J. Eur. Math. Soc. 25, 2573–2639 (2023) DOI 10.4171/JEMS/1246

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Geometry of geodesics through Busemann measures in directed last-passage percolation

Received April 17, 2020

Abstract. We consider planar directed last-passage percolation on the square lattice with general i.i.d. weights and study the geometry of the full set of semi-infinite geodesics in a typical realization of the random environment. The structure of the geodesics is studied through the properties of the Busemann functions viewed as a stochastic process indexed by the asymptotic direction. Our results are further connected to the ergodic program for and stability properties of random Hamilton–Jacobi equations. In the exactly solvable exponential model, our results specialize to give the first complete characterization of the uniqueness and coalescence structure of the entire family of semi-infinite geodesics for any model of this type. Furthermore, we compute statistics of locations of instability, where we discover an unexpected connection to simple symmetric random walk.

Keywords. Busemann functions, coalescence, corner growth model, Hamilton–Jacobi equations, instability, KPZ, last-passage percolation, random dynamical system

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Mathematics Subject Classification (2020): Primary 60K35; Secondary 60K37

1. Introduction

1.1. Random growth models

Irregular or random growth is a ubiquitous phenomenon in nature, from the growth of tumors, crystals, and bacterial colonies to the propagation of forest fires and the spread of water through a porous medium. Models of random growth have been a driving force in probability theory over the last sixty years and a wellspring of important ideas [2].

The mathematical analysis of such models began in the early 1960s with the introduction of the *Eden model* by Eden [21] and *first-passage percolation* (FPP) by Hammersley and Welsh [31]. About two decades later, early forms of a directed variant of FPP, *directed last-passage percolation* (LPP), appeared in a paper by Muth [42] in connection with *series of queues in tandem*. Soon after, Rost [47] introduced a random growth model, now known as the *corner growth model* (CGM), in connection with the *totally asymmetric simple exclusion process* (TASEP), a model of interacting particles. A decade later, the CGM arose naturally from LPP in queueing theory in the work of Szczotka and Kelly [50] and Glynn and Whitt [29]. Around the same time, the third author [48] connected the CGM and LPP to Hamilton–Jacobi equations and Hopf–Lax–Oleinik semigroups.

Much of this early work was primarily concerned with the deterministic asymptotic shape and large deviations of the randomly growing interface. The breakthrough of Baik, Deift, and Johansson [3] showed that the fluctuations of the Poissonian LPP model have the same limit as the fluctuations of the largest eigenvalue of the *Gaussian unitary ensemble* derived by Tracy and Widom [51]. This result was extended to the exactly solvable versions of the CGM by Johansson [37]. These results marked the CGM and the related LPP and TASEP models as members of the *Kardar–Parisi–Zhang* (KPZ) universality class. This universality class is conjectured to describe the statistics of a growing interface observed when a rapidly mixing stable state invades a rapidly mixing metastable state. This subject has been a major focus of probability theory and statistical physics over the last three decades. Recent surveys appear in [14, 15, 30, 44, 45].

1.2. Geodesics

A common feature of many models of random growth is a natural metric-like interpretation in which there exist paths that can be thought of as geodesics. In these interpretations, the growing interface can be viewed as a sequence of balls of increasing radius and centered at the origin. This connection is essentially exact in the case of FPP, which describes a random pseudo-metric on $\mathbb{Z}^d$. Related models like the CGM and stochastic Hamilton–Jacobi equations have natural extremizers through their Hopf–Lax–Oleinik semigroups, which share many of the properties of geodesics. For this reason and following the convention in the field, we will call all such paths geodesics.

Considerable effort has been devoted to understanding the geometric structure of semi-infinite geodesics in models of random growth. In the mathematical literature, this program was largely pioneered in the seminal work of Newman and co-authors [33, 34, 40, 43], beginning with his paper in the 1994 Proceedings of the ICM [43]. Under strong hypotheses on the curvature of the limit shape, that early work showed that all such geodesics must be asymptotically directed and that for Lebesgue-almost every fixed direction, from each site of the lattice, there exists a unique semi-infinite geodesic with that asymptotic direction and all these geodesics coalesce. In special cases where the curvature hypotheses are met, Newman's program was subsequently implemented in LPP models [11–13, 26, 52] and certain stochastic Hamilton–Jacobi equations [4, 5, 7]. In all the results of the last twenty-five years, the obstruction of needing to work on direction-dependent events of full probability has been a persistent issue. A description of the overall geometric structure of semi-infinite geodesics has remained elusive.

It is known that the picture described by these now-classical methods cannot be complete, because uniqueness fails for countably infinitely many random directions [16, 25, 27]. In the CGM, these special directions are the asymptotic directions of *competition interfaces*. These are dual lattice paths that separate geodesics rooted at a fixed site. Competition interface directions are distinguished by the existence of (at least) two geodesics that emanate from the same site, have the same asymptotic direction, but separate immediately in their first step. Once these two geodesics separate they never intersect again. So in these directions coalescence also fails.

Borrowing ideas from classical metric geometry, Newman [43] introduced the tool of Busemann functions into the field. In Newman's work, these Busemann functions are defined as directional limits of differences of metric distances or passage times. Following Newman's work and the subsequent seminal work of Hoffman [32], Busemann functions have become a principal tool for studying semi-infinite geodesics. The existence of the

Busemann limits, however, relies on strong hypotheses on the limit shape. Modern work primarily uses generalized Busemann functions, which exist without assumptions on the limit shape [1, 17, 18, 27, 28].

1.3. Busemann measures

The present paper introduces a new framework that relates geometric properties of geodesics to analytic properties of a measure-valued stochastic process called the *Busemann process* or *Busemann measures*. These Busemann measures are Lebesgue–Stieltjes measures of generalized Busemann functions on the space of spatial directions, and the Busemann process is the associated family of distribution functions. This approach enables a study of the entire family of semi-infinite geodesics on a single event of full probability.

We describe, in terms of the supports of the Busemann measures, the random exceptional directions in which uniqueness or coalescence of geodesics fails. Many of these results hold without further assumptions on the weight distribution. This work also identifies key hypotheses that are equivalent to desirable coalescence and uniqueness properties of geodesics. We expect that our methods will apply in related models including FPP and stochastic Hamilton–Jacobi equations.

In the exactly solvable case with i.i.d. exponential weights, when the new results are combined with previous work from [16, 26, 27], this yields a complete characterization of the uniqueness and coalescence structure of all semi-infinite geodesics on a single event of full probability. Here is a summary:

(i) Every semi-infinite geodesic has an asymptotic direction.

(ii) There exists a random countably infinite dense set of interior directions in which there are *exactly two* geodesics from each lattice site, a left geodesic and a right geodesic. These two families of left and right geodesics can be constructed from the Busemann process. Each family forms a tree of coalescing geodesics.

(iii) In every other interior direction there is a unique geodesic from each lattice point, which again can be constructed from the Busemann process. In each such direction these geodesics coalesce to form a tree.

(iv) The countable set of directions of non-uniqueness is exactly the set of asymptotic directions of competition interfaces from all lattice points, in addition to being the set of discontinuity directions of the Busemann process.

(v) In a direction ξ of non-uniqueness, finite geodesics out of a site x with endpoints going in direction ξ converge to the left (resp. right) semi-infinite geodesic out of x with asymptotic direction ξ if and only if the endpoints eventually stay to the left (resp. right) of the competition interface rooted at the point where the left and right semi-infinite geodesics out of x split.

(vi) In a direction ξ of uniqueness, finite geodesics out of a site x with endpoints going in direction ξ converge to the semi-infinite geodesic out of x with asymptotic direction ξ.

This gives the first complete accounting of semi-infinite geodesics in a model which lies in the KPZ class.

1.4. Instability points

Passage times in LPP solve a variational problem that is a discrete version of the stochastic Burgers Hopf–Lax–Oleinik semigroup. Through this connection, this paper is also related to the ergodic program for the stochastic Burgers equation initiated by Sinai [49]. As mentioned in point (iv) above, the exceptional directions in which coalescence fails correspond to directions at which the Busemann process has jump discontinuities. This means that the Cauchy problem at time $-\infty$ is not well-posed for certain initial conditions that correspond to these exceptional directions. In this case, it is reasonable to expect that solving the Cauchy problem with the initial condition given at time t_0 and letting $t_0 \to -\infty$ gives multiple limits at the space-time locations where the Busemann process has jump discontinuities. Thus we call these locations points of instability. In situations where the Cauchy problem is well-posed, points of instability correspond to shock locations. The structure of shocks in connection with the Burgers program has been a major line of research [4, 10, 20], with a conjectured relationship between shock statistics and the KPZ universality phenomenon (Bakhtin and Khanin [6]). These conjectures are open.

Past works [4, 6, 10, 20] considered shocks in fixed deterministic directions, where the Cauchy problem at time $-\infty$ is shown to be well-posed almost surely and these shocks are the only points of instability. Our model is in a non-compact space setting, where these problems have been especially difficult to study. In exceptional directions, points of instability turn out to have a markedly different structure from what has been seen previously in fixed directions. Among the new phenomena are that points of instability form bi-infinite paths that both branch and coalesce. Bi-infinite shock paths have previously been observed only when the space is compact and the asymptotic direction is fixed. Branching shocks have not been observed.

In the exponential model we compute non-trivial statistics of points of instability. Among our results is an unexpected connection with simple symmetric random walk: conditional on a ξ-directed path of instability points passing through the origin, the distribution of the locations of ξ-points of instability on the x-axis has the same law as the zero set of simple symmetric random walk sampled at even times.

1.5. Organization of the paper

Section 2 defines the model and summarizes the currently known results on Busemann functions and existence, uniqueness, and coalescence of geodesics. Section 3 contains our main results on Busemann measures and the geometry of geodesics for general weight distributions. Section 4 connects our general results to dynamical systems and studies the web of instability defined by the discontinuities of the Busemann process. Section 5 specializes to the exponential case to compute non-trivial statistics of the Busemann process.

Proofs come in Sections 7–9, with some auxiliary results relegated to Appendices B–D. Appendix A collects the inputs we need from previous work.

1.6. Setting and notation

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ is a Polish probability space equipped with a group $T = \{T_x\}_{x\in\mathbb{Z}^2}$ of $\mathcal{F}$-measurable $\mathbb{P}$-preserving bijections $T_x : \Omega \to \Omega$ such that $T_0 =$ identity and $T_x T_y = T_{x+y}$, and $\mathbb{E}$ is expectation relative to $\mathbb{P}$. A generic point in this space is denoted by $\omega \in \Omega$. We assume that there exists a family $\{\omega_x(\omega) : x \in \mathbb{Z}^2\}$ of real-valued random variables called *weights* such that

$$\begin{aligned}&\{\omega_x\} \text{ are i.i.d. with a continuous distribution under } \mathbb{P},\\ &\mathbb{V}\mathrm{ar}(\omega_0) > 0, \text{ and } \exists p > 2 : \mathbb{E}[|\omega_0|^p] < \infty.\end{aligned} \tag{1.1}$$

We require further that $\omega_y(T_x\omega) = \omega_{x+y}(\omega)$ for all $x, y \in \mathbb{Z}^2$. Moreover, $\mathbb{P}_0$ denotes the marginal distribution of $\{\omega_x : x \in \mathbb{Z}^2\}$ under $\mathbb{P}$. Continuous distribution means that $\mathbb{P}_0(X \le r)$ is a continuous function of $r \in \mathbb{R}$. $X \sim \mathrm{Exp}(\alpha)$ means that the random variable X satisfies $P(X > t) = e^{-\alpha t}$ for $t > 0$ (rate α exponential distribution).

The canonical setting is the one where $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ is endowed with the product topology, Borel σ-algebra $\mathcal{F}$, and the natural shifts, ω_x are the coordinate projections, and $\mathbb{P} = \mathbb{P}_0$ is a product shift-invariant measure.

The standard basis vectors of $\mathbb{R}^2$ are $e_1 = e_+ = (1, 0)$ and $e_2 = e_- = (0, 1)$. The $e_\pm$ notation will conveniently shorten some statements. Additional special vectors are $\widehat{e}_1 = e_1 + e_2$, $\widehat{e}_1^* = \widehat{e}_1/2$, $\widehat{e}_2 = e_2 - e_1$, and $\widehat{e}_2^* = \widehat{e}_2/2$. In the dynamical view of LPP, $\widehat{e}_1$ is the time coordinate and $\widehat{e}_2$ the space coordinate. See Figure 1.1. The spatial level at time $t \in \mathbb{Z}$ is denoted by $\mathbb{L}_t = \{x \in \mathbb{Z}^2 : x \cdot \widehat{e}_1 = t\}$. The half-vectors $\widehat{e}_1^*$ and $\widehat{e}_2^*$ connect $\mathbb{Z}^2$ with its dual lattice $\mathbb{Z}^{2*} = \widehat{e}_1^* + \mathbb{Z}^2$.

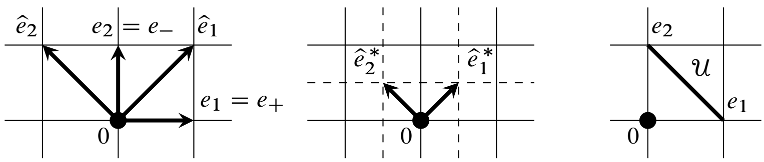

Fig. 1.1. An illustration of the vectors $e_1, e_2, e_\pm, \widehat{e}_1, \widehat{e}_2, \widehat{e}_1^*, \widehat{e}_2^*$, and the set $\mathcal{U}$. The dashed lines in the middle plot are edges of the dual lattice $\mathbb{Z}^{2*} = \mathbb{Z}^2 + \widehat{e}_1^*$.

A statement with $\pm$ and possibly also $\mp$ is a conjunction of two statements: one for the top signs, and another one for the bottom signs. We employ $\square$ to represent an arbitrary element of $\{-, +\}$.

We use $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ and $\mathbb{N} = \{1, 2, 3, \dots\}$. For $x, y \in \mathbb{R}^2$, inequalities such as $x \le y$ and $x < y$, and operations such as $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$ are understood coordinatewise. (In particular, $x < y$ means $x \cdot e_i < y \cdot e_i$ for both $i = 1, 2$.) For $x \le y$ in $\mathbb{Z}^2$, $[\![x, y]\!]$ denotes the rectangle $\{z \in \mathbb{Z}^2 : x \le z \le y\}$. For integers $i \le j$, $[\![i, j]\!]$ denotes the interval $[i, j] \cap \mathbb{Z}$. For $m \le n$ in $\mathbb{Z} \cup \{-\infty, \infty\}$ we denote a sequence $\{a_i : m \le i \le n\}$ by $a_{m,n}$.

A path $\pi_{m,n}$ in $\mathbb{Z}^2$ with $\pi_{i+1} - \pi_i \in \{e_1, e_2\}$ for all i is called an *up-right* path. Throughout, paths are indexed so that $\pi_k \cdot \hat{e}_1 = k$.

For vectors $\zeta, \eta \in \mathbb{R}^2$, denote open and closed line segments by $]\zeta, \eta[= \{t\zeta + (1-t)\eta : 0 < t < 1\}$ and $[\zeta, \eta] = \{t\zeta + (1-t)\eta : 0 \le t \le 1\}$, with the consistent definitions for $]\zeta, \eta]$ and $[\zeta, \eta[$. Set $\mathcal{U} = [e_2, e_1]$ with relative interior ri $\mathcal{U} =]\,e_2, e_1[$. See Figure 1.1. A left-to-right ordering of points $\zeta, \eta \in \mathbb{R}^2$ with $\zeta \cdot \hat{e}_1 = \eta \cdot \hat{e}_1$ is defined by $\zeta \prec \eta$ if $\zeta \cdot e_1 < \eta \cdot e_1$ and $\zeta \preceq \eta$ if $\zeta \cdot e_1 \le \eta \cdot e_1$. This leads to notions of left and right limits: if $\zeta_n \to \xi$ in $\mathcal{U}$, then $\zeta_n \nearrow \xi$ if $\zeta_n \prec \zeta_{n+1}$ for all n, while $\zeta_n \searrow \xi$ if $\zeta_{n+1} \prec \zeta_n$ for all n.

The support supp μ of a signed Borel measure μ is the smallest closed set whose complement has zero measure under the total variation measure $|\mu|$.

2. Preliminaries on last-passage percolation

This section introduces the background required for the main results in Sections 3–5. To avoid excessive technical detail at this point, precise statements of previous results needed for the proofs later in the paper are deferred to Appendix A.

2.1. *The shape function*

Recall the assumption (1.1). For $x \le y$ in $\mathbb{Z}^2$ satisfying $x \cdot \hat{e}_1 = k$ and $y \cdot \hat{e}_1 = m$, denote by Π_x^y the collection of up-right paths $\pi_{k,m}$ which satisfy $\pi_k = x$ and $\pi_m = y$. The *last-passage time* from x to y is defined by

$$G_{x,y} = G(x, y) = \max_{\pi_{k,m} \in \Pi_x^y} \sum_{i=k}^{m-1} \omega_{\pi_i}. \tag{2.1}$$

A maximizing path is called a (point-to-point or finite) *geodesic* and denoted by $\gamma^{x,y}$. Under the i.i.d. continuous distribution assumption (1.1), $\gamma^{x,y}$ is almost surely unique.

The shape theorem [41] says there exists a non-random function $g : \mathbb{R}_+^2 \to \mathbb{R}$ such that with probability 1,

$$\lim_{n\to\infty} \max_{x \in \mathbb{Z}_+^2 : |x|_1 = n} \frac{|G_{0,x} - g(x)|}{n} = 0. \tag{2.2}$$

This *shape function* g is symmetric, concave, and homogeneous of degree one. By homogeneity, g is determined by its values on $\mathcal{U}$. Concavity implies the existence of one-sided derivatives:

$$\nabla g(\xi\pm) \cdot e_1 = \lim_{\varepsilon \searrow 0} \frac{g(\xi \pm \varepsilon e_1) - g(\xi)}{\pm\varepsilon}, \quad \nabla g(\xi\pm) \cdot e_2 = \lim_{\varepsilon \searrow 0} \frac{g(\xi \mp \varepsilon e_2) - g(\xi)}{\mp\varepsilon}.$$

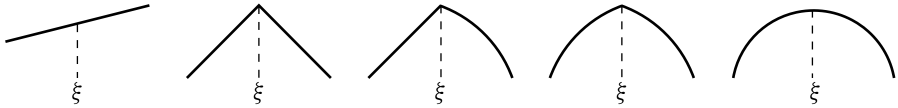

Fig. 2.1. In the first three graphs, g is not strictly concave at ξ while in the last two it is.

By [35, Lemma 4.7 (c)] differentiability of g at $\xi \in \operatorname{ri} \mathcal{U}$ is the same as $\nabla g(\xi+) = \nabla g(\xi-)$. Denote the directions of differentiability by

$$\mathcal{D} = \{\xi \in \operatorname{ri} \mathcal{U} : g \text{ is differentiable at } \xi\}. \tag{2.3}$$

For $\xi \in \operatorname{ri} \mathcal{U}$, define the maximal linear segments of g with slopes given by the right ($\square = +$) and the left ($\square = -$) derivatives of g at ξ to be

$$\mathcal{U}_{\xi\square} = \{\zeta \in \operatorname{ri} \mathcal{U} : g(\zeta) - g(\xi) = \nabla g(\xi\square) \cdot (\zeta - \xi)\}, \quad \square \in \{-, +\}.$$

We say g is *strictly concave* at $\xi \in \operatorname{ri} \mathcal{U}$ if $\mathcal{U}_{\xi-} = \mathcal{U}_{\xi+} = \{\xi\}$. Geometrically this means that ξ does not lie on a non-degenerate closed linear segment of g. The usual notion of strict concavity on an open subinterval of $\mathcal{U}$ is the same as having this pointwise strict concavity at all ξ in the interval.

For a given $\xi \in \operatorname{ri} \mathcal{U}$, let $\underline{\xi} \preceq \overline{\xi}$ denote the endpoints of the (possibly degenerate) interval

$$\mathcal{U}_\xi = \mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+} = [\underline{\xi}, \overline{\xi}].$$

If $\xi \in \mathcal{D}$ then $\mathcal{U}_{\xi-} = \mathcal{U}_{\xi+} = \mathcal{U}_\xi$ while if $\xi \notin \mathcal{D}$ then $\mathcal{U}_{\xi-} \cap \mathcal{U}_{\xi+} = \{\xi\}$. Set $\mathcal{U}_{e_i} = \{e_i\}$ for $i \in \{1, 2\}$.

Additional control over the geometry of geodesics is provided by this regularity condition:

$$\begin{aligned}&\text{The shape function } g \text{ is strictly concave at all } \xi \notin \mathcal{D}, \text{ or equivalently}\\ &g \text{ is differentiable at the endpoints of its linear segments.}\end{aligned} \tag{2.4}$$

Condition (2.4) holds obviously if g is either differentiable or strictly concave. Both of these latter properties are true for exponential weights and are conjectured to be valid more generally for continuously distributed weights. Under (2.4), if both $\mathcal{U}_{\xi-}$ and $\mathcal{U}_{\xi+}$ are non-degenerate intervals, then $\mathcal{U}_{\xi-} = \mathcal{U}_{\xi+} = \mathcal{U}_\xi$ (leftmost graph in Figure 2.1).

2.2. *The Busemann process*

Under regularity condition (2.4), it is known that for each fixed $\xi \in \mathcal{D}$ and $x, y \in \mathbb{Z}^2$, there is a ξ-dependent event of full probability on which the limit

$$B^\xi(x, y) = \lim_{n\to\infty} (G_{x,v_n} - G_{y,v_n}) \tag{2.5}$$

exists and agrees for all sequences $v_n \in \mathbb{Z}^2$ such that $|v_n| \to \infty$ and $v_n/n \to \xi$. Similar limits appear in metric geometry under the name of *Busemann functions*.

The goal of this paper is to study the LPP model without a priori hypotheses on the shape function. Hence the limit in (2.5) cannot serve as a starting point. Instead we work with a stochastic process of *generalized* Busemann functions, indexed by $\xi \in \operatorname{ri} \mathcal{U}$, constructed through a weak limit procedure on an extended probability space. See Remark A.2 for a brief discussion of the construction of this process in [36], which is based in part on ideas from [17,28]. This process agrees with (2.5) when the limit in (2.5) exists.

The construction in [36] produces a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a group of shifts $T = \{T_x : x \in \mathbb{Z}^2\}$ that satisfies the requirements of Section 1.6 and a stochastic process $\{B^{\xi\square}(x, y) : x, y \in \mathbb{Z}^2, \xi \in \operatorname{ri} \mathcal{U}, \square \in \{-, +\}\}$ on Ω, which we call the *Busemann process*. We record here those properties of this process that are needed for Sections 3–5.

In general, there is a T-invariant full probability event on which the following hold. For all $\xi \in \operatorname{ri} \mathcal{U}$, $x, y, z \in \mathbb{Z}^2$, and $\square \in \{-, +\}$,

$$B^{\xi\square}(x + z, y + z, \omega) = B^{\xi\square}(x, y, T_z\omega), \tag{2.6}$$

$$B^{\xi\square}(x, y, \omega) + B^{\xi\square}(y, z, \omega) = B^{\xi\square}(x, z, \omega), \tag{2.7}$$

$$\min\{B^{\xi\square}(x, x + e_1, \omega), B^{\xi\square}(x, x + e_2, \omega)\} = \omega_x, \tag{2.8}$$

$$\mathbb{E}[B^{\xi\square}(x, x + e_i)] = \nabla g(\xi\square) \cdot e_i. \tag{2.9}$$

Properties (2.6)–(2.7) express that each $B^{\xi\square}$ is a *covariant cocycle*. The *weights recovery* property (2.8) is the key that relates these cocycles to the LPP process. (2.9) shows that the Busemann process is naturally parametrized by the superdifferential of the shape function g. The following monotonicity is inherited from the path structure: for all $x \in \mathbb{Z}^2$ and $\xi, \xi' \in \operatorname{ri} \mathcal{U}$ with $\xi \prec \xi'$,

$$\begin{aligned} B^{\xi-}(x, x + e_1, \omega) &\geq B^{\xi+}(x, x + e_1, \omega) \geq B^{\xi'-}(x, x + e_1, \omega) \geq B^{\xi'+}(x, x + e_1, \omega), \\ B^{\xi-}(x, x + e_2, \omega) &\leq B^{\xi+}(x, x + e_2, \omega) \leq B^{\xi'-}(x, x + e_2, \omega) \leq B^{\xi'+}(x, x + e_2, \omega). \end{aligned} \tag{2.10}$$

As a consequence of monotonicity and the cocycle property (2.7), left and right limits exist. The signs in $B^{\xi\pm}$ correspond to left and right continuity: for all $x, y \in \mathbb{Z}^2$, $\xi \in \operatorname{ri} \mathcal{U}$, and $\square \in \{-, +\}$,

$$B^{\xi-}(x, y, \omega) = \lim_{\operatorname{ri} \mathcal{U} \ni \zeta \nearrow \xi} B^{\zeta\square}(x, y, \omega), \quad B^{\xi+}(x, y, \omega) = \lim_{\operatorname{ri} \mathcal{U} \ni \zeta \searrow \xi} B^{\zeta\square}(x, y, \omega). \tag{2.11}$$

When $B^{\xi+}(x, y, \omega) = B^{\xi-}(x, y, \omega)$ we drop the $+/-$ distinction and write $B^{\xi}(x, y, \omega)$.

Theorem A.1 in Appendix A contains the complete list of the properties of the Busemann process that are used in the proofs in Sections 7–8.

2.3. *Semi-infinite geodesics*

A path $\pi_{k,\infty}$ with $\pi_{i+1} - \pi_i \in \{e_1, e_2\}$ for all $i \geq k$ is called a *semi-infinite geodesic emanating from*, or *rooted at*, x if $\pi_k = x$ and for any $m, n \in \mathbb{Z}_+$ with $k \leq m \leq n$, the restricted path $\pi_{m,n}$ is a geodesic between π_m and π_n. A path $\pi_{-\infty,\infty}$ with $\pi_{i+1} - \pi_i \in \{e_1, e_2\}$ for all i is called a *bi-infinite geodesic* if $\pi_{m,n}$ is a geodesic for any $m \le n$ in $\mathbb{Z}$. Due to the fact that the set of admissible steps is $\{e_1, e_2\}$, from each site x, there are always two trivial semi-infinite geodesics, namely $x + \mathbb{Z}_+ e_1$, which we denote by $\boldsymbol{\gamma}^{x,e_1}$, and $x + \mathbb{Z}_+ e_2$, which we denote by $\boldsymbol{\gamma}^{x,e_2}$. There are two trivial bi-infinite geodesics going through x, namely $x + \mathbb{Z}e_1$ and $x + \mathbb{Z}e_2$, which we do not introduce notation for.

A semi-infinite geodesic $\pi_{k,\infty}$, or a bi-infinite geodesic $\pi_{-\infty,\infty}$, is *directed* into a set $\mathcal{A} \subset \mathcal{U}$ if the limit points of π_n/n as $n \to \infty$ are all in $\mathcal{A}$. When $\mathcal{A} = \{\xi\}$ the condition becomes $\lim_{n\to\infty} \pi_n/n = \xi$ and we say $\pi_{k,\infty}$ is ξ*-directed*.

Using the Busemann process, we construct a semi-infinite path $\boldsymbol{\gamma}^{x,\xi\square}$ for each $\xi \in \operatorname{ri}\mathcal{U}$, both signs $\square \in \{-,+\}$, and all $x \in \mathbb{Z}^2$, via these rules: the initial point is $\boldsymbol{\gamma}_m^{x,\xi\square} = x$ where $m = x \cdot \widehat{e}_1$, and for $n \ge m$,

$$\boldsymbol{\gamma}_{n+1}^{x,\xi\square} = \begin{cases} \boldsymbol{\gamma}_n^{x,\xi\square} + e_1 & \text{if } B^{\xi\square}(\boldsymbol{\gamma}_n^{x,\xi\square}, \boldsymbol{\gamma}_n^{x,\xi\square} + e_1) < B^{\xi\square}(\boldsymbol{\gamma}_n^{x,\xi\square}, \boldsymbol{\gamma}_n^{x,\xi\square} + e_2), \\ \boldsymbol{\gamma}_n^{x,\xi\square} + e_2 & \text{if } B^{\xi\square}(\boldsymbol{\gamma}_n^{x,\xi\square}, \boldsymbol{\gamma}_n^{x,\xi\square} + e_1) > B^{\xi\square}(\boldsymbol{\gamma}_n^{x,\xi\square}, \boldsymbol{\gamma}_n^{x,\xi\square} + e_2), \\ \boldsymbol{\gamma}_n^{x,\xi\square} + e_\square & \text{if } B^{\xi\square}(\boldsymbol{\gamma}_n^{x,\xi\square}, \boldsymbol{\gamma}_n^{x,\xi\square} + e_1) = B^{\xi\square}(\boldsymbol{\gamma}_n^{x,\xi\square}, \boldsymbol{\gamma}_n^{x,\xi\square} + e_2). \end{cases} \tag{2.12}$$

As above, we dispense with the $\pm$ distinction when $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$. These geodesics inherit an ordering from (2.10): for all $x \in \mathbb{Z}^2$, $n \ge x \cdot \widehat{e}_1$, and $\zeta \prec \eta$ in $\operatorname{ri}\mathcal{U}$,

$$\boldsymbol{\gamma}_n^{x,\zeta-} \preceq \boldsymbol{\gamma}_n^{x,\zeta+} \preceq \boldsymbol{\gamma}_n^{x,\eta-} \preceq \boldsymbol{\gamma}_n^{x,\eta+}. \tag{2.13}$$

Similarly, the geodesics inherit one-sided continuity from (2.11) in the sense of convergence of finite length segments: for all $x \in \mathbb{Z}^2$, $\xi \in \operatorname{ri}\mathcal{U}$ and $\square \in \{-,+\}$, if $k = x \cdot \widehat{e}_1$ and $m \ge k$ is an integer, then

$$\lim_{\operatorname{ri}\mathcal{U}\ni\eta\searrow\xi} \boldsymbol{\gamma}_{k,m}^{x,\eta\square} = \boldsymbol{\gamma}_{k,m}^{x,\xi+} \quad \text{and} \quad \lim_{\operatorname{ri}\mathcal{U}\ni\zeta\nearrow\xi} \boldsymbol{\gamma}_{k,m}^{x,\zeta\square} = \boldsymbol{\gamma}_{k,m}^{x,\xi-}. \tag{2.14}$$

An elementary argument given in [27, Lemma 4.1] shows that properties (2.7) and (2.8) combine to imply that these paths are all semi-infinite geodesics and that moreover for all choices of $x \in \mathbb{Z}^2$, $n \ge x \cdot \widehat{e}_1$, $\square \in \{-,+\}$, and $\xi \in \operatorname{ri}\mathcal{U}$, we have

$$G(x, \boldsymbol{\gamma}_n^{x,\xi\square}) = B^{\xi\square}(x, \boldsymbol{\gamma}_n^{x,\xi\square}). \tag{2.15}$$

Below are the main properties of these Busemann geodesics $\boldsymbol{\gamma}^{x,\xi\square}$ under assumption (1.1), from article [27]. (Theorem A.4 provides a more precise accounting.)

(i) Every semi-infinite geodesic is $\mathcal{U}_\xi$-directed for some $\xi \in \mathcal{U}$.

(ii) $\boldsymbol{\gamma}^{x,\xi\square}$ is $\mathcal{U}_{\xi\square}$-directed for each $x \in \mathbb{Z}^2$ and each $\xi \in \mathcal{U}$.

(iii) If $\xi, \underline{\xi}, \overline{\xi} \in \mathcal{D}$, then there is a ξ-dependent event of full probability on which $\boldsymbol{\gamma}^{x,\xi-} = \boldsymbol{\gamma}^{x,\xi+}$ for all $x \in \mathbb{Z}^2$.

(iv) There is a ξ-dependent event of full probability on which $\boldsymbol{\gamma}^{x,\xi\square}$ and $\boldsymbol{\gamma}^{y,\xi\square}$ coalesce for each $\square \in \{+,-\}$. That is, for each $x, y \in \mathbb{Z}^2$, there exists an ω-dependent $K \in \mathbb{N}$ such that for all $k \ge K$, $\boldsymbol{\gamma}_{k,\infty}^{x,\xi\square} = \boldsymbol{\gamma}_{k,\infty}^{y,\xi\square}$.

The regularity condition (2.4) guarantees that $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{x,\xi+}$ are extreme among the $\mathcal{U}_\xi$-directed geodesics out of x in the sense that for any $x \in \mathbb{Z}^2$, $\xi \in \operatorname{ri} \mathcal{U}$, and any $\mathcal{U}_\xi$-directed semi-infinite geodesic π emanating from x, we have

$$\boldsymbol{\gamma}_n^{x,\xi-} \preceq \pi_n \preceq \boldsymbol{\gamma}_n^{x,\xi+} \tag{2.16}$$

for all $n \geq x \cdot \widehat{e}_1$. We record this fact as Theorem A.7.

Under the regularity condition (2.4) and $\xi, \underline{\xi}, \overline{\xi} \in \mathcal{D}$, part (iii) combined with (2.16) implies that there is a ξ-dependent event of full probability on which there is a unique $\mathcal{U}_\xi$-directed geodesic from each $x \in \mathbb{Z}^2$. Moreover, by part (iv), all of these geodesics coalesce. On the other hand, under the same condition, it is known that there are exceptional random directions at which both uniqueness and coalescence fail. We discuss these directions in the next subsection.

2.4. *Non-uniqueness of directed semi-infinite geodesics*

For a fixed site $x \in \mathbb{Z}^2$, a natural direction in which non-uniqueness occurs is the *competition interface direction*, which we denote by $\xi_*(T_x\omega)$. At the origin, $\xi_*(\omega) \in \operatorname{ri} \mathcal{U}$ is the unique direction such that

$$B^{\zeta\pm}(e_1, e_2) < 0 \quad \text{if } \zeta \prec \xi_*(\omega), \qquad B^{\zeta\pm}(e_1, e_2) > 0 \quad \text{if } \zeta \succ \xi_*(\omega). \tag{2.17}$$

Theorem A.8 records the main properties of competition interface directions, including the existence and uniqueness of such a direction.

Under the regularity condition (2.4), we also have the following alternative description of $\xi_*(\omega)$. Fix a site $x \in \mathbb{Z}^2$. The uniqueness of finite geodesics implies that the collection of geodesics from x to all points $y \in x + \mathbb{Z}_+^2$ forms a tree $\mathcal{T}_x$ rooted at x and spanning $x + \mathbb{Z}_+^2$. The subtree rooted at $x + e_1$ is separated from the subtree rooted at $x + e_2$ by a path $\{\varphi_n^x : n \geq x \cdot \widehat{e}_1\}$ on the dual lattice $\widehat{e}_1^* + \mathbb{Z}^2$, known as the *competition interface*. See Figure 2.2.

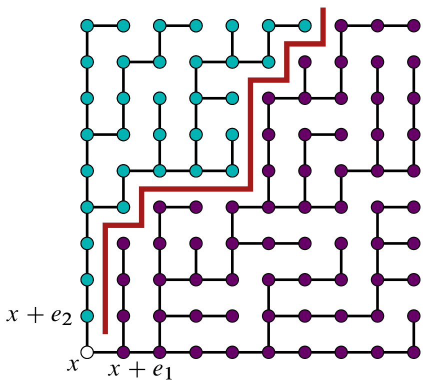

Fig. 2.2. The geodesic tree $\mathcal{T}_x$ rooted at x. The competition interface (solid line) emanates from $x + \widehat{e}_1^*$ and separates the subtrees of $\mathcal{T}_x$ rooted at $x + e_1$ and at $x + e_2$.

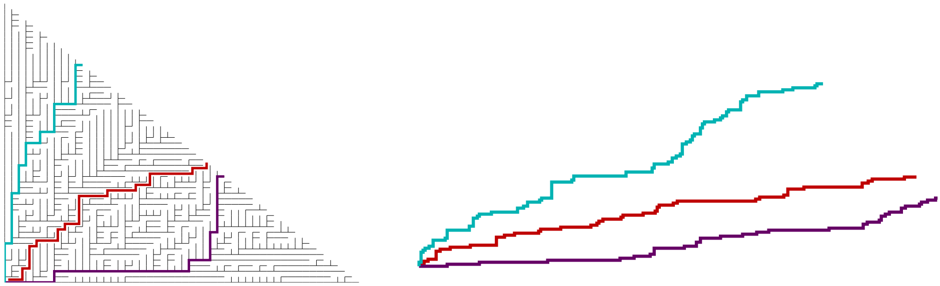

Fig. 2.3. The competition interface (middle path) separating the two ξ_*-directed geodesics. The left picture is a small portion of the right one. In the picture on the right the x-axis appears to be stretched, but the scales of the axes are in fact identical.

Under condition (2.4), the competition interface satisfies $\varphi_n^x/n \to \xi_*(T_x\omega)$, given by (2.17). Moreover, each of these two trees contains at least one semi-infinite geodesic with asymptotic direction $\xi_*(T_x\omega)$. Indeed, $\xi_*(T_x\omega)$ is the unique direction with the property that there exist at least two semi-infinite geodesics rooted at x, with asymptotic direction $\xi_*(T_x\omega)$, and which differ in their first step. See Figure 2.3. Theorem A.9 records the fact that when the weights are exponentially distributed, there are no directions ξ with three ξ-directed geodesics emanating from the same point.

3. Busemann measures, exceptional directions, and coalescence points

The central theme of this paper is the relationship between analytic properties of the Busemann process and the geometric properties of the geodesics $\boldsymbol{\gamma}^{\bullet,\xi\square}$ for $\xi \in \operatorname{ri}\mathcal{U}$ and $\square \in \{-,+\}$. It will be convenient in what follows to have a bookkeeping tool for the locations at which the Busemann processes are not locally constant. A natural way to record this information is through the supports of the associated Lebesgue–Stieltjes measures.

As functions of the direction parameter ξ, $B^{\xi-}_{x,x+e_i}$ and $B^{\xi+}_{x,x+e_i}$ are respectively left- and right-continuous versions of the same monotone function and satisfy the cocycle property (2.7). As a consequence, for each $x, y \in \mathbb{Z}^2$, $\square \in \{-,+\}$, $\xi \mapsto B^{\xi\square}(x,y)$ has locally bounded total variation. Hence on each compact subset K of $\operatorname{ri}\mathcal{U}$ there exists a signed Lebesgue–Stieltjes measure $\mu^K_{x,y}$ with the property that whenever $\zeta \prec \eta$ and $[\zeta,\eta] \subset K$,

$$\mu^K_{x,y}(]\zeta,\eta]) = B^{\eta+}_{x,y} - B^{\zeta+}_{x,y} \quad \text{and} \quad \mu^K_{x,y}([\zeta,\eta[) = B^{\eta-}_{x,y} - B^{\zeta-}_{x,y}. \tag{3.1}$$

The restriction to compact sets is a technical point: in general, $B^{\xi+}_{x,y}$ and $B^{\xi-}_{x,y}$ are signed sums of monotone functions and thus correspond to formal linear combinations of positive measures. By the limit in (A.1), each of these positive measures assigns infinite mass to the interval $\operatorname{ri}\mathcal{U}$ and if any two of the measures come with different signs, the formal linear combination will not define a signed measure on all of $\operatorname{ri}\mathcal{U}$. We will ignore this technical point in what follows and write $\mu_{x,y}(\bullet)$ for the value of this measure and $|\mu_{x,y}|(\bullet)$ for the value of the total variation measure whenever they are unambiguously defined. In that vein, we define the *support of the measure* $\mu_{x,y}$ on $\operatorname{ri}\mathcal{U}$ as

$$\operatorname{supp}\mu_{x,y} = \bigcup_{\zeta,\eta\in\operatorname{ri}\mathcal{U}:\,\zeta\prec\eta} \operatorname{supp}\mu_{x,y}^{[\zeta,\eta]}, \tag{3.2}$$

where $\operatorname{supp}\mu_{x,y}^{[\zeta,\eta]}$ is, as usual, the support of the (well-defined) total variation measure $|\mu_{x,y}^{[\zeta,\eta]}|$. Naturally, this definition agrees with the standard notion of the support of a measure when $\mu_{x,y}$ is a well-defined positive or negative measure on $\mathcal{U}$.

3.1. Coalescence and the Busemann measures

The first result below relates membership in the support to the existence of disjoint Busemann geodesics.

Theorem 3.1. *With $\mathbb{P}$-probability 1, for all $x \neq y$ in $\mathbb{Z}^2$ and $\xi \in \operatorname{ri}\mathcal{U}$ statements (i) and (ii) below are equivalent:*

(i) $\xi \in \operatorname{supp}\mu_{x,y}$.

(ii) *Either $\boldsymbol{\gamma}^{x,\xi-} \cap \boldsymbol{\gamma}^{y,\xi+} = \emptyset$ or $\boldsymbol{\gamma}^{x,\xi+} \cap \boldsymbol{\gamma}^{y,\xi-} = \emptyset$.*

Under the regularity condition (2.4), (i) and (ii) are equivalent to

(iii) *There exist $\mathcal{U}_\xi$-directed semi-infinite geodesics π^x and π^y out of x and y, respectively, such that $\pi^x \cap \pi^y = \emptyset$.*

The difference between statements (ii) and (iii) is that if $\xi \notin \operatorname{supp}\mu_{x,y}$ then (ii) leaves open the possibility that even though $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{y,\xi+}$ intersect and $\boldsymbol{\gamma}^{x,\xi+}$ and $\boldsymbol{\gamma}^{y,\xi-}$ intersect, there may be other $\mathcal{U}_\xi$-directed geodesics out of x and y that do not intersect. This is because without the regularity condition (2.4), we currently do not know whether (2.16) holds, that is, whether $\boldsymbol{\gamma}^{x,\xi+}$ is the rightmost and $\boldsymbol{\gamma}^{x,\xi-}$ the leftmost $\mathcal{U}_\xi$-directed geodesic out of x.

The subsequent several results relate the support of Busemann measures to the coalescence geometry of geodesics. For $x, y \in \mathbb{Z}^2$, $\xi \in \operatorname{ri}\mathcal{U}$, and signs $\square \in \{-,+\}$, define the *coalescence point* of the geodesics $\boldsymbol{\gamma}^{x,\xi\square}$ and $\boldsymbol{\gamma}^{y,\xi\square}$ by

$$\mathbf{z}^{\xi\square}(x,y) = \begin{cases} \text{first point in } \boldsymbol{\gamma}^{x,\xi\square} \cap \boldsymbol{\gamma}^{y,\xi\square} & \text{if } \boldsymbol{\gamma}^{x,\xi\square} \cap \boldsymbol{\gamma}^{y,\xi\square} \neq \emptyset, \\ \infty & \text{if } \boldsymbol{\gamma}^{x,\xi\square} \cap \boldsymbol{\gamma}^{y,\xi\square} = \emptyset. \end{cases} \tag{3.3}$$

The first point z in $\boldsymbol{\gamma}^{x,\xi\square} \cap \boldsymbol{\gamma}^{y,\xi\square}$ is identified uniquely by choosing the common point $z = \boldsymbol{\gamma}_k^{x,\xi\square} = \boldsymbol{\gamma}_k^{y,\xi\square}$ that minimizes k. In the expression above, ∞ is the point added in the one-point compactification of $\mathbb{Z}^2$. If the two geodesics $\boldsymbol{\gamma}^{x,\xi\square}$ and $\boldsymbol{\gamma}^{y,\xi\square}$ ever meet, they coalesce due to the local rule in (2.12). We write $\mathbf{z}^{\xi}(x,y)$ when $\mathbf{z}^{\xi-}(x,y) = \mathbf{z}^{\xi+}(x,y)$.

As $\mathbb{Z}^2 \cup \{\infty\}$-valued functions, $\xi \mapsto \mathbf{z}^{\xi+}(x,y)$ is right-continuous and $\xi \mapsto \mathbf{z}^{\xi-}(x,y)$ is left-continuous. Namely, a consequence of (2.14) is that for $\xi \in \operatorname{ri}\mathcal{U}$ and $\square \in \{-,+\}$,

$$\lim_{\operatorname{ri}\mathcal{U}\ni\eta\searrow\xi} \mathbf{z}^{\eta\square}(x,y) = \mathbf{z}^{\xi+}(x,y). \tag{3.4}$$

If $\mathbf{z}^{\xi+}(x,y) = \infty$ this limit still holds in the sense that then $|\mathbf{z}^{\eta\square}(x,y)| \to \infty$. The analogous statement holds for convergence from the left to $\mathbf{z}^{\xi-}(x,y)$.

The next theorem states that an interval of directions outside the support of a Busemann measure corresponds to geodesics following common initial segments to a common coalescence point.

Theorem 3.2. *With probability* 1*, simultaneously for all $\zeta \prec \eta$ in* ri $\mathcal{U}$ *and all $x, y \in \mathbb{Z}^2$, statements* (i)–(iii) *below are equivalent:*

(i) $|\mu_{x,y}|(]\zeta, \eta[) = 0$.

(ii) *Letting $k = x \cdot \widehat{e}_1$ and $\ell = y \cdot \widehat{e}_1$, there exist a point z with $z \cdot \widehat{e}_1 = m \geq k \vee \ell$ and path segments $\pi_{k,m}$ and $\widetilde{\pi}_{\ell,m}$ with these properties: $\pi_k = x$, $\widetilde{\pi}_\ell = y$, $\pi_m = \widetilde{\pi}_m = z$, and for all $\xi \in]\zeta, \eta[$ and $\square \in \{-, +\}$ we have $\boldsymbol{\gamma}^{x,\xi\square}_{k,m} = \pi_{k,m}$ and $\boldsymbol{\gamma}^{y,\xi\square}_{\ell,m} = \widetilde{\pi}_{\ell,m}$.*

(iii) *Letting $k = x \cdot \widehat{e}_1$ and $\ell = y \cdot \widehat{e}_1$, there exists a point z with $z \cdot \widehat{e}_1 = m \geq k \vee \ell$ such that for all $\xi \in]\zeta, \eta[$ and $\square \in \{-, +\}$, $\mathbf{z}^{\xi\square}(x, y) = z$.*

The next lemma shows that intervals that satisfy statement (i) of Theorem 3.2 almost surely make up a random dense open subset of ri $\mathcal{U}$.

Lemma 3.3. *Let $\mathcal{U}_0 \subset$ ri $\mathcal{U}$ be a fixed countable dense set of points of differentiability of g. Then with $\mathbb{P}$-probability* 1*, for every $x, y \in \mathbb{Z}$ and every $\xi \in \mathcal{U}_0$, there exist $\zeta \prec \xi \prec \eta$ in* ri $\mathcal{U}$ *such that $|\mu_{x,y}|(]\zeta, \eta[) = 0$.*

A natural question is whether the measure is Cantor-like with no isolated points of support, or if the support consists entirely of isolated points, or if both are possible. These features also turn out to have counterparts in coalescence properties. For a set $\mathcal{A} \subset \mathcal{U}$, say that ξ is a *limit point of $\mathcal{A}$ from the right* if $\mathcal{A}$ intersects $]\xi, \eta[$ for each $\eta \succ \xi$, with a similar definition for limit points from the left.

Theorem 3.4. *With probability* 1*, for all $x, y \in \mathbb{Z}^2$ and $\xi \in$* ri $\mathcal{U}$*:*

(a) $\xi \notin \operatorname{supp} \mu_{x,y} \Leftrightarrow \mathbf{z}^{\xi+}(x, y) = \mathbf{z}^{\xi-}(x, y) \in \mathbb{Z}^2$.

(b) *ξ is an isolated point of* $\operatorname{supp} \mu_{x,y} \Leftrightarrow \mathbf{z}^{\xi+}(x, y) \neq \mathbf{z}^{\xi-}(x, y)$ *but both $\mathbf{z}^{\xi\pm}(x, y)$ are in $\mathbb{Z}^2$.*

(c) *ξ is a limit point of* $\operatorname{supp} \mu_{x,y}$ *from the right* $\Leftrightarrow \mathbf{z}^{\xi+}(x, y) = \infty$. *Similarly, ξ is a limit point of* $\operatorname{supp} \mu_{x,y}$ *from the left* $\Leftrightarrow \mathbf{z}^{\xi-}(x, y) = \infty$.

This motivates the following condition on the Busemann process which will be invoked in some results in the sequel:

$$\text{There exists a full } \mathbb{P}\text{-probability event on which every point of } \operatorname{supp} \mu_{x,y} \text{ is isolated, for all } x, y \in \mathbb{Z}^2. \tag{3.5}$$

Equivalently, condition (3.5) says that $\xi \mapsto B^{\xi\pm}(x, y)$ is a jump process whose jumps do not accumulate on ri $\mathcal{U}$. For this reason, we refer to (3.5) as the *jump process condition*. It is shown in [22, Theorem 3.4] that (3.5) holds when the weights ω_x are i.i.d. exponential random variables. In addition to Lemma 3.3, this is a further reason to expect that (3.5) holds very generally.

Under condition (3.5) Theorem 3.4 extends to a global coalescence statement.

Theorem 3.5. *Statements* (i) *and* (ii) *below are equivalent:*

(i) *The jump process condition* (3.5) *holds.*

(ii) *With* $\mathbb{P}$*-probability* 1*, for all* $x, y \in \mathbb{Z}^2$*, all* $\xi \in \operatorname{ri} \mathcal{U}$*, and both signs* $\square \in \{-, +\}$*, the geodesics* $\boldsymbol{\gamma}^{x,\xi\square}$ *and* $\boldsymbol{\gamma}^{y,\xi\square}$ *coalesce.*

We introduce the random set of exceptional directions obtained by taking the union of the supports of the Busemann measures:

$$\mathcal{V}^\omega = \bigcup_{x, y \in \mathbb{Z}^2} \operatorname{supp} \mu_{x,y} \subset \operatorname{ri} \mathcal{U}. \tag{3.6}$$

It turns out that not all pairs x, y are necessary for the union. It suffices to take pairs of adjacent points along horizontal or vertical lines, or along any bi-infinite path with non-positive local slopes.

Lemma 3.6. *The following holds for* $\mathbb{P}$*-almost every* ω*. Let* $x_{-\infty,\infty}$ *be any bi-infinite path in* $\mathbb{Z}^2$ *such that for all* $i \in \mathbb{Z}$*,* $(x_{i+1} - x_i) \cdot e_1 \geq 0$ *and* $(x_{i+1} - x_i) \cdot e_2 \leq 0$ *and are not both zero. Then*

$$\mathcal{V}^\omega = \bigcup_{i \in \mathbb{Z}} \operatorname{supp} \mu_{x_i, x_{i+1}}.$$

The remainder of this section addresses (i) characterizations of $\mathcal{V}^\omega$ and (ii) its significance for uniqueness and coalescence of geodesics. The first item relates the exceptional directions to asymptotic directions of competition interfaces.

Theorem 3.7. *The following hold for* $\mathbb{P}$*-almost every* ω*:*

(a) *For all* $x \in \mathbb{Z}^2$*,* $\operatorname{supp} \mu_{x,x+e_1} \cap \operatorname{supp} \mu_{x,x+e_2} = \{\xi_*(T_x\omega)\}$*. In particular,* $\mathcal{V}^\omega \supset \{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$*.*

(b) *Under the jump process condition* (3.5)*,* $\mathcal{V}^\omega = \{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$*.*

The next issue is the relationship between $\mathcal{V}^\omega$ and regularity properties of g. Recall the definition (2.3) of $\mathcal{D}$ as the set of differentiability points of g. Let $\mathcal{H}$ be the subset of $\operatorname{ri} \mathcal{U}$ that remains after removal of all open linear segments of g and removal of those endpoints of linear segments that are differentiability points. Equivalently, $\mathcal{H}$ consists of those $\xi \in \operatorname{ri} \mathcal{U}$ at which g is either non-differentiable or strictly concave.

Theorem 3.8. (a) *Let* $\xi \in \operatorname{ri} \mathcal{U}$*. Then* $\xi \in \mathcal{D}$ *if and only if* $\mathbb{P}(\xi \in \mathcal{V}^\omega) = 0$*. If* $\xi \notin \mathcal{D}$ *then*

$$\mathbb{P}(\exists x : \xi_*(T_x\omega) = \xi) = \mathbb{P}(\xi \in \mathcal{V}^\omega) = 1.$$

(b) *For* $\mathbb{P}$*-almost every* ω*, the set* $\{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$ *and the set* $\mathcal{V}^\omega$ *are dense subsets of* $\mathcal{H}$*.*

The next theorem identifies $\mathcal{V}^\omega$ as the set of directions with multiple semi-infinite geodesics. As before, the regularity condition (2.4) allows us to talk about general $\mathcal{U}_\xi$-directed semi-infinite geodesics, instead of only the Busemann geodesics $\boldsymbol{\gamma}^{x,\xi\square}$.

Theorem 3.9. *The following hold for $\mathbb{P}$-almost every ω:*

(a) $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{V}^\omega$ *if and only if the following is true:* $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$ *for all* $x \in \mathbb{Z}^2$ *and all these geodesics coalesce.*

(b) *Under the regularity condition* (2.4), $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{V}^\omega$ *if and only if the following is true: there exists a unique* $\mathcal{U}_\xi$*-directed semi-infinite geodesic out of every* $x \in \mathbb{Z}^2$ *and all these geodesics coalesce.*

(c) *Under the jump process condition* (3.5) *the existence of* $x \in \mathbb{Z}^2$ *such that* $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$ *implies that* $\boldsymbol{\gamma}^{y,\xi+} = \boldsymbol{\gamma}^{y,\xi-}$ *for all* $y \in \mathbb{Z}^2$*, all these geodesics coalesce, and* $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{V}^\omega$.

(d) *Assume both the regularity condition* (2.4) *and the jump process condition* (3.5). *Suppose there exists* $x \in \mathbb{Z}^2$ *such that* $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$*. Then there is a unique* $\mathcal{U}_\xi$*-directed semi-infinite geodesic out of every* $x \in \mathbb{Z}^2$*, all these geodesics coalesce, and* $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{V}^\omega$.

By the uniqueness of finite geodesics, two geodesics emanating from the same site x cannot intersect after they separate. Consequently, non-uniqueness of semi-infinite directed geodesics implies the existence of non-coalescing semi-infinite directed geodesics. When both conditions (2.4) and (3.5) hold, Theorem 3.9 (d) shows the converse: uniqueness implies coalescence.

We close this section with a theorem that collects those previously established properties of geodesics which hold when both the regularity condition (2.4) and the jump process condition (3.5) are in force. Lemma 7.4 justifies that the geodesics in part (d) are ξ-directed rather than merely $\mathcal{U}_\xi$-directed.

Theorem 3.10. *Assume the regularity condition* (2.4) *and the jump process condition* (3.5)*. The following hold for* $\mathbb{P}$*-almost every* ω*:*

(a) $\xi \in \mathcal{V}^\omega$ *if and only if there exist* $x, y \in \mathbb{Z}^2$ *with* $B^{\xi-}(x,y) \neq B^{\xi+}(x,y)$.

(b) $\xi \in \mathcal{V}^\omega$ *if and only if there exists* $x \in \mathbb{Z}^2$ *such that* $\xi = \xi_*(T_x\omega)$.

(c) *If* $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{V}^\omega$*, then for each* $x \in \mathbb{Z}^2$, $\boldsymbol{\gamma}^{x,\xi} = \boldsymbol{\gamma}^{x,\xi-} = \boldsymbol{\gamma}^{x,\xi+}$ *and this is the unique* $\mathcal{U}_\xi$*-directed semi-infinite geodesic out of* x*. For any* $x, y \in \mathbb{Z}^2$, $\boldsymbol{\gamma}^{x,\xi}$ *and* $\boldsymbol{\gamma}^{y,\xi}$ *coalesce.*

(d) *If* $\xi \in \mathcal{V}^\omega$*, then from each* $x \in \mathbb{Z}^2$ *there exist at least two* ξ*-directed semi-infinite geodesics that separate eventually, namely* $\boldsymbol{\gamma}^{x,\xi-}$ *and* $\boldsymbol{\gamma}^{x,\xi+}$*. For each pair* $x, y \in \mathbb{Z}^2$, $\boldsymbol{\gamma}^{x,\xi-}$ *and* $\boldsymbol{\gamma}^{y,\xi-}$ *coalesce and* $\boldsymbol{\gamma}^{x,\xi+}$ *and* $\boldsymbol{\gamma}^{y,\xi+}$ *coalesce.*

3.2. Exponential case

We specialize to the case where

$$\{\omega_x : x \in \mathbb{Z}^2\} \text{ are i.i.d. mean-1 exponential random variables.} \tag{3.7}$$

Rost's classical result [47] gives the shape function

$$g(\xi) = \left(\sqrt{\xi \cdot e_1} + \sqrt{\xi \cdot e_2}\right)^2, \quad \xi \in \mathbb{R}^2_+. \tag{3.8}$$

The regularity condition (2.4) is satisfied as g is strictly concave and differentiable on $\mathrm{ri}\,\mathcal{U}$. The supports $\mathrm{supp}\,\mu_{x,y}$ are unions of inhomogeneous Poisson processes and hence the jump process condition (3.5) is satisfied. This comes from [22, Theorem 3.4] and is described in Section 9.1 below. These two observations imply that the conclusions of Theorem 3.10 hold. With some additional work, we can go beyond the conclusions of Theorem 3.10 in this solvable setting.

Let $s_\xi(x)$ denote the location where the $\xi+$ and $\xi-$ geodesics out of x split:

$$s_\xi(x) = \begin{cases} \text{last point in } \boldsymbol{\gamma}^{x,\xi-} \cap \boldsymbol{\gamma}^{x,\xi+} & \text{if } \boldsymbol{\gamma}^{x,\xi-} \neq \boldsymbol{\gamma}^{x,\xi+}, \\ \infty & \text{if } \boldsymbol{\gamma}^{x,\xi-} = \boldsymbol{\gamma}^{x,\xi+}. \end{cases} \tag{3.9}$$

For part (c) in the next theorem, recall the finite geodesic $\gamma^{x,y}$ defined below (2.1) and the competition interface path φ^x introduced in Section 2.4. Convergence of paths means that any finite segments eventually coincide.

Theorem 3.11. *Assume (3.7). Then the conclusions of Theorem 3.10 hold with $\mathcal{U}_\xi = \{\xi\}$ for all $\xi \in \mathrm{ri}\,\mathcal{U}$. Additionally, the following hold $\mathbb{P}$-almost surely:*

(a) *If $\xi \in \mathcal{V}^\omega$ then from each $x \in \mathbb{Z}^2$ there emanate exactly two semi-infinite ξ-directed geodesics that eventually separate, namely $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{x,\xi+}$.*

(b) *For any $\xi \in \mathrm{ri}\,\mathcal{U}$ and any three ξ-directed semi-infinite geodesics rooted at any three points, at least two of the geodesics coalesce.*

(c) *Let $x \in \mathbb{Z}^2$, $\xi \in \mathcal{V}^\omega$, and let $\{v_n\}_{n \geq m}$ be any sequence on $\mathbb{Z}^2$ such that $v_n \cdot \widehat{e}_1 = n$ and $v_n/n \to \xi$. If $v_n \prec \varphi_n^{s_\xi(x)}$ for all sufficiently large n, then $\gamma^{x,v_n} \to \gamma^{x,\xi-}$ as $n \to \infty$. If $\varphi_n^{s_\xi(x)} \prec v_n$ for all sufficiently large n then $\gamma^{x,v_n} \to \gamma^{x,\xi+}$ as $n \to \infty$.*

(d) *For each $x \in \mathbb{Z}^2$, the entire collection of semi-infinite geodesics emanating from x is exactly $\{\boldsymbol{\gamma}^{x,e_1}, \boldsymbol{\gamma}^{x,e_2}, \boldsymbol{\gamma}^{x,\xi\square} : \xi \in \mathrm{ri}\,\mathcal{U}, \square \in \{+,-\}\}$.*

Theorem 3.11 resolves a number of previously open problems on the geometry of geodesics in the exponential model. It shows that in all but countably many exceptional directions, the collection of geodesics with that asymptotic direction coalesce and form a tree. These exceptional directions are identified both with the directions of discontinuity of the Busemann process and the asymptotic directions of competition interfaces. Moreover, in each exceptional direction $\xi \in \mathcal{V}^\omega$, ahead of each lattice site x, there is a ξ-directed competition interface at which the $\xi-$ and $\xi+$ geodesics out of x split. These are the only two ξ-directed geodesics rooted at x. Strikingly, each of the two families of $\xi-$ and $\xi+$ geodesics has the same structure as the collection of geodesics in a typical direction: each family forms a tree of coalescing semi-infinite paths.

Theorem 3.11 utilizes Theorem A.9, due to Coupier [16], that rules out three geodesics that have the same direction, emanate from a common vertex, and eventually separate. It appears that the modification argument of [16] cannot rule out three non-coalescing geodesics from distinct roots, and so Theorem 3.11 (b) significantly extends Theorem A.9.

Finally, Theorem 3.11 gives a complete description of the coalescence structure of finite geodesics to semi-infinite geodesics in the exponential model. Part (c) says that if we consider a sequence of lattice sites v_n with asymptotic direction ξ, then the geodesic from x to v_n will converge to the $\xi-$ geodesic out of x if and only if v_n eventually stays to the left of the competition interface emanating from the site $s_\xi(x)$ where the $\xi-$ and $\xi+$ geodesics out of x separate. Similarly it will converge to the $\xi+$ geodesic if and only if it stays to the right of that path. The competition interface lives on the dual lattice, so for large n every point v_n is either to the left or to the right of the competition interface. The coalescence structure of semi-infinite geodesics to arbitrary sequences v_n with $v_n/n \to \xi$ then follows by passing to subsequences.

The results of Section 3 are proved in Section 7, except Lemma 3.6 which is proved at the end of Section 8.1.

4. Last-passage percolation as a dynamical system

After the general description of uniqueness and coalescence of Section 3, we take a closer look at the spatial structure of the set of lattice points where particular values or ranges of values from the set $\mathcal{V}^\omega$ of exceptional directions appear. (Recall its definition (3.6).) As mentioned in the introduction, there is a connection to instability in noise-driven conservation laws. The next section explains this point of view.

4.1. Discrete Hamilton–Jacobi equations

We take a dynamical point of view of LPP. Time proceeds in the negative diagonal direction $-\widehat{e}_1 = -e_1 - e_2$ and the spatial axis is $\widehat{e}_2 = e_2 - e_1$. For each $t \in \mathbb{Z}$, the spatial level at time t is $\mathbb{L}_t = \{x \in \mathbb{Z}^2 : x \cdot \widehat{e}_1 = t\}$. For $x \in \mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$ let Π_x^A denote the set of up-right paths $\pi_{k,m}$ such that $\pi_k = x$ and $\pi_m \in A$, where $k = x \cdot \widehat{e}_1$ and m is any integer $\ge k$ such that $A \cap \mathbb{L}_m \neq \emptyset$. For each $\xi \in \operatorname{ri} \mathcal{U}$ and sign $\square \in \{-,+\}$, the Busemann function $B^{\xi\square}$ satisfies the following equation: for all $t \le t_0$ and $x \in \mathbb{L}_t$,

$$B^{\xi\square}(x,0) = \max\left\{\sum_{i=t}^{t_0-1} \omega_{\pi_i} + B^{\xi\square}(\pi_{t_0},0) : \pi \in \Pi_x^{\mathbb{L}_{t_0}}\right\}. \tag{4.1}$$

The unique maximizing path in (4.1) is the geodesic segment $\boldsymbol{\gamma}_{t,t_0}^{x,\xi\square}$.

Equation (4.1) can be viewed as a discrete Hopf–Lax–Oleinik semigroup. For example, equation (4.1) is an obvious discrete analogue of the variational formula (1.3) of [5]. At first blush the two formulas appear different because (1.3) of [5] contains a kinetic energy term. However, this term is not needed in (4.1) above because all admissible steps are of size 1 and all paths between levels $\mathbb{L}_t$ and $\mathbb{L}_{t_0}$ have equal length (number of steps).

Through this analogy with a Hopf–Lax–Oleinik semigroup we can regard $B^{\xi\square}(\bullet,0)$ as a global solution of a discrete stochastic Hamilton–Jacobi equation started in the infinite past ($t_0 \to \infty$) and driven by the noise ω. The spatial difference $B^{\xi\square}(x+e_1,x+e_2) = B^{\xi\square}(x+e_1,0) - B^{\xi\square}(x+e_2,0)$ can then be viewed as a global solution of a discretized stochastic Burgers equation.

By Lemma B.1, if g is differentiable on ri $\mathcal{U}$, then $B^{\xi+}$ and $B^{\xi-}$ both satisfy, for each $x \in \mathbb{Z}^2$,

$$\lim_{|n|\to\infty} \frac{B^{\xi\pm}(x, x + n\widehat{e}_2)}{n} = \nabla g(\xi) \cdot \widehat{e}_2.$$

Thus, $B^{\xi\pm}$ are two solutions with the same value of the conserved quantity. Under the jump process condition (3.5), $\xi \in \operatorname{supp} \mu_{x+e_1,x+e_2}$ if and only if $B^{\xi+}(x + e_1, x + e_2) \neq B^{\xi-}(x + e_1, x + e_2)$. This means that the locations x where $\xi \in \operatorname{supp} \mu_{x+e_1,x+e_2}$ are precisely the space-time points at which the two solutions $B^{\xi\pm}$ differ. It is reasonable to expect then that these points are locations of instability in the following sense. The spatial difference of the solution to the stochastic Hamilton–Jacobi equation started at time t_0 with a linear initial condition dual to ξ,

$$\max\{G_{x+e_1,y} - y \cdot \nabla g(\xi) : y \cdot \widehat{e}_1 = t_0\} - \max\{G_{x+e_2,y} - y \cdot \nabla g(\xi) : y \cdot \widehat{e}_1 = t_0\},$$

has at least two limit points $B^{\xi\pm}(x + e_1, x + e_2) + (e_1 - e_2) \cdot \nabla g(\xi)$ as $t_0 \to \infty$. This is supported by simulations and is hinted at by Theorem 3.11 (c).

With these points in mind, we now define what we mean by instability points and then turn to studying their geometric structure. Proofs of the results of this section appear in Section 8.

4.2. *Webs of instability*

For a direction $\xi \in$ ri $\mathcal{U}$ and a sign $\square \in \{-, +\}$, let $\mathcal{G}_{\xi\square}$ be the directed graph whose vertex set is $\mathbb{Z}^2$ and whose edge set includes $(x, x + e_i)$ whenever $\boldsymbol{\gamma}^{x,\xi\square}_{m+1} = x + e_i$. Here $m = x \cdot \widehat{e}_1$ and we consider both $i \in \{1, 2\}$. These are the directed graphs of $\xi\square$ geodesics defined by (2.12). By construction, each $\mathcal{G}_{\xi\square}$ is a disjoint union of trees, i.e. a forest, and for each $x \in \mathbb{Z}^2$, the geodesic $\boldsymbol{\gamma}^{x,\xi\square}$ follows the directed edges of $\mathcal{G}_{\xi\square}$.

Recall the vectors $\widehat{e}_1^* = \widehat{e}_1/2 = (e_1 + e_2)/2$ and $\widehat{e}_2^* = \widehat{e}_2/2 = (e_2 - e_1)/2$. Let $\mathcal{G}^*_{\xi\square}$ be the directed graph whose vertex set is the dual lattice $\mathbb{Z}^{2*} = \widehat{e}_1^* + \mathbb{Z}^2$ and whose edge set is defined by this rule: for each $x \in \mathbb{Z}^2$, on the dual lattice $x + \widehat{e}_1^*$ points to $x + \widehat{e}_1^* - e_i$ in $\mathcal{G}^*_{\xi\square}$ if and only if on the original lattice x points to $x + e_i$ in $\mathcal{G}_{\xi\square}$. Pictorially this means that $\mathcal{G}^*_{\xi\square}$ contains all the south and west directed nearest-neighbor edges of $\mathbb{Z}^{2*}$ that do not cross an edge of $\mathcal{G}_{\xi\square}$. See Figure 4.1 for an illustration.

For $\zeta \preceq \eta$ in ri $\mathcal{U}$ let the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$ be the union of the graphs $\mathcal{G}^*_{\xi\square}$ over $\xi \in [\zeta, \eta]$ and $\square \in \{-, +\}$. That is, the vertex set of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ is $\mathbb{Z}^{2*}$, and the edge set of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ is the union of the edge sets of $\mathcal{G}^*_{\xi\pm}$ over $\xi \in [\zeta, \eta]$. From each point $x^* \in \mathbb{Z}^{2*}$ a directed edge of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ points to $x^* - e_1$ or $x^* - e_2$ or both. Due to the monotonicity (2.10) of the Busemann functions, $\mathcal{G}^*_{\cup[\zeta,\eta]}$ is the union of just the two graphs $\mathcal{G}^*_{\zeta-}$ and $\mathcal{G}^*_{\eta+}$. In particular, x^* points to $x^* - e_2$ in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ if and only if $x^* - \widehat{e}_1^*$ points to $x^* + \widehat{e}_2^*$ in $\mathcal{G}_{\zeta-}$, and x^* points to $x^* - e_1$ in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ if and only if $x^* - \widehat{e}_1^*$ points to $x^* - \widehat{e}_2^*$ in $\mathcal{G}_{\eta+}$.

Identify the space-time point $x + \widehat{e}_1^* \in \mathbb{Z}^{2*}$ on the dual lattice with the diagonal edge that connects $x + e_1$ and $x + e_2$ on the primal lattice (see Figure 4.2). Call the dual lattice

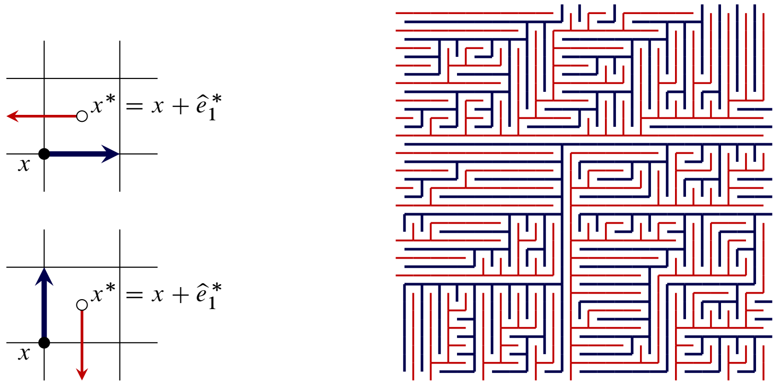

Fig. 4.1. Left plot: An illustration of the duality relation between the edges of $\mathcal{G}_{\xi_\square}$ (black/thick) and those of $\mathcal{G}^*_{\xi_\square}$ (red/thin). Right plot: An illustration of a (blue/thick) north-east directed geodesic graph $\mathcal{G}_{\xi_\square}$ and its (red/thin) south-west directed dual $\mathcal{G}^*_{\xi_\square}$.

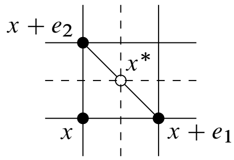

Fig. 4.2. The edge $(x + e_1, x + e_2)$ is identified with the dual point $x^* = x + \hat{e}_1^*$.

point $x^* = x + \hat{e}_1^*$ a $[\zeta, \eta]$-*instability point* if $[\zeta, \eta] \cap \operatorname{supp} \mu_{x+e_1, x+e_2} \neq \emptyset$. If $\zeta = \eta = \xi$, call x^* a ξ-*instability point*. Denote the set of $[\zeta, \eta]$-instability points by $\mathbf{S}^*_{[\zeta,\eta]}$, with $\mathbf{S}^*_\xi = \mathbf{S}^*_{[\xi,\xi]}$. Then $\mathbf{S}^*_{[\zeta,\eta]}$ is the union of $\mathbf{S}^*_\xi$ over $\xi \in [\zeta, \eta]$. Theorem 3.1 and the ordering (2.13) of geodesics give the following characterization in terms of disjoint geodesics, alluded to in Section 4.1.

Lemma 4.1. *The following holds for $\mathbb{P}$-almost every ω. Let $\zeta \preceq \eta$, including the case $\zeta = \eta = \xi$. Let $x \in \mathbb{Z}^2$ and $x^* = x + \hat{e}_1^*$. Then $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ if and only if $\boldsymbol{\gamma}^{x+e_2,\zeta-} \cap \boldsymbol{\gamma}^{x+e_1,\eta+} = \emptyset$.*

Let the *instability graph* $\mathcal{S}^*_{[\zeta,\eta]}$ be the subgraph of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ with vertex set $\mathbf{S}^*_{[\zeta,\eta]}$ and those directed edges of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ that point from some $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ to a point $x^* - e_i \in \mathbf{S}^*_{[\zeta,\eta]}$, for either $i \in \{1, 2\}$. (The proof of Theorem 4.3 in Section 8.1 shows that *every* edge of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ that emanates from a point of $\mathbf{S}^*_{[\zeta,\eta]}$ is in fact an edge of $\mathcal{S}^*_{[\zeta,\eta]}$.)

In the case $\zeta = \eta = \xi$ write $\mathcal{S}^*_\xi$ for $\mathcal{S}^*_{[\xi,\xi]}$. Explicitly, the vertices of $\mathcal{S}^*_\xi$ are dual points $x + \hat{e}_1^*$ such that $\xi \in \operatorname{supp} \mu_{x+e_1,x+e_2}$ and the edges are those of $\mathcal{G}^*_{\xi-} \cup \mathcal{G}^*_{\xi+}$ that connect these points.

The graph $\mathcal{S}^*_{[\zeta,\eta]}$ is also the edge union of the graphs $\mathcal{S}^*_\xi$ over $\xi \in [\zeta, \eta]$. To see this, let $x^* = x + \hat{e}_1^*$. If $\xi = \xi_*(T_x\omega) \in [\zeta, \eta]$ then $\mathcal{S}^*_\xi$ contains both edges from x^* to $x^* - e_1$ and $x^* - e_2$, as does $\mathcal{S}^*_{[\zeta,\eta]}$. If $\xi_*(T_x\omega) \notin [\zeta, \eta]$ then in $\mathcal{S}^*_\xi$ and in $\mathcal{S}^*_{[\zeta,\eta]}$, x^* points to the same vertex $x^* - e_i$.

Remark 4.2. By the continuity (2.14) and the fact that the support of a measure is a closed set we find that almost surely, for any $\zeta \preceq \eta$ in ri $\mathcal{U}$ and for any finite box $[-L, L]^2 \cap \mathbb{Z}^2$, $\mathcal{S}^*_{[\zeta',\eta']} = \mathcal{S}^*_{[\zeta,\eta]}$ on the entire box, for $\zeta' \prec \zeta$ close enough to ζ and $\eta' \succ \eta$ close enough to η. This explains why the two top graphs in Figure 4.4 are identical and are in fact equal to $\mathcal{S}^*_{\xi}$.

The message of the next theorem is that instability points exist for all exceptional directions in $\mathcal{V}^{\omega}$, and these instability points arrange themselves on bi-infinite directed paths in the instability graphs.

Theorem 4.3. *The following holds for $\mathbb{P}$-almost every ω. Pick any $\zeta \preceq \eta$ in* ri $\mathcal{U}$ *such that $[\zeta, \eta] \cap \mathcal{V}^{\omega} \neq \emptyset$, including the case $\zeta = \eta = \xi$. Then the instability graph $\mathcal{S}^*_{[\zeta,\eta]}$ is an infinite directed graph. Furthermore, $\mathcal{S}^*_{[\zeta,\eta]}$ equals the union of the bi-infinite directed paths of the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$. In the backward* (*north and east*) *orientation, each such path is $[\underline{\zeta}, \overline{\eta}]$-directed.*

In particular, if $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ and $m = x^* \cdot \widehat{e}_1$, there exists a bi-infinite sequence $\{x^*_n\}_{n\in\mathbb{Z}} \subset \mathbf{S}^*_{[\zeta,\eta]}$ such that $x^*_m = x^*$ and for each n, $x^*_n \cdot \widehat{e}_1 = n$ and x^*_n points to x^*_{n-1} in the graph $\mathcal{S}^*_{[\zeta,\eta]}$. As $n \to \infty$, the limit points of $n^{-1} x^*_n$ lie in $[\underline{\zeta}, \overline{\eta}]$.

Next we describe the branching and coalescing of the bi-infinite directed paths that make up the graph $\mathcal{S}^*_{[\zeta,\eta]}$. If there is a directed path in the graph $\mathcal{S}^*_{[\zeta,\eta]}$ from y^* to x^*, then y^* is an *ancestor* of x^* and equivalently x^* is a *descendant* of y^*. Let $\mathbf{A}^*_{[\zeta,\eta]}(x^*)$ denote the set of ancestors of x^* in the graph $\mathcal{S}^*_{[\zeta,\eta]}$. Abbreviate again $\mathbf{A}^*_{\xi}(x^*) = \mathbf{A}^*_{[\xi,\xi]}(x^*)$.

A point $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ is a *branch point* in the graph $\mathcal{S}^*_{[\zeta,\eta]}$ if x^* is an ancestor of both $x^* - e_1$ and $x^* - e_2$. Branch points are dual to those where $\zeta-$ and $\eta+$ geodesics separate. Similarly, $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ is a *coalescence point* if both $x^* + e_1$ and $x^* + e_2$ are ancestors of x^*. Figures 4.3 and 4.4 display simulations that illustrate the branching and coalescing.

For the sharpest branching and coalescing properties in the next theorem, we invoke again the regularity condition (2.4) and the jump process condition (3.5), and additionally the non-existence of non-trivial bi-infinite geodesics:

$$\text{There exists a full } \mathbb{P}\text{-probability event on which the only bi-infinite geodesics are the trivial ones: } x + \mathbb{Z}e_i \text{ for } x \in \mathbb{Z}^2 \text{ and } i \in \{1, 2\}. \tag{4.2}$$

Condition (4.2) is known to hold in the exponential case [8, 9].

Theorem 4.4. *The following hold for $\mathbb{P}$-almost every ω and all $\zeta \preceq \eta$ in* ri $\mathcal{U}$ *such that $[\zeta, \eta] \cap \mathcal{V}^{\omega} \neq \emptyset$* (*the case $\zeta = \eta = \xi$ is included unless otherwise stated*)*:*

(a) *x^* is a branch point in $\mathcal{S}^*_{[\zeta,\eta]}$ if and only if $\xi_*(T_{x^*-\widehat{e}^*_1}\omega) \in [\zeta, \eta]$.*

(b) *If $\zeta \prec \eta$, then any $x^*, y^* \in \mathbf{S}^*_{[\zeta,\eta]}$ have a common descendant: there isz$^* \in \mathbf{S}^*_{[\zeta,\eta]}$ such that $x^*, y^* \in \mathbf{A}^*_{[\zeta,\eta]}(z^*)$. If we assume the no bi-infinite geodesics condition* (4.2)*, then the same statement also holds for the case $\zeta = \eta = \xi$.*

(c) *Assume the jump process condition* (3.5)*. Then any $x^*, y^* \in \mathbf{S}^*_{[\zeta,\eta]}$ have a common ancestor $z^* \in \mathbf{A}^*_{[\zeta,\eta]}(x^*) \cap \mathbf{A}^*_{[\zeta,\eta]}(y^*)$.*

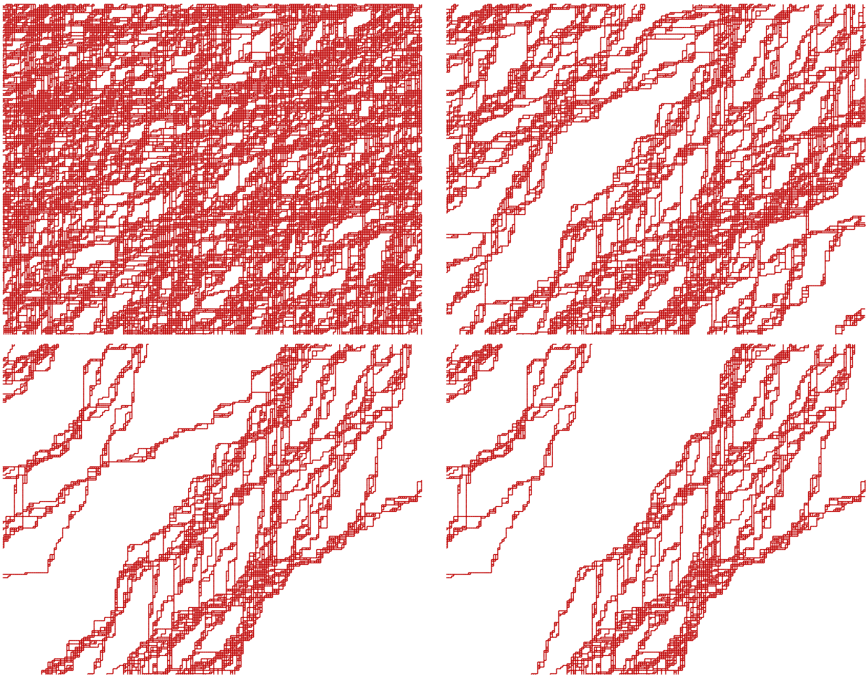

Fig. 4.3. Four nested down-left pointing $\mathcal{S}^*_{[\zeta,\eta]}$ graphs in the square $[-100, 100]^2$. Top to bottom, left to right, in reading order, $[\zeta \cdot e_1, \eta \cdot e_1]$ equals $[0.096, 0.772]$, $[0.219, 0.595]$, $[0.318, 0.476]$, and $[0.355, 0.436]$. Two further nested subgraphs appear in Figure 4.4. In the simulation the weights were exponentially distributed and we chose the direction ξ to be a jump point of the Busemann process on the edge $(0, e_1)$.

(d) *Suppose $\zeta \prec \eta$ are such that $]\zeta, \eta[\cap \mathcal{V}^\omega \neq \emptyset$. Then for any $z \in \mathbb{Z}^2$ there is a coordinatewise strictly ordered infinite sequence $z < z_1^* < z_2^* < \cdots$ such that each z_n^* is a branch point in $\mathcal{S}^*_{[\zeta,\eta]}$. There are also infinitely many coalescence points in $\mathcal{S}^*_{[\zeta,\eta]}$.*

(e) *If the jump process condition (3.5) holds and $\xi \in \mathcal{V}^\omega$, then for any $z \in \mathbb{Z}^2$ there is a coordinatewise strictly ordered infinite sequence $z < z_1^* < z_2^* < \cdots$ such that each z_n^* is a branch point in $\mathcal{S}^*_\xi$. If additionally the no bi-infinite geodesics condition (4.2) holds, then there are infinitely many coalescence points in $\mathcal{S}^*_\xi$.*

Remark 4.5. If the regularity condition (2.4) holds, then part (d) holds for $\zeta \prec \eta$ with $[\zeta, \eta] \cap \mathcal{V}^\omega \neq \emptyset$. The proof of this is given right after that of Theorem 4.4 in Section 8.1.

Given that there are infinitely many instability points when instability points exist, it is natural to wonder what their density on the lattice is. We identify the following trichotomy.

Proposition 4.6. *Assume the regularity condition (2.4). Then for $\mathbb{P}$-almost every ω and all $\xi \in \operatorname{ri} \mathcal{U}$, exactly one of the following three scenarios happens:*

(a) *$\xi \notin \mathcal{V}^\omega$ and hence there are no ξ-instability points.*

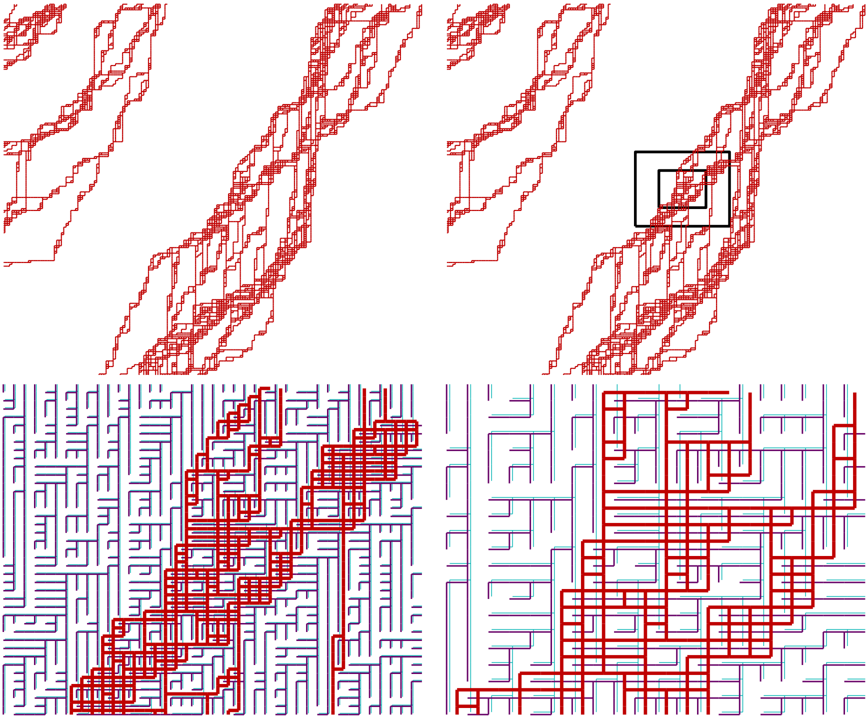

Fig. 4.4. Continuing with the simulation setting of Figure 4.3, the top two pictures are $\mathcal{S}^*_{[\zeta,\eta]}$ graphs in $[-100, 100]^2$ with $[\zeta \cdot e_1, \eta \cdot e_1] = [0.374, 0.417]$ (left) and $[0.393, 0.397]$ (right). The two graphs are in fact identical. The pictures on the second row zoom into the framed squares of the top right picture, the left one into the square $[-20, 20]^2$ and the right one into $[-10, 10]^2$. Besides the down-left pointing red $\mathcal{S}^*_{[\zeta,\eta]}$ graphs, the bottom pictures include the up-right pointing graphs $\mathcal{G}_{\zeta-}$ (green/lighter) and $\mathcal{G}_{\eta+}$ (purple/darker). Whenever $\mathcal{G}_{\zeta-}$ and $\mathcal{G}_{\eta+}$ separate at x, green points up and purple points right, and $\mathcal{S}^*_{[\zeta,\eta]}$ has a branch point at $x + \widehat{e}_1^*$. The blue/green trees that occupy the islands surrounded by red paths are described in Section 4.3.

(b) *$\xi \in \mathcal{V}^\omega \cap \mathcal{D}$ and there are infinitely many ξ-instability points but they have zero density.*

(c) *$\xi \notin \mathcal{D}$ and the ξ-instability points have positive density.*

We return to this question in Section 5 in the solvable case of exponential weights, where we can say significantly more.

4.3. Flow of Busemann measure

This section views the instability graph $\mathcal{S}^*_{[\zeta,\eta]}$ as a description of the south-west directed flow of Busemann measure on the dual lattice. As discussed in Section 4.1, we can think of the function $B^{\xi\square}(x + e_1, x + e_2)$ as a global solution of a discretized stochastic Burgers equation. We can assign the value $B^{\xi\square}(x + e_1, x + e_2)$ to the dual point $x^* = x + \widehat{e}_1^*$

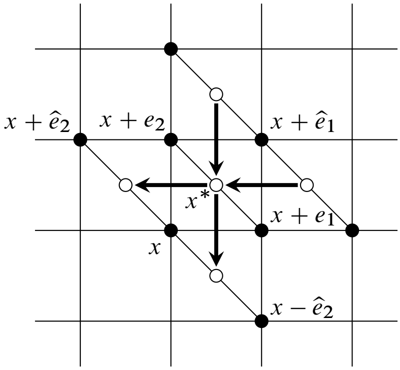

Fig. 4.5. The flow of Busemann measure follows the arrows. The antidiagonal edge $(x+e_1, x+e_2)$ is identified with the dual point $x^* = x + \hat e_1^*$. The Busemann measure $\mu_{x+e_1,x+e_2}$ on this edge is composed of the mass flowing from the north and east, and it in turns divides its mass between the flows south and west.

that represents the diagonal edge $(x+e_1, x+e_2)$. Then the cocycle property (2.7) gives us a flow of Busemann measure along the south and west pointing edges of the dual lattice $\mathbb{Z}^{2*}$. First decompose the Busemann measure of the edge $(x+e_1, x+e_2)$ as a sum $\mu_{x+e_1,x+e_2} = \mu_{x+e_1,x} + \mu_{x,x+e_2}$ of two positive measures. This is justified by the cocycle property (2.7). Then stipulate that the measure $\mu_{x+e_1,x}$ flows south from x^* to $x^* - e_2$ and contributes to the Busemann measure $\mu_{x-\hat e_2,x}$, while the measure $\mu_{x,x+e_2}$ flows west from the x^* to $x^* - e_1$ and contributes to the Busemann measure $\mu_{x,x+\hat e_2}$. See Figure 4.5.

The cocycle property also tells us that $\mu_{x+e_1,x+e_2} = \mu_{x+e_1,x+\hat e_1} + \mu_{x+\hat e_1,x+e_2}$. This represents $\mu_{x+e_1,x+e_2}$ as the sum of the contributions it receives from the next level up: $\mu_{x+e_1,x+\hat e_1}$ comes from the east from the dual vertex $x + e_1 + \hat e_1^*$, while $\mu_{x+\hat e_1,x+e_2}$ comes from the north from the dual vertex $x + e_2 + \hat e_1^*$.

Now pick a pair of directions $\zeta \preceq \eta$ in ri $\mathcal{U}$, and consider the graph $\mathcal{B}^*_{[\zeta,\eta]}$ on the dual lattice $\mathbb{Z}^{2*}$ obtained as follows. Include the vertex $x^* = x + \hat e_1^*$ if $[\zeta, \eta] \cap \operatorname{supp} \mu_{x+e_1,x+e_2} \neq \emptyset$. For $i \in \{1, 2\}$, include the dual edge $(x^*, x^* - e_i)$ if $[\zeta, \eta]$ intersects $\operatorname{supp} \mu_{x,x+e_{3-i}}$, or somewhat pictorially, if some of the support in $[\zeta, \eta]$ flows along the dual edge $(x^*, x^* - e_i)$.

The results of this section hold $\mathbb{P}$-almost surely simultaneously for all $\zeta \preceq \eta$ in ri $\mathcal{U}$, including the case $\zeta = \eta = \xi$.

Theorem 4.7. *The graphs $\mathcal{B}^*_{[\zeta,\eta]}$ and $\mathcal{S}^*_{[\zeta,\eta]}$ are the same.*

Under the jump condition (3.5), a closed set cannot intersect the support without actually having non-zero measure. Thus under (3.5), Theorem 4.7 tells us that $\mathcal{S}^*_{[\zeta,\eta]}$ is precisely the graph along which positive Busemann measure in the interval $[\zeta, \eta]$ flows.

Next we describe the "islands" on $\mathbb{Z}^2$ carved out by the paths of the graph $\mathcal{S}^*_{[\zeta,\eta]}$ (islands surrounded by red paths in Figures 4.3 and 4.4). These islands are trees, they are the connected components of an intersection of geodesic graphs, and they are the equivalence classes of an equivalence relation defined in terms of the supports of Busemann measures.

Define the graph $\mathcal{G}_{\cap[\zeta,\eta]} = \bigcap_{\xi\in[\zeta,\eta]}(\mathcal{G}_{\xi-} \cap \mathcal{G}_{\xi+})$ on the vertex set $\mathbb{Z}^2$ by keeping only those edges that lie in each geodesic graph $\mathcal{G}_{\xi\square}$ as ξ varies over $[\zeta,\eta]$ and $\square$ over $\{-,+\}$. Also, directly from the definitions it follows that an edge of $\mathbb{Z}^2$ lies in $\mathcal{G}_{\cap[\zeta,\eta]}$ if and only if the dual edge it crosses does not lie in the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$ introduced in Section 4.2. Since each $\mathcal{G}_{\xi\square}$ is a forest, $\mathcal{G}_{\cap[\zeta,\eta]}$ is a forest, that is, a union of disjoint trees.

Define an equivalence relation $\overset{[\zeta,\eta]}{\sim}$ on $\mathbb{Z}^2$ by $x \overset{[\zeta,\eta]}{\sim} y$ if and only if $\operatorname{supp}\mu_{x,y} \cap [\zeta,\eta] = \emptyset$. It is an equivalence relation because $\mu_{x,x}$ is the identically zero measure, and $B^{\xi\square}_{x,z} = B^{\xi\square}_{x,y} + B^{\xi\square}_{y,z}$ implies that $|\mu_{x,z}| \le |\mu_{x,y}| + |\mu_{y,z}|$. In terms of coalescence, $x \overset{[\zeta,\eta]}{\sim} y$ if and only if the coalescence points $\mathbf{z}^{\xi\square}(x,y)$ remain constant in $\mathbb{Z}^2$ as ξ varies across $[\zeta,\eta]$ and $\square$ over $\{-,+\}$. (This follows from Propositions 7.1 and 7.2 proved below.) As usual, replace $\overset{[\zeta,\eta]}{\sim}$ with $\overset{\xi}{\sim}$ when $[\zeta,\eta] = [\xi,\xi]$.

Proposition 4.8. *The equivalence classes of the relation $\overset{[\zeta,\eta]}{\sim}$ are exactly the connected components* (*subtrees*) *of $\mathcal{G}_{\cap[\zeta,\eta]}$.*

Lemma 8.6 proved below shows that nearest-neighbor points of $\mathbb{Z}^2$ are in distinct $\overset{[\zeta,\eta]}{\sim}$ equivalence classes if and only if the edge between them is bisected by an edge of the instability graph $\mathcal{S}^*_{[\zeta,\eta]}$. Together with Proposition 4.8 this tells us that the paths of $\mathcal{S}^*_{[\zeta,\eta]}$ are precisely the boundaries that separate distinct connected components of $\mathcal{G}_{\cap[\zeta,\eta]}$ and the equivalence classes of $\overset{[\zeta,\eta]}{\sim}$.

The next two lemmas indicate how the structure of the subtrees of $\mathcal{G}_{\cap[\zeta,\eta]}$ is constrained by the fact that they are intersections of geodesic trees. These properties are clearly visible in the bottom pictures of Figure 4.4 where these subtrees are the blue/green trees in the islands separated by red paths.

Lemma 4.9. *Let $\mathcal{K}$ be a subtree of $\mathcal{G}_{\cap[\zeta,\eta]}$ and let x and y be two distinct vertices of $\mathcal{K}$. Assume that neither strictly dominates the other in the coordinatewise ordering, that is, both coordinatewise strict inequalities $x < y$ and $y < x$ fail. Then the entire rectangle $[\![x \wedge y, x \vee y]\!]$ is a subset of the vertex set of $\mathcal{K}$.*

In particular, if for some integers $\{t,k,\ell\}$, level-t lattice points $(k,t-k)$ and $(\ell,t-\ell)$ are vertices of a subtree $\mathcal{K}$, the entire discrete interval $\{(i,t-i) : i \in [\![k,\ell]\!]\}$ is a subset of the vertex set of $\mathcal{K}$. Similarly, points on horizontal and vertical line segments between vertices of a subtree $\mathcal{K}$ are again vertices of $\mathcal{K}$.

Lemma 4.10. *Let $\mathcal{K}$ be a subtree of $\mathcal{G}_{\cap[\zeta,\eta]}$. There is at most one vertex x in $\mathcal{K}$ such that $\{x-e_1, x-e_2\} \cap \mathcal{K} = \emptyset$. Such a point x exists if and only if* $\inf\{t \in \mathbb{Z} : \mathcal{K} \cap \mathbb{L}_t \ne \emptyset\} > -\infty$. *In that case $\mathcal{K}$ lies in $\{y : y \ge x\}$.*

Note that Lemma 4.10 does not say that a subtree has a single leaf. Both x and $x - e_i$ can be leaves of a subtree when the edge $(x - e_i, x)$ is not present in $\mathcal{G}_{\cap[\zeta,\eta]}$.

For the remainder of this section assume the jump condition (3.5), in order to give a sharper description of the subtrees of $\mathcal{G}_{\cap[\zeta,\eta]}$. Let $\mathbf{D}_{[\zeta,\eta]} = \{z \in \mathbb{Z}^2 : \xi_*(T_z\omega) \in [\zeta,\eta]\}$. By Theorem 4.4 (a), $z \in \mathbf{D}_{[\zeta,\eta]}$ if and only if $z + \widehat{e}_1^*$ is a branch point of the instability graph $\mathcal{S}^*_{[\zeta,\eta]}$. It follows then that both $z \pm \widehat{e}_2^*$ are also $[\zeta,\eta]$-instability points.

Assume for the moment that $\mathbf{D}_{[\zeta,\eta]} \neq \emptyset$. By Theorem 3.8, under the jump condition (3.5) this is equivalent to $[\zeta,\eta] \cap \mathcal{V}^\omega \neq \emptyset$.

The graph $\mathcal{G}_{\cap[\zeta,\eta]}$ has no outgoing up or right edges from a point $z \in \mathbf{D}_{[\zeta,\eta]}$ because geodesics split: $\boldsymbol{\gamma}^{z,\xi_*(T_z\omega)-}$ and $\{\boldsymbol{\gamma}^{z,\xi\pm} : \zeta \preceq \xi \prec \xi_*(T_z\omega)\}$ take the e_2-step at z, while $\boldsymbol{\gamma}^{z,\xi_*(T_z\omega)+}$ and $\{\boldsymbol{\gamma}^{z,\xi\pm} : \xi_*(T_z\omega) \prec \xi \preceq \eta\}$ take the e_1-step at z. For each $z \in \mathbf{D}_{[\zeta,\eta]}$, let the tree $\mathcal{K}(z)$ consist of all directed paths in $\mathcal{G}_{\cap[\zeta,\eta]}$ that terminate at z. $\mathcal{K}(z)$ can consist of z alone.

These properties come from previously established facts:

- Each $x \in \mathbb{Z}^2 \setminus \mathbf{D}_{[\zeta,\eta]}$ lies in a unique $\mathcal{K}(z)$ determined by following the common path of the geodesics $\{\boldsymbol{\gamma}^{x,\xi\square} : \xi \in [\zeta,\eta], \square \in \{-,+\}\}$ until the first point z at which a split happens. A split must happen eventually because for any $u \in \mathbf{D}_{[\zeta,\eta]}$ the two geodesics $\boldsymbol{\gamma}^{u,\xi_*(T_u\omega)\pm}$ separate immediately at u, while by Theorem 3.5 the geodesic $\boldsymbol{\gamma}^{x,\xi_*(T_u\omega)\square}$ coalesces with $\boldsymbol{\gamma}^{u,\xi_*(T_u\omega)\square}$ for both $\square \in \{-,+\}$.
- If $\zeta \prec \eta$ then each tree $\mathcal{K}(z)$ is finite. Same holds also for the case $\zeta = \eta = \xi$ under the no bi-infinite geodesics condition (4.2). This follows from Theorem 4.4 (b) because the $[\zeta,\eta]$-instability points $z \pm \widehat{e}_2^*$ that flank z have a common descendant u^* in the graph $\mathcal{S}^*_{[\zeta,\eta]}$. The two directed paths of $\mathcal{S}^*_{[\zeta,\eta]}$ that connect $z + \widehat{e}_1^*$ to u^* surround $\mathcal{K}(z)$.

The final theorem of this section decomposes $\mathcal{G}_{\cap[\zeta,\eta]}$ into its connected components.

Theorem 4.11. *Assume the jump condition* (3.5).

(a) $\mathcal{G}_{\cap[\zeta,\eta]}$ *is a single tree if and only if* $[\zeta,\eta] \cap \mathcal{V}^\omega = \emptyset$.

(b) *If* $[\zeta,\eta] \cap \mathcal{V}^\omega \neq \emptyset$, *the connected components of* $\mathcal{G}_{\cap[\zeta,\eta]}$ *are the trees* $\{\mathcal{K}(z) : z \in \mathbf{D}_{[\zeta,\eta]}\}$.

We finish by reminding the reader that all the hypotheses and hence all the conclusions hold in the case of i.i.d. exponential weights. The results of Section 4.3 are proved in Section 8.3.

5. Statistics of instability points in the exponential model

Under condition (3.7), i.e. when the weights are exponentially distributed, we derive explicit statistics of the instability graphs. For $\xi \in \operatorname{ri}\mathcal{U}$, $k \in \mathbb{Z}$, and $\square \in \{-,+\}$, abbreviate $B_k^{\xi\square} = B^{\xi\square}(ke_1,(k+1)e_1)$ and write B_k^ξ when there is no $\pm$ distinction. For $\zeta \preceq \eta$ in $\operatorname{ri}\mathcal{U}$ let

$$\cdots < \tau^{\zeta,\eta}(-1) < 0 \le \tau^{\zeta,\eta}(0) < \tau^{\zeta,\eta}(1) < \cdots$$

be the ordered indices such that

$$B_k^{\zeta-} > B_k^{\eta+} \quad \text{if and only if} \quad k \in \{\tau^{\zeta,\eta}(i) : i \in \mathbb{Z}\}. \tag{5.1}$$

If $B_k^{\zeta-} > B_k^{\eta+}$ happens for only finitely many indices k, then some $\tau^{\zeta,\eta}(i)$ are set equal to $-\infty$ or ∞.

By Theorem 3.11, under condition (3.7), (5.1) is equivalent to

$$\mathbf{z}^{\zeta-}(ke_1,(k+1)e_1) \neq \mathbf{z}^{\eta+}(ke_1,(k+1)e_1).$$

It is worth keeping this geometric implication of (5.1) in mind in this section to provide some context for the results that follow.

It will be convenient in what follows to parametrize directions in $\operatorname{ri}\mathcal{U}$ through the increasing bijection

$$\zeta = \zeta(\alpha) = \left(\frac{\alpha^2}{(1-\alpha)^2+\alpha^2}, \frac{(1-\alpha)^2}{(1-\alpha)^2+\alpha^2}\right) \iff \alpha = \alpha(\zeta) = \frac{\sqrt{\zeta\cdot e_1}}{\sqrt{\zeta\cdot e_1}+\sqrt{1-\zeta\cdot e_1}} \tag{5.2}$$

between $\zeta \in \operatorname{ri}\mathcal{U}$ and $\alpha \in (0,1)$. Recall the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. By (C.6) from Appendix C, the conditioning event in the theorem below has probability $\mathbb{P}(B_0^\zeta > B_0^\eta) = \frac{\alpha(\eta)-\alpha(\zeta)}{\alpha(\eta)}$. Since $\zeta \prec \eta$ are fixed, with probability 1 no $\pm$ distinction appears in the Busemann functions.

Theorem 5.1. *Assume* (3.7). *Fix $\zeta \prec \eta$ in* $\operatorname{ri}\mathcal{U}$. *Conditional on $B_0^\zeta > B_0^\eta$,*

$$\{\tau^{\zeta,\eta}(i+1)-\tau^{\zeta,\eta}(i),\, B^\zeta_{\tau^{\zeta,\eta}(i)} - B^\eta_{\tau^{\zeta,\eta}(i)} : i \in \mathbb{Z}\}$$

is an i.i.d. sequence with marginal distribution

$$\begin{aligned}\mathbb{P}\{\tau^{\zeta,\eta}(i+1)-\tau^{\zeta,\eta}(i) = n,\ & B^\zeta_{\tau^{\zeta,\eta}(i)} - B^\eta_{\tau^{\zeta,\eta}(i)} > r \mid B_0^\zeta > B_0^\eta\} \\ &= C_{n-1}\frac{\alpha(\zeta)^{n-1}\alpha(\eta)^n}{(\alpha(\zeta)+\alpha(\eta))^{2n-1}}\, e^{-\alpha(\zeta)r}, \quad \forall\, i \in \mathbb{Z},\, n \in \mathbb{N},\, r \in \mathbb{R}_+. \end{aligned} \tag{5.3}$$

Abbreviate $\tau^\xi(i) = \tau^{\xi,\xi}(i)$. Our next goal is to describe the joint distribution of the process

$$\{\tau^\xi(i),\ B^{\xi-}_{\tau^\xi(i)} - B^{\xi+}_{\tau^\xi(i)} : i \in \mathbb{Z}\}$$

of locations and sizes of jumps in direction ξ, conditional on $\{B_0^{\xi-} > B_0^{\xi+}\}$. However, for a fixed ξ, $B^{\xi+} = B^{\xi-}$ almost surely and so this conditioning has to be understood in the Palm sense. This is natural for conditioning on a jump of a point process at a particular location.

In the theorem below, Lebesgue measure on $\mathcal{U}$ refers to one-dimensional Lebesgue measure (length of a line segment). The Lebesgue-almost every qualifier is in the theorem because the Palm kernel is defined only up to Lebesgue-null sets of the points ξ. We denote Palm conditioning with two vertical lines $\|$ to distinguish it from ordinary conditioning. The definition of the Palm conditioning used in (5.4) below appears in (9.5) at the end of Section 9.1. For references, see [38, 39].

Theorem 5.2. *Assume* (3.7)*. For Lebesgue-almost every* $\xi \in \operatorname{ri} \mathcal{U}$*, under the Palm kernel, conditional on* $B_0^{\xi-} > B_0^{\xi+}$*,* $\{\tau^\xi(i+1) - \tau^\xi(i), B_{\tau^\xi(i)}^{\xi-} - B_{\tau^\xi(i)}^{\xi+} : i \in \mathbb{Z}\}$ *is an i.i.d. sequence with marginal distribution*

$$\begin{aligned}\mathbb{P}\{\tau^\xi(i+1) - \tau^\xi(i) = n,\ B_{\tau^\xi(i)}^{\xi-} - B_{\tau^\xi(i)}^{\xi+} > r \parallel B_0^{\xi-} > B_0^{\xi+}\} \\ = C_{n-1}\frac{1}{2^{2n-1}}e^{-\alpha(\xi)r}, \quad \forall\, i \in \mathbb{Z}, n \in \mathbb{N}, r \in \mathbb{R}_+. \qquad (5.4)\end{aligned}$$

Equation (5.4) connects the Palm distribution of the locations of jumps of the Busemann process with the zero set of simple symmetric random walk (SSRW). Let S_n denote a two-sided SSRW, that is, $S_0 = 0$ and $S_n - S_m = \sum_{i=m+1}^n Z_i$ for all $m < n$ in $\mathbb{Z}$ where $\{Z_i\}_{i\in\mathbb{Z}}$ are i.i.d. with $P(Z_i = \pm 1) = 1/2$. Set $\rho_n = \mathbb{1}_{\{S_{2n}=0\}}$ and let $\mathbf{P}$ be the distribution of $\rho = \{\rho_n\}_{n\in\mathbb{Z}}$ on the sequence space $\{0,1\}^{\mathbb{Z}}$. That is, $\mathbf{P}$ is the law of the zero set of simple symmetric random walk sampled at even times. The classical inter-arrival distribution of this renewal process is (Feller [23, III.3(3.7), p. 78])

$$\mathbf{P}(\rho_1 = 0, \dots, \rho_{n-1} = 0,\ \rho_n = 1) = C_{n-1}\frac{1}{2^{2n-1}}. \qquad (5.5)$$

Comparison of (5.4) and (5.5) reveals that for Lebesgue-almost every ξ, the Palm distribution of the locations of ξ-instability points on a line is the same as the law of the zero set of SSRW sampled at even times. (We record this fact precisely as Lemma 9.2.) The next result applies this to show that any translation invariant event which holds with probability 1 for the zero set of SSRW holds for all of the instability graphs simultaneously almost surely.

Theorem 5.3. *Assume* (3.7)*. Suppose A is a translation-invariant Borel subset of* $\{0,1\}^{\mathbb{Z}}$ *that satisfies* $\mathbf{P}(A) = 1$*. Then*

$$\mathbb{P}\big\{\forall \xi \in \mathcal{V}^\omega : (\mathbb{1}\{B_\ell^{\xi-} > B_\ell^{\xi+}\} : \ell \in \mathbb{Z}) \in A\big\} = 1. \qquad (5.6)$$

From (5.6) and known facts about random walk, we can derive corollaries. From [46, (10.8)], we deduce that

$$\mathbb{P}\left\{\forall \xi \in \mathcal{V}^\omega : \varlimsup_{n\to\infty} \frac{\sum_{i=0}^n \mathbb{1}\{B_i^{\xi-} > B_i^{\xi+}\}}{\sqrt{8n \log\log n}} = 1\right\} = 1. \qquad (5.7)$$

From [46, Theorem 11.1] we also find that for a non-increasing δ_n,

$$\mathbb{P}\left\{\forall \xi \in \mathcal{V}^\omega : n^{-1/2}\sum_{i=0}^n \mathbb{1}\{B_i^{\xi-} > B_i^{\xi+}\} \geq \delta_n \text{ for all sufficiently large } n\right\} = 1 \qquad (5.8)$$

if $\sum_n \delta_n/n < \infty$, and

$$\mathbb{P}\left\{\forall \xi \in \mathcal{V}^\omega : n^{-1/2}\sum_{i=0}^n \mathbb{1}\{B_i^{\xi-} > B_i^{\xi+}\} \leq \delta_n \text{ infinitely often}\right\} = 1 \qquad (5.9)$$

otherwise. Similar statements hold for the sums $\sum_{i=-n}^{0}$. This implies that for $\mathbb{P}$-almost every ω and any $\xi \in \mathcal{V}^\omega$, the number of horizontal edges $(ke_1, (k+1)e_1)$ with $\xi \in \operatorname{supp} \mu_{ke_1,(k+1)e_1}$ and $-n \le k \le n$ is of order $n^{1/2}$. It suggests that the number of such horizontal edges (and thus also vertical edges and ξ-instability points) in an $n \times n$ box should be of order $n^{3/2}$. The next theorem gives an upper bound. The lower bound is left for future work.

Theorem 5.4. *Assume* (3.7) *and fix* $i \in \{1,2\}$. *Then for any* $\zeta \in \operatorname{ri} \mathcal{U}$,

$$\mathbb{P}\Big\{\exists n_0 : \forall \xi \in [\zeta, e_2[, \forall n \ge n_0 : \sum_{x \in [\![0,n]\!]^2} \mathbb{1}\{\xi \in \operatorname{supp} \mu_{x,x+e_i}\} \le 2n^{3/2}\sqrt{\log n}\Big\} = 1.$$

The same holds when $[\![0,n]\!]^2$ *is replaced by any of* $[\![-n,0]\!]^2$, $[\![0,n]\!] \times [\![-n,0]\!]$, *or* $[\![-n,0]\!] \times [\![0,n]\!]$.

This completes the presentation of the main results. After a list of open problems, the remaining sections cover the proofs. The results of Section 5 are proved in Section 9.

6. Open problems

The list below contains some immediate open questions raised by the results of this paper.

1. Find tail estimates for the coalescence points $\mathbf{z}^\xi(x, y)$.
2. Theorem 3.7 (b) showed that the jump process condition (3.5) implies that $\mathcal{V}^\omega = \{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$. Is this implication an equivalence?
3. Prove the jump process condition (3.5) for any model other than the exactly solvable exponential and geometric cases.
4. Does the web of instability have a scaling limit?
5. Does the web of instability, with branching and coalescing in exceptional directions, have any analogue in stochastic equations in continuous space and/or continuous time?
6. Extend the statistics of instability points in the exponential model beyond a single line on the lattice.

7. Busemann measures: proofs

The rest of the paper relies on Appendix A where prior results from the literature are collected. The reader may wish to look through that appendix before proceeding; in particular, we will work on the T-invariant full-measure event Ω_0 constructed in (A.7).

Fix a countable dense set $\mathcal{U}_0 \subset \mathcal{D}$ of points of differentiability of the shape function g (recall (2.2)). These play a role in the definition of the event Ω_0 in (A.7). Recall the definition (3.3) of the coalescence point $\mathbf{z}^{\xi\square}(x, y)$. When $\mathbf{z}^{\xi\square}(x, y) \in \mathbb{Z}^2$, equation (2.15) leads to the following identity, which is fundamental to the analysis that follows:

$$B^{\xi\square}(x,y) = G(x,\mathbf{z}^{\xi\square}(x,y)) - G(y,\mathbf{z}^{\xi\square}(x,y)) = \sum_{i=k}^{n-1} \omega_{\boldsymbol{\gamma}_i^{x,\xi\square}} - \sum_{i=\ell}^{n-1} \omega_{\boldsymbol{\gamma}_i^{y,\xi\square}}, \tag{7.1}$$

where $k = x\cdot\hat{e}_1$, $\ell = y\cdot\hat{e}_1$, and $n = \mathbf{z}^{\xi\square}(x,y)\cdot\hat{e}_1$. By Theorem A.4 (b), for all $\omega\in\Omega_0$, all $\xi\in\mathcal{U}_0$, and all $x,y\in\mathbb{Z}^2$, both $\mathbf{z}^{\xi+}(x,y)$ and $\mathbf{z}^{\xi-}(x,y)$ are in $\mathbb{Z}^2$.

We begin with results linking analytic properties of the Busemann process and coalescence points.

Proposition 7.1. *For all $\omega\in\Omega_0$, for any $\zeta\prec\eta$ in* ri $\mathcal{U}$*, and any $x,y\in\mathbb{Z}^2$, the following statements are equivalent:*

(i) $|\mu_{x,y}|(]\zeta,\eta[) = 0$.

(ii) $B^{\zeta+}(x,y) = B^{\eta-}(x,y)$ *and* $\mathbf{z}^{\zeta+}(x,y), \mathbf{z}^{\eta-}(x,y)\in\mathbb{Z}^2$.

(iii) $\mathbf{z}^{\zeta+}(x,y) = \mathbf{z}^{\eta-}(x,y)\in\mathbb{Z}^2$.

(iv) *There exists $z\in\mathbb{Z}^2$ such that the following holds. For any $\pi\in\{\boldsymbol{\gamma}^{x,\xi\square} : \xi\in\,]\zeta,\eta[, \square\in\{-,+\}\}$ and any $\pi'\in\{\boldsymbol{\gamma}^{y,\xi\square} : \xi\in\,]\zeta,\eta[, \square\in\{-,+\}\}$, $\pi\cap\pi'\neq\emptyset$ and z is the first point where π and π' intersect: $z\cdot\hat{e}_1 = \min\{z'\cdot\hat{e}_1 : z'\in\pi\cap\pi'\}$.*

Proof. (i)⇒(ii). Under (i) the functions $\xi\mapsto B^{\xi\square}_{x,y}$ match for $\square\in\{-,+\}$ and are constant on the open interval $]\zeta,\eta[$. The equality $B^{\zeta+}(x,y) = B^{\eta-}(x,y)$ follows by taking limits $\xi\searrow\zeta$ and $\xi\nearrow\eta$.

Since on $]\zeta,\eta[\,\cap\,\mathcal{U}_0$, $\xi\mapsto B^{\xi}_{x,y}$ is constant and $\mathbf{z}^{\xi}(x,y)\in\mathbb{Z}^2$ (Theorem A.4 (b)), (7.1) and condition (A.5) imply that $\mathbf{z}^{\xi}(x,y)$ is constant in $\mathbb{Z}^2$ for all $\xi\in\,]\zeta,\eta[\,\cap\,\mathcal{U}_0$. Since $\mathcal{U}_0$ is dense in $]\zeta,\eta[$, limits (3.4) as $\xi\searrow\zeta$ and $\xi\nearrow\eta$ imply that $\mathbf{z}^{\zeta+}(x,y), \mathbf{z}^{\eta-}(x,y)\in\mathbb{Z}^2$.

(ii)⇒(iii). Set $k = x\cdot\hat{e}_1$ and $\ell = y\cdot\hat{e}_1$. With both $\mathbf{z}^{\zeta+}(x,y)$ and $\mathbf{z}^{\eta-}(x,y)$ in $\mathbb{Z}^2$, we also set $m = \mathbf{z}^{\zeta+}(x,y)\cdot\hat{e}_1$ and $n = \mathbf{z}^{\eta-}(x,y)\cdot\hat{e}_1$. By (7.1),

$$B^{\zeta+}_{x,y} = G(x,\mathbf{z}^{\zeta+}(x,y)) - G(y,\mathbf{z}^{\zeta+}(x,y)) = \sum_{i=k}^{m-1} \omega_{\boldsymbol{\gamma}_i^{x,\zeta+}} - \sum_{i=\ell}^{m-1} \omega_{\boldsymbol{\gamma}_i^{y,\zeta+}},$$

$$B^{\eta-}_{x,y} = G(x,\mathbf{z}^{\eta-}(x,y)) - G(y,\mathbf{z}^{\eta-}(x,y)) = \sum_{i=k}^{n-1} \omega_{\boldsymbol{\gamma}_i^{x,\eta-}} - \sum_{i=\ell}^{n-1} \omega_{\boldsymbol{\gamma}_i^{y,\eta-}}.$$

By condition (A.5), the vanishing of $B^{\zeta+}_{x,y} - B^{\eta-}_{x,y}$ forces $m=n$, $\boldsymbol{\gamma}^{x,\zeta+}_{k,m} = \boldsymbol{\gamma}^{x,\eta-}_{k,m}$, and $\boldsymbol{\gamma}^{y,\zeta+}_{\ell,m} = \boldsymbol{\gamma}^{y,\eta-}_{\ell,m}$, and hence in particular $\mathbf{z}^{\zeta+}(x,y) = \mathbf{z}^{\eta-}(x,y)$.

(iii)⇒(iv). With $m = \mathbf{z}^{\zeta+}(x,y)\cdot\hat{e}_1 = \mathbf{z}^{\eta-}(x,y)\cdot\hat{e}_1$, uniqueness of finite geodesics implies $\boldsymbol{\gamma}^{x,\zeta+}_{k,m} = \boldsymbol{\gamma}^{x,\eta-}_{k,m}$ and $\boldsymbol{\gamma}^{y,\zeta+}_{\ell,m} = \boldsymbol{\gamma}^{y,\eta-}_{\ell,m}$. Then monotonicity (2.13) gives $\boldsymbol{\gamma}^{x,\zeta+}_{k,m} = \boldsymbol{\gamma}^{x,\xi\square}_{k,m} = \boldsymbol{\gamma}^{x,\eta-}_{k,m}$ and $\boldsymbol{\gamma}^{y,\zeta+}_{\ell,m} = \boldsymbol{\gamma}^{y,\xi\square}_{\ell,m} = \boldsymbol{\gamma}^{y,\eta-}_{\ell,m}$ for all $\xi\in\,]\zeta,\eta[$ and $\square\in\{-,+\}$. The point z is $\boldsymbol{\gamma}^{x,\zeta+}_m = \boldsymbol{\gamma}^{y,\zeta+}_m = \boldsymbol{\gamma}^{x,\eta-}_m = \boldsymbol{\gamma}^{y,\eta-}_m$.

(iv)⇒(i). Let $m = z\cdot\hat{e}_1$. It follows from uniqueness of finite geodesics that all of the paths $\boldsymbol{\gamma}^{x,\xi\pm}_{k,m}$ must be the same, for all $\xi\in\,]\zeta,\eta[$, and similarly all of the paths $\boldsymbol{\gamma}^{y,\xi\pm}_{\ell,m}$ must be the same. Letting $\xi \searrow \zeta$ and $\xi \nearrow \eta$, we find that for all $\xi \in]\zeta,\eta[$, $\boldsymbol{\gamma}^{x,\zeta+}_{k,m} = \boldsymbol{\gamma}^{x,\xi\pm}_{k,m} = \boldsymbol{\gamma}^{x,\eta-}_{k,m}$ and $\boldsymbol{\gamma}^{y,\zeta+}_{\ell,m} = \boldsymbol{\gamma}^{y,\xi\pm}_{\ell,m} = \boldsymbol{\gamma}^{y,\eta-}_{\ell,m}$. Recall that (7.1) applies for any $\xi \in \mathcal{U}_0$. Thus, the functions $\xi \mapsto B^{\xi\pm}_{x,y}$ match and are constant when restricted to the dense set $\mathcal{U}_0 \cap]\zeta,\eta[$. Combining this with the left-continuity of $\xi \mapsto B^{\xi-}_{x,y}$ and the right-continuity of $\xi \mapsto B^{\xi+}_{x,y}$, we see that the functions $\xi \mapsto B^{\xi\pm}_{x,y}$ match and are constant on $]\zeta,\eta[$. This implies (i). ∎

Proposition 7.1 has a counterpart in terms of fixed directions lying in the support of $\mu_{x,y}$.

Proposition 7.2. *For all $\omega \in \Omega_0$ and all $x, y \in \mathbb{Z}^2$, the following are equivalent:*

(i) $\xi \notin \operatorname{supp} \mu_{x,y}$.

(ii) $\mathbf{z}^{\xi-}(x,y) = \mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$.

(iii) $B^{\xi-}(x,y) = B^{\xi+}(x,y)$ *and* $\mathbf{z}^{\xi-}(x,y), \mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$.

Proof. Let $x \cdot \hat{e}_1 = k$ and $y \cdot \hat{e}_1 = \ell$. Take sequences $\zeta_n, \eta_n \in \mathcal{U}_0$ with $\zeta_n \nearrow \xi$ and $\eta_n \searrow \xi$. Since $\zeta_n, \eta_n \in \mathcal{U}_0$ we have $\mathbf{z}^{\zeta_n}(x,y), \mathbf{z}^{\eta_n}(x,y) \in \mathbb{Z}^2$ for all n. Furthermore, $B^{\zeta_n}(x,y) \to B^{\xi-}(x,y)$ and $B^{\eta_n}(x,y) \to B^{\xi+}(x,y)$ as $n \to \infty$.

(i)⇒(ii). If $\xi \notin \operatorname{supp} \mu_{x,y}$, then $\zeta \mapsto B^{\zeta\pm}(x,y)$ is constant on some neighborhood of ξ. Then (i)⇒(iii) from Proposition 7.1 gives (ii).

(ii)⇒(iii). Let $m = \mathbf{z}^{\xi-}(x,y) \cdot \hat{e}_1 = \mathbf{z}^{\xi+}(x,y) \cdot \hat{e}_1$. Then uniqueness of finite geodesics implies that $\boldsymbol{\gamma}^{x,\xi-}_{k,m} = \boldsymbol{\gamma}^{x,\xi+}_{k,m}$ and $\boldsymbol{\gamma}^{y,\xi-}_{\ell,m} = \boldsymbol{\gamma}^{y,\xi+}_{\ell,m}$. (2.14) implies that for sufficiently large n, $\boldsymbol{\gamma}^{x,\xi-}_{k,m} = \boldsymbol{\gamma}^{x,\zeta_n}_{k,m}$, $\boldsymbol{\gamma}^{x,\xi+}_{k,m} = \boldsymbol{\gamma}^{x,\eta_n}_{k,m}$, $\boldsymbol{\gamma}^{y,\xi-}_{\ell,m} = \boldsymbol{\gamma}^{y,\zeta_n}_{\ell,m}$, and $\boldsymbol{\gamma}^{y,\xi+}_{\ell,m} = \boldsymbol{\gamma}^{y,\eta_n}_{\ell,m}$. For these large n,

$$\begin{aligned}
B^{\zeta_n}(x,y) &= G(x, \mathbf{z}^{\zeta_n}(x,y)) - G(y, \mathbf{z}^{\zeta_n}(x,y)) \\
&= \sum_{i=k}^{m-1} \omega_{\boldsymbol{\gamma}_i^{x,\zeta_n}} - \sum_{i=\ell}^{m-1} \omega_{\boldsymbol{\gamma}_i^{y,\zeta_n}} = \sum_{i=k}^{m-1} \omega_{\boldsymbol{\gamma}_i^{x,\xi-}} - \sum_{i=\ell}^{m-1} \omega_{\boldsymbol{\gamma}_i^{y,\xi-}} \\
&= \sum_{i=k}^{m-1} \omega_{\boldsymbol{\gamma}_i^{x,\xi+}} - \sum_{i=\ell}^{m-1} \omega_{\boldsymbol{\gamma}_i^{y,\xi+}} = \sum_{i=k}^{m-1} \omega_{\boldsymbol{\gamma}_i^{x,\eta_n}} - \sum_{i=\ell}^{m-1} \omega_{x_i^{y,\eta_n}} \\
&= G(x, \mathbf{z}^{\eta_n}(x,y)) - G(y, \mathbf{z}^{\eta_n}(x,y)) = B^{\eta_n}(x,y).
\end{aligned}$$

Taking $n \to \infty$ gives $B^{\xi+}(x,y) = B^{\xi-}(x,y)$. Claim (iii) is proved.

(iii)⇒(i). The assumption $\mathbf{z}^{\xi-}(x,y), \mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$ allows us to use (7.1). Together with the convergence of geodesics (2.14), this implies that $B^{\zeta_n}(x,y) = B^{\xi-}(x,y) = B^{\xi+}(x,y) = B^{\eta_n}(x,y)$ for sufficiently large n. The equivalence between (ii) and (i) in Proposition 7.1 implies that for such n, both processes are constant on the interval $]\zeta_n,\eta_n[$. Therefore $\xi \notin \operatorname{supp} \mu_{x,y}$. ∎

With these results in hand, we next turn to the proofs of our main results.

Proof of Theorem 3.1. Fix $\omega \in \Omega_0$, $x, y \in \mathbb{Z}^2$, and $\xi \in \operatorname{ri} \mathcal{U}$. Suppose that (i) does not hold, i.e. $\xi \notin \operatorname{supp} \mu_{x,y}$. By Proposition 7.2, we have $\mathbf{z}^{\xi-}(x,y) = \mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$, in which case both $\boldsymbol{\gamma}^{x,\xi-} \cap \boldsymbol{\gamma}^{y,\xi+}$ and $\boldsymbol{\gamma}^{x,\xi+} \cap \boldsymbol{\gamma}^{y,\xi-}$ include this common point and thus (ii) is false. This proves that (ii) implies (i).

Now, suppose that $\xi \in \operatorname{supp} \mu_{x,y}$ and that $\boldsymbol{\gamma}^{x,\xi-} \cap \boldsymbol{\gamma}^{y,\xi+} \neq \emptyset$ and $\boldsymbol{\gamma}^{x,\xi+} \cap \boldsymbol{\gamma}^{y,\xi-} \neq \emptyset$. Without loss of generality assume that $x \cdot \widehat{e}_1 = k \leq m = y \cdot \widehat{e}_1$. Let z_1 denote the first point at which $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{y,\xi+}$ meet and let z_2 be the first point at which $\boldsymbol{\gamma}^{x,\xi+}$ and $\boldsymbol{\gamma}^{y,\xi-}$ meet. Let $\ell_1 = z_1 \cdot \widehat{e}_1$ and $\ell_2 = z_2 \cdot \widehat{e}_1$. We denote by u the leftmost (i.e. with smallest e_1 coordinates) of the three points $\boldsymbol{\gamma}_m^{x,\xi+}$, y, $\boldsymbol{\gamma}_m^{x,\xi-}$ and by v the rightmost of these three points. Note that if $u = v$, then $\mathbf{z}^{\xi+}(x, y) = \mathbf{z}^{\xi-}(x, y) = y$, which would imply that $\xi \notin \operatorname{supp} \mu_{x,y}$. Thus $u \neq v$ and there are two cases: either $y \in \{u, v\}$ or not. We show a contradiction in both cases.

First, we work out the case $y = v$, with the case of $y = u$ being similar. See the left picture in Figure 7.1 for an illustration. In this case we have, for all $n \geq m$, $\boldsymbol{\gamma}_n^{x,\xi-} \preceq \boldsymbol{\gamma}_n^{x,\xi+} \preceq \boldsymbol{\gamma}_n^{y,\xi+}$ and $\boldsymbol{\gamma}_n^{x,\xi-} \preceq \boldsymbol{\gamma}_n^{y,\xi-} \preceq \boldsymbol{\gamma}_n^{y,\xi+}$. In words, $\boldsymbol{\gamma}^{y,\xi+}$ is the rightmost geodesic and $\boldsymbol{\gamma}^{x,\xi-}$ is the leftmost geodesic among the four geodesics $\boldsymbol{\gamma}^{x,\xi\pm}$, $\boldsymbol{\gamma}^{y,\xi\pm}$. By the path ordering (2.13) and planarity, z_1 must lie on all four geodesics. Then by the uniqueness of finite geodesics, $\boldsymbol{\gamma}_{k,\ell_1}^{x,\xi+} = \boldsymbol{\gamma}_{k,\ell_1}^{x,\xi-}$ and $\boldsymbol{\gamma}_{m,\ell_1}^{y,\xi+} = \boldsymbol{\gamma}_{m,\ell_1}^{y,\xi-}$. It follows that $z_1 = \mathbf{z}^{\xi+}(x, y) = \mathbf{z}^{\xi-}(x, y)$, contradicting $\xi \in \operatorname{supp} \mu_{x,y}$.

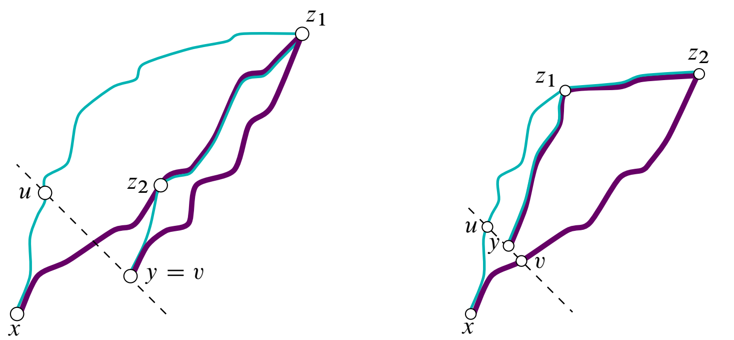

Fig. 7.1. Proof of Theorem 3.1; $\xi+$ geodesics are in purple with medium thickness, and $\xi-$ geodesics are in green and thin.

If $y \notin \{u, v\}$, then $u = \boldsymbol{\gamma}_m^{x,\xi-} \prec y \prec v = \boldsymbol{\gamma}_m^{x,\xi+}$ (right picture in Figure 7.1). The geodesics $\boldsymbol{\gamma}^{x,\xi+}$ and $\boldsymbol{\gamma}^{x,\xi-}$ have already split and so cannot meet again by the uniqueness of finite geodesics. For all $n \geq m$, $\boldsymbol{\gamma}_n^{x,\xi-} \preceq \boldsymbol{\gamma}_n^{y,\xi-} \preceq \boldsymbol{\gamma}_n^{y,\xi+} \preceq \boldsymbol{\gamma}_n^{x,\xi+}$. Due to this ordering, the meeting of $\boldsymbol{\gamma}_n^{x,\xi-}$ and $\boldsymbol{\gamma}_n^{y,\xi+}$ at z_1 implies that $\boldsymbol{\gamma}_n^{x,\xi-}$ and $\boldsymbol{\gamma}_n^{y,\xi-}$ coalesce at or before z_1. By the uniqueness of finite geodesics again, $\boldsymbol{\gamma}^{y,\xi-}$ and $\boldsymbol{\gamma}^{y,\xi+}$ agree from y to z_1. The same reasoning applies to z_2 and gives that $\boldsymbol{\gamma}^{y,\xi-}$ and $\boldsymbol{\gamma}^{y,\xi+}$ agree from y to z_2 and that $\boldsymbol{\gamma}^{y,\xi+}$ and $\boldsymbol{\gamma}^{x,\xi+}$ coalesce at z_2. Thus now $\boldsymbol{\gamma}^{y,\xi-}$ and $\boldsymbol{\gamma}^{y,\xi+}$ agree from y through both z_1 and z_2. The coalescence of $\boldsymbol{\gamma}^{x,\xi-}$ with $\boldsymbol{\gamma}^{y,\xi-}$ and the coalescence of $\boldsymbol{\gamma}^{y,\xi+}$ with $\boldsymbol{\gamma}^{x,\xi+}$ then force $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{x,\xi+}$ to meet again, contradicting what was said above. We have now shown that (i) implies (ii).

(ii) implies (iii) by the directedness in Theorem A.4 (a). It remains to prove the reverse implication under the regularity condition (2.4). Without loss of generality we can assume that $x \cdot \widehat{e}_1 \le y \cdot \widehat{e}_1 = k$. If $\pi^x_k \prec y$, then the extremality of the geodesics $\boldsymbol{\gamma}^{\bullet,\xi\pm}$ in Theorem A.7 and the fact that $\pi^x \cap \pi^y = \emptyset$ imply that $\boldsymbol{\gamma}^{x,\xi-} \cap \boldsymbol{\gamma}^{y,\xi+} = \emptyset$. Similarly, if $\pi^x_k \succ y$, then we get $\boldsymbol{\gamma}^{x,\xi+} \cap \boldsymbol{\gamma}^{y,\xi-} = \emptyset$. ■

Proof of Theorem 3.2. The equivalence (i)⇔(iv) of Proposition 7.1, together with the uniqueness of finite geodesics, gives Theorem 3.2. ■

Proof of Lemma 3.3. For $\xi \in \mathcal{U}_0$, almost surely $\mathbf{z}^{\xi+}(x,y) = \mathbf{z}^{\xi-}(x,y) = \mathbf{z}^{\xi}(x,y) \in \mathbb{Z}^2$. Proposition 7.2 implies that ξ lies in the complement of the closed set $\operatorname{supp} \mu_{x,y}$. ■

Next, we prove Theorem 3.4 about the relation between the coalescence points and properties of the support of Busemann measures.

Proof of Theorem 3.4. Take $\omega \in \Omega_0$. Equivalence (a) follows from Proposition 7.2. Equivalence (b) follows from the equivalences in (a) and (c).

The two equivalences of (c) are proved the same way. We prove the first equivalence in this form: there exists $\eta \succ \xi$ such that $|\mu_{x,y}|(]\xi,\eta[) = 0 \Leftrightarrow \mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$.

The implication ⇒ is contained in (i)⇒(ii) of Proposition 7.1.

To prove ⇐, let $k = x \cdot \widehat{e}_1$ and $\ell = y \cdot \widehat{e}_1$, suppose $\mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$, and let $m = \mathbf{z}^{\xi+}(x,y) \cdot \widehat{e}_1$. Take a sequence $\eta_n \in \mathcal{U}_0$ with $\eta_n \searrow \xi$ as $n \to \infty$. For sufficiently large n, $\boldsymbol{\gamma}^{x,\xi+}_{k,m} = \boldsymbol{\gamma}^{x,\eta_n}_{k,m}$ and $\boldsymbol{\gamma}^{y,\xi+}_{\ell,m} = \boldsymbol{\gamma}^{y,\eta_n}_{\ell,m}$, and hence $\mathbf{z}^{\eta_n}(x,y) = \mathbf{z}^{\xi+}(x,y)$. The implication (iii)⇒(i) of Proposition 7.1 gives $|\mu_{x,y}|(]\xi,\eta_n[) = 0$. ■

When the jump process condition (3.5) holds, call the event in the statement of that condition Ω_0^5. As noted when it was introduced, Theorem 3.5, which gives the equivalence between (3.5) and coalescence of $\xi\square$ geodesics, is essentially an immediate consequence of Theorem 3.4.

Proof of Theorem 3.5. Assume the jump process condition (3.5). Fix $\omega \in \Omega_0 \cap \Omega_0^5$, $x, y \in \mathbb{Z}^2$, and $\xi \in \operatorname{ri} \mathcal{U}$. If $\xi \notin \operatorname{supp} \mu_{x,y}$, then Proposition 7.2 says that $\mathbf{z}^{\xi-}(x,y) = \mathbf{z}^{\xi+}(x,y) \in \mathbb{Z}^2$. In particular, $\boldsymbol{\gamma}^{x,\xi+}$ coalesces with $\boldsymbol{\gamma}^{y,\xi+}$ and $\boldsymbol{\gamma}^{x,\xi-}$ coalesces with $\boldsymbol{\gamma}^{y,\xi-}$. If, on the other hand, $\xi \in \operatorname{supp} \mu_{x,y}$, then it is an isolated point and now Theorem 3.4 says that $\mathbf{z}^{\xi\pm}(x,y) \in \mathbb{Z}^2$ (although now the two points are not equal). Again, $\boldsymbol{\gamma}^{x,\xi\pm}$ coalesces with $\boldsymbol{\gamma}^{y,\xi\pm}$, respectively. Statement (ii) is proved.

Now, assume (ii) holds and let Ω_0^6 be a full measure event on which statement (ii) holds. Let $\omega \in \Omega_0 \cap \Omega_0^6$, $x, y \in \mathbb{Z}^2$, and $\xi \in \operatorname{supp} \mu_{x,y}$. The fact that $\boldsymbol{\gamma}^{x,\xi\pm}$ and $\boldsymbol{\gamma}^{y,\xi\pm}$ coalesce, respectively, says that $\mathbf{z}^{\xi\pm}(x,y) \in \mathbb{Z}^2$. Since we assumed $\xi \in \operatorname{supp} \mu_{x,y}$, Proposition 7.2 implies that the two coalescence points $\mathbf{z}^{\xi\pm}(x,y)$ are not equal. Theorem 3.4 implies that ξ is isolated. ■

The proof of Lemma 3.6 is delayed to the end of Section 8.1. When the jump process condition (3.5) holds, define

$$\Omega_0^{\text{jump}} = \Omega_0 \cap \Omega_0^5. \tag{7.2}$$

Proof of Theorem 3.7. (a) Take $\omega \in \Omega_0$. Let $x \cdot \widehat{e}_1 = k$ and $\xi = \xi_*(T_x\omega)$. Take $T_x\omega$ in place of ω in (2.17), let $\zeta \to \xi_*(T_x\omega)$, and use (2.6), (2.7), and (2.11), to get $B^{\xi-}(x, x+e_2) \le B^{\xi-}(x, x+e_1)$ and $B^{\xi+}(x, x+e_1) \le B^{\xi+}(x, x+e_2)$. Then by definition $\boldsymbol{\gamma}_k^{x,\xi+} = \boldsymbol{\gamma}_k^{x,\xi-} = x$, $\boldsymbol{\gamma}_{k+1}^{x,\xi+} = x + e_1$, and $\boldsymbol{\gamma}_{k+1}^{x,\xi-} = x + e_2$. Therefore we cannot have $\mathbf{z}^{\xi+}(x, x+e_i) = \mathbf{z}^{\xi-}(x, x+e_i) \in \mathbb{Z}^2$ for either $i \in \{1, 2\}$ by uniqueness of finite geodesics. By Proposition 7.2, $\xi \in \operatorname{supp} \mu_{x,x+e_1} \cap \operatorname{supp} \mu_{x,x+e_2}$.

For the converse, for $\zeta \prec \xi_*(T_x\omega) \prec \eta$ we see that $B^{\zeta\pm}(x, x+e_2) = \omega_x = B^{\eta\pm}(x, x+e_1)$. Thus

$$\operatorname{supp} \mu_{x,x+e_1} \subset \left]e_2, \xi_*(T_x\omega)\right] \quad \text{and} \quad \operatorname{supp} \mu_{x,x+e_2} \subset \left[\xi_*(T_x\omega), e_1\right[.$$

Consequently, $\operatorname{supp} \mu_{x,x+e_1} \cap \operatorname{supp} \mu_{x,x+e_2} \subset \{\xi_*(T_x\omega)\}$.

(b) It remains to show $\mathcal{V}^\omega \subset \{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$. Assume the jump process condition (3.5) holds and $\omega \in \Omega_0^{\text{jump}}$. Suppose $\zeta \in \operatorname{supp} \mu_{x,y}$. By Theorem 3.4 (b) the coalescence points $\mathbf{z}^{\zeta\pm}(x, y)$ are distinct lattice points. Hence the geodesics $\boldsymbol{\gamma}^{x,\zeta+}$ and $\boldsymbol{\gamma}^{x,\zeta-}$ separate at some point z where then $\xi_*(T_z\omega) = \zeta$. ∎

The next results relate $\mathcal{V}^\omega$ to regularity properties of the shape function g.

Lemma 7.3. *The following holds for all $\omega \in \Omega_0$: for all $\zeta \prec \eta$ in* ri $\mathcal{U}$, $]\zeta, \eta[\cap \mathcal{V}^\omega \ne \emptyset$ *if and only if $\nabla g(\zeta+) \ne \nabla g(\eta-)$.*

Proof. If $\nabla g(\zeta+) \ne \nabla g(\eta-)$, Theorem A.8 (c) says that $]\zeta, \eta[$ contains some $\xi_*(T_x\omega)$, which by Theorem 3.7 (a) is a member of $\mathcal{V}^\omega$.

If $\nabla g(\zeta+) = \nabla g(\eta-)$, then by concavity, $\nabla g(\zeta+) = \nabla g(\eta-) = \nabla g(\xi\square)$ for all $\xi \in]\zeta, \eta[$ and $\square \in \{-, +\}$. By Theorem A.1 (d), $B^{\xi\square}(x, y, \omega)$ is constant over $\xi \in]\zeta, \eta[$ and $\square \in \{-, +\}$, for any $x, y \in \mathbb{Z}^2$ and $\omega \in \Omega$. Consequently, for any given x, the geodesics $\boldsymbol{\gamma}^{x,\xi\square}$ match. By Theorem A.4 (b), all these geodesics coalesce on the event Ω_0. Hence the coalescence points $\mathbf{z}^{\xi\square}(x, y)$ also match. By Theorem 3.4 (a), no point $\xi \in]\zeta, \eta[$ is a member of $\mathcal{V}^\omega$. ∎

Proof of Theorem 3.8. (a) Let $\xi \in \mathcal{D}$. Theorem A.4 (b) says that almost surely $\mathbf{z}^{\xi\pm}(x, x+e_i) \in \mathbb{Z}^2$ for $x \in \mathbb{Z}^2$ and $i \in \{1, 2\}$. Theorem A.1 (k) says that there is no $\pm$ distinction. Hence $\mathbb{P}(\mathbf{z}^{\xi-}(x, x+e_i) = \mathbf{z}^{\xi+}(x, x+e_i) \in \mathbb{Z}^2) = 1$ and therefore $\mathbb{P}(\xi \in \operatorname{supp} \mu_{x,x+e_i}) = 0$ by Proposition 7.2. A union bound implies that $\mathbb{P}(\exists x \in \mathbb{Z}^2, i \in \{1, 2\} : \xi \in \operatorname{supp} \mu_{x,x+e_i}) = 0$. The cocycle property (2.7) then implies that $\mathbb{P}(\xi \in \mathcal{V}^\omega) = 0$.

Let $\xi \in (\text{ri}\, \mathcal{U}) \setminus \mathcal{D}$. Theorem A.8 (d) implies that ξ is among $\{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$, and these lie in $\mathcal{V}^\omega$ by Theorem 3.7 (a).

(b) By definition, the complement of $\mathcal{H}$ is the union of the (at most countably many) maximal open intervals $]\zeta, \eta[$ such that $\nabla g(\zeta+) = \nabla g(\eta-)$ and their differentiable endpoints. Lemma 7.3 together with part (a) proved above implies that $\mathbb{P}(\mathcal{V}^\omega \cap \mathcal{H}^c \ne \emptyset) = 0$. Hence almost surely, $\{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\} \subset \mathcal{V}^\omega \subset \mathcal{H}$.

For the density claim it is enough to prove that $\{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$ is dense in $\mathcal{H}$. Suppose first that $\xi \in \mathcal{H} \cap \mathcal{D}$. Then ξ is not on a closed linear segment of g, and hence for any $\zeta \prec \xi \prec \eta$ we have $\nabla g(\zeta+) \neq \nabla g(\eta-)$. By Theorem A.8 (c) the open interval $]\zeta, \eta[$ contains a value $\xi_*(T_x\omega)$. The other case is $\xi \in \mathscr{H} \setminus \mathcal{D}$. Then $\xi \in \{\xi_*(T_x\omega) : x \in \mathbb{Z}^2\}$ by Theorem A.8 (d). ∎

The next proof, of Theorem 3.9, identifies $\mathcal{U}\setminus\mathcal{V}^\omega$ in terms of directions in which (Busemann) geodesic uniqueness holds.

Proof of Theorem 3.9. (a) Fix $\omega \in \Omega_0$ and $\xi \in \operatorname{ri} \mathcal{U}$. Suppose first that there exists an $x \in \mathbb{Z}^2$ with the property that $\boldsymbol{\gamma}^{x,\xi+} \neq \boldsymbol{\gamma}^{x,\xi-}$. These geodesics separate at some point z where then $\xi = \xi_*(T_z\omega) \in \mathcal{V}^\omega$. If, on the other hand, $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$ for all $x \in \mathbb{Z}^2$, but there exist x and y for which $\boldsymbol{\gamma}^{x,\xi}$ and $\boldsymbol{\gamma}^{y,\xi}$ do not coalesce, then Proposition 7.2 implies that $\xi \in \operatorname{supp} \mu_{x,y} \subset \mathcal{V}^\omega$.

Conversely, suppose $\xi \in \mathcal{V}^\omega$ and let x, y be such that $\xi \in \operatorname{supp} \mu_{x,y}$. Then by Theorem 3.1, possibly after interchanging the roles of x and y, we have $\boldsymbol{\gamma}^{x,\xi+} \cap \boldsymbol{\gamma}^{y,\xi-} = \emptyset$. In particular, these two geodesics do not coalesce. Part (a) is proved.

(c) Assume the jump process condition (3.5) and let $\omega \in \Omega_0^{\text{jump}}$. Suppose that $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-} = \boldsymbol{\gamma}^{x,\xi}$. By Theorem 3.5, $\boldsymbol{\gamma}^{y,\xi\square}$ coalesces with $\boldsymbol{\gamma}^{x,\xi}$ for all $y \in \mathbb{Z}^2$ and both signs $\square \in \{-,+\}$. By the uniqueness of finite geodesics, $\boldsymbol{\gamma}^{y,\xi+} = \boldsymbol{\gamma}^{y,\xi-}$. Now all these geodesics coalesce. Part (a) implies $\xi \notin \mathcal{V}^\omega$.

Parts (b) and (d) follow from (a) and (c), respectively, because under the regularity condition (2.4), Theorem A.7 implies that the uniqueness of a $\mathcal{U}_\xi$-directed geodesic out of x is equivalent to $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$. ∎

The next lemma completes the proof of Theorem 3.10. Recall the event Ω_0 defined in (A.7).

Lemma 7.4. *Assume the regularity condition* (2.4). *If $\omega \in \Omega_0$ and $\xi \in \mathcal{V}^\omega$, then $\mathcal{U}_\xi = \{\xi\}$.*

Proof. Take $\omega \in \Omega_0$ and suppose $\mathcal{U}_\xi \neq \{\xi\}$. Recall the dense set of differentiability directions $\mathcal{U}_0$ introduced just before (A.7). Because $\mathcal{U}_\xi$ is a line segment in $\mathcal{U}$, there exists a $\zeta \in \mathcal{U}_0 \cap \mathcal{U}_\xi$. By its definition, $\Omega_0 \subset \Omega_\zeta^3$, where Ω_ζ^3 was introduced in Theorem A.4. Theorem A.4 (e) then implies that $\mathbf{z}^{\zeta-}(x, y) = \mathbf{z}^{\zeta+}(x, y) \in \mathbb{Z}^2$ for each pair x, y. Since $\zeta, \xi \in \mathcal{U}_\xi$ and we assumed (2.4), Theorem A.1 (d) implies that for all $x, y \in \mathbb{Z}^2$, $B^{\zeta-}(x, y) = B^{\zeta+}(x, y) = B^{\xi-}(x, y) = B^{\xi+}(x, y)$. Consequently, $\mathbf{z}^{\xi-}(x, y) = \mathbf{z}^{\xi+}(x, y) \in \mathbb{Z}^2$ for all $x, y \in \mathbb{Z}^2$ and Theorem 3.4 (a) shows that $\xi \notin \mathcal{V}^\omega$. ∎

Proof of Theorem 3.11. (a) Theorem A.9 implies that for $\xi \in \mathcal{V}^\omega$, $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{x,\xi+}$ are the only ξ-directed geodesics out of x.

(b) Consider any three geodesics with the same asymptotic direction $\xi \in \operatorname{ri} \mathcal{U}$. If $\xi \in (\operatorname{ri} \mathcal{U}) \setminus \mathcal{V}^\omega$ then by Theorem 3.10 (c) all three coalesce. If $\xi \in \mathcal{V}^\omega$ then by part (a) at least two of these three geodesics must have the same sign $+$ or $-$. By Theorem 3.10 (d) these two coalesce.

(c) Consider a sequence v_n as in the first part of the statement and set $k = x \cdot \hat{e}$. From an arbitrary subsequence, extract a further subsequence n_ℓ so that $\boldsymbol{\gamma}^{x,v_{n_\ell}}$ converges to a semi-infinite geodesic $\pi_{k,\infty}$ vertex-by-vertex. Let $\zeta \prec \xi \prec \eta$. Using the fact that

$v_n/n \to \xi$ and directedness of $\boldsymbol{\gamma}^{x,\zeta+}$ and $\boldsymbol{\gamma}^{x,\eta-}$, for all sufficiently large n we must have $\boldsymbol{\gamma}_n^{x,\zeta+} \prec v_n \prec \boldsymbol{\gamma}_n^{x,\eta-}$. By uniqueness of finite geodesics, we must then have $\boldsymbol{\gamma}_m^{x,\zeta+} \preceq \boldsymbol{\gamma}_m^{x,v_n} \preceq \boldsymbol{\gamma}_m^{x,\eta-}$ for all $m \geq x \cdot \widehat{e}$ and all such n. It then follows by letting $\zeta, \eta \to \xi$ that π must be ξ-directed. Therefore, by part (a), $\pi \in \{\boldsymbol{\gamma}^{x,\xi+}, \boldsymbol{\gamma}^{x,\xi-}\}$. Let $r = s_\xi(x) \cdot \widehat{e}$ and let n_ℓ be sufficiently large that $\boldsymbol{\gamma}_{k,r+1}^{x,v_{n_\ell}} = \pi_{k,r+1}$. By definition of the competition interface, since $v_{n_\ell} \prec \varphi_{n_\ell}^{s_\xi(x)}$ we must have $\pi_{r+1} = s_\xi(x) + e_2$, which identifies π as $\boldsymbol{\gamma}^{x,\xi-}$. As the subsequence was arbitrary, the result follows. The second claim is similar.

(d) By Theorem A.4 (d), any semi-infinite geodesic emanating from x is ξ-directed for some $\xi \in \mathcal{U}$. Combining part (a) and Theorem 3.10 (c), the only claim which remains to be shown is that $\boldsymbol{\gamma}^{x,e_i}$ is the only e_i-directed geodesic. This comes from Lemma A.6. ∎

8. Webs of instability: proofs

Recall again the event Ω_0 constructed in (A.7) and fix $\omega \in \Omega_0$ throughout this section. Unless otherwise indicated, an assumption of the form $\zeta \preceq \eta$ includes the case $\zeta = \eta = \xi$.

8.1. Instability points and graphs

Proof of Lemma 4.1. Suppose x^* is a $[\zeta, \eta]$-instability point. Then there exists a direction $\xi \in [\zeta, \eta] \cap \operatorname{supp} \mu_{x+e_1, x+e_2}$, which by Theorem 3.1 implies $\boldsymbol{\gamma}^{x+e_1,\xi-} \cap \boldsymbol{\gamma}^{x+e_2,\xi+} = \emptyset$. Then the ordering of geodesics implies $\boldsymbol{\gamma}^{x+e_1,\zeta-} \cap \boldsymbol{\gamma}^{x+e_2,\eta+} = \emptyset$.

If x^* is not a $[\zeta, \eta]$-instability point, then combining Propositions 7.1 and 7.2 we have

$$\begin{aligned} z &= \mathbf{z}^{\zeta-}(x+e_1, x+e_2) = \mathbf{z}^{\zeta+}(x+e_1, x+e_2) = \mathbf{z}^{\eta-}(x+e_1, x+e_2) \\ &= \mathbf{z}^{\eta+}(x+e_1, x+e_2) \in \mathbb{Z}^2, \end{aligned}$$

$\boldsymbol{\gamma}^{x+e_1,\zeta\pm}$ and $\boldsymbol{\gamma}^{x+e_1,\eta\pm}$ all match until z is reached, and $\boldsymbol{\gamma}^{x+e_2,\zeta\pm}$ and $\boldsymbol{\gamma}^{x+e_2,\eta\pm}$ also all match until z is reached. In particular, $z \in \boldsymbol{\gamma}^{x+e_1,\zeta-} \cap \boldsymbol{\gamma}^{x+e_2,\eta+}$. ∎

The following is immediate from the definitions and monotonicity.

Lemma 8.1. *Let $\zeta \preceq \eta$ in $\operatorname{ri} \mathcal{U}$. A directed path in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ can never cross a directed path in $\mathcal{G}_{\eta+}$ from right to left (i.e. along a dual edge in the $-e_1$ direction) nor a directed path in $\mathcal{G}_{\zeta-}$ from above to below (i.e. along a dual edge in the $-e_2$ direction).*

The next lemma characterizes the ancestors of an instability point in the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$.

Lemma 8.2. *Let $\zeta \preceq \eta$ in $\operatorname{ri} \mathcal{U}$ and $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$. The following statements (i) and (ii) are equivalent for any point $y^* \in \mathbb{Z}^{2*}$:*

(i) *There is a directed path from y^* to x^* in the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$.*

(ii) *$y^* \geq x^*$ and y^* is between the two geodesics $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$, embedded as paths on $\mathbb{R}^2$.*

Proof. (i)⇒(ii). By Lemma 8.1 no directed path in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ can go from y^* to x^* unless y^* lies between $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$.

(ii)⇒(i). We prove this by induction on $|y^*-x^*|_1$. The claim is trivial if $y^*=x^*$. Suppose $y^*\geq x^*$ is such that $y^*\neq x^*$ and y^* is between $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$. If y^* points to both y^*-e_1 and y^*-e_2 in $\mathcal{G}^*_{\cup[\zeta,\eta]}$, then since y^*-e_i is between the two geodesics for at least one $i\in\{1,2\}$, the induction hypothesis implies that there is a directed path from y^* to x^* through this y^*-e_i.

Suppose next that y^* points to y^*-e_1 in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ but y^*-e_1 is not between the two geodesics. Then, on the one hand, y^*-e_2 must be between the geodesics and the induction hypothesis implies that there is a path from y^*-e_2 to x^*. On the other hand, $y^*-\widehat{e}_1^*$ must point to $y^*+\widehat{e}_2^*$ in $\mathcal{G}_{\zeta-}$ to prevent y^*-e_1 from falling between the two geodesics. This implies that y^* points to y^*-e_2 in $\mathcal{G}^*_{\cup[\zeta,\eta]}$. Now we have a path from y^* to x^* through y^*-e_2. See the left plot in Figure 8.1. The case when y^* points to y^*-e_2 and the latter is not between the two geodesics is similar. ∎

The next lemma characterizes $[\zeta,\eta]$-instability points as the *endpoints* of semi-infinite directed paths in $\mathcal{G}^*_{\cup[\zeta,\eta]}$. Furthermore, such paths consist entirely of instability points.

Lemma 8.3. *Let $\zeta\preceq\eta$ in* ri $\mathcal{U}$.

(a) *Let $\{x_k^*\}_{k\geq m}$ be any semi-infinite path on $\mathbb{Z}^{2*}$ such that x_{k+1}^* points to x_k^* in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ for each $k\geq m$. Then $\{x_k^*\}_{k\geq m}\subset\mathbf{S}^*_{[\zeta,\eta]}$ and as $k\to\infty$, the limit points of x_k^*/k lie in the interval $[\underline{\zeta},\overline{\eta}]$.*

(b) *Let $x^*\in\mathbb{Z}^{2*}$ and $m=x^*\cdot\widehat{e}_1$. Then $x^*\in\mathbf{S}^*_{[\zeta,\eta]}$ if and only if there is a path $\{x_k^*\}_{k\geq m}$ on $\mathbb{Z}^{2*}$ such that $x_m^*=x^*$ and for each $k\geq m$, $x_k^*\cdot\widehat{e}_1=k$ and x_{k+1}^* points to x_k^* in $\mathcal{G}^*_{\cup[\zeta,\eta]}$. When this happens, the path $\{x_k^*\}_{k\geq m}$ satisfies part* (a) *above.*

Proof. (a) For each k, Lemma 8.1 implies that the geodesics $\boldsymbol{\gamma}^{x_k^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x_k^*-\widehat{e}_2^*,\eta+}$ are disjoint because they remain forever separated by the path $\{x_k^*\}_{k\geq m}$. Since the backward path $\{x_k^*\}_{k\geq m}$ is sandwiched between the geodesics $\boldsymbol{\gamma}^{x_m^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x_m^*-\widehat{e}_2^*,\eta+}$, Theorem A.4 (a) implies that as $k\to\infty$ the limit points of x_k^*/k lie in the interval $[\underline{\zeta},\overline{\eta}]$.

(b) The "if" claim follows from part (a). To prove the "only if" claim, suppose $x^*\in\mathbf{S}^*_{[\zeta,\eta]}$. Then the geodesics $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$ are disjoint. At every level $k>x^*\cdot\widehat{e}_1$ we can choose a point y_k^* between the geodesics $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$, that is, a point $y_k^*\in\mathbb{Z}^{2*}$ such that $y^*\cdot\widehat{e}_1=k$ and $\boldsymbol{\gamma}_k^{x^*+\widehat{e}_2^*,\zeta-}\prec y_k^*\prec\boldsymbol{\gamma}_k^{x^*-\widehat{e}_2^*,\eta+}$. By Lemma 8.2 there is a directed path in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ from each y_k^* to x^*. Along some subsequence these directed paths converge to a semi-infinite directed path to x^*. ∎

Proof of Theorem 4.3. **Step 1.** We show that $\mathbf{S}_\xi^*\neq\emptyset$ for any $\xi\in\mathcal{V}^\omega$. Since $\xi\in\operatorname{supp}\mu_{z,y}$ for some $z,y\in\mathbb{Z}^2$, the cocycle property (2.7) implies that $\xi\in\operatorname{supp}\mu_{x,x+e_i}$ for some nearest-neighbor edge $(x,x+e_i)$. Since $\mu_{x+e_1,x+e_2}=\mu_{x+e_1,x}+\mu_{x,x+e_2}$ is a sum of two positive measures there can be no cancellation, and hence $\xi\in\operatorname{supp}\mu_{x+e_1,x+e_2}$ and thereby $x+\widehat{e}_1^*\in\mathbf{S}_\xi^*$.

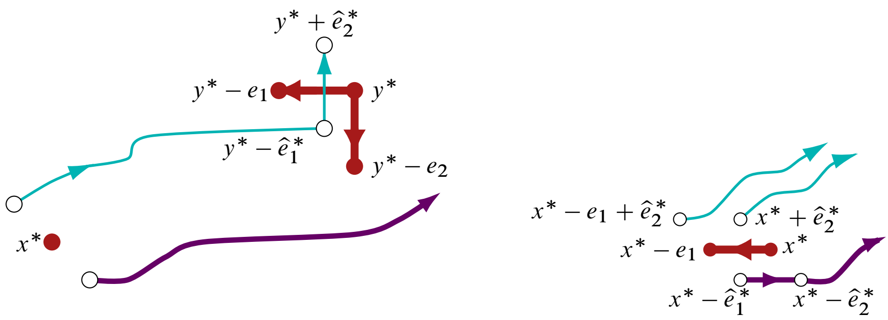

Fig. 8.1. The proofs of Lemma 8.2 (left) and Theorem 4.3 (right). $\zeta-$ geodesics are in green and thin. $\eta+$ geodesics are in purple with medium thickness. Directed edges in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ are in red/thick. White circles are points in $\mathbb{Z}^2$ while points in $\mathbb{Z}^{2*}$ are filled in (red).

Step 2. We show that every edge of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ that emanates from a point of $\mathbf{S}^*_{[\zeta,\eta]}$ is an edge of $\mathcal{S}^*_{[\zeta,\eta]}$. Take $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$. Then $\boldsymbol{\gamma}^{x^*+\hat e_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{x^*-\hat e_2^*,\eta+}$ are disjoint. Suppose x^* points to $x^* - e_1$ in $\mathcal{G}^*_{\cup[\zeta,\eta]}$. Then $x^* - \hat e_1^*$ points to $x^* - \hat e_2^*$ in $\mathcal{G}_{\eta+}$. The geodesic $\boldsymbol{\gamma}^{x^*-\hat e_1^*,\eta+}$ takes first an e_1 step and then follows $\boldsymbol{\gamma}^{x^*-\hat e_2^*,\eta+}$. Since $\boldsymbol{\gamma}^{x^*-e_1+\hat e_2^*,\zeta-}$ must always stay to the left of $\boldsymbol{\gamma}^{x^*+\hat e_2^*,\zeta-}$, it is prevented from touching $\boldsymbol{\gamma}^{x^*-\hat e_1^*,\eta+} = \boldsymbol{\gamma}^{x^*-e_1-\hat e_2^*,\eta+}$ and we see that $x^* - e_1 \in \mathbf{S}^*_{[\zeta,\eta]}$. See the right plot in Figure 8.1. The case when x^* points to $x^* - e_2$ in $\mathcal{G}^*_{\cup[\zeta,\eta]}$ is similar.

Step 3. We conclude the proof. Combining Lemma 8.3 (a) with Step 2 implies that every bi-infinite directed path of the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$ is in fact a directed path of the graph $\mathcal{S}^*_{[\zeta,\eta]}$.

Conversely, let $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$. Lemma 8.3 together with Step 2 implies that x^* is the endpoint of a semi-infinite directed path in $\mathcal{S}^*_{[\zeta,\eta]}$ which is inherited from $\mathcal{G}^*_{\cup[\zeta,\eta]}$. Step 2 implies that by following the edges of $\mathcal{G}^*_{\cup[\zeta,\eta]}$ from x^* creates an infinite down-left directed path in the graph $\mathcal{G}^*_{\cup[\zeta,\eta]}$, and this path is a directed path also in $\mathcal{S}^*_{[\zeta,\eta]}$. In other words, every instability point $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ lies on a bi-infinite directed path of the graph $\mathcal{S}^*_{[\zeta,\eta]}$ that was inherited from $\mathcal{G}^*_{\cup[\zeta,\eta]}$.

The $[\underline{\zeta}, \overline{\eta}]$-directedness of these paths comes from Lemma 8.3 (a). ∎

Proof of Theorem 4.4. (a) Let $x = x^* - \hat e_1^*$. If x^* is a branch point in $\mathcal{S}^*_{[\zeta,\eta]}$, then $\boldsymbol{\gamma}^{x,\zeta-}$ goes from x to $x + e_2$ and $\boldsymbol{\gamma}^{x,\eta+}$ goes from x to $x + e_1$, which is equivalent to $B^{\zeta-}(x + e_1, x + e_2) \le 0 \le B^{\eta+}(x + e_1, x + e_2)$, which in turn is equivalent to $\xi_*(T_x\omega) \in [\zeta, \eta]$.

Conversely, suppose $\xi_*(T_x\omega) \in [\zeta, \eta]$. Reversing the above equivalences we see that $x^* \in \mathbf{S}^*_{[\zeta,\eta]}$ and points to both $x^* - e_1$ and $x^* - e_2$ in $\mathcal{G}^*_{\cup[\zeta,\eta]}$. By Step 2 of the proof of Theorem 4.3 these edges are in $\mathcal{S}^*_{[\zeta,\eta]}$, and hence x^* is a branch point.

(b) Start with the case $\zeta \prec \eta$. Let $\xi \in [\zeta, \eta] \cap \mathcal{U}_0$. Then $\Omega^3_\xi \subset \Omega_0$ and parts (b) and (c) of Theorem A.4 imply that $\mathcal{G}_\xi$ is a tree that does not contain any bi-infinite up-right paths. (Recall that for $\xi \in \mathcal{U}_0$ there is no $\pm$ distinction.) This implies that $\mathcal{G}^*_\xi$ is a tree as well, i.e. all down-left paths of $\mathcal{G}_\xi^*$ coalesce. Since $\mathcal{G}_\xi^* \subset \mathcal{G}_{\cup[\zeta,\eta]}^*$, one can follow the edges e.g. in $\mathcal{G}_\xi^*$ starting from x^* and from y^* to get to a coalescence point z^* that will then be a descendant of both points in $\mathcal{S}_{[\zeta,\eta]}^*$. The same argument can be repeated if $\zeta = \eta = \xi \in \mathcal{V}^\omega$ when condition (4.2) holds, since then both $\mathcal{G}_{\xi\pm}^*$ are trees. Claim (b) is proved.

(c) Observe that for any $x^*, y^* \in \mathbf{S}_{[\zeta,\eta]}^*$, Theorem 3.5 says that under the jump process condition (3.5), if $\omega \in \Omega_0^{\text{jump}}$ (defined in (7.2)), then the geodesics $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ and $\boldsymbol{\gamma}^{y^*+\widehat{e}_2^*,\zeta-}$ coalesce, as do $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$ and $\boldsymbol{\gamma}^{y^*-\widehat{e}_2^*,\eta+}$. By Lemma 8.2, any point in $\mathbf{S}_{[\zeta,\eta]}^*$ that is between the two $+$ and $-$ coalesced geodesics is an ancestor to both x^* and y^*. Such a point exists. For example, take a point z on $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$ above the coalescence levels, in other words, such that $z \cdot \widehat{e}_1 \geq (\mathbf{z}^{\zeta-}(x^* + \widehat{e}_2^*, y^* + \widehat{e}_2^*) \cdot \widehat{e}_1) \vee (\mathbf{z}^{\eta+}(x^* - \widehat{e}_2^*, y^* - \widehat{e}_2^*) \cdot \widehat{e}_1)$. Since $\boldsymbol{\gamma}^{z,\eta+}$ coalesces with $\boldsymbol{\gamma}^{x^*-\widehat{e}_2^*,\eta+}$, which does not touch $\boldsymbol{\gamma}^{z,\zeta-}$ (because this latter is part of $\boldsymbol{\gamma}^{x^*+\widehat{e}_2^*,\zeta-}$), $\boldsymbol{\gamma}^{z,\eta+}$ must separate from $\boldsymbol{\gamma}^{z,\zeta-}$ at some point z'. The dual point $z' + \widehat{e}_1^*$ is then in $\mathbf{S}_{[\zeta,\eta]}^*$ and is an ancestor to both x^* and y^*. Part (c) is proved.

(d) The assumption is that $\zeta \prec \eta$ and $]\zeta, \eta[\cap \mathcal{V}^\omega \neq \emptyset$. By Lemma 7.3, $\nabla g(\zeta+) \neq \nabla g(\eta-)$. For any $z \in \mathbb{Z}^2$, Theorem A.8 (c) gives a strictly increasing sequence $z < z_1 < z_2 < \cdots$ such that $\xi_*(T_{z_k}\omega) \in]\zeta, \eta[$ for each k. Then by (2.17), $B^{\zeta+}(z_k + e_1, z_k + e_2) < 0 < B^{\eta-}(z_k + e_1, z_k + e_2)$, which implies that $z_k^* = z_k + \widehat{e}_1^*$ is a $[\zeta, \eta]$-instability point. Each such point is a branch point in $\mathcal{S}_{[\zeta,\eta]}^*$ because z_k points to $z_k + e_2$ in $\mathcal{G}_{\zeta+}$, and hence also in $\mathcal{G}_{\zeta-}$, and to $z_k + e_1$ in $\mathcal{G}_{\eta-}$, and hence also in $\mathcal{G}_{\eta+}$.

The proof of the existence of infinitely many coalescence points in $\mathcal{S}_{[\zeta,\eta]}^*$ follows from this and the first claim in part (b) in a way similar to the proof below for the case of $\mathcal{S}_\xi^*$ (but without the need for any extra conditions) and is therefore omitted.

(e) Fix $\xi \in \mathcal{V}^\omega$ for the duration of the proof. Assume the jump process condition (3.5). By Theorem 3.1 there exist $x, y \in \mathbb{Z}^2$ such that $\boldsymbol{\gamma}^{x,\xi-} \cap \boldsymbol{\gamma}^{y,\xi+} = \emptyset$ and then Theorem 3.5 says that for any $z \in \mathbb{Z}^2$, the two geodesics $\boldsymbol{\gamma}^{z,\xi\pm}$ must separate at some point z_1 (in order to coalesce with $\boldsymbol{\gamma}^{x,\xi-}$ and $\boldsymbol{\gamma}^{y,\xi+}$, respectively). Uniqueness of finite geodesics implies that $\boldsymbol{\gamma}^{z_1+e_i,\xi\pm}$, $i \in \{1, 2\}$, cannot touch. Thus, $z_1 + \widehat{e}_1^* \in \mathbf{S}_\xi^*$. Now define inductively z_{n+1} to be the point where the geodesics $\boldsymbol{\gamma}^{z_n+\widehat{e}_1,\xi\pm}$ separate. Then for each n, $z_{n+1} > z_n$ coordinatewise and $z_n^* = z_n + \widehat{e}_1^*$ is a point in $\mathbf{S}_\xi^*$.

Next, assume both the jump process condition (3.5) and the no bi-infinite geodesic condition (4.2). We prove the second claim of part (e) about infinitely many coalescence points by mapping branch points injectively to coalescence points as follows.

Given a branch point x^*, let π^* and $\bar{\pi}^*$ be the two innermost down-left paths out of x^* along the directed graph $\mathcal{S}_\xi^*$, defined by these rules:

(i) π^* starts with edge $(x^*, x^* - e_1)$, follows the arrows of $\mathcal{S}_\xi^*$, and at vertices where both $-e_1$ and $-e_2$ steps are allowed, it takes the $-e_2$ step;

(ii) $\bar{\pi}^*$ starts with edge $(x^*, x^* - e_2)$, follows the arrows of $\mathcal{S}_\xi^*$, and whenever both steps are available takes the $-e_1$ step. (8.1)

By part (b), $x^* - e_1$ and $x^* - e_2$ have a common descendant (this is where assumption (4.2) is used). By planarity, the paths π^* and $\bar{\pi}^*$ must then meet at some point after x^*.

Let z^* be their first common point after x^*, that is, the point $z^* \in (\pi^* \cap \bar\pi^*) \setminus \{x^*\}$ that maximizes $z^* \cdot \widehat{e}_1$. This z^* is the coalescence point that the branch point x^* is mapped to.

We argue that the map $x^* \mapsto z^*$ thus defined is one-to-one. Two observations that help:

- There cannot be any $\mathbf{S}^*_\xi$-points strictly inside the region bounded by π^* and $\bar\pi^*$ between x^* and z^*. By Theorem 4.3 such a point would lie on an $\mathcal{S}^*_\xi$ path, which contradicts the choice of π^* and $\bar\pi^*$ as the innermost paths from x^* to z^*.
- The last step that π^* takes to reach z^* is $-e_2$ and the last step of $\bar\pi^*$ is $-e_1$. Otherwise π^* and $\bar\pi^*$ would have met before z^*.

Suppose another branch point $y^* \in \mathbf{S}^*_\xi$ distinct from x^* maps to the same coalescence point z^*. Let the innermost paths from y^* to z^* be γ^* and $\bar\gamma^*$, defined by the same rules (8.1) but with x^* replaced by y^*. As observed, γ^* and $\bar\gamma^*$ cannot enter the region strictly between π^* and $\bar\pi^*$.

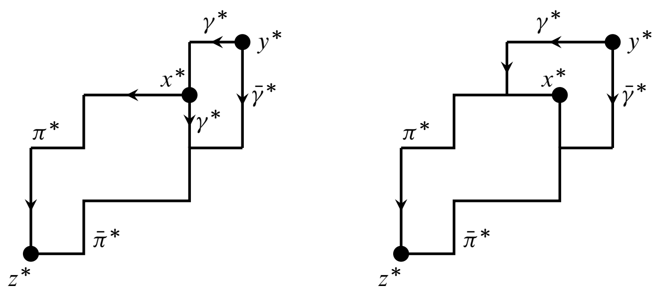

Fig. 8.2. Illustration of the proof that the map $x^* \mapsto z^*$ is one-to-one.

Since γ^* uses the edge $(z^* + e_2, z^*)$, it must coalesce at some point with π^*. The point x^* itself cannot lie on γ^* because otherwise (8.1) forces γ^* to take the edge $(x^*, x^* - e_2)$ and γ^* cannot follow π^* to z^*. This scenario is depicted by the left drawing in Figure 8.2. Thus γ^* meets π^* after x^*, at which point rule (8.1) forces them to coalesce (right drawing in Figure 8.2).

Similarly, x^* cannot lie on $\bar\gamma^*$, and $\bar\gamma^*$ meets $\bar\pi^*$ after x^* at which point these coalesce (right drawing in Figure 8.2).

Paths from y^* cannot meet both π^* and $\bar\pi^*$ while avoiding x^* unless $y^* > x^*$ holds coordinatewise. It follows now that x^* must lie strictly inside the region bounded by γ^* and $\bar\gamma^*$ between y^* and z^*, as illustrated by the right drawing in Figure 8.2. But we already ruled out such a possibility. These contradictions show that the map is one-to-one.

Since we have already proved that under the jump process condition (3.5) there are infinitely many branch points in $\mathcal{S}^*_\xi$, it now follows that there are also infinitely many coalescence points and part (e) is proved. ∎

Proof of the claim in Remark 4.5. It suffices to consider the case where $]\zeta, \eta[\cap \mathcal{V}^\omega = \emptyset$ but $\{\zeta, \eta\} \cap \mathcal{V}^\omega \neq \emptyset$. By Theorem 3.8 (a), the differentiable endpoints of the (countably many) linear segments of g are all outside $\mathcal{V}^\omega$. By Theorem 3.8 (b) we know $]\zeta,\eta[$ must be inside a linear segment. Thus, it must be the case that $\{\zeta,\eta\}\cap\mathcal{V}^\omega\setminus\mathcal{D}\neq\emptyset$. Suppose, without loss of generality, that ζ is in this intersection. Then Theorem A.8 (d) implies the existence of infinitely many $x\in\mathbb{Z}^2$ with $\xi_*(T_x\omega)=\zeta\in\mathcal{V}^\omega$, and Theorem 4.4 (a) says that the corresponding dual points x^* are all branch points in $\mathcal{S}^*_\zeta\subset\mathcal{S}^*_{[\zeta,\eta]}$. The claim about coalescence points follows from the just proved infinite number of branch points, combined with the first claim in part (b), similarly to the way the corresponding claim is proved in Theorem 4.4 (e). ∎

In words, the next result says that there are no semi-infinite horizontal or vertical paths in any of the instability graphs $\mathbf{S}^*_{[\zeta,\eta]}$. The idea behind the proof is that the existence of such a path would force the existence of a semi-infinite horizontal or vertical path in one of the geodesic graphs $\mathcal{G}_{\xi\square}$ for some $\square\in\{+,-\}$ and $\xi\in\operatorname{ri}\mathcal{U}$. This is ruled out by the law of large numbers behavior of the Busemann functions.

Lemma 8.4. *For any $\omega\in\Omega_0$, $\zeta\preceq\eta$, and $i\in\{1,2\}$, there does not exist an $x^*\in\mathbf{S}^*_{[\zeta,\eta]}$ such that $x^*-ne_i\in\mathbf{A}^*_{[\zeta,\eta]}(x^*-(n+1)e_i)$ for all $n\in\mathbb{Z}_+$; nor does there exist an $x^*\in\mathbf{S}^*_{[\zeta,\eta]}$ such that $x^*+(n+1)e_i\in\mathbf{A}^*_{[\zeta,\eta]}(x^*+ne_i)$ for all $n\in\mathbb{Z}_+$.*

Proof. We prove the result for $i=1$, the case $i=2$ being similar. We also only work with paths of the first type; the other type can be treated similarly.

The existence of a path of the first type, with $i=1$, implies that $x^*-ne_1-\widehat{e}_1^*$ points to $x^*-(n-1)e_1-\widehat{e}_1^*$ in $\mathcal{G}_{\eta+}$ for all $n\in\mathbb{Z}_+$. But this implies that

$$B^{\eta+}(x^*-ne_1-\widehat{e}_1^*,x^*-(n-1)e_1-\widehat{e}_1^*)=\omega_{x^*-ne_1-\widehat{e}_1^*}$$

for all $n\in\mathbb{Z}_+$. Take any sequence $\eta_m\in\mathcal{U}_0$ such that $\eta_m\searrow\eta$. Then (2.10) and (2.7) imply that

$$\sum_{k=1}^n\omega_{x^*-ke_1-\widehat{e}_1^*}=B^{\eta+}(x^*-ne_1-\widehat{e}_1^*,x^*-\widehat{e}_1^*)\geq B^{\eta_m}(x^*-ne_1-\widehat{e}_1^*,x^*-\widehat{e}_1^*).$$

Divide by n and apply the ergodic theorem on the left-hand side and (A.3) on the right-hand side to get $\mathbb{E}[\omega_0]\geq e_1\cdot\nabla g(\eta_m)$ for all m. Take $m\to\infty$ to get $\mathbb{E}[\omega_0]\geq e_1\cdot\nabla g(\eta+)$. It follows from Martin's estimate of the asymptotic behavior of the shape function near the boundary of $\mathcal{U}$, [41, Theorem 2.4], along with concavity that this cannot happen. ∎

Proof of Lemma 3.6. A general step of the path can be decomposed as $x_{i+1}-x_i=\sum_k(y_{k+1}-y_k)$ where $y_{k+1}-y_k\in\{e_1,-e_2\}$ for each k. Then each $\mu_{y_k,y_{k+1}}$ is a negative measure, and consequently $\operatorname{supp}\mu_{x_i,x_{i+1}}=\bigcup_k\operatorname{supp}\mu_{y_k,y_{k+1}}$. Thus we may assume that the path satisfies $x_{i+1}-x_i\in\{e_1,-e_2\}$ for all i.

One direction is clear: $\bigcup_{i\in\mathbb{Z}}\operatorname{supp}\mu_{x_i,x_{i+1}}\subset\mathcal{V}^\omega$.

For the other direction, take $\xi\in\mathcal{V}^\omega$. By Theorem 4.3, there is a bi-infinite up-right reverse-directed path $x^*_{-\infty,\infty}$ through x^* in $\mathcal{S}^*_\xi$ with increments in $\{e_1,e_2\}$. By Lemma 8.4 this path must cross any down-right lattice path $x_{-\infty,\infty}$. This means that there exists an $i \in \mathbb{Z}$ such that either $x_{i+1} - x_i = e_1$ and $x_i + \widehat{e}_1^*$ points to $x_i - \widehat{e}_2^*$ in $\mathcal{S}_\xi^*$, i.e. x_i points to $x_i + e_2$ in $\mathcal{G}_{\xi-}$, or $x_{i+1} - x_i = -e_2$ and $x_i - \widehat{e}_2^*$ points to $x_i - \widehat{e}_1^*$ in $\mathcal{S}_\xi^*$, i.e. x_i points to $x_i + e_1$ in $\mathcal{G}_{\xi+}$. In the former case, $\boldsymbol{\gamma}^{x_i,\xi-}$ goes from x_i to $x_i + e_2$ and from there it never touches $\boldsymbol{\gamma}^{x_{i+1},\xi+} = \boldsymbol{\gamma}^{x_i+e_1,\xi+}$, since $x_i + \widehat{e}_1^* \in \mathbf{S}^*_{x i}$. Consequently, in this case Theorem 3.1 says that $\xi \in \operatorname{supp} \mu_{x_i,x_{i+1}}$. The other case is similar and again gives $\xi \in \operatorname{supp} \mu_{x_i,x_{i+1}}$. This proves Lemma 3.6. ∎

8.2. *Density of instability points on the lattice*

For $\zeta \preceq \eta$ in $\operatorname{ri} \mathcal{U}$, $x \in \mathbb{Z}^2$, and $i \in \{1,2\}$ let

$$\rho_x^i(\zeta,\eta) = \mathbb{1}\{[\zeta,\eta] \cap \operatorname{supp} \mu_{x,x+e_i} \neq \emptyset\}. \tag{8.2}$$

We write $\rho_x^i(\xi)$ for $\rho_x^i(\xi,\xi)$. By definition, $x + \widehat{e}_1^* \in \mathbf{S}^*_{[\zeta,\eta]}$ if and only if $\rho_x^1(\zeta,\eta)$ and $\rho_x^2(\zeta,\eta)$ are not both 0. By Lemma 8.6 below, $\rho_x^i(\zeta,\eta) = 1$ is equivalent to $x + \widehat{e}_1^*$ pointing to $x + \widehat{e}_1^* - e_{3-i}$ in $\mathcal{S}^*_{[\zeta,\eta]}$. Also, $\rho_x^1(\xi) = \rho_x^2(\xi) = 1$ if and only if $\xi_*(T_x\omega) = \xi$. Let

$$\kappa_i(\zeta,\eta) = \mathbb{P}\{\rho_0^i(\zeta,\eta) = 1\}.$$

Since $\operatorname{supp} \mu_{x,x+e_i}$ is by definition closed, κ_i is left-continuous in ζ and right-continuous in η. Furthermore, by Theorem 3.8, κ_i is continuous in each argument at points of differentiability of g. Again, we write $\kappa_i(\xi)$ for $\kappa_i(\xi,\xi)$. We thus have

$$\kappa_i(\xi) = \lim_{\zeta \nearrow \xi,\, \eta \searrow \xi} \kappa_i(\zeta,\eta).$$

By Theorem 3.8, $\xi \in \mathcal{D}$ if and only if $\kappa_i(\xi) = 0$ for some (and hence both) $i \in \{1,2\}$. Let

$$\kappa_{12}(\zeta,\eta) = \mathbb{P}\{\rho_0^1(\zeta,\eta) = \rho_0^2(\zeta,\eta) = 1\} = \mathbb{P}\{\xi_* \in [\zeta,\eta]\}$$

and write $\kappa_{12}(\xi)$ for $\kappa_{12}(\xi,\xi)$. The last equality above follows because if $\xi_* \notin [\zeta,\eta]$, then by recovery (2.8) and by the Busemann characterization (2.17) of ξ_*, one of the processes $\xi \mapsto B^{\xi\pm}(0,e_i)$ for $i \in \{1,2\}$ is constant for $\xi \in [\zeta,\eta]$.

The next result essentially follows from the ergodic theorem and gives the density of horizontal and vertical edges, instability points, branch points, and coalescence points.

Lemma 8.5. *Assume the regularity condition (2.4). There exists a T-invariant event $\Omega_0' \subset \Omega_0$ with $\mathbb{P}(\Omega_0') = 1$ and such that for all $\omega \in \Omega_0'$, $i, j \in \{1,2\}$, $a, b, a', b' \in \{0,1\}$ with $(b+a)(b'+a') \neq 0$, and for all $\zeta \preceq \eta$ in $\operatorname{ri} \mathcal{U}$, we have*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{|b+a|n} \sum_{k=-an}^{bn} \rho_{ke_i}^j(\zeta,\eta) \\
&\quad = \lim_{n\to\infty} \frac{1}{|(b+a)(b'+a')|n^2} \sum_{x\in[-an,bn]\times[-a'n,b'n]} \rho_x^j(\zeta,\eta) = \kappa_j(\zeta,\eta),
\end{aligned} \tag{8.3}$$

$$\lim_{n\to\infty}\frac{1}{|b+a|n}\sum_{k=-an}^{bn}\mathbb{1}\{ke_i+\hat{e}_1^*\in\mathbf{S}^*_{[\zeta,\eta]}\}$$
$$=\lim_{n\to\infty}\frac{1}{(b+a)(b'+a')n^2}\sum_{x\in[-an,bn]\times[-a'n,b'n]}\mathbb{1}\{x+\hat{e}_1^*\in\mathbf{S}^*_{[\zeta,\eta]}\}$$
$$=\kappa_1(\zeta,\eta)+\kappa_2(\zeta,\eta)-\kappa_{12}(\zeta,\eta), \tag{8.4}$$

$$\lim_{n\to\infty}\frac{1}{|b+a|n}\sum_{k=-an}^{bn}\rho^1_{ke_i}(\zeta,\eta)\rho^2_{ke_i}(\zeta,\eta)$$
$$=\lim_{n\to\infty}\frac{1}{|(b+a)(b'+a')|n^2}\times\sum_{x\in[-an,bn]\times[-a'n,b'n]}\rho^1_x(\zeta,\eta)\rho^2_x(\zeta,\eta)=\kappa_{12}(\zeta,\eta), \tag{8.5}$$

and

$$\lim_{n\to\infty}\frac{1}{|b+a|n}\sum_{k=-an}^{bn}\rho^1_{ke_i}(\zeta,\eta)\rho^2_{(k+1)e_i-e_{3-i}}(\zeta,\eta)$$
$$=\lim_{n\to\infty}\frac{1}{|(b+a)(b'+a')|n^2}\times\sum_{x\in[-an,bn]\times[-a'n,b'n]}\rho^1_{x-e_1}(\zeta,\eta)\rho^2_{x-e_2}(\zeta,\eta)=\kappa_{12}(\zeta,\eta). \tag{8.6}$$

All of the above limits are positive if and only if $\nabla g(\zeta+)\neq\nabla g(\eta-)$.

Proof. As explained in Remark A.3, under the regularity condition (2.4), the Busemann process is a measurable function of $\{\omega_x : x\in\mathbb{Z}^2\}$. Thus, by the ergodic theorem, there exists a T-invariant event $\Omega_0'\subset\Omega_0$ with $\mathbb{P}(\Omega_0')=1$ and such that for $\omega\in\Omega_0'$ the limits (8.3)–(8.6) hold for all $\zeta,\eta\in\mathcal{U}_0\cup((\mathrm{ri}\,\mathcal{U})\setminus\mathcal{D})$.

To justify the equality of the limit in (8.6) with the one in (8.5) observe that since every instability point must have at least one descendant and at least one ancestor, we have

$$\mathbb{P}\{-\hat{e}_1^*\in\mathbf{S}^*_{[\zeta,\eta]}\}$$
$$=\mathbb{P}\{\rho^1_{-e_1}(\zeta,\eta)=1\}+\mathbb{P}\{\rho^2_{-e_2}(\zeta,\eta)=1\}-\mathbb{P}\{\rho^1_{-e_1}(\zeta,\eta)=\rho^2_{-e_2}(\zeta,\eta)=1\}$$

and

$$\mathbb{P}\{\hat{e}_1^*\in\mathbf{S}^*_{[\zeta,\eta]}\}=\mathbb{P}\{\rho^1_0(\zeta,\eta)=1\}+\mathbb{P}\{\rho^2_0(\zeta,\eta)=1\}-\mathbb{P}\{\rho^1_0(\zeta,\eta)=\rho^2_0(\zeta,\eta)=1\}.$$

By shift invariance, the first three probabilities in the first display match the corresponding three probabilities in the second display. Thus,

$$\mathbb{P}\{\rho^1_{-e_1}(\zeta,\eta)=\rho^2_{-e_2}(\zeta,\eta)=1\}=\kappa_{12}(\zeta,\eta).$$

We now prove the first limit in (8.3), the rest of the limits in the statement of the lemma being similar. Take $\omega \in \Omega_0'$ and any $\zeta \prec \eta$ in ri $\mathcal{U}$. Suppose first g is differentiable at both ζ and η. Take sequences $\zeta_m' \prec \zeta \prec \zeta_m \prec \eta_m \prec \eta \prec \eta_m'$ with $\zeta_m', \zeta_m, \eta_m, \eta_m' \in \mathcal{U}_0$ and use monotonicity and the continuity of κ_j to get

$$\begin{aligned}
\kappa_j(\zeta_m, \eta_m) &= \lim_{n\to\infty} \frac{1}{(b-a)n} \sum_{k=-an}^{bn} \rho^j_{ke_i}(\zeta_m, \eta_m) \\
&\le \varliminf_{n\to\infty} \frac{1}{(b-a)n} \sum_{k=-an}^{bn} \rho^j_{ke_i}(\zeta, \eta) \le \varlimsup_{n\to\infty} \frac{1}{(b-a)n} \sum_{k=-an}^{bn} \rho^j_{ke_i}(\zeta, \eta) \\
&\le \lim_{n\to\infty} \frac{1}{(b-a)n} \sum_{k=-an}^{bn} \rho^j_{ke_i}(\zeta_m', \eta_m') = \kappa_j(\zeta_m', \eta_m').
\end{aligned}$$

Taking $m \to \infty$ and using continuity of κ_j at ζ and η gives that the above liminf and limsup are equal to $\kappa_j(\zeta, \eta)$. The same proof works if $\zeta = \eta$ is a point of differentiability of g. In this case, we can use 0 as a lower bound and for the upper bound we have $\kappa_j(\zeta) = \kappa_j(\eta) = 0$.

Next, suppose ζ is a point of non-differentiability of g, but η is still a point of differentiability. We can repeat the same argument as above, but this time only using the sequences η_m and η_m' and the intervals $[\zeta, \eta_m]$ and $[\zeta, \eta_m']$ for the upper and lower bounds, because ζ has been included in the set $\mathcal{U}_0 \cup ((\text{ri}\,\mathcal{U}) \setminus \mathcal{D})$. A similar argument works if ζ is a point of differentiability but η is not. When g is not differentiable at both ζ and η, the claimed limits follow from the choice of Ω_0'. ∎

Proof of Proposition 4.6. The claim follows from Lemma 8.5. ∎

8.3. *Flow of Busemann measure*

Proof of Theorem 4.7. The vertex set of $\mathcal{B}^*_{[\zeta,\eta]}$ is by definition the same as that of $\mathcal{S}^*_{[\zeta,\eta]}$. That the edges also agree follows from Lemma 8.6 below. ∎

Lemma 8.6. *For* $i \in \{1, 2\}$, $[\zeta, \eta] \cap \operatorname{supp} \mu_{x,x+e_i} \neq \emptyset$ *if and only if* $(x + \hat{e}_1^*, x + \hat{e}_1^* - e_{3-i})$ *is a directed edge in the graph* $\mathcal{S}^*_{[\zeta,\eta]}$.

Proof. We prove the case of $i = 1$. Assume first that $[\zeta, \eta] \cap \operatorname{supp} \mu_{x,x+e_1} \neq \emptyset$. From $\mu_{x+e_1,x+e_2} = \mu_{x+e_1,x} + \mu_{x,x+e_2}$ and $\mu_{x-\hat{e}_2,x} = \mu_{x-\hat{e}_2,x+e_1} + \mu_{x+e_1,x}$ (sums of positive measures) we see that both $x + \hat{e}_1^*, x + \hat{e}_1^* - e_2 \in \mathbf{S}^*_{[\zeta,\eta]}$.

Suppose $\xi \in [\zeta, \eta] \cap \operatorname{supp} \mu_{x,x+e_1}$. By Theorem 3.1, x must point to $x + e_2$ in $\mathcal{G}_{\xi-}$, which forces the same in $\mathcal{G}_{\zeta-}$. Thus $x + \hat{e}_1^*$ points to $x + \hat{e}_1^* - e_2$ in $\mathcal{G}^*_{\zeta-}$ and hence also in $\mathcal{G}^*_{\cup[\zeta,\eta]}$.

Conversely, if $x + \hat{e}_1^* \in \mathbf{S}^*_{[\zeta,\eta]}$ then $\boldsymbol{\gamma}^{x+e_2,\zeta-}$ and $\boldsymbol{\gamma}^{x+e_1,\eta+}$ do not intersect. If furthermore $x + \hat{e}_1^*$ points to $x + \hat{e}_1^* - e_2$ in $\mathcal{G}^*_{\cup[\zeta,\eta]}$, then x points to $x + e_2$ in $\mathcal{G}_{\zeta-}$ and hence $\boldsymbol{\gamma}^{x,\zeta-}$ joins $\boldsymbol{\gamma}^{x+e_2,\zeta-}$ and does not intersect $\boldsymbol{\gamma}^{x+e_1,\eta+}$.

Let $\zeta' \prec \zeta$ and $\eta' \succ \eta$. By geodesic ordering (2.13), $\boldsymbol{\gamma}^{x,\zeta'+}$ and $\boldsymbol{\gamma}^{x+e_1,\eta'-}$ are disjoint. In particular, the coalescence points $\mathbf{z}^{\zeta'+}(x, x+e_1)$ and $\mathbf{z}^{\eta'-}(x, x+e_1)$ cannot coincide on $\mathbb{Z}^2$. By Proposition 7.1, $]\zeta', \eta'[$ intersects $\operatorname{supp} \mu_{x,x+e_1}$. Since this holds for every choice of $]\zeta', \eta'[\supset [\zeta, \eta]$, it follows that also $[\zeta, \eta]$ intersects $\operatorname{supp} \mu_{x,x+e_1}$. ∎

Proof of Proposition 4.8. Suppose $x \overset{[\zeta,\eta]}{\sim} y$. Since $\operatorname{supp} \mu_{x,y}$ is a closed subset of $\operatorname{ri} \mathcal{U}$ and $[\zeta, \eta]$ a compact set, we can find $\zeta' \prec \zeta$ and $\eta' \succ \eta$ such that $|\mu_{x,y}|(]\zeta', \eta'[) = 0$. Then by Proposition 7.1, there exists $z \in \mathbb{Z}^2$ such that all geodesics $\boldsymbol{\gamma}^{x,\xi\square}$ and $\boldsymbol{\gamma}^{y,\xi\square}$ for $\xi \in [\zeta, \eta]$ and $\square \in \{-, +\}$ meet at z. Thus x and y are in the same subtree of the graph $\mathcal{G}_{\cap[\zeta,\eta]}$.

Conversely, suppose x and y are two distinct points in the same subtree $\mathcal{K}$ of the graph $\mathcal{G}_{\cap[\zeta,\eta]}$. In this tree the following holds.

In $\mathcal{K}$ there is a point z and a path π from x to z and a path π' from y to z such that z is the first common point of π and π'. For each $\xi \in [\zeta, \eta]$ and both signs $\square \in \{-, +\}$, all the geodesics $\boldsymbol{\gamma}^{x,\xi\square}$ follow π from x to z, and all the geodesics $\boldsymbol{\gamma}^{y,\xi\square}$ follow π' from y to z. (8.7)

Consequently, each $\xi \in [\zeta, \eta]$ satisfies $\mathbf{z}^{\xi-}(x, y) = \mathbf{z}^{\xi+}(x, y) = z$. By Proposition 7.2 each $\xi \in [\zeta, \eta]$ lies outside $\operatorname{supp} \mu_{x,y}$. ∎

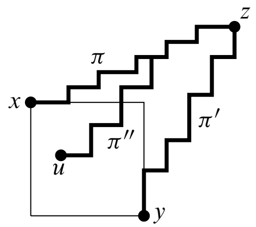

Fig. 8.3. Proof of Lemma 4.9.

Proof of Lemma 4.9. The hypotheses imply that, by switching x and y around if necessary, $x \cdot e_1 \le y \cdot e_1$ and $x \cdot e_2 \ge y \cdot e_2$. Let z, π, π' be as in (8.7). Let u be any point of $[\![x \wedge y, x \vee y]\!]$. By planarity, each geodesic $\boldsymbol{\gamma}^{u,\xi\square}$ for $\xi \in [\zeta, \eta]$ and $\square \in \{-, +\}$ must eventually intersect π or π' and then follow this to z. See Figure 8.3. By uniqueness of finite geodesics, all these geodesics $\boldsymbol{\gamma}^{u,\xi\square}$ follow the same path π'' from u to z. Thus π'' is part of the graph $\mathcal{G}_{\cap[\zeta,\eta]}$, and since it comes together with π and π' at z, it is part of the same subtree $\mathcal{K}$. ∎

Proof of Lemma 4.10. Suppose x is such a vertex but $\mathcal{K} \subset \{y : y \ge x\}$ fails. We claim that then there necessarily exists a vertex $y \in \mathcal{K}$ such that x and y satisfy the hypotheses of Lemma 4.9 and one of $\{x - e_1, x - e_2\}$ lies in $[\![x \wedge y, x \vee y]\!]$. This leads to a contradiction.

To verify the claim, pick $y \in \mathcal{K}$ such that $y \ge x$ fails. If $y < x$ also fails, there are two possible cases:

(i) $y \cdot e_1 < x \cdot e_1$ and $y \cdot e_2 \ge x \cdot e_2$, in which case $x - e_1 \in [\![x \wedge y, x \vee y]\!] \subset \mathcal{K}$;

(ii) $y \cdot e_1 \ge x \cdot e_1$ and $y \cdot e_2 < x \cdot e_2$, in which case $x - e_2 \in [\![x \wedge y, x \vee y]\!] \subset \mathcal{K}$.

If $y < x$ does not fail, follow the geodesics $\{\boldsymbol{\gamma}^{y,\xi\pm} : \xi \in [\zeta, \eta]\}$ until they hit the level $\mathbb{L}_{x\cdot\widehat{e}_1}$ at some point y'. The assumption that neither $x - e_1$ nor $x - e_2$ lies in $\mathcal{K}$ implies that $y' \neq x$. Thus y' is a point of $\mathcal{K}$ that fails both $y' \geq x$ and $y' < x$. Replace y with y' and apply the previous argument.

We have shown that the existence of $x \in \mathcal{K}$ such that $\{x - e_1, x - e_2\} \cap \mathcal{K} = \emptyset$ implies that $\mathcal{K} \subset \{y : y \geq x\}$. That such an x must be unique follows since x lies outside $\{y : y \geq x'\}$ for any $x' \neq x$ that satisfies $x' \geq x$.

Assuming that $\inf\{t \in \mathbb{Z} : \mathcal{K} \cap \mathbb{L}_t \neq \emptyset\} > -\infty$, pick $x \in \mathcal{K}$ to minimize the level $x \cdot \widehat{e}_1$. ■

Proof of Theorem 4.11. (a) If $[\zeta, \eta] \cap \mathcal{V}^\omega = \emptyset$ then the interval $[\zeta, \eta]$ is strictly on one side of $\xi_*(T_x\omega)$ at every x. Hence the graphs $\{\mathcal{G}_{\xi\square} : \xi \in [\zeta, \eta], \square \in \{-, +\}\}$ are all identical. This common graph is a tree by Theorem 3.5.

Conversely, if $\xi \in [\zeta, \eta] \cap \mathcal{V}^\omega$, then there exist x, y such that $\xi \in \operatorname{supp} \mu_{x,y}$ and by Theorem 3.1 there are disjoint geodesics in $\mathcal{G}_{\cap[\zeta,\eta]}$.

(b) It follows from what was already said that $\{\mathcal{K}(z) : z \in \mathbf{D}^{[\zeta,\eta]}\}$ are disjoint subtrees of $\mathcal{G}_{\cap[\zeta,\eta]}$ and their vertex sets cover $\mathbb{Z}^2$. Suppose $(x, x + e_i)$ is an edge in $\mathcal{G}_{\cap[\zeta,\eta]}$. Then all geodesics $\{\boldsymbol{\gamma}^{x,\xi\square} : \xi \in [\zeta, \eta], \square \in \{-, +\}\}$ go through this edge. Thus this edge must be an edge of the tree $\mathcal{K}(z)$ that contains both x and $x + e_i$. Hence each edge of $\mathcal{G}_{\cap[\zeta,\eta]}$ is an edge of one of the trees $\mathcal{K}(z)$, and no such edge can connect two trees $\mathcal{K}(z)$ and $\mathcal{K}(z')$ for distinct z and z'. ■

9. Instability points in the exponential model: proofs

We turn to the proofs of the results in Section 5, beginning with a discussion of Palm kernels, which are needed in order to prove Theorems 5.2 and 5.3.

9.1. Palm kernels

Let $\mathcal{M}_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}}$ denote the space of locally bounded positive Borel measures on the locally compact space $\mathbb{Z} \times \mathrm{ri}\,\mathcal{U}$. Consider $\mathbb{Z} \times \mathrm{ri}\,\mathcal{U}$ as the disjoint union of copies of $\mathrm{ri}\,\mathcal{U}$, one copy for each horizontal edge $(ke_1, (k+1)e_1)$ on the x-axis. Recall that $B_k^{\xi\square} = B^{\xi\square}(ke_1, (k+1)e_1)$. We define two random measures $\boldsymbol{\nu}$ and $\mathbf{n}$ on $\mathbb{Z} \times \mathrm{ri}\,\mathcal{U}$ in terms of the Busemann functions $\xi \mapsto B_k^{\xi\pm}$ attached to these edges.

On each subset $\{k\} \times \mathrm{ri}\,\mathcal{U}$ of $\mathbb{Z} \times \mathrm{ri}\,\mathcal{U}$ we (slightly abuse notation and) define the measure $\boldsymbol{\nu}_k$ by

$$\boldsymbol{\nu}_k(\{k\} \times]\zeta, \eta]) = \boldsymbol{\nu}_k(]\zeta, \eta]) = B_k^{\zeta+} - B_k^{\eta+}$$

for $\zeta \prec \eta$ in $\mathrm{ri}\,\mathcal{U}$. In terms of definition (3.1), $\boldsymbol{\nu}_k = \mu_{(k+1)e_1, ke_1}$ is a positive measure due to monotonicity (2.10). On $\mathbb{Z} \times \mathrm{ri}\,\mathcal{U}$, define the measure $\boldsymbol{\nu} = \sum_k \boldsymbol{\nu}_k$. In other words, for Borel sets $A_k \subset \mathrm{ri}\,\mathcal{U}$, $\boldsymbol{\nu}(\bigcup_k \{k\} \times A_k) = \sum_k \boldsymbol{\nu}_k(A_k)$.

Let $\mathbf{n}_k$ denote the simple point process on $\{k\} \times \operatorname{ri} \mathcal{U}$ that records the locations of the jumps of the Busemann function $\xi \mapsto B_k^{\xi\pm}$: for Borel $A \subset \operatorname{ri} \mathcal{U}$,

$$\mathbf{n}_k(\{k\} \times A) = \mathbf{n}_k(A) = \sum_{\xi \in A} \mathbb{1}\{B_k^{\xi-} > B_k^{\xi+}\}.$$

We describe the probability distributions of the component measures $\boldsymbol{\nu}_k$ and $\mathbf{n}_k$, given in [22, Theorem 3.4]. Marginally, for each k, $\mathbf{n}_k$ is a Poisson point process on $\operatorname{ri} \mathcal{U}$ with intensity measure

$$\lambda(]\zeta, \eta]) = \lambda_k(]\zeta, \eta]) = \mathbb{E}[\mathbf{n}_k(]\zeta, \eta])] = \int_{\alpha(\zeta)}^{\alpha(\eta)} \frac{ds}{s} = \log \frac{\alpha(\eta)}{\alpha(\zeta)}. \tag{9.1}$$

In particular, almost every realization of $\mathbf{n}_k$ satisfies $\mathbf{n}_k[\zeta, \eta] < \infty$ for all $\zeta \prec \eta$ in $\operatorname{ri} \mathcal{U}$.

Create a marked Poisson process by attaching an independent $\operatorname{Exp}(\alpha(\xi))$-distributed weight Y_ξ to each point ξ in the support of $\mathbf{n}_k$. Then the distribution of $\boldsymbol{\nu}_k$ is that of the purely atomic measure defined by

$$\boldsymbol{\nu}_k(]\zeta, \eta]\,) = \sum_{\xi \in \operatorname{ri} \mathcal{U}:\, \mathbf{n}_k(\xi)=1} Y_\xi \, \mathbb{1}_{]\zeta,\eta]}(\xi) \quad \text{for } \zeta \prec \eta \text{ in } \operatorname{ri} \mathcal{U}. \tag{9.2}$$

The random variable $\boldsymbol{\nu}_k(]\zeta, \eta])$ has distribution $\operatorname{Ber}(1 - \frac{\alpha(\zeta)}{\alpha(\eta)}) \otimes \operatorname{Exp}(\alpha(\zeta))$ (product of a Bernoulli and an independent exponential) and expectation

$$\mathbb{E}[\boldsymbol{\nu}_k(]\zeta, \eta])] = \frac{1}{\alpha(\zeta)} - \frac{1}{\alpha(\eta)}. \tag{9.3}$$

Note the following technical point. The jumps of $B_k^{\xi\pm}$ concentrate at e_2 and $B_k^{e_2-} = \infty$. To define $\boldsymbol{\nu}$ and $\mathbf{n}$ as locally finite measures, the standard Euclidean topology of $\operatorname{ri} \mathcal{U}$ has to be metrized so that $]e_2, \eta]$ is an unbounded set for any $\eta \succ e_2$. This point makes no difference to our calculations and we already encountered this issue around definition (3.1) of the Busemann measures. With this convention we can regard $\mathbf{n} = \sum_k \mathbf{n}_k$ as a simple point process on $\mathbb{Z} \times \operatorname{ri} \mathcal{U}$ with mean measure $\widetilde{\lambda} =$ (counting measure on $\mathbb{Z}$) $\otimes\, \lambda$.

For $(k, \xi) \in \mathbb{Z} \times \operatorname{ri} \mathcal{U}$, let $Q_{(k,\xi)}$ be the Palm kernel of $\boldsymbol{\nu}$ with respect to $\mathbf{n}$. That is, $Q_{(k,\xi)}$ is the stochastic kernel from $\mathbb{Z} \times \operatorname{ri} \mathcal{U}$ into $\mathcal{M}_{\mathbb{Z} \times \operatorname{ri} \mathcal{U}}$ that gives the distribution of $\boldsymbol{\nu}$, conditional on $\mathbf{n}$ having a point at (k, ξ), understood in the Palm sense. Rigorously, the kernel is defined by disintegrating the Campbell measure of the pair $(\mathbf{n}, \boldsymbol{\nu})$ with respect to the mean measure $\widetilde{\lambda}$ of $\mathbf{n}$ (this is developed in [39, Section 6.1]): for any non-negative Borel function $f : (\mathbb{Z} \times \operatorname{ri} \mathcal{U}) \times \mathcal{M}_{\mathbb{Z} \times \operatorname{ri} \mathcal{U}} \to \mathbb{R}_+$,

$$\begin{aligned} &\mathbb{E}\left[\int_{\mathbb{Z} \times \operatorname{ri} \mathcal{U}} f(k, \xi, \boldsymbol{\nu})\, \mathbf{n}(dk \otimes d\xi)\right] \\ &\qquad = \int_{\mathbb{Z} \times \operatorname{ri} \mathcal{U}} \int_{\mathcal{M}_{\mathbb{Z} \times \operatorname{ri} \mathcal{U}}} f(k, \xi, \nu)\, Q_{(k,\xi)}(d\nu)\, \widetilde{\lambda}(dk \otimes d\xi). \end{aligned} \tag{9.4}$$

Now we consider the indices $\tau^\xi(i) = \tau^{\xi,\xi}(i)$ of jumps at ξ, defined in (5.1). In terms of the random measures introduced above, for $(k,\xi) \in \mathbb{Z} \times \operatorname{ri} \mathcal{U}$,

$$\boldsymbol{\nu}\{(k,\xi)\} > 0 \iff \mathbf{n}\{(k,\xi)\} = 1 \iff B_k^{\xi-} > B_k^{\xi+} \iff k \in \{\tau^\xi(i) : i \in \mathbb{Z}\}.$$

We condition on the event $\{\mathbf{n}(0,\xi) = 1\}$, in other words, consider the distribution of $\{\tau^\xi(i)\}$ under $Q_{(0,\xi)}$. For this to be well-defined, we define these functions also on the space $\mathcal{M}_{\mathbb{Z}\times\operatorname{ri}\mathcal{U}}$ in the obvious way: for $\nu \in \mathcal{M}_{\mathbb{Z}\times\operatorname{ri}\mathcal{U}}$, the $\mathbb{Z} \cup \{\pm\infty\}$-valued functions $\tau^\xi(i) = \tau^\xi(i,\nu)$ are defined by the order requirement

$$\cdots < \tau^\xi(-1,\nu) < 0 \le \tau^\xi(0,\nu) < \tau^\xi(1,\nu) < \cdots$$

and the condition

$$\text{for } k \in \mathbb{Z}, \quad \nu\{(k,\xi)\} > 0 \quad \text{if and only if} \quad k \in \{\tau^\xi(i,\nu) : i \in \mathbb{Z}\}.$$

Since $\boldsymbol{\nu}$ is $\mathbb{P}$-almost surely a purely atomic measure, it follows from general theory that $Q_{(0,\xi)}$ is also supported on such measures. Furthermore, the conditioning itself forces $Q_{(0,\xi)}\{\nu : \tau^\xi(0,\nu) = 0\} = 1$. Thus the random integer points $\tau^\xi(i,\nu)$ are not all trivially $\pm\infty$ under $Q_{(0,\xi)}$. Connecting back to the notation of Section 5, for each $k \in \mathbb{Z}$, $\xi \in \operatorname{ri}\mathcal{U}$, each finite $A \subset \mathbb{Z}$ and $n_i \in \mathbb{Z}_+, r_i \in \mathbb{R}_+$ with $i \in A$, the Palm kernel introduced in that section is defined by

$$\begin{aligned}
&\mathbb{P}\{\tau^\xi(i+1) - \tau^\xi(i) = n_i,\, B^{\xi-}_{\tau^\xi(i)} - B^{\xi+}_{\tau^\xi(i)} > r_i \ \forall i \in A \,\|\, B_k^{\xi-} > B_k^{\xi+}\} \\
&\quad = Q_{(k,\xi)}\big\{\nu : \tau^\xi(i+1,\nu) - \tau^\xi(i,\nu) = n_i,\, \nu\{(\tau^\xi(i,\nu),\xi)\} > r_i \ \forall i \in A\big\}. \qquad (9.5)
\end{aligned}$$

9.2. *Statistics of instability points*

We turn to the proofs of the theorems of Section 5. These proofs make use of results from Appendices C and D.

Proof of Theorem 5.1. By Corollary C.2, the process $\{B_k^\zeta - B_k^\eta\}_{k\in\mathbb{Z}}$ has the same distribution as $\{W_k^+\}_{k\in\mathbb{Z}}$ defined in (D.6). An application of the appropriate mapping to these sequences produces the sequence $\{B_0^\zeta - B_0^\eta,\ \tau^{\zeta,\eta}(i+1) - \tau^{\zeta,\eta}(i),\ B^\zeta_{\tau^{\zeta,\eta}(i)} - B^\eta_{\tau^{\zeta,\eta}(i)} : i \in \mathbb{Z}\}$ that appears in Theorem 5.1 and the sequence $\{W_0^+,\ \sigma_{i+1} - \sigma_i,\ W^+_{\sigma_i} : i \in \mathbb{Z}\}$ that appears in Theorem D.2. Hence these sequences also have identical distribution. (We have $W^+_{\sigma_i} = W_{\sigma_i}$ by (D.9).) The distributions remain equal when these sequences are conditioned on the positive probability events $B_0^\zeta - B_0^\eta > 0$ and $W_0^+ > 0$. ∎

It will be convenient to have notation for the conditional joint distribution that appears in (5.3) in Theorem 5.1. For $0 < \alpha \le \beta \le 1$ define probability distributions $q^{\alpha,\beta}$ on the product space $\mathbb{Z}^\mathbb{Z} \times [0,\infty)^\mathbb{Z}$ as follows. Denote the generic variables on this product space by $(\{\tau_i\}_{i\in\mathbb{Z}}, \{\Delta_k\}_{k\in\mathbb{Z}})$ with $\tau_i \in \mathbb{Z}$ and $0 \le \Delta_k < \infty$. Given an integer $L > 0$, integers $n_{-L} < \cdots < n_{-2} < n_{-1} < n_0 = 0 < n_1 < n_2 < \cdots < n_L$, and positive reals $r_{-L}, \ldots, r_L$, abbreviate $b_i = n_{i+1} - n_i$. The measure $q^{\alpha,\beta}$ is defined by

$$q^{\alpha,\beta}\big\{\tau_i = n_i \text{ and } \Delta_{n_i} > r_i \text{ for } i \in [\![-L, L]\!],\ \Delta_k = 0 \text{ for } k \in [\![n_{-L}, n_L]\!] \setminus \{n_j\}_{j\in[\![-L,L]\!]}\big\}$$
$$= \bigg(\prod_{i=-L}^{L-1} C_{b_i-1} \frac{\alpha^{b_i-1}\beta^{b_i}}{(\alpha+\beta)^{2b_i-1}}\bigg) \cdot \bigg(\prod_{i=-L}^{L} e^{-\alpha r_i}\bigg). \tag{9.6}$$

To paraphrase the definition, the following holds under $q^{\alpha,\beta}$: $\tau_0 = 0$, $\Delta_k = 0$ for $k \notin \{\tau_i\}_{i\in\mathbb{Z}}$, and the variables $\{\tau_{i+1} - \tau_i, \Delta_{\tau_i}\}_{i\in\mathbb{Z}}$ are mutually independent with marginal distribution

$$q^{\alpha,\beta}\{\tau_{i+1} - \tau_i = n, \Delta_{\tau_i} > r\} = C_{n-1} \frac{\alpha^{n-1}\beta^n}{(\alpha+\beta)^{2n-1}} e^{-\alpha r} \quad \text{for } i \in \mathbb{Z},\ n \geq 1,\ r \geq 0. \tag{9.7}$$

Abbreviate $q^\alpha = q^{\alpha,\alpha}$ which has marginal $q^\alpha\{\tau_{i+1} - \tau_i = n, \Delta_{\tau_i} > r\} = C_{n-1}(\frac{1}{2})^{2n-1} e^{-\alpha r}$. As $\beta \to \alpha$, $q^{\alpha,\beta}$ converges weakly to q^α.

Theorem 5.1 can now be restated by saying that, conditional on $B_0^\zeta > B_0^\eta$, the variables

$$(\{\tau^{\zeta,\eta}(i)\}_{i\in\mathbb{Z}}, \{B_k^\zeta - B_k^\eta\}_{k\in\mathbb{Z}})$$

have joint distribution $q^{\alpha(\zeta),\alpha(\eta)}$. Consequently, for a measurable set $A \subset \mathbb{Z}^{\mathbb{Z}} \times [0,\infty)^{\mathbb{Z}}$,

$$\begin{aligned}
&\mathbb{P}[B_0^\zeta > B_0^\eta, (\{\tau^{\zeta,\eta}(i)\}_{i\in\mathbb{Z}}, \{B_k^\zeta - B_k^\eta\}_{k\in\mathbb{Z}}) \in A] \\
&\quad = \mathbb{P}(B_0^\zeta > B_0^\eta)\, \mathbb{P}[(\{\tau^{\zeta,\eta}(i)\}_{i\in\mathbb{Z}}, \{B_k^\zeta - B_k^\eta\}_{k\in\mathbb{Z}}) \in A \mid B_0^\zeta > B_0^\eta] \\
&\quad = \frac{\alpha(\eta) - \alpha(\zeta)}{\alpha(\eta)} \cdot q^{\alpha(\zeta),\alpha(\eta)}(A).
\end{aligned} \tag{9.8}$$

The first probability on the last line came from (C.6) and the second from Theorem 5.1.

Proof of Theorem 5.2. Define $\mathbb{Z} \cup \{\pm\infty\}$-valued ordered indices $\cdots < \tau_{-1}^{\zeta,\eta} < 0 \leq \tau_0^{\zeta,\eta} < \tau_1^{\zeta,\eta} < \cdots$ as measurable functions of a locally finite measure $\nu \in \mathcal{M}_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}}$ by the rule

$$\nu(\{k\} \times [\zeta, \eta]) > 0 \iff k \in \{\tau_i^{\zeta,\eta} : i \in \mathbb{Z}\}. \tag{9.9}$$

If $\nu(\{k\} \times [\zeta, \eta]) > 0$ does not hold for infinitely many $k > 0$ then $\tau_i^{\zeta,\eta} = \infty$ for large enough i, and analogously for $k < 0$. Definition (9.9) applied to the random measure $\boldsymbol{\nu} = \sum_k \boldsymbol{\nu}_k$ reproduces (5.1).

Fix integers $K, N \in \mathbb{N}$ and $\ell_{-N} \leq \cdots \leq \ell_{-1} \leq \ell_0 = 0 \leq \ell_1 \leq \cdots \leq \ell_N$ and strictly positive reals $r_{-K}, \ldots, r_K$. Define the event

$$\begin{aligned}
H^{\zeta,\eta} = H(\zeta, \eta) = &\bigcap_{1\leq i\leq N} \{\nu : \tau_{-i}^{\zeta,\eta} \leq \ell_{-i} \text{ and } \tau_i^{\zeta,\eta} \geq \ell_i\} \\
&\cap \bigcap_{-K\leq k\leq K} \{\nu : \nu(\{k\} \times]\zeta, \eta]) < r_k\}
\end{aligned} \tag{9.10}$$

on the space $\mathcal{M}_{\mathbb{Z}\times \mathrm{ri}\,\mathcal{U}}$. Note the monotonicity

$$H^{\zeta,\eta} \subset H^{\zeta',\eta'} \quad \text{for } [\zeta',\eta'] \subset [\zeta,\eta]. \tag{9.11}$$

Abbreviate $H^{\xi} = H^{\xi,\xi}$. Recall the measures $q^{\alpha,\beta}$ defined in (9.6). The analogous event under the measures $q^{\alpha,\beta}$ on the space $\mathbb{Z}^{\mathbb{Z}} \times [0,\infty)^{\mathbb{Z}}$ is denoted by

$$\begin{aligned} H_q = \big\{(\{\tau_i\}_{i\in\mathbb{Z}}, \{\Delta_k\}_{k\in\mathbb{Z}}) \in \mathbb{Z}^{\mathbb{Z}} \times [0,\infty)^{\mathbb{Z}} : \tau_{-i} \le \ell_{-i} \text{ and } \tau_i \ge \ell_i \text{ for } i \in [\![1,N]\!], \\ \Delta_k < r_k \text{ for } k \in [\![-K,K]\!]\big\}. \end{aligned} \tag{9.12}$$

Fix $\zeta \prec \eta$ in ri $\mathcal{U}$. We prove the theorem by showing that

$$Q_{(0,\xi)}(H^{\xi}) = q^{\alpha(\xi)}(H_q) \quad \text{for Lebesgue-almost every } \xi \in\,]\zeta,\eta]. \tag{9.13}$$

This equality comes from separate arguments for upper and lower bounds.

Upper bound proof. Define a sequence of nested partitions $\zeta = \zeta_0^n \prec \zeta_1^n \prec \cdots \prec \zeta_n^n = \eta$. For each n and $\xi \in\,]\zeta,\eta]$, let $]\zeta^n(\xi),\eta^n(\xi)]$ denote the unique interval $]\zeta_i^n,\zeta_{i+1}^n]$ that contains ξ. Assume that, as $n \nearrow \infty$, the mesh size $\max_i |\zeta_{i+1}^n - \zeta_i^n|$ tends to 0. Consequently, for each $\xi \in\,]\zeta,\eta]$, the intervals $]\zeta^n(\xi),\eta^n(\xi)]$ decrease to the singleton $\{\xi\}$.

The key step of this upper bound proof is that for all m and i and Lebesgue-a.e. $\xi \in\,]\zeta,\eta]$,

$$Q_{(0,\xi)}(H^{\zeta_i^m,\zeta_{i+1}^m}) = \lim_{n\to\infty} \mathbb{P}\{\boldsymbol{\nu} \in H^{\zeta_i^m,\zeta_{i+1}^m} \mid \mathbf{n}_0(]\zeta^n(\xi),\eta^n(\xi)]) \ge 1\}. \tag{9.14}$$

This limit is a special case of Theorem 6.32 (iii) in Kallenberg [39], for the simple point process $\mathbf{n}$ and the sets $B_n = \{0\} \times (\zeta^n(\xi),\eta^n(\xi)] \searrow \{(0,\xi)\}$. The proof of [38, Theorem 12.8] can also be used to establish this limit; the result of [38] by itself is not quite suitable because we use the Palm kernel for the measure $\boldsymbol{\nu}$ which is not the same as $\mathbf{n}$.

If we take $\xi \in\,]\zeta_i^m,\zeta_{i+1}^m]$, then for $n \ge m$, $]\zeta^n(\xi),\eta^n(\xi)] \subset\,]\zeta^m(\xi),\eta^m(\xi)] =\,]\zeta_i^m,\zeta_{i+1}^m]$. Considering all ξ in the union $]\zeta,\eta] = \bigcup_i\,]\zeta_i^m,\zeta_{i+1}^m]$, for any fixed m and Lebesgue-a.e. $\xi \in\,]\zeta,\eta]$ we have

$$\begin{aligned} Q_{(0,\xi)}(H^{\zeta^m(\xi),\eta^m(\xi)}) &= \lim_{n\to\infty} \mathbb{P}\{\boldsymbol{\nu} \in H^{\zeta^m(\xi),\eta^m(\xi)} \mid \mathbf{n}_0(]\zeta^n(\xi),\eta^n(\xi)]) \ge 1\} \\ &\le \varliminf_{n\to\infty} \mathbb{P}\{\boldsymbol{\nu} \in H^{\zeta^n(\xi),\eta^n(\xi)} \mid \mathbf{n}_0(]\zeta^n(\xi),\eta^n(\xi)]) \ge 1\}. \end{aligned}$$

The inequality is due to (9.11).

Interpreting (9.8) in terms of the random measures $\boldsymbol{\nu}$ and $\mathbf{n}$ and referring to (9.10) and (9.12) gives the identity

$$\mathbb{P}\{\boldsymbol{\nu} \in H^{\zeta^n(\xi),\eta^n(\xi)} \mid \mathbf{n}_0(]\zeta^n(\xi),\eta^n(\xi)]) \ge 1\} = q^{\alpha(\zeta^n(\xi)),\alpha(\eta^n(\xi))}(H_q).$$

As $(\zeta^n(\xi),\eta^n(\xi)] \searrow \{\xi\}$, the parameters converge: $\alpha(\zeta^n(\xi)), \alpha(\eta^n(\xi)) \to \alpha(\xi)$. Consequently, the distribution $q^{\alpha(\zeta^n(\xi)),\alpha(\eta^n(\xi))}$ converges to $q^{\alpha(\xi)}$. Hence

$$\lim_{n\to\infty} \mathbb{P}\{\boldsymbol{\nu} \in H^{\zeta^n(\xi),\eta^n(\xi)} \mid \mathbf{n}_0(]\zeta^n(\xi),\eta^n(\xi)]) \ge 1\} = q^{\alpha(\xi)}(H_q).$$

In summary, for all m and Lebesgue-a.e. $\xi \in]\zeta, \eta]$ we have

$$Q_{(0,\xi)}(H^{\zeta^m(\xi),\eta^m(\xi)}) \le q^{\alpha(\xi)}(H_q).$$

Let $m \nearrow \infty$ so that $H^{\zeta^m(\xi),\eta^m(\xi)} \nearrow H^\xi$, to obtain the upper bound

$$Q_{(0,\xi)}(H^\xi) \le q^{\alpha(\xi)}(H_q) \tag{9.15}$$

for Lebesgue-a.e. $\xi \in]\zeta, \eta]$.

Lower bound proof. Let $\zeta = \zeta_0 \prec \zeta_1 \prec \cdots \prec \zeta_\ell = \eta$ be a partition of the interval $[\zeta, \eta]$ and set $\alpha_j = \alpha(\zeta_j)$.

In order to get an estimate below, let $\mathbf{m} = (m_i)_{1\le|i|\le N}$ be a $2N$-vector of integers such that $m_i < \ell_i$ for $-N \le i \le -1$ and $m_i > \ell_i$ for $1 \le i \le N$. Define the subset $H_q^{\mathbf{m}}$ of H_q from (9.12) by truncating the coordinates τ_i:

$$\begin{aligned} H_q^{\mathbf{m}} = \big\{(\{\tau_i\}_{i\in\mathbb{Z}}, \{\Delta_k\}_{k\in\mathbb{Z}}) \in \mathbb{Z}^{\mathbb{Z}} \times [0,\infty)^{\mathbb{Z}} : m_{-i} \le \tau_{-i} \le \ell_{-i} \text{ and } \ell_i \le \tau_i \le m_i \\ \text{for } i \in [\![1, N]\!],\ \Delta_k < r_k \text{ for } k \in [\![-K, K]\!]\big\}. \end{aligned} \tag{9.16}$$

On the last line in the following computation, c_1 is a constant that depends on the parameters $\alpha(\zeta)$ and $\alpha(\eta)$ and on the quantities in (9.16):

$$\begin{aligned} \int_{]\zeta,\eta]} Q_{(0,\xi)}(H^\xi)\,\lambda_0(d\xi) &= \sum_{j=0}^{\ell-1} \int_{]\zeta_j,\zeta_{j+1}]} Q_{(0,\xi)}(H^\xi)\,\lambda_0(d\xi) \\ &\ge \sum_{j=0}^{\ell-1} \int_{]\zeta_j,\zeta_{j+1}]} Q_{(0,\xi)}(H^{\zeta_j,\zeta_{j+1}})\,\lambda_0(d\xi) = \sum_{j=0}^{\ell-1} \mathbb{E}[\mathbf{n}_0(]\zeta_j,\zeta_{j+1}]) \cdot \mathbb{1}_{H^{\zeta_j,\zeta_{j+1}}}(\boldsymbol{\nu})] \\ &\ge \sum_{j=0}^{\ell-1} \mathbb{P}\{\mathbf{n}_0(]\zeta_j,\zeta_{j+1}]) \ge 1,\ \boldsymbol{\nu} \in H^{\zeta_j,\zeta_{j+1}}\} \\ &= \sum_{j=0}^{\ell-1} \frac{\alpha_{j+1}-\alpha_j}{\alpha_{j+1}} q^{\alpha_j,\alpha_{j+1}}(H_q) \ \ge\ \sum_{j=0}^{\ell-1} \frac{\alpha_{j+1}-\alpha_j}{\alpha_{j+1}} q^{\alpha_j,\alpha_{j+1}}(H_q^{\mathbf{m}}) \\ &\ge \sum_{j=0}^{\ell-1} \frac{\alpha_{j+1}-\alpha_j}{\alpha_{j+1}} q^{\alpha_{j+1}}(H_q^{\mathbf{m}}) \cdot (1 - c_1(\alpha_{j+1}-\alpha_j)). \end{aligned}$$

The steps above come as follows. The second equality uses the characterization (9.4) of the kernel $Q_{(0,\xi)}$. The third equality is from (9.8). The second last inequality is from $H_q^{\mathbf{m}} \subset H_q$. The last inequality is from Lemma 9.1 below, which is valid once the mesh size $\max(\alpha_{j+1}-\alpha_j)$ is small enough relative to the numbers $\{m_i, \ell_i\}$.

The function $\alpha \mapsto q^\alpha(H_q^{\mathbf{m}})$ is continuous in the Riemann sum approximation on the last line of the calculation above. Let $\max(\alpha_{j+1}-\alpha_j) \to 0$ to obtain the inequality

$$\int_{]\zeta,\eta]} Q_{(0,\xi)}(H^\xi)\,\lambda_0(d\xi) \ge \int_{\alpha(\zeta)}^{\alpha(\eta)} q^\alpha(H_q^{\mathbf{m}})\,\frac{d\alpha}{\alpha} = \int_{]\zeta,\eta]} q^{\alpha(\xi)}(H_q^{\mathbf{m}})\,\lambda_0(d\xi).$$

Let $m_i \searrow -\infty$ for $-N \le i \le -1$ and $m_i \nearrow \infty$ for $1 \le i \le N$. The above turns into

$$\int_{]\zeta,\eta]} Q_{(0,\xi)}(H^\xi)\,\lambda_0(d\xi) \ge \int_{]\zeta,\eta]} q^{\alpha(\xi)}(H_q)\,\lambda_0(d\xi). \tag{9.17}$$

The upper bound (9.15) and the lower bound (9.17) together imply (9.13). ∎

The proof of Theorem 5.2 is complete once we verify the auxiliary lemma used in the calculation above.

Lemma 9.1. *Let the event $H_q^{\mathbf{m}}$ be as defined in* (9.16)*. Fix $0 < \underline{\alpha} < \overline{\alpha} < 1$. Then there exist constants $\varepsilon, c_1 \in (0,\infty)$ such that*

$$q^{\alpha,\beta}(H_q^{\mathbf{m}}) \ge q^\beta(H_q^{\mathbf{m}}) \cdot (1 - c_1(\beta-\alpha))$$

for all $\alpha, \beta \in [\underline{\alpha}, \overline{\alpha}]$ such that $\alpha \le \beta \le \alpha + \varepsilon$. The constants $\varepsilon, c_1 \in (0,\infty)$ depend on $\underline{\alpha}$, $\overline{\alpha}$, and the parameters ℓ_i, m_i and r_k in (9.16)*.*

Proof. Let

$$\begin{aligned}\mathcal{A} = \bigl\{\mathbf{p} = (p_i)_{-N\le i\le N} \in \mathbb{Z}^{2N+1} : p_0 = 0,\ p_i < p_j \text{ for } i < j,\\ m_{-i} \le p_{-i} \le \ell_{-i} \text{ and } \ell_i \le p_i \le m_i\ \forall i \in [\![1,N]\!]\bigr\}\end{aligned}$$

be the relevant finite set of integer-valued $(2N+1)$-vectors for the decomposition below. For each $\mathbf{p} \in \mathcal{A}$ let $\mathcal{K}(\mathbf{p}) = \{p_i : i \in [\![-N,N]\!],\ p_i \in [\![-K,K]\!]\}$ be the set of coordinates of $\mathbf{p}$ in $[\![-K,K]\!]$. Abbreviate $b_i = p_{i+1} - p_i$. Recall that, under $q^{\alpha,\beta}$, $\tau_0 = 0$ and $\Delta_k < r_k$ holds with probability 1 if $k \notin \{\tau_i\}$; recall also the independence in (9.7). The factors $d_k > 0$ below that satisfy $1 - e^{-\alpha r_k} \ge (1 - e^{-\beta r_k})(1 - d_k(\beta - \alpha))$ can be chosen uniformly for $\alpha \le \beta$ in $[\underline{\alpha}, \overline{\alpha}]$, as functions of $\underline{\alpha}$, $\overline{\alpha}$, and $\{r_k\}$. Now compute:

$$\begin{aligned}
&q^{\alpha,\beta}(H_q^{\mathbf{m}})\\
&\quad= q^{\alpha,\beta}\{m_i \le \tau_{-i} \le \ell_{-i} \text{ and } \ell_i \le \tau_i \le m_i \text{ for } i \in [\![1,N]\!],\ \Delta_k < r_k \text{ for } k \in [\![-K,K]\!]\}\\
&\quad= \sum_{\mathbf{p}\in\mathcal{A}} q^{\alpha,\beta}\{\tau_i = p_i \text{ for } i \in [\![-N,N]\!],\ \Delta_k < r_k \text{ for } k \in [\![-K,K]\!]\}\\
&\quad= \sum_{\mathbf{p}\in\mathcal{A}} q^{\alpha,\beta}\{\tau_{i+1} - \tau_i = b_i \text{ for } i \in [\![-N,N-1]\!]\} \cdot \prod_{k\in\mathcal{K}(\mathbf{p})} (1 - e^{-\alpha r_k})\\
&\quad\ge \sum_{\mathbf{p}\in\mathcal{A}} \Biggl(\prod_{i=-N}^{N-1} C_{b_i-1} \frac{\alpha^{b_i-1}\beta^{b_i}}{(\alpha+\beta)^{2b_i-1}}\Biggr) \cdot \prod_{k\in\mathcal{K}(\mathbf{p})} (1 - e^{-\beta r_k})(1 - d_k(\beta-\alpha))\\
&\quad\ge \sum_{\mathbf{p}\in\mathcal{A}} \Biggl(\prod_{i=-N}^{N-1} C_{b_i-1} \Bigl(\frac{1}{2}\Bigr)^{2b_i-1}\Biggr)\Biggl(\prod_{k\in\mathcal{K}(\mathbf{p})} (1 - e^{-\beta r_k})\Biggr) \cdot (1 - c_1(\beta-\alpha))\\
&\quad= \sum_{\mathbf{p}\in\mathcal{A}} q^\beta\{\tau_{i+1} - \tau_i = b_i \text{ for } i \in [\![-N,N-1]\!],\ \Delta_k < r_k \text{ for } k \in [\![-K,K]\!]\}\\
&\qquad\cdot (1 - c_1(\beta-\alpha))\\
&\quad= q^\beta(H_q^{\mathbf{m}}) \cdot (1 - c_1(\beta-\alpha)).
\end{aligned}$$

To get the inequality above, (i) apply Lemma B.2 to the first factor in parentheses with ε chosen so that $0 < \varepsilon < \underline{\alpha}/b_i$ for all $\mathbf{p} \in \mathcal{A}$, and (ii) set $c_1 = \sum_{k=-K}^{K} d_k$. ■

In the proofs that follow, we denote the indicators of the locations of the positive atoms of a measure $\nu \in \mathcal{M}_{\mathbb{Z}\times \mathrm{ri}\,\mathcal{U}}$ by $u_k(\nu,\xi) = u_k(\xi) = \mathbb{1}[\nu\{(k,\xi)\} > 0]$ for $(k,\xi) \in \mathbb{Z} \times \mathrm{ri}\,\mathcal{U}$. Applied to the random measure $\boldsymbol{\nu}$, this gives $u_k(\boldsymbol{\nu},\xi) = \mathbf{n}_k(\xi)$.

Lemma 9.2. *For Lebesgue-almost every $\xi \in \mathrm{ri}\,\mathcal{U}$ and all $m \in \mathbb{Z}$,*

$$Q_{(m,\xi)}[\nu : \{u_{m+k}(\nu,\xi)\}_{k\in\mathbb{Z}} \in A] = \mathbf{P}(A) \tag{9.18}$$

for all Borel sets $A \subset \{0,1\}^{\mathbb{Z}}$.

Proof. For $m = 0$, (9.18) comes from a comparison of (5.4) and (5.5). For general m it then follows from the shift-invariance of the weights ω. ■

Proof of Theorem 5.3. Take $A \subset \{0,1\}^{\mathbb{Z}}$ as in the statement of Theorem 5.3. Fix $\zeta \prec \eta$ in $\mathrm{ri}\,\mathcal{U}$ and let $N \in \mathbb{N}$. We restrict the integrals below to the compact set $[-N,N]\times[\zeta,\eta]$ with the indicator

$$g(k,\xi,\nu) = \mathbb{1}_{[-N,N]\times[\zeta,\eta]}(k,\xi)$$

and then define on $\mathbb{Z} \times \mathrm{ri}\,\mathcal{U} \times \mathcal{M}_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}}$,

$$f(k,\xi,\nu) = g(k,\xi,\nu)\cdot \mathbb{1}_{\{(u_\ell\{\xi\}:\,\ell\in\mathbb{Z})\in A\}}(\xi,\nu).$$

By the definition (9.4) of the Palm kernel,

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}} f(k,\xi,\boldsymbol{\nu})\,\mathbf{n}(dk\otimes d\xi)\Big] \\
&\qquad = \int_{[-N,N]\times[\zeta,\eta]} Q_{(k,\xi)}\{(u_\ell\{\xi\} : \ell\in\mathbb{Z}) \in A\}\,\widetilde{\lambda}(dk\otimes d\xi) \\
&\qquad = \int_{[-N,N]\times[\zeta,\eta]} Q_{(k,\xi)}\{(u_{k+\ell}\{\xi\} : \ell\in\mathbb{Z}) \in A\}\,\widetilde{\lambda}(dk\otimes d\xi) \\
&\qquad = \int_{[-N,N]\times[\zeta,\eta]} \widetilde{\lambda}(dk\otimes d\xi) = \mathbb{E}\Big[\int_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}} g(k,\xi,\boldsymbol{\nu})\,\mathbf{n}(dk\otimes d\xi)\Big].
\end{aligned}$$

The second equality uses shift-invariance of A and the third equality uses (9.18) and $\mathbf{P}(A) = 1$. The left-hand side and the right-hand side are both finite because the integrals are restricted to the compact set $[-N,N]\times[\zeta,\eta]$. Since $\mathbf{n}$ is a positive random measure, it follows that

$$\mathbb{P}\left(\int_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}} f(k,\xi,\boldsymbol{\nu})\,\mathbf{n}(dk\otimes d\xi) = \int_{\mathbb{Z}\times\mathrm{ri}\,\mathcal{U}} g(k,\xi,\boldsymbol{\nu})\,\mathbf{n}(dk\otimes d\xi)\right) = 1.$$

As ζ,η, and N were arbitrary, we conclude that $\mathbb{P}$-almost surely $(\mathbf{n}_\ell\{\xi\} : \ell\in\mathbb{Z}) \in A$ for all $(k,\xi) \in \mathbb{Z}\times\mathrm{ri}\,\mathcal{U}$ such that $\mathbf{n}\{(k,\xi)\} = 1$. Lemma 3.6 applied to the x-axis ($x_i = ie_1$) then shows that $\xi \in \mathcal{V}^{\omega}$ if and only if $\mathbf{n}\{(k,\xi)\} = 1$ for some k. ■

Lemma 9.3. *Assume* (3.7). *Then for any* $\delta \in (0,1)$, $n \in \mathbb{N}$, *and* $\zeta \in \operatorname{ri}\mathcal{U}$ *we have*

$$\mathbb{P}\big\{\exists \xi \in [\zeta, e_1[\,:\, \mathbf{n}([\![0,n]\!] \times \{\xi\}) > 2\delta n + 1\big\} \le 2(n+1)\bigg(\frac{(1-\delta/2)^{2-\delta}}{(1-\delta)^{1-\delta}}\bigg)^n \log \alpha(\zeta)^{-1}.$$

Proof. Let $\{\Delta_j\}_{j\in\mathbb{N}}$ be i.i.d. random variables with probability mass function $p(n) = C_{n-1}2^{1-2n}$ for $n \in \mathbb{N}$. For $k \in [\![0,n]\!]$ and $\xi \in \operatorname{ri}\mathcal{U}$ use a union bound, translation, and (5.4) to write

$$\begin{aligned}
Q_{(k,\xi)}\Big\{\sum_{i=0}^{n} u_i(\xi) > 2\delta n + 1\Big\} &\le Q_{(k,\xi)}\Big\{\sum_{i=k-n}^{k+n} u_i(\xi) > 2\delta n + 1\Big\} \\
&= Q_{(0,\xi)}\Big\{\sum_{i=-n}^{n} u_i(\xi) > 2\delta n + 1\Big\} \\
&\le Q_{(0,\xi)}\Big\{\sum_{i=1}^{n} u_i(\xi) > \delta n\Big\} + Q_{(0,\xi)}\Big\{\sum_{i=-n}^{-1} u_i(\xi) > \delta n\Big\} \le 2P\Big\{\sum_{j=1}^{\lceil \delta n\rceil} \Delta_j \le n\Big\}.
\end{aligned}$$

Using the generating function $f(s) = \sum_{n\ge 0} C_n s^n = \frac{1}{2}(1-\sqrt{1-4s})$ of Catalan numbers we obtain, for $0 < s < 1$,

$$\begin{aligned}
P\Big\{\sum_{j=1}^{\lceil \delta n\rceil} \Delta_j \le n\Big\} &\le s^{-n} E[s^{\Delta}]^{\delta n} = s^{-n}\Big(2\sum_{n=1}^{\infty} C_{n-1}\,(s/4)^n\Big)^{\delta n} \\
&= s^{-n}\bigg(\frac{s}{2}\sum_{k=0}^{\infty} C_k\,(s/4)^k\bigg)^{\delta n} = s^{-n}(1-\sqrt{1-s})^{\delta n}.
\end{aligned}$$

Take $s = \frac{4(1-\delta)}{(2-\delta)^2} < 1$ in the upper bound above to get

$$Q_{(k,\xi)}\Big\{\sum_{i=0}^{n} u_i(\xi) > 2\delta n + 1\Big\} \le 2\bigg(\frac{(1-\delta/2)^{2-\delta}}{(1-\delta)^{1-\delta}}\bigg)^n.$$

Apply (9.4) to write

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{\operatorname{ri}\mathcal{U}} \mathbb{1}\{\xi \in [\zeta, e_1[\} \cdot \mathbb{1}\big\{\mathbf{n}([\![0,n]\!] \times \{\xi\}) > 2\delta n + 1\big\}\, \mathbf{n}_k(d\xi)\bigg] \\
&\qquad = \int_{\operatorname{ri}\mathcal{U}} \mathbb{1}\{\xi \in [\zeta, e_1[\}\, Q_{(k,\xi)}\Big\{\sum_{i=0}^{n} u_i(\xi) > 2\delta n + 1\Big\}\, \lambda_k(d\xi) \\
&\qquad \le 2\bigg(\frac{(1-\delta/2)^{2-\delta}}{(1-\delta)^{1-\delta}}\bigg)^n \int_{\operatorname{ri}\mathcal{U}} \mathbb{1}\{\xi \in [\zeta, e_1[\}\lambda_k(d\xi) \\
&\qquad \overset{(9.1)}{=} 2\bigg(\frac{(1-\delta/2)^{2-\delta}}{(1-\delta)^{1-\delta}}\bigg)^n \log \alpha(\zeta)^{-1}.
\end{aligned}$$

To complete the proof, sum over $k \in [\![0,n]\!]$ and observe that

$$\int_{\mathrm{ri}\,\mathcal{U}} \mathbb{1}\{\xi \in [\zeta, e_1[\}\mathbb{1}\{\mathbf{n}([\![0,n]\!] \times \{\xi\}) > 2\delta n + 1\} \sum_{k=0}^{n} \mathbf{n}_k(d\xi) \geq \mathbb{1}\{\exists \xi \in [\zeta, e_1[\, : \mathbf{n}([\![0,n]\!] \times \{\xi\}) > 2\delta n + 1\}. \quad \blacksquare$$

Proof of Theorem 5.4. The result follows from Theorem 9.4 below and the observation that for any $\varepsilon > 0$, $\delta_n = 2\sqrt{n^{-1}\log n}$ satisfies the summability condition in that theorem. $\blacksquare$

Theorem 9.4. *Assume* (3.7) *and fix* $i \in \{1,2\}$. *Consider a sequence* $\delta_n \in (0,1)$ *with* $\sum n^2 e^{-n\delta_n^2} < \infty$. *Then for any* $\zeta \in \mathrm{ri}\,\mathcal{U}$,

$$\mathbb{P}\Big\{\exists n_0 : \forall \xi \in [\zeta, e_2[,\ \forall n \geq n_0 : \sum_{x \in [\![0,n]\!]^2} \mathbb{1}\{\xi \in \operatorname{supp} \mu_{x,x+e_i}\} \leq n^2\delta_n\Big\} = 1. \tag{9.19}$$

The same result holds when $[\![0,n]\!]^2$ *is replaced by any one of* $[\![-n,0]\!]^2$, $[\![0,n]\!] \times [\![-n,0]\!]$, *or* $[\![-n,0]\!] \times [\![0,n]\!]$.

Proof. Apply Lemma 9.3 and a union bound to deduce that for any $j \in \{1,2\}$, $\delta \in (0,1)$, $n \in \mathbb{N}$, and $\zeta \in \mathrm{ri}\,\mathcal{U}$,

$$\mathbb{P}\Big\{\exists \xi \in [\zeta, e_1[\, : \sum_{x \in [\![0,n]\!]^2} \rho_x^j(\xi) \geq (2\delta n + 1)(n+1)\Big\} \leq 2(n+1)^2 \bigg(\frac{(1-\delta/2)^{2-\delta}}{(1-\delta)^{1-\delta}}\bigg)^n \log \alpha(\zeta)^{-1}.$$

A Taylor expansion gives

$$\log\bigg(\frac{(1-\delta/2)^{2-\delta}}{(1-\delta)^{1-\delta}}\bigg) = -\delta^2/4 + \mathcal{O}(\delta^3).$$

Thus, we see that for any $\zeta \in \mathrm{ri}\,\mathcal{U}$, and any sequence $\delta_n \in (0,1)$ such that $\sum n^2 e^{-n\delta_n^2} < \infty$,

$$\mathbb{P}\Big\{\exists n_0 \in \mathbb{N} : \forall \xi \in [\zeta, e_1[,\ \forall n \geq n_0 : \sum_{x \in [\![0,n]\!]^2} \rho_x^j(\xi) \leq n^2\delta_n\Big\} = 1.$$

The result for the other three sums comes similarly. $\blacksquare$

Appendix A. The geometry of geodesics: previously known results

This appendix states the properties of Busemann functions, geodesics, and competition interfaces which were discussed informally in Section 2.2. Theorem A.1 introduces the *Busemann process* with its main properties. It combines results that follow from [36, Theorems 4.4 and 4.7, Lemmas 4.5 (c) and 4.6 (c), and Remark 4.11] and [27, Lemmas 4.7 and 5.1].

Theorem A.1 ([27, 36]). *Let $\mathbb{P}_0$ be a probability measure on $\mathbb{R}^{\mathbb{Z}^2}$ under which the coordinate projections are i.i.d., have positive variance, and have $p > 2$ finite moments. There exists a Polish probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with*

(1) *a group $T = \{T_x\}_{x\in\mathbb{Z}^2}$ of $\mathcal{F}$-measurable $\mathbb{P}$-preserving bijections $T_x : \Omega \to \Omega$,*

(2) *a family $\{\omega_x(\omega) : x \in \mathbb{Z}^2\}$ of real-valued random variables $\omega_x : \Omega \to \mathbb{R}$ such that $\omega_y(T_x\omega) = \omega_{x+y}(\omega)$ for all $x, y \in \mathbb{Z}^2$,*

(3) *real-valued measurable functions $B^{\xi+}(x, y, \omega) = B^{\xi+}_{x,y}(\omega)$ and $B^{\xi-}(x, y, \omega) = B^{\xi-}_{x,y}(\omega)$ of $(x, y, \omega, \xi) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \Omega \times \operatorname{ri}\mathcal{U}$,*

(4) *and T-invariant events $\Omega_0^1 \subset \Omega$ and $\Omega_\xi^1 \subset \Omega_0^1$ for each $\xi \in \operatorname{ri}\mathcal{U}$, with $\mathbb{P}(\Omega_0^1) = \mathbb{P}(\Omega_\xi^1) = 1$,*

such that properties (a)–(k) listed below hold:

(a) *$\{\omega_x : x \in \mathbb{Z}^2\}$ has distribution $\mathbb{P}_0$ under $\mathbb{P}$.*

(b) *For any $I \subset \mathbb{Z}^2$, the variables*

$$\{(\omega_x, B^{\xi\square}(x, y, \omega)) : x \in I,\ y \geq x,\ \square \in \{-,+\},\ \xi \in \operatorname{ri}\mathcal{U}\}$$

are independent of $\{\omega_x : x \in I^<\}$ where $I^< = \{x \in \mathbb{Z}^2 : x \not\geq z\ \forall z \in I\}$.

(c) *For each $\xi \in \operatorname{ri}\mathcal{U}$, $x, y \in \mathbb{Z}^2$, and $\square \in \{-,+\}$, $B^{\xi\square}(x, y)$ are integrable and (2.9) holds.*

(d) *For each $\omega \in \Omega$, $x, y \in \mathbb{Z}^2$, and $\square \in \{-,+\}$, if $\zeta, \eta \in \operatorname{ri}\mathcal{U}$ are such that $\nabla g(\zeta\square) = \nabla g(\eta\square)$, then $B^{\zeta\square}(x, y, \omega) = B^{\eta\square}(x, y, \omega)$.*

(e) *For each $\omega \in \Omega_0^1$, $x, y, z \in \mathbb{Z}^2$, $\xi \in \operatorname{ri}\mathcal{U}$, and $\square \in \{-,+\}$ properties (2.6)–(2.8) hold.*

(f) *For each $\omega \in \Omega_0^1$, monotonicity (2.10) holds.*

(g) *For each $\omega \in \Omega_0^1$, one-sided limits (2.11) hold.*

(h) *For each $\omega \in \Omega_0^1$ and each $x \in \mathbb{Z}^2$,*

$$B^{\xi\square}(x, x + e_i) \to \infty \quad \text{as } \xi \to e_{3-i},\ \text{ for } i \in \{1, 2\}. \tag{A.1}$$

(i) *If $\mathbb{P}(\omega_0 \leq r)$ is continuous in r, then for all $\xi \in \operatorname{ri}\mathcal{U}$, $\omega \in \Omega_\xi^1$, $x \in \mathbb{Z}^2$, and $\square \in \{-,+\}$,*

$$B^{\xi\square}(x, x + e_1) \neq B^{\xi\square}(x, x + e_2). \tag{A.2}$$

(j) *For all $\xi \in \operatorname{ri}\mathcal{U}$, $\omega \in \Omega_\xi^1$, and $\square \in \{-,+\}$,*

$$\lim_{n\to\infty} \max_{x\in n\mathcal{U}\cap\mathbb{Z}^2_+} n^{-1}|B^{\xi\square}(0, x) - x \cdot \nabla g(\xi\square)| = 0. \tag{A.3}$$

(k) *For all $\xi \in \mathcal{D}$, $\omega \in \Omega_\xi^1$, and $x, y \in \mathbb{Z}^2$,*

$$B^{\xi+}(x, y, \omega) = B^{\xi-}(x, y, \omega) = B^{\xi}(x, y, \omega). \tag{A.4}$$

(l) *If $\xi, \underline{\xi}, \overline{\xi} \in \mathcal{D}$ then for all $\omega \in \Omega_\xi^1$, the Busemann limit (2.5) holds.*

Remark A.2 (Weak limit construction of the Busemann process). Both articles [27, 36] on which we rely for Theorem A.1 construct the process B as a weak limit point of Cesàro averages of probability distributions of pre-limit objects. This gives existence of the process on a probability space Ω that is larger than the product space $\mathbb{R}^{\mathbb{Z}^2}$ of the i.i.d. weights $\{\omega_x\}$. Article [27] takes the outcome of the weak limit from existing literature in the form of a queueing fixed point, while [36] builds the weak limit from scratch by considering the distribution of increments of point-to-line passage times, following the approach introduced in [17].

To appeal to queueing literature, [27] assumes that $\mathbb{P}(\omega_0 \geq c) = 1$ for some real c. A payoff is that each process $\{B^{\xi\square}(x, y) : x, y \in \mathbb{Z}^2\}$ is ergodic under either shift T_{e_i} [27, Theorem 5.2 (i)]. The construction in [36] does not need the lower bound assumption but gives only the T-invariance stated above in Theorem A.1 (a).

Theorem A.4 below quotes results from [27] that were proved with the help of ergodicity. Remark A.5 explains how the required properties can be obtained without ergodicity.

Remark A.3 (Strong existence and ergodicity of the Busemann process). The regularity condition (2.4) is equivalent to the existence of a countable dense set $\mathcal{D}_0 \subset \mathcal{D}$ such that $\underline{\zeta}, \overline{\zeta} \in \mathcal{D}$ for each $\zeta \in \mathcal{D}_0$. When (2.4) holds, [28, Theorem 3.1] shows that for ζ in $\mathcal{D}_0$, $B^{\zeta}(x, y) = B^{\zeta\pm}(x, y)$ can be realized as an almost sure limit of $G_{x,v_n} - G_{y,v_n}$ when $v_n/n \to \xi$. The remaining values $B^{\xi\square}(x, y)$ can be obtained as left and right limits from $\{B^{\zeta}(x, y)\}_{\zeta\in\mathcal{D}_0}$ as $\zeta \to \xi$. This way the entire process $\{B^{\xi\square}(x, y) : x, y \in \mathbb{Z}^2, \xi \in \operatorname{ri}\mathcal{U}, \square \in \{-, +\}\}$ becomes a measurable function of the i.i.d. weights $\{\omega_x : x \in \mathbb{Z}^2\}$. We can take $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ and the Busemann process is ergodic under any shift T_x for $x \neq 0$.

We record a simple observation here, valid under the continuous i.i.d. weights assumption (1.1): there exists an event Ω_0^2 with $\mathbb{P}(\Omega_0^2) = 1$ such that for all $\omega \in \Omega_0^2$,

$$\begin{aligned}&\text{for every non-empty finite subset } I \subset \mathbb{Z}^2 \text{ and non-zero}\\ &\text{integer coefficients } \{a_x\}_{x\in I}, \text{ we have } \textstyle\sum_{x\in I} a_x\omega_x \neq 0.\end{aligned} \tag{A.5}$$

This condition implies the uniqueness of point-to-point geodesics mentioned under (2.1).

The following theorem summarizes previous knowledge of the structure of semi-infinite geodesics under assumption (1.1). These results were partly summarized in Section 2.3.

Theorem A.4 ([27, Theorems 2.1, 4.3, 4.5, and 4.6]). *There exist T-invariant events Ω_0^3 and $\Omega_\xi^3 \subset \Omega_0^3$ for each $\xi \in \operatorname{ri}\mathcal{U}$, with $\mathbb{P}(\Omega_0^3) = 1$, $\mathbb{P}(\Omega_\xi^3) = 1$, and such that the following hold:*

(a) *For every $\omega \in \Omega_0^3$ and all $x \in \mathbb{Z}^2$, $\square \in \{-, +\}$, and $\xi \in \operatorname{ri}\mathcal{U}$, $\boldsymbol{\gamma}^{x,\xi\square}$ is $\mathcal{U}_{\xi\square}$-directed, and every semi-infinite geodesic is $\mathcal{U}_\xi$-directed for some $\xi \in \mathcal{U}$.*

(b) *For every $\xi \in \operatorname{ri}\mathcal{U}$ and all $\omega \in \Omega_\xi^3$, $x, y \in \mathbb{Z}^2$, and $\square \in \{-, +\}$, $\boldsymbol{\gamma}^{x,\xi\square}$ and $\boldsymbol{\gamma}^{y,\xi\square}$ coalesce, i.e. there exists an integer $k \geq x \cdot \hat{e}_1 \vee y \cdot \hat{e}_1$ such that $\boldsymbol{\gamma}^{x,\xi\square}_{k,\infty} = \boldsymbol{\gamma}^{y,\xi\square}_{k,\infty}$.*

(c) *For all $\xi \in \operatorname{ri}\mathcal{U}$, $\omega \in \Omega_\xi^3$, $x \in \mathbb{Z}^2$, and $\square \in \{-, +\}$, there exist at most finitely many $z \in \mathbb{Z}^2$ such that $\boldsymbol{\gamma}^{z,\xi\square}$ goes through x.*

(d) *If g is strictly concave, then for any $\omega \in \Omega_0^3$ every semi-infinite geodesic is ξ-directed for some $\xi \in \mathcal{U}$.*

(e) *If $\xi \in \operatorname{ri}\mathcal{U}$ is such that $\mathcal{U}_\xi = [\underline{\xi}, \overline{\xi}]$ satisfies $\underline{\xi}, \xi, \overline{\xi} \in \mathcal{D}$, then for any $\omega \in \Omega_\xi^3$ and $x \in \mathbb{Z}^2$ we have $\boldsymbol{\gamma}^{x,\xi+} = \boldsymbol{\gamma}^{x,\xi-}$. This is the unique $\mathcal{U}_\xi$-directed semi-infinite geodesic out of x and, by part (b), all these geodesics coalesce. By part (c), there are no bi-infinite $\mathcal{U}_\xi$-directed geodesics.*

Remark A.5 (Ergodicity in the proof of Theorem A.4). As mentioned in Remark A.2, [27] uses ergodicity of cocycles. But the results quoted above in Theorem A.4 can be obtained with stationarity, which comes from [36] without the restrictive assumption $\omega_x \geq c$.

The proof of directedness (Theorem A.4 (a) above) given in [27, Theorem 4.3] uses the shape theorem of ergodic cocycles stated in [27, Theorem A.1]. This shape theorem also holds in the stationary setting, as stated above in (A.3). This result comes from [36, Theorem 4.4] and it is proved in detail in [35, Appendix B]. Now the proof of [27, Theorem 4.3] goes through line-by-line after switching its references and applying [36, Lemma 4.5 (c)] to identify the correct centering for the cocycle.

Similarly, the non-existence of directed bi-infinite geodesics (Theorem A.4 (c) above) proved in [27, Theorem 4.6] needs only stationarity after minor changes. Essentially the same argument is given in [36, Lemma 6.1] in positive temperature, assuming only stationarity.

We next record an easy consequence of the previous results, ruling out the existence of non-trivial semi-infinite geodesics which are either e_1- or e_2-directed.

Lemma A.6. *For $\omega \in \Omega_0^1 \cap \Omega_0^2 \cap \Omega_0^3$, if γ^x is a semi-infinite geodesic emanating from x with $\gamma_n^x/n \to e_i$ for some $i \in \{1,2\}$, then $\gamma^x = \boldsymbol{\gamma}^{x,e_i}$.*

Proof. We consider the case of $i = 1$, with the case of $i = 2$ being similar. Set $x \cdot \widehat{e} = k$ and fix a sequence $\zeta_n \in \operatorname{ri}\mathcal{U}$ with $\zeta_n \to e_1$ as $n \to \infty$. By Theorem A.4 (a), $\boldsymbol{\gamma}^{x,\zeta_n}$ is $\mathcal{U}_{\zeta_n}$-directed. [41, Theorem 2.4] implies that $e_1 \notin \mathcal{U}_{\zeta_n}$. Then, by (A.5), if γ^x is as in the statement, we must have $\boldsymbol{\gamma}_\ell^{x,\zeta_n} \preceq \gamma_\ell^x \preceq x + (k-\ell)e_1 = \boldsymbol{\gamma}_\ell^{x,e_1}$ for all $n \in \mathbb{N}$ and $\ell \geq k$. But Theorem A.1 (h) implies that for each fixed $\ell \geq k$, $\boldsymbol{\gamma}_\ell^{x,\zeta_n} = x + (k-\ell)e_1 = \boldsymbol{\gamma}_\ell^{x,e_1}$ holds for all large enough n. The result follows. ∎

Under the assumption that g is differentiable on $\operatorname{ri}\mathcal{U}$, Theorem A.4 (e) holds for all $\xi \in \operatorname{ri}\mathcal{U}$. An application of the Fubini–Tonelli theorem gives that the claims in Theorem A.4 (b, c) in fact hold on a single full $\mathbb{P}$-measure event simultaneously for Lebesgue-almost all directions $\xi \in \operatorname{ri}\mathcal{U}$. It is conjectured that the claim in part (c) holds in fact on a single full-measure event, simultaneously, for all $\xi \in \operatorname{ri}\mathcal{U}$.

The next result is a small extension of [27, Lemma 4.4], achieved by an application of the monotonicity in (2.13).

Theorem A.7. *Assume the regularity condition (2.4). Then for any $\omega \in \Omega_0^1 \cap \Omega_0^2$, condition (2.16) holds.*

The next theorem says that there are multiple geodesics that are directed in the same asymptotic direction ξ_* as the competition interface, which itself can be characterized using the Busemann process. See Figure 2.3.

Theorem A.8 ([27, (5.2) and Theorems 2.6, 2.8, and 5.3]). *There exists a T-invariant event Ω_0^4 such that $\mathbb{P}(\Omega_0^4) = 1$ and the following hold for all $\omega \in \Omega_0^4$:*

(a) *There exists a unique point $\xi_*(\omega) \in \operatorname{ri} \mathcal{U}$ such that (2.17) holds.*

(b) *For any $\xi \in \operatorname{ri} \mathcal{U}$, $\mathbb{P}(\xi_* = \xi) > 0$ if and only if $\xi \in (\operatorname{ri} \mathcal{U}) \setminus \mathcal{D}$.*

(c) *For any $\zeta \prec \eta$ in $\operatorname{ri} \mathcal{U}$ with $\nabla g(\zeta+) \neq \nabla g(\eta-)$, and any $x \in \mathbb{Z}^2$, there exists $y \geq x$ such that $\xi_*(T_y\omega) \in]\zeta, \eta[$. Consequently, any open interval outside the closed linear segments of g contains ξ_* with positive probability.*

(d) *For any $\xi \in (\operatorname{ri} \mathcal{U}) \setminus \mathcal{D}$ and for any $x \in \mathbb{Z}^2$, there exists $y \geq x$ such that $\xi_*(T_y\omega) = \xi$.*

If the regularity condition (2.4) holds then the following also hold:

(e) *We have the limit*

$$\xi_*(\omega) = \lim_{n\to\infty} n^{-1}\varphi_n^0(\omega). \tag{A.6}$$

(f) *$\xi_*(T_x\omega)$ is the unique direction ξ such that there are at least two $\mathcal{U}_\xi$-directed semi-infinite geodesics from x, namely $\boldsymbol{\gamma}^{x,\xi\pm}$, that separate at x and never intersect thereafter.*

Remark A.5 applies here as well. Ergodicity is invoked in the proofs of parts (b), (c) and (d) in [27, Theorem 5.3 (iii)–(iv)] to apply the cocycle shape theorem. In our stationary setting this can be replaced with the combination of [36, Theorem 4.4 and Lemma 4.5 (c)].

The following result for exponential weights, due to Coupier, states that there are no directions ξ with three ξ-directed geodesics emanating from the same site.

Theorem A.9 ([16, Theorem 1 (2)]). *Assume that under $\mathbb{P}$, the weights $\{\omega_x : x \in \mathbb{Z}^2\}$ are exponentially distributed i.i.d. random variables. Then there exists a T-invariant event Ω_0^{no3geo} with $\mathbb{P}(\Omega_0^{\text{no3geo}}) = 1$ and such that for any $\omega \in \Omega_0^{\text{no3geo}}$, any $\xi \in \operatorname{ri} \mathcal{U}$, and any $x \in \mathbb{Z}^2$, there exist at most two ξ-directed semi-infinite geodesics out of x.*

Fix a countable dense set $\mathcal{U}_0 \subset \mathcal{D}$. The following event of full $\mathbb{P}$-probability is the basic setting for the proofs in Sections 7–8:

$$\Omega_0 = \Omega_0^1 \cap \Omega_0^2 \cap \Omega_0^4 \cap \Big(\bigcap_{\xi\in\mathcal{U}_0} [\Omega_\xi^1 \cap \Omega_\xi^3]\Big) \cap \Big(\bigcap_{\xi\in(\operatorname{ri}\mathcal{U})\setminus\mathcal{D}} \Omega_\xi^3\Big). \tag{A.7}$$

When additional assumptions are needed, Ω_0 will be further restricted.

Appendix B. Auxiliary lemmas

The next lemma follows from the shape theorem for cocycles (A.3).

Lemma B.1. *Suppose g is differentiable on* $\mathrm{ri}\,\mathcal{U}$. *For any $\omega \in \Omega_0$, $\xi \in \mathrm{ri}\,\mathcal{U}$, and any $v \in \mathbb{R}^2$, $n^{-1}B^{\xi\pm}(0, \lfloor nv \rfloor)$ both converge to $v \cdot \nabla g(\xi)$ as $n \to \infty$.*

Proof. The claim is obvious for $v = 0$. Suppose that $v \in \mathbb{R}^2_+ \setminus \{0\}$, the other cases being similar. Take $\omega \in \Omega_0$ and $\zeta, \eta \in \mathcal{U}_0$ with $\zeta \cdot e_1 < \xi \cdot e_1 < \eta \cdot e_1$. Let $x_n = \lfloor nv \rfloor = m_n e_1 + \ell_n e_2$. Then

$$\begin{aligned}
B^{\xi+}(0, x_n) &= B^{\xi+}(0, m_n e_1) + B^{\xi+}(m_n e_1, x_n) \\
&\le B^{\eta}(0, m_n e_1) + B^{\zeta}(m_n e_1, x_n) \\
&\le B^{\eta}(0, m_n e_1) + B^{\zeta}(0, x_n) - B^{\zeta}(0, m_n e_1).
\end{aligned}$$

Divide by n, take it to ∞, and apply (A.3) to B^{ζ} and B^{η} to get

$$\overline{\lim_{n\to\infty}}\, n^{-1} B^{\xi+}(0, x_n) \le (v \cdot e_1) e_1 \cdot \nabla g(\eta) + v \cdot \nabla g(\zeta) - (v \cdot e_1) e_1 \cdot \nabla g(\zeta).$$

Take ζ and η to ξ to get

$$\overline{\lim_{n\to\infty}}\, n^{-1} B^{\xi+}(0, x_n) \le v \cdot \nabla g(\xi).$$

The lower bound on the liminf holds similarly and so we have proved the claim for $B^{\xi+}$. The same argument works for $B^{\xi-}$. ∎

The lemma below is proved by calculus.

Lemma B.2. *Fix $c > 0$. Then for all $n \ge 1$ and all a, b such that $c \le a \le b \le a + \frac{c}{n}$,*

$$\frac{a^{n-1} b^n}{(a+b)^{2n-1}} \ge \left(\frac{1}{2}\right)^{2n-1}. \tag{B.1}$$

Appendix C. M/M/1 queues and Busemann functions

This appendix summarizes results from [22] that are needed for the proofs of the results of Section 5. Fix parameters $0 < \alpha < \beta$. We formulate a stationary M/M/1 queue in a particular way. The inputs are two independent i.i.d. sequences: an *inter-arrival process* $I = (I_i)_{i \in \mathbb{Z}}$ with marginal distribution $I_i \sim \mathrm{Exp}(\alpha)$ and a *service process* $Y = (Y_i)_{i \in \mathbb{Z}}$ with marginal distribution $Y_i \sim \mathrm{Exp}(\beta)$. Out of these inputs are produced two outputs: an *inter-departure process* $\widetilde{I} = (\widetilde{I}_k)_{k \in \mathbb{Z}}$ and a *sojourn process* $J = (J_k)_{k \in \mathbb{Z}}$, through the following formulas. Let $G = (G_k)_{k \in \mathbb{Z}}$ be any function on $\mathbb{Z}$ with $I_k = G_k - G_{k+1}$. Define the function $\widetilde{G} = (\widetilde{G}_k)_{k \in \mathbb{Z}}$ by

$$\widetilde{G}_k = \sup_{m:\, m \ge k} \Big\{ G_m + \sum_{i=k}^{m} Y_i \Big\} = G_k + Y_k + \sup_{m:\, m \ge k} \sum_{i=k}^{m-1} (Y_{i+1} - I_i). \tag{C.1}$$

The convention for the empty sum is $\sum_{i=k}^{k-1} = 0$. Under the assumption on I and Y, the supremum in (C.1) is almost surely assumed at some finite m. Then define the outputs by

$$\widetilde{I}_k = \widetilde{G}_k - \widetilde{G}_{k+1}, \tag{C.2}$$

$$J_k = \widetilde{G}_k - G_k = Y_k + \sup_{m:\, m \ge k} \sum_{i=k}^{m-1} (Y_{i+1} - I_i). \tag{C.3}$$

The outputs satisfy the useful iterative equations

$$\widetilde{I}_k = Y_k + (I_k - J_{k+1})^+ \quad \text{and} \quad J_k = Y_k + (J_{k+1} - I_k)^+. \tag{C.4}$$

In particular, this implies the inequality $\widetilde{I}_k \ge Y_k$.

It is a basic fact about M/M/1 queues that $\widetilde{I}$ and J are i.i.d. sequences with marginals $\widetilde{I}_k \sim \mathrm{Exp}(\alpha)$ and $J_k \sim \mathrm{Exp}(\beta - \alpha)$. Furthermore, the three variables (Y_k, I_k, J_{k+1}) on the right-hand sides of equations (C.4) are independent. (See for example [22, Appendix A].) But $\widetilde{I}$ and J are not independent of each other.

The queueing interpretation goes as follows. A service station processes a bi-infinite sequence of customers. Queueing time runs backwards on the lattice $\mathbb{Z}$. Further, I_i is the time between the arrivals of customers $i+1$ and i ($i+1$ arrived before i) and Y_i is the service time required by customer i; $\widetilde{I}_k$ is the time between the departures of customers $k+1$ and k, with $k+1$ departing before k; and J_k is the sojourn time of customer k, that is, the total time customer k spent in the system from arrival to departure. Then J_k is the sum of the service time Y_k and the waiting time of customer k, represented by the last member of (C.3). Because of our unusual convention with backward indexing, even if G_k is the arrival time of customer k, $\widetilde{G}_k$ is not the time of departure. The definition of $\widetilde{G}$ in (C.1) is natural in the present setting because it immediately ties in with LPP. The convention in [22] is different because in [22] geodesics go south and west instead of north and east.

The joint distribution of successive nearest-neighbor increments of two Busemann functions on a horizontal or vertical line can now be described as follows. This is a special case of [22, Theorem 3.2].

Theorem C.1. *Let $\zeta \prec \eta$ in $\mathrm{ri}\,\mathcal{U}$ with parameters $\alpha = \alpha(\zeta) < \alpha(\eta) = \beta$ given by (5.2). Let $I = (I_i)_{i\in\mathbb{Z}}$ and $Y = (Y_i)_{i\in\mathbb{Z}}$ be two independent i.i.d. sequences and define $\widetilde{I} = (\widetilde{I}_k)_{k\in\mathbb{Z}}$ as above through (C.1)–(C.2).*

(a) *Let $I_i \sim \mathrm{Exp}(\alpha)$ and $Y_i \sim \mathrm{Exp}(\beta)$. Then the sequence $(B^{\zeta}_{ke_1,(k+1)e_1}, B^{\eta}_{ke_1,(k+1)e_1})_{k\in\mathbb{Z}}$ has the same joint distribution as the pair $(\widetilde{I}, Y)$.*

(b) *Let $I_i \sim \mathrm{Exp}(1-\beta)$ and $Y_i \sim \mathrm{Exp}(1-\alpha)$. Then $(B^{\zeta}_{ke_2,(k+1)e_2}, B^{\eta}_{ke_2,(k+1)e_2})_{k\in\mathbb{Z}}$ has the same joint distribution as $(Y, \widetilde{I})$.*

Next we derive a random walk representation for the sequence $\{B^{\zeta}_{ke_1,(k+1)e_1} - B^{\eta}_{ke_1,(k+1)e_1}\}_{k\in\mathbb{Z}}$ of (non-negative) differences. By Theorem C.1 this sequence is equal in distribution to $\{\widetilde{I}_k - Y_k\}_{k\in\mathbb{Z}}$. Define a two-sided randon walk S with positive drift $E[I_{i-1} - Y_i] = \alpha^{-1} - \beta^{-1}$ by

$$S_n = \begin{cases} -\sum_{i=n+1}^{0}(I_{i-1} - Y_i), & n < 0, \\ 0, & n = 0, \\ \sum_{i=1}^{n}(I_{i-1} - Y_i), & n > 0. \end{cases} \tag{C.5}$$

Then from (C.4) and (C.3),

$$\widetilde{I}_k - Y_k = (I_k - J_{k+1})^+ = \Big\{\inf_{n:\,n>k} \sum_{i=k+1}^{n} (I_{i-1} - Y_i)\Big\}^+ = \Big\{\inf_{n:\,n>k} (S_n - S_k)\Big\}^+.$$

From the above we can record that for $r > 0$,

$$\mathbb{P}(B^{\zeta}_{0,e_1} > B^{\eta}_{0,e_1}) = P(I_k > J_{k+1}) = \frac{\beta - \alpha}{\beta}. \tag{C.6}$$

Corollary C.2. *Let $\zeta \prec \eta$ in* ri $\mathcal{U}$ *with parameters $\alpha = \alpha(\zeta) < \alpha(\eta) = \beta$ given by* (5.2). *Let S be the random walk in* (C.5) *with step distribution* $\mathrm{Exp}(\alpha) - \mathrm{Exp}(\beta)$. *Then the sequence $\{B^{\zeta}_{ke_1,(k+1)e_1} - B^{\eta}_{ke_1,(k+1)e_1}\}_{k\in\mathbb{Z}}$ has the same distribution as the sequence $\{(\inf_{n:\,n>k} S_n - S_k)^+\}_{k\in\mathbb{Z}}$.*

Appendix D. Random walk

Let $0 < \alpha < \beta$ and let $\{X_i\}_{i\in\mathbb{Z}}$ be a doubly infinite sequence of i.i.d. random variables with marginal distribution $X_i \sim \mathrm{Exp}(\alpha) - \mathrm{Exp}(\beta)$ (difference of two independent exponential random variables). Let θ denote the shift on the underlying canonical sequence space so that $X_j = X_k \circ \theta^{j-k}$. Let $\{S_n\}_{n\in\mathbb{Z}}$ be the two-sided random walk such that $S_0 = 0$ and $S_n - S_m = \sum_{i=m+1}^{n} X_i$ for all $m < n$ in $\mathbb{Z}$. Let $(\lambda_i)_{i\geq 1}$ be the strict ascending ladder epochs of the forward walk. That is, begin with $\lambda_0 = 0$, and for $i \geq 1$ let

$$\lambda_i = \inf\{n > \lambda_{i-1} : S_n > S_{\lambda_{i-1}}\}.$$

The positive drift of S_n ensures that these variables are finite almost surely. For $i \geq 1$ define the increments $L_i = \lambda_i - \lambda_{i-1}$ and $H_i = S_{\lambda_i} - S_{\lambda_{i-1}}$. The variables $\{L_i, H_i\}_{i\geq 1}$ are mutually independent with marginal distribution

$$P(L_1 = n, H_1 > r) = C_{n-1} \frac{\alpha^{n-1}\beta^n}{(\alpha+\beta)^{2n-1}} e^{-\alpha r}, \quad n \in \mathbb{N},\ r \geq 0. \tag{D.1}$$

Above $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$ are the Catalan numbers. A small extension of the proof of [22, Lemma B.3] yields (D.1).

Let

$$W_0 = \inf_{m>0} S_m. \tag{D.2}$$

Note that $W_0 \circ \theta^n > 0$ if and only if $S_n < \inf_{m>n} S_m$, that is, n is a last exit time for the random walk. Define successive last exit times (in the language of Doney [19]) by

$$\begin{aligned} \sigma_0 &= \inf\{n \geq 0 : S_n < \inf_{m>n} S_m\}, \\ \sigma_i &= \inf\{n > \sigma_{i-1} : S_n < \inf_{m>n} S_m\} \quad \text{for } i \geq 1. \end{aligned} \tag{D.3}$$

Proposition D.1. *Conditionally on $W_0 > 0$ (equivalently, on $\sigma_0 = 0$), the pairs $\{(\sigma_i - \sigma_{i-1}, S_{\sigma_i} - S_{\sigma_{i-1}})\}_{i\geq 1}$ are i.i.d. with marginal distribution*

$$P(\sigma_i - \sigma_{i-1} = n,\ S_{\sigma_i} - S_{\sigma_{i-1}} > r \mid W_0 > 0) = C_{n-1} \frac{\alpha^{n-1}\beta^n}{(\alpha+\beta)^{2n-1}} e^{-\alpha r} \tag{D.4}$$

for all $i \in \mathbb{N}$, $n \in \mathbb{N}$, and $r \geq 0$.

Proof. Let $0 = n_0 < n_1 < \cdots < n_\ell$ and $r_1, \ldots, r_\ell > 0$. The dual random walk

$$S_k^* = S_{n_\ell} - S_{n_\ell - k} \quad \text{for } 0 \leq k \leq n_\ell$$

(Feller [24, p. 394]) satisfies $(S_k^*)_{0\leq k\leq n_\ell} \overset{d}{=} (S_k)_{0\leq k\leq n_\ell}$ and is independent of $W_0 \circ \theta^{n_\ell}$. We have

$$\begin{aligned} &P(\forall i \in [\![1,\ell]\!] : \sigma_i - \sigma_{i-1} = n_i - n_{i-1} \text{ and } S_{\sigma_i} - S_{\sigma_{i-1}} > r_i,\ W_0 > 0) \\ &= P(\forall i \in [\![1,\ell]\!] : \sigma_i = n_i \text{ and } S_{\sigma_i} - S_{\sigma_{i-1}} > r_i,\ W_0 > 0) \\ &= P(\forall i \in [\![1,\ell]\!] : S_k > S_{n_i} > S_{n_{i-1}} + r_i \text{ for } k \in]\!]n_{i-1}, n_i[\![,\ W_0 \circ \theta^{n_\ell} > 0) \\ &= P(\forall i \in [\![1,\ell]\!] : S_j^* < S^*_{n_\ell - n_i} < S^*_{n_\ell - n_{i-1}} - r_i \text{ for } j \in]\!]n_\ell - n_i, n_\ell - n_{i-1}[\![,\ W_0 \circ \theta^{n_\ell} > 0) \\ &= P(\forall k \in [\![1,\ell]\!] : \lambda_k = n_\ell - n_{\ell-k} \text{ and } H_k > r_{\ell-k+1}) P(W_0 > 0) \\ &= P(\forall k \in [\![1,\ell]\!] : L_k = n_{\ell-k+1} - n_{\ell-k} \text{ and } H_k > r_{\ell-k+1}) P(W_0 > 0). \end{aligned}$$

The claim follows from the independence of $\{L_k, H_k\}$ and (D.1). ∎

From σ_0 as defined in (D.3), extend σ_i to negative indices by defining, for $i = -1, -2, -3, \ldots$,

$$\sigma_i = \max\{k < \sigma_{i+1} : S_k < S_{\sigma_{i+1}}\}. \tag{D.5}$$

For each $k \in \mathbb{Z}$ set

$$W_k = \inf_{n:n>k} S_n - S_k. \tag{D.6}$$

Then one can check that $\sigma_{-1} < 0 \leq \sigma_0$, and for all $i, k \in \mathbb{Z}$,

$$S_{\sigma_i} = \inf_{n:n>\sigma_{i-1}} S_n, \tag{D.7}$$

$$W_{\sigma_i} = \inf_{n:n>\sigma_i} S_n - S_{\sigma_i} = S_{\sigma_{i+1}} - S_{\sigma_i}, \tag{D.8}$$

and

$$W_k > 0 \iff k \in \{\sigma_i : i \in \mathbb{Z}\}. \tag{D.9}$$

Theorem D.2. *Conditionally on $\sigma_0 = 0$, equivalently, on $W_0 > 0$, $\{\sigma_{i+1} - \sigma_i, W_{\sigma_i} : i \in \mathbb{Z}\}$ is an i.i.d. sequence with marginal distribution*

$$P(\sigma_{i+1} - \sigma_i = n,\, W_{\sigma_i} > r \mid W_0 > 0) = C_{n-1} \frac{\alpha^{n-1}\beta^n}{(\beta+\alpha)^{2n-1}} e^{-\alpha r} \tag{D.10}$$

for all $i \in \mathbb{N}$, $n \in \mathbb{N}$, and $r \geq 0$.

Proof. Define the processes $\Psi_+ = \{\sigma_{i+1} - \sigma_i, W_{\sigma_i} : i \geq 0\}$ and $\Psi_- = \{\sigma_{i+1} - \sigma_i, W_{\sigma_i} : i \leq -1\}$. Then Ψ_+ and the conditioning event $W_0 > 0$ depend only on $(S_n)_{n\geq 1}$, while $W_0 > 0$ implies for $n < 0$ that $\inf_{m: m>n} S_m = \inf_{m: n<m\leq 0} S_m$. Thus Ψ_+ and Ψ_- have been decoupled.

Define another forward walk with the same step distribution by $\widetilde{S}_k = -S_{-k}$ for $k \geq 0$. Let $\lambda_0 = 0$, $(\lambda_i)_{i\geq 1}$ be the successive ladder epochs and $H_i = \widetilde{S}_{\lambda_i} - \widetilde{S}_{\lambda_{i-1}}$ the successive ladder height increments for the $\widetilde{S}$ walk.

We claim that on the event $\sigma_0 = 0$,

$$\lambda_{-i} = -\sigma_i \quad \text{and} \quad W_{\sigma_i} = H_{-i} \quad \text{for } i \leq -1. \tag{D.11}$$

First by definition, $\lambda_0 = 0 = -\sigma_0$. By the definitions and by induction, for $i \leq -1$,

$$\begin{aligned}\lambda_{-i} &= \min\{k > \lambda_{-i-1} : \widetilde{S}_k > \widetilde{S}_{\lambda_{-i-1}}\} = \min\{k > -\sigma_{i+1} : S_{-k} < S_{\sigma_{i+1}}\} \\ &= -\max\{n < \sigma_{i+1} : S_n < S_{\sigma_{i+1}}\} = -\sigma_i\end{aligned}$$

where the last equality came from (D.5). Then from (D.8),

$$W_{\sigma_i} = S_{\sigma_{i+1}} - S_{\sigma_i} = -\widetilde{S}_{-\sigma_{i+1}} + \widetilde{S}_{-\sigma_i} = -\widetilde{S}_{\lambda_{-i-1}} + \widetilde{S}_{\lambda_{-i}} = H_{-i}.$$

Claim (D.11) has been verified.

Let $\Psi' = \{\lambda_{-i} - \lambda_{-i-1}, H_{-i} : i \leq -1\}$, a function of $(S_n)_{n\leq -1}$. By (D.11), $\Psi_- = \Psi'$ on the event $\sigma_0 = 0$.

Let A and B be suitable measurable sets of infinite sequences.

$$\begin{aligned}P(\Psi_+ \in A,\, \Psi_- \in B \mid W_0 > 0) &= \frac{1}{P(W_0 > 0)} P(\Psi_+ \in A,\, \Psi' \in B,\, W_0 > 0) \\ &= \frac{P(\Psi_+ \in A,\, W_0 > 0)}{P(W_0 > 0)} P(\Psi' \in B) = P(\Psi_+ \in A \mid W_0 > 0) P(\Psi' \in B).\end{aligned}$$

The conclusion follows. By Proposition D.1, conditional on $W_0 > 0$, Ψ_+ has the i.i.d. distribution (D.10), which is the same as the i.i.d. distribution (D.1) of Ψ'. ∎

Acknowledgments. The authors thank Yuri Bakhtin for helpful conversations.

Funding. C. Janjigian was partially supported by National Science Foundation grant DMS-2125961. F. Rassoul-Agha was partially supported by NSF grants DMS-1407574 and DMS-1811090. T. Seppäläinen was partially supported by National Science Foundation grants DMS-1602486 and DMS-1854619, and by the Wisconsin Alumni Research Foundation.

References

[1] Ahlberg, D., Hoffman, C.: Random coalescing geodesics in first-passage percolation. arXiv:1609.02447 (2016)

[2] Auffinger, A., Damron, M., Hanson, J.: 50 Years of First-Passage Percolation. Univ. Lecture Ser. 68, Amer. Math. Soc., Providence, RI (2017) Zbl 1452.60002 MR 3729447

[3] Baik, J., Deift, P., Johansson, K.: On the distribution of the length of the longest increasing subsequence of random permutations. J. Amer. Math. Soc. **12**, 1119–1178 (1999) Zbl 0932.05001 MR 1682248

[4] Bakhtin, Y.: Inviscid Burgers equation with random kick forcing in noncompact setting. Electron. J. Probab. **21**, art. 37, 50 pp. (2016) Zbl 1338.37117 MR 3508684

[5] Bakhtin, Y., Cator, E., Khanin, K.: Space-time stationary solutions for the Burgers equation. J. Amer. Math. Soc. **27**, 193–238 (2014) Zbl 1296.37051 MR 3110798

[6] Bakhtin, Y., Khanin, K.: On global solutions of the random Hamilton–Jacobi equations and the KPZ problem. Nonlinearity **31**, R93–R121 (2018) Zbl 1387.37073 MR 3816628

[7] Bakhtin, Y., Li, L.: Thermodynamic limit for directed polymers and stationary solutions of the Burgers equation. Comm. Pure Appl. Math. **72**, 536–619 (2019) Zbl 1455.60134 MR 3911894

[8] Balázs, M., Busani, O., Seppäläinen, T.: Non-existence of bi-infinite geodesics in the exponential corner growth model. Forum Math. Sigma **8**, art. e46, 34 pp. (2020) MR 4176750

[9] Basu, R., Hoffman, C., Sly, A.: Nonexistence of bigeodesics in planar exponential last passage percolation. Comm. Math. Phys. **389**, 1–30 (2022) Zbl 07463703 MR 4365136

[10] Bec, J., Khanin, K.: Burgers turbulence. Phys. Rep. **447**, 1–66 (2007) MR 2318582

[11] Cator, E., Pimentel, L. P. R.: A shape theorem and semi-infinite geodesics for the Hammersley model with random weights. Latin Amer. J. Probab. Math. Statist. **8**, 163–175 (2011) Zbl 1276.60109 MR 2783936

[12] Cator, E., Pimentel, L. P. R.: Busemann functions and equilibrium measures in last passage percolation models. Probab. Theory Related Fields **154**, 89–125 (2012) Zbl 1262.60094 MR 2981418

[13] Cator, E., Pimentel, L. P. R.: Busemann functions and the speed of a second class particle in the rarefaction fan. Ann. Probab. **41**, 2401–2425 (2013) Zbl 1276.60108 MR 3112921

[14] Corwin, I.: The Kardar–Parisi–Zhang equation and universality class. Random Matrices Theory Appl. **1**, art. 1130001, 76 pp. (2012) Zbl 1247.82040 MR 2930377

[15] Corwin, I.: Kardar–Parisi–Zhang universality. Notices Amer. Math. Soc. **63**, 230–239 (2016) Zbl 1342.82098 MR 3445162

[16] Coupier, D.: Multiple geodesics with the same direction. Electron. Comm. Probab. **16**, 517–527 (2011) Zbl 1244.60093 MR 2836758

[17] Damron, M., Hanson, J.: Busemann functions and infinite geodesics in two-dimensional first-passage percolation. Comm. Math. Phys. **325**, 917–963 (2014) Zbl 1293.82014 MR 3152744

[18] Damron, M., Hanson, J.: Bigeodesics in first-passage percolation. Comm. Math. Phys. **349**, 753–776 (2017) Zbl 1361.60089 MR 3595369

[19] Doney, R. A.: Last exit times for random walks. Stochastic Process. Appl. **31**, 321–331 (1989) Zbl 0672.60072 MR 998121

[20] E, W., Khanin, K., Mazel, A., Sinai, Y.: Invariant measures for Burgers equation with stochastic forcing. Ann. of Math. (2) **151**, 877–960 (2000) Zbl 0972.35196 MR 1779561

[21] Eden, M.: A two-dimensional growth process. In: Proc. 4th Berkeley Sympos. Math. Statist. Probab., Vol. IV, Univ. California Press, Berkeley, CA, 223–239 (1961) Zbl 0104.13801 MR 0136460

[22] Fan, W.-T. L., Seppäläinen, T.: Joint distribution of Busemann functions in the exactly solvable corner growth model. Probab. Math. Phys. **1**, 55–100 (2020) Zbl 07485833

[23] Feller, W.: An Introduction to Probability Theory and its Applications. Vol. I. 3rd ed., Wiley, New York (1968) Zbl 0155.23101 MR 0228020

[24] Feller, W.: An Introduction to Probability Theory and its Applications. Vol. II. 2nd ed., Wiley, New York (1971) Zbl 0219.60003 MR 0270403

[25] Ferrari, P. A., Martin, J. B., Pimentel, L. P. R.: A phase transition for competition interfaces. Ann. Appl. Probab. **19**, 281–317 (2009) Zbl 1185.60109 MR 2498679

[26] Ferrari, P. A., Pimentel, L. P. R.: Competition interfaces and second class particles. Ann. Probab. **33**, 1235–1254 (2005) Zbl 1078.60083 MR 2150188

[27] Georgiou, N., Rassoul-Agha, F., Seppäläinen, T.: Geodesics and the competition interface for the corner growth model. Probab. Theory Related Fields **169**, 223–255 (2017) Zbl 1407.60123 MR 3704769

[28] Georgiou, N., Rassoul-Agha, F., Seppäläinen, T.: Stationary cocycles and Busemann functions for the corner growth model. Probab. Theory Related Fields **169**, 177–222 (2017) Zbl 1407.60122 MR 3704768

[29] Glynn, P. W., Whitt, W.: Departures from many queues in series. Ann. Appl. Probab. **1**, 546–572 (1991) Zbl 0749.60090 MR 1129774

[30] Halpin-Healy, T., Takeuchi, K. A.: A KPZ cocktail—shaken, not stirred...: toasting 30 years of kinetically roughened surfaces. J. Statist. Phys. **160**, 794–814 (2015) Zbl 1327.82065 MR 3373641

[31] Hammersley, J. M., Welsh, D. J. A.: First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In: Proc. Internat. Res. Semin., Statist. Lab., Univ. California, Berkeley, CA, Springer, New York, 61–110 (1965) Zbl 0143.40402 MR 0198576

[32] Hoffman, C.: Geodesics in first passage percolation. Ann. Appl. Probab. **18**, 1944–1969 (2008) Zbl 1153.60055 MR 2462555

[33] Howard, C. D., Newman, C. M.: Euclidean models of first-passage percolation. Probab. Theory Related Fields **108**, 153–170 (1997) Zbl 0883.60091 MR 1452554

[34] Howard, C. D., Newman, C. M.: Geodesics and spanning trees for Euclidean first-passage percolation. Ann. Probab. **29**, 577–623 (2001) Zbl 1062.60099 MR 1849171

[35] Janjigian, C., Rassoul-Agha, F.: Busemann functions and Gibbs measures in directed polymer models on $\mathbb{Z}^2$. arXiv:1810.03580v2 (2019)

[36] Janjigian, C., Rassoul-Agha, F.: Busemann functions and Gibbs measures in directed polymer models on $\mathbb{Z}^2$. Ann. Probab. **48**, 778–816 (2020) Zbl 1444.60083 MR 4089495

[37] Johansson, K.: Shape fluctuations and random matrices. Comm. Math. Phys. **209**, 437–476 (2000) Zbl 0969.15008 MR 1737991

[38] Kallenberg, O.: Random Measures. 3rd ed., Akademie-Verlag, Berlin, and Academic Press, London (1983) Zbl 0544.60053 MR 818219

[39] Kallenberg, O.: Random Measures, Theory and Applications. Probab. Theory Stoch. Modelling 77, Springer, Cham (2017) Zbl 1376.60003 MR 3642325

[40] Licea, C., Newman, C. M.: Geodesics in two-dimensional first-passage percolation. Ann. Probab. **24**, 399–410 (1996) Zbl 0863.60097 MR 1387641

[41] Martin, J. B.: Limiting shape for directed percolation models. Ann. Probab. **32**, 2908–2937 (2004) Zbl 1065.60149 MR 2094434

[42] Muth, E. J.: The reversibility property of production lines. Management Sci. **25**, 152–158 (1979/80) Zbl 0419.90044 MR 537320

[43] Newman, C. M.: A surface view of first-passage percolation. In: Proc. International Congress of Mathematicians (Zürich, 1994, Vol. II), Birkhäuser, Basel, 1017–1023 (1995) Zbl 0848.60089 MR 1404001

[44] Quastel, J.: Introduction to KPZ. In: Current Developments in Mathematics, 2011, Int. Press, Somerville, MA, 125–194 (2012) Zbl 1316.60019 MR 3098078

[45] Quastel, J., Spohn, H.: The one-dimensional KPZ equation and its universality class. J. Statist. Phys. **160**, 965–984 (2015) Zbl 1327.82069 MR 3373647

[46] Révész, P.: Random Walk in Random and Non-Random Environments. 2nd ed., World Sci., Hackensack, NJ (2005) Zbl 1090.60001 MR 3060348

[47] Rost, H.: Nonequilibrium behaviour of a many particle process: density profile and local equilibria. Z. Wahrsch. Verw. Gebiete **58**, 41–53 (1981) Zbl 0451.60097 MR 635270

[48] Seppäläinen, T.: Hydrodynamic scaling, convex duality and asymptotic shapes of growth models. Markov Process. Related Fields **4**, 1–26 (1998) Zbl 0906.60082 MR 1625007

[49] Sinaĭ, Y. G.: Statistics of shocks in solutions of inviscid Burgers equation. Comm. Math. Phys. **148**, 601–621 (1992) Zbl 0755.60105 MR 1181071

[50] Szczotka, W., Kelly, F. P.: Asymptotic stationarity of queues in series and the heavy traffic approximation. Ann. Probab. **18**, 1232–1248 (1990) Zbl 0726.60092 MR 1062067

[51] Tracy, C. A., Widom, H.: Level-spacing distributions and the Airy kernel. Comm. Math. Phys. **159**, 151–174 (1994) Zbl 0789.35152 MR 1257246

[52] Wüthrich, M. V.: Asymptotic behaviour of semi-infinite geodesics for maximal increasing subsequences in the plane. In: In and Out of Equilibrium (Mambucaba, 2000), Progr. Probab. 51, Birkhäuser Boston, Boston, MA, 205–226 (2002) Zbl 1011.60085 MR 1901954