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Corrigendum: Long gaps in sieved sets

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Abstract. This corrigendum fixes a number of small errors/omissions in [J. Eur. Math. Soc. 23, 667–700 (2021)], which in particular affect the numerical values of the exponents of $\log\log x$ in Theorem 1 and its corollaries.

The authors are grateful to Mikhail Gabdullin for pointing out a number of errors/typos in the paper. The most serious are errors in the exponents of H on pages 685–686, which, when corrected, force the parameter M to be somewhat larger than claimed, namely $M > 6$. This affects the numerical estimates for the exponents of $\log\log x$ in Theorem 1 and corollaries. Below we enumerate the specific errors and corrections. A version of the paper incorporating all of these corrections is posted at arXiv:1802.07604.

(1) P. 669: In Theorem 1, the definition of $C(\rho)$, the factor $4+\delta$ should be 6. Likewise, the corrected lower bound is $C(\rho) > e^{-1-6/\rho}$. Correct (2.3) and the following display accordingly. The corrected asymptotic, five lines after (2.3), should be $C(\rho) \sim \frac{1}{2}e^{-6/\rho}$ as $\rho \to 0^+$.

(2) P. 669: In Example 1, the corrected bound is $C(1) > 1/835$.

(3) P. 670: In Corollary 1, the corrected lower bound is $C(1/d) > e^{-(6d+1)}$.

(4) P. 671: In (1.7) and Corollary 2, the corrected bound is $C(1/2) > 1/325565$.

(5) P. 675: Six lines after (2.3), state that M is a fixed number slightly larger than 6.

(6) P. 678: In (2.10), write $6 < M \le 7$. Three lines before Remark 9 (on p. 687), write "M sufficiently close to 6".

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(7) P. 680: The hypotheses of Theorem 2 need a small adjustment. With δ fixed satisfying (2.3), M should be taken sufficiently close to 6, ξ sufficiently close to 1, K sufficiently large (all depending on δ) and ε satisfying $M < 6 + 6\varepsilon$, with x sufficiently large in terms of all of these parameters.

(8) P. 682, last line: a factor of K is missing from all terms, thus it should read

$$|\mathbf{e}_q| \le KH_q \le \frac{Ky}{z} = \frac{K(\log x)^{1/2}}{\log\log x} \le \frac{K(\log y)^{1/2}}{\log\log y}.$$

Consequently, add a factor K to the right side of (3.5) and stipulate in Lemma 3.1 that $y \ge y_0(\delta, K)$ with $y_0(\delta, K)$ sufficiently large, and add a factor of K to the definition of r on page 698.

(9) P. 683: In the final two-line display of Section 3, change the conclusion to

$$C_2 + O((\log x)^{-\delta(1+\varepsilon)}).$$

(10) P. 683: In (4.4), the factor KH on the right side should read $\lfloor KH \rfloor$, since KH need not be an integer. This induces other changes: change $|\mathcal{Q}_H|KH$ to $|\mathcal{Q}_H| \cdot \lfloor KH \rfloor$ twice in the display preceding (4.10), twice in (4.10), on the right side of (4.11), and on the right side of the display on line 15 of page 686. Change the definition of C_2 on page 686 to

$$C_2 = \frac{1}{(K+1)y} \sum_{H \in \mathfrak{H}} \frac{|\mathcal{Q}_H| \cdot \lfloor KH \rfloor}{\sigma_2}.$$

On page 694, change KH to $\lfloor KH \rfloor$ on the right side of the display before (5.9), and change K^2H^2 to $\lfloor KH \rfloor^2$ on the right side of the display before (5.11).

(11) P. 685: In (4.10), the denominator on the right side should be $\sigma_2 H^{1+\varepsilon}$. The four lines following (4.10) are corrected as follows: "then, recalling that $M > 6$ and ε is very small,

$$\mathbb{E}|\mathcal{E}_H| \ll \frac{\sigma y}{H^{1+2\varepsilon}}.$$

By Markov's inequality, we conclude that $|\mathcal{E}_H| \le \sigma y / H^{1+\varepsilon}$ with probability $1 - O(H^{-\varepsilon})$."

(12) P. 685: In three places, the summation $\sum_{n=1}^{y}$ should read $\sum_{-Ky<n-hq\le y}$.

(13) P. 685–686: Change the denominator on the right side of (4.11) to $H^{1+\varepsilon}\sigma_2$. The following lines are then corrected as: "Then

$$\mathbb{E}|\mathcal{E}'_H| \ll \frac{yH^{1+\varepsilon}\sigma_2}{H^{M-4-2\varepsilon}} \ll \sigma y \frac{\log H}{H^{M-5-3\varepsilon}}.$$

By Markov's inequality, $|\mathcal{E}'_H| \le \sigma y / H^{1+\varepsilon}$ with probability $1 - O(1/H^{M-6-5\varepsilon})$. By (2.10) again, if ε is small enough then $M - 6 - 5\varepsilon > \varepsilon$. Consider the event that (4.5) holds, and that for every H, we have (4.9), $|\mathcal{E}_H| \le \sigma y/H^{1+\varepsilon}$ and $|\mathcal{E}'_H| \le \sigma y / H^{1+\varepsilon}$."

(14) P. 686, line 15: The big-O term should read $O(\frac{1}{H^{1+\varepsilon}})$.

(15) P. 688, line −7: The inequality $k \le 10H$ should read $k \le 10KH$.

(16) P. 690, last line: An error term has been omitted. The line should read

$$\mathbb{P}(n_1, n_2 \in \mathbf{S}_2) = (1 + O(H^{-M}) + O(E_{8B}(n_1 - n_2; H)))\sigma_2^2.$$

Consequently, a factor $(1 + O(1/\log y))$ is needed at the beginning of the second line of (5.4).

(17) P. 691: Two lines after (5.5), the relation $d \in \mathcal{D}_{H+}$ should read $d \in \mathcal{D}_H \setminus \{1\}$.

(18) P. 691: The final fraction in (5.6) needs an additional factor P_1^2 in the denominator.

(19) P. 691–693: In several places, we wrote that variables are $\ge -Ky$ and it should be $> -Ky$. This occurs four lines after the statement of Theorem 3, in the last line on page 691, line 7 on page 692, and in five places on page 693.

(20) P. 693: In the definition of $\mathcal{V}$, it should read $1 \le h \le KH$ rather than $1 \le H \le KH$.

(21) P. 693: In the proof of the $j = 2$ case of Theorem 3 (ii), the argument as written works unless $n_1 \equiv n_2 \pmod{q}$. To take this case into account, replace the two lines following the definition of $\mathcal{V}$ with the following: "so that $|\mathcal{V}| = \ell \ge \lfloor KH \rfloor$. When $n_1 \not\equiv n_2 \pmod{q}$, $\mathbf{AP}(KH; q, n_1)$ and $\mathbf{AP}(KH; q, n_2)$ are disjoint. There are $O(y^2/q) = O(yH)$ pairs (n_1, n_2) with $n_1 \equiv n_2 \pmod{q}$, and for each such pair, $|\mathbf{AP}(KH; q, n_1)| + |\mathbf{AP}(KH; q, n_2)| \le |\mathcal{U}| + KH$. We also have $\sigma_2^{-KH} \ll y^{o(1)}$. Noting that $\mathbf{S}_2$ is independent of both $\mathbf{AP}(KH; q, n_1)$ and $\mathbf{AP}(KH; q, n_2)$, we see that the previous expectation is $O(y^{1+o(1)} H |\mathcal{Q}_H|)$ plus".

(22) P. 694–695: In the proof of Theorem 3 (iii), $j = 2$ case, the case $q_1 = q_2$ requires special analysis. By (2.8) these terms contribute $\ll H^2 y |\mathcal{Q}_H| \sigma_2^{-2KH} \ll |\mathcal{Q}_H|^2 y^{o(1)}$, which is negligible. Also, "we may write (5.11) as" is better understood as "we may write the sum of (5.11) over h_1, h_2 as", and add a factor $\lfloor KH \rfloor^2$ to the following display. The reason for this change is that $E'(q_1)$, $E'(q_2)$ and $E''(q_1, q_2)$ already incorporate sums over h_1, h_2.

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