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Derivation of strain-gradient plasticity from a generalized Peierls–Nabarro model

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Abstract. We derive strain-gradient plasticity from a nonlocal phase-field model of dislocations in a plane. After scaling, from the nonlocal elastic interaction we derive a continuous energy with linear growth depending on a measure which characterizes the macroscopic dislocation density as well as a nonlocal effective energy representing the far-field interaction between dislocations. Relaxation and formation of microstructures at intermediate scales are automatically incorporated in the limiting procedure based on Γ-convergence.

Keywords. Peierls–Nabarro model, strain gradient plasticity, Gamma-convergence, phase-field model

1. Introduction

Crystal plasticity and dislocations are a fundamental theme in the mechanics of solids. Whereas the calculus of variations has been very helpful in the study of nonlinear elasticity and phase transitions, the study of plasticity and dislocations has proven much more demanding. Dislocations are topological singularities of the strain field, which share many features with other important classes of topological defects, such as Ginzburg–Landau vortices, defects in liquid crystals, harmonic maps, models of superconductivity. Their importance for the understanding of the yield behavior of crystals motivated a large interest, and indeed in the last decade tools have been developed to study individual dislocations, both in reduced two-dimensional formulations in which one deals with point singularities [5, 24, 52], and, with some geometrical restrictions, in the three-dimensional setting in which singularities are lines [21].

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In this paper we go beyond the scale of single dislocation lines and derive macroscopic strain-gradient plasticity, starting from a generalization of the classical Peierls–Nabarro model for dislocations. Here the plastic slips are confined to a single slip plane and the three-dimensional elastic problem can then be solved (implicitly) and results in a nonlocal energy induced by the phase field of this model, which represents slip on a plane. In addition, the crystalline nature of the material implies that slips that are not lattice preserving generate a large short-range, local energy. This results in a local, nonconvex term that penalizes the L^2 distance of the phase field from a discrete set. The nonconvex term is minimized by piecewise constant, integer-valued phase fields, while the convex nonlocal term does not allow for discontinuities along lines and regularizes the interfaces. A more detailed interpretation of the model will be given in Section 2. Specifically, we consider an energy of this type:

$$\frac{1}{\varepsilon}\int_\Omega \mathrm{dist}^2(u(x),\mathbb{Z}^N)\,dx+[u]^2_{\hat\Gamma,1/2} \tag{1.1}$$

where $u\in L^2(\Omega;\mathbb{R}^N)$, with $\Omega\subset\mathbb{R}^2$ open and bounded. The parameter $\varepsilon>0$ represents, ideally, the lattice spacing and reveals the semi-discrete nature of this type of dislocation model, while $[u]_{\hat\Gamma,1/2}$ is an anisotropic seminorm equivalent to the $H^{1/2}$ seminorm, defined by

$$[f]^2_{H^{1/2}(\Omega)}:=\int_{\Omega\times\Omega}\frac{|f(x)-f(y)|^2}{|x-y|^3}\,dx\,dy. \tag{1.2}$$

The anisotropy of the interaction kernel $\hat\Gamma$ reflects the anisotropy of the underlying lattice and, possibly, the boundary conditions imposed at the level of the local three-dimensional problem. Note that, even with the simplest case of isotropic elasticity, invariance under (linearized) rotations requires a nontrivial and anisotropic interaction between the components of u.

This model was proposed and studied numerically in [42,43,50], in a regime in which a few individual dislocations are present. In the same regime, a scalar simplification of the model was studied analytically in [32, 33]. In the limit of small lattice spacing ε, the energy concentrates along the dislocations and the problem reduces to a line-tension model; the relevant independent variable is a measure concentrated on a line and the energetic cost of single dislocations is logarithmic in ε. One important tool was the study of variational convergence for phase transitions with nonlocal interactions (see also [3, 4, 29, 55]). The extension to the physically relevant vectorial situation in [15, 18] (see also [17] for a generalization to multiple slip planes) leads to the discovery of unexpected microstructures at intermediate scales. This effect together with the singularity of the kernel in the nonlocal energy makes the asymptotic analysis very demanding. A key tool is a multiscale self-similar decomposition of the singular kernel performed in [18].

Here we consider a scale for which the total length of the singularities diverges and we derive a macroscopic strain-gradient theory, where the macroscopic effect of the singularities is captured by a density, which is a measure in $H^{-1/2}$. In this critical energetic regime the limiting model combines effects at different scales, it includes both long-range interactions of singularities and a short-range term which arises from the self-interaction of singularities. The study of a single limiting process is crucial to obtain the limiting behavior of both the local term and the nonlocal one; if the various homogenization and relaxation steps are taken separately then the nonlocal (interaction) term disappears [20]. This critical phase field model does not show equipartition of energy, at variance with many other results in the theory of phase transitions. A related effect is that there is no characteristic length scale at which the energy concentrates.

The main difficulty in the proof is to obtain a joint treatment of the many different scales present in the problem. The discrete nature of the dislocations leads to localization and to slip fields in $BV(\Omega;\mathbb{Z}^N)$, at the same time the nonlocal part requires slip fields in $H^{1/2}(\Omega;\mathbb{R}^N)$. These two spaces are, except for constants, disjoint (see Lemma 3.1 below), hence both requirements can only be realized approximately. This is performed introducing a number of well-separated scales, regularizations and cutoffs, as discussed in the introduction to Section 7 for the upper bound and Section 8 for the lower bound. Indeed, both the original functional and the limiting functional are finite on $H^{1/2}(\Omega;\mathbb{R}^N)$, whereas the relaxation steps occur at intermediate scales, and is best described using functions in $BV(\Omega;\mathbb{Z}^N)$.

The critical scaling considered here corresponds to a total length of dislocations that increases logarithmically. Correspondingly, the energy scales as $|\log\varepsilon|^2$. This scaling has already been considered in the literature for cylindrical configurations (i.e., in two-dimensional models where defects are point singularities). The first result in this context is the one for Ginzburg–Landau vortices by Sandier and Serfaty [53]. As for point (edge) dislocations in two dimensions this regime was first studied in [31] in a dilute geometrically linear setting. The latter was then generalized to dilute geometrically nonlinear models [47,48], the same result without the diluteness restriction was obtained in [34,35]. Along this line of thought, with a different energy scaling, a similar model in two dimensions was used in [45] in order to derive the Read–Shockley formula for the energy of small-angle grain boundaries.

Our result with the kinematic restriction of lines confined to one single plane is an important step toward the understanding of the full three-dimensional problem, which is substantially more subtle: firstly, because the geometry plays an important role in the interaction between line singularities, and secondly, because it is a higher-order tensorial problem, in which the energy only controls some components of the relevant strain field. For these reasons, we expect that any extension to three dimensions will, at least initially, require strong diluteness assumptions, as used for example for the line-tension scaling in [21]. In contrast, the present work does not have any restriction on the admissible configurations to defects (other than the confinement to a single slip plane).

In the context of Ginzburg–Landau for filament singularities and isotropic models of superconductivity in dimension 3 the critical regime has been studied in [2,10,11,53,54]. Whereas some ideas are closely related, the vectorial and anisotropic nature of the dislocation problem renders a direct transfer difficult and requires new techniques, in particular for treating the microstructures at intermediate scales. As is clear from the summary above, our argument makes a strong use of the existence of a lifting of the dislocation density to a function of bounded variation which represents the slip on the plane that contains Ω. Therefore extension to the unconstrained three-dimensional case will probably require a different functional framework, which could possibly be formulated via cartesian currents [16, 40, 56].

Our results have interesting implications for modeling in continuum mechanics. On the one hand, it provides a justification for strain-gradient plasticity models, which have been widely used in the literature [12, 22, 27, 28, 30, 44, 49], and gives a characterization of the strain-gradient energy density. In particular, we show that it has linear growth in the strain gradient, whereas in many phenomenological models expressions with quadratic growth are used. On the other hand, our result implies that dislocation microstructures form at intermediate scales, in the form of dislocation networks, the study of which has long been an important problem in mechanics [1, 13, 27, 36–39, 46, 51]. We refer to [9, 19] for a more precise discussion of the mechanical implications of our result and its connection to strain-gradient theories of plasticity.

2. Model and main results

We now formulate the model we study. The total energy associated to a phase field $u \in L^2(\Omega;\mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^2$ open and bounded, is

$$E_\varepsilon[u,\Omega] := \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \int_{\Omega\times\Omega} \Gamma(x-y)(u(x)-u(y))\cdot(u(x)-u(y))\,dx\,dy. \tag{2.1}$$

The nonlinear potential $W:\mathbb{R}^N \to [0,\infty)$ satisfies

$$\frac{1}{c}\mathrm{dist}^2(\xi,\mathbb{Z}^N) \le W(\xi) \le c\,\mathrm{dist}^2(\xi,\mathbb{Z}^N) \tag{2.2}$$

for some $c > 0$. The elasticity kernel $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^{N\times N}_{\mathrm{sym}})$ is defined by

$$\Gamma(z) := \frac{1}{|z|^3}\hat\Gamma\left(\frac{z}{|z|}\right), \tag{2.3}$$

where $\mathbb{R}^{N\times N}_{\mathrm{sym}}$ denotes the set of symmetric $N\times N$ matrices and $\hat\Gamma \in L^\infty(S^1;\mathbb{R}^{N\times N}_{\mathrm{sym}})$ obeys, for some $c > 0$,

$$\hat\Gamma(z) = \hat\Gamma(-z) \quad \text{and} \quad \frac{1}{c}|\xi|^2 \le \hat\Gamma(z)\xi\cdot\xi \le c|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N,\ z \in S^1. \tag{2.4}$$

The specific form of $\hat\Gamma$ depends on the elastic constants and the Burgers vectors of the crystal, for example for an elastically isotropic cubic crystal one obtains $\Gamma = \Gamma^{\mathrm{cubic}}$, where

$$\Gamma^{\mathrm{cubic}}(z) := \frac{\mu}{16\pi(1-\overline{\nu})|z|^3}\begin{pmatrix} \overline{\nu}+1-3\overline{\nu}\frac{z_2^2}{|z|^2} & 3\overline{\nu}\frac{z_1 z_2}{|z|^2} \\ 3\overline{\nu}\frac{z_1 z_2}{|z|^2} & \overline{\nu}+1-3\overline{\nu}\frac{z_1^2}{|z|^2} \end{pmatrix}. \tag{2.5}$$

Here $\overline{\nu}$ and μ denote the material's Poisson's ratio and shear modulus, respectively (see [15]). It is easy to see that for $\mu > 0$ and $\overline{\nu} \in (-1, 1/2)$ the kernel Γ^{cubic} fulfills the assumption (2.4).

In order to present our main result we first introduce several effective energy densities which are generated by the rescaling procedure. Detailed explanations of the physical significance of the different steps are given in Section 4. In a first step the nonlocal kernel Γ generates an unrelaxed line-tension energy $\psi : \mathbb{Z}^N \times S^1 \to [0, \infty)$ by

$$\psi(b, n) := 2 \int_{\{x \cdot n = 1\}} \Gamma(x) b \cdot b \, d\mathcal{H}^1(x). \tag{2.6}$$

Relaxation at the line-tension scale leads to the BV-elliptic envelope $\psi_{\text{rel}} : \mathbb{Z}^N \times S^1 \to [0, \infty)$ of $\psi : \mathbb{Z}^N \times S^1 \to [0, \infty)$, defined by

$$\psi_{\text{rel}}(b, n) := \inf \left\{ \frac{1}{2} \int_{J_u \cap B_1} \psi([u], \nu) \, d\mathcal{H}^1 : u \in BV_{\text{loc}}(\mathbb{R}^2; \mathbb{Z}^N), \right. \\ \left. \operatorname{supp}(u - u^0_{b,n}) \subset\subset B_1 \right\}, \tag{2.7}$$

where $u^0_{b,n}(x) := b\chi_{\{x \cdot n > 0\}}$ and ν is the normal to the jump set J_u of u. We recall that the concept of BV-elliptic envelope was introduced and studied in [6, 7] (see also [8, Sect. 5.3]).

Finally, in the second relaxation step one obtains a continuous energy density $g : \mathbb{R}^{N \times 2} \to [0, \infty)$ defined as the convex envelope of

$$g_0(A) := \begin{cases} 0 & \text{if } A = 0, \\ \psi_{\text{rel}}(b, n) & \text{if } A = b \otimes n \text{ for } b \in \mathbb{Z}^N,\ n \in S^1, \\ \infty & \text{otherwise.} \end{cases} \tag{2.8}$$

In particular, the function g turns out to be positively 1-homogeneous [20].

Our main result is the following.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain, and let $E_\varepsilon[\cdot, \Omega]$ be defined as in (2.1), with W and Γ which satisfy (2.2)–(2.4).*

We say that a family of functions $u_\varepsilon \in L^2(\Omega; \mathbb{R}^N)$, $\varepsilon > 0$, converges to u if

$$\frac{u_\varepsilon}{\ln(1/\varepsilon)} \to u \quad \text{in } L^2(\Omega; \mathbb{R}^N) \text{ as } \varepsilon \to 0. \tag{2.9}$$

With respect to this convergence we have

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \frac{1}{(\ln(1/\varepsilon))^2} E_\varepsilon[\cdot, \Omega] = F_0[\cdot, \Omega],$$

where F_0 is defined by

$$F_0[u, \Omega] := F_{\text{self}}[u, \Omega] + \int_{\Omega \times \Omega} \Gamma(x - y)(u(x) - u(y)) \cdot (u(x) - u(y)) \, dx \, dy \tag{2.10}$$

and

$$F_{\rm self}[u,\Omega] := \int_\Omega g(\nabla u)\,dx + \int_\Omega g\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| \tag{2.11}$$

if $u \in BV(\Omega;\mathbb{R}^N) \cap H^{1/2}(\Omega;\mathbb{R}^N)$*, and* $F_0[u,\Omega] = \infty$ *otherwise. Here* $g : \mathbb{R}^{N\times 2} \to [0,\infty)$ *is the convex envelope of the function* g_0 *defined from the kernel* Γ *in* (2.6)–(2.8)*.*

Further, the functionals $(\ln(1/\varepsilon))^{-2} E_\varepsilon[\cdot,\Omega]$ *are, with respect to the stated convergence, equicoercive, in the sense that if* $u_\varepsilon \in L^2(\Omega;\mathbb{R}^N)$ *are such that* $E_\varepsilon[u_\varepsilon,\Omega] \le C(\ln(1/\varepsilon))^2$ *for all* ε *then there is a subsequence* $\varepsilon_k \to 0$ *such that, for some* $d_k \in \mathbb{Z}^N$ *and some* $u \in L^2(\Omega;\mathbb{R}^N)$*, one has*

$$\lim_{k\to\infty} \frac{u_{\varepsilon_k} - d_k}{\ln(1/\varepsilon_k)} = u \quad \text{in } L^2(\Omega;\mathbb{R}^N). \tag{2.12}$$

We remark that $\psi_{\rm rel}$ coincides with the line-tension energy density obtained in the subcritical regime [18]; see Theorem 4.1 below.

The proof of the above theorem is a combination of various results proved in the rest of the paper. The compactness assertion follows from Proposition 5.1, the upper bound from Proposition 7.2, and the lower bound from Proposition 8.1.

To close this introduction we briefly recall the connection between E_ε and the classical Peierls–Nabarro model which contains the elastic energy over a three-dimensional domain. The Peierls–Nabarro model, as generalized in [42, 43, 50] to three dimensions, expresses the free energy in terms of the slip $v : \Omega \to \mathbb{R}^3$ on a two-dimensional cross-section $\Omega \subset \mathbb{R}^2$ as

$$E_{\rm free}[v] := E_{\rm elastic}[v] + E_{\rm interfacial}[v].$$

Here the first term represents the long-range elastic distortion due to the slip, and the second term penalizes slips that are not integer multiples of the Burgers vectors of the crystal lattice. One denotes by $b_1,\dots,b_N \in \mathbb{R}^3$ the relevant Burgers vectors and considers slips of the form

$$v = u_1 b_1 + \cdots + u_N b_N$$

where $u : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^N$. Typically, $N = 2$ and the vectors b_i are a basis of $\mathbb{R}^2 \times \{0\}$, but this is not relevant for the mathematical analysis. The term $E_{\rm interfacial}$ penalizes values of u far from $\mathbb{Z}^N$, so that v is close to the lattice generated by $\{b_1,\dots,b_N\}$. A simple model is

$$E_{\rm interfacial}[v] = \frac{1}{\varepsilon}\int_\Omega {\rm dist}^2(u,\mathbb{Z}^N)\,dx,$$

where u is related to v as stated above. We observe that the specific functional form does not contribute to the limit. At variance with many classical results on Γ-convergence for phase-field models of phase transitions, there is no equipartition of energy, and the only role of the interfacial energy is to force u to jump on a scale ε. The limiting energy arises then completely from the elastic term, as is apparent from the characterization of g and ψ in terms of the kernel Γ in (2.6)–(2.8).

The elastic interaction is given by

$$E_{\text{elastic}}[v] = \inf\left\{\int_{\Omega\times\mathbb{R}} \frac{1}{2}\mathbb{C}\nabla U\cdot\nabla U\; dx\right\},$$

where the displacement $U:\Omega\times\mathbb{R}\to\mathbb{R}^3$ is required to have a discontinuity of $v=\sum u_i b_i$ across $\Omega\times\{0\}$. Minimizing out U leads to a nonlocal functional of u of the kind of (2.1) up to boundary effects which do not influence the leading-order behavior; see [32, 33] for a discussion. The factor ε in $E_{\text{interfacial}}$ is proportional to the lattice spacing and arises from the different scaling of the bulk and the interfacial term. We refer to [19, 32] for a more detailed discussion of this relation.

Remark 2.2. We discuss in this paper the case that Ω is a bounded Lipschitz set. Similar results can be obtained, with the same proofs, for Ω being a torus. In the latter case it is easy to see that the elastic energy E_{elastic} coincides with the nonlocal energy in E_ε, up to lower-order terms which are continuous in the topology considered here. This leads to the model described in [19].

3. Functional setting

In this section we briefly recall the main properties and the standard notation for the function spaces used in the paper.

The elements of $BV(\Omega;\mathbb{R}^N)$ are the functions with bounded variation in $\Omega\subset\mathbb{R}^2$, which are the functions in $L^1(\Omega;\mathbb{R}^N)$ whose distributional derivative Du is a bounded measure on Ω. We denote by $D^s u$ the part of this measure which is singular with respect to the Lebesgue measure. In turn, $SBV(\Omega;\mathbb{R}^N)$ denotes the space of special functions of bounded variation, which are the functions in $BV(\Omega;\mathbb{R}^N)$ whose distributional gradient can be characterized as $Du=\nabla u\mathcal{L}^2+[u]\otimes\nu\mathcal{H}^1\llcorner J_u$. Here J_u is a 1-rectifiable set called the *jump set* of u, and it is defined as the set of points for which u does not have an approximate limit. The normal to this set is denoted by ν, and $[u]=u^+-u^-$ denotes the jump of the function u across the set J_u. For any $\mathcal{H}^1$-a.e. $x\in J_u$ one has

$$\lim_{r\to 0}\frac{\mathcal{H}^1(J_u\cap B_r(x))}{2r}=1$$

and

$$\lim_{r\to 0}\frac{1}{r^2}\int_{B_r(x)\cap\{\pm(y-x)\cdot\nu>0\}} |u-u^{\pm}(x)|\,dy=0.$$

We refer to [8] for details.

The fractional Sobolev space $H^{1/2}(\Omega)$ is equipped with the homogeneous $H^{1/2}$ semi-norm, which as stated in (1.2) is given by

$$[f]^2_{H^{1/2}(\Omega)} := \int_{\Omega\times\Omega}\frac{|f(x)-f(y)|^2}{|x-y|^3}\,dx\,dy \tag{3.1}$$

for a measurable function $f : \Omega \to \mathbb{R}$, and for an open set $\Omega \subset \mathbb{R}^2$. We observe that if Ω is bounded then for any f there is $a_f \in \mathbb{R}$ such that

$$\|f - a_f\|_{L^2(\Omega)} \le c_\Omega [f]_{H^{1/2}(\Omega)}.$$

This can be proven directly from the definition of $[f]_{H^{1/2}(\Omega)}$, letting a_f be the average of f over Ω. If Ω is bounded and Lipschitz, then any sequence f_j which converges weakly in L^2 and is bounded in the $H^{1/2}$ seminorm converges strongly in L^2 (see for example [25, Section 7]). One can see that this norm is equivalent to the one obtained by the trace method.

We next recall that $BV(\Omega; \mathbb{Z}^N)$ is (up to constants) disjoint from $H^{1/2}(\Omega; \mathbb{R}^N)$, which only contains functions that “do not jump”. This fact can be made precise using the following lemma.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^2$ be open and $u \in BV(\Omega) \cap H^{1/2}(\Omega)$. Then $\mathcal{H}^1(J_u) = 0$.*

In particular, if $u \in BV(\Omega; \mathbb{Z}^N)$ then $Du = [u] \otimes \nu \mathcal{H}^1 \llcorner J_u$, hence in this case $\mathcal{H}^1(J_u) = 0$ implies $Du = 0$.

Proof of Lemma 3.1. We claim that for any $\delta > 0$ we have

$$\mathcal{H}^1(\{x \in J_u : |u^+ - u^-|(x) > \delta\}) = 0. \tag{3.2}$$

Since δ is arbitrary, this will imply the assertion.

We write $J_u^{(\delta)} := \{x \in J_u : |u^+ - u^-|(x) > \delta\}$. Fix $\eta > 0$, and choose $\rho > 0$ such that

$$\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^3} \chi_{B_\rho(x)}(y)\, dx\, dy \le \eta. \tag{3.3}$$

This is possible, since $u \in H^{1/2}(\Omega)$ and by dominated convergence this integral converges to zero as $\rho \to 0$. For $\mathcal{H}^1$-a.e. $x \in J_u^{(\delta)}$ one has

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r(x)} |u(y) - u_x(y)|\, dy = 0,$$

where

$$u_x(y) := \begin{cases} u^+(x) & \text{if } (y - x) \cdot \nu > 0, \\ u^-(x) & \text{if } (y - x) \cdot \nu \le 0, \end{cases}$$

with $u^\pm(x)$ the traces and ν the normal to the jump set in x. Therefore for $\mathcal{H}^1$-a.e. $x \in J_u^{(\delta)}$,

$$\begin{aligned} \lim_{r\to 0} \frac{1}{\pi r^2} \int_{B_r(x)} |u(y) - (u)_r|\, dy &= \lim_{r\to 0} \frac{1}{\pi r^2} \int_{B_r(x)} |u_x(y) - (u_x)_r|\, dy \\ &= \frac{1}{2}|u^+ - u^-|(x) > \frac{\delta}{2}, \end{aligned}$$

where $(f)_r$ denotes the average of f over the ball $B_r(x)$. Moreover, from the 1-rectifiability of J_u, and hence of $J_u^{(\delta)}$, we have

$$\lim_{r\to 0}\frac{1}{2r}\mathcal{H}^1(J_u^{(\delta)}\cap B_r(x)) = 1.$$

By Vitali–Besicovitch's covering lemma (see for instance [8, Theorem 2.19]) we can choose countably many disjoint balls $B_j := B(x_j, r_j)$ such that for all j one has $r_j \in (0, \rho/2)$, $x_j \in J_u^{(\delta)}$,

$$\frac{1}{\pi r_j^2}\int_{B_j}|u(y)-u_j|\,dy \geq \frac{\delta}{4}, \tag{3.4}$$

where u_j is the average of u over B_j, with $r_j \leq \mathcal{H}^1(J_u^{(\delta)}\cap \overline{B}_j) \leq 3r_j$, and satisfying $\mathcal{H}^1(J_u^{(\delta)}\setminus \bigcup_j \overline{B}_j) = 0$.

For each B_j we estimate, as in the proof of Poincaré's inequality for $H^{1/2}$, using Jensen's inequality,

$$\begin{aligned}\int_{B_j}|u(y)-u_j|^2\,dy &\leq \frac{1}{\pi r_j^2}\int_{B_j\times B_j}|u(y)-u(x)|^2\,dx\,dy\\ &\leq \frac{8r_j}{\pi}\int_{B_j\times B_j}\frac{|u(y)-u(x)|^2}{|x-y|^3}\,dx\,dy.\end{aligned}$$

In particular, recalling (3.4),

$$\frac{\delta^2}{16}\pi r_j^2 \leq \int_{B_j}|u(y)-u_j|^2\,dy \leq \frac{8r_j}{\pi}\int_{B_j\times B_j}\frac{|u(x)-u(y)|^2}{|x-y|^3}\,dx\,dy.$$

We divide by r_j, sum over j, and obtain

$$\begin{aligned}\mathcal{H}^1(J_u^{(\delta)}) &\leq \sum_j \mathcal{H}^1(J_u^{(\delta)}\cap\overline{B}_j) \leq \sum_j 3r_j\\ &\leq C_\delta\int_{\bigcup_j B_j\times B_j}\frac{|u(x)-u(y)|^2}{|x-y|^3}\,dx\,dy \leq C_\delta\eta,\end{aligned}$$

where C_δ depends only on δ and in the last step we have used $2r_j < \rho$ and (3.3). Since η was arbitrary, the proof of (3.2) and therefore of the lemma is concluded. ■

4. Limits at separated scales

In this section we briefly review two previous results on different scalings which have been mentioned in the introduction and that will be used in the proofs.

If a sequence u_ε has energy proportional to $\ln(1/\varepsilon)$, in the sense that $E_\varepsilon[u_\varepsilon,\Omega] \leq C\ln(1/\varepsilon)$, then asymptotically u_ε describes dislocations with finite total length, and the limiting energy is given by an integral over the line. This is called the *line-tension approximation* and was studied in [18, 32].

Theorem 4.1 ([18, Theorem 1.1]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain, and let $E_\varepsilon[\cdot,\Omega]$ be defined as in (2.1), with W and Γ which satisfy (2.2)–(2.4).*

The functionals $(\ln(1/\varepsilon))^{-1}E_\varepsilon[\cdot,\Omega]$ are L^1-equicoercive, in the sense that if $E_\varepsilon[u_\varepsilon,\Omega] \le C\ln(1/\varepsilon)$ then there are $d_\varepsilon \in \mathbb{Z}^N$ and $u \in BV(\Omega;\mathbb{Z}^N)$ such that $u_\varepsilon - d_\varepsilon$ has a subsequence that converges to u in $L^1(\Omega;\mathbb{R}^N)$.

Further, we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\frac{1}{\ln(1/\varepsilon)}E_\varepsilon[\cdot,\Omega] = E^{\mathrm{LT,rel}}[\cdot,\Omega]$$

with respect to strong L^1 convergence, where the relaxed line-tension fuctional $E^{\mathrm{LT,rel}}$ is defined by

$$E^{\mathrm{LT,rel}}[u,\Omega] := \begin{cases} \displaystyle\int_{J_u\cap\Omega} \psi_{\mathrm{rel}}([u],\nu)\, d\mathcal{H}^1 & \text{if } u \in SBV(\Omega;\mathbb{Z}^N), \\ \infty & \text{otherwise,} \end{cases} \tag{4.1}$$

with ψ_{rel} obtained from Γ as in (2.6) and (2.7).

The main difficulty in the proof of Theorem 4.1 is that this problem has no natural rescaling, since infinitely many scales asymptotically contribute to the energy. The proof is based on a dyadic decomposition of the interaction kernel, which is also used in Section 8 below, and on an iterative mollification technique which permits one to show that microstructure can only appear at few scales, and therefore on the average scale there is no microstructure. This permits passing from the nonlocal functional E_ε to a line-tension functional with the unrelaxed energy ψ; the relaxation from ψ to ψ_{rel} then takes place at the line-tension level and does not couple to the nonlocality of E_ε. We refer to [18] for details and we remark that a similar formula for the subcritical regime also holds without the geometric restriction to a single plane, if the dislocations are dilute [21].

For later reference we observe that (2.4) and the definition in (2.6) imply that $c|b| \le \psi(b,n)$ for all $b \in \mathbb{Z}^N$, $n \in S^1$, and by (2.7) we obtain, with Jensen's inequality,

$$c|b| \le \psi_{\mathrm{rel}}(b,n) \le \psi(b,n) \quad \text{for all } b \in \mathbb{Z}^N, n \in S^1. \tag{4.2}$$

The transition from scaled line-tension functionals to a functional with a continuous distribution of dislocations was studied in [20]. Here two effects are present. Firstly, by rescaling the discrete nature of the dislocations is lost, and macroscopically one only sees the effective dislocation density, passing from ψ_{rel} to g_0. This corresponds to recovering continuous slips from superposition of many atomic-scale plastic slips, and naturally relates to strain-gradient plasticity models. Secondly, and already at the macroscopic scale, one relaxes g_0 (which is finite only on certain rank-one matrices) to the macroscopic energy g. As usual in problems with linear growth, the gradient constraint does not affect the effective energy density, which turns out to be convex (see [41] for a general statement).

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain, and let $\psi : \mathbb{Z}^N \times S^1 \to [0, \infty)$ obey $\frac{1}{c}|b| \le \psi(b, n)$ for all $b \in \mathbb{Z}^N$ and $n \in S^1$. The functionals*

$$E_\sigma^{\mathrm{LT}}[u, \Omega] := \begin{cases} \displaystyle\int_{J_u \cap \Omega} \sigma \psi([u]/\sigma, \nu)\, d\mathcal{H}^1 & \text{if } u \in SBV(\Omega; \sigma\mathbb{Z}^N), \\ \infty & \text{otherwise,} \end{cases} \tag{4.3}$$

are equicoercive with respect to the strong L^1 topology, in the sense that if $\sigma_k > 0$, $\sigma_k \to 0$ and $E_{\sigma_k}^{\mathrm{LT}}[u_k, \Omega] \le C$ then there are $d_k \in \mathbb{R}^N$ and $u \in BV(\Omega; \mathbb{R}^N)$ such that $u_k - d_k$ has a subsequence that converges to u in $L^1(\Omega; \mathbb{R}^N)$.

Further, with respect to this topology,

$$\Gamma\text{-} \lim_{\sigma \to 0} E_\sigma^{\mathrm{LT}}[\cdot, \Omega] = E_0^{\mathrm{LT}}[\cdot, \Omega],$$

where

$$E_0^{\mathrm{LT}}[u, \Omega] := \begin{cases} \displaystyle\int_\Omega g\left(\frac{dDu}{d|Du|}\right) d|Du| & \text{if } u \in BV(\Omega; \mathbb{R}^N), \\ \infty & \text{otherwise.} \end{cases} \tag{4.4}$$

The function g is positively 1-homogeneous, obeys $\frac{1}{c}|A| \le g(A) \le c|A|$, and coincides with the convex envelope of g_0, which is defined from ψ via ψ_{rel} as in (2.7) and (2.8).

We observe that in [20, (1.4)] there is a typo: the integral should be (as in (4.4) above) over Ω, not J_u.

Proof of Theorem 4.2. This statement reduces to [20, Theorem 1.1] if ψ is BV-elliptic (i.e., $\psi = \psi_{\mathrm{rel}}$), after a change in notation.

In the general case we observe that for any (fixed) $\sigma > 0$ by [16] the functional

$$E_\sigma^{\mathrm{LT,rel}}[u, \Omega] := \begin{cases} \displaystyle\int_{J_u \cap \Omega} \sigma \psi_{\mathrm{rel}}([u]/\sigma, \nu)\, d\mathcal{H}^1 & \text{if } u \in SBV(\Omega; \sigma\mathbb{Z}^N), \\ \infty & \text{otherwise,} \end{cases} \tag{4.5}$$

is the relaxation of E_σ^{LT}, and that ψ_{rel} is BV-elliptic and has linear growth, in the sense that

$$\frac{1}{c}|b| \le \psi_{\mathrm{rel}}(b, n) \le c|b| \quad \text{for all } b \in \mathbb{Z}^N,\ n \in S^1.$$

Therefore the Γ-limit of the sequence E_σ^{LT} is the same as the Γ-limit of $E_\sigma^{\mathrm{LT,rel}}$; we refer to [23, Proposition 6.11] for details.

Finally, as mentioned above, [20, Theorem 1.1] implies that the sequence $E_\sigma^{\mathrm{LT,rel}}$ Γ-converges to E_0^{LT}. Coercivity is also inherited, since $\psi_{\mathrm{rel}} \le \psi$. This concludes the proof. ∎

In proving our main result we shall have to take into account both these results, but also include the effects of the long-range elastic energy, which scales as the squared $H^{1/2}$ norm of u. We remark that $H^{1/2}$ is singular with respect to the natural spaces of piecewise constant functions entering the above results, hence one cannot recover Theorem 2.1 from a direct combination of Theorems 4.1 and 4.2.

5. Compactness

The functions $u_\varepsilon/\ln(1/\varepsilon)$ belong to the space $H^{1/2}(\Omega;\mathbb{R}^N)$, the limit however will also belong to $BV(\Omega;\mathbb{R}^N)$. The key step in the proof of the compactness result is to produce a new sequence of functions, called v_ε in the proof below, which belong to $BV(\Omega;\mathbb{Z}^N)$ and are close to u_ε.

Proposition 5.1 (Compactness). *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain, and let $E_\varepsilon[\cdot,\Omega]$ be defined as in (2.1), with W and Γ which satisfy (2.2)–(2.4). Let u_ε be a family with $E_\varepsilon[u_\varepsilon,\Omega] \le M(\ln(1/\varepsilon))^2$ for some $M > 0$. Then there are a function $u \in BV(\Omega;\mathbb{R}^N) \cap H^{1/2}(\Omega;\mathbb{R}^N)$, vectors $d_\varepsilon \in \mathbb{Z}^N$ and a subsequence $\varepsilon_k \to 0$ such that*

$$\frac{1}{\ln(1/\varepsilon_k)}(u_{\varepsilon_k} - d_{\varepsilon_k}) \to u \quad \text{in } L^2(\Omega;\mathbb{R}^N). \tag{5.1}$$

In order to prove the compactness result we recall some notation and a result from [18]. We define the truncated kernels by

$$\Gamma_{[0,k]} := \sum_{i=0}^{k} \Gamma_i$$

where (see Figure 1)

$$\Gamma_k(x) := \begin{cases} \hat{\Gamma}(x/|x|)(2^{3(k+1)} - 2^{3k}) & \text{if } 0 < |x| \le 2^{-k-1}, \\ \hat{\Gamma}(x/|x|)(|x|^{-3} - 2^{3k}) & \text{if } 2^{-k-1} < |x| \le 2^{-k}, \\ 0 & \text{if } |x| > 2^{-k} \end{cases}$$

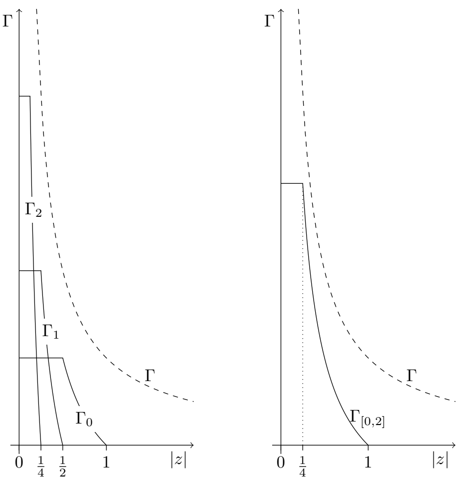

Fig. 1. Sketch of Γ, Γ_0, Γ_1 and Γ_2 (left) and of Γ, $\Gamma_{[0,2]}$ (right). For clarity we have plotted $1/|z|$ instead of $1/|z|^3$.

and the corresponding truncated energies by

$$E_k^*[u,\Omega] := \int_{\Omega\times\Omega} \Gamma_k(x-y)(u(x)-u(y))\cdot(u(x)-u(y))\,dx\,dy. \tag{5.2}$$

The result from [18] that we use concerns approximation of regular fields by BV phase fields. We observe that the symbol E_ε is used in [18] for the energy already divided by $\ln(1/\varepsilon)$, i.e., for the quantity $E_\varepsilon^{2011} := E_\varepsilon/\ln(1/\varepsilon)$.

Proposition 5.2 ([18, Proposition 4.1]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $E_\varepsilon[\cdot,\Omega]$ be defined as in (2.1), with W and Γ which satisfy (2.2)–(2.4). Assume that $\omega \subset\subset \Omega$ and $\delta \in (0,1/2)$. Then there exists a constant $C > 0$ such that for every sufficiently small $\varepsilon > 0$ (on a scale set by δ and $\operatorname{dist}(\omega,\partial\Omega)$) and every $u \in L^2(\Omega;\mathbb{R}^N)$ there are $k \in \mathbb{N}$ and $v \in BV(\omega;\mathbb{Z}^N)$ such that*

$$\sum_{h=0}^{k} E_h^*[v,\omega] \le E_\varepsilon[u,\Omega]\left(1+\frac{C}{\delta(\ln(1/\varepsilon))^{1/2}}\right), \tag{5.3}$$

$$|Dv|(\omega) \le \frac{C}{\delta}\,\frac{E_\varepsilon[u,\Omega]}{\ln(1/\varepsilon)}, \tag{5.4}$$

$$\varepsilon^{1-\delta/2} \le 2^{-k} \le \varepsilon^{1-\delta}.$$

Furthermore,

$$\|u-v\|_{L^1(\omega)} \le C2^{-k/2}\left(\frac{E_\varepsilon[u,\Omega]}{\ln(1/\varepsilon)}\right)^{1/2}.$$

The constants depend only on W and Γ.

Proof of Proposition 5.1. We start by proving that the sequence $(u_\varepsilon - d_\varepsilon)/\ln(1/\varepsilon)$, for a suitable choice of $d_\varepsilon \in \mathbb{Z}^N$, converges in L^2 along a subsequence to a limit which is contained in $H^{1/2}$. By coercivity of Γ,

$$E_\varepsilon[u_\varepsilon,\Omega] \ge c[u_\varepsilon]^2_{H^{1/2}(\Omega)}.$$

Therefore the sequence $u_\varepsilon/\ln(1/\varepsilon)$ is bounded in the homogeneous $H^{1/2}$ seminorm. By the Poincaré inequality we can find vectors $\hat{d}_\varepsilon \in \mathbb{R}^N$ such that $(u_\varepsilon - \hat{d}_\varepsilon)/\ln(1/\varepsilon)$ is bounded in $H^{1/2}$ and has a subsequence which converges weakly in $H^{1/2}$ and strongly in L^2 to a limit u. We choose $d_\varepsilon \in \mathbb{Z}^N$ such that $|d_\varepsilon - \hat{d}_\varepsilon| \le N^{1/2}$ and observe that $(d_\varepsilon - \hat{d}_\varepsilon)/\ln(1/\varepsilon) \to 0$.

It remains to show that the limit u is in $BV(\Omega;\mathbb{R}^N)$. Let $\omega \subset\subset \Omega$. By Proposition 5.2 with $\delta = 1/4$, for sufficiently small ε there are $k_\varepsilon \in \mathbb{N}$ and $v_\varepsilon \in BV(\omega;\mathbb{Z}^N)$ such that $\varepsilon^{7/8} \le 2^{-k_\varepsilon} \le \varepsilon^{3/4}$,

$$\|v_\varepsilon - u_\varepsilon\|_{L^1(\omega)} \le CM^{1/2}2^{-k_\varepsilon/2}(\ln(1/\varepsilon))^{1/2} \le CM^{1/2}\varepsilon^{3/8}(\ln(1/\varepsilon))^{1/2}$$

and

$$|Dv_\varepsilon|(\omega) \le c\,\frac{E_\varepsilon[u_\varepsilon,\Omega]}{\ln(1/\varepsilon)} \le cM\ln(1/\varepsilon).$$

In particular, after extracting the same subsequence as above, $(v_\varepsilon - d_\varepsilon)/\ln(1/\varepsilon)$ converges to u in $L^1(\omega;\mathbb{R}^N)$ and $(v_\varepsilon - d_\varepsilon)/\ln(1/\varepsilon)$ is bounded in $BV(\omega;\mathbb{R}^N)$. Therefore, after possibly extracting a further subsequence, we obtain

$$\frac{v_\varepsilon - d_\varepsilon}{\ln(1/\varepsilon)} \rightharpoonup u \quad \text{weakly in } BV(\omega;\mathbb{R}^N)$$

and

$$|Du|(\omega) \le cM$$

with c not depending on ω. Since the bound does not depend on ω we conclude that $u \in BV(\Omega;\mathbb{R}^N)$. ∎

6. Density and approximation

We give here a refinement of Theorem 4.2 that will be needed in the proof of the upper bound. The main difference is that we can approximate with functions which are at the same time polyhedral and uniformly bounded in L^∞. This is clearly only possible if the limit is contained in L^∞. The refined upper bound requires an extra assumption on the energy density which is fulfilled by the function ψ defined in (2.6) (see Lemma 6.4).

We recall that $v \in BV_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^2$ open, is *polyhedral* if $Dv = \sum_{h=0}^{H} [v_h] \otimes n_h \mathcal{H}^1 \llcorner S_h$, where $H \in \mathbb{N}$, $S_h = [a_h, b_h]$ is a segment in $\mathbb{R}^2$, and $n_h \in S^1$ is normal to S_h.

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and let $\psi : \mathbb{Z}^N \times S^1 \to [0,\infty)$ obey*

$$\frac{1}{c}|b| \le \psi(b,n) \le (1 + c|n - n'|)\psi(b,n') \quad \text{for all } b \in \mathbb{Z}^N,\ n, n' \in S^1.$$

For any $u \in L^\infty(\Omega;\mathbb{R}^N) \cap BV(\Omega;\mathbb{R}^N)$, any $\delta > 0$, and any sequence $\sigma_j \to 0$ there is a sequence of polyhedral functions $v_j \in SBV(\Omega;\sigma_j\mathbb{Z}^N)$ such that $v_j \to u$ in L^1 and

$$\limsup_{j\to\infty} E^{\mathrm{LT}}_{\sigma_j}[v_j,\Omega] \le E^{\mathrm{LT}}_0[u,\Omega] + \delta$$

with $\sup_j \|v_j\|_{L^\infty(\Omega)} + |Dv_j|(\Omega) < \infty$ and E^{LT}_σ, E^{LT}_0 as in (4.3) and (4.4).

The proof of Proposition 6.1 is based on the following density result, which was proven in [21, Lemma. 6.4], building on [16, Corollary 2.2]. The key ingredient in this construction is the scalar result in [26, Theorem 4.2.20]. The related situation for partition problems was studied in [14, Theorem 2.1].

Lemma 6.2 ([21, Lemma 6.4]). *Assume that $\psi : \mathbb{Z}^N \times S^1 \to \mathbb{R}$ satisfies*

$$\psi(b,n) \le (1 + c|n - n'|)\psi(b,n') \quad \text{for all } b \in \mathbb{Z}^N,\ n, n' \in S^1.$$

Assume that $u \in BV(\mathbb{R}^2;\mathbb{Z}^N)$ and let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz set with $|Du|(\partial\Omega) = 0$. Then for any $\eta \in (0,1)$ there are $r > 0$, a polyhedral $v \in BV(\mathbb{R}^2;\mathbb{Z}^N)$ and a bijective *map* $f \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ *such that*

$$|D(u\circ f) - Dv|(\mathbb{R}^2) \le \eta, \tag{6.1}$$

$$|Df(x) - \mathrm{Id}| + |f(x) - x| \le \eta \quad \text{for all } x \in \mathbb{R}^2, \tag{6.2}$$

and

$$\int_{J_v\cap\Omega_r} \psi([v],\nu)\, d\mathcal{H}^1 \le (1+c\eta)\int_{J_u\cap\Omega} \psi([u],\nu)\, d\mathcal{H}^1 + c\eta, \tag{6.3}$$

where $\Omega_r := \{x \in \mathbb{R}^2 : \mathrm{dist}(x,\Omega) < r\}$. *Further, the restriction of* v *to* Ω *is polyhedral.*

We start by deriving a variant of this lemma.

Lemma 6.3. *Assume that* $\psi : \mathbb{Z}^N \times S^1 \to \mathbb{R}$ *satisfies*

$$\psi(b,n) \le (1 + c|n-n'|)\psi(b,n') \quad \text{for all } b \in \mathbb{Z}^N,\ n,n' \in S^1.$$

Assume that $u \in BV(\Omega;\sigma\mathbb{Z}^N)$ *for some* $\sigma > 0$ *with* $\Omega \subset \mathbb{R}^2$ *bounded and Lipschitz. Then for any* $\eta \in (0,1)$ *there is a polyhedral* $v \in BV(\Omega;\sigma\mathbb{Z}^N)$ *such that*

$$|Dv|(\Omega) \le 5|Du|(\Omega), \tag{6.4}$$

$$\|u - v\|_{L^1(\Omega)} \le c\sigma + c\eta|Du|(\Omega), \tag{6.5}$$

and

$$\int_{J_v\cap\Omega} \sigma\psi([v]/\sigma,\nu)\, d\mathcal{H}^1 \le (1+c\eta)\int_{J_u\cap\Omega} \sigma\psi([u]/\sigma,\nu)\, d\mathcal{H}^1 + c\eta\sigma. \tag{6.6}$$

Proof. Replacing u by u/σ and v by v/σ we see that it suffices to consider the case $\sigma = 1$. We can also assume $|Du|(\Omega) > 0$ (otherwise u is constant and $v = u$ will do). We extend u to a function $u \in BV(\mathbb{R}^2;\mathbb{Z}^N)$ such that $|Du|(\partial\Omega) = 0$, for instance, by reflection (see [8]). Possibly reducing η we can assume $\eta \le |Du|(\Omega)$ and $|Du|(\Omega_\eta) \le 2|Du|(\Omega)$, where Ω_η is defined as in Lemma 6.2.

We apply Lemma 6.2 to obtain a polyhedral $v \in BV(\mathbb{R}^2;\mathbb{Z}^N)$ and a diffeomorphism f satisfying (6.1)–(6.3). We define

$$d := \frac{1}{|\Omega|}\int_\Omega (v - u\circ f)\, dx \in \mathbb{R}^N$$

and choose $\tilde d \in \mathbb{Z}^N$ such that $|d - \tilde d| \le N^{1/2}$. We replace v by $v - \tilde d$, so that

$$\left|\frac{1}{|\Omega|}\int_\Omega (v - u\circ f)\, dx\right| \le N^{1/2}, \tag{6.7}$$

while (6.1) and (6.3) are not affected. We then estimate using (6.1) and (6.2):

$$\begin{aligned}
|Dv|(\Omega) &\le \eta + |D(u\circ f)|(\Omega) \le \eta + \int_{J_{u\circ f}\cap\Omega} |[u]|\circ f\, d\mathcal{H}^1 \\
&\le \eta + \int_{J_u\cap f(\Omega)} |[u]|\,|Df^{-1}\nu^\perp|\, d\mathcal{H}^1 \\
&\le \eta + |Du|(\Omega_\eta)\sup\{|Df^{-1}e|(x) : x \in \mathbb{R}^2,\ e \in S^1\} \\
&\le \eta + |Du|(\Omega_\eta)(1+\eta) \le 5|Du|(\Omega).
\end{aligned}$$

This proves (6.4). Since (6.6) follows immediately from (6.3), it remains to prove (6.5). By Poincaré, (6.1) and (6.7),

$$\|u \circ f - v\|_{L^1(\Omega)} \le N^{1/2}|\Omega| + c|D(u \circ f) - Dv|(\Omega) \le N^{1/2}|\Omega| + c\eta \le c.$$

Now if we prove that there is $c_* > 0$ such that

$$\|u - u \circ f\|_{L^1(\Omega)} \le 2c_*\eta|Du|(\Omega_\eta), \tag{6.8}$$

then with a triangular inequality we obtain $\|v - u\|_{L^1(\Omega)} \le c + 2c_*\eta|Du|(\Omega_\eta)$ and conclude the proof.

It remains to prove (6.8). We start by proving that for any $z \in C^1(\mathbb{R}^2;\mathbb{R}^N)$ we have

$$\|z - z \circ f\|_{L^1(\Omega)} \le 2c_*\eta|Dz|(\Omega_\eta), \tag{6.9}$$

for some c_* chosen below. Indeed,

$$\begin{aligned}\int_\Omega |z(x) - z(f(x))|\,dx &= \int_\Omega \left|\int_0^1 Dz\big(x + t(f(x) - x)\big)(f(x) - x)\,dt\right| dx \\ &\le \|f(x) - x\|_{L^\infty(\Omega)} \int_0^1 \int_\Omega |Dz|\big(x + t(f(x) - x)\big)\,dx\,dt.\end{aligned}$$

For any $t \in [0,1]$, we define $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_t(x) := x + t(f(x) - x)$ and estimate by a change of variables and (6.2):

$$\int_\Omega |Dz|(F_t(x))\,dx = \int_{F_t(\Omega)} |Dz|\,|\det DF_t^{-1}|\,dx \le \|Dz\|_{L^1(\Omega_\eta)}(1 + c_*\eta).$$

Therefore (6.9) holds for any $z \in C^1$ and, by density, (6.8) is proven. ■

In what follows we prove that the unrelaxed line-tension energy density ψ defined in (2.6) satisfies the assumptions of Lemma 6.3.

Lemma 6.4. *Let Γ obey (2.3) and (2.4), and let ψ be defined by (2.6). Then*

$$\psi(b,n) = \int_{S^1} |y \cdot n|\hat\Gamma(y)b \cdot b\; d\mathcal{H}^1(y) \tag{6.10}$$

and

$$\psi(b,n) \le (1 + c|n - n'|)\psi(b,n') \quad \textit{for all } b \in \mathbb{Z}^N, n, n' \in S^1.$$

Proof. To prove the first equality, for a fixed $n \in S^1$, we write both sides in polar coordinates, measuring the angles θ with respect to the vector n. Precisely, we let $n^\perp := (-n_2, n_1)$ and write $y = n\cos\theta + n^\perp \sin\theta$, for $\theta \in [0, 2\pi)$. Then

$$\int_{S^1} |y \cdot n|\hat\Gamma(y)b \cdot b\,d\mathcal{H}^1(y) = \int_0^{2\pi} |\cos\theta|\hat\Gamma(n\cos\theta + n^\perp \sin\theta)b \cdot b\,d\theta.$$

At the same time, if $x \cdot n = 1$ we write $x = n + tn^\perp$ with $t = \tan\theta$, and using (2.3) and the first condition in (2.4),

$$\begin{aligned} 2\int_{\{x\cdot n=1\}} \Gamma(x)b\cdot b\, d\mathcal{H}^1(x) &= 2\int_{-\infty}^{\infty} \Gamma(n+n^\perp t)b\cdot b\, dt \\ &= 2\int_{-\pi/2}^{\pi/2} \Gamma(n+n^\perp \tan\theta)b\cdot b\frac{1}{\cos^2\theta}\, d\theta \\ &= 2\int_{-\pi/2}^{\pi/2} \Gamma\left(\frac{n\cos\theta + n^\perp\sin\theta}{\cos\theta}\right)b\cdot b\,\frac{1}{\cos^2\theta}\, d\theta \\ &= 2\int_{-\pi/2}^{\pi/2} \hat\Gamma(n\cos\theta + n^\perp \sin\theta)b\cdot b\cos\theta\, d\theta \\ &= \int_0^{2\pi} \hat\Gamma(n\cos\theta + n^\perp\sin\theta)b\cdot b|\cos\theta|\, d\theta. \end{aligned}$$

This concludes the proof of (6.10).

To prove the estimate we then write, with $|y\cdot n| - |y\cdot n'| \le |y\cdot(n-n')|$ and (2.4),

$$\begin{aligned} \psi(b,n) - \psi(b,n') &= \int_{S^1} (|y\cdot n| - |y\cdot n'|)\hat\Gamma(y)b\cdot b\, d\mathcal{H}^1(y) \\ &\le |n-n'| \int_{S^1} \hat\Gamma(y)b\cdot b\, d\mathcal{H}^1(y) \le c|n-n'|\,|b|^2 \end{aligned}$$

and, again from (2.4), $|b|^2 \le c\psi(b,n')$. This concludes the proof. ■

Proof of Proposition 6.1. By Theorem 4.2 there are functions $u_j \in SBV(\Omega;\sigma_j\mathbb{Z}^N)$ such that u_j converges to u strongly in $L^1(\Omega;\mathbb{R}^N)$ and

$$\limsup_{j\to\infty} \int_{J_{u_j}\cap\Omega} \sigma_j\psi([u_j]/\sigma_j,\nu_j)\, d\mathcal{H}^1 \le E_0^{\mathrm{LT}}[u,\Omega].$$

From $\frac{1}{c}|b| \le \psi(b,n)$ we infer that u_j is bounded in $BV(\Omega;\mathbb{R}^N)$. We apply Lemma 6.3 to u_j with $\eta_j := 1/j$ and obtain a polyhedral map $z_j \in SBV(\Omega;\sigma_j\mathbb{Z}^N)$ such that

$$|Dz_j|(\Omega) \le 5|Du_j|(\Omega) \le C_*, \quad \limsup_{j\to\infty} \|z_j - u_j\|_{L^1(\Omega)} = 0,$$

for some $C_* > 0$ and

$$\limsup_{j\to\infty} \int_{J_{z_j}\cap\Omega} \sigma_j\psi([z_j]/\sigma_j,\nu_j)\, d\mathcal{H}^1 \le E_0^{\mathrm{LT}}[u,\Omega].$$

Since z_j is polyhedral, there are finitely many segments $[a_h,b_h]$ such that

$$Dz_j = \sum_h (z_h^+ - z_h^-)\otimes n_h\mathcal{H}^1 \llcorner [a_h,b_h], \tag{6.11}$$

where for simplicity we do not indicate the index j on the traces, the normal, and the points.

Let $f_j := |z_j|$. Then $f_j \in BV(\Omega;[0,\infty))$ with $|Df_j|(\Omega) \le |Dz_j|(\Omega) \le C_*$. Possibly increasing C_* we can assume $2^{C_*} > \|u\|_{L^\infty(\Omega)}$. By the coarea formula,

$$|Df_j|(\Omega) = \int_0^\infty \mathcal{H}^1(\partial_*\{f_j > t\})\,dt = \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \mathcal{H}^1(\partial_*\{f_j > t\})\,dt \le C_*$$

(we use the short notation $\partial_*\{f_j > t\}$ for $\Omega \cap \partial_*\{x \in \Omega : f_j(x) > t\}$, where ∂_* denotes the essential boundary). For fixed $\delta > 0$, for any j we choose $k_j \in \mathbb{N}\cap(C_*, C_* + C_*/\delta + 1)$ such that

$$\int_{2^{k_j}}^{2^{k_j+1}} \mathcal{H}^1(\partial_*\{f_j > t\})\,dt \le \delta$$

and then pick $M_j \in (2^{k_j}, 2^{k_j+1})$ such that

$$2^{k_j}\mathcal{H}^1(\partial_*\{f_j > M_j\}) \le \delta. \tag{6.12}$$

We now define $\hat z_j : \Omega \to \sigma_j\mathbb{Z}^N$ by

$$\hat z_j(x) := \begin{cases} z_j(x) & \text{if } f_j(x) \le M_j, \\ 0 & \text{otherwise.} \end{cases}$$

From $z_j \to u$ pointwise almost everywhere and $M_j > 2^{C_*} > \|u\|_{L^\infty(\Omega)}$ we deduce that $\hat z_j \to u$ pointwise almost everywhere. It is easy to check that $\hat z_j \in BV(\Omega;\sigma_j\mathbb{Z}^N)$ (indeed, $J_{\hat z_j} \subseteq J_{z_j}$), and that

$$\|\hat z_j\|_{L^\infty(\Omega)} \le M_j \le C_\delta$$

(where $C_\delta := 2^{2+C_*+C_*/\delta}$). Further, by (6.12)

$$\begin{aligned} |D\hat z_j|(\Omega) &\le |Dz_j|(\Omega) + M_j\mathcal{H}^1(\partial_*\{f_j > M_j\}) \\ &\le |Dz_j|(\Omega) + 2\delta \le C^* + 2\delta. \end{aligned} \tag{6.13}$$

Therefore $\hat z_j$ converges to u weakly in $BV(\Omega;\mathbb{R}^N)$.

It remains to estimate the energy. The natural bound

$$E^{\mathrm{LT}}_{\sigma_j}[\hat z_j,\Omega] \le E^{\mathrm{LT}}_{\sigma_j}[z_j,\Omega] + \int_{\partial_*\{f_j > M_j\}} \sigma_j\psi([\hat z_j]/\sigma_j,\nu)\,d\mathcal{H}^1$$

does not give the stated result since we do not assume linear control on ψ from above (indeed, in the specific application of interest here ψ is quadratic in the first argument, as is apparent from (2.6)). Therefore we need another construction, to separate big jumps into many small jumps, which corresponds to the fact that the relaxed energy ψ_{rel} has linear growth in the first argument. We shall use the fact that the assumption $\psi(b,n) \le (1 + c|n - n'|)\psi(b,n')$ for all $b \in \mathbb{Z}^N$ and $n, n' \in S^1$ clearly implies that there exists a constant $\hat c > 0$ such that

$$\psi(b,n) \le \hat c|b| \quad \text{for all } b \in [-1,1]^N \cap \mathbb{Z}^N \text{ and all } n \in S^1. \tag{6.14}$$

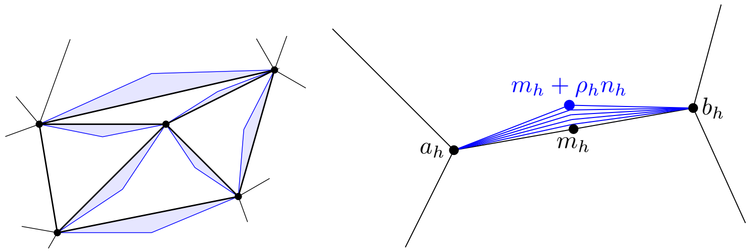

Fig. 2. Sketch of the construction in Proposition 6.1. Left panel: construction of the triangles around the segments on which $\hat{z}_j$ jumps. Right panel: separation of one "large" jump into many smaller jumps.

Recalling (6.11) we see that

$$D\hat{z}_j = \sum_h (\hat{z}_h^+ - \hat{z}_h^-) \otimes n_h \mathcal{H}^1 \llcorner [a_h, b_h], \tag{6.15}$$

where the segments are the same as in (6.11), and $\hat{z}_h^+ = z_h^+$ if $|z_h^+| \le M_j$ and 0 otherwise, and correspondingly for $\hat{z}_h^-$. The segments for which both traces are unchanged, or both new traces are zero, need not be treated. The critical set is

$$H := \{h : 0 = \hat{z}_h^+ \ne z_h^+ \text{ and } \hat{z}_h^- \ne 0\} \cup \{h : 0 = \hat{z}_h^- \ne z_h^- \text{ and } \hat{z}_h^+ \ne 0\}.$$

As in the computation in (6.13), we obtain

$$\sum_{h \in H} |D\hat{z}_j|([a_h, b_h]) \le M_j \mathcal{H}^1(\partial_* \{f_j > M_j\}) \le 2\delta.$$

For these segments we need to separate the jump into many smaller jumps. For any $h \in H$ we let $m_h := (a_h + b_h)/2$ be the midpoint of $[a_h, b_h]$, and choose $\rho_h \in (0, |b_h - a_h|)$ such that the triangles $T^h := \mathrm{conv}(a_h, b_h, m_h + \rho_h n_h)$ are, apart from the vertices, all disjoint and their total area is less than 2^{-j} (see Figure 2). This is possible since there are finitely many segments.

Choose now one $h \in H$, and assume for definiteness that $\hat{z}_h^- = 0$. Since $|\hat{z}_h^+| \le M_j$ there are $L_h \le M_j/\sigma_j$ and $\alpha_l \in \sigma_j(\mathbb{Z}^N \cap [-1, 1]^N)$, with $l = 1, \dots, L_h$, such that $\hat{z}_h^+ = \sum_{l=1}^{L_h} \alpha_l$. For $l = 0, \dots, L_h$ we define the triangles

$$T_l^h := \mathrm{conv}\left(a_h, b_h, m_h + \frac{l+1}{L_h + 1} \rho_h n_h\right)$$

and $v_j : \Omega \to \sigma_j \mathbb{Z}^N$ by setting $v_j = \sum_{l'=1}^{l} \alpha_{l'}$ on each $T_l^h \setminus T_{l-1}^h$, and $v_j = \hat{z}_j$ outside the union of the triangles. Then $[v_j] \in \sigma_j(\mathbb{Z}^N \cap [-1, 1]^N)$ on each of the closed triangles T^h, $v_j = \hat{z}_j$ on the outer boundary of T^h, and $|Dv_j|(T^h) \le c|a_h - b_h||\hat{z}_h^+ - \hat{z}_h^-| \le c|D\hat{z}_j|([a_h, b_h])$. Therefore, recalling (6.14),

$$\begin{aligned} E^{\mathrm{LT}}_{\sigma_j}[v_j,\Omega] &\le E^{\mathrm{LT}}_{\sigma_j}[z_j,\Omega] + \sum_{h\in H}\int_{\cup_l \partial T_l^h} \sigma_j\psi([v_j]/\sigma_j,\nu)\,d\mathcal{H}^1 \\ &\le E^{\mathrm{LT}}_{\sigma_j}[z_j,\Omega] + \sum_{h\in H}\int_{\cup_l \partial T_l^h} \hat c|[v_j]|\,d\mathcal{H}^1 \\ &= E^{\mathrm{LT}}_{\sigma_j}[z_j,\Omega] + c\sum_{h\in H}|Dv_j|(T^h) \\ &\le E^{\mathrm{LT}}_{\sigma_j}[z_j,\Omega] + c\sum_{h\in H}|D\hat z_j|([a_h,b_h]) \le E^{\mathrm{LT}}_{\sigma_j}[z_j,\Omega] + c\delta. \end{aligned}$$

The same computation also shows that v_j is bounded in BV. Since $|\{v_j \neq \hat z_j\}| \le 2^{-j}$ we obtain $v_j \to u$ pointwise almost everywhere. ■

7. Upper bound

The upper bound is obtained by an explicit but involved construction that combines several rescaling steps. Due to the incompatibility of the two constraints of being BV with values in a scaled copy of $\mathbb{Z}^N$ and being in $H^{1/2}$ we cannot use density and separate the two scales. Instead we need to use a joint construction, which depends on both scales.

We start from the sequence constructed in Section 6, which takes values in $SBV(\Omega;\sigma\mathbb{Z}^N)$ for a scale σ and converges slowly to 0 with respect to ε. The key step is the construction in the following lemma.

Lemma 7.1. *Let $\Omega \subset\subset \Omega'$ be two bounded Lipschitz domains. Let $\sigma > 0$, $v \in SBV(\Omega';\sigma\mathbb{Z}^N)\cap L^\infty(\Omega';\sigma\mathbb{Z}^N)$ polyhedral, $\alpha \in (0,1/2)$ and $\rho > 0$ with $3\rho^\alpha < \mathrm{dist}(\Omega,\partial\Omega')$. Then for any $\varepsilon > 0$ there are $w_\varepsilon \in L^2(\Omega;\mathbb{R}^N)$ and $\zeta \in \overline{B}_1$ such that*

$$\left\| \frac{w_\varepsilon}{\ln(1/\varepsilon)} \right\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)},$$
$$\left\| \frac{w_\varepsilon}{\ln(1/\varepsilon)} - v \right\|_{L^1(\Omega)} \le \rho^\alpha |Dv|(\Omega'),$$

and

$$\limsup_{\varepsilon\to 0} \frac{E_\varepsilon[w_\varepsilon,\Omega]}{(\ln(1/\varepsilon))^2} \le \int_{J_v\cap\Omega'} \sigma\psi([v]/\sigma,n)\,d\mathcal{H}^1 + f(\rho)(|Dv|(\Omega'))^{4/3}\|v\|^{2/3}_{L^\infty(\Omega')} + \int_\Omega\int_{\Omega\setminus B_\rho(x)} \Gamma(x-y)(v^\zeta_\infty(x)-v^\zeta_\infty(y))\cdot(v^\zeta_\infty(x)-v^\zeta_\infty(y))\,dy\,dx,$$

where $v^\zeta_\infty : \Omega \to \mathbb{R}^N$ is defined by

$$v^\zeta_\infty(x) := \int_0^1 v(x+\rho^\alpha\zeta t)\,dt, \tag{7.1}$$

and $f(\rho) \to 0$ as $\rho \to 0$. The function f depends on Γ and α, but not on v.

The proof will be given at the end of this section. The main point is to replicate each interface $\sigma \ln(1/\varepsilon)$ times, and then mollify on a scale ε. This modifies the function only on a small set, of area proportional to $\sigma\varepsilon \ln(1/\varepsilon)$, which ensures that the nonlinear term $\varepsilon^{-1}\|\mathrm{dist}(u_\varepsilon, \mathbb{Z}^N)\|_2^2$ vanishes in the limit, for an appropriate scaling of σ. Since the separation between the interfaces is much larger than the scale of the mollification, their interaction is small. For each interface, the energy is estimated by an explicit computation in Lemma 7.3. Care must be taken in undoing the several relaxation steps, both at the line-tension and at the continuous scale, and in several truncation steps in order to estimate the various error terms. For this construction in the upper bound we fix a mollifier $\varphi_1 \in C_c^\infty(B_1)$ and set $\varphi_\lambda := \lambda^{-2}\varphi_1(\lambda x)$.

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected Lipschitz domain, and let $E_\varepsilon[\cdot, \Omega]$ be defined as in (2.1), with W and Γ which satisfy (2.2)–(2.4). Let $u \in BV(\Omega; \mathbb{R}^N) \cap H^{1/2}(\Omega; \mathbb{R}^N)$. For any $\varepsilon > 0$ there is $u_\varepsilon \in L^2(\Omega; \mathbb{R}^N)$ with $u_\varepsilon/\ln(1/\varepsilon) \to u$ in L^2 and*

$$\limsup_{\varepsilon\to 0} \frac{E_\varepsilon[u_\varepsilon, \Omega]}{(\ln(1/\varepsilon))^2} \le F_0[u, \Omega]. \tag{7.2}$$

We recall that F_0 was defined in (2.10).

Proof. We start by reducing to the case that u is smooth and defined on a domain Ω' larger than Ω.

To see this, observe that since Ω is Lipschitz there are an open set ω with $\partial\Omega \subset \omega$ and a bilipschitz map $\Phi : \omega \to \omega$ such that $\Phi(x) = x$ for $x \in \partial\Omega$ and $\Phi(\Omega \cap \omega) = \omega \setminus \overline{\Omega}$. We define $\hat{u} : \Omega \cup \omega \to \mathbb{R}^N$ by reflection:

$$\hat{u} := \begin{cases} u & \text{in } \Omega, \\ u \circ \Phi & \text{in } \omega \setminus \Omega. \end{cases}$$

Then $\hat{u} \in BV(\Omega \cup \omega; \mathbb{R}^N) \cap H^{1/2}(\Omega \cup \omega; \mathbb{R}^N)$ with $|D\hat{u}|(\partial\Omega) = 0$. We fix $\delta > 0$ and let $\Omega_\delta := \{x : \mathrm{dist}(x, \Omega) < \delta\}$, so that

$$\limsup_{\delta\to 0} |D\hat{u}|(\Omega_\delta) = |D\hat{u}|(\Omega) = |Du|(\Omega)$$

and

$$\limsup_{\delta\to 0} [\hat{u}]_{H^{1/2}(\Omega_\delta)} = [\hat{u}]_{H^{1/2}(\Omega)} = [u]_{H^{1/2}(\Omega)}.$$

In particular,

$$\limsup_{\delta\to 0} F_0[\hat{u}, \Omega_\delta] = F_0[\hat{u}, \Omega] = F_0[u, \Omega].$$

Now for sufficiently small δ define $u_\delta := \varphi_\delta * \hat{u} \in C^\infty(\overline{\Omega_\delta}; \mathbb{R}^N)$, with φ_δ the mollification kernel. Since F_0 is convex, we have

$$\limsup_{\delta\to 0} F_0[u_\delta, \Omega_\delta] \le \limsup_{\delta\to 0} F_0[\hat{u}, \Omega_{2\delta}] = F_0[u, \Omega]$$

and $u_\delta \to u$ in $L^2(\Omega; \mathbb{R}^N)$.

Therefore in the rest of the proof we assume that $u \in C^\infty(\overline{\Omega'};\mathbb{R}^N)$ is given, with $\Omega \subset\subset \Omega'$ and Ω' Lipschitz. We shall show that for any $\eta > 0$ and any $\varepsilon > 0$ there is $w_\varepsilon \in L^2(\Omega;\mathbb{R}^N)$ such that $w_\varepsilon/\ln(1/\varepsilon) \to u$ in $L^2(\Omega;\mathbb{R}^N)$ and

$$\limsup_{\varepsilon\to 0} \frac{1}{(\ln(1/\varepsilon))^2} E_\varepsilon[w_\varepsilon,\Omega] \le F_0[u,\Omega'] + \eta. \tag{7.3}$$

Since η is arbitrary, taking a diagonal subsequence will conclude the proof of (7.2).

It remains to prove (7.3). Let $\sigma_j \in (0,1)$ be such that $\sigma_j \downarrow 0$ as $j \to 0$. By Proposition 6.1 (which can be applied thanks to Lemma 6.4) there are polyhedral functions $v_j \in SBV(\Omega';\sigma_j\mathbb{Z}^N)$ such that v_j converges to u strongly in $L^1(\Omega';\mathbb{R}^N)$,

$$\limsup_{j\to\infty} \int_{J_{v_j}\cap\Omega'} \sigma_j \psi([v_j]/\sigma_j, n_j)\, d\mathcal{H}^1 \le F_{\text{self}}[u,\Omega'] + \eta,$$

and $C_\eta := \sup \|v_j\|_{L^\infty(\Omega)} + |Dv_j|(\Omega') < \infty$. We recall that since u is smooth, in particular $u \in L^\infty(\Omega';\mathbb{R}^N)$ and $F_{\text{self}}[u,\Omega'] = E_0^{\text{LT}}[u,\Omega']$ (see Theorems 2.1 and 4.2).

Since v_j is polyhedral, by Lemma 7.1 for $\alpha := 1/3$, and ε and ρ small enough, there are functions $w_\varepsilon^{j,\rho} \in L^2(\Omega;\mathbb{R}^N)$ and vectors $\zeta_j \in \overline{B}_1$ such that

$$\begin{aligned} \left\| \frac{w_\varepsilon^{j,\rho}}{\ln(1/\varepsilon)} \right\|_{L^\infty(\Omega)} &\le \|v_j\|_{L^\infty(\Omega)} \le C_\eta, \\ \left\| \frac{w_\varepsilon}{\ln(1/\varepsilon)} - v \right\|_{L^1(\Omega)} &\le \rho^\alpha |Dv|(\Omega'), \end{aligned} \tag{7.4}$$

and

$$\begin{aligned} \limsup_{\varepsilon\to 0} & \frac{1}{(\ln(1/\varepsilon))^2} E_\varepsilon[w_\varepsilon^{j,\rho},\Omega] \\ &\le \int_{J_{v_j}\cap\Omega'} \sigma_j \psi([v_j]/\sigma_j,n_j)\, d\mathcal{H}^1 + f(\rho)(|Dv_j|(\Omega'))^{4/3} \|v_j\|_{L^\infty(\Omega')}^{2/3} \\ &\quad + \int_\Omega \int_{\Omega\setminus B_\rho(x)} \Gamma(x-y)(v_{j,\infty}^{\zeta_j}(x) - v_{j,\infty}^{\zeta_j}(y)) \cdot (v_{j,\infty}^{\zeta_j}(x) - v_{j,\infty}^{\zeta_j}(y))\, dy\, dx, \end{aligned}$$

where $v_{j,\infty}^{\zeta_j} = (v_j)_\infty^{\zeta_j}$ is an average of v_j in direction ζ_j at a scale set by ρ as defined in (7.1). Further, from (7.4),

$$\begin{aligned} \limsup_{\varepsilon\to 0} \left\| \frac{w_\varepsilon^{j,\rho}}{\ln(1/\varepsilon)} - u \right\|_{L^1(\Omega)} &\le \|v_j - u\|_{L^1(\Omega)} + \limsup_{\varepsilon\to 0} \left\| \frac{w_\varepsilon^{j,\rho}}{\ln(1/\varepsilon)} - v_j \right\|_{L^1(\Omega)} \\ &\le \|v_j - u\|_{L^1(\Omega)} + \rho^\alpha |Dv_j|(\Omega') \\ &\le \|v_j - u\|_{L^1(\Omega)} + C_\eta \rho^\alpha. \end{aligned}$$

We now take $j \to \infty$, and extract a subsequence such that $\zeta_{j_k} \to \zeta_\infty$ and

$$\lim_{k\to\infty} \limsup_{\varepsilon\to 0} \frac{1}{(\ln(1/\varepsilon))^2} E_\varepsilon[w_\varepsilon^{j_k,\rho},\Omega] = \limsup_{j\to\infty} \limsup_{\varepsilon\to 0} \frac{1}{(\ln(1/\varepsilon))^2} E_\varepsilon[w_\varepsilon^{j,\rho},\Omega].$$

By dominated convergence, $(v_j)_\infty^{\xi_j} \to u_\infty^{\xi_\infty}$ pointwise and hence in $L^2(\Omega;\mathbb{R}^N)$ and

$$\limsup_{j\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{(\ln(1/\varepsilon))^2}E_\varepsilon[w_\varepsilon^{j,\rho},\Omega] \le F_{\mathrm{self}}[u,\Omega'] + \eta + C_\eta^2 f(\rho) + \int_\Omega\int_{\Omega\setminus B_\rho(x)} \Gamma(x-y)(u_\infty^{\xi_\infty}(x)-u_\infty^{\xi_\infty}(y))\cdot(u_\infty^{\xi_\infty}(x)-u_\infty^{\xi_\infty}(y))\,dy\,dx.$$

As $\rho\to 0$ we see that, since $u\in C^\infty(\Omega';\mathbb{R}^N)$, we have $u_\infty^{\xi_\infty}\to u$ in $L^2(\Omega;\mathbb{R}^N)$ and in $H^{1/2}(\Omega;\mathbb{R}^N)$ and therefore

$$\begin{aligned}\limsup_{\rho\to 0}\limsup_{j\to\infty}\limsup_{\varepsilon\to 0}&\frac{1}{(\ln(1/\varepsilon))^2}E_\varepsilon[w_\varepsilon^{j,\rho},\Omega]\\ &\le F_{\mathrm{self}}[u,\Omega'] + \eta + \int_{\Omega'\times\Omega'}\Gamma(x-y)(u(x)-u(y))\cdot(u(x)-u(y))\,dy\,dx\\ &= F_0[u,\Omega']+\eta\end{aligned}$$

with

$$\limsup_{\rho\to 0}\limsup_{j\to\infty}\limsup_{\varepsilon\to 0}\left\|\frac{w_\varepsilon^{j,\rho}}{\ln(1/\varepsilon)}-u\right\|_{L^2(\Omega)}=0.$$

Taking a diagonal sequence concludes the proof of (7.3). ■

It remains to show the detailed construction of the functions $w_\varepsilon^{j,\rho}$ given in Lemma 7.1. First we recall that the unrelaxed line-tension energy for polyhedral interfaces can be obtained with a direct computation starting from the nonlocal energy.

Lemma 7.3. *Let $\Omega\subset\subset\Omega'$ be two bounded open sets, $v\in SBV(\Omega';\mathbb{Z}^N)$ polyhedral, and assume that Γ obeys (2.3) and (2.4). Let $\varphi_\varepsilon\in C_c^\infty(B_\varepsilon)$ be a mollifier and $\varepsilon<\mathrm{dist}(\Omega,\partial\Omega')$. Then $w_\varepsilon:=\varphi_\varepsilon * v$ obeys*

$$\limsup_{\varepsilon\to 0}\frac{1}{\ln(1/\varepsilon)}\int_{\Omega\times\Omega}\Gamma(x-y)(w_\varepsilon(x)-w_\varepsilon(y))\cdot(w_\varepsilon(x)-w_\varepsilon(y))\,dx\,dy \le \int_{J_v\cap\Omega'}\psi([v],n)\,d\mathcal{H}^1,$$

where ψ is as in (2.6).

Proof. See [33, Section 6]. ■

Proof of Lemma 7.1. We choose Ω'' such that $\Omega\subset\subset\Omega''\subset\subset\Omega'$, $\rho^\alpha<\mathrm{dist}(\Omega,\partial\Omega'')$, and $\rho^\alpha<\mathrm{dist}(\Omega'',\partial\Omega')$, and for $\zeta\in\overline{B}_1\subset\mathbb{R}^2$ and $L>0$ we define the functions $v_L^\zeta:\Omega''\to\frac{\sigma}{L}\mathbb{Z}^N$ by

$$v_L^\zeta(x):=\sum_{j=1}^{\lfloor L\rfloor}\frac{1}{L}v\left(x+\rho^\alpha\frac{j}{L}\zeta\right).$$

Note that $v_L^\zeta\in SBV(\Omega'';\frac{\sigma}{L}\mathbb{Z}^N)$ has jump set which is obtained by $\lfloor L\rfloor$ copies of the jump set of v, translated in the direction of ζ, and that $\|v_L^\zeta\|_{L^\infty(\Omega'')}\le\|v\|_{L^\infty(\Omega')}$.

We set $L_\varepsilon := \sigma \ln(1/\varepsilon)$. For $\varepsilon \le \mathrm{dist}(\Omega, \partial\Omega'')$ we define

$$w_\varepsilon := \ln(1/\varepsilon) v_{L_\varepsilon}^{\zeta_\varepsilon} * \varphi_\varepsilon \quad \text{and} \quad \hat{w}_\varepsilon := v_{L_\varepsilon}^{\zeta_\varepsilon} * \varphi_\varepsilon$$

(if $\varepsilon > \mathrm{dist}(\Omega, \partial\Omega'')$ we can set $w_\varepsilon = 0$), and the vectors $\zeta_\varepsilon \in \overline{B}_1$ will be chosen below.

We remark that $v \in \sigma\mathbb{Z}^N$ almost everywhere implies $v_{L_\varepsilon}^{\zeta_\varepsilon} \in \frac{1}{L_\varepsilon}\sigma\mathbb{Z}^N = \frac{1}{\ln(1/\varepsilon)}\mathbb{Z}^N$ a.e., therefore $w_\varepsilon(x) \in \mathbb{Z}^N$ for any x at distance at least ε from $J_{v_{L_\varepsilon}^{\zeta_\varepsilon}}$. Since J_v is a finite union of segments, and $\mathrm{dist}(w_\varepsilon, \mathbb{Z}^N) \le N^{1/2}$ everywhere, we have

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon(\ln(1/\varepsilon))^2} \int_\Omega \mathrm{dist}^2(w_\varepsilon, \mathbb{Z}^N)\, dx \le \lim_{\varepsilon\to 0} \frac{\lfloor L_\varepsilon \rfloor \, N \, |\{x \in \Omega : \mathrm{dist}(x, J_v \cap \Omega') < \varepsilon\}|}{\varepsilon(\ln(1/\varepsilon))^2} = 0.$$

We write the long-range elastic energy as a bilinear form, $B_{LR}^\rho : L^2(\Omega; \mathbb{R}^N)^2 \to \mathbb{R}$, where

$$B_{LR}^\rho(u, u') := \int_\Omega \int_{\Omega \setminus B_\rho(x)} \Gamma(x-y)(u(x)-u(y)) \cdot (u'(x)-u'(y))\, dy\, dx,$$

and choose a sequence $\varepsilon_i \to 0$, $\varepsilon_i > 0$, such that

$$\lim_{i\to\infty} B_{LR}^\rho(\hat{w}_{\varepsilon_i}, \hat{w}_{\varepsilon_i}) = \limsup_{\varepsilon\to 0} B_{LR}^\rho(\hat{w}_\varepsilon, \hat{w}_\varepsilon). \tag{7.5}$$

After extracting a further subsequence, we can additionally assume that $\zeta_{\varepsilon_i} \to \zeta$ for some $\zeta \in \overline{B}_1$. This defines the vector ζ in the statement (in terms of the vectors ζ_ε chosen below). We now show that

$$\hat{w}_{\varepsilon_i} \to v_\infty^\zeta \quad \text{strongly in } L^2(\Omega; \mathbb{R}^N). \tag{7.6}$$

To see this, we first observe that since $\|\hat{w}_{\varepsilon_i}\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega')}$ and $\|v_\infty^\zeta\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega')}$ it suffices to prove convergence in $L^1(\Omega; \mathbb{R}^N)$. We write

$$\hat{w}_{\varepsilon_i} - v_\infty^\zeta = \varphi_{\varepsilon_i} * (v_{L_{\varepsilon_i}}^{\zeta_{\varepsilon_i}} - v_\infty^{\zeta_{\varepsilon_i}}) + \varphi_{\varepsilon_i} * (v_\infty^{\zeta_{\varepsilon_i}} - v_\infty^\zeta) + (\varphi_{\varepsilon_i} * v_\infty^\zeta - v_\infty^\zeta)$$

and estimate the three terms separately. Convergence of the last one is immediate. Performing an explicit computation one can show that

$$\|v_\infty^{\zeta_{\varepsilon_i}} - v_\infty^\zeta\|_{L^1(\Omega'')} \le \rho^\alpha |Dv|(\Omega') |\zeta_{\varepsilon_i} - \zeta|,$$

which implies $\|\varphi_{\varepsilon_i} * (v_\infty^{\zeta_{\varepsilon_i}} - v_\infty^\zeta)\|_{L^1(\Omega)} \le \|v_\infty^{\zeta_{\varepsilon_i}} - v_\infty^\zeta\|_{L^1(\Omega'')} \to 0$. Analogously, from

$$\|v_{L_{\varepsilon_i}}^{\zeta_{\varepsilon_i}} - v_\infty^{\zeta_{\varepsilon_i}}\|_{L^1(\Omega'')} \le \frac{\rho^\alpha}{L_{\varepsilon_i}} |Dv|(\Omega') + |\Omega''| \frac{L_\varepsilon - \lfloor L_\varepsilon \rfloor}{L_\varepsilon} \|v\|_{L^\infty(\Omega'')}$$

and $\lim_{i\to\infty} L_{\varepsilon_i} = \infty$ we obtain $\|\varphi_{\varepsilon_i} * (v_{L_{\varepsilon_i}}^{\zeta_{\varepsilon_i}} - v_\infty^{\zeta_{\varepsilon_i}})\|_{L^1(\Omega)} \to 0$. This concludes the proof of (7.6).

By continuity of B^ρ_{LR}, (7.5) and (7.6) imply

$$\limsup_{\varepsilon\to 0} B^\rho_{LR}(\hat w_\varepsilon, \hat w_\varepsilon) = \lim_{i\to\infty} B^\rho_{LR}(\hat w_{\varepsilon_i}, \hat w_{\varepsilon_i}) = B^\rho_{LR}(v^\zeta_\infty, v^\zeta_\infty).$$

The short-range elastic energy can be correspondingly written, for a Borel set $E \subseteq \mathbb{R}^2$, as the bilinear form $B^\rho_{SR}(\cdot,\cdot,E) : L^2(E;\mathbb{R}^N)^2 \to \mathbb{R}$, where

$$B^\rho_{SR}(u,u',E) := \int_E \int_{E\cap B_\rho(x)} \Gamma(x-y)(u(x)-u(y))\cdot(u'(x)-u'(y))\,dy\,dx.$$

This term will lead us to the choice of ζ_ε. We are interested in showing that for any ε there is a choice of $\zeta \in \overline{B_1}$ which permits control of the quantity

$$B^\rho_{SR}(\hat w_\varepsilon, \hat w_\varepsilon, \Omega) = \frac{1}{L_\varepsilon^2} \sum_{j,j'=1}^{\lfloor L_\varepsilon \rfloor} B^\rho_{SR}(T^\zeta_j v * \varphi_\varepsilon, T^\zeta_{j'} v * \varphi_\varepsilon, \Omega)$$

where T^ζ_j is the translation operator, $(T^\zeta_j f)(x) := f(x + j\zeta\rho^\alpha/L_\varepsilon)$. The separation introduced by the translations is on a length scale much larger than ε, but still infinitesimal (the choice of ζ_ε below shall implicitly ensure that it is not too small), therefore it is appropriate to treat the diagonal ($j = j'$) terms separately. Using translation invariance we can see that the diagonal contribution is

$$\begin{aligned} B^{\text{diag}}_{SR}(\zeta) &:= \frac{1}{L_\varepsilon^2} \sum_{j=1}^{\lfloor L_\varepsilon \rfloor} B^\rho_{SR}(T^\zeta_j v * \varphi_\varepsilon, T^\zeta_j v * \varphi_\varepsilon, \Omega) \\ &\le \frac{\lfloor L_\varepsilon \rfloor}{L_\varepsilon^2} B^\rho_{SR}(v * \varphi_\varepsilon, v * \varphi_\varepsilon, \Omega'') = \frac{\lfloor L_\varepsilon \rfloor \sigma}{L_\varepsilon \ln(1/\varepsilon)} B^\rho_{SR}(\sigma^{-1} v * \varphi_\varepsilon, \sigma^{-1} v * \varphi_\varepsilon, \Omega'') \end{aligned}$$

and in particular that the latter expression does not depend on the choice of ζ. Since $\sigma^{-1} v \in SBV(\Omega';\mathbb{Z}^N)$ is polyhedral and $\Gamma \ge 0$ pointwise, recalling Lemma 7.3, we obtain

$$\limsup_{\varepsilon\to 0} B^{\text{diag}}_{SR}(\zeta) \le \int_{J_v\cap\Omega'} \sigma\psi([v]/\sigma, n)\,d\mathcal{H}^1 \quad \text{for any } \zeta \in \overline{B_1}.$$

The off-diagonal contributions reduce to

$$\begin{aligned} B^{\text{cross}}_{SR}(\zeta) &:= \frac{1}{L_\varepsilon^2} \sum_{j\ne j'} B^\rho_{SR}(T^\zeta_j v * \varphi_\varepsilon, T^\zeta_{j'} v * \varphi_\varepsilon, \Omega) \\ &\le \frac{1}{L_\varepsilon^2} \sum_{j\ne j'} B^\rho_{SR}(v * \varphi_\varepsilon, T^\zeta_{j-j'} v * \varphi_\varepsilon, \Omega''). \end{aligned}$$

We average over all possible choices of the shifts ζ. Precisely, we compute, using linearity of B^ρ_{SR} in the second argument,

$$\begin{aligned} A_\varepsilon &:= \int_{B_1} B^{\text{cross}}_{SR}(\zeta)\varphi_1(\zeta)\,d\zeta \le \int_{B_1} \frac{1}{L_\varepsilon^2} \sum_{j\ne j'} B^\rho_{SR}(v * \varphi_\varepsilon, \varphi_1(\zeta) T^\zeta_{j-j'} v * \varphi_\varepsilon, \Omega'')\,d\zeta \\ &= B^\rho_{SR}(v * \varphi_\varepsilon, V * \varphi_\varepsilon, \Omega''), \end{aligned}$$

with

$$V(x) := \int_{B_1} \frac{1}{L_\varepsilon^2} \sum_{j \neq j'} \varphi_1(\zeta) v(x + \frac{j - j'}{L_\varepsilon} \rho^\alpha \zeta)\, d\zeta.$$

By a change of variables we obtain $V = \Phi * v$, where

$$\Phi(x) := \frac{1}{L_\varepsilon^2} \sum_{j \neq j'} \frac{L_\varepsilon^2}{(j - j')^2} \varphi_{\rho^\alpha}\left(\frac{L_\varepsilon}{j - j'} x\right),$$

and then

$$A_\varepsilon \leq B_{SR}^\rho(v * \varphi_\varepsilon, \Phi * v * \varphi_\varepsilon, \Omega'').$$

We fix $p \in (2, \infty)$ and denote by $q := p/(p-1)$ the dual exponent. Then

$$\begin{aligned} A_\varepsilon \leq &\int_{B_\rho} \frac{c}{|z|^3} \int_{\Omega''} |(\varphi_\varepsilon * v)(x) - (\varphi_\varepsilon * v)(x+z)| \\ &\qquad \times |(\varphi_\varepsilon * \Phi * v)(x) - (\varphi_\varepsilon * \Phi * v)(x+z)|\, dx\, dz \\ \leq &\int_{B_\rho} \frac{c}{|z|^{2-1/p}} \frac{\|v(\cdot) - v(\cdot + z)\|_{L^p(\Omega''_\varepsilon)}}{|z|^{1/p}} \frac{\|(\Phi * v)(\cdot) - (\Phi * v)(\cdot + z)\|_{L^q(\Omega''_\varepsilon)}}{|z|}\, dz, \end{aligned}$$

with $\Omega''_\varepsilon := \{x : \mathrm{dist}(x, \Omega'') < \varepsilon\}$. We estimate, for small z,

$$\begin{aligned} \frac{\|v(\cdot) - v(\cdot + z)\|_{L^p(\Omega''_\varepsilon)}^p}{|z|} &\leq 2^{p-1} \|v\|_{L^\infty(\Omega')}^{p-1} \int_{\Omega''_\varepsilon} \frac{|v(x) - v(x+z)|}{|z|}\, dx \\ &\leq 2^{p-1} \|v\|_{L^\infty(\Omega')}^{p-1} |Dv|(\Omega') \end{aligned}$$

and

$$\frac{\|(\Phi * v)(\cdot) - (\Phi * v)(\cdot + z)\|_{L^q(\Omega''_\varepsilon)}}{|z|} \leq \|\Phi\|_{L^q(\mathbb{R}^2)} |Dv|(\Omega'),$$

so that, with $\int_{B_\rho} \frac{1}{|z|^{2-1/p}} dz \leq c\rho^{1/p}$, we conclude

$$A_\varepsilon \leq c\rho^{1/p} \|v\|_{L^\infty(\Omega')}^{1/q} (|Dv|(\Omega'))^{1+1/p} \|\Phi\|_{L^q(\mathbb{R}^2)}.$$

Finally, recalling that $p > 2$,

$$\begin{aligned} \|\Phi\|_{L^q(\mathbb{R}^2)} &\leq 2L_\varepsilon \sum_{j=1}^{\lfloor L_\varepsilon \rfloor} \frac{1}{j^2} \left\| \varphi_{\rho^\alpha}\left(\frac{L_\varepsilon}{j} x\right) \right\|_{L^q(\mathbb{R}^2)} \leq cL_\varepsilon \sum_{j=1}^{\lfloor L_\varepsilon \rfloor} \frac{1}{j^2} \left(\frac{j}{L_\varepsilon}\right)^{2/q} \rho^{-2\alpha \frac{q-1}{q}} \\ &\leq cL_\varepsilon^{1-2/q} \sum_{j=1}^{\lfloor L_\varepsilon \rfloor} \frac{1}{j^{2/p}} \rho^{-2\alpha/p} \leq cL_\varepsilon^{1-2/q} L_\varepsilon^{1-2/p} \rho^{-2\alpha/p} = c\rho^{-2\alpha/p}. \end{aligned}$$

Therefore $A_\varepsilon \leq c\rho^{(1-2\alpha)/p} \|v\|_{L^\infty(\Omega')}^{1/q} (|Dv|(\Omega'))^{1+1/p}$. We finally choose $p = 3$, and ζ_ε so that it is as good as on average, in the sense that $B_{SR}^{\mathrm{cross}}(\zeta_\varepsilon) \leq A_\varepsilon$, and conclude the proof. ∎

8. Lower bound

In this section we prove the lower bound. The idea is that the limit is given by two terms, arising from short-range and long-range contributions to the nonlocal interaction, respectively. Indeed, one key idea in the proof of Proposition 8.1 is to localize the limiting energy and view it as a measure on $\Omega \times \Omega \subset \mathbb{R}^4$. One then shows that this measure can be written as the sum of two mutually singular terms, one supported on the diagonal and one supported outside the diagonal (see Figure 3). The lower bound arises from estimating these two terms separately. In the estimate of the diagonal term, which is local in the limit, we build upon techniques obtained for a different scaling in [18]; see Proposition 8.2 below. One important step is to iteratively mollify the functions along the sequence and to show that on most scales the mollification does not reduce the BV norm significantly, which implies that the functions are approximately one-dimensional at that scale. The proof is done by showing that one can choose a scale that contains, up to higher-order terms, as much energy as the average scale, and that at the same time it has a small loss of BV norm; see (8.17) and (8.18) below.

Proposition 8.1. *Under the assumptions of Theorem 2.1, for any $u \in BV(\Omega; \mathbb{R}^N)$ and any sequences $\varepsilon_i \to 0$ and $u_i \in L^2(\Omega; \mathbb{R}^N)$ with $u_i/\ln(1/\varepsilon_i) \to u$ in $L^2(\Omega; \mathbb{R}^N)$ one has*

$$F_0[u, \Omega] \le \liminf_{i\to\infty} \frac{E_{\varepsilon_i}[u_i, \Omega]}{(\ln(1/\varepsilon_i))^2}, \tag{8.1}$$

where F_0 was defined in (2.10).

The proof is based on the following local lower bound, which relates the short-range part of the energy to F_{self}.

Proposition 8.2. *Under the assumptions of Theorem 2.1, for any $u \in BV(\Omega; \mathbb{R}^N)$ and any sequences $\varepsilon_i \to 0$ and $u_i \in L^2(\Omega; \mathbb{R}^N)$ with $u_i/\ln(1/\varepsilon_i) \to u$ in $L^2(\Omega; \mathbb{R}^N)$, and any open set $\omega \subset \Omega$, one has*

$$F_{\mathrm{self}}[u, \omega] \le \liminf_{i\to\infty} \frac{E_{\varepsilon_i}[u_i, \omega]}{(\ln(1/\varepsilon_i))^2},$$

where F_{self} was defined in (2.11).

We postpone the proof of Proposition 8.2, and first show that it implies Proposition 8.1.

Proof of Proposition 8.1. We can assume that the lim inf in (8.1) is finite and, after passing to a subsequence, that it is a limit. By Proposition 5.1 we can assume that $u \in BV(\Omega; \mathbb{R}^N) \cap H^{1/2}(\Omega; \mathbb{R}^N)$.

We start by localizing the energy. We denote by $\Delta := \{(x, x) : x \in \mathbb{R}^2\} \subset \mathbb{R}^4$ the diagonal set in $\mathbb{R}^4$ and by $P : \mathbb{R}^4 \to \mathbb{R}^2$, $P(x_1, x_2, y_1, y_2) := (x_1, x_2)$, the projection on the first two components. For any Borel set $E \subseteq \Omega \times \Omega$ we define

$$\mu_i(E) := \frac{1}{c_W \varepsilon_i (\ln(1/\varepsilon_i))^2} \int_{P(E\cap\Delta)} \operatorname{dist}^2(u_i(x), \mathbb{Z}^N)\, d\mathcal{L}^2(x) + \int_E \Gamma(x-y)\frac{u_i(x)-u_i(y)}{\ln(1/\varepsilon_i)} \cdot \frac{u_i(x)-u_i(y)}{\ln(1/\varepsilon_i)}\, d\mathcal{L}^4(x,y),$$

where c_W is the constant entering (2.2), so that $\mu_i(\Omega\times\Omega) \le (\ln(1/\varepsilon_i))^{-2} E_{\varepsilon_i}[u_i,\Omega]$. We observe that μ_i is a Radon measure, and after extracting a further subsequence we can assume that μ_i converges weakly in the space of measures to some measure μ, which implies $\mu(A\times A) \le \lim_{i\to\infty} \mu_i(A\times A)$ for any open set $A \subseteq \Omega$. To conclude it suffices to prove that

$$F_0[u,\Omega] \le \mu(\Omega\times\Omega). \tag{8.2}$$

In order to treat the long-range part of the interaction we define a measure μ_{LR} on $\Omega\times\Omega$ by

$$\mu_{\mathrm{LR}}(E) := \int_E \Gamma(x-y)(u(x)-u(y))\cdot(u(x)-u(y))\, d\mathcal{L}^4(x,y)$$

for any Borel set $E \subseteq \Omega\times\Omega$. Since $u \in H^{1/2}(\Omega;\mathbb{R}^N)$, we have $\mu_{\mathrm{LR}}(\Omega\times\Omega) < \infty$. Since $\Gamma(x-y)\xi\cdot\xi \ge 0$ for any $\xi \in \mathbb{R}^N$ and $(x,y) \in \Omega\times\Omega$, and (possibly extracting a further subsequence) $(u_i(x)-u_i(y))/\ln(1/\varepsilon_i)$ converges pointwise to $u(x)-u(y)$ for $\mathcal{L}^4$-almost every (x,y), by Fatou's lemma we obtain

$$\mu_{\mathrm{LR}}(E) \le \liminf_{i\to\infty} \mu_i(E)$$

for any Borel set $E \subseteq \Omega\times\Omega$ and in particular $\mu_{\mathrm{LR}}(B_r^{(4)}(x)) \le \mu(B_R^{(4)}(x))$ if $r < R$ and $B_R^{(4)}(x) \subset \Omega\times\Omega$, where $B_r^{(4)}(x)$ is the four-dimensional ball of radius r centered at $x \in \mathbb{R}^4$. Since μ_{LR} is absolutely continuous with respect to $\mathcal{L}^4$, we conclude

$$\mu_{\mathrm{LR}}(E) \le \mu(E) \quad \text{for any Borel set } E \subseteq \Omega\times\Omega. \tag{8.3}$$

We now deal with the short-range part of the energy, which concentrates on the diagonal set. We define the measure

$$\lambda := g\left(\frac{dDu}{d|Du|}\right)|Du|$$

so that $F_{\mathrm{self}}[u,E] = \lambda(E)$ for any Borel set $E \subseteq \Omega$ (we recall that F_{self} has been defined in (2.11)). Since $u \in BV(\Omega;\mathbb{R}^N)$, we have $\lambda(\Omega) < \infty$. Let $\eta > 0$. For each $x \in \Omega$ there are arbitrarily small $r > 0$ such that $B_{2r}(x) \subset \Omega$, $2r < \eta$, $\mu(\partial(B_r(x)\times B_r(x))) = 0$ and $\lambda(\partial B_r(x)) = 0$. By Vitali's covering theorem we can find countably many such balls, denoted by $(B_j)_{j\in\mathbb{N}}$, so that they are pairwise disjoint, have centers in Ω, and there exists a Borel set $N_* \subseteq \Omega\times\Omega$ such that

$$(\Omega\times\Omega)\cap\Delta \subseteq N_* \cup \bigcup_j (B_j\times B_j) \quad \text{with } \mu(N_*) = \lambda(PN_*) = 0$$

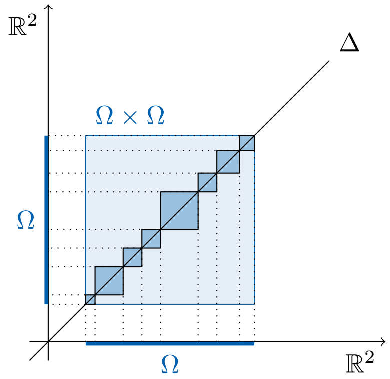

Fig. 3. Sketch of the set $(\Omega \times \Omega) \cap \Delta$. The set Ω is covered (up to a null set) by finitely many balls B_j, and $(\Omega \times \Omega) \cap \Delta$ is correspondingly covered (up to a null set) by the products $B_j \times B_j$.

(see Figure 3), we recall that $P : \mathbb{R}^4 \to \mathbb{R}^2$ denotes projection onto the first two components. By Proposition 8.2 applied with $\omega = B_j$, and using $\mu(\partial(B_j \times B_j)) = 0$, we have

$$\lambda(B_j) \le \mu(B_j \times B_j). \tag{8.4}$$

Then, with $\Delta_\eta := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x - y| < 2\eta\}$,

$$\lambda(\Omega) = \sum_{j\in\mathbb{N}} \lambda(B_j) \le \sum_{j\in\mathbb{N}} \mu(B_j \times B_j) \le \mu((\Omega \times \Omega) \cap \Delta_\eta).$$

Since this holds for any $\eta > 0$, and $\mu(\Omega \times \Omega) < \infty$, we conclude

$$\lambda(\Omega) \le \mu((\Omega \times \Omega) \cap \Delta).$$

Recalling (8.3) and $\mu_{\mathrm{LR}} \ll \mathcal{L}^4$, we obtain $\mu_{\mathrm{LR}}((\Omega \times \Omega) \cap \Delta) = 0$ and

$$\begin{aligned} F_0[u, \Omega] &= \lambda(\Omega) + \mu_{\mathrm{LR}}(\Omega \times \Omega) = \lambda(\Omega) + \mu_{\mathrm{LR}}((\Omega \times \Omega) \setminus \Delta) \\ &\le \mu((\Omega \times \Omega) \cap \Delta) + \mu((\Omega \times \Omega) \setminus \Delta) = \mu(\Omega \times \Omega). \end{aligned}$$

This concludes the proof of (8.2) and therefore of the proposition. ■

It remains to prove the local lower bound stated in Proposition 8.2. The proof uses a result from [18] that the nonlocal energy of almost-one-dimensional BV phase fields controls the line-tension energy of a similar field, which we recall in Proposition 8.3. We start by fixing a mollifier $\varphi_0 \in C_c^\infty(B_1; [0, \infty))$ with $\int_{B_1} \varphi_0 \, dx = 1$ and $\varphi \ge 1$ on $B_{1/2}$, and scaling it to $\varphi_h(x) := 2^{2h} \varphi_0(2^h x)$. We remark that the index h in φ_h denotes the exponent, at variance with the usage in the previous part of this paper, and recall the definition of the truncated energy in (5.2).

Proposition 8.3 ([18, Proposition 7.1]). *Let $\omega \subset\subset \Omega$ be two bounded open sets, $u \in W^{1,1}(\Omega;\mathbb{R}^N)$, $M > 1$, $h, t \in \mathbb{N}$ with $t \geq 3$, $\zeta \in (0,1)$. Assume* $\operatorname{dist}(\omega, \partial\Omega) \geq 2^{-h+1}$. *Then there is $w = w_{h,\zeta,t,M} \in BV(\omega;\mathbb{Z}^N)$ such that*

$$\begin{aligned}(\ln 2) \int_{J_w \cap \omega} & \psi_{\mathrm{rel}}([w], \nu)\, d\mathcal{H}^1 \\ & \leq (1 + \zeta + c2^{-t}) E^*_{h+t}[u, \Omega] + \frac{C_M}{\zeta} 2^{h+t} \|\operatorname{dist}(u, \mathbb{Z}^N)\|_{L^1(\Omega)} \\ & \quad + \frac{C_M}{\zeta} 2^t A^{5/6} \big(|Du|(\Omega) - |D(u * \varphi_h)|(\omega)\big)^{1/6} + \frac{c}{M^{1/2}} 2^{t/2} A \end{aligned} \tag{8.5}$$

and

$$\begin{aligned}\|u - w\|_{L^1(\omega)} \leq & \frac{c}{M^{1/2}} 2^{-h+t/2} A + C_M \|\operatorname{dist}(u, \mathbb{Z}^N)\|_{L^1(\Omega)} \\ & + C_M 2^{-h} A^{2/3} \big(|Du|(\Omega) - |D(u * \varphi_h)|(\omega)\big)^{1/3}. \end{aligned} \tag{8.6}$$

Here $A := \max\{|Du|(\Omega), E^*_{h+t}[u, \Omega]\}$. The constant c may depend on N and Γ, and the constant C_M also on M.

We remark that the statement of Proposition 7.1 in [18] contains the unnecessary assumption that both sets are Lipschitz. The proof is based on covering ω with squares contained in Ω and performing a separate estimate on each square, in particular it never uses this assumption.

We finally prove the lower bound in Proposition 8.2. The following argument is a modification of [18, Proposition 8.1]. It is here used only in the case where ω is a ball.

Proof of Proposition 8.2. It suffices to prove the estimate in the case $\omega = \Omega$ (otherwise we restrict all functions to ω, and then relabel ω as Ω). We can also assume that the right-hand side is finite, and extract a subsequence such that the lim inf is a limit. We fix $\hat\omega \subset\subset \Omega$ and prove that

$$F_{\mathrm{self}}[u, \hat\omega] \leq \liminf_{i\to\infty} \frac{E_{\varepsilon_i}[u_i, \Omega]}{(\ln(1/\varepsilon_i))^2}. \tag{8.7}$$

Taking the supremum over all such $\hat\omega$ will conclude the proof.

To prove (8.7) we choose a Lipschitz set Ω' such that $\hat\omega \subset\subset \Omega' \subset\subset \Omega$ and fix $\delta > 0$. By Proposition 5.2, for i sufficiently large there are $k_i \in \mathbb{N}$ with

$$\varepsilon_i^{1-\delta/2} \leq 2^{-k_i} \leq \varepsilon_i^{1-\delta}, \tag{8.8}$$

which implies $(1-\delta)\ln(1/\varepsilon_i) \leq k_i \ln 2$, and a function $v_{k_i,\delta} \in BV(\Omega';\mathbb{Z}^N)$ such that

$$\lim_{i\to\infty} \|v_{k_i,\delta} - u_i\|_{L^1(\Omega')} \leq \lim_{i\to\infty} C 2^{-k_i/2} (\ln(1/\varepsilon_i))^{1/2} = 0 \tag{8.9}$$

and, with (5.3),

$$\liminf_{i\to\infty} \frac{1}{k_i^2} \sum_{h=0}^{k_i} E^*_h[v_{k_i,\delta}, \Omega'] \leq (\ln 2)^2 \frac{1}{(1-\delta)^2} \liminf_{i\to\infty} \frac{E_{\varepsilon_i}[u_i, \Omega]}{(\ln(1/\varepsilon_i))^2} < \infty. \tag{8.10}$$

With (5.4) we see that there is $A_\delta > 0$ such that

$$\frac{1}{k_i^2}\sum_{h=0}^{k_i} E_h^*[v_{k_i,\delta},\Omega'] + \frac{1}{k_i}|Dv_{k_i,\delta}|(\Omega') \le A_\delta \quad \text{for all } i \in \mathbb{N}. \tag{8.11}$$

For simplicity in the following we write k for k_i, and $\liminf_{k\to\infty}$ for $\liminf_{i\to\infty}$.

One important idea in the proof is to define an iterated mollification of the function $v_{k,\delta}$ using a family of length scales ranging from 1 to 2^{-k}. We use scales separated by a factor 2^m, in order to apply Proposition 8.3 between each pair of consecutive scales. The key idea is that each mollification step eliminates the structure present in the function on a certain length scale, as measured by the BV norm. Since we have a BV bound on the original function, and a large number of mollification steps, most of them will result in a very small reduction of the BV norm, which means that on many scales the function will have an essentially one-dimensional structure. To make this precise, we fix $m \ge 3$ and define for $h \in \mathbb{N}$ the sets $\Omega_h := \{x \in \mathbb{R}^2 : B_{2^{2-h}}(x) \subseteq \Omega'\}$, so that

$$\operatorname{dist}(\Omega_h, \partial\Omega_{h+m}) \ge \operatorname{dist}(\Omega_h, \partial\Omega') - \operatorname{dist}(\Omega_{h+m}, \partial\Omega') \ge 2^{1-h}. \tag{8.12}$$

We then define for $h \in \mathbb{N}$ the function $z_h \in L^1(\Omega_h;\mathbb{R}^N)$ (implicitly depending also on k, δ, and m) by

$$z_h := \begin{cases} v_{k,\delta} & \text{if } h \ge k,\\ z_{h+m} * \varphi_h & \text{otherwise,}\end{cases}$$

where φ_h is the mollifier that enters Proposition 8.3.

One key estimate, which is obtained by summing the m telescoping series and using (8.11), is

$$\sum_{h=0}^{k}[|Dz_{h+m}|(\Omega_{h+m}) - |Dz_h|(\Omega_h)] = \sum_{h=k+1}^{k+m}|Dz_h|(\Omega_h) - \sum_{h=0}^{m-1}|Dz_h|(\Omega_h)$$
$$\le m|Dv_{k,\delta}|(\Omega') \le kmA_\delta. \tag{8.13}$$

By the properties of the mollification we also obtain, for $h < k$,

$$\|z_h - z_{h+m}\|_{L^1(\Omega_h)} = \|z_{h+m} * \varphi_h - z_{h+m}\|_{L^1(\Omega_h)}$$
$$\le 2^{-h}|Dz_{h+m}|(\Omega_{h+m}) \le 2^{-h}|Dv_{k,\delta}|(\Omega') \le k2^{-h}A_\delta,$$

and therefore

$$\|z_h - v_{k,\delta}\|_{L^1(\Omega_h)} \le \sum_{j=0}^{\infty}\|z_{h+jm} - z_{h+(j+1)m}\|_{L^1(\Omega_h)} \le 2k2^{-h}A_\delta. \tag{8.14}$$

Since $v_{k,\delta} \in \mathbb{Z}^N$ a.e., this implies

$$\|\operatorname{dist}(z_h, \mathbb{Z}^N)\|_{L^1(\Omega_h)} \le 2k2^{-h}A_\delta. \tag{8.15}$$

We next show that convexity and translation invariance of the nonlocal energy imply that mollification decreases the energy. Indeed, the map $v \mapsto E_s^*[v, \Omega_h]$ is a quadratic and nonnegative map from $L^2(\Omega_h; \mathbb{R}^N)$ to $\mathbb{R}$, and therefore convex. Let $u_z(x) := u(x-z)$, so that $(u * \varphi_h)(x) = \int_{\mathbb{R}^n} \varphi_h(z) u_z(x)\, dz$. By Jensen's inequality,

$$E_s^*[u * \varphi_h, \Omega_h] \le \int_{R^2} \varphi_h(z) E_s^*[u_z, \Omega_h] dz.$$

By translation invariance, and monotonicity of E_s^* in the second argument, using (8.12) we have $E_s^*[u_z, \Omega_h] = E_s^*[u, \Omega_h - z] \le E_s^*[u, \Omega_{h+m}]$. Therefore

$$E_s^*[u * \varphi_h, \Omega_h] \le E_s^*[u, \Omega_{h+m}] \quad \text{for any } s \in \mathbb{N} \text{ and } u \in L^2(\Omega_{h+m}; \mathbb{R}^N),$$

and therefore, iterating this inequality,

$$E_s^*[z_h, \Omega_h] \le E_s^*[v_{k,\delta}, \Omega']$$

for any s and h. In particular,

$$E_{h+t}^*[z_{h+m}, \Omega_{h+m}] \le E_{h+t}^*[v_{k,\delta}, \Omega']. \tag{8.16}$$

At this point we choose $\zeta \in (0, 1/4)$ and $t \in \mathbb{N}$ with $m \ge t \ge 3$. Since we shall take the limit $k \to \infty$ first, we can assume that $k \ge m/\zeta$. We now choose a good value for $h \in (\zeta k, k - \zeta k) \cap \mathbb{N}$. Specifically, let

$$J := \left\{ h \in (\zeta k, k - \zeta k) \cap \mathbb{N} : E_{h+t}^*[v_{k,\delta}, \Omega'] > (1 + 5\zeta) \frac{1}{k} \sum_{j=0}^{k} E_j^*[v_{k,\delta}, \Omega'] \right\},$$

$$H := \left\{ h \in (\zeta k, k - \zeta k) \cap \mathbb{N} : |Dz_{h+m}|(\Omega_{h+m}) - |Dz_h|(\Omega_h) > \frac{m}{\zeta} A_\delta \right\}.$$

One easily verifies that

$$\#J \, \frac{1 + 5\zeta}{k} \le 1$$

and, recalling (8.13),

$$\#H \, \frac{1}{\zeta} \le k.$$

We assume $\zeta \le 1/20$, which implies $1/(1 + 5\zeta) \le 1 - 4\zeta$, and obtain, since $\zeta k \ge m \ge 3$,

$$\#J + \#H \le \frac{k}{1 + 5\zeta} + k\zeta \le (1 - 2\zeta)k - 3.$$

Since $\#((\zeta k, k - \zeta k) \cap \mathbb{N}) \ge (1 - 2\zeta)k - 2$, this implies that we can choose $h \in (\zeta k, k - \zeta k) \cap \mathbb{N} \setminus (J \cup H)$. This value will be fixed for the rest of the argument (depending on the other parameters) and satisfies, recalling (8.16),

$$E_{h+t}^*[z_{h+m}, \Omega_{h+m}] \le E_{h+t}^*[v_{k,\delta}, \Omega'] \le (1 + 5\zeta) \frac{1}{k} \sum_{j=0}^{k} E_j^*[v_{k,\delta}, \Omega'], \tag{8.17}$$

$$|Dz_{h+m}|(\Omega_{h+m}) - |Dz_h|(\Omega_h) \le \frac{m}{\zeta} A_\delta. \tag{8.18}$$

We apply Proposition 8.3 to z_{h+m}, for some $M > 1$ chosen below, on the sets $\Omega_h \subset\subset \Omega_{h+m}$, and denote the result by $w = w_{k,\delta,m,t,M}$. We obtain

$$
\begin{aligned}
(\ln 2) &\int_{J_w \cap \Omega_h} \psi_{\mathrm{rel}}([w], \nu)\, d\mathcal{H}^1 \\
&\le (1 + \zeta + c2^{-t}) E^*_{h+t}[z_{h+m}, \Omega_{h+m}] + \frac{C_M}{\zeta} 2^{h+t} \|\mathrm{dist}(z_{h+m}, \mathbb{Z}^N)\|_{L^1(\Omega_{h+m})} \\
&\quad + \frac{C_M}{\zeta} 2^t A_*^{5/6} (|Dz_{h+m}|(\Omega_{h+m}) - |Dz_h|(\Omega_h))^{1/6} + \frac{c}{M^{1/2}} 2^{t/2} A_*
\end{aligned} \tag{8.19}
$$

and

$$
\begin{aligned}
\|z_{h+m} - w\|_{L^1(\Omega_h)} \le{}& \frac{c}{M^{1/2}} 2^{-h+t/2} A_* + C_M \|\mathrm{dist}(z_{h+m}, \mathbb{Z}^N)\|_{L^1(\Omega_{h+m})} \\
&+ C_M 2^{-h} A_*^{2/3} (|Dz_{h+m}|(\Omega_{h+m}) - |Dz_h|(\Omega_h))^{1/3},
\end{aligned} \tag{8.20}
$$

where $A_* := \max\{|Dz_{h+m}|(\Omega_{h+m}), E^*_{h+t}[z_{h+m}, \Omega_{h+m}]\}$. Using (8.17), the condition $|Dz_{h+m}|(\Omega_{h+m}) \le |Dv_{k,\delta}|(\Omega')$ and then (8.11), we obtain

$$
A_* \le |Dv_{k,\delta}|(\Omega') + \frac{2}{k} \sum_{j=0}^{k} E^*_j[v_{k,\delta}, \Omega'] \le 2kA_\delta. \tag{8.21}
$$

Then (8.20) becomes, using (8.15) and (8.18),

$$
\begin{aligned}
\|z_{h+m} - w\|_{L^1(\Omega_h)} \le{}& \frac{c}{M^{1/2}} 2^{-h+t/2} k A_\delta + C_M k 2^{-h-m} A_\delta \\
&+ C_M 2^{-h} k^{2/3} A_\delta m^{1/3} \zeta^{-1/3}.
\end{aligned} \tag{8.22}
$$

We recall that $\hat{\omega} \subset \Omega_h$ for sufficiently large k, since we chose $h \ge k\zeta$. From (8.19), (8.17), (8.15), and (8.18),

$$
\begin{aligned}
\frac{\ln 2}{k} &\int_{J_w \cap \hat{\omega}} \psi_{\mathrm{rel}}([w], \nu)\, d\mathcal{H}^1 \\
&\le (1 + \zeta + c2^{-t})(1 + 5\zeta) \frac{1}{k^2} \sum_{j=0}^{k} E^*_j[v_{k,\delta}, \Omega'] + \frac{C_M}{\zeta} 2^{h+t} 2^{-h-m} A_\delta \\
&\quad + \frac{C_M}{k\zeta} 2^t A_*^{5/6} (m\zeta^{-1} A_\delta)^{1/6} + \frac{c}{kM^{1/2}} 2^{t/2} A_*
\end{aligned} \tag{8.23}
$$

We notice that this expression does not depend any more explicitly on the choice of h, since $2^{h+t} 2^{-h-m} = 2^{t-m}$.

We set $u^k := \frac{1}{k \ln 2} w$, where $w = w_{k,\delta,m,t,M}$ is the function constructed in Proposition 8.3, so that the relaxed line-tension functional $E^{\mathrm{LT,rel}}_\sigma$ defined in (4.5) reads

$$
E^{\mathrm{LT,rel}}_{1/(k \ln 2)}[u^k, \hat{\omega}] = \frac{1}{k \ln 2} \int_{J_w \cap \hat{\omega}} \psi_{\mathrm{rel}}([w], \nu)\, d\mathcal{H}^1.
$$

Then (8.23), together with (8.11) and (8.21) yields, for sufficiently large k,

$$E^{\mathrm{LT,rel}}_{1/(k\ln 2)}[u^k,\hat\omega] \le \frac{1}{k^2(\ln 2)^2}\sum_{j=0}^{k} E_j^*[v_{k,\delta},\Omega'] + (c\zeta + c2^{-t})A_\delta + \frac{C_M}{\zeta}2^{t-m}A_\delta + \frac{C_M}{\zeta}2^t A_\delta^{5/6}\left(\frac{m}{k\zeta}A_\delta\right)^{1/6} + \frac{c}{M^{1/2}}2^{t/2}A_\delta. \tag{8.24}$$

Correspondingly, from (8.22) a similar procedure leads to

$$\left\|\frac{1}{k\ln 2}z_{h+m} - u^k\right\|_{L^1(\hat\omega)} \le \frac{c}{M^{1/2}}2^{-h+t/2}A_\delta + C_M 2^{-h-m}A_\delta + C_M 2^{-h}k^{-1/3}A_\delta m^{1/3}\zeta^{-1/3}. \tag{8.25}$$

By (8.24) and (8.11) we obtain $\limsup_{k\to\infty} E^{\mathrm{LT,rel}}_{1/(k\ln 2)}[u^k,\hat\omega] < \infty$. Recalling the inequality $E^{\mathrm{LT,rel}}_\sigma \le E^{\mathrm{LT}}_\sigma$ and the compactness statement in Theorem 4.2, there are $d_k \in \mathbb{R}^N$ such that, after extracting a subsequence, $u^k - d_k$ converges as $k\to\infty$ to some $u^{\delta,m,t,M}$ in $L^1(\hat\omega;\mathbb{R}^N)$. Taking the limit $k\to\infty$, and recalling Theorem 4.2 and (8.10), we obtain

$$E_0^{\mathrm{LT}}[u^{\delta,m,t,M},\hat\omega] \le \frac{1}{(1-\delta)^2}\liminf_{i\to\infty}\frac{E_{\varepsilon_i}[u_i,\Omega]}{(\ln(1/\varepsilon_i))^2} + (c\zeta + c2^{-t})A_\delta + \frac{C_M}{\zeta}2^{t-m}A_\delta + \frac{c}{M^{1/2}}2^{t/2}A_\delta. \tag{8.26}$$

At the same time by (8.25) we have

$$\limsup_{k\to\infty}\left\|\frac{1}{k\ln 2}z_{h+m} - u^k\right\|_{L^1(\hat\omega)} \le \frac{c}{M^{1/2}}2^{-h+t/2}A_\delta + C_M 2^{-h-m}A_\delta.$$

By (8.14) we have

$$\left\|\frac{1}{k\ln 2}z_{h+m} - \frac{1}{k\ln 2}v_{k,\delta}\right\|_{L^1(\hat\omega)} \le C2^{-h-m}A_\delta,$$

and therefore

$$\limsup_{k\to\infty}\left\|u^k - \frac{1}{k\ln 2}v_{k,\delta}\right\|_{L^1(\hat\omega)} \le \frac{c}{M^{1/2}}2^{-h+t/2}A_\delta + C_M 2^{-h-m}A_\delta.$$

With (8.9), and going back to the notation where the index i is explicit, we obtain

$$\limsup_{i\to\infty}\left\|u^{k_i} - \frac{1}{k_i\ln 2}u_i\right\|_{L^1(\hat\omega)} \le \frac{c}{M^{1/2}}2^{-h+t/2}A_\delta + C_M 2^{-h-m}A_\delta,$$

so that (8.8) gives

$$\limsup_{i\to\infty}\left\|u^{k_i} - \frac{1}{\ln(1/\varepsilon_i)}u_i\right\|_{L^1(\hat\omega)} \le \frac{c}{M^{1/2}}2^{-h+t/2}A_\delta + C_M 2^{-h-m}A_\delta + c\delta\limsup_{i\to\infty}\left\|\frac{1}{\ln(1/\varepsilon_i)}u_i\right\|_{L^1(\hat\omega)},$$

and since $u_i/\ln(1/\varepsilon_i) \to u$ and $u^{k_i} \to u^{\delta,m,t,M}$ in $L^2(\hat{\omega})$,

$$\|u^{\delta,m,t,M} - u\|_{L^1(\hat{\omega})} \le \frac{c}{M^{1/2}} 2^{-h+t/2} A_\delta + C_M 2^{-h-m} A_\delta + c\delta \|u\|_{L^1(\hat{\omega})}.$$

The argument is then concluded by recalling (8.26) and taking a suitable diagonal subsequence. Indeed, as δ, m, ζ, M and t were arbitrary, and since $u^{\delta,m,t,M} \to u$, by lower semicontinuity of F_{self}, taking first $m \to \infty$, then $\zeta \to 0$, then $M \to \infty$, then $t \to \infty$, and finally $\delta \to 0$, we conclude that

$$F_{\text{self}}[u_0, \hat{\omega}] = E_0^{\text{LT}}[u, \hat{\omega}] \le \liminf_{i \to \infty} \frac{E_{\varepsilon_i}[u_i, \Omega]}{(\ln(1/\varepsilon_i))^2}.$$

This finishes the proof of (8.7) and therefore of the proposition. ■

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