only if $r_t(\theta) < 1$. From Proposition 4.16, we see that $r_t(\theta) < 1$ if and only if $T(e^{i\theta}) < t$. Now, (4.43) and (4.44) hold.

Recall from Proposition 2.9 that

$$\partial\Omega_{t,\bar{\mu}} = \{r_t(\theta)e^{i\theta} : -\pi \leq \theta \leq \pi\},$$

which yields

$$\partial\Omega_{t,\bar{\mu}} \cap \mathbb{T} = \{e^{i\theta} : r_t(\theta) = 1\} = \mathbb{T} \setminus \{e^{i\theta} : \theta \in U_{t,\bar{\mu}}\}. \quad \blacksquare$$

We are now ready to prove our main result in this section.

Proof of Theorem 4.10. Recall from Proposition 2.9 that $\Omega_{t,\bar{\mu}} = \{re^{i\theta} : 0 \leq r < r_t(\theta), \theta \in [-\pi, \pi]\}$. Hence, by (4.43) in Proposition 4.17,

$$\overline{\mathbb{D}} \setminus \overline{\Omega_{t,\bar{\mu}}} = \{re^{i\theta} : r_t(\theta) < r \leq 1, r_t(\theta) < 1\} = \{re^{i\theta} : r_t(\theta) < r \leq 1, \theta \in U_{t,\bar{\mu}}\},$$

which yields (4.36).

We discuss the function $T(\lambda)$ according to the argument of λ in the following cases:

- (1) When $\theta \in U_{t,\bar{\mu}}$, for λ with $\arg(\lambda) = \theta$ and $r_t(\theta) < |\lambda| \leq 1$, we know that $T(\lambda) < t$ from Proposition 4.16 (1b). Hence, $\Delta_{t,\mu} \subset \{\lambda \in \mathbb{C} : T(\lambda) < t\}$ by the definition of $\Delta_{t,\mu}$ and the symmetry property $T(z) = T(1/\bar{z})$ as in Lemma 4.15. Since $r \mapsto T(re^{i\theta})$ is continuous and decreasing for $0 < r < 1$, it follows that

$$r_t(\theta)e^{i\theta} \in \partial\Omega_{t,\bar{\mu}} \cap \partial\Delta_{t,\mu}$$

and $T(r_t(\theta)e^{i\theta}) = t$ by Proposition 4.16. Moreover, $\lambda \notin \overline{\Delta_{t,\mu}}$ and $T(\lambda) > t$ if $|\lambda| < r_t(\theta)$ or $|\lambda| > 1/r_t(\theta)$.

- (2) When $\theta \in [-\pi, \pi] \setminus \overline{U_{t,\bar{\mu}}}$, we know that $T(\lambda) > t$ for all λ in the ray with angle θ by Proposition 4.16. It is also clear that $\lambda \notin \overline{\Delta_{t,\mu}}$ for such λ .
- (3) Finally, when $\theta \in \partial U_{t,\bar{\mu}}$, we have $r_t(\theta) = 1$, $e^{i\theta} \in \partial\Omega_{t,\bar{\mu}} \cap \partial\Delta_{t,\mu}$ and $T(e^{i\theta}) \geq t$. Since $r \mapsto T(re^{i\theta})$ is decreasing for $0 < r < 1$, and increasing for $r > 1$, we have $\lambda \notin \overline{\Delta_{t,\mu}}$ and $T(\lambda) > t$ for any other λ that is in the ray with angle θ except $e^{i\theta}$.

By the above discussion, we see that $T(\lambda) > t$ if $\lambda \notin \overline{\Delta_{t,\mu}}$ for any θ . In addition, $T(\lambda) \geq t$ if $\lambda \in \partial\Delta_{t,\mu}$, and $T(\lambda) = t$ if $\lambda \in \partial\Delta_{t,\mu} \setminus \mathbb{T}$. Therefore, $\lambda \in \Delta_{t,\mu}$ if and only if $T(\lambda) < t$. \blacksquare

Remark 4.18. For $\lambda \in \partial\Delta_{t,\mu} \cap \mathbb{T}$ (i.e., $\arg \lambda \in [-\pi, \pi] \setminus \overline{U_{t,\bar{\mu}}}$), it is not always true that $T(\lambda) = t$ as discussed in Remark 4.12 (recall that $T(\lambda) = t_*(\lambda, 0)$). When this is true, then the boundary of $\Delta_{t,\mu}$ may be expressed as

$$\partial\Delta_{t,\mu} = \{\lambda \in \mathbb{C} : T(\lambda) = t\}.$$

We end this section with the bound of $d\phi/d\theta$, where ϕ is defined in (4.38), which will be used to give a bound of the density of the Brown measure of ub_t (see Theorem 4.28). We first establish the following lemma that will be used in establishing the bounds.

Lemma 4.19. *The function*

$$r \mapsto \frac{-r \log r}{1 - r^2}$$

is strictly increasing for $0 < r < 1$.

Proof. We will prove that $r \mapsto (1 - r^2)/(-r \log r)$ for $0 < r < 1$ is strictly decreasing, which holds, by putting $x = -\log r$, if and only if the function

$$x \mapsto \frac{1 - e^{-2x}}{xe^{-x}} = \frac{2 \sinh x}{x}$$

for $x > 0$ is strictly increasing. Since the Taylor coefficients of

$$\frac{2 \sinh x}{x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n + 1)!}$$

are all nonnegative, the function $x \mapsto 2 \sinh x/x$ is strictly increasing. ■

Recall from Section 2.4 that ω_t is a function on \mathbb{D} whose left inverse is $\Phi_{t,\bar{\mu}}$. The function ϕ defined in (4.38) is a function on \mathbb{R} , and is differentiable at θ where $r_t(\theta) < 1$. The next lemma gives an upper and a lower bound for $d\phi/d\theta$.

Lemma 4.20. *Given any $a \in \mathbb{R}$, the function ϕ defined in (4.38) is a strictly increasing continuous version of $\arg \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$ for any $\theta \in [a, a + 2\pi)$ onto some interval of the form $[b, b + 2\pi)$ for some $b \in \mathbb{R}$. If $r_t(\theta) < 1$, then ϕ is differentiable at θ , and we have*

$$0 < \frac{d\phi}{d\theta} < 2.$$

Proof. The first assertion follows from the paragraph following (4.38). Clearly, ϕ is differentiable at $\theta \in \mathbb{R}$ where $r_s(\theta) < 1$, since μ is a measure on the unit circle.

Write $w = r_s(\theta)e^{i\theta}$. Recall from Remark 2.8 that r_t is analytic at θ where $r_t(\theta) < 1$. We take the derivative with respect to θ of the identity $e^{i\phi} = \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$; we have

$$\begin{aligned} i e^{i\phi} \frac{d\phi}{d\theta} &= \Phi'_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})(r'_t(\theta)e^{i\theta} + i r_t(\theta)e^{i\theta}) \\ &= i \Phi'_{t,\bar{\mu}}(w)w \left(-i \frac{r'_t(\theta)}{r_t(\theta)} + 1 \right). \end{aligned}$$

If we divide both sides by $i \Phi'_{t,\bar{\mu}}(w)w$ and recall that $e^{i\phi} = \Phi_{t,\bar{\mu}}(w)$, we find that

$$\frac{\Phi_{t,\bar{\mu}}(w)}{w \Phi'_{t,\bar{\mu}}(w)} \frac{d\phi}{d\theta} = -i \frac{r'_t(\theta)}{r_t(\theta)} + 1.$$

Since $d\phi/d\theta$ is real, we may take the real part of both sides to get

$$\operatorname{Re} \left[\frac{\Phi_{t,\bar{\mu}}(w)}{w\Phi'_{t,\bar{\mu}}(w)} \right] \frac{d\phi}{d\theta} = 1.$$

Therefore,

$$\frac{d\phi}{d\theta} = \frac{1}{\operatorname{Re} \left[\frac{\Phi_{t,\bar{\mu}}(w)}{w\Phi'_{t,\bar{\mu}}(w)} \right]}. \quad (4.45)$$

We will show that $0 < d\phi/d\theta < 2$ by estimating $\operatorname{Re} \left[\frac{\Phi_{t,\bar{\mu}}(w)}{w\Phi'_{t,\bar{\mu}}(w)} \right]$.

We compute

$$\frac{w\Phi'_{t,\bar{\mu}}(w)}{\Phi_{t,\bar{\mu}}(w)} = 1 + \frac{t}{2} \int \frac{2w\xi}{(\xi - w)^2} d\mu(\xi),$$

so that if $r_t(\theta) < 1$, we have

$$\begin{aligned} \left| 1 - \frac{w\Phi'_{t,\bar{\mu}}(w)}{\Phi_{t,\bar{\mu}}(w)} \right| &= \left| \frac{t}{2} \int \frac{2w\xi}{(\xi - w)^2} d\mu(\xi) \right| \\ &\leq tr_t(\theta) \int \frac{1}{|\xi - w|^2} d\mu(\xi) \\ &= \frac{-2r_t(\theta) \log(r_t(\theta))}{1 - r_t(\theta)^2} < 1, \end{aligned} \quad (4.46)$$

where the last inequality follows from $r_t(\theta) < 1$ and that $r \mapsto -2(\log r)/(1 - r^2)$ is strictly increasing by Lemma 4.19.

The Möbius transform $z \mapsto 1/z$ maps $2, 1 + i, 1 - i$ to the vertical line with real part equal to $1/2$; it also maps 1 to 1 . Thus, it maps the unit disk centered at 1 to $\{z \in \mathbb{C} : \operatorname{Re} z > 1/2\}$. We conclude by (4.46) that

$$\operatorname{Re} \left[\frac{\Phi_{t,\bar{\mu}}(w)}{w\Phi'_{t,\bar{\mu}}(w)} \right] > \frac{1}{2}.$$

By (4.45), this shows that $0 < d\phi/d\theta < 2$. ■

4.5. Surjectivity

The goal of this subsection is to prove the following surjectivity result which extends the result in [20, Theorem 6.17] to our general setting. This result allows us to find unique initial conditions $\lambda_0 \in \Delta_{t,\mu}$ and $\varepsilon_0 > 0$ for each $\lambda \in \Delta_{t,\mu}$ such that $\lim_{u \uparrow t} \lambda(u) = \lambda$ and $\lim_{u \uparrow t} \varepsilon(u) = 0$. These initial conditions will help us apply the Hamilton–Jacobi formulas to compute S and ΔS at $\varepsilon = 0$.

Theorem 4.21. *Given $t > 0$, for all $\lambda \in \Delta_{t,\mu}$, there exist unique $\lambda_0 \in \mathbb{C}$ and $\varepsilon_0 > 0$ such that the solution to (4.6) with these initial conditions exists on $[0, t)$ where $t = t_*(\lambda_0, \varepsilon_0)$, with $\lim_{u \uparrow t} \varepsilon(u) = 0$ and $\lim_{u \uparrow t} \lambda(u) = \lambda$. For all $\lambda \in \Delta_{t,\mu}$, the corresponding λ_0 also belongs to $\Delta_{t,\mu}$.*

Define functions $\lambda_0^t : \Delta_{t,\mu} \rightarrow \Delta_{t,\mu}$ and $\varepsilon_0^t : \Delta_{t,\mu} \rightarrow (0, \infty)$ by letting $\lambda_0^t(\lambda)$ and $\varepsilon_0^t(\lambda)$ be the corresponding values of λ_0 and ε_0 for $\lambda \in \Delta_{t,\mu}$ respectively. Then λ_0^t and ε_0^t extend to continuous maps from $\overline{\Delta_{t,\mu}}$ into $\overline{\Delta_{t,\mu}}$ and $[0, \infty)$, respectively, with the continuous extensions satisfying $\lambda_0^t(\lambda) = \lambda$ and $\varepsilon_0^t(\lambda) = 0$ for λ in the boundary of $\Delta_{t,\mu}$.

Given $\lambda_0 \in \Delta_{t,\mu}$, we know from Corollary 4.11 that $t_*(\lambda_0, 0) < t$. On the other hand, by Proposition 4.8, the function $\varepsilon \mapsto t_*(\lambda_0, \varepsilon)$ is strictly increasing and $\lim_{\varepsilon_0 \rightarrow +\infty} t_*(\lambda_0, \varepsilon_0) = +\infty$. Hence, there is a unique value $\varepsilon_0^t(\lambda_0)$ such that $t_*(\lambda_0, \varepsilon_0^t(\lambda_0)) = t$. We set

$$\lambda_t(\lambda_0) = \lim_{u \uparrow t} \lambda(u)$$

where $\lambda(u)$ is computed with initial conditions $\lambda(0) = \lambda_0$ and $\varepsilon(0) = \varepsilon_0^t(\lambda_0)$. This defines maps

$$\begin{aligned} \varepsilon_0^t : \Delta_{t,\mu} &\rightarrow [0, \infty), & \lambda_0 &\mapsto \varepsilon_0^t(\lambda_0), \\ \lambda_t : \Delta_{t,\mu} &\rightarrow \Delta_{t,\mu}, & \lambda_0 &\mapsto \lambda_t(\lambda_0). \end{aligned}$$

Proposition 4.22. *The function ε_0^t extends continuously from $\Delta_{t,\mu}$ to $\overline{\Delta_{t,\mu}}$. The extended map satisfies $\varepsilon_0^t(\lambda_0) = 0$ for $\lambda_0 \in \partial\Delta_{t,\mu}$. Moreover, the function ε_0^t is expressed as*

$$\varepsilon_0^t(\lambda_0) = r_0 \left(\frac{r_t(\theta)^2 + 1}{r_t(\theta)} - \frac{r_0^2 + 1}{r_0} \right).$$

The formula for $\varepsilon_0^t(\lambda_0)$ is similar to [20, (6.67)].

Proof. Our strategy is to find an explicit formula for $\varepsilon_0^t(\lambda_0)$ in terms of $r_0 = |\lambda_0|$ and θ . Recall that $\theta = \arg \lambda_0$, and that $r_t(\theta)e^{i\theta} \in \partial\Delta_{t,\mu} \cap \mathbb{D}$ satisfies $T(r_t(\theta)e^{i\theta}) = t$ by Theorem 4.10. In light of Proposition 4.9, we have $t_*(r_t(\theta)e^{i\theta}, 0) = t$. Thus, we have the following relation between f_{λ_0} and $f_{r_t(\theta)e^{i\theta}}$:

$$f_{\lambda_0}(\delta_1) = \int_{-\pi}^{\pi} \frac{1}{g_{\theta_0-\alpha}(\delta_1)} d\mu(e^{i\alpha}) = \frac{1}{t} = \int_{-\pi}^{\pi} \frac{1}{g_{\theta_0-\alpha}(\delta_2)} d\mu(e^{i\alpha}) = f_{r_t(\theta)e^{i\theta}}(\delta_2),$$

where

$$\delta_1 = \frac{r_0^2 + 1 + \varepsilon_0^t(\lambda_0)}{r_0} \quad \text{and} \quad \delta_2 = \frac{r_t(\theta)^2 + 1}{r_t(\theta)}.$$

This implies that we must have $\delta_1 = \delta_2$ thanks to Proposition 4.8. That is,

$$\delta = \frac{r_0^2 + 1 + \varepsilon_0^t(\lambda_0)}{r_0} = \frac{r_t(\theta)^2 + 1}{r_t(\theta)}. \tag{4.47}$$

Hence,

$$\varepsilon_0^t(\lambda_0) = r_0 \left(\frac{r_t(\theta)^2 + 1}{r_t(\theta)} - \frac{r_0^2 + 1}{r_0} \right). \tag{4.48}$$

The desired result then follows. ■

Proposition 4.23. *The function λ_t extends continuously from $\Delta_{t,\mu}$ to $\overline{\Delta_{t,\mu}}$, with the extended map satisfying $\lambda_t(\lambda_0) = \lambda_0$ for $\lambda_0 \in \partial\Delta_{t,\mu}$. The extended map λ_t is a homeomorphism of $\overline{\Delta_{t,\mu}}$ onto itself.*

Proof. To find the formula for $\lambda_t(\lambda_0)$, we use the facts that $t_*(\lambda_0, \varepsilon_0^t(\lambda_0)) = t$ and that the argument θ of λ remains unchanged along the path. We apply (4.28) to obtain

$$\lambda_t(\lambda_0) = \frac{\lambda_0}{|\lambda_0|} e^{Ct/2}. \quad (4.49)$$

By formula (4.18) for C and formula (4.25) for $t_*(\lambda_0, \varepsilon_0)$, we have

$$Ct = Ct_*(\lambda_0, \varepsilon_0^t(\lambda_0)) = \frac{\delta - 2/|\lambda_0|}{\sqrt{\delta^2 - 4}} \log\left(\frac{\delta + \sqrt{\delta^2 - 4}}{\delta - \sqrt{\delta^2 - 4}}\right), \quad (4.50)$$

where $\delta = (r_t(\theta)^2 + 1)/r_t(\theta)$ as in (4.47). We remark that the formula for $\lambda_t(\lambda_0)$ is similar to [20, (6.64)].

By (4.47), δ only depends on θ but not on r_0 ; hence, Ct is strictly increasing in r_0 by (4.50). From the expression (4.49), we then deduce that, fixing a $\theta \in U_{t,\mu}$, the function $|\lambda_0| \mapsto |\lambda_t(\lambda_0)|$ is strictly increasing for $\lambda_0 \in \{re^{i\theta} : r_t(\theta) < r < 1/r_t(\theta)\}$. Moreover, when $\lambda_0 = r_t(\theta)e^{i\theta}$, we have

$$\lambda_t(r_t(\theta)e^{i\theta}) = e^{i\theta} e^{Ct/2} = r_t(\theta)e^{i\theta}$$

and $\lambda_t(e^{i\theta}/r_t(\theta)) = e^{i\theta}/r_t(\theta)$ due to (4.28) and (4.34) (which can also be verified directly). In other words, λ_t maps the interval $\{re^{i\theta} : r_t(\theta) < r < 1/r_t(\theta)\}$ bijectively to itself and fixes the endpoints. Since this holds for any $\theta \in U_{t,\mu}$, we conclude that λ_t maps $\overline{\Delta_{t,\mu}} \setminus (\partial\Delta_{t,\mu} \cap \mathbb{T})$ bijectively to itself and fixes any $\lambda \in \partial\Delta_{t,\mu} \setminus \mathbb{T}$. As λ_t is continuous and $r_t(\theta) \rightarrow 1$ as $e^{i\theta}$ approaches $\partial\Delta_{t,\mu} \cap \mathbb{T}$, we then conclude that the inverse of λ_t extends to a homeomorphism of $\overline{\Delta_{t,\mu}}$. ■

Proof of Theorem 4.21. It is a direct consequence of Propositions 4.22 and 4.23. ■

4.6. The Brown measure of g_t and its connection to uu_t

We will show in Theorem 4.28 that the Brown measure of ub_t is supported on the closure $\overline{\Delta_{t,\mu}}$ by showing that the Brown measure puts mass 1 on $\Delta_{t,\mu}$. We prove this by first showing that a certain measure on $\Delta_{t,\mu}$ can be pushed forward to the spectral distribution of uu_t in Lemma 4.25; hence this measure is a probability measure. Then, in Theorem 4.28, we show that the Brown measure has the same formula on $\Delta_{t,\mu}$ as the one defined in Lemma 4.25. It then follows that the Brown measure puts mass 1 on $\Delta_{t,\mu}$. Thus, it suffices to focus on the computation for $\lambda \in \Delta_{t,\mu}$ in the section.

Our first goal is to calculate the Laplacian of the function $s_t(\lambda)$ defined in Section 4.1 for fixed t by using Hamilton–Jacobi analysis. Recall that

$$s_t(\lambda) := \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon)$$

for t fixed. By Theorem 4.21, for each $t > 0$ and $\lambda \in \Delta_{t,\mu}$, we can choose $\varepsilon_0 > 0$ and $\lambda_0 \in \Delta_{t,\mu}$ such that $\lim_{u \rightarrow t} \varepsilon(u) = 0$, $\lim_{u \rightarrow t} \lambda(u) = \lambda$ where $t = t_*(\lambda_0, \varepsilon_0)$. Moreover, as the argument of λ remains unchanged along the flow by Proposition 4.4, we have $\arg \lambda = \arg \lambda_0$. We note that the Laplacian in logarithmic polar coordinates has the form

$$\Delta = \Delta_\lambda = \frac{1}{r^2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2} \right),$$

where we recall that $r = |\lambda|$, $\rho = \log |\lambda|$, and $\theta = \arg \lambda$.

We consider the function on $U_{t,\bar{\mu}} = \Delta_{t,\mu} \cap \mathbb{T}$ defined by

$$w_t(\theta) = \frac{1}{4\pi} \left(\frac{2}{t} + \frac{d}{d\theta} m_t(\theta) \right) \tag{4.51}$$

where

$$m_t(\theta) = \int_{-\pi}^\pi \frac{2r_t(\theta) \sin(\theta - \alpha)}{r_t(\theta)^2 + 1 - 2r_t(\theta) \cos(\theta - \alpha)} d\mu(e^{i\alpha}). \tag{4.52}$$

Recall that $r_t(\theta)$ is defined in (2.16), which is the smaller of the radii where the ray with angle θ intersects the boundary of $\Delta_{t,\mu}$, and μ is the spectral distribution of u . The functions w_t and m_t will play the main role in computing the Brown measure of ub_t .

Proposition 4.24. *For any $\theta \in \Delta_{t,\mu} \cap \mathbb{T}$,*

$$w_t(\theta) = \frac{1}{2\pi t} \frac{d\phi}{d\theta}, \tag{4.53}$$

where $\phi = \arg(\Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta}))$ is defined by (4.38).

Proof. Recall that by (4.38),

$$\phi = \theta + t \int_{-\pi}^\pi \frac{r_t(\theta) \sin(\theta - x)}{|1 - r_t(\theta)e^{i(\theta-x)}|^2} d\mu(e^{ix}).$$

Hence,

$$\begin{aligned} d\phi &= \left(1 + t \frac{d}{d\theta} \int_{-\pi}^\pi \frac{r_t(\theta) \sin(\theta - x)}{|1 - r_t(\theta)e^{i(\theta-x)}|^2} d\mu(e^{ix}) \right) d\theta \\ &= \left(1 + \frac{t}{2} \frac{d}{d\theta} m_t(\theta) \right) d\theta = 2\pi t w_t(\theta) d\theta. \end{aligned}$$

and the formula (4.53) follows. ■

We next show that the measure $w_t(\theta)/r^2$ puts mass 1 on $\Delta_{t,\mu}$. Later, we will show that $w_t(\theta)/r^2$ is indeed the density of the Brown measure of ub_t inside $\Delta_{t,\mu}$. That it puts mass 1 on $\Delta_{t,\mu}$ will be used to show that the Brown measure is 0 outside $\Delta_{t,\mu}$.

Lemma 4.25. *We have*

$$\int_{\Delta_{t,\mu}} \frac{1}{r^2} w_t(\theta) r dr d\theta = 1.$$

Proof. Using the characterization of $\Delta_{t,\mu}$ in Theorem 4.10, we have

$$\begin{aligned}
 \int_{\Delta_{t,\mu}} \frac{1}{r^2} w_t(\theta) r \, dr \, d\theta &= \int_{U_{t,\bar{\mu}}} \int_{r_t(\theta)}^{1/r_t(\theta)} \frac{1}{r^2} w_t(\theta) r \, dr \, d\theta \\
 &= \int_{U_{t,\bar{\mu}}} -2 \log(r_t(\theta)) w_t(\theta) \, d\theta \\
 &= \int_{U_{t,\bar{\mu}}} -2 \log(r_t(\theta)) \frac{1}{2\pi t} \frac{d\phi}{d\theta} \, d\theta \quad (\text{by Proposition 4.24}) \\
 &= \int_{\mathbb{T}} -\frac{1}{\pi t} \log(r_t(\theta)) \, d\phi \\
 &= \int_{\mathbb{T}} p_t(e^{i\phi}) \, d\phi \quad (\text{by Theorem 2.11}) \\
 &= 1. \quad \blacksquare
 \end{aligned}$$

Given any $\lambda \in \Delta_{t,\mu}$, by Theorem 4.21, there are unique λ_0 and ε_0 such that

$$\lim_{u \rightarrow t} (u, \lambda(u), \varepsilon(u)) = (t, \lambda, 0).$$

We attempt to compute the Brown measure as the limit of $\frac{1}{4\pi} \Delta S(u, \lambda(u), \varepsilon(u))$ as $u \uparrow t$. However, the definition of the Brown measure of ub_t is

$$\frac{1}{4\pi} \Delta \left(\lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon) \right),$$

with t and λ fixed in the limiting process. We want to show that this limit is the same as

$$\lim_{u \uparrow t} \frac{1}{4\pi} \Delta S(u, \lambda(u), \varepsilon(u)).$$

To achieve this, we intend to show that the limit is independent of the path approaching $(t, \lambda, \varepsilon)$. We will see that the analogue of [20, Theorem 7.4] holds for our S . More precisely, given any $(\sigma, \omega) \in \mathbb{R}^+ \times \Delta_{t,\mu}$, the function

$$\tilde{S}(t, \lambda, z) = S(t, \lambda, z^2), \quad z > 0, \quad (4.54)$$

extends to a real analytic function in a neighborhood of $(\sigma, \lambda, 0)$ inside $\mathbb{R} \times \mathbb{C} \times \mathbb{R}$. The key here is that the regularity holds even in the triple (t, λ, z) , not just in the pair (λ, z) ; the Laplacian of S at $(t, \lambda, 0)$ is then equal to the limit of the Laplacian along the path $(u, \lambda(u), \varepsilon(u))$ since there is no partial derivative with respect to ε involved. We will give the main lines below. For more details, the readers are encouraged to consult [20, Section 7.4].

Theorem 4.26. *The function \tilde{S} extends to a real analytic function in a neighborhood of $(t, \lambda, 0)$ in $\mathbb{R}^+ \times \Delta_{t,\mu}$.*

Remark 4.27. The function S itself does not have a smooth extension of that sort. Indeed, since $\sqrt{\varepsilon} p_\varepsilon$ is a constant of motion, the second Hamilton–Jacobi formula (3.12) tells us that $\partial S/\partial \varepsilon$ must blow up like $1/\sqrt{\varepsilon}$ as we approach $(t, \lambda, 0)$ along a solution of the ODEs. The same reasoning tells us that the extended \tilde{S} does not satisfy $\tilde{S}(t, \lambda, z) = S(t, \lambda, z^2)$ for $z < 0$. Because $\sqrt{\varepsilon} p_\varepsilon$ is a constant of motion,

$$\frac{\partial \tilde{S}}{\partial z}(t, \lambda, z) = 2\sqrt{\varepsilon} \frac{\partial S}{\partial \varepsilon}(t, \lambda, z^2)$$

has a nonzero limit as $z \rightarrow 0$. Thus, \tilde{S} cannot have a smooth extension that is even in z .

Proof of Theorem 4.26. Denote by $\lambda(t; \lambda_0, \varepsilon_0)$ λ -component of the solution of the ODEs at time t given initial conditions λ_0 and ε_0 . Write $z(t; \lambda_0, \varepsilon_0) = \sqrt{\varepsilon(t; \lambda_0, \varepsilon_0)}$ where $\varepsilon(t; \lambda_0, \varepsilon_0)$ is ε -component of the solution of the ODEs at time t given initial conditions λ_0 and ε_0 .

Define the map

$$V(t, \lambda_0, \varepsilon_0) = (t, \lambda(t; \lambda_0, \varepsilon_0), z(t; \lambda_0, \varepsilon_0)).$$

We first show that V can be extended to $t \in \mathbb{R}$, given any λ_0, ε_0 . The main observation is that, given any $\varepsilon_0 > 0$, $1/p_\varepsilon(t; \lambda_0, \varepsilon_0)$ extends to a real analytic function for all $t \in \mathbb{R}$ (see (4.22) for the formula for p_ε). Given any $\lambda_0 \in \Delta_{t, \mu}$ and $\varepsilon_0 > 0$, we can extend $z(t; \lambda_0, \varepsilon_0)$ by the same formula

$$z(t; \lambda_0, \varepsilon_0) = \sqrt{\varepsilon_0} p_0 e^{-Ct/2} \frac{1}{p_\varepsilon(t; \lambda_0, \varepsilon_0)} \tag{4.55}$$

to all $t \in \mathbb{R}$. By (4.55), $z(t; \lambda_0, \varepsilon_0)^2 = \varepsilon(t; \lambda_0, \varepsilon_0)$ and so $z(t; \lambda, \varepsilon_0) = 0$ when $t = t_*(\lambda_0, \varepsilon_0)$. The function $z(t; \lambda_0, \varepsilon_0)$ is positive when $t < t_*(\lambda_0, \varepsilon_0)$; it is negative when $t > t_*(\lambda_0, \varepsilon_0)$. Since, $1/p_\varepsilon(t; \lambda_0, \varepsilon_0)$ extends to a real analytic function for all $t \in \mathbb{R}$, by (4.15) and Proposition 4.5 the function

$$\lambda(t) = \lambda_0 \exp\left(\int_0^t \varepsilon(s) p_\varepsilon(s) ds\right) = \lambda_0 \exp\left(\int_0^t \frac{\varepsilon_0 p_0^2 e^{-Cs}}{p_\varepsilon(s)} ds\right)$$

also extends to an analytic function of $t \in \mathbb{R}$.

Fix any $\lambda \in \Delta_{t, \mu}$. Next, we show that we can extend \tilde{S} to a neighborhood of $(t, \lambda, 0)$, where $\lambda \in \Delta_{t, \mu}$, by applying the inverse function theorem to the map

$$V(t, \lambda_0, \varepsilon_0) = (t, \lambda(t; \lambda_0, \varepsilon_0), z(t; \lambda_0, \varepsilon_0)),$$

which was shown to be extendable to $t \in \mathbb{R}$ in the preceding paragraph. Once we show that the Jacobian matrix of V at $(t, \lambda_0^t(\lambda), \varepsilon_0^t(\lambda))$ is invertible (see Theorem 4.21 for the definitions of λ_0^t and ε_0^t), there exists a local inverse V^{-1} defined around a neighborhood of $(t, \lambda, 0)$ which satisfies

$$\tilde{S}(\sigma, \omega, z) = HJ(V^{-1}(\sigma, \omega, z)) \tag{4.56}$$

where HJ is the right hand side of (4.13). Note that (4.56) gives a (real) analytic extension of \tilde{S} around a neighborhood of $(t, \lambda, 0)$; recall that \tilde{S} is originally defined only for $z > 0$ in (4.54).

Thus, it remains to show that the Jacobian matrix of V at $(t, \lambda_0^t(\lambda), \varepsilon_0^t(\lambda))$ is invertible. The trick here is to do a change of variable to view the map V as a function of $(t, \theta, \rho, \delta)$, since, by (4.19), the map $(t, \lambda, \varepsilon) \mapsto (t, \theta, \rho, \delta)$ is smoothly invertible. Because the formula for $t_*(\lambda_0, \varepsilon_0)$ is independent of $\rho_0 = \log r_0$ when δ and θ are fixed, it follows that when $t_*(\lambda_0, \varepsilon_0) = t$, $z(t; \lambda_0, \varepsilon_0)$ remains 0 if r_0 varies with δ and θ fixed; this shows $\partial z / \partial r_0 = 0$. Furthermore, by (4.49) and (4.50),

$$\lim_{t \rightarrow t_*(\lambda_0, \varepsilon_0)} \rho(t) = \frac{\delta - 2/r_0}{2\sqrt{\delta^2 - 4}} \log \left(\frac{\delta + \sqrt{\delta^2 - 4}}{\delta - \sqrt{\delta^2 - 4}} \right)$$

whose partial derivative with respect to r_0 is positive. Since $\frac{\partial \rho}{\partial r_0} > 0$ as shown in the proof of Proposition 4.23, it remains to check $\frac{\partial z}{\partial \delta} > 0$ to prove that the Jacobian matrix

$$\begin{pmatrix} I_{2 \times 2} & 0 & 0 \\ * & \frac{\partial \rho}{\partial r_0} & \frac{\partial \rho}{\partial \delta} \\ * & \frac{\partial z}{\partial r_0} & \frac{\partial z}{\partial \delta} \end{pmatrix} = \begin{pmatrix} I_{2 \times 2} & 0 & 0 \\ * & \frac{\partial \rho}{\partial r_0} & \frac{\partial \rho}{\partial \delta} \\ * & 0 & \frac{\partial z}{\partial \delta} \end{pmatrix}$$

is invertible. To this end, we write

$$\frac{\partial z}{\partial \delta} = \frac{\partial z}{\partial \varepsilon_0} \frac{\partial \varepsilon_0}{\partial \delta} = -r_0 \frac{\partial z}{\partial t} \frac{\partial t_*(\lambda_0, \varepsilon_0)}{\partial \varepsilon_0}$$

where the last equality comes from differentiating $z(t_*(\lambda_0, \varepsilon_0), \lambda_0, \varepsilon_0) = 0$ with respect to ε_0 using the chain rule and $\frac{\partial \varepsilon_0}{\partial \delta} = r_0$. Now, by (4.22) and (4.55), we have

$$z(t) = \frac{\sqrt{\varepsilon_0} e^{Ct/2}}{\cosh(kt) + \frac{2r_0 - \delta}{\sqrt{\delta^2 - 4}} \sinh(kt)} \left(\cosh(kt) - \frac{\delta}{\sqrt{\delta^2 - 4}} \sinh(kt) \right).$$

Recall that t_* given in (4.24) is chosen such that the denominator in (4.22) is zero. When $t = t_*$, we have

$$\begin{aligned} & \left. \frac{\partial z(t, \lambda_0, \varepsilon_0)}{\partial t} \right|_{t=t_*(\lambda_0, \varepsilon_0)} \\ &= \frac{\sqrt{\varepsilon_0} e^{Ct/2}}{\cosh(kt) + \frac{2r_0 - \delta}{\sqrt{\delta^2 - 4}} \sinh(kt)} k \left(\sinh(kt) - \frac{\delta}{\sqrt{\delta^2 - 4}} \cosh(kt) \right) < 0 \end{aligned}$$

because the denominator is positive (see discussion after (4.22) for the definition of $p_\varepsilon(t)$) and $\delta / \sqrt{\delta^2 - 4} > 0$. Finally, using $t_*(\lambda_0, \varepsilon_0) = 1 / \int_{-\pi}^{\pi} \frac{1}{g_{\theta_0 - \alpha}(\delta)} d\mu(e^{i\alpha})$, Proposition 4.8 and the definition (4.19) of δ , we obtain

$$\frac{\partial t_*(\lambda_0, \varepsilon_0)}{\partial \varepsilon_0} = \frac{\partial t_*(\lambda_0, \varepsilon_0)}{\partial \delta} \frac{\partial \delta}{\partial \varepsilon_0} > 0.$$

We conclude that $\frac{\partial z}{\partial \delta} > 0$ and our proof is complete. \blacksquare

Recall that we want to compute the distributional Laplacian of the function

$$s_t(\lambda) = \lim_{\varepsilon \downarrow 0} S(t, \lambda, \varepsilon).$$

Theorem 4.26 shows that s_t is indeed analytic on $\Delta_{t,\mu}$ and hence the distributional Laplacian of s_t is the ordinary Laplacian.

The following is our main theorem in the multiplicative case, which generalizes [20, Theorem 2.2].

Theorem 4.28. *Given any $t > 0$, for any $\lambda \in \Delta_{t,\mu}$ with $\lambda = e^\rho e^{i\theta} = r e^{i\theta}$, we have*

$$\frac{\partial s_t}{\partial \rho}(t, \lambda) = \frac{2\rho}{t} + 1. \tag{4.57}$$

Furthermore, $\partial s_t / \partial \theta$ is independent of ρ and can be expressed as

$$\frac{\partial s_t}{\partial \theta} = m_t(\theta)$$

where m_t is defined in (4.52).

The Brown measure μ_{g_t} of ub_t is supported on $\overline{\Delta_{t,\mu}}$ and can be expressed as

$$W_t(r, \theta) = \frac{1}{4\pi} \Delta s_t(\lambda) = \frac{1}{r^2} w_t(\theta). \tag{4.58}$$

Moreover, in $\Delta_{t,\mu}$, the density W_t of μ_{g_t} with respect to the Lebesgue measure is strictly positive and real analytic. We also have

$$w_t(\theta) < \frac{1}{\pi t}. \tag{4.59}$$

Proof. For any $\lambda \in \Delta_{t,\mu}$, choose $\lambda_0 = \lambda_0^t(\lambda)$ and $\varepsilon_0 = \varepsilon_0^t(\lambda)$ as in Theorem 4.21. Hence $t = t_*(\lambda_0, \varepsilon_0)$. Recall that S is a function of t, ρ , and θ . Also recall that while the Hamiltonian H does not depend on θ or p_θ , we can still regard θ and p_θ as constants of motion. Over the trajectory of S which solves the system (4.6) over the interval $[0, t]$, the momentum p_θ remains unchanged by Proposition 4.4. Write $\lambda_0 = r_0 e^{i\theta}$. Then

$$\begin{aligned} \frac{\partial S}{\partial \theta}(u, \lambda(u), \varepsilon(u)) &= p_\theta = \int_{-\pi}^\pi \frac{2r_0 \sin(\theta - \alpha)}{1 + r_0^2 - 2r_0 \cos(\theta - \alpha) + \varepsilon_0} d\mu(e^{i\alpha}) \\ &= 2 \int_{-\pi}^\pi \frac{\sin(\theta_0 - \alpha)}{\delta - 2 \cos(\theta_0 - \alpha)} d\mu(e^{i\alpha}). \end{aligned}$$

Using the fact that ε_0 is chosen so that $\delta = (r_t(\theta)^2 + 1)/r_t(\theta)$ as in (4.47), the above expression is independent of $|\lambda|$ and only depends on θ . Hence, the function m_t defined in (4.52) can be written as

$$m_t(\theta) = 2 \int_{-\pi}^\pi \frac{r_t(\theta) \sin(\theta - \alpha)}{r_t(\theta)^2 + 1 - 2 \cos(\theta - \alpha)} d\mu(e^{i\alpha}) = 2 \int_{-\pi}^\pi \frac{\sin(\theta_0 - \alpha)}{\delta - 2 \cos(\theta_0 - \alpha)} d\mu(e^{i\alpha}),$$

which shows

$$\frac{\partial^2 S}{\partial \theta^2}(u, \lambda(u), \varepsilon(u)) = \frac{d}{d\theta} m_t(\theta).$$

Because $s_t(\lambda) = \lim_{z \rightarrow 0} \tilde{S}(t, \lambda, z)$ (see (4.56) for the definition of \tilde{S}), Theorem 4.26 implies that taking the limit $u \rightarrow t$ of the above displayed equation gives us

$$\frac{\partial^2 s_t}{\partial \theta^2} = \frac{d}{d\theta} m_t(\theta). \quad (4.60)$$

Similarly, we use (4.29) and Theorem 4.26 to get

$$\frac{\partial s_t}{\partial \rho} = \lim_{u \rightarrow t} \frac{2 \log |\lambda(t)|}{t} + 1 = \frac{2\rho}{t} + 1.$$

It follows that

$$\frac{\partial^2 s_t}{\partial \rho^2} = \frac{2}{t}.$$

So, the restriction of the Brown measure μ_{g_t} to $\Delta_{t,\mu}$ is given by

$$d(\mu_{g_t}|_{\Delta_{t,\mu}})(\lambda) = \frac{1}{4\pi} \Delta s_t(\lambda) = \frac{1}{4\pi} \frac{1}{r^2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2} \right) s_t(\lambda) = \frac{1}{r^2} w_t(\theta). \quad (4.61)$$

As μ_{g_t} is a probability measure, it then follows from Lemma 4.25 that μ_{g_t} is supported on $\Delta_{t,\mu}$. Recall that $r_t(\theta) < 1$ and is analytic for all $\theta \in U_{t,\bar{\mu}}$. We conclude that W_t is strictly positive (by Proposition 4.24 and $\frac{d\phi}{d\theta} > 0$) and analytic for all $\lambda \in \Delta_{t,\mu}$. Finally, the upper bound (4.59) follows from Lemma 4.20 and Proposition 4.24. ■

We point out that it is possible to express the formula (4.51) for w_t in an alternative way so that there is no derivative involved. Indeed, when $\theta \in U_{t,\bar{\mu}}$, recall that $r_t(\theta) < 1$ and $r_t(\theta)$ is analytic by Proposition 2.9. We first calculate

$$\begin{aligned} \frac{1}{2} \frac{d}{d\theta} m_t(\theta) &= \frac{dr_t(\theta)}{d\theta} \cdot \int_{-\pi}^{\pi} \frac{\sin(\theta - x)(1 - r_t(\theta)^2)}{(r_t(\theta)^2 + 1 - 2r_t(\theta) \cos(\theta - x))^2} d\mu(e^{ix}) \\ &\quad + \int_{-\pi}^{\pi} \frac{\cos(\theta - x)(r_t(\theta)^3 + r_t(\theta)) - 2r_t(\theta)^2}{(r_t(\theta)^2 + 1 - 2r_t(\theta) \cos(\theta - x))^2} d\mu(e^{ix}). \end{aligned} \quad (4.62)$$

Next, we recall, from Proposition 2.9 and the definition of $f(r, \theta)$ given in (4.40), that when $r_t(\theta) < 1$, $r_t(\theta)$ satisfies the identity

$$f(r_t(\theta), \theta) = \frac{1 - r_t(\theta)^2}{-2 \log r_t(\theta)} \int_{-\pi}^{\pi} \frac{1}{|1 - r_t(\theta) e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) = \frac{1}{t}. \quad (4.63)$$

Then $\frac{dr_t(\theta)}{d\theta}$ can be calculated by the differential of the above implicit function:

$$\frac{dr_t(\theta)}{d\theta} = - \frac{\partial f(r, \theta) / \partial \theta}{\partial f(r, \theta) / \partial r} \Big|_{r=r_t(\theta)},$$

where the denominator is strictly positive whenever $r_t(\theta) < 1$, by Corollary 4.14. The expression in the above displayed equation is rather complicated. The numerator can be computed as

$$\frac{\partial f(r, \theta)}{\partial \theta} = \frac{1 - r^2}{-2 \log r} \int_{-\pi}^{\pi} \frac{2r \sin(\theta + x)}{|1 - r_t(\theta)e^{i(\theta+x)}|^4} d\bar{\mu}(e^{ix}),$$

while the denominator can be computed as

$$\begin{aligned} \frac{\partial f(r, \theta)}{\partial r} &= \left(\frac{1 - r^2}{2r(\log r)^2} + \frac{r}{\log r} \right) \int_{-\pi}^{\pi} \frac{1}{|1 - r_t(\theta)e^{i(\theta+x)}|^2} d\bar{\mu}(e^{ix}) \\ &\quad - \frac{1 - r^2}{-2 \log r} \int_{-\pi}^{\pi} \frac{2r + 2 \cos(\theta + x)}{|1 - r_t(\theta)e^{i(\theta+x)}|^4} d\bar{\mu}(e^{ix}). \end{aligned}$$

Plugging the above formulas for $\frac{dr_t(\theta)}{d\theta}$ into (4.62) and (4.51), we can obtain an alternative expression for w_t . We remark that in the special case when $u = I$, this expression is a very elegant formula (see [20, Theorem 8.2]).

Corollary 4.29. *The support of the Brown measure μ_{g_t} of $g_t = ub_t$ is the closure of the open set $\Delta_{t,\mu}$. The number of connected components of $\Delta_{t,\mu}$ is a nonincreasing function of t .*

Proof. The first assertion follows from μ_{g_t} putting mass 1 on $\Delta_{t,\mu}$. The second assertion follows from the fact that $\Delta_{t,\mu}$ has the same number of connected components as $U_{t,\bar{\mu}}$. (See (2.15) for the definition of $U_{t,\bar{\mu}}$.) Now, the number of connected components of $U_{t,\bar{\mu}}$ is nonincreasing by Proposition 2.12. ■

We now describe the connection between the Brown measure of $g_t = ub_t$ with the density function of the spectral distribution of uu_t obtained in [45] by the second author. The following two results generalize [20, Propositions 2.5 and 2.6].

Corollary 4.30. *The distribution of the argument of λ with respect to μ_{g_t} has a density given by*

$$a_t(\theta) = -2 \log[r_t(\theta)]w_t(\theta). \tag{4.64}$$

Furthermore, the push-forward of μ_{g_t} under the map $\lambda \mapsto \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$ is the distribution of uu_t .

Proof. The Brown measure is computed in polar coordinates as $(1/r^2)w_t(\theta)r dr d\theta$ in the domain. Integrating with respect to r from $r_t(\theta)$ to $1/r_t(\theta)$ then gives the claimed density for θ . The last assertion follows from a computation similar to Lemma 4.25. ■

Proposition 4.31. *The Brown measure of ub_t is the unique measure σ on $\overline{\Delta_{t,\mu}}$ with the following two properties: (1) the push-forward of σ by the map $\lambda \mapsto \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$ is the distribution of uu_t where $\theta = \arg \lambda$, and (2) σ is absolutely continuous with respect to the Lebesgue measure and its density is given by*

$$W(r, \theta) = \frac{1}{r^2}g(\theta)$$

in polar coordinates, where g is a continuous function.

Proof. Suppose σ is a measure on $\overline{\Delta_{t,\mu}}$ satisfying the above two properties. We have

$$\int_{r_t(\theta)}^{1/r_t(\theta)} \frac{1}{r^2} g(\theta) dr = -2 \log[r_t(\theta)]g(\theta).$$

Hence, using (4.53), we deduce the density of the push-forward measure of σ as

$$-2 \log[r_t(\theta)]g(\theta) \frac{d\theta}{d\phi} = \frac{-\log[r_t(\theta)]}{\pi t} \frac{g(\theta)}{w_t(\theta)} d\phi.$$

By comparing the density formula for uu_t in Theorem 2.11, and noticing that $r_t(\theta) < 1$ for $\theta \in U_{t,\bar{\mu}}$, we must have $g(\theta) = w_t(\theta)$. ■

We include the matrix simulations of ub_t at time $t = 0.2$ and $t = 0.8$ where u has spectral distribution

$$\mu = \frac{1}{3}\delta_{e^{-4\pi i/15}} + \frac{2}{3}\delta_{e^{4\pi i/15}}.$$

We again emphasize that it is still an open problem to prove mathematically that the Brown measure of ub_t is the weak limit of the empirical eigenvalue distribution of $U_N G_N(t)$ where $G_N(t)$ is the Brownian motion on $GL(N, \mathbb{C})$ and U_N is a deterministic (or random but independent of $G_N(t)$) unitary matrix whose empirical eigenvalue distribution has weak limit μ .

In each of Figures 6 and 7, part (a) plots the eigenvalues of ub_t against the unit circle, and the curves $\omega_t(e^{i\phi})$, $1/\overline{\omega_t(e^{i\phi})}$. Part (b) of the figures shows the eigenvalues of ub_t and the curves $\omega_t(e^{i\phi})$, $1/\overline{\omega_t(e^{i\phi})}$ after pushing forward by the principal logarithm. We can see that the density of the points is constant along the horizontal direction. Part (c) shows the histogram of the argument of eigenvalues after pushing forward by the map $\lambda \mapsto \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$ against the theoretical distribution—the spectral distribution of uu_t —in argument. We remark that the spectral distribution of uu_t is established in Theorem 2.11.

4.7. The Brown measure of hb_t

In this section, we calculate the Brown measure of hb_t where h is Haar unitary as an example. Recall from Section 2.4 that when u is a Haar unitary h , we have

$$\Sigma_{t,\bar{h}}(z) = e^{t/2} \quad \text{and} \quad \Phi_{t,\bar{h}}(z) = ze^{t/2}, \quad (4.65)$$

and $r_t(\theta) = e^{-t/2}$ for all θ . Hence $\Phi_{t,\bar{h}}(r_t(\theta)e^{i\theta}) = e^{i\theta}$ and

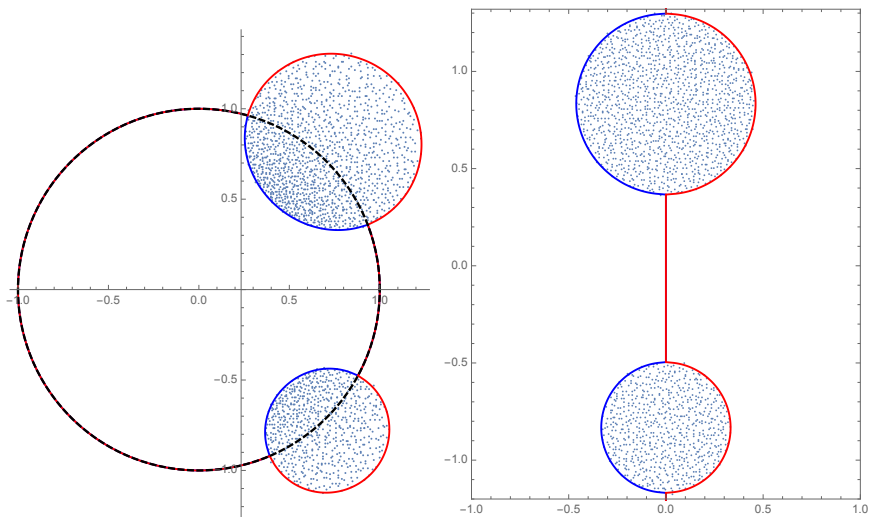
$$\phi = \phi(\theta) = \arg(\Phi_{t,\bar{h}}(r_t(\theta)e^{i\theta})) = \theta.$$

In addition,

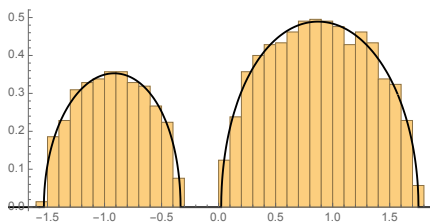
$$\Omega_{t,\bar{h}} = \{z : |z| < e^{-t/2}\},$$

and the set $\Delta_{t,\bar{h}}$ is the annulus

$$\Delta_{t,\bar{h}} = \{z : e^{-t/2} < |z| < e^{t/2}\}. \quad (4.66)$$



(a) Eigenvalue plot for ub_t and the curves $\omega_t(e^{i\phi})$ (blue), $1/\omega_t(e^{i\phi})$ (red) and the unit circle (dashed black) (b) Push-forward of eigenvalues ub_t by principal logarithm and the curves $\log(\omega_t(e^{i\phi}))$, $-\log(\omega_t(e^{i\phi}))$ and the unit circle



(c) Histogram of the argument of the eigenvalues ub_t pushed forward by $\lambda \mapsto \Phi_{t, \bar{\mu}}(r_t(\theta))e^{i\theta}$. The distribution of uu_t is superimposed.

Fig. 6. 2100×2100 matrix simulations for uu_t at $t = 0.3$, where u is distributed as $\frac{1}{3}\delta_{e^{-4\pi i/15}} + \frac{2}{3}\delta_{e^{4\pi i/15}}$.

Therefore, by (4.53), we have

$$w_t(\theta) = \frac{1}{2\pi t}.$$

Finally, by Theorem 4.28, we have the following formula for the Brown measure of hb_t .

Theorem 4.32. *The Brown measure μ_{hb_t} of hb_t is the annulus law. It is supported in the annulus $A_t = \{e^{-t/2} \leq |z| \leq e^{t/2}\}$, rotationally invariant, and the density W_t of μ_{hb_t} in polar coordinates is given by*

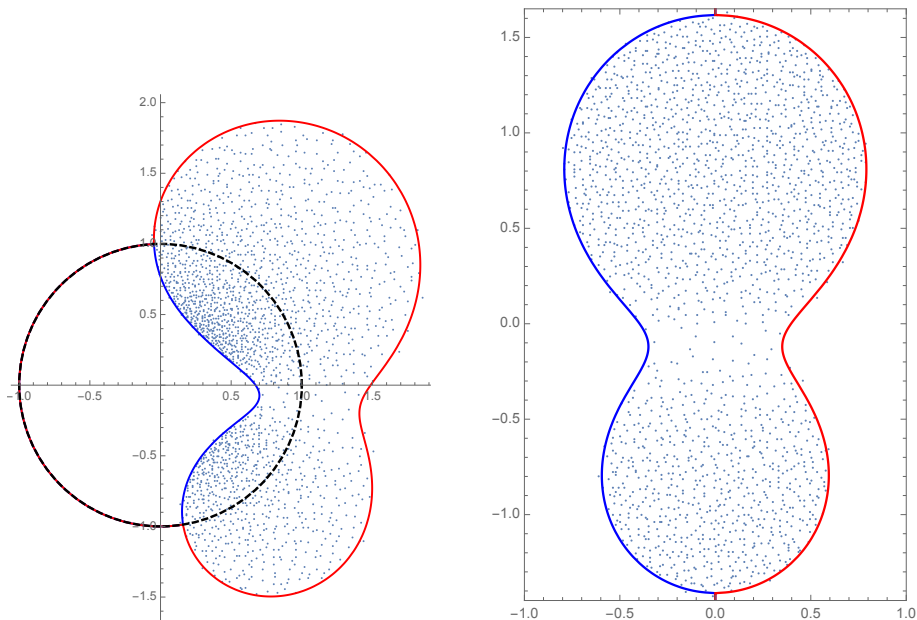
$$W_t(r, \theta) = \frac{1}{2\pi t} \frac{1}{r^2}, \tag{4.67}$$

which is independent of θ .

Remark 4.33. The name *annulus law* was coined by Driver, Hall and Kemp. It is expected that the solution of

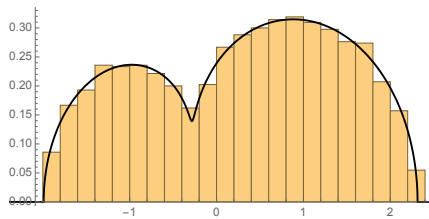
$$dG_N(t) = G_N(t) dZ_N(t)$$

under the initial condition that $G_N(0)$ is the Haar measure on the unitary group has the limiting eigenvalue distribution equal to the Brown measure of hb_t . One can run a computer program to see that the eigenvalues of $G_N(t)$ are distributed on an annulus A_t with inner radius $e^{-t/2}$ and outer radius $e^{t/2}$. The support A_t also occurs in the discussion of the free Hall transform as $s \rightarrow \infty$ [31, Corollary 4.26].



(a) Eigenvalue plot for ub_t and the curves $\omega_t(e^{i\phi})$ (blue), $1/\overline{\omega_t(e^{i\phi})}$ (red) and the unit circle (dashed black)

(b) Push-forward of eigenvalues ub_t by logarithm (with branch cut $(0, \infty)$) and the curves $\log(\omega_t(e^{i\phi}))$, $-\log(\overline{\omega_t(e^{i\phi})})$ and the unit circle



(c) Histogram of the argument of the eigenvalues ub_t pushed forward by $\lambda \mapsto \Phi_{t,\bar{\mu}}(r_t(\theta)e^{i\theta})$. The distribution of uu_t is superimposed.

Fig. 7. 2100×2100 matrix simulations for uu_t at $t = 0.8$, where u is distributed as $\frac{1}{3}\delta_{e^{-4\pi i/15}} + \frac{2}{3}\delta_{e^{4\pi i/15}}$.

Appendix A. The Brown measure of hb_t and R -diagonal operators

In this section, we provide an alternative way to calculate the Brown measure of hb_t , the free multiplicative Brownian motion with the Haar unitary as the initial condition, using Haagerup and Larsen’s formula. The operator hb_t is an R -diagonal operator [36, Corollary 15.9] and its Brown measure can be characterized by the S -transform of $b_t b_t^*$ [24, Theorem 4.4]. (See (2.4) for the definition of S -transform.) The S -transform of $b_t b_t^*$ has been calculated by Biane [11] using a different (but equivalent) normalization. We put these work together to give an explicit formula for the Brown measure of hb_t .

For every $t \in \mathbb{R}$, it is known by [7, Lemmas 6.3 and 7.1] that the functions

$$S(z) = \exp(t(z + 1/2))$$

are the S -transforms of probability measures γ_t (see (2.4) for the definition of S -transform). The corresponding probability measures are supported on the unit circle \mathbb{T} for $t > 0$, and the corresponding probability measures are supported on the positive half-line \mathbb{R}^+ for $t < 0$. For a probability measure μ on \mathbb{T} or on \mathbb{R}^+ , the S -transform S_μ of μ is equivalent to the Σ -transform Σ_μ

$$\Sigma_\mu(z) = S_\mu\left(\frac{z}{1-z}\right).$$

Converting the S -transform to the Σ -transform, we have a family of probability measures γ_t ($t \in \mathbb{R}$) such that

$$\Sigma_{\gamma_t}(z) = \exp\left(\frac{t}{2} \frac{1+z}{1-z}\right). \tag{A.1}$$

When $t > 0$, the measure γ_t is known [9] to be the spectral distribution of the free unitary Brownian motion u_t defined in (2.5). When $t = 0$, the measure γ_0 is the Dirac measure at $x = 1$. When $t < 0$, the measure is known as the positive Brownian motion on \mathbb{R}^+ in free probability literature. The formulas for the densities and moments of γ_t can be found in [9, 11, 45].

Let us recall Haagerup–Larsen’s formula [24] for the Brown measure of R -diagonal operators reformulated in [6, Theorem 2].

Theorem A.1 (Haagerup–Larsen [24]). *Let a be an operator that is $*$ -free from the Haar unitary h in a W^* -probability space (\mathcal{A}, τ) . The Brown measure μ_x of the R -diagonal operator $x = ha$ is the unique rotationally invariant probability measure such that*

$$\mu_x(\{z \in \mathbb{C} : |z| \leq r\}) = \begin{cases} 0 & \text{for } r \leq [\tau(xx^*)^{-1}]^{-1/2}, \\ 1 + S_{xx^*}^{(-1)}(r^{-2}) & \text{for } [\tau(xx^*)^{-1}]^{-1/2} \leq r \leq [\tau(xx^*)]^{1/2}, \\ 1 & \text{for } r \geq [\tau(xx^*)]^{1/2}. \end{cases}$$

We then have to calculate the S -transform of $b_t b_t^*$. It is known from Biane’s work [9, 11] (see also [32]) that the probability measures γ_t that connect to the spectral distribution of $b_t b_t^*$ are the case $t < 0$. As the parametrizations in [9, 11] are slightly different from ours, we provide details here for convenience.

For $t > 0$, we denote $Q_t = e^{-t} b_t b_t^*$ and set the generating function

$$\psi(t, z) = \sum_{n=1}^{\infty} z^n \tau(Q_t^n).$$

The following result is essentially from [11, Lemma 13].

Lemma A.2. *The generating function $\psi(t, z)$ satisfies*

$$\frac{\partial \psi(t, z)}{\partial t} = 2z \psi(t, z) \frac{\partial \psi(t, z)}{\partial z}. \quad (\text{A.2})$$

Proof. Recalling that b_t is the solution of (2.6), by the free Itô formula (3.3) we have

$$dQ_t = e^{-t} b_t (dc_t + dc_t^*) b_t^*.$$

By induction, from the free Itô formula we obtain

$$dQ_t^n = \sum_{k=0}^{n-1} Q_t^k dQ_t Q_t^{n-k-1} + 2 \sum_{k=1}^{n-1} k Q_t^k \tau(Q_t^{n-k}) dt.$$

Integrating and take trace, we deduce

$$\tau(Q_t^n) = 2 \int_0^t \sum_{k=1}^{n-1} k \tau(Q_s^k) \tau(Q_s^{n-k}) ds + 1, \quad (\text{A.3})$$

which yields

$$\psi(t, z) = 2 \int_0^t z \psi(t, z) \frac{\partial \psi(t, z)}{\partial z} ds + \frac{z}{1-z}.$$

The conclusion then follows by taking the derivative with respect to t . ■

The following result was mentioned implicitly in [9] and the proof can be adapted from [9, proof of Lemma 1].

Proposition A.3. *The spectral distribution of $b_t b_t^*$ is the measure γ_{-2t} defined in (A.1). The S -transform of $b_t b_t^*$ is given by*

$$S_{b_t b_t^*}(z) = e^{-t-2tz}.$$

Proof. Let $\rho(t, z) = \psi_{\gamma_{-2t}}(e^{-t} z)$; we shall prove that $\rho(t, z) = \psi(t, z)$. Fix $t > 0$. By (A.1) and the definition of Σ -transform, a simple calculation yields

$$\frac{\psi_{\gamma_{-2t}}(z)}{1 + \psi_{\gamma_{-2t}}(z)} \exp(-t - 2t \psi_{\gamma_{-2t}}(z)) = z.$$

By replacing z by $e^{-t} z$, we obtain

$$\frac{\rho(t, z)}{1 + \rho(t, z)} \exp(-2t \rho(t, z)) = z. \quad (\text{A.4})$$

We next show that $\rho(t, z)$ satisfies the same differential equation (A.2) as $\psi(t, z)$. To this end, from (A.4), we calculate the implicit differentials

$$\frac{\partial \rho(t, z)}{\partial t} = \frac{2z\rho(t, z)(1 + \rho(t, z))e^{2t\rho(t, z)}}{1 - ze^{2t\rho(t, z)} - 2tz(1 + \rho(t, z))e^{2t\rho(t, z)}},$$

and

$$\frac{\partial \rho(t, z)}{\partial z} = \frac{(1 + \rho(t, z))e^{2t\rho(t, z)}}{1 - ze^{2t\rho(t, z)} - 2tz(1 + \rho(t, z))e^{2t\rho(t, z)}},$$

which implies that $\rho(t, z)$ satisfies the same differential equation (A.2) as $\psi(t, z)$. Since ψ is a power series in z , the differential equation (A.2) is equivalent to the differential equations (A.3) for the coefficients $\{\tau(Q_t^n)\}_{n=1}^\infty$ of $\psi(t, z)$. Because ρ is an analytic function in z satisfying the same differential equation as ψ , the coefficients $a_{t,n}$ of

$$\rho(t, z) = \sum_{n=1}^\infty a_{t,n}z^n$$

satisfy the same differential equations as $\tau(Q_t^n)$ in (A.3). Since $\rho(0, z) = z/(1 - z) = \psi(0, z)$, we have the initial conditions $\tau(Q_0^n) = a_{0,n}$. The differential equations for $\tau(Q_t^n)$ (and $a_{t,n}$) given in (A.3) have a unique solution. Consequently, $\tau(Q_t^n) = a_{t,n}$ and $\rho = \psi$. Therefore,

$$S_{b_t b_t^*}(z) = S_{\gamma_{-2t}}(z) = e^{-t-2tz},$$

which concludes the proof. ■

Remark A.4. Indeed, the moments $\tau(Q_t^n)$ can be computed explicitly. The interested readers are referred to [9, Lemma 1 and Proposition 5].

We can calculate $\tau(b_t b_t^*)$ directly from the S -transform of $b_t b_t^*$ as

$$\tau(b_t b_t^*) = \psi'_{b_t b_t^*}(0) = \frac{1}{S_{b_t b_t^*}(0)} = e^t.$$

It is known [45] that $\text{supp}(\gamma_{-2t}) = [r_-(2t), r_+(2t)]$, where

$$r_\pm(2t) = [(1 - t) \pm \sqrt{t(t - 2)}e^{-\sqrt{t(t-2)}}].$$

In particular, $r_-(2t)r_+(2t) = 1$ and so $b_t b_t^*$ is strictly positive. By the free Itô formula, we can check that

$$db_t^{-1} = -dc_t b_t^{-1}.$$

Hence it is routine to check that $(b_t b_t^*)^{-1}$ has the same distribution as $b_t b_t^*$. Therefore, $\tau((b_t b_t^*)^{-1}) = \tau(b_t b_t^*) = e^t$. We are now ready to apply Theorem A.1 to deduce the formula for the Brown measure of hb_t . For any $e^{-t/2} \leq r \leq e^{t/2}$, we have

$$\mu_{hb_t}(\{|z| \leq r\}) = 1 + S_{b_t b_t^*}^{(-1)}\left(\frac{1}{r^2}\right) = \frac{1}{2} + \frac{\log r}{t}. \tag{A.5}$$

This leads to an alternative derivation of the density for the Brown measure of hb_t computed in Theorem 4.32.

Theorem A.5. *The Brown measure μ_{hb_t} of hb_t is the annulus law. It is supported in the annulus $A_t = \{e^{-t/2} \leq |z| \leq e^{t/2}\}$, rotationally invariant, and the density W_t of μ_{hb_t} in polar coordinates is given by*

$$W_t(r, \theta) = \frac{1}{2\pi t} \frac{1}{r^2}, \quad (\text{A.6})$$

which is independent of θ .

Proof. Since the density of the Brown measure of the R -diagonal operator hb_t is rotationally invariant, we assume that the density of the Brown measure is $f(r)$. Then (A.5) shows that

$$\frac{1}{2} + \frac{\log r}{t} = \mu_{hb_t}(\{|z| \leq r\}) = \int_0^r \int_0^{2\pi} f(R) R dR d\theta = 2\pi \int_0^r f(R) R dR.$$

Differentiating the above equation with respect to r gives $1/(rt) = 2\pi f(r)r$. This proves

$$f(r) = \frac{1}{2\pi t} \frac{1}{r^2},$$

which is the claimed density (A.6) of the Brown measure of hb_t . ■

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