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Chemotaxis and reactions in biology

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Abstract. Chemotaxis plays a crucial role in a variety of processes in biology and ecology. Quite often it acts to improve efficiency of biological reactions. One example is the immune system signalling, where infected tissues release chemokines attracting monocytes to fight invading bacteria. Another example is reproduction, where eggs release pheromones that attract sperm. A macro scale example is flower scent appealing to pollinators. In this paper we consider a system of PDEs designed to model such processes. Our interest is to quantify the effect of chemotaxis on reaction rates compared to pure reaction-diffusion. We limit consideration to surface chemotaxis, which is well motivated from the point of view of many applications. Our results provide the first insight into situations where chemotaxis can be crucial for reaction success, and where its effect is likely to be limited. The proofs are based on new analytical tools; a significant part of the paper is dedicated to building up the linear machinery that can be useful in more general settings. In particular, we establish precise estimates on the rates of convergence to the ground state for a class of Fokker–Planck operators with potentials that grow at a logarithmic rate at infinity. These estimates are made possible by a new sharp weak weighted Poincaré inequality.

Keywords. Chemotaxis, reaction enhancement, reaction-diffusion equations, Fokker–Planck operators, convergence to equilibrium, logarithmic potential

1. Introduction

Chemotaxis describes the motion of cells or species that sense and attempt to move towards higher (or lower) concentration of some chemical. Its first mathematical studies go back to Patlak [58] and Keller–Segel [42, 43]. The Keller–Segel system introduced in the latter work describes a population of bacteria or mold secreting an attractive chemical

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substance, and remains the most studied model of chemotaxis. In the simplified parabolic-elliptic form, this equation can be written as (see, e.g., [60])

$$\partial_t \rho - \Delta\rho + \chi\nabla\cdot(\rho\nabla(-\Delta)^{-1}\rho) = 0, \quad \rho(x,0) = \rho_0(x). \tag{1.1}$$

The last term on the left side describes the attraction of ρ by a chemical with concentration $c(x,t) = (-\Delta)^{-1}\rho(x,t)$. This is an approximation to the diffusion equation

$$\partial_t c = \kappa\Delta c + R\rho,$$

under the assumption that $\kappa \sim R \gg 1$, so that the chemical is both produced and diffuses on faster typical time scales than those for the rest of the dynamics of (1.1). The literature on the Keller–Segel equation is very extensive. In particular, a number of different variants of (1.1) have been derived from more basic kinetic models (see, e.g., [22,33,38,56,62]). It is known that in dimensions larger than 1 solutions to (1.1) can concentrate and become singular in finite time. We refer to [36,37,60] for more details and further references.

In many settings in biology where chemotaxis is present, it facilitates and enhances success rates of reaction-like processes. One example is reproduction for many species, where eggs secrete chemicals that attract sperm and help improve fertilization rates. This is especially well studied for marine life such as corals, sea urchins, mollusks, etc. (see [35, 65, 80] for further references), but the role of chemotaxis in fertilization extends to a great number of species, including humans [64]. In the same vein, many plants appeal primarily to the insects' sense of smell to attract pollinators. Another process where chemotaxis plays an important role is mammal immune systems fighting bacterial infections. Inflamed tissues release special proteins, called chemokines, that serve to chemically attract monocytes, blood killer cells, to the source of infection [21, 67]. Chemotaxis can also be involved when things go awry, for instance, playing a role in tumor growth [70].

In the mathematical literature, the studies of equations including both chemotaxis and reactions focused mainly on existence and regularity of solutions as well as general features of long-time dynamics (see [16,24,25,54,55,68,74,75,77,79] for further references). To the best of our knowledge, there are very few works where the question of how chemotaxis affects the reaction rates has been studied rigorously or even modeled computationally. As far as we know, the first step in this direction has been taken in [45,46] where a generalization of (1.1) including an absorbing reaction and a fluid flow has been considered:

$$\partial_t \rho + (u\cdot\nabla)\rho - \Delta\rho + \chi\nabla\cdot(\rho\nabla(-\Delta)^{-1}\rho) = -\epsilon\rho^q, \quad \nabla\cdot u = 0, \quad \rho(x,0) = \rho_0(x) \ge 0. \tag{1.2}$$

This work was motivated by modeling the life cycle of corals. Corals, and many other marine species, reproduce by broadcast spawning. It is a fertilization strategy whereby males and females release sperm and egg gametes that rise to the surface of the ocean. As they are initially separated by the ambient water, an effective surface mixing is necessary for a successful fertilization. For coral spawning, field measurements of fertilization rates are usually around 50%, and are often as high as 90% [48, 59]. On the other hand, numerical simulations based on purely reaction-diffusion models [20] predict fertilization rates of less than 1% due to the strong dilution of gametes. A more sophisticated model, taking into account the instantaneous details of the advective transport, was proposed in [17, 18]. Adding fluid flow to the model can account for part of the gap between simulations and field measurements, but appears unlikely to completely explain it [46]. However, as already mentioned, there is also experimental evidence that chemotaxis plays a role in coral and other marine animals fertilization: eggs release a chemical that attracts sperm [14, 15, 52, 53].

The results of [45, 46] show, in the framework of (1.2), that the role of chemotaxis in reaction enhancement can be quite significant – especially when reaction is weak, as is known to be the case in many biological processes [73]. The efficiency of the reaction can be measured by the decay of the total mass of the remaining density,

$$m(t) = \int \rho(x,t)\,dx.$$

If $\chi = 0$, then the decay of $m(t)$ is very slow if ϵ is small, uniformly in the incompressible fluid velocity u [46]. On the other hand, if $\chi \neq 0$, then in dimension 2, relevant for the corals application, the extent of decay and time scales of decay of $m(t)$ are independent of ϵ, and the decay can be very significant and fast if the chemotactic coupling is sufficiently strong. While the results of [45, 46] are suggestive, taking (1.2) as a model makes a strong simplifying assumption that the densities of male and female species are equal and are both chemotactic on each other. In reality, only the male density is chemotactic, hence (1.2) can be expected to overestimate the effect of chemotaxis on the reaction rates.

Although there are certainly examples of mold and bacteria that are chemotactic on the chemicals they themselves release, significantly more numerous situations in biology involve species that are chemotactic on a chemical secreted by other agents. Most of the examples mentioned above are of this kind. In this paper, we would like to initiate qualitative analysis of a more realistic system of equations modeling chemotaxis enhanced reaction processes, of the form

$$\begin{aligned} &\partial_t \rho_1 - \kappa\Delta\rho_1 + \chi\nabla\cdot(\rho_1\nabla(-\Delta)^{-1}\rho_2) = -\epsilon\rho_1\rho_2, \\ &\partial_t \rho_2 = -\epsilon\rho_1\rho_2. \end{aligned} \tag{1.3}$$

There is no ambient fluid advection: as the first step, we assume that the fluid flow is adequately modelled by effective diffusion. The chemically attracted density is ρ_1; the density ρ_2 that produces the attractant is assumed to be immobile, which is a realistic assumption in many interesting problems: for example, the inflamed tissue releasing chemokines and attracting monocytes, plants attracting insects, or immobile eggs attracting sperm in the mammal reproduction tract are in this category. We also maintain the parabolic-elliptic structure, with the assumption that the signaling chemical diffusion time is much shorter than other relevant time scales. The system (1.3) is one of the most natural first step models in analyzing any situation where a fixed target aims to attract, by using a fast diffusing chemical, a diffusing and mobile species which is involved in some kind of reaction with the target. Systems of this type have been certainly analyzed in the literature – for example, in [16] a system of a very similar form but with a different chemotactic term has been considered as a model of angiogenesis. However, the focus of most such studies has been on proving global regularity, finite-time blow-up, asymptotic behavior, and finding special classes of self-similar solutions. Moreover, in general, there are few rigorous results that detect specific taxis-driven effects, and these have focused almost exclusively on finite-time blow-up [29–32, 76] and on transient growth [40, 78]. Perhaps the closest to our aim here are the papers [13, 23] that yield some estimates on the effect of chemotaxis on reaction in a related setting. However, to the best of our knowledge, our paper is the first attempt at sharp qualitative estimates for the scaling rules of the effect of chemotaxis on the reaction rates in a chemotaxis system involving two distinct densities. Here, we will limit consideration to two spatial dimensions and to the classical form of the Keller–Segel chemotaxis flux. We make comments on some possible extensions and generalizations in Section 9.

The purpose of this paper is twofold. First, we provide a careful analysis of the linear problem corresponding to (1.3). This analysis is interesting in its own right, and focuses on a class of Fokker–Planck operators with logarithmic potentials that is very natural, especially in dimension 2. This linear problem models convergence of a density attracted by a fast diffusing chemical to a target that releases it. Secondly, we present an initial nonlinear application of the techniques we develop which also involves the reaction term. In the nonlinear case, this paper focuses on the radial setting and develops a general framework for applying the linear techniques to analysis of reaction rates. Generalizations to more general settings will be addressed in future work; Section 9 outlines some of the avenues that we expect to pursue. An interesting by-product of our work is a suggestion that the traditional Keller–Segel term may be ill-suited to accurately modeling reaction enhancement effects, and a so-called flux-limited version may be more appropriate. This is also discussed in more detail below and in Section 9.

To describe our main results, we begin from the nonlinear application that will motivate the linear problem. For the sake of simplicity, we assume that the initial condition for ρ_2 is compactly supported and smooth: $\rho_2(x,0) = \theta\eta(x)$, where θ is a coupling constant, and $\eta \in C_0^\infty(\mathbb{R}^2)$ is close to the characteristic function of the disc B_R centered at the origin in the L^1 norm – obviously, we can make it as close as we want. It is useful to rescale (1.3); by a space-time rescaling we can normalize the parameters κ and R, so that (1.3) becomes

$$\begin{aligned}
&\partial_t\rho_1 - \Delta\rho_1 + \chi\nabla\cdot(\rho_1\nabla(-\Delta)^{-1}\rho_2) = -\epsilon\rho_1\rho_2, \\
&\partial_t\rho_2 = -\epsilon\rho_1\rho_2,
\end{aligned} \tag{1.4}$$

where for simplicity we keep the same notation for variables and parameters. The connection between parameters before and after rescaling will be documented after Theorem 1.1 below. The initial condition for ρ_2 has the form $\rho_2(x,0) = \theta\eta(x)$, with some $\theta > 0$ and radial $\eta \in C_0^\infty(\mathbb{R}^2)$, such that η is close in L^1 to the characteristic function χ_{B_1} of the unit disk, with

$$0 \le \chi_{B_1}(x) \le \eta(x) \le 1.$$

In particular, we can think of the constant θ as a measure of total initial mass of ρ_2:

$$\theta\pi \leq \int \rho_2(x,0)\,dx \leq 2\theta\pi. \tag{1.5}$$

It is not difficult to extend our results to more general radial initial data $\rho_2(\cdot,0) \in C_c^\infty(\mathbb{R}^2)$ or just rapidly decaying – though at the expense of less sharp constants and more technicalities. Nevertheless, to reduce technicalities, it will be convenient for us to think of η as very close to χ_{B_1}.

For the initial condition $\rho_1(x,0) \geq 0$ for (1.4), we assume that it is smooth and quickly decaying at infinity, and is located at a distance $\sim L$ from the origin. Specifically, we will assume that its mass in a ball $B_L(0)$ is at least M_0 while the mass inside B_1 is much smaller than M_0:

$$\int_{|x|\leq L} \rho_1(x,0)\,dx \geq M_0, \quad \int_{|x|\leq 1} \rho_1(x,0)\,dx \ll M_0. \tag{1.6}$$

Thus, M_0, L, θ, χ and ϵ are the parameters of the problem, and it is convenient to combine the mass of ρ_2 that is $\sim \theta$ and χ into a single parameter $\gamma := \theta\chi$. We are primarily interested in the situations where M_0 is large, so that $M_0\epsilon \gg \gamma \gg 1$ and $M_0 \gg \theta$; the motivation for such a relationship between the parameters will be discussed below. Our goal is to compare the efficiency of reaction, that is, the decay rate of the integral

$$\int_{\mathbb{R}^2} \rho_2(x,t)\,dx,$$

with and without chemotaxis. A reasonable measure of the reaction rate is a typical "half-time" scale during which about half of the initial mass $\sim \theta$ of ρ_2 will react. More precisely, the half-time τ_C will be the time by which the mass of ρ_2 decreases by $\pi\theta/2$. Our main nonlinear application is

Theorem 1.1. *Let the constants χ and ϵ describe the chemotactic mobility and the reaction strength as in* (1.4). *Suppose that M_0 and $L > 1$ satisfy* (1.6). *Let θ be as above and in particular satisfy* (1.5) *and $\gamma = \theta\chi$. Assume that the initial conditions $\rho_1(\cdot,0)$ and $\rho_2(\cdot,0)$ are as above and, in addition, radially symmetric. Suppose $\chi\gamma/\epsilon \geq \tilde{c} > 0$. There exists $B > 0$ sufficiently large, depending only on $\tilde{c}$, such that if*

$$M_0\epsilon/\gamma, \gamma, M_0/\theta \geq B, \tag{1.7}$$

then the half-time for the solution of the system (1.4) *satisfies*

$$\tau_C \leq C_1(L^2/\gamma + \log\gamma) \tag{1.8}$$

with a constant C_1 only depending on $\tilde{c}$ and B. On the other hand, if $\chi = 0$ and $\rho_1(\cdot,0)$ is supported in $\{|x| \geq L/2\}$, then the pure reaction-diffusion half-time satisfies $\tau_D \geq C_2L^2/\log(\epsilon M_0)$, where C_2 is a universal constant.

Remarks. 1. Note that time-space rescaling leading from (1.3) to (1.4) is given by $x' = x/R$, $t' = tR^2/\kappa$. The new parameters are given by $\chi' = \chi R^2/\kappa$, $\epsilon' = \epsilon R^2/\kappa$, $M_0' = M_0/R^2$, $L' = L/R$, and $\gamma' = \theta\chi' = \theta R^2\chi/\kappa$. As mentioned above, after the change of variables, we denote the new parameters without primes. Conditions (1.7) in the original parameters take the form $M_0\epsilon/(\theta R^2\chi) \geq B$, $\theta R^2\chi/\kappa \geq B$, $M_0/(\theta R^2) \geq B$. Here $\theta R^2 \sim$ initial mass of ρ_2.

2. The assumption (1.7) is reasonable in many applications. For example, in coral spawning, a typical number of sperms is of the order $\sim 10^{10}$, the number of eggs $\sim 10^6$, and $\epsilon \sim 10^{-2}$. It is difficult to find data on the measurements of strength of chemotactic coupling in biological literature.

3. In Section 7, we prove Theorem 7.3, a variant of Theorem 1.1 that eliminates the $\log\gamma$ term in (1.8) at the price of providing slightly less precise information about the dynamics of the system.

We believe that, possibly up to a correction logarithmic in γ, the result of Theorem 1.1 is sharp. It indicates that the presence of chemotaxis can significantly improve reaction rates if $\gamma \gg \log(M_0\epsilon)$. In particular, in the framework of (1.4), one can expect chemotaxis to provide significant improvement only if γ is sufficiently large.

There are natural further questions, discussed in some detail in Section 9. Here, let us just comment on the radial assumption on the initial data. The technical reason behind this condition is an artifact of the Keller–Segel form of the chemotaxis term. As the chemical concentration is $(-\Delta)^{-1}\rho_2$, the ρ_1 species concentrates near the center of the support of ρ_2, and, in general, it may arrive there without ever meeting ρ_2, so that reaction is not enhanced at all. This is prohibited in the radial geometry where the ρ_1 species will have to see ρ_2 as they move toward the origin. We expect that the techniques developed in this paper should apply to other chemotactic models and to a broader class of initial data configurations, with Theorem 1.1 as an initial application.

The proof of Theorem 1.1 relies on several ideas. We expect that the main positive effect of chemotaxis is in speeding up transport of the ρ_1 species towards the origin where the ρ_2 species is concentrated. To capture this, we estimate the transport stage by comparing the solutions of the coupled system to the solutions of the linear Fokker–Planck equation with a properly chosen time independent potential,

$$\partial_t\rho - \Delta\rho + \nabla\cdot(\rho\nabla H) = 0. \tag{1.9}$$

One would wish to take $\rho(x,0) = \rho_1(x,0)$, and $H = \chi(-\Delta)^{-1}\rho_2$. However, the time dependence of H would complicate the analysis. Instead, we use a comparison to the solution to (1.9) with the "weakest" attractive potential $H(x)$ in an appropriate class. The operator

$$F_H\phi = -\Delta\phi + \nabla\cdot(\phi\nabla H)$$

appearing in (1.9) is self-adjoint and nonnegative on the weighted space $L^2(e^{-H}, dx)$ and, if γ is sufficiently large, has a ground state e^H. The rate of convergence of the solution to the ground state for large times corresponds to transport of the density ρ from the far field towards the region with higher values of $H(x)$. As we will see, the worst case potential is

$$H(x) = \gamma(-\Delta)^{-1}\big(\chi_{B_1}(x) - \chi_{B_{1/\sqrt{2}}}(x)\big). \tag{1.10}$$

It is not difficult to deduce that in dimension 2, $H(x) \approx -(\gamma\pi/2)\log|x|$ for $|x| \gg 1$, and we need to deal with a Fokker–Planck equation with a logarithmic potential. We stress that all estimates we prove for the linear problem (1.9) apply in full generality, without radial constraint on f.

Thus, our principal goal in this paper is to provide very precise bounds on the rate of convergence to the ground state for this class of Fokker–Planck operators, and to develop a comparison scheme to use these estimates in the analysis of nonlinear problems. The rate of convergence to equilibrium for Fokker–Planck operators is a classical subject, and the literature on this question is vast. The uniformly convex case $-D^2H(x) \geq \lambda\,\mathrm{Id}$ with $\lambda > 0$ can be viewed as a direct application of Brascamp–Lieb ideas [9], and the operator F_H has a spectral gap, so that convergence to the ground state is exponential in time. There has been much work on generalizations of these results. An extension to (in particular) $H(x) = |x|^\beta$ with $1 < \beta < 2$ and further references can be found in [1]. For slower growth potentials there may be no spectral gap. Röckner and Wang [66] have initiated analysis of convergence to equilibrium for $H(x) = |x|^\beta$ with $0 < \beta < 1$ which are subexponential in time (see also [69]), as well as algebraic in time convergence bounds for a logarithmic potential – which is precisely our case. However, the dependence of these bounds on the coupling constant is not sufficiently sharp for the applications that motivate us. There is also related work based on probabilistic techniques by Veretennikov [57, 72]. These estimates are designed with different applications in mind, and are also insufficient for our purpose.

While weighted Poincaré inequalities can be used to prove exponential in time convergence to equilibrium for Fokker–Planck operators, the tools that can be deployed when the rate of convergence is slower are called weak Poincaré or Poincaré-type inequalities. An inequality of this kind involving Cauchy-type power weights has been proved by Bobkov and Ledoux [7] (see also [4]). That paper contains, in particular, the following inequality for every $f \in C_0^\infty(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} |f - \bar{f}|^2 v(x)\,dx \leq \frac{C}{\gamma}\int_{\mathbb{R}^d} |\nabla f|^2(1 + |x|^2)v(x)\,dx, \tag{1.11}$$

with the weight $v(x) = (1 + |x|^2)^{-\gamma/2}$ for some sufficiently large γ, and

$$\bar{f} = \int_{\mathbb{R}^d} f(x)v(x)\,dx.$$

The proof of Bobkov and Ledoux is based on convexity techniques, and builds on generalizations of the Brascamp–Lieb inequality [9]. For our application, we need a version of (1.11) with the weight $w(x) = e^{H(x)}$. While the behavior of $w(x)$ and $v(x)$ near infinity is virtually identical, the weight $w(x)$ does not seem to satisfy the convexity assumptions needed for the techniques of [7] to work. Moreover, the factor C/γ on the right side of (1.11) would lead to suboptimal estimates on the rate of convergence to the ground state. One could verify that such estimates could only yield $\tau_C \lesssim L^2$ in Theorem 1.1. This is not very interesting, since pure reaction-diffusion is not outperformed in relevant regimes.

In recent years, there has been more work focusing on weak weighted Poincaré inequalities for Cauchy-type measures [8, 12]. The paper [8] proves a one-dimensional estimate similar to [7] by estimating the spectral gap of a related transformed operator. The paper [12] applies the elegant Lyapunov function method which in application to the weight $v(x)$ above yields an estimate similar to (1.11). The paper [12] also treats a non-smooth weight $(1+|x|)^{-d-\gamma}$, and in this case the constant C/γ in (1.11) is replaced by a sharp (for the far range) constant C/γ^2. The proof of this sharp bound relies on the mass transportation method reducing the analysis to a spectral gap estimate for a related operator, which has been analyzed in [6]. The latter work relies on a variational approach. It is not clear how to extend it to the smooth weight $v(x)$ or the weight e^H that features different behavior (and so different scaling of constants) in different regions.

Here, we prove precise weak weighted Poincaré estimates for weights with Cauchy-type behavior at infinity which can be flat near the origin. The exact result that we need for the sharpest estimates on convergence to equilibrium, Theorem 5.2, is a bit too technical to state in the introduction, but it implies in particular the following improved weighted Poincaré-type inequality by distinguishing the regions where behavior of the weight w is different.

Theorem 1.2. *Let $\gamma > 2$, $f \in C_0^\infty(\mathbb{R}^2)$, and $w(x) = e^{H(x)}$, with H given by (1.10). Then the following weak weighted Poincaré inequality holds:*

$$\int_{\mathbb{R}^2} |f-\bar{f}|^2 w(x)\,dx \le C\int_{B_1} |\nabla f|^2 w(x)\,dx + \frac{C}{\gamma^2}\int_{B_1^c} |\nabla f|^2(1+|x|^2)w(x)\,dx. \tag{1.12}$$

The bound (1.12) provides an improvement from γ^{-1} to γ^{-2} in the far field, which is absolutely crucial for our application. It is not difficult to build examples to show that such scaling is sharp. For the straight power weight $v(x)$, our results imply that

$$\int_{\mathbb{R}^d} |f-\bar{f}|^2 v(x)\,dx \le \frac{C(d)}{\gamma}\int_{B_1} |\nabla f|^2 v(x)\,dx + \frac{C(d)}{\gamma^2}\int_{(B_1)^c} |\nabla f|^2(1+|x|^2)v(x)\,dx \tag{1.13}$$

for all $\gamma > d$. Our proof of Theorem 1.2 is based on direct analytic estimates.

The weak weighted Poincaré estimates that we show allow us to derive quite sharp bounds on convergence to equilibrium for the Fokker–Planck equation (1.9). Here is a sample result for the weighted L^2 norm. Define

$$Z(t) = \int_{\mathbb{R}^2} (\rho(x,t)-\rho_s(x))^2 e^{-H(x)}\,dx.$$

Theorem 1.3. *Fix any $\sigma \in (0,1)$. Let $\gamma \ge \gamma_0(w)$ be sufficiently large, and let $\rho(x,t)$ be the solution to (1.9) with initial condition $\rho_0 \in L^\infty(e^{-H}) \cap L^1(\mathbb{R}^2)$. Set $t_1 :=$*

$C(1 + \log \sigma^{-1} + \log \gamma)$, *where* C *is a sufficiently large universal constant, and* $Z^\sigma := \sigma e^{-H(0)} \|\rho_0\|_1^2$. *Then for all* $t \geq t_1$, *we have*

$$Z(t) \leq \max \{Z^\sigma, (c\gamma(t - t_1))^{-(\gamma-8)/8} \|\rho_0 e^{-H}\|_\infty^2\}, \tag{1.14}$$

where $c > 0$ *is a universal constant.*

The inequality (1.14) provides very fast, optimal in t and γ (modulo t_1 which arises for technical reasons and constant factors), rate of decay when $Z(t)$ is large. One cannot expect such decay rate for all times, since as solution concentrates near the flat part of the weight the dynamics changes. Also, we will need to use a duality argument to get a faster decay rate in order to rigorously reach the natural heuristic laws in the nonlinear application.

We note that there have been several papers that analyzed convergence rates to equilibrium for Fokker–Planck operators with Cauchy-type equilibrium measures [2, 41, 66]. Specifically, the paper [2] contains the sharpest earlier estimate for the logarithmic potential case which has the correct scaling in time, but not in γ; the reason for that is similar to the lack of sharp far-range constant in (1.11) and is essentially a consequence of the analysis not separating the weight into qualitatively different regions. We believe that our work is the first one that provides virtually sharp (in t and γ) estimates on convergence to equilibrium for a Fokker–Planck operator with a potential which has qualitatively different behavior in different regions: flat near the origin and logarithmic in the far range. This is exactly what one needs to understand the parabolic-elliptic chemical attraction in two dimensions: a density attracted by a fast diffusing chemical secreted by a given fixed target. We believe therefore that the linear problem estimates are of independent interest, and to make these estimates relevant in applications they need to be quite precise.

The paper is organized as follows. In Section 2, we provide a heuristic motivation for the main application result. In Section 3, we sketch the proof of the global well-posedness for (1.4), along with an L^∞ bound on the density ρ_1. In Section 4, we discuss the mass comparison principles, which will allow the estimates for linear Fokker–Planck equations with a time independent potential to be useful for the nonlinear analysis. In Section 5, we derive new weak weighted Poincaré inequalities, in particular proving Theorem 1.2, and in Section 6 we use these inequalities to obtain estimates on the rates of convergence to the ground state for Fokker–Planck operators with logarithmic-type potentials. In Section 7, we provide a brief detour and show how to set up a version of Theorem 1.1, Theorem 7.3, using only comparison principles and avoiding the analysis of Fokker–Planck equations. This argument is much simpler, and generates a result similar to our main application here. However, it provides limited information on the distribution of ρ_1 near the target support, which may be useful in other applications, and does not yield intuition explaining limitations of the standard Keller–Segel chemotaxis term that led to the radial assumption. In Section 8, we apply the results proved in previous sections to finalize the proofs of Theorems 1.1 and 7.3. In Section 9, we provide a preview of more advanced applications that we believe may be possible using the techniques developed.

Throughout the paper, we will denote by $\|f\|_p$ the $L^p(\mathbb{R}^d)$ norm of the function f with respect to Lebesgue measure. The notation $\lesssim, \gtrsim$ and $\sim$ means, as usual, bounds with universal constants independent of the key parameters of the problem. The constants C, c appearing in the estimates are universal constants that may change from step to step.

2. Heuristics

In order to decide whether the chemotaxis term can enhance reaction, it suffices to compare the half-times τ_C, τ_D in the two systems, with and without chemotaxis, respectively. In Section 2.1, we will derive a *rigorous* lower bound for τ_D in the absence of chemotaxis. We then give a *heuristic* argument for the full system in Section 2.2, formally deriving an upper bound for τ_C in the presence of the chemotaxis term. Comparing with the estimate without chemotaxis, it suggests that in a certain parameter regime, chemotaxis should significantly shorten the half-time, thus meaningfully enhancing the reaction between the two densities. Of course, the upper bound for τ_C in the system with chemotaxis is just formal at this moment, but it will be made rigorous in the rest of this paper in the radially symmetric case.

2.1. Estimates in the purely diffusive case

Consider the system without chemotaxis,

$$\begin{aligned} &\partial_t \rho_1 - \Delta\rho_1 = -\epsilon\rho_1\rho_2, \\ &\partial_t \rho_2 = -\epsilon\rho_1\rho_2, \end{aligned} \tag{2.1}$$

where the initial conditions are the same as for the original system (1.4). The time τ_D it takes for $\|\rho_2(\cdot,t)\|_{L^1}$ to drop by half obeys a lower bound

$$\tau_D \geq \tau. \tag{2.2}$$

Here, τ is the time it takes for $\|g_2\|_{L^1}$ to drop by half, where g_2 is the solution to

$$\begin{cases} \partial_t g_1 = \Delta g_1, \\ \partial_t g_2 = -\epsilon g_1 g_2, \end{cases} \tag{2.3}$$

where g_1 and g_2 have the same initial data as ρ_1 and ρ_2 respectively. Indeed, the comparison principle implies that $\rho_1(\cdot,t) \leq g_1(\cdot,t)$ for all $t \geq 0$, so that $\rho_2(\cdot,t) \geq g_2(\cdot,t)$, and (2.2) follows.

Recall that $g_1(\cdot,0) = \rho_1(\cdot,0)$ is concentrated at a distance $L \gg 1$ from the origin, in the sense of (1.6), and $\rho_1(x,0)$ is supported inside $|x| \geq L/2$. This gives an upper bound

$$g_1(x,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4t}}\rho_1(y,0)dy \leq \frac{M_0}{4\pi t}e^{-CL^2/t} \quad \text{for all } x \in B(0,1).$$

One can plug this estimate in the equation for g_2 and obtain

$$\partial_t \log g_2 \geq -\frac{M_0\epsilon}{4\pi t}e^{-CL^2/t}.$$

Hence, τ_D satisfies

$$M_0\epsilon \int_0^{\tau_D} \frac{1}{4\pi t}e^{-CL^2/t}\,dt \geq \log 2,$$

which, after a change of variable $y = CL^2/t$, is equivalent to

$$\int_{CL^2/\tau_D}^{\infty} \frac{e^{-y}}{y}\,dy \geq \frac{4\pi \log 2}{M_0\epsilon}. \tag{2.4}$$

To estimate τ_D, we consider two cases.

Case 1. $M_0\epsilon \ll 1$, which is a very weak reaction regime, or fairly small M_0 regime. Then (2.4) is equivalent to

$$\int_{CL^2/\tau_D}^{1} \frac{1}{y}\,dy \gtrsim \frac{1}{M_0\epsilon},$$

or $-\log(CL^2/\tau_D) \gtrsim 1/(M_0\epsilon)$. Thus τ_D has to satisfy

$$\tau_D \gtrsim L^2 e^{\frac{C'}{M_0\epsilon}}, \tag{2.5}$$

which is a very long time due to the large exponent.

Case 2. $M_0\epsilon \gg 1$, the reaction regime that appears more relevant to the applications we have in mind. In this case we have $CL^2/\tau_D \gg 1$, hence for a lower bound for τ_D, one can find τ such that

$$\int_{CL^2/\tau_D}^{\infty} e^{-y}\,dy \gtrsim \frac{1}{M_0\epsilon},$$

which reduces to $CL^2/\tau_D \lesssim \log(M_0\epsilon)$, and gives a bound

$$\tau_D \gtrsim \frac{L^2}{\log(M_0\epsilon)}. \tag{2.6}$$

2.2. Formal heuristics with the chemotaxis term

Now we come back to the full system (1.4), including the chemotaxis term. Again, let τ_C denote the half-time of ρ_2. The following formal argument suggests that adding this term may significantly reduce the half-time in the regime $M_0\epsilon \gg 1$, where we will formally argue that $\tau_C \sim L^2/\gamma \ll \tau_D \sim L^2/\log(M_0\epsilon)$ as long as $\log(M_0\epsilon) \ll \gamma$.

To this end, note that due to chemotaxis, ρ_1 is advected by the velocity field

$$v(x,t) = \chi\nabla((-\Delta)^{-1}\rho_2)(x,t) = -\frac{\chi}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\rho_2(y,t)\,dy.$$

Since τ_C is the half-time for ρ_2, for any $t \le \tau_C$ we have $\|\rho_2(\cdot,t)\|_{L^1} \sim \theta$, and $\rho_2(\cdot,t)$ is supported near the origin. Therefore, for all $|x| \ge 2$ and $t \le \tau_C$, we have the following lower bound for the inward drift:

$$v(x,t) \cdot \frac{-x}{|x|} \sim \chi \int_{\mathbb{R}^2} \frac{\rho_2(y,t)}{|x-y|}\,dy \sim \frac{\gamma}{|x|}.$$

Recall that initially all of ρ_1 starts at distance L from the origin. Hence, in time $t \sim L^2/\gamma$, the chemotactic transport should bring a significant portion (say, half) of ρ_1 into $B_1(0)$, and then $\rho_1 \sim M_0$ in this ball. This enables the mass of ρ_2 to decrease exponentially at the rate $M_0\epsilon \gg 1$, and the half-time is quickly reached; thus one formally expects $\tau_C \lesssim L^2/\gamma$.

In the "risky" regime $M_0\epsilon \ll 1$, we need to add nontrivial reaction time, which is now of the order $\sim 1/(M_0\epsilon)$. Then, one expects

$$\tau_C \sim \frac{L^2}{\gamma} + \frac{1}{M_0\epsilon},$$

which can be quite a dramatic improvement compared to (2.5).

Note that this heuristic argument ignores many essential points, such as effect of diffusion, or close field dynamics. There are indications that for the Keller–Segel chemotaxis term, reaction time may be longer due to "overconcentration" of ρ_1. We discuss this point further in Section 9.

3. Global regularity and an L^∞ bound

In order to get a uniform bound for the solutions to (1.4), let us first consider an equation with a prescribed drift,

$$\rho_t - \Delta\rho + \nabla\cdot(\rho\nabla\Phi(x,t)) = -h(x,t)\rho, \tag{3.1}$$

where $h \in L^\infty(\mathbb{R}^d \times [0,\infty))$ is nonnegative, Φ is H^2_{loc} in space for all time, and such that $\nabla\Phi \in L^\infty(L^\infty(\mathbb{R}^d), [0,\infty))$. The proof of the following a priori L^1-L^∞ bound for (3.1) is very close to that of [10, Theorem 5]. We recall it in Appendix A.1 for completeness.

Theorem 3.1. *Let the initial condition ρ_0 for (3.1) satisfy $\rho_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Assume that $h \in L^\infty(\mathbb{R}^d \times [0,\infty))$ is nonnegative, and Φ is H^2_{loc} in space for all time and $\nabla\Phi \in L^\infty(L^\infty(\mathbb{R}^d), [0,\infty))$. If there exists $\gamma > 0$ such that $\Delta\Phi(\cdot,t) \ge -\gamma$ for all $t \ge 0$, then*

$$\|\rho(\cdot,t)\|_\infty \le C(d)\max\{t^{-d/2}, \gamma^{d/2}\}\,\|\rho_0\|_1 \quad \text{for all } t \ge 0. \tag{3.2}$$

The assumption that $\rho_0 \in L^\infty(\mathbb{R}^d)$ in Theorem 3.1 is not necessary, and is made simply because we always consider solutions with bounded initial conditions. We also discuss well-posedness and regularity of solutions to (3.1) briefly in Section 5 and Appendix A.2.

Note that the ρ_1-equation in (1.4) is of the form (3.1) with $h = \epsilon\rho_2 \ge 0$ and the potential $\Phi(\cdot,t) = \chi(-\Delta)^{-1}\rho_2(\cdot,t)$. The potential Φ grows at a logarithmic rate at infinity, and minimal beyond L^∞ regularity of ρ_2 would ensure that $\Phi \in H^2_{\mathrm{loc}}$. This extra regularity is established below in Theorem 3.3. Also, from the explicit formula for the inverse Laplacian it is not hard to see that $\nabla\Phi \in L^\infty(L^\infty(\mathbb{R}^d), [0,\infty))$. We will therefore be able to apply Theorem 3.1 to obtain an a priori bound for $\|\rho_1(\cdot,t)\|_{L^\infty}$.

The global regularity of solutions to (1.4) in all dimensions $d \geq 1$ follows from a standard argument, which we briefly sketch below. The following lemma contains the key estimates.

Lemma 3.2. *Suppose $f \in L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$, $g \in L^1(\mathbb{R}^d) \cap H^{m-1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with an integer $m > d/2$. Then*

$$\|f\nabla(-\Delta)^{-1}g\|_1 \leq C\|f\|_1(\|g\|_1 + \|g\|_\infty), \tag{3.3}$$

$$\|f\nabla(-\Delta)^{-1}g\|_{H^m} \leq C(\|g\|_1 + \|g\|_\infty)\|f\|_{H^m} + C\|f\|_\infty\|g\|_{H^{m-1}}. \tag{3.4}$$

Proof. The inequality (3.3) follows from the estimates

$$\|f\nabla(-\Delta)^{-1}g\|_1 \leq \|f\|_1\|\nabla(-\Delta)^{-1}g\|_\infty,$$

and

$$\|\nabla(-\Delta)^{-1}g\|_\infty \leq C\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^{-d+1}|g(y)|\,dy \leq C(\|g\|_1 + \|g\|_\infty). \tag{3.5}$$

To estimate the H^m norm in (3.4), let us start with the L^2 norm which is controlled similarly to (3.5):

$$\|f\nabla(-\Delta)^{-1}g\|_2 \leq \|f\|_2\|\nabla(-\Delta)^{-1}g\|_\infty \leq C\|f\|_{H^m}(\|g\|_1 + \|g\|_\infty).$$

Any other term that we need to estimate to control the H^m norm squared from the right hand side of (3.4) is of the form

$$\int_{\mathbb{R}^d} D^j(f\nabla(-\Delta)^{-1}g)\cdot D^j(f\nabla(-\Delta)^{-1}g)\,dx,$$

where D^j is some partial derivative of order $j \leq m$. It suffices to control any term of the form

$$\int_{\mathbb{R}^d}|D^{j-s}f|^2|D^s\nabla(-\Delta)^{-1}g|^2\,dx, \tag{3.6}$$

where the integer s satisfies $0 \leq s \leq j$. If $s = 0$, then (3.6) is bounded by $\|\nabla(-\Delta)^{-1}g\|_\infty^2\|f\|_{H^j}^2$, and using (3.5) and $j \leq m$ leads to the estimate we seek. If $s = j$, then (3.6) is bounded by $\|f\|_\infty^2\|g\|_{H^{m-1}}^2$. If $j > s > 1$, we can estimate (3.6) by

$$C\|D^{j-s}f\|_p^2\|D^{s-1}g\|_q^2, \tag{3.7}$$

where $p^{-1} + q^{-1} = 1/2$, and $1 < p, q < \infty$. Specifically, let us choose $p = \frac{2(j-1)}{j-s}$ and $q = \frac{2(j-1)}{s-1}$. In this step we have used only that the Riesz transforms are bounded in L^r if $1 < r < \infty$. Recall a Gagliardo–Nirenberg inequality

$$\|D^k f\|_{2n/k} \leq C\|f\|_\infty^{1-k/n}\|f\|_{H^n}^{k/n}, \tag{3.8}$$

valid in any dimension for integers k, n such that $0 < k < n$ (see e.g. [51]). Applying it to the norms in (3.7) with $n = j - 1$, and $k = j - s$ and $k = s - 1$, respectively, we get the bound

$$C\|f\|_\infty^{2-2\frac{j-s}{j-1}}\|f\|_{H^{j-1}}^{2\frac{j-s}{j-1}}\|g\|_\infty^{2-2\frac{s-1}{j-1}}\|g\|_{H^{j-1}}^{2\frac{s-1}{j-1}} \le C(\|f\|_\infty^2\|g\|_{H^{j-1}}^2 + \|g\|_\infty^2\|f\|_{H^{j-1}}^2).$$

Here we have used the inequality $a^\beta b^{1-\beta} \le a + b$ for $a, b \ge 0$ and $0 \le \beta \le 1$. Finally, if $s = 1$, note that we can assume $j > 1$ since otherwise $s = j$ and this is covered above. In this case, we estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |D^{j-1}f|^2|D\nabla(-\Delta)^{-1}g|^2\,dx &\le \|D^{j-1}f\|_{\frac{2j}{j-1}}^2\|D\nabla(-\Delta)^{-1}g\|_{2j}^2\\ &\le C\|D^{j-1}f\|_{\frac{2j}{j-1}}^2\|g\|_{2j}^2.\end{aligned}$$

Due to (3.8),

$$\|D^{j-1}f\|_{\frac{2j}{j-1}} \le C\|f\|_\infty^{\frac{1}{j}}\|f\|_{H^j}^{\frac{j-1}{j}},$$

while

$$\|g\|_{2j} \le \|g\|_\infty^{\frac{2j-1}{2j}}\|g\|_1^{\frac{1}{2j}}.$$

By Young's inequality,

$$\begin{aligned}\|f\|_\infty^{\frac{1}{j}}\|f\|_{H^j}^{\frac{j-1}{j}}\|g\|_\infty^{\frac{2j-1}{2j}}\|g\|_1^{\frac{1}{2j}} &\le C(\|f\|_{H^j}\|g\|_\infty^{\frac{2j-3}{2j-2}}\|g\|_1^{\frac{1}{2j-2}} + \|f\|_\infty\|g\|_\infty)\\ &\le C\|f\|_{H^j}(\|g\|_1 + \|g\|_\infty) + C\|f\|_{H^m}\|g\|_\infty.\end{aligned}$$

Here in the last step we have used $m > d/2$. Since also $m \ge j$, the lemma follows. ∎

Theorem 3.3. *If the initial conditions $\rho_1(\cdot, 0)$, $\rho_2(\cdot, 0)$ for (1.4) are nonnegative, and lie in $L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d)$ with an integer $m > d/2$, then there is a unique global-in-time solution $(\rho_1(\cdot, t), \rho_2(\cdot, t)) \in C(L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d), [0, \infty))$ to (1.4).*

Proof. We assume that $\rho_1(\cdot, 0)$, $\rho_2(\cdot, 0)$ are nonnegative purely for simplicity since this is the case in all our applications. This assumption is not hard to remove. We note that the standard comparison principle implies that nonnegativity is conserved in time for all sufficiently regular solutions.

The local-in-time well-posedness in $C(L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d), [0, T])$ can be shown by a standard argument, using the Duhamel formula and the contraction mapping theorem. In our case the Duhamel formula takes the form

$$\begin{aligned}\rho_1(x, t) &= e^{t\Delta}\rho_1(x, 0)\\ &\quad + \int_0^t e^{(t-s)\Delta}\big(-\chi\nabla\cdot(\rho_1(s)\nabla(-\Delta)^{-1}\rho_2(s)) - \epsilon\rho_1(s)\rho_2(s)\big)\,ds, \qquad (3.9)\\ \rho_2(x, t) &= e^{-\epsilon\int_0^t \rho_1(x,s)\,ds}\rho_2(x, 0). \qquad (3.10)\end{aligned}$$

We can reduce this system to a single equation substituting (3.10) into (3.9). Using the fact that H^m is an algebra if $m > d/2$, Lemma 3.2, and simple estimates of the type

$$\|e^{-\epsilon \int_0^t \rho_1(\cdot,s)\,ds}\rho_2(\cdot,0)\|_{H^m} \le \|\rho_2(\cdot,0)\|_{H^m} e^{\epsilon t \|\rho_1\|_{C(H^m(\mathbb{R}^d),[0,t])}}, \tag{3.11}$$

it is not hard to obtain local well-posedness in $C(L^1(\mathbb{R}^d) \cap H^m(\mathbb{R}^d), [0,T])$ for sufficiently small T using the contraction mapping theorem. We refer for details to [46, Appendix I] where a very similar argument can be found.

As usual with parabolic equations, the solution is actually more regular for every $t > 0$; however, in our setting the solution does not become C^∞ due to the lack of a smoothing mechanism for ρ_2 (unless we assume $\rho_2(\cdot,0) \in C^\infty$, in which case the solution will be C^∞ for all $t > 0$). In the general setting, using (3.11), the fact that H^m is an algebra, and the elementary estimate

$$\|\nabla e^{t\Delta} f\|_{H^{m+1-\epsilon}} \le C t^{-1+\epsilon/2} \|f\|_{H^m}$$

for any $\epsilon > 0$, we can deduce from (3.9) and (3.10) that $\rho_1 \in H^{m+1-\epsilon}(\mathbb{R}^d)$ for every $t > 0$. Using (3.4) to bootstrap this stronger regularity, we can then find that for any $1 > \epsilon > 0$, $\rho_1 \in H^{m+2-\epsilon}(\mathbb{R}^d)$ for all $t > 0$ (in fact, $t = 0$ is excluded only because of the first linear heat evolution term in (3.9)). This regularity implies that for a local solution we have $\partial_t \rho_1 \in H^{m-\epsilon}(\mathbb{R}^d)$ for all $t > 0$, and this along with higher regularity in spatial variables can be used to justify the calculation of $\partial_t \|\partial^\alpha \rho_1\|_2^2$ and integrations by parts below; the justification for the latter uses trace theorems for fractional Sobolev spaces (see e.g. [19]).

Now let us prove global regularity. By integrating the equations, we find that the L^1 norms of $\rho_1(\cdot,t)$ and $\rho_2(\cdot,t)$ (which are equal to their integrals due to nonnegativity) are nonincreasing in time. Hence to improve the local well-posedness result to a global-in-time one, it suffices to obtain an a priori bound on

$$I(t) := \|\rho_1(t)\|_{H^m}^2 + \|\rho_2(t)\|_{H^m}^2$$

on any given finite time interval $[0,T]$. Fix any multi-index α with $0 \le |\alpha| \le m$ and write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha \rho_2\|_2^2 &= -\epsilon \int_{\mathbb{R}^2} (\partial^\alpha \rho_2)\partial^\alpha(\rho_1\rho_2)\,dx \le \epsilon \|\rho_2\|_{H^m}\|\rho_1\rho_2\|_{H^m} \\ &\le C\|\rho_2\|_{H^m}(\|\rho_1\|_\infty \|\rho_2\|_{H^m} + \|\rho_2\|_\infty \|\rho_1\|_{H^m}) \\ &\le C(\|\rho_1(\cdot,t)\|_\infty + 1)(\|\rho_1(\cdot,t)\|_{H^m}^2 + \|\rho_2(\cdot,t)\|_{H^m}^2).\end{aligned} \tag{3.12}$$

Here, the second line is obtained by the inequality (see, e.g., [50, Lemma 3.4])

$$\|uv\|_{H^m} \le C(\|u\|_\infty \|v\|_{H^m} + \|v\|_\infty \|u\|_{H^m}) \quad \text{for } m > d/2, \tag{3.13}$$

and in the last line we use the fact that $\|\rho_2(t)\|_\infty \le \|\rho_2(0)\|_\infty \le C$. As for $\|\rho_1\|_{H^m}$, for any multi-index α as above, integration by parts gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha \rho_1\|_2^2 = -\|\nabla \partial^\alpha \rho_1\|_2^2 &+ \chi \int_{\mathbb{R}^2} \nabla(\partial^\alpha \rho_1)\cdot \partial^\alpha(\rho_1 \nabla(-\Delta)^{-1}\rho_2)\,dx \\ &- \epsilon \int_{\mathbb{R}^2} (\partial^\alpha \rho_1)\partial^\alpha(\rho_1\rho_2)\,dx.\end{aligned} \tag{3.14}$$

The last integral on the right side can be bounded by the right side of (3.12), while the first one can be estimated by

$$
\begin{aligned}
&\frac{1}{\chi}\|\nabla\partial^\alpha\rho_1\|_2^2+C\|\rho_1\nabla(-\Delta)^{-1}\rho_2\|_{H^m}^2\\
&\qquad\le\frac{1}{\chi}\|\nabla\partial^\alpha\rho_1\|_2^2+C\big(\|\rho_1\|_\infty^2\|\rho_2\|_{H^m}^2+\|\rho_1\|_{H^m}^2(\|\rho_2\|_1^2+\|\rho_2\|_\infty^2)\big).
\end{aligned}
\tag{3.15}
$$

We have used the Cauchy–Schwarz inequality and the Young inequality in the first line and Lemma 3.2 in the second line; the constants C depend on χ and may change from line to line. Combining the above estimates and taking into account that $\|\rho_2\|_1$ and $\|\rho_2\|_\infty$ are nonincreasing gives

$$
\frac{d}{dt}(\|\rho_1(\cdot,t)\|_{H^m}^2+\|\rho_2(\cdot,t)\|_{H^m}^2)\le C(\|\rho_1(\cdot,t)\|_\infty+1)^2(\|\rho_1(\cdot,t)\|_{H^m}^2+\|\rho_2(\cdot,t)\|_{H^m}^2).
\tag{3.16}
$$

The first equation in (1.4) gives the bound

$$
\|\rho_1(\cdot,t)\|_\infty\le\|\rho_1(\cdot,0)\|_\infty\exp\{\chi\|\rho_2(\cdot,t)\|_\infty t\}\le\|\rho_1(\cdot,0)\|_\infty\exp\{\chi\|\rho_2(\cdot,0)\|_\infty t\}.
\tag{3.17}
$$

Thus $\|\rho_1(\cdot,t)\|_\infty$ remains finite for all times, and then (3.16) leads to global regularity. To get a more precise bound, we may use (3.17) for $0\le t\le 1$, while for $t\ge 1$ we may deploy the uniform bound from Theorem 3.1. Therefore, there exists $C>0$ such that $\|\rho_1(\cdot,t)\|_\infty\le C$ for all $t\ge 0$. Then (3.16) gives exponential-in-time control of the H^m norms of the solution, for all times. ■

4. The mass comparison principle

We now obtain a comparison principle that allows us to compare ρ_1 to the solution ρ of the Fokker–Planck equation

$$
\partial_t\rho-\Delta\rho+\nabla\cdot(\rho\nabla H)=0
\tag{4.1}
$$

with a certain prescribed H. The comparison will be in a mass concentration sense that will be clarified in Proposition 4.3. The results of this section are valid in arbitrary space dimension – the proof below is given for $d=2$ for notational convenience since it is our setting in this paper, but the argument can be generalized in a straightforward manner. Let us assume that $H=(-\Delta)^{-1}g$, with a radially symmetric function $g=g(|x|)$ supported in a ball $B_{R_0}(0)$. The explicit form of g and H that we will use is given in (4.10) and (4.11). The function H is radially symmetric as well, and the divergence theorem gives

$$
\partial_r H(r)=\frac{1}{|\partial B_r|}\int_{B_r}\Delta H(x)\,dx=\frac{1}{2\pi r}\int_{B_r}(-g(x))\,dx=-\frac{1}{r}\int_0^r g(s)s\,ds.
\tag{4.2}
$$

Integrating in r gives an expression

$$
H(r)=-(\log r)\int_0^r g(s)s\,ds-\int_r^\infty(\log s)g(s)s\,ds+\text{const}.
\tag{4.3}
$$

Since g is compactly supported, taking the arbitrary constant in (4.3) to be zero gives

$$H(r) = -\frac{1}{2\pi}\|g\|_1 \log r, \quad r \geq R_0.$$

As a direct consequence of (4.2), we have the following.

Lemma 4.1. *Assume that g_1 and g_2 are both radially symmetric and compactly supported. Suppose that g_1 is more concentrated than g_2 in the sense that*

$$\int_0^r g_1(s)s\,ds \geq \int_0^r g_2(s)s\,ds \quad \textit{for all } r \geq 0.$$

Then the functions $H_i := (-\Delta)^{-1} g_i$, $i = 1, 2$, satisfy $\partial_r H_1 \leq \partial_r H_2 \leq 0$ for all $r > 0$. In addition, if $g_i \in L^\infty(\mathbb{R}^2)$, then $\partial_r H_i(0) = 0$ for $i = 1, 2$.

We now compare the mass concentration of solutions to the Fokker–Planck equations.

Proposition 4.2. *Suppose that u_1 and u_2 are nonnegative solutions to*

$$\partial_t u_i - \Delta u_i + \nabla \cdot (u_i \nabla H_i) = 0$$

for $i = 1, 2$, and u_1 is more concentrated than u_2 at $t = 0$, so that

$$\int_{B_r} u_1(x,0)\,dx \geq \int_{B_r} u_2(x,0)\,dx \quad \textit{for all } r \geq 0. \tag{4.4}$$

If in addition $H_2(\cdot, t)$ is radially symmetric, and

$$\partial_r H_2(r,t) \geq \max_\phi \partial_r H_1(r,\phi,t) \quad \textit{for all } t \geq 0 \textit{ and } r > 0, \tag{4.5}$$

then $u_1(\cdot, t)$ is more concentrated than $u_2(\cdot, t)$ for all $t \geq 0$.

Note that $u_{1,2}$ are not necessarily radially symmetric.

Proof. The masses

$$M_i(r,t) := \int_{B_r} u_i(x,t)\,dx$$

satisfy

$$\begin{aligned} \partial_t M_i(r,t) &= \int_{B_r} \Delta u_i\,dx - \int_{B_r} \nabla \cdot (u_i \nabla H_i)\,dx = \int_{\partial B_r} \partial_r u_i\,d\sigma - \int_{\partial B_r} u_i \partial_r H_i\,d\sigma \\ &= r\int_0^{2\pi} \partial_r u_i(r,\phi,t)\,d\phi - r\int_0^{2\pi} u_i(r,\phi,t)\partial_r H_i(r,\phi,t)\,d\phi. \end{aligned} \tag{4.6}$$

Here, $d\sigma = r\,d\phi$ is the surface measure on the boundary. Note that

$$\partial_r M_i = \int_{\partial B_r} u_i\,d\sigma = r\int_0^{2\pi} u_i(r,\phi,t)\,d\phi,$$

so that

$$\int_0^{2\pi} \partial_r u_i \, d\phi = \partial_r \left(\frac{\partial_r M_i}{r} \right) = \frac{\partial_{rr} M_i}{r} - \frac{\partial_r M_i}{r^2}.$$

Substituting the above two equations into (4.6) gives

$$\partial_t M_i(r,t) = \partial_{rr} M_i - \frac{1}{r} \partial_r M_i - r \int_0^{2\pi} u_i(r,\phi,t) \partial_r H_i(r,\phi,t) \, d\phi. \tag{4.7}$$

Subtracting the two equations and using the radial symmetry of H_2, we obtain

$$\begin{aligned} \partial_t (M_1 - M_2) - \partial_{rr}(M_1 - M_2) &+ \frac{1}{r} \partial_r (M_1 - M_2) \\ &\geq \partial_r H_2 \partial_r M_2 - (\partial_r M_1) \max_\phi \partial_r H_1(r,\phi,t) \\ &= -(\partial_r H_2) \partial_r (M_1 - M_2) + \Big(\partial_r H_2 - \max_\phi \partial_r H_1(r,\phi,t) \Big) \partial_r M_1 \\ &\geq -(\partial_r H_2) \partial_r (M_1 - M_2). \end{aligned} \tag{4.8}$$

We have used (4.5) as well as $\partial_r M_1 \geq 0$ in the last inequality above. Now, the standard parabolic comparison principle (see e.g. [49, 63]) and (4.4) imply that

$$M_1(r,t) \geq M_2(r,t) \quad \text{for all } r, t \geq 0.$$

To make the application completely routine one can consider $M_1^\epsilon(r,t) = M_1(r,t) + \epsilon$ with $\epsilon > 0$ (note that M_1^ϵ satisfies the same equation as M_1). Then in view of the definition of M_i and the upper bound of Theorem 3.1, we have $M_1^\epsilon(r,t) - M_2(r,t) > 0$ in some small neighborhood of $r = 0$ uniformly in t. Larger values of r are controlled by the standard comparison principle. Letting $\epsilon \to 0$ yields the result. ∎

Let us now go back to (1.4). Let us recall that

$$\frac{1}{2\pi} \|\rho_2(\cdot,0)\|_1 \leq \theta \leq \frac{1}{\pi} \|\rho_2(\cdot,0)\|_1, \quad M_0 = \|\rho_1(\cdot,0)\|_1, \quad \gamma = \chi\theta, \tag{4.9}$$

and that we are interested in the regime $M_0 \gg \theta$. To simplify the technicalities we assume that $\rho_2(\cdot,0)$ is smooth but very close to χ_{B_1} in L^1 norm, and $\rho_2(x,0) \geq \chi_{B_1}(x)$, but in the argument below we think of $\rho_2(x,0)$ as equal to $\theta\chi_{B_1}(x)$. To make this argument completely rigorous, while still using exactly the function g in (4.10), and keeping $\rho_2(\cdot,0)$ smooth, one may work with a time τ_α by which the mass of ρ_2 drops by a factor of α with $\alpha < 1/2$, rather than τ_C, as the discrepancy between $\rho_2(\cdot,0)$ and χ_{B_1} can be made arbitrarily small in $L^1(\mathbb{R}^2)$.

Observe that any radial function $f(x) \geq 0$ supported on B_1, and such that

$$0 \leq f(x) \leq \rho_2(x,0) \quad \text{and} \quad \|f\|_1 \geq \tfrac{1}{2} \|\rho_2(\cdot,0)\|_1,$$

is more concentrated than

$$g(x) := \theta(\chi_{B_1}(x) - \chi_{B_{1/\sqrt{2}}}(x)). \tag{4.10}$$

In particular, g is less concentrated than $\rho_2(\cdot,t)$ for all $t \le \tau_C$. One may use (4.3) and (4.9) to obtain

$$H(x) := \chi(-\Delta)^{-1}g = \begin{cases} (\gamma/8)(1-\log 2) & \text{for } 0 \le r < 1/\sqrt{2}, \\ (\gamma/4)(\log r + 1 - r^2) & \text{for } 1/\sqrt{2} \le r < 1, \\ -(\gamma/4)\log r & \text{for } r \ge 1. \end{cases} \tag{4.11}$$

We can now compare ρ_1 to the solution to the Fokker–Planck equation with the drift potential H, and conclude the following:

Proposition 4.3. *Let $\rho_1(x,t), \rho_2(x,t)$ solve (1.4) with radially symmetric initial conditions, where $\rho_2(\cdot,0) = \theta\eta$, η smooth, radial and $\eta(x) \ge \chi_{B_1}(x)$, and suppose $\rho(x,t)$ solves the Fokker–Planck equation (4.1) with the drift potential H given by (4.11) and the same initial condition as ρ_1. Let τ_C be the time it takes for the L^1 norm of ρ_2 to decrease by $\theta\pi/2$. Then*

$$\int_{B_r} \rho_1(x,t)\,dx \ge \int_{B_r} \rho(x,t)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \rho_2(x,0)\,dx \quad \textit{for all } t \le \tau_C \textit{ and } r \ge 0. \tag{4.12}$$

Proof. Let $\tilde\rho$ solve the equation for ρ_1 without the reaction term:

$$\partial_t \tilde\rho - \Delta\tilde\rho + \chi\nabla\cdot(\tilde\rho\nabla(-\Delta)^{-1}\rho_2) = 0, \tag{4.13}$$

with the same initial condition as ρ_1. Note that $\tilde\rho(\cdot,t)$ is more concentrated than $\rho(\cdot,t)$ for all $t \le \tau_C$. Indeed, the function g defined in (4.10) is less concentrated than $\rho_2(\cdot,t)$ for all $t \le \tau_C$, hence Lemma 4.1 implies that

$$\chi\partial_r(-\Delta)^{-1}\rho_2(\cdot,t) \le \partial_r H \le 0 \quad \text{for all } t \le \tau_C,$$

where H as in (4.11). Thus, Proposition 4.2 gives

$$\int_{B_r} \tilde\rho(x,t)\,dx \ge \int_{B_r} \rho(x,t)\,dx \quad \text{for all } t \le \tau_C \text{ and } r \ge 0. \tag{4.14}$$

To prove (4.12), it now suffices to compare ρ_1 and $\tilde\rho$ and show that

$$\int_{B_r} \rho_1(x,t)\,dx \ge \int_{B_r} \tilde\rho(x,t)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \rho_2(x,0)\,dx \quad \text{for all } t \le \tau_C \text{ and } r \ge 0. \tag{4.15}$$

Note that

$$\begin{aligned} \int_{\mathbb{R}^2} \rho_1(x,0)\,dx - \int_{\mathbb{R}^2} \rho_1(x,t)\,dx &= \int_{\mathbb{R}^2} \rho_2(x,0)\,dx - \int_{\mathbb{R}^2} \rho_2(x,t)\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^2} \rho_2(x,0)\,dx \quad \text{for all } t \le \tau_C, \end{aligned} \tag{4.16}$$

and the comparison principle implies that

$$\tilde\rho(x,t) \ge \rho_1(x,t) \quad \text{for all } x, t. \tag{4.17}$$

Hence, we may write

$$\begin{aligned}\int_{B_r} \rho_1(x,t)\,dx &= \int_{\mathbb{R}^2} \rho_1(x,t)\,dx - \int_{\mathbb{R}^2\setminus B_r} \rho_1(x,t)\,dx \\ &\geq \int_{\mathbb{R}^2} \rho_1(x,0)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \rho_2(x,0)\,dx - \int_{\mathbb{R}^2\setminus B_r} \tilde{\rho}(x,t)\,dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^2} \rho_2(x,0)\,dx + \int_{\mathbb{R}^2} \tilde{\rho}(x,0)\,dx - \int_{\mathbb{R}^2\setminus B_r} \tilde{\rho}(x,t)\,dx \\ &= \int_{B_r} \tilde{\rho}(\cdot,t)\,dx - \frac{1}{2}\int_{\mathbb{R}^2} \rho_2(x,0)\,dx,\end{aligned}$$

which is (4.15). Here we have used (4.16) and (4.17) in the first step, and conservation of mass for $\tilde{\rho}$ in the last step. ■

5. Weak weighted Poincaré-type inequalities

In this section, we develop some analytical tools that we will need to derive sufficiently sharp estimates on the convergence to equilibrium rates for solutions to Fokker–Planck equations with a logarithmic potential. To motivate these results, consider the Fokker–Planck equation

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho \nabla H) = 0 \quad \text{in } \mathbb{R}^2 \times [0,\infty), \tag{5.1}$$

where $H = \chi(-\Delta)^{-1} g$ is time independent, and g is the radially symmetric function supported in $B(0,1)$ defined in (4.10). As outlined in the previous section, we plan to use the solution ρ as a comparison tool to control the behavior of ρ_1.

Before setting the stage for the main arguments of this section, for completeness, let us briefly discuss well-posedness and regularity properties of the solutions to (5.1). Equation (5.1) is linear, and the result we are going to state is definitely known – but a convenient reference does not appear to be easily available.

Theorem 5.1. *Suppose that H in (5.1) satisfies $\Delta H, \nabla H \in L^\infty(\mathbb{R}^2)$, and the initial data $\rho_0 \in W^{2,\infty}(\mathbb{R}^2) \cap W^{2,1}(\mathbb{R}^2)$. Then there exists a unique solution $\rho(x,t)$ such that for any $1 \leq p < \infty$ and any $T \geq 0$, we have*

$$\|\partial_t \rho\|_{L^p(\mathbb{R}^2)} + \|\rho\|_{W^{2,p}(\mathbb{R}^2)} \leq C(p, \rho_0, H, T) < \infty \tag{5.2}$$

for all $0 \leq t \leq T$.

Remark. 1. The result holds in any dimension d without changes.

2. In our case, H is smooth away from two concentric circles. Standard parabolic regularity theory (see e.g. [47, Corollary 2.4.3]) in this case implies that $\rho(x,t)$ is smooth away from those circles. Theorem 5.1 implies that $\rho \in C^{1,\gamma}(\mathbb{R}^2)$ for any $\gamma < 1$; such regularity near singular interfaces is certainly sufficient for any estimates below.

We will sketch a simple proof of Theorem 5.1 in Appendix A.2.

The operator L given by

$$L\rho = -\Delta\rho + \nabla\cdot(\rho\nabla H)$$

is self-adjoint in the weighted space $L^2(e^{-H}dx)$ (when defined on a natural weighted Sobolev space), and is nonnegative. Its unique ground state corresponding to the zero eigenvalue is a multiple of e^H, provided that

$$\int e^H\,dx < \infty,$$

otherwise there is no ground state. In our situation, H is given by (4.11), so that

$$e^{H(x)} = \begin{cases} (e/2)^{\gamma/8}, & |x| < 1/\sqrt{2}, \\ e^{\gamma/4}|x|^{\gamma/4}e^{-\gamma|x|^2/4}, & 1/\sqrt{2} \le |x| \le 1, \\ |x|^{-\gamma/4}, & |x| \ge 1. \end{cases} \tag{5.3}$$

As the evolution (5.1) conserves the integral of ρ, we expect that

$$\rho(t,x) \to e^{H(x)}\Big(\int \rho\,dx\Big)\Big(\int e^H\,dx\Big)^{-1} \quad \text{as } t \to +\infty.$$

The dual operator L^* with respect to the standard $L^2(dx)$ inner product, given by

$$L^* f = -\Delta f - \nabla H\cdot\nabla f,$$

is self-adjoint in $L^2(e^H\,dx)$, with ground state equal to a constant. The corresponding dual evolution is

$$\partial_t f - \Delta f - \nabla H\cdot\nabla f = 0. \tag{5.4}$$

Note that $\rho(\cdot,t)$ solving (5.1) is equivalent to

$$f(x,t) := \rho(x,t)e^{-H(x)}$$

solving (5.4). The evolution (5.4) conserves the integral of $f(x)\exp(H(x))$ so we expect that

$$f(x,t) \to \bar{f} := \Big(\int f_0 e^H\,dx\Big)\Big(\int e^H\,dx\Big)^{-1} \quad \text{as } t \to +\infty,$$

where $f_0(x) = f(x,0)$. Note that

$$\frac{d}{dt}\underbrace{\int_{\mathbb{R}^2}(f(x,t)-\bar{f})^2 e^{H(x)}\,dx}_{=:Z(t)} = -2\underbrace{\int_{\mathbb{R}^2}|\nabla f(x,t)|^2 e^{H(x)}\,dx}_{=:W(t)}. \tag{5.5}$$

If we can bound $Z(t)$ from above as

$$Z(t) \le g(W(t), \|f_0\|_\infty),$$

with some function g that increases in W, that would allow us to bound $W(t)$ from below in terms of $Z(t)$ and $\|f_0\|_\infty$. Then (5.5) would give us a differential inequality for $Z(t)$ leading to an explicit decay estimate on $Z(t)$. In the simplest case, the bound $Z \le CW$ applies, which is a standard Poincaré inequality. Then there is a spectral gap for L^*, and exponential-in-time convergence to the ground state in $L^2(e^H dx)$. This is true for uniformly concave potentials, as in the Brascamp–Lieb inequality. However, it is not difficult to verify that in the case of logarithmic potential (or even $|x|^\alpha$ with $\alpha < 1$) there is no spectral gap, that is, the ground state zero is not an isolated point of the spectrum. Then the usual Poincaré inequality cannot hold, and one needs what is called a weak Poincaré version that manifests itself in a different, stronger weight deployed for the gradient norm.

We will prove the weak weighted Poincaré inequality for a more general family of radial weights $w(r) \ge 0$, which depend on a parameter $\gamma > 0$, than the specific choice (5.3), since the argument is essentially the same. We will assume that the weights have the following properties: there exist $0 < r_1 < r_2 < \infty$ and constants C_0, C_1, C_2 independent of γ such that

$$
\begin{aligned}
&C_0^{-1} w(s) \le w(r) \le C_0 w(s) && \text{for all } s, r \in [0, r_1], && (5.6)\\
&w'(r) \le -C_1 \gamma (r - r_1) w(r) && \text{for } r \in [r_1, r_2], && (5.7)\\
&w'(r) \le -C_2 \gamma r^{-1} w(r) && \text{for } r \in [r_2, \infty). && (5.8)
\end{aligned}
$$

An elementary computation shows that for the weight $w(r) = \exp(H(r))$ given by (5.3), assumptions (5.6)–(5.8) hold with

$$
r_1 = 1/\sqrt{2}, \quad r_2 = 3/4, \tag{5.9}
$$

where the choice of r_2 is rather arbitrary; any number larger than r_1 would do. The power weight $v(r) = (1 + r^2)^{-\gamma/2}$ analyzed by Bobkov and Ledoux [7] does not directly fit the above assumptions; as we will see below, the natural choice of r_1 in this case does depend on γ, the difference with our case being the lack of a plateau near zero. We will indicate changes necessary to accommodate the power weight in Theorem 5.5.

It will be convenient for us to derive a slightly stronger version of the standard Poincaré estimate. Given any $f(x)$, let

$$
\tilde{f}(r) := \frac{1}{2\pi} \int_0^{2\pi} f(r, \phi)\, d\phi.
$$

Instead of directly looking for an upper bound for

$$
Z = \int_{\mathbb{R}^2} (f(x) - \bar{f})^2 w(x)\, dx, \quad \bar{f} = \left(\int_{\mathbb{R}^2} w(x)\, dx \right)^{-1} \int_{\mathbb{R}^2} f(x) w(x)\, dx.
$$

it turns out to be easier to control the following integral that is closely related to $Z(t)$:

$$
I := \int_{\mathbb{R}^2} (f(x) - \tilde{f}(r_1))^2 w(x)\, dx =: I_1 + I_2 + I_3.
$$

Here, I_1, I_2, I_3 are given by

$$\begin{aligned} I_1 &:= \int_{B_{r_1}} (f(x)-\tilde{f}(r_1))^2 w(x)\,dx, \quad I_2 := \int_{B_{r_2}\setminus B_{r_1}} (f(x)-\tilde{f}(r_1))^2 w(x)\,dx \\ I_3 &:= \int_{(B_{r_2})^c} (f(x)-\tilde{f}(r_1))^2 w(x)\,dx, \end{aligned} \tag{5.10}$$

with r_1 and r_2 given by (5.9). Note that

$$Z = \int_{\mathbb{R}^2} (f(x)-\bar{f})^2 w(x)\,dx = \inf_a \int (f-a)^2 w(x)\,dx \le I.$$

Let us also define

$$\begin{aligned} J_1 &:= \int_{B_{r_1}} |\nabla f|^2 w(x)\,dx, \quad J_2 := \int_{B_{r_2}\setminus B_{r_1}} |\nabla f|^2 w(x)\,dx, \\ J_3 &:= \int_{(B_{r_2})^c} |\nabla f|^2 |x|^2 w(x)\,dx. \end{aligned} \tag{5.11}$$

Note that J_1 and J_2 are directly related to

$$W = \int |\nabla f|^2 w(x)\,dx,$$

but J_3 has an extra factor $|x|^2$ in the integrand.

Theorem 5.2. *Suppose that the weight $w(x) \ge 0$ is radial and satisfies (5.6)–(5.8). Let I_k, J_k be defined by (5.10) and (5.11) respectively. Then there exists a universal constant C such that for all sufficiently large $\gamma \ge \gamma_0(w)$ and every f in the weighted Sobolev class $W^{1,2}(w\,dx)$ the following inequalities hold:*

$$I_1 \le C J_1, \tag{5.12}$$

$$I_2 \le \frac{C}{\gamma} J_2 + \frac{C}{\gamma} J_1 + \frac{1}{4} I_1, \tag{5.13}$$

$$I_3 \le \frac{C}{\gamma^2} J_3 + \frac{C}{\gamma^2} J_2 + \frac{1}{4} I_2. \tag{5.14}$$

Remarks. 1. As usual, it suffices to prove the inequalities for $f \in C_0^\infty(\mathbb{R}^2)$.

2. The factors $1/4$ in estimates (5.13) and (5.14) are needed (any factor less than 1 would work) to derive the sharpest version of the convergence to equilibrium estimate.

3. Here and in the estimates that follow, C and c stand for universal constants (in particular independent of γ) that may change from expression to expression. These constants may depend on r_1, r_2, C_0, C_1 and C_2 – that is, on w.

4. The proof extends to all dimensions with a minor adjustment of the constants. While in dimensions $d \ne 2$ the logarithmic behavior of H does not correspond to the Green function of the Laplacian, the behavior of a particle in such slowly growing potential is of independent interest.

Proof of Theorem 5.2. We are interested here in the large γ regime, so we assume that γ is sufficiently large; we do not try to optimize the constants. For the reader's convenience, we record a bound on $\gamma_0(w)$ that is going to be sufficient for the proof:

$$\gamma_0(w) = \max\left\{\frac{4}{r_1^2 C_1^2}, \frac{2}{C_2}, \frac{1}{(r_2-r_1)^2}, \frac{r_1^2}{9}, 4096 c_0^2\left(1+\frac{2}{C_1}\right)^2, \frac{4}{r_1(r_2-r_1)C_1}\right\}. \quad (5.15)$$

Since we will be working in polar coordinates, it is convenient to incorporate the Jacobian into the weight, setting $u(r) = rw(r)$. Let us restate our assumptions on w in terms of u. On the interval $[r_1, r_2]$ we have

$$u'(r) = rw'(r) + w(r) \le \left(-C_1\gamma(r-r_1) + \frac{1}{r}\right)u(r). \quad (5.16)$$

Thus if γ is large, u is increasing at most for only a small distance past r_1, and reaches its maximum no further than $r_{\max} = r_1 + O(\gamma^{-1})$. In particular, there is γ_0 large enough so that

$$u'(r) \le -c\sqrt{\gamma}\, u(r) \quad \text{for all } r \in [r_1 + 1/\sqrt{\gamma}, r_2], \text{ for } \gamma > \gamma_0,$$

with some $c > 0$ (we could take $c = C_1/2$, so that the above inequality holds for all $\gamma \ge \frac{4}{r_1^2 C_1^2}$). For $r \in [r_2, \infty)$, we have

$$u'(r) \le (-C_2\gamma + 1)w(r) \le -\frac{C_2}{2}\gamma r^{-1} u(r) \quad \text{for all } \gamma \ge 2/C_2.$$

Altogether, u satisfies the following differential inequalities, with some $C, c > 0$, and $\tilde{r}_1 := r_1 + 1/\sqrt{\gamma}$:

$$u'(r) \le \begin{cases} Cu(r) & \text{for } r \in [r_1, \tilde{r}_1), & (5.17a) \\ -c\sqrt{\gamma}\, u(r) & \text{for } r \in [\tilde{r}_1, r_2), & (5.17b) \\ -c\gamma r^{-1} u(r) & \text{for } r \in [r_2, \infty). & (5.17c) \end{cases}$$

These are the inequalities that we will use in the analysis below, along with (5.16).

We first note that (5.12) is a direct consequence of a slight variation of the standard proof of the Poincaré inequality (see, e.g., [26]), so we only need to estimate I_2 and I_3. We will first show the inequalities for the radially symmetric f, where we only need the first term of the right side of (5.13) and the first two terms of the right side of (5.14), respectively, and then consider a general f.

• **Control of I_2: radial estimates.** Let f be radial, and $h > 0$ an arbitrary function of single variable, then

$$\begin{aligned} \int_{r_1}^{r_2} (f(r) - f(r_1))^2 u(r)\, dr &= \int_{r_1}^{r_2} \left(\int_{r_1}^{r} f'(s)\, ds\right)^2 u(r)\, dr \\ &\le \int_{r_1}^{r_2} \left(\int_{r_1}^{r} f'(s)^2 h(s)\, ds\right)\left(\int_{r_1}^{r} h(t)^{-1} dt\right) u(r)\, dr \\ &= \int_{r_1}^{r_2} f'(s)^2 h(s) \int_{s}^{r_2} u(r) \left(\int_{r_1}^{r} h(t)^{-1} dt\right) dr\, ds. \end{aligned} \quad (5.18)$$

We choose $h = u^{1/2}$, and claim that

$$\int_s^{r_2} u(r)\Big(\int_{r_1}^{r} u(t)^{-1/2}\,dt\Big)\,dr \le \frac{C}{\gamma} u(s)^{1/2} \qquad \text{for all } s \in [r_1, r_2]. \tag{5.19}$$

Once this claim is proved, plugging it into (5.18) yields

$$I_2 = \int_{r_1}^{r_2} (f - f(r_1))^2 u\,dr \le \frac{C}{\gamma}\int_{r_1}^{r_2} f'(s)^2 u(s)\,ds = \frac{C}{\gamma} J_2. \tag{5.20}$$

Now, let us prove (5.19). To this end, we will show that

$$\int_{r_1}^{r} u(t)^{-1/2}\,dt \le \frac{C}{\sqrt{\gamma}} u(r)^{-1/2} \quad \text{for all } r \in [r_1, r_2], \tag{5.21}$$

$$\int_{s}^{r_2} u(r)^{1/2}\,dr \le \frac{C}{\sqrt{\gamma}} u(s)^{1/2} \quad \text{for all } s \in [r_1, r_2], \tag{5.22}$$

which together imply (5.19) immediately. To prove (5.21), we note that if $r \in [r_1, \tilde{r}_1]$, then (5.16) implies

$$u(t)^{-1/2} \le C u(r)^{-1/2} \quad \text{for any } t \in [r_1, r],$$

hence (5.21) holds for $r \le \tilde{r}_1$. If $r > \tilde{r}_1$, we split the integration domain in (5.21) as

$$\int_{r_1}^{r} u(t)^{-1/2}\,dt = \int_{r_1}^{\tilde{r}_1} u(t)^{-1/2}\,dt + \int_{\tilde{r}_1}^{r} u(t)^{-1/2}\,dt = A + B. \tag{5.23}$$

Again by (5.16), we have

$$A \le \frac{C}{\sqrt{\gamma}} u(\tilde{r}_1)^{-1/2} \le \frac{C}{\sqrt{\gamma}} u(r)^{-1/2},$$

as $u(r)$ is decreasing for $r > \tilde{r}_1$. For the second integral in (5.23), note that (5.17b) gives

$$u(t)^{-1/2} \le e^{-c\sqrt{\gamma}(r-t)/2} u(r)^{-1/2} \quad \text{for } t \in [\tilde{r}_1, r],$$

thus

$$B \le \frac{C}{\sqrt{\gamma}} u(r)^{-1/2}. \tag{5.24}$$

To prove (5.22), note that if $s > \tilde{r}_1$, then (5.22) follows directly from (5.17b), as in (5.24). If $s < \tilde{r}_1$, we again split the integration domain

$$\int_{s}^{r_2} u(t)^{1/2}\,dt = \int_{s}^{\tilde{r}_1} u(t)^{1/2}\,dt + \int_{\tilde{r}_1}^{r_2} u(t)^{1/2}\,dt = A + B. \tag{5.25}$$

The first integral on the right side can be controlled by

$$A \le \frac{C}{\sqrt{\gamma}} u(s)^{1/2},$$

because $u(t) \le Cu(s)$ on this interval due to (5.16), and $|\tilde{r}_1 - s| \le 1/\sqrt{\gamma}$. The second integral can be controlled by

$$B \le \frac{C}{\sqrt{\gamma}} u(\tilde{r}_1)^{1/2} \le \frac{C}{\sqrt{\gamma}} u(s)^{1/2},$$

by (5.17b) and (5.16).

• **Control of I_2 for a nonradial function.** For the general nonradial case, we decompose a function $f = f(r,\phi)$ into the Fourier series

$$f(r,\phi) = \tilde{f}(r) + \sum_{n=1}^{\infty} (\psi_n(r)\cos(n\phi) + \xi_n(r)\sin(n\phi)). \tag{5.26}$$

Using this decomposition, I_2 becomes

$$I_2 = \int_{r_1}^{r_2} (\tilde{f}(r) - \tilde{f}(r_1))^2 u(r)\,dr + \pi \sum_{n=1}^{\infty} \int_{r_1}^{r_2} (\psi_n(r)^2 + \xi_n(r)^2) u(r)\,dr, \tag{5.27}$$

whereas J_2 becomes

$$\begin{aligned} J_2 = &\int_{r_1}^{r_2} \tilde{f}'(r)^2 u(r)\,dr \\ &+ \pi \sum_{n=1}^{\infty} \int_{r_1}^{r_2} \left(\frac{n^2}{r^2}\psi_n(r)^2 + \frac{n^2}{r^2}\xi_n(r)^2 + \psi_n'(r)^2 + \xi_n'(r)^2 \right) u(r)\,dr. \end{aligned} \tag{5.28}$$

Note that I_1 and J_1 can be written in the same form as I_2 and J_2 with the domain of integration replaced by $[0, r_1]$. To bound I_2, we will prove the following estimate for each $n \ge 1$:

$$\begin{aligned} \int_{r_1}^{r_2} \psi_n(r)^2 u(r)\,dr \le &\frac{C}{\gamma} \int_{r_1}^{r_2} \psi_n'(r)^2 u(r)\,dr + \frac{C}{\gamma} \int_0^{r_1} \psi_n'(r)^2 u(r)\,dr \\ &+ \frac{1}{4} \int_0^{r_1} \psi_n(r)^2 u(r)\,dr, \end{aligned} \tag{5.29}$$

with an identical estimate holding for ξ_n. With (5.29) in hand, adding (5.20) for $\tilde{f}$ and (5.29) for both ψ_n and ξ_n, we arrive at (5.13).

To prove (5.29), first note that

$$\begin{aligned} \int_{r_1}^{r_2} \psi_n(r)^2 u(r)\,dr &\le 2\int_{r_1}^{r_2} (\psi_n(r) - \psi_n(r_1))^2 u(r)\,dr + 2\psi_n(r_1)^2 \int_{r_1}^{r_2} u(r)\,dr \\ &\le \frac{C}{\gamma} \int_{r_1}^{r_2} \psi_n'(r)^2 u(r)\,dr + 2\psi_n(r_1)^2 \int_{r_1}^{r_2} u(r)\,dr. \end{aligned} \tag{5.30}$$

We have used (5.20) applied to $\psi_n(r)$ in the last inequality above. To bound the last integral on the right side, we use (5.16) to observe that $u(\tilde{r}_1) \le 2u(r_1)$ for γ sufficiently large, and then also (5.17b) to get

$$\int_{r_1}^{r_2} u(r)\,dr = \int_{r_1}^{\tilde r_1} u(r)\,dr + \int_{\tilde r_1}^{r_2} u(r)\,dr \le \frac{2}{\sqrt{\gamma}}u(r_1) + \int_{\tilde r_1}^{r_2} 2e^{-c\sqrt{\gamma}(r-\tilde r_1)}u(r_1)\,dr$$
$$\le \frac{2}{\sqrt{\gamma}}\Big(1+\frac{1}{c}\Big)u(r_1). \qquad (5.31)$$

Note that if

$$4\Big(1+\frac{1}{c}\Big)\psi_n(r_1)^2\frac{1}{\sqrt{\gamma}}u(r_1) \le \frac{1}{4}\int_0^{r_1}\psi_n(r)^2u(r)\,dr, \qquad (5.32)$$

then (5.29) follows from (5.30)–(5.32). If (5.32) does not hold, and γ is sufficiently large, there exist C, independent of γ, and $r_3 \in [r_1 - C/\sqrt{\gamma}, r_1]$ such that $|\psi_n(r_3)| \le |\psi_n(r_1)|/2$ (specifically, $C = 32C_0(1+1/C)$ would work here). Here, we have used the fact that $u(r) \ge C_0^{-1}u(r_1)/2$ for all $r \in [r_1/2, r_1]$ due to (5.6). Thus, we have

$$\int_0^{r_1}\psi_n'(r)^2u(r)\,dr \ge \int_{r_3}^{r_1}\psi_n'(r)^2u(r)\,dr \ge \Big(\int_{r_3}^{r_1}\psi_n'(r)\,dr\Big)^2\Big(\int_{r_3}^{r_1}\frac{1}{u(r)}\,dr\Big)^{-1}$$
$$\ge \frac{|\psi_n(r_1)|^2}{4}\,\frac{u(r_1)}{2C_0(r_1-r_3)} \ge \frac{|\psi_n(r_1)|^2}{4}\,\frac{\sqrt{\gamma}\,u(r_1)}{2C_0C}, \qquad (5.33)$$

which, using (5.31), gives that if (5.32) fails, then

$$\psi_n(r_1)^2\int_{r_1}^{r_2}u(r)\,dr \le \frac{C}{\sqrt{\gamma}}\psi_n(r_1)^2u(r_1) \le \frac{C}{\gamma}\int_0^{r_1}\psi_n'(r)^2u(r)\,dr.$$

This finishes the proof of (5.29), and hence also of (5.13).

• **Control of I_3: radial estimates.** To control I_3 for a radial function f, first note that

$$\int_{r_2}^{\infty}(f(r)-f(r_1))^2u(r)\,dr$$
$$\le 2\int_{r_2}^{\infty}(f(r)-f(r_2))^2u(r)\,dr + 2(f(r_2)-f(r_1))^2\int_{r_2}^{\infty}u(r)\,dr. \qquad (5.34)$$

We start with the second term on the right side, and claim that

$$(f(r_2)-f(r_1))^2\int_{r_2}^{\infty}u(r)\,dr \le \frac{C}{\gamma^2}J_2. \qquad (5.35)$$

To see this, note that (5.17c) implies that for all $r > s \ge r_2$ we have

$$u(r) \le u(s)\Big(\frac{s}{r}\Big)^{c\gamma}. \qquad (5.36)$$

Applying this with $s = r_2$ we get

$$\int_{r_2}^{\infty}u(r)\,dr \le \frac{C}{\gamma}u(r_2). \qquad (5.37)$$

Also note that

$$(f(r_2) - f(r_1))^2 = \left(\int_{r_1}^{r_2} f'(r)\,dr\right)^2 \le \left(\int_{r_1}^{r_2} f'(r)^2 u(r)\,dr\right)\left(\int_{r_1}^{r_2} \frac{1}{u(r)}\,dr\right). \tag{5.38}$$

Next, we will show that

$$\int_{r_1}^{r_2} \frac{1}{u(r)}\,dr \le \frac{C}{\gamma u(r_2)}.$$

By (5.16), we have

$$\left(\frac{1}{u(r)}\right)' = -\frac{u'(r)}{u(r)^2} \ge \left(C_1\gamma(r - r_1) - \frac{1}{r}\right)\frac{1}{u(r)} \quad \text{for } r \in [r_1, r_2]. \tag{5.39}$$

Hence, provided that γ is sufficiently large, we have

$$\left(\frac{1}{u(r)}\right)' \ge \frac{c\gamma}{u(r)} \quad \text{for } r \in \left[\frac{r_1 + r_2}{2}, r_2\right],$$

with some $c > 0$, implying that

$$\frac{1}{u(r)} \le e^{c\gamma(r - r_2)}\frac{1}{u(r_2)} \quad \text{for } r \in \left[\frac{r_1 + r_2}{2}, r_2\right]. \tag{5.40}$$

By (5.39), we also have

$$\left(\frac{1}{u(r)}\right)' \ge -\frac{C}{u(r)}, \quad \text{thus} \quad \frac{1}{u(r)} \le \frac{C}{u(\frac{r_1+r_2}{2})} \le \frac{Ce^{-c\gamma(r_2 - r_1)/2}}{u(r_2)} \quad \text{for } r \in \left[r_1, \frac{r_1 + r_2}{2}\right].$$

We have used (5.40) with $r = (r_1 + r_2)/2$ in the last inequality above. Putting these estimates together yields

$$\int_{r_1}^{r_2} \frac{1}{u(r)}\,dr \le C\int_{r_1}^{\frac{r_1+r_2}{2}} e^{-c\gamma(\frac{r_2-r_1}{2})}\frac{1}{u(r_2)}\,dr + \int_{\frac{r_1+r_2}{2}}^{r_2} e^{c\gamma(r-r_2)}\frac{1}{u(r_2)}\,dr \le \frac{C}{\gamma u(r_2)}.$$

Combining this bound with (5.38) and (5.37) gives us (5.35).

For the first integral on the right side of (5.34), a computation identical to (5.18), but with r_1 replaced by r_2, and r_2 replaced by ∞, yields

$$\int_{r_2}^{\infty} (f(r) - f(r_2))^2 u(r)\,dr \le \int_{r_2}^{\infty} f'(s)^2 h(s) \int_s^{\infty} u(r)\left(\int_{r_2}^{r} h(t)^{-1}\,dt\right) dr\,ds \tag{5.41}$$

for any function $h > 0$. We again choose $h = u^{1/2}$, and claim that

$$\int_s^{\infty} u(r)\left(\int_{r_2}^{r} u(t)^{-1/2} dt\right) dr \le \frac{C}{\gamma^2} s^2 u(s)^{1/2} \quad \text{for all } s \ge r_2, \tag{5.42}$$

with some $C > 0$ (to be shown below). Substituting this into (5.41) gives

$$\int_{r_2}^{\infty} (f(r) - f(r_2))^2 u(r)\,dr \le \frac{C}{\gamma^2}\int_{r_2}^{\infty} f'(s)^2 s^2 u(s)\,ds, \tag{5.43}$$

and combining it with (5.35) and (5.34) yields

$$\int_{r_2}^{\infty} (f(r) - f(r_1))^2 u(r)\, dr \le \frac{C}{\gamma^2} \int_{r_2}^{\infty} f'(s)^2 s^2 u(s) ds + \frac{C}{\gamma^2} \int_{r_1}^{r_2} f'(r)^2 u(r)\, dr. \tag{5.44}$$

That is, we have

$$I_3 \le \frac{C}{\gamma^2} J_3 + \frac{C}{\gamma^2} J_2$$

for all radially symmetric f.

To show (5.42), we consider the inner integral first. Using (5.36), we find that if $r_2 \le t < r$ then

$$u(r)^{-1/2} \left(\frac{t}{r}\right)^{c\gamma} \ge u(t)^{-1/2},$$

so that

$$\int_{r_2}^{r} u(t)^{-1/2}\, dt \le u(r)^{-1/2} r^{-c\gamma} \int_{r_2}^{r} t^{c\gamma}\, dt \le \frac{C}{\gamma} u(r)^{-1/2} r.$$

Thus the left hand side of (5.42) is bounded from above by

$$\frac{C}{\gamma} \int_{s}^{\infty} u(r)^{1/2} r\, dr \le \frac{C}{\gamma^2} s^2 u(s)^{1/2} \quad \text{for all } s > r_2.$$

The last inequality follows from (5.36) with r_2 replaced by s and a direct computation.

• **Control of I_3 for a nonradial function.** For a general function f, using the decomposition (5.26), we can write I_3 and J_3 as

$$I_3 = \int_{r_2}^{\infty} (\tilde{f}(r) - \tilde{f}(r_1))^2 u(r)\, dr + \sum_{n=1}^{\infty} \int_{r_2}^{\infty} \pi(\psi_n(r)^2 + \xi_n(r)^2) u(r)\, dr, \tag{5.45}$$

$$\begin{aligned} J_3 = &\int_{r_2}^{\infty} \tilde{f}'(r)^2 r^2 u(r)\, dr \\ &+ \pi \sum_{n=1}^{\infty} \int_{r_2}^{\infty} \left(\frac{n^2}{r^2}\psi_n(r)^2 + \frac{n^2}{r^2}\xi_n(r)^2 + \psi_n'(r)^2 + \xi_n'(r)^2\right) r^2 u(r)\, dr. \end{aligned} \tag{5.46}$$

We now aim to show the following estimate for each ψ_n, $n \ge 1$:

$$\begin{aligned} \int_{r_2}^{\infty} \psi_n^2(r) u(r)\, dr \le \frac{C}{\gamma^2} \int_{r_2}^{\infty} \psi_n'(r)^2 r^2 u(r)\, dr + \frac{C}{\gamma^2} \int_{r_1}^{r_2} \psi_n'(r)^2 u(r)\, dr \\ + \frac{1}{4} \int_{r_1}^{r_2} \psi_n(r)^2 u(r)\, dr. \end{aligned} \tag{5.47}$$

Combining (5.47) with the analogous estimate for ξ_n and the radial estimate (5.44), we will have (5.14).

First, we write

$$\int_{r_2}^{\infty} \psi_n(r)^2 u(r)\, dr \le 2 \int_{r_2}^{\infty} (\psi_n(r) - \psi_n(r_2))^2 u(r)\, dr + 2\psi_n(r_2)^2 \int_{r_2}^{\infty} u(r)\, dr. \tag{5.48}$$

Applying (5.43) to the first integral on the right side gives

$$\int_{r_2}^{\infty} (\psi_n(r) - \psi_n(r_2))^2 u(r)\, dr \leq \frac{C}{\gamma^2} \int_{r_2}^{\infty} \psi_n'(r)^2 r^2 u(r)\, dr.$$

For the second term on the right side of (5.48), by (5.37) we have

$$2\psi_n(r_2)^2 \int_{r_2}^{\infty} u(r)\, dr \leq \frac{C}{\gamma} \psi_n(r_2)^2 u(r_2).$$

Thus, if

$$\frac{C}{\gamma} \psi_n(r_2)^2 u(r_2) \leq \frac{1}{4} \int_{r_1}^{r_2} \psi_n(r)^2 u(r)\, dr, \tag{5.49}$$

we are done. If not, since u is decreasing in $(\tilde{r}_1, r_2)$, there exists $r_3 \in [r_2 - 16C/\gamma, r_2)$ such that $\psi_n(r_3) \leq \psi_n(r_2)/2$. Then

$$\begin{aligned} \int_{r_1}^{r_2} \psi_n'(r)^2 u(r)\, dr \geq \int_{r_3}^{r_2} \psi_n'(r)^2 u(r)\, dr \geq \left(\int_{r_3}^{r_2} |\psi_n'(r)|\, dr \right)^2 \left(\int_{r_3}^{r_2} \frac{dr}{u(r)} \right)^{-1} \\ \geq C\gamma \psi_n(r_2)^2 u(r_2). \end{aligned} \tag{5.50}$$

In the last step we have used the inequality

$$\int_{r_3}^{r_2} \frac{1}{u(r)}\, dr \leq \frac{C}{\gamma u(r_2)},$$

which follows from the decay of u on $[r_3, r_2]$ and $r_2 - r_3 \leq 16C/\gamma$. Thus, if (5.49) fails, then

$$2\psi_n(r_2)^2 \int_{r_2}^{\infty} u(r)\, dr \leq \frac{C}{\gamma} \psi_n(r_2)^2 u(r_2) \leq \frac{C}{\gamma^2} \int_{r_1}^{r_2} (\psi_n')^2 u\, dr,$$

which finishes the proof of (5.47). ■

Theorem 5.2 leads to the following two corollaries. Note that adding the inequalities in the theorem together, we get

$$I = I_1 + I_2 + I_3 \leq C\left(J_1 + \frac{1}{\gamma} J_2 + \frac{1}{\gamma^2} J_3 \right). \tag{5.51}$$

We also recall, as already noted in the remarks to Theorem 5.2, that the arguments above generalize to an arbitrary dimension $d > 2$ in a straightforward manner. This implies

Corollary 5.3. *Suppose that $d \geq 2$. For the weight w satisfying (5.6)–(5.8), we have*

$$\begin{aligned} &\int_{\mathbb{R}^d} |f - \bar{f}|^2 w\, dx \leq \int_{\mathbb{R}^d} |f - \tilde{f}(r_1)|^2 w\, dx \\ &\leq C\left(\int_{B_{r_1}} |\nabla f|^2 w\, dx + \frac{1}{\gamma} \int_{B_{r_2} \setminus B_{r_1}} |\nabla f|^2 w\, dx + \frac{1}{\gamma^2} \int_{B_{r_2}^c} |\nabla f|^2 |x|^2 w\, dx \right) \end{aligned} \tag{5.52}$$

for all sufficiently large γ.

Remarks. 1. The one-dimensional results are in fact stronger and will be considered elsewhere.

2. It is not hard to adapt the arguments in the proof so that, once we get the $\frac{1}{4}$ factor as in (5.13), (5.14) is not necessary and an arbitrary constant would suffice, and the result can be extended to all $\gamma > d$ with an adjustment of the constant C.

Tracing the proof of Theorem 5.2, it is straightforward to check that the result remains true for truncated integrals. For $R > r_2$, let us define

$$\begin{aligned} I_R &= \int_{B_R} (f(x) - \tilde{f}(r_1))^2 w(x)\,dx, \quad I_3^R = \int_{B_R \setminus B_{r_2}} (f(x) - \tilde{f}(r_1))^2 w(x)\,dx, \\ J_3^R &= \int_{B_R \setminus B_{r_2}} |\nabla f|^2 |x|^2 w(x)\,dx. \end{aligned} \tag{5.53}$$

Corollary 5.4. *For any $R > r_2$, let I_3^R, J_3^R, and I_R be as in (5.53), and recall that I_2, J_1 and J_2 are defined in (5.10), (5.11). Then for all $\gamma \geq \gamma_0(w)$ we have*

$$I_3^R \leq \frac{C}{\gamma^2} J_3^R + \frac{C}{\gamma^2} J_2 + \frac{1}{4} I_2, \tag{5.54}$$

$$I_R \leq C\left(J_1 + \frac{1}{\gamma} J_2 + \frac{1}{\gamma^2} J_3^R\right). \tag{5.55}$$

We now pause to indicate a result that can be obtained with similar techniques for the power weight $v(x) = (1 + |x|^2)^{-\gamma/2}$ with a sufficiently large γ.

Theorem 5.5. *Let $v(x) = (1 + |x|^2)^{-\gamma/2}$. Then the following weak weighted Poincaré inequality holds for all dimensions $d \geq 2$ for $\gamma \geq d$:*

$$\begin{aligned} \int_{\mathbb{R}^d} |f - \bar{f}|^2 v(x)\,dx \leq \frac{C(d)}{\gamma} &\int_{B_1} |\nabla f|^2 v(x)\,dx \\ &+ \frac{C(d)}{\gamma^2} \int_{(B_1)^c} |\nabla f|^2 (1 + |x|^2) v(x)\,dx. \end{aligned} \tag{5.56}$$

Proof. In two dimensions, the only essential difference is that for the weight $v(x)$ the condition (5.6) holds if we choose $r_1 \lesssim \gamma^{-1/2}$ that depends on γ. Specifically, we could take $r_1 = 2/\sqrt{\gamma}$. With that choice, direct computations show that for the weight $u(r) = rv(r)$ the inequality (5.16) remains valid, while (5.17b) and (5.17c) hold with $\tilde{r}_1 = r_1$ and $r_2 = 1$. The standard Poincaré inequality becomes

$$I_1 \leq C r_1^2 J_1 = \frac{C}{\gamma} J_1, \tag{5.57}$$

as it is important to keep track of the r_1^2 factor which now depends on γ. The rest of the proof goes through. One place that requires attention and minor adjustment is the control of I_2 for a nonradial function, namely the estimates (5.32) and (5.33), since we need to "step back" a distance $\tilde{C}/\sqrt{\gamma}$ into the $[0, r_1]$ region, and we may not have that much space. However, the factor $1/4$ in (5.13) is not crucial for establishing (5.56) given that we can control I_1 via (5.57); any constant would do. Then we can choose a sufficiently large constant C instead of $1/4$ in (5.32) so that r_3 with the needed properties can be found in $[r_1 - 1/\sqrt{\gamma}, r_1]$. With this modification, the rest of the argument goes through. We leave the details to the interested reader. Finally, as already noted above, the proof generalizes to an arbitrary dimension d with minor adjustments. ■

6. Convergence to equilibrium estimates for Fokker–Planck operators

6.1. *Weighted L^2 norm decay*

With the weak weighted Poincaré inequalities in hand, we may now go back to the dual evolution (5.4) and the dissipation inequality (5.5):

$$\frac{dZ}{dt} = -2W(t) \tag{6.1}$$

with

$$Z(t) = \int_{\mathbb{R}^2} (f(x,t) - \bar{f})^2 e^{H(x)}\, dx, \quad W(t) = \int_{\mathbb{R}^2} |\nabla f(x,t)|^2 e^{H(x)}\, dx.$$

We are going to focus on the specific weight in (5.3); we will need fairly sharp estimates to get close to the heuristic bounds. Our analysis in this section will be driven by the nonlinear application we have in mind: to derive sharp bounds on the time required to transport a significant part of density towards the center of the attracting potential, the ball B_{r_1}. We stress that in this section, we do not need the initial data $f_0(x)$ or $\rho_0(x)$ to be radial: the bounds on convergence to equilibrium in the linear setting with a fixed potential apply in full generality.

Although Corollary 5.3 with $w(x) = e^{H(x)}$ already gives us an upper bound for $Z(t)$, we cannot directly control the right side of (5.52) by $W(t)$, due to the extra factor $|x|^2$ in the integrand of J_3. To overcome this issue, we follow a general scheme introduced in [66]. Let us take a truncation at radius $R \geq r_2$ in $I(t)$ and apply Corollary 5.4:

$$\begin{aligned} I(t) &\leq I_R(t) + \frac{C}{\gamma} R^{-\gamma/4+2} \|f(\cdot,t)\|_\infty^2 \\ &\leq C\left(J_1 + \frac{1}{\gamma} J_2 + \frac{1}{\gamma^2} J_3^R\right) + \frac{C}{\gamma} R^{-\gamma/4+2} \|f_0\|_\infty^2 \\ &\leq C W(t) + C\frac{R^2}{\gamma^2} W(t) + \frac{C}{\gamma} R^{-\gamma/4+2} \|f_0\|_\infty^2. \end{aligned} \tag{6.2}$$

In the first inequality above we used (5.3), and in the second the fact that $\|f(\cdot,t)\|_\infty$ is nonincreasing in time, as well as (5.55). To optimize the right side of (6.2) over $R \geq r_2$, we take

$$R = \gamma^{4/\gamma} W(t)^{-4/\gamma} \|f_0\|_\infty^{8/\gamma}.$$

Note that if $R < 1$, then $I(t) \leq C W(t)$, fitting the scheme below. As

$$\gamma^{8/\gamma} \lesssim 1, \tag{6.3}$$

this leads to

$$I(t) \le C\,W(t) + C\gamma^{-2}W(t)^{\frac{\gamma-8}{\gamma}}\|f_0\|_\infty^{16/\gamma} \le 2C \max\{W(t), \gamma^{-2}W(t)^{\frac{\gamma-8}{\gamma}}\|f_0\|_\infty^{16/\gamma}\}. \tag{6.4}$$

Since $Z(t) \le I(t)$, it follows that (6.1) and (6.4) together with (6.3) imply

$$Z'(t) \le -c \min\{Z(t), \gamma^2 Z(t)^{\frac{\gamma}{\gamma-8}}\|f_0\|_\infty^{-\frac{16}{\gamma-8}}\}. \tag{6.5}$$

Let us now discuss how this differential inequality relates to the heuristic bound $\tau_C \sim L^2/\gamma$ for the reaction time we have informally derived in Section 2, to give context and outline the main ideas behind the technical estimates that follow. Let us think for now of the linear Fokker–Planck operator (5.1) with the potential H given by (4.11). Consider an initial condition ρ_0 that has total mass M_0 and is concentrated at a distance L from the origin. Then $f = \rho e^{-H}$ solves the dual Fokker–Planck equation (5.4), and (6.5) is applicable. If we drop the term $Z(t)$ from the minimum in (6.5) (which of course strengthens the differential inequality compared to what we really have), then a direct computation, with yet another use of (6.3), gives

$$Z(t) \le \left(Z(0)^{-\frac{8}{\gamma-8}} + c\gamma t\|f_0\|_\infty^{-\frac{16}{\gamma-8}}\right)^{-\frac{\gamma-8}{8}} \le (c\gamma t)^{-\frac{\gamma-8}{8}}\|f_0\|_\infty^2. \tag{6.6}$$

In our situation, we have $\|f_0\|_\infty \sim M_0 L^{\gamma/4}$ according to the assumptions on ρ_0 and (5.3). Also, using the relationship between ρ and f, we see that

$$Z(t) = \int_{\mathbb{R}^2} |\rho(x,t) - \rho_s(x)|^2 e^{-H(x)}\,dx. \tag{6.7}$$

Here,

$$\rho_s(x) = e^H \frac{\int \rho_0\,dx}{\int e^H\,dx}$$

is the stationary state of the same mass as ρ_0 to which the solution ρ converges. From (6.7) it is clear that transport of ρ to the origin corresponds to decay of $Z(t)$. Intuitively, from (5.3) it looks likely that we need $Z(t) \ll M_0^2$ in order to be sure that a significant portion of ρ is inside B_{r_1} (we will make these arguments precise later). Going back to (6.6) and the estimate on $\|f_0\|_\infty$, we find that to ensure the needed bound on $Z(t)$, we need $t \gtrsim L^4/\gamma$, which is quite a bit off the heuristic estimate. The situation is similar to the usual heat equation, where the L^1 to L^2 estimate decays only as $t^{-d/4}$, while the faster decay rate $t^{-d/2}$ is realized for the L^1 to L^∞ estimate. A standard way to attain the latter estimate if an explicit heat kernel is not available (like for diffusions with incompressible drift, see e.g. [27]) is to combine the L^1 to L^2 bound with its dual L^2 to L^∞ bound. We will need to follow a similar route in what follows. The L^∞ to $L^2(e^H dx)$ bound (6.6) provides a decay estimate for $Z(t)$, which via (6.7) leads to the $L^\infty(e^{-H}dx)$ to $L^2(e^{-H}dx)$ bound for ρ. We will also derive a variant of the bound dual to (6.6), which is an $L^2(e^{-H}dx)$ to L^1 estimate for ρ. Combining them leads to an $L^\infty(e^{-H}dx)$ to L^1 bound for ρ which will have the needed decay and also will provide control in the L^1 space most convenient for measuring mass transport.

Before we go to the duality estimates, however, there is one more issue to take care of. The presence of the term $Z(t)$ under minimum in (6.5) affects the bound (6.6). The balance of the two terms depends on the initial data; the second term is smaller if $Z(t)$ is sufficiently small, namely if

$$Z(t) \lesssim \|f_0\|_\infty^2 \gamma^{-(\gamma-8)/4}.$$

Our assumptions on ρ_0 give $Z(0) \sim M_0^2 L^{\gamma/4}$, and the above condition at $t = 0$ translates into an additional constraint $L \gtrsim \gamma$. For some configurations of parameters, say when $1 \ll L \ll \gamma$, the time delay before the second term in (6.5) becomes smaller can be up to order $\gamma \log \gamma$. We would like to avoid these additional constraints and significant losses in the estimate of the transport time, as they appear to be of technical nature. The idea is to use the L^∞ norm time decay estimate proved in Theorem 3.1. This gives an outline for the rest of this section. First, we deploy the L^∞ norm decay bound to improve the weighted L^2 control on f and ρ, and then use a duality argument to obtain optimal convergence to equilibrium bounds for ρ in L^1.

The differential inequality (6.5) can be improved in the following way for $t \gtrsim 1$. In the second inequality of (6.2), instead of using (5.55) to bound the whole I_R, we can instead split

$$I_R = I_1 + I_2 + I_3^R,$$

and directly control I_1 and I_2 as follows. The bound in Theorem 3.1 implies that

$$\|f(\cdot,t)e^H\|_\infty = \|\rho(\cdot,t)\|_\infty \le C\gamma\|\rho_0\|_1 = C\gamma\|f_0 e^H\|_1$$

for all $t \ge 1$. Two immediate consequences are

$$I_1(t) = \int_{B_{r_1}} |f - \tilde{f}(r_1)|^2 e^H \, dx \le C\gamma^2 \|f_0 e^H\|_1^2 e^{-H(0)} =: Q_1 \quad \text{for all } t \ge 1, \tag{6.8}$$

$$\begin{aligned} I_1(t) + I_2(t) = \int_{B_{r_2}} |f - \tilde{f}(r_1)|^2 e^H \, dx &\le C\gamma^2 \|f_0 e^H\|_1^2 e^{-H(r_2)} \\ &=: Q_2 \quad \text{for all } t \ge 1. \end{aligned} \tag{6.9}$$

Recall that $r_1 = 1/\sqrt{2}$ and $r_2 = 3/4$, as defined in (5.9). Note that $Q_2 \gg Q_1$ due to $\gamma \gg 1$ and (5.3); hence, for $t \ge 1$, if $I(t) \ge 4Q_2$, then we can bound I_R by (6.9) and (5.54) as follows:

$$I_R(t) \le Q_2 + I_3^R \le Q_2 + \frac{C}{\gamma^2} J_2 + \frac{C}{\gamma^2} J_3^R + \frac{1}{4} I_2.$$

Substituting this into the second inequality of (6.2), and then absorbing Q_2 and $\frac{1}{4} I_2$ into the left side, we obtain

$$I(t) \le \frac{C}{\gamma^2} W(t) + C\frac{R^2}{\gamma^2} W(t) + \frac{C}{\gamma} R^{-\gamma/4+2} \|f_0\|_\infty^2 \le \frac{C}{\gamma^2} W(t) + \frac{C}{\gamma^2} W(t)^{\frac{\gamma-8}{\gamma}} \|f_0\|_\infty^{16/\gamma},$$

where the last inequality comes from choosing the same optimal R as before, since the terms containing R are the same as in (6.2) (and again, if we get $R \le r_2$ then $I(t) \le \frac{C}{\gamma^2} W$ fitting the scheme below). The γ^{-2} factor in the first term then leads to a stronger differential inequality:

$$Z'(t) \le -c\gamma^2 \min\left\{Z(t), Z(t)^{\frac{\gamma}{\gamma-8}} \|f_0\|_\infty^{-\frac{16}{\gamma-8}}\right\}. \tag{6.10}$$

Here $c > 0$ is a universal constant (corresponding to our fixed weight $w = e^H$).

Likewise, for $t \ge 1$, if $I(t) \in [4Q_1, 4Q_2]$, then we control I_R using (6.8), (5.13) and (5.54):

$$I_R(t) \le Q_1 + I_2 + I_3^R \le Q_1 + \frac{C}{\gamma^2} J_3^R + \frac{C}{\gamma} J_2 + \frac{1}{4} I_1 + \frac{1}{4} I_2.$$

and a similar argument leads to the differential inequality

$$Z'(t) \le -c \min\left\{\gamma Z(t), \gamma^2 Z(t)^{\frac{\gamma}{\gamma-8}} \|f_0\|_\infty^{-\frac{16}{\gamma-8}}\right\}. \tag{6.11}$$

For all $t \ge 1$, the inequalities (6.5), (6.10), (6.11) control convergence of $Z(t)$ to zero. The above results are summarized in the following proposition.

Proposition 6.1. *Suppose that $\gamma \ge \gamma_0(w)$ is sufficiently large. For all $t \ge 1$, $Z(t)$ given by (6.7) satisfies the differential inequality*

$$Z'(t) \le -c \min\left\{\eta(Z(t))Z(t), \gamma^2 Z(t)^{\frac{\gamma}{\gamma-8}} \|f_0\|_\infty^{-\frac{16}{\gamma-8}}\right\}, \tag{6.12}$$

where

$$\eta(Z) := \begin{cases} 1 & \text{for } Z(t) \le 4Q_1, \\ \gamma & \text{for } Z(t) \in (4Q_1, 4Q_2), \\ \gamma^2 & \text{for } Z(t) \ge 4Q_2, \end{cases} \tag{6.13}$$

and $c > 0$ is a universal constant.

Due to the minimum taken in (6.12), which part will dominate depends on the initial data, or, more precisely, on the relationship between $\|f_0\|_\infty$, Q_1, and Q_2. A careful accounting is needed to take care of several cases; however, it turns out that for the sake of the application at hand, we only need to track the decay of $Z(t)$ until it drops to Z^σ, defined as

$$Z^\sigma := \sigma e^{-H(0)} \|f_0 e^H\|_1^2, \tag{6.14}$$

where $\sigma < 1$ is sufficiently small. The definition of Z^σ is motivated by Proposition 6.4 below. Basically, we will see that by the time $Z(t)$ reaches Z^σ, a significant portion of the mass of $\rho = f e^H$ has already moved into B_{r_1}, which will be sufficient to prove that significant reaction took place.

The following theorem says that even with the first item in the minimum in (6.12), the decay of $Z(t)$ is not too much worse than in (6.6) – namely, as long as $Z(t)$ is above Z^σ, the presence of the first item in the min function introduces at most an extra time delay t_1 which is estimated below (and is much better than γ).

Theorem 6.2. *Suppose that $\gamma \geq \gamma_0(w)$ is sufficiently large. Let $f(x,t)$ be the solution to (5.4) with initial condition $f_0 \in L^\infty(\mathbb{R}^2) \cap L^2(e^H)$, let σ be any number in $(0,1)$, and let $Z(t)$ and Z^σ be as given in (5.5) and (6.14) respectively. Let $t_1 := C(1 + \log \sigma^{-1} + \log \gamma)$, where C is a sufficiently large universal constant. Then for all $t \geq t_1$, we have*

$$Z(t) \leq \max\{Z^\sigma, (c\gamma(t-t_1))^{-(\gamma-8)/8}\|f_0\|_\infty^2\}. \tag{6.15}$$

Proof. Note that $Z(t)$ is decreasing in time, and in every regime where the form of η in (6.13) stays fixed, once the second term becomes the smaller one, this continues for all subsequent times. Let us first estimate the total time in the interval $t \geq 1$ where the first term under minimum in (6.12) is smaller, while $Z(t) \geq 4Q_2$. Comparing the two terms in the min function of (6.12), we see that the minimum is achieved by the first term as long as $Z(t) \geq \|f_0\|_\infty^2$. Thus, $Z(t)$ decays exponentially not more slowly than $\exp(-c\gamma^2 t)$. Note that at $t = 1$, we have

$$Z(1) \leq Z(0) = \int_{\mathbb{R}^2} (f_0 - \bar{f})^2 e^H \, dx \leq \int_{\mathbb{R}^2} f_0^2 e^H \, dx \leq \|f_0\|_\infty^2 \int_{\mathbb{R}^2} e^H \, dx,$$

hence the total time $t \geq 1$ when $Z(t) \geq 4Q_2$ and the first term in (6.12) is the smaller one is bounded by

$$t_{11} := \frac{1}{c\gamma^2} \log\left(\frac{Z(1)}{\|f_0\|_\infty^2}\right) \leq \frac{1}{c\gamma^2} \log\left(\int_{\mathbb{R}^2} e^H \, dx\right) \leq \frac{C}{\gamma}.$$

Hence, in the $Z(t) \geq 4Q_2$ regime, the presence of the first term at most introduces a time delay of the order $\gamma^{-1} \leq 1$.

Likewise, if the first term in (6.12) is smaller and $Z(t) \in [4Q_1, 4Q_2]$, then $Z(t)$ has exponential decay not slower than $\exp(-c\gamma t)$. Hence, in this case the time with the first term active is bounded by

$$t_{12} := \frac{1}{c\gamma} \log\left(\frac{Q_2}{Q_1}\right) \leq \frac{1}{c\gamma}(H(0) - H(r_2)) \leq C.$$

Thus the presence of the first term also at most introduces a time delay of order 1 in this regime.

Finally, in the $Z(t) \in [Z^\sigma, 4Q_1]$ regime and when the first term in (6.12) is smaller, $Z(t)$ has exponential decay not slower than $\exp(-ct)$. So the time with the first term active is bounded by

$$t_{13} := \frac{1}{c} \log\left(\frac{Q_1}{Z^\sigma}\right) \leq \frac{1}{c} \log\left(\frac{\gamma^2}{\sigma}\right) \leq C(1 + \log \sigma^{-1} + \log \gamma).$$

Combining these estimates, we see that the total time delay caused by the first term in the minimum function is bounded by

$$t_1 := t_{11} + t_{12} + t_{13} = C(1 + \log \sigma^{-1} + \log \gamma). \qquad \blacksquare$$

Remark 6.3. The appearance of $\log\gamma$ in the definition of t_1 is likely not optimal. In fact, as far as pure transport of the density goes (without estimate on the rate of convergence to equilibrium), in Section 7 we outline a different method that yields a bound on transport without extra delay terms. In the context of convergence to equilibrium estimates, this extra correction comes from the γ^2 factor in Z_1 and Z_2, which is due to the γ factor in our L^∞ estimate of ρ in Theorem 3.1,

$$\|\rho(\cdot,t)\|_\infty \le C\gamma\|\rho_0\|_1 \quad \text{for all } t \ge 1.$$

Such a bound would be optimal if we had $H = (-\Delta)^{-1}\chi_{B(0,1)}$, and our argument can also be adapted to this case. But for the weight e^H in (5.3), the top is flat and $\|e^H\|_1 \sim \|e^H\|_\infty$, which suggests that there should not be a γ factor, and we should have

$$\|f(\cdot,t)e^H\|_\infty \le C\|f_0e^H\|_1 \quad \text{for } t \ge 1.$$

We have not been able to show this and settle here for the $\log\gamma$ correction that in most situations is not very significant.

We now translate the above weighted L^2 bounds to ρ. Let $\rho(x,t)$ be a solution to (5.1) with initial condition $\rho_0 \in L^\infty(e^{-H}) \cap L^1(\mathbb{R}^2)$. Recall that

$$\rho_s := e^H \frac{\int \rho_0\,dx}{\int e^H\,dx}$$

is a stationary solution to (5.1) with the same mass as ρ. Also recall that $Z(t)$ can be written as in (6.7):

$$Z(t) = \int_{\mathbb{R}^2} (\rho(x,t) - \rho_s(x))^2 e^{-H(x)}\,dx,$$

and that $f(x,t) := \rho(x,t)e^{-H(x)}$ satisfies (5.4) with initial condition

$$f_0 = \rho_0 e^{-H} \in L^\infty(\mathbb{R}^2) \cap L^1(e^H).$$

Applying Theorem 6.2 to $f = \rho e^{-H}$, we get Theorem 1.3. It implies, in particular, that $Z(t) \le Z^\sigma$ for all

$$t \ge t_2 := t_1 + \frac{C}{\gamma}\left(\frac{\|\rho_0 e^{-H}\|_\infty}{\sqrt{\sigma}\,\|\rho_0\|_1}\right)^{\frac{16}{\gamma-8}}.$$

On the other hand, once $Z(t)$ drops below Z^σ, the following proposition shows that $\rho(\cdot,t)$ is sufficiently close to ρ_s in B_{r_1}.

Proposition 6.4. *Let $Z^\sigma = \sigma e^{-H(0)}\|\rho_0\|_1^2$ with $\sigma < 1$, and let $r \le r_1 = 1/\sqrt{2}$. If $Z(t) \le AZ^\sigma$, then*

$$\int_{B_r} |\rho(x,t) - \rho_s(x)|\,dx \le \sqrt{\pi\sigma A}\,r\|\rho_0\|_1. \tag{6.16}$$

Moreover, if we assume in addition that $\rho_0 \ge 0$, then

$$\int_{B_r} \rho(x,t)\,dx \ge \left(2r^2 - \frac{C}{\sqrt{\gamma}} - r\sqrt{\pi\sigma A}\right)\|\rho_0\|_1. \tag{6.17}$$

Proof. If $Z(t) \leq AZ^\sigma$, the definitions of $Z(t)$ and Z^σ give

$$\int_{B_r} |\rho - \rho_s|^2 e^{-H}\, dx \leq \int_{\mathbb{R}^2} |\rho - \rho_s|^2 e^{-H}\, dx \leq \sigma A e^{-H(0)} \|\rho_0\|_1^2.$$

Using the fact that $e^{-H} \equiv e^{-H(0)}$ in B_r, the above inequality becomes

$$\int_{B_r} |\rho - \rho_s|^2\, dx \leq \sigma A \|\rho_0\|_1^2.$$

Then a direct application of the Cauchy–Schwarz inequality gives (6.16).

A direct computation using (5.3) shows that

$$\frac{\int_{(B_{r_1})^c} e^{H(x)}\, dx}{\int_{B_{r_1}} e^{H(x)}\, dx} \leq \frac{C}{\sqrt{\gamma}}.$$

Then, if $\rho_0 \geq 0$, we have, since ρ_s is constant on B_{r_1} and $r \leq r_1$,

$$\int_{B_r} \rho_s(x)\, dx \geq \frac{r^2}{r_1^2} \|\rho_0\|_1 \left(1 - \frac{C}{\sqrt{\gamma}}\right).$$

Combining this inequality with (6.16), we obtain (6.17). ■

Inequality (6.17) gives us a way to ensure that much of the mass of ρ_1 has been transported into the support of ρ_2, provided we choose σ sufficiently small and γ is sufficiently large. However, as mentioned above, the weighted L^2 decay estimates we have for $Z(t)$ lead to bounds on the transport time that are far from the heuristic ones. We now discuss this issue in more detail and use duality to rectify the situation.

6.2. *Duality and L^1 control*

Theorem 1.3 and Proposition 6.4 give us an explicit upper bound for the time it takes for a large portion of mass to enter B_{r_1}, but this is not sufficient for our application. Let us recap the reason: Consider a special case where $\rho_0(x)$ is a bump of mass M_0, located at distance L from the origin. In this case, we have

$$\|\rho_0 e^{-H}\|_\infty \sim M_0 L^{\gamma/4}.$$

Then (1.14) requires the time

$$t \sim 1 + \log \gamma + \frac{L^4}{\gamma}$$

to ensure transport of a significant portion of ρ to B_{r_1}, which is at odds with the heuristic bound of the order L^2/γ. To get control at a time scale close to heuristic, we employ a duality procedure which is somewhat delicate in our case since we may have different regimes in differential inequalities. A direct computation leads to the following auxiliary duality lemma.

Lemma 6.5. *Let ρ and f be solutions to (5.1) and (5.4) respectively, with initial conditions ρ_0 and f_0, where $\rho_0 \in L^\infty(\mathbb{R}^2, e^{-H}dx) \cap L^1(\mathbb{R}^2, dx)$, and $f_0 \in L^\infty(\mathbb{R}^2, dx) \cap L^1(\mathbb{R}^2, e^H dx)$, and set*

$$\rho_s(x) := \frac{e^{H(x)} \int \rho_0\, dx}{\int e^H\, dx}, \quad \bar{f} := \frac{\int f_0 e^H\, dx}{\int e^H\, dx}.$$

Then, for any $t > 0$ and $s \in [0, t]$, the integral

$$\int_{\mathbb{R}^2} (\rho(x,s) - \rho_s(x))(f(x,t-s) - \bar{f})\, dx \tag{6.18}$$

does not depend on s for all $s \in [0, t]$.

Note that the term $\bar{f}$ on the right side can always be dropped since

$$\int \rho(x,s)\, dx = \int \rho_s(x)\, dx \quad \text{for all } s \geq 0.$$

Proof of Lemma 6.5. By standard approximation arguments, it suffices to show the result for smooth, sufficiently quickly decaying ρ and f. Denote the integral in (6.18) by $U(s)$. Taking the derivative in s gives

$$\begin{aligned}\frac{d}{ds}U(s) &= \int_{\mathbb{R}^2} \partial_t \rho(x,s)(f(x,t-s) - \bar{f})\, dx - \int_{\mathbb{R}^2} (\rho(x,s) - \rho_s(x))\partial_t f(x,t-s)\, dx \\ &=: T_1 - T_2,\end{aligned}$$

where

$$\begin{aligned} T_1 &= \int_{\mathbb{R}^2} \big(\Delta\rho(x,s) - \nabla \cdot (\rho(x,s)\nabla H)\big)(f(x,t-s) - \bar{f})\, dx, \\ T_2 &= \int_{\mathbb{R}^2} (\rho(x,s) - \rho_s(x))\big(\Delta f(x,t-s) + \nabla f(x,t-s) \cdot \nabla H\big)\, dx. \end{aligned}$$

Now one can check that $T_1 = T_2$ by the divergence theorem – using, in particular, the fact that ρ_s and $\bar{f}$ are eigenfunctions of $\Delta - \nabla(\cdot \nabla H)$ and $\Delta + \nabla H \nabla$, respectively, with zero eigenvalue. ∎

We can now prove the following theorem.

Theorem 6.6. *Fix any $0 < r \leq r_1$. For all $\sigma \in (0,1)$, let t_1 be as in Theorem 6.2. Define*

$$t_3 := C\left(t_1 + \frac{1}{\gamma}\left(\frac{\|\rho_0 e^{-H}\|_\infty}{\sigma\|\rho_0\|_1}\right)^{\frac{8}{\gamma-8}}\right) \tag{6.19}$$

with some sufficiently large constant C that will be fixed in the proof. Then, for all $t \geq t_3$,

$$\int_{B_r} |\rho(x,t) - \rho_s(x)|\, dx \leq (4\sqrt{\sigma}\, r + 4\sigma)\|\rho_0\|_1. \tag{6.20}$$

In particular, if σ is sufficiently small, γ is sufficiently large, and $\rho_0 \geq 0$, then

$$\int_{B_r} \rho(x,t)\, dx \geq (2r^2 - 0.1)\|\rho_0\|_1 \quad \text{for all } t \geq t_3. \tag{6.21}$$

Proof. Consider first what happens at time $t_3/3$. If $Z(t_3/3)$ drops below $Z^\sigma = \sigma e^{-H(0)}\|\rho_0\|_1^2$, we are done due to Proposition 6.4. Otherwise, we have the second bound in (1.14) for $Z(t_3/3)$ with $t = t_3/3$. To obtain better L^1 control of $\rho(\cdot, t_3) - \rho_s$ in the latter case, we use the following duality argument. For any $f_0 \in L^\infty(\mathbb{R}^2) \cap L^1(e^H)$, let $f(x,t)$ be the solution to the dual equation (5.4) with initial condition f_0. Applying Lemma 6.5 with $t = 2t_3/3$, $s = 2t_3/3$ and then $s = t_3/3$, we obtain

$$\int_{\mathbb{R}^2}\left(\rho\left(x, \frac{2t_3}{3}\right) - \rho_s(x)\right) f_0(x)\,dx = \int_{\mathbb{R}^2}\left(\rho\left(x, \frac{t_3}{3}\right) - \rho_s\right)\left(f\left(x, \frac{t_3}{3}\right) - \bar{f}\right)dx. \tag{6.22}$$

We dropped the term involving $\bar{f}$ on the left side using the remark after Lemma 6.5. We can then bound the left side in (6.22) as

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\left(\rho\left(x, \frac{2t_3}{3}\right) - \rho_s(x)\right) f_0(x)\,dx\right| \le \left\|\rho\left(\frac{t_3}{3}\right) - \rho_s\right\|_{L^2(e^{-H})}\left\|f\left(\frac{t_3}{3}\right) - \bar{f}\right\|_{L^2(e^H)} \\
&\le \left\|\rho\left(\frac{t_3}{3}\right) - \rho_s\right\|_{L^2(e^{-H})} \max\left\{(\sigma e^{-H(0)}\|f_0 e^H\|_1^2)^{1/2}, \left(c\gamma\left(\frac{t_3}{3} - t_1\right)\right)^{-\frac{\gamma-8}{16}}\|f_0\|_\infty\right\} \\
&\qquad\qquad \text{(by Theorem 6.2)} \\
&\le \left(c\gamma\left(\frac{t_3}{3} - t_1\right)\right)^{-\frac{\gamma-8}{16}}\|\rho_0 e^{-H}\|_\infty \\
&\quad \times\left(\sigma^{1/2}e^{-H(0)/2}\|f_0 e^H\|_1 + \left(c\gamma\left(\frac{t_3}{3} - t_1\right)\right)^{-\frac{\gamma-8}{16}}\|f_0\|_\infty\right) \qquad \text{(by Theorem 1.3)} \\
&\le \alpha\|f_0 e^H\|_1 + \beta\|f_0\|_\infty,
\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}
\alpha &:= \left(c\gamma\left(\frac{t_3}{3} - t_1\right)\right)^{-\frac{\gamma-8}{16}}\sigma^{1/2}e^{-H(0)/2}\|\rho_0 e^{-H}\|_\infty, \\
\beta &:= \left(c\gamma\left(\frac{t_3}{3} - t_1\right)\right)^{-\frac{\gamma-8}{8}}\|\rho_0 e^{-H}\|_\infty.
\end{aligned} \tag{6.24}$$

Now let us apply the following lemma, the proof of which is postponed till the end of this subsection.

Lemma 6.7. *Suppose that for some $G \in L^\infty(e^{-H}) \cap L^1(\mathbb{R}^2)$, there exist $\alpha, \beta > 0$ such that*

$$\left|\int_{\mathbb{R}^2} G(x) f(x)\,dx\right| \le \alpha\|f e^H\|_1 + \beta\|f\|_\infty \quad \textit{for all } f \in L^\infty(\mathbb{R}^2) \cap L^1(e^H). \tag{6.25}$$

Then G can be decomposed as $G = G_1 + G_2$, where $G_1, G_2 \in L^\infty(e^{-H}) \cap L^1(\mathbb{R}^2)$ satisfy the estimates $\|G_1 e^{-H}\|_\infty \le 2\alpha$, $\|G_2\|_1 \le 2\beta$.

Applying this lemma to (6.23), we can decompose

$$\rho(x, 2t_3/3) - \rho_s(x) = G_1(x) + G_2(x),$$

where using (6.24),

$$\|G_1 e^{-H}\|_\infty \le 2\alpha = 2(c\gamma(t_3/3 - t_1))^{-(\gamma-8)/16} \sigma^{1/2} e^{-H(0)/2} \|\rho_0 e^{-H}\|_\infty, \tag{6.26}$$

and

$$\|G_2\|_1 \le 2\beta = 2(c\gamma(t_3/3 - t_1))^{-(\gamma-8)/8} \|\rho_0 e^{-H}\|_\infty \le 2\sigma \|\rho_0\|_1,$$

where the last inequality comes from choosing a sufficiently large universal constant C in the definition (6.19) of t_3.

Let $\zeta_1(x,t)$ and $\zeta_2(x,t)$ denote the solutions to (5.1) starting at $t = 2t_3/3$ with initial conditions $\zeta_1(\cdot, 2t_3/3) = G_1$, $\zeta_2(\cdot, 2t_3/3) = G_2$, respectively. Since (5.1) is linear, we have

$$\rho(\cdot, t_3) - \rho_s = \zeta_1(\cdot, t_3) + \zeta_2(\cdot, t_3).$$

Note that $\|\zeta_2(\cdot, t)\|_1$ is nonincreasing in time, hence

$$\|\zeta_2(\cdot, t_3)\|_1 \le \|G_2\|_1 \le 2\sigma \|\rho_0\|_1. \tag{6.27}$$

To control $\zeta_1(\cdot, t_3)$, set

$$\zeta_1^s := e^H \frac{\int G_1\, dx}{\int e^H\, dx}.$$

By Theorem 1.3, we have

$$\begin{aligned} &\|\zeta_1(\cdot, t_3) - \zeta_1^s\|_{L^2(e^{-H})} \\ &\quad \le \max\{\sigma^{1/2} e^{-H(0)/2} \|G_1\|_1, (c\gamma(t_3/3 - t_1))^{-(\gamma-8)/16} \|G_1 e^{-H}\|_\infty\} \end{aligned} \tag{6.28}$$

If the first term in the max function is larger, using the fact that

$$\|G_1\|_1 \le \|\rho\|_1 + \|\rho_s\|_1 + \|G_2\|_1 \le 3\|\rho_0\|_1,$$

we obtain

$$\|\zeta_1(\cdot, t_3) - \zeta_1^s\|_{L^2(e^{-H})} \le 3\sigma^{1/2} e^{-H(0)/2} \|\rho_0\|_1 \le 3\sqrt{Z^\sigma}.$$

And if the second term is larger, combining (6.28) with (6.26) we get

$$\begin{aligned} \|\zeta_1(\cdot, t_3) - \zeta_1^s\|_{L^2(e^{-H})} &\le 2(c\gamma(t_3/3 - t_1))^{-(\gamma-8)/8} \sigma^{1/2} e^{-H(0)/2} \|\rho_0 e^{-H}\|_\infty \\ &\le 2\sigma^{3/2} e^{-H(0)/2} \|\rho_0\|_1 = 2\sigma\sqrt{Z^\sigma}. \end{aligned}$$

In both cases, applying Proposition 6.4 yields

$$\int_{B_r} |\zeta_1(x, t_3) - \zeta_1^s(x)|\, dx \le 4\sqrt{\sigma}\, r \|\rho_0\|_1.$$

Finally, combining the above estimate with (6.27), we have

$$\begin{aligned} \|\rho(t_3) - \rho_s\|_{L^1(B_r)} &\le \|\zeta_1(t_3) - \zeta_1^s\|_{L^1(B_r)} + \|\zeta_2(t_3) + \zeta_1^s\|_{L^1(B_r)} \\ &\le \|\zeta_1(t_3) - \zeta_1^s\|_{L^1(B_r)} + \|\zeta_2(t_3)\|_1 + \|G_2\|_1 \\ &\le (4\sqrt{\sigma}\, r + 4\sigma)\|\rho_0\|_1, \end{aligned}$$

where in the second inequality we have used the mean zero property

$$\int (G_1 + G_2)\,dx = 0,$$

which gives

$$\|\zeta_1^s\|_1 = \left|\int G_1\,dx\right| = \left|\int G_2\,dx\right|.$$

The above argument shows that (6.20) holds at $t = t_3$. For $t > t_3$, the same argument works by replacing t_3 with t.

Estimate (6.21) follows from a simple computation similar to that in the proof of (6.17). ∎

Proof of Lemma 6.7. Let $S = \{x : |G(x)| \geq 2\alpha e^H\}$, and define $G_1 := G\chi_{S^c}(x)$, so that

$$\|G_1 e^{-H}\|_\infty \leq 2\alpha.$$

To show that $G_2 := G - G_1 = G\chi_S$ satisfies $\|G_2\|_1 \leq 2\beta$, we use (6.25) with $f = (\operatorname{sgn} G)\chi_S$:

$$\|G_2\|_{L^1(\mathbb{R}^2)} = \int_S G_2 f\,dx \leq \alpha \int_S e^H\,dx + \beta \leq \frac{1}{2}\|G_2\|_1 + \beta,$$

and the proof is complete. ∎

7. Transport estimates based on comparison principles

In this section we take a quick detour to provide a simple alternative proof that a significant portion of the initial mass of ρ_0 gets transported inside a certain ball of radius less than 1 under the action of the potential H in time $\tau \sim L^2/\gamma$. As mentioned in the introduction, this result can be used to obtain a simpler proof of a result similar to Theorem 1.1 if one is willing to compromise and settle for an estimate that provides little information on the closeness to the ground state.

The main step is the analysis of the dual equation (5.4). Recall that the dual operator L^* is given by

$$L^* f = -\Delta f - \nabla H \cdot \nabla f,$$

and the dual evolution by

$$\partial_t f = \Delta f + \nabla H \cdot \nabla f = -L^* f. \tag{7.1}$$

We will prove the following theorem.

Theorem 7.1. *Let $f(x,t)$ solve (7.1) with H given by (4.11). Suppose that the radial initial data $f_0 \in C_0^\infty$ satisfies $1 \geq f \geq 0$, f nonincreasing in the radial direction, and $f_0(x) \geq \chi_{B_{d_1}}(x)$ where $1 \geq d_1 > r_1 = 1/\sqrt{2}$. Then there exists a constant $c > 0$ such that for all sufficiently large γ we have*

$$f(x,t) \geq c\chi_{B_{c\sqrt{1+\gamma t}}}(x) \quad \textit{for all } t \geq 0. \tag{7.2}$$

Proof. Fix some d_0 such that $1 \geq d_1 > d_0 > r_1$. For simplicity, in the argument that follows, we can think for instance of $d_0 = 5/7$ and $d_1 = 6/7$, but any other choice satisfying the above relationship works as well (the constant c will depend on this choice). Due to parabolic comparison principles and since H is radial, we know that the solution $f(x,t)$ remains radial, nonincreasing in the radial direction, and satisfies $1 \geq f(x,t) \geq 0$ for all times.

Observe that

$$\begin{aligned} f(x,t)|_{\mathbb{S}_{d_0}} \int_{\mathbb{R}^2\setminus B_{d_0}} e^H\,dx &\geq \int_{\mathbb{R}^2\setminus B_{d_0}} fe^H\,dx \geq \int_{\mathbb{R}^2} f_0 e^H\,dx - \int_{B_{d_0}} fe^H\,dx \\ &\geq \int_{\mathbb{R}^2\setminus B_{d_0}} f_0 e^H\,dx \geq \int_{\mathbb{R}^2\setminus B_{d_0}} \chi_{B_{d_1}} e^H = \int_{B_{d_1}\setminus B_{d_0}} e^H\,dx. \end{aligned} \tag{7.3}$$

Here $\mathbb{S}_{d_0}$ is the circle of radius d_0; in the first step we use monotonicity of f in the radial variable, in the second step conservation of $\int f(x,t)e^{H(x)}\,dx$ and in the third step

$$\int_{B_{d_0}} f_0 e^H\,dx \geq \int_{B_{d_0}} fe^H\,dx$$

due to $1 = f_0(x) \geq f(x,t)$ in B_{d_0}. However, since $|\nabla H(x)| = -\partial_r H(x) \geq c_0\gamma|x|^{-1}$ if $|x| \geq d_0$, we have

$$e^{H(tx)} \leq t^{-c_0\gamma} e^{H(x)}.$$

Set $q = d_1/d_0$. Then

$$\int_{B_{d_1 q^k}\setminus B_{d_0 q^k}} e^{H(x)}\,dx = q^{kd} \int_{B_{d_1}\setminus B_{d_0}} e^{H(q^k x)}\,dx \leq q^{k(d-c_0\gamma)} \int_{B_{d_1}\setminus B_{d_0}} e^{H(x)}\,dx.$$

Therefore,

$$\int_{\mathbb{R}^2\setminus B_{d_0}} e^{H(x)}\,dx \leq 2\int_{B_{d_1}\setminus B_{d_0}} e^{H(x)}\,dx$$

for γ large enough. In this case from (7.3) we conclude that $f(x,t)|_{\mathbb{S}_{d_0}} \geq 1/2$ for all times.

Now fix any convex C^2 function ω on $[d_0,\infty)$ such that $\omega(d_0) = 1/2$, $\omega(r) > 0$ for $r \in [d_0, d_1)$, and $\omega(r) = 0$ if $r \geq d_1$. For $\varphi \in [0,1]$ define $\omega_\varphi(r) = \omega(d_0 + \varphi(r - d_0))$; we will abuse notation by also writing $\omega_\varphi(x) = \omega_\varphi(|x|)$. Note that

$$\begin{aligned} L^*\omega_\varphi(x) &= \omega''_\varphi(r) + \frac{1}{r}\omega'_\varphi(r) + \partial_r H(r)\omega'_\varphi(r) \\ &\geq \frac{c_0\gamma - 1}{r}|\omega'_\varphi(r)| \geq \frac{c_0\gamma}{2r}|\omega'(d_0 + \varphi(r - d_0))|\varphi, \end{aligned}$$

where we have used $\omega''_\varphi(r) \geq 0$, $\omega'(r) < 0$, and the last step holds if γ is sufficiently large. Choose a decreasing $\varphi(t)$ defined for $t \geq 0$ such that $\varphi(0) = 1$. Consider $F(x,t) = \omega_{\varphi(t)}(x)$. Since we always have $f(x,t)|_{\mathbb{S}_{d_0}} \geq 1/2 = F|_{\mathbb{S}_{d_0}}$ and $f_0(x) \geq \chi_{B_{d_1}}(x) \geq \omega(|x|)$, we can be sure that $f(x,t) \geq F(x,t)$ in $\mathbb{R}^2 \setminus B_{d_0}$ for all times if

$\partial_t F \le L^* F$. However, $\partial_t F = (r-d_0)\omega'(d_0+\varphi(r-d_0))\varphi'(t)$, and $\partial_t F = L^* F = 0$ if $d_0 + \varphi(r-d_0) \ge 1$. Hence we just need to check the inequality

$$-(r-d_0)\varphi'(t) \le \frac{c_0\gamma}{2r}\varphi \quad \text{when } r \le d_0 + \frac{1-d_0}{\varphi} \le \frac{1}{\varphi}.$$

Thus, it suffices to ensure that

$$-\frac{1-d_0}{\varphi}\varphi'(t) \le \frac{c_0\gamma}{2}\varphi^2,$$

which would follow from $\partial_t(1/\varphi(t)^2) \le c_0\gamma$. Therefore

$$\varphi(t) = \frac{1}{\sqrt{1+c_0\gamma t}}$$

is acceptable. Now fix a constant $a < d_1 - d_0$. Then we can make

$$d_0 + \frac{1}{\sqrt{1+c_0\gamma t}}(r-d_0) \le d_0 + a$$

for $r \le d_0 + c\sqrt{1+\gamma t}$ by choosing small enough c. In this case, if $d_0 \le r \le d_0 + c\sqrt{1+\gamma t}$, we have

$$f(x,t) \ge \omega\left(d_0 + \frac{1}{\sqrt{1+c_0\gamma t}}(r-d_0)\right) \ge \omega(d_0+a) \ge c > 0,$$

where we may have to make our constant c smaller if necessary. ∎

Here is the corollary for the behavior of the density $\rho(x,t)$ satisfying (1.9).

Corollary 7.2. *Let $\rho(x,t)$ solve (1.9) with a potential H given by (4.11). Suppose that the initial data ρ_0 satisfies $\rho_0(x) \ge 0$ and $\int_{1\le|x|\le L} \rho_0(x)\,dx = M_0$. Then for all sufficiently large γ, there exists a constant C_1 such that if $t \ge C_1 L^2/\gamma$, we have*

$$\int_{B_{\sqrt{3}/2}} \rho(x,t)\,dx \ge cM_0. \tag{7.4}$$

Remark. For simplicity, we picked a fixed constant as a radius of the ball in (7.4). It is not hard to run the argument for an arbitrary radius greater than $1/\sqrt{2}$ (adjusting d_0 and d_1), but then all constants and the range of validity in γ will depend on the choice of radius.

Proof of Corollary 7.2. Recall that we took $d_0 = 5/7$ and $d_1 = 6/7$, and note that $6/7 < \sqrt{3}/2$. Take $f_0 \in C_0^\infty(B_{\sqrt{3}/2})$ as in Theorem 7.1. By duality, we have

$$\int_{\mathbb{R}^2} f_0(x)\rho(x,t)\,dx = \int_{\mathbb{R}^2} f(x,t)\rho_0(x)\,dx.$$

Therefore, applying Theorem 7.1 we find that if C_1 is sufficiently large then

$$\int_{B_{\sqrt{3}/2}} \rho(x,t)\,dx \ge \int_{\mathbb{R}^2} f_0(x)\rho(x,t)\,dx = \int_{\mathbb{R}^2} f(x,t)\rho_0(x)\,dx \ge cM_0. \qquad ∎$$

Corollary 7.2 and (7.4) can replace Theorem 6.6 and (6.21) in the nonlinear argument of the next section. We state here the theorem alternative to Theorem 1.1 that this would yield.

Theorem 7.3. *Under the assumptions of Theorem 1.1, with chemotaxis present, a quarter of the initial mass of ρ_2 will react by time $\tau_C \le C_1 L^2/\gamma$.*

Remark. It is not difficult to design an additional argument that will show, under the assumptions of Theorem 1.1, that more than half of the initial mass of ρ_2 will react if we wait an additional time ~ 1. Basically, once mass $\sim M_0$ has entered B_1, arguments similar to the ones we have used above and employing mass comparison with the simple heat equation lead to the conclusion that after an additional unit time, mass $\sim M_0$ can be found inside $B_{1/\sqrt{2}}$ (or in fact in a ball of smaller radius, with a constant of proportionality depending on the radius). Then the pass-through argument of the following section would yield consumption of the larger fraction of ρ_2.

8. Decay for ρ_2 based on a "pass-through" argument

Let us now consider the nonlinear system (1.4):

$$\begin{aligned} &\partial_t \rho_1 - \Delta \rho_1 + \chi \nabla \cdot (\rho_1 \nabla (-\Delta)^{-1} \rho_2) = -\epsilon \rho_1 \rho_2, \\ &\partial_t \rho_2 = -\epsilon \rho_1 \rho_2. \end{aligned} \tag{8.1}$$

We focus on the case when the initial conditions $\rho_1(\cdot, 0)$ and $\rho_2(\cdot, 0)$ are radially symmetric, so that radial symmetry is preserved for all time. Assume that $\rho_1(\cdot, 0)$ is initially concentrated near $r = L$ with total mass M_0, while $\rho_2(\cdot, 0) = \gamma\eta$ with $\eta \in C_0^\infty$. We think of η as very close to χ_{B_1} in the L^1 norm. As in the introduction, we assume that $\epsilon M_0 \gg \gamma \gg 1$, and $M_0 \gg \theta$. As we will see, the constant B involved in $\gg$ will depend on the value of the ratio $\chi\gamma/\epsilon$ and would have to be larger if the ratio is small (but can be taken uniform for all larger values of the ratio). Combining Proposition 4.3 and Theorem 6.6 shows that if $t_3 \le \tau_C$, with t_3 given in Theorem 6.6, and τ_C the half-time of ρ_2, then at least $1/4$ of the mass of ρ_1 must have entered $B_{1/2}$ by time t_3.

In this section, we will use this result to obtain decay estimates on the mass of ρ_2 which will show that, in fact, $\tau_C \le t_3$. Let us start with a heuristic argument to see how much of ρ_2 should react by time t_3. Since the drift velocity is $\partial_r(-\Delta)^{-1}\rho_2 \sim -\gamma$ for all $r \in (1/2, 1)$, a generic particle of ρ_1 should take about $\sim \gamma^{-1}$ time to pass through the region $(1/2, 1)$. It will react with ρ_2 during this time with coupling coefficient ϵ, so that approximately the $\epsilon M_0/\gamma$ portion of the mass of ρ_2 originally situated in $B_1 \setminus B_{1/2}$ should be gone by time t_3. In other words, if $\epsilon M_0/\gamma \gg 1$, then we should have $\tau_C \le t_3$.

We will discuss below why we have to resort to this "pass-through" argument to get an estimate on the reaction time. The reason has to do with the form of the Keller–Segel chemotaxis term that leads to the possibility of an excessive concentration of ρ_1.

The goal of this section is to rigorously justify the above heuristics. The key step is the following proposition.

Proposition 8.1. *Let ρ_1, ρ_2 be a solution to (1.4) with radially symmetric initial conditions. Assume that $\rho_1(\cdot, 0)$ is concentrated near $r = L$ with total mass M_0, and $\rho_2(\cdot, 0) = \theta\eta(x)$ as described above. Assume that $\epsilon M_0 \gg \gamma \gg 1$, and suppose $\tau_C \geq t_3 + 1$. Then the following holds with some universal constant $c > 0$, where $t_3 > 0$ is as given by (6.19):*

$$\int_0^{t_3+1} \rho_1(r,t)\,dt \geq \frac{cM_0}{\gamma} \quad \textit{for all } r \in (1/2, 1). \tag{8.2}$$

Before we prove the proposition, let us point out that it implies Theorem 1.1.

Proof of Theorem 1.1. Suppose that $\tau_C \geq t_3 + 1$. The second equation in (8.1) implies that

$$\rho_2(r,t) = \rho_2(r,0)\exp\left\{-\epsilon \int_0^t \rho_1(r,s)\,ds\right\},$$

so that if (8.2) holds, then

$$\frac{\rho_2(r, t_3+1)}{\rho_2(r,0)} = \exp\left\{-\epsilon \int_0^{t_3+1} \rho_1(r,t)\,dt\right\} \leq e^{-\epsilon c M_0/\gamma} \quad \text{for all } r \in (1/2, 1).$$

Thus, if $\epsilon M_0/\gamma \gg 1$, then most of the mass of ρ_2 originally supported in $B_1 \setminus B_{1/2}$ will react away by time $t_3 + 1$ and the half-time τ_C satisfies $\tau_C \leq t_3 + 1$, a contradiction. ∎

Recall that in the pure diffusion case, we have

$$\tau_D \gtrsim \frac{L^2}{\log(M_0\epsilon)}.$$

Comparing this with $t_3 + 1$, and assuming that $L^2/\gamma \gtrsim \log\gamma$, we see that chemotaxis would significantly reduce the half-time of reaction in the regime

$$1 \ll \gamma \ll M_0\epsilon \ll e^\gamma.$$

As mentioned in the introduction, such a relationship between parameters is natural in some applications.

The rest of this section contains the proof of Proposition 8.1. Let us denote

$$\tilde{H}(\cdot, t) := \chi(-\Delta)^{-1}\rho_2(\cdot, t).$$

Since $\tilde{H}(\cdot, t)$ is radial, we denote it by $\tilde{H}(r, t)$.

Recall from (4.7) that

$$M(r,t) = \int_{B_r} \rho_1(x,t)\,dx,$$

satisfies

$$\partial_t M - \partial_{rr}^2 M + \frac{1}{r}\partial_r M + (\partial_r M)(\partial_r \tilde{H}) + \epsilon \int_{B_r} \rho_1\rho_2\,dx = 0. \tag{8.3}$$

Since $\rho_1(r,t) = (2\pi r)^{-1}\partial_r M(r,t)$, to prove (8.2) it suffices to show that

$$\int_0^{t_3+1} \partial_r M(r,t)\,dt \geq \frac{cM_0}{\gamma} \quad \text{for all } r \in (1/2, 1). \tag{8.4}$$

Let $I = (a,b) \subset (1/2,1)$ be an arbitrary interval. For any $s \in (a,b)$, integrating (8.3) over (a,s) in r gives

$$\begin{aligned}\int_a^s \partial_t M(r,t)\,dr &= \partial_r M(s,t) - \partial_r M(a,t) - \int_a^s \left(\frac{1}{r}\partial_r M(r,t) + \partial_r M(r,t)\partial_r \tilde{H}(r,t)\right) dr \\ &\quad - \int_a^s \epsilon \int_{B_r} \rho_1\rho_2\,dx\,dr \leq \partial_r M(s,t) + C\gamma \int_a^s \partial_r M(r,t)\,dr,\end{aligned}$$

since $\partial_r M \geq 0$, $\partial_r \tilde{H} \geq -C\gamma$, and $\rho_1, \rho_2 \geq 0$. As $M(r,0) = 0$ for all $r < 1$, since ρ_1 is initially concentrated near $r = L$, integrating this inequality in time from $t = 0$ to $t = t_3$ gives

$$\int_a^s M(r,t_3)\,dr \leq \int_0^{t_3} \partial_r M(s,t)\,dt + C\gamma \int_0^{t_3}\int_a^s \partial_r M(r,t)\,dr\,dt.$$

Combining Proposition 4.3 and Theorem 6.6, we have $M(r,t_3) \geq M_0/4$ for all $r \in (1/2,1)$, and the above inequality becomes

$$\int_0^{t_3} \partial_r M(s,t)\,dt + C\gamma \int_0^{t_3}\int_a^s \partial_r M(r,t)\,dr\,dt \geq \frac{(s-a)M_0}{4}.$$

Integrating this inequality over $s \in I$ gives

$$\int_0^{t_3}\int_I \partial_r M(s,t)\,ds\,dt + C\gamma \int_0^{t_3} (b-a)\int_I \partial_r M(r,t)\,dr\,dt \geq \frac{(b-a)^2 M_0}{8},$$

so that

$$\int_0^{t_3} \frac{1}{|I|}\int_I \partial_r M(s,t)\,ds\,dt \geq \frac{M_0}{8(|I|^{-1} + C\gamma)}.$$

Therefore, for any interval $I \subset (1/2,1)$ with $|I| = \gamma^{-1}$, we have

$$\frac{1}{|I|}\int_I \int_0^{t_3} \partial_r M(s,t)\,dt\,ds \geq \frac{cM_0}{\gamma}. \tag{8.5}$$

This inequality shows that (8.4) holds in each such interval I in an average sense. To finish the proof of Proposition 8.1, we need to rule out the possibility that $\int_0^{t_3} \partial_r M(s,t)\,dt$ is distributed very nonuniformly among $s \in I$. We are going to show that this cannot happen since ρ_1 satisfies a parabolic PDE.

Taking a derivative of (8.3), we deduce a parabolic equation

$$\partial_t u - \partial_{rr}^2 u + \left(\frac{1}{r} + \partial_r \tilde{H}\right)\partial_r u + \left(-\frac{1}{r^2} + \partial_{rr}^2 \tilde{H} + \epsilon\rho_2(r,t)\right)u = 0 \tag{8.6}$$

for $u(r,t) := \partial_r M(r,t) = 2\pi r\rho_1(r,t)$.

Lemma 8.2. *There exists a universal constant $c > 0$ such that any nonnegative solution to* (8.6) *satisfies*

$$u(r,t) \geq c\gamma^3 \int_I \int_{t_0}^{t_0+\gamma^{-2}} u(r,t)\,dt\,dr \quad \textit{for all } r \in I,\ t \in [t_0+\gamma^{-2}, t_0+2\gamma^{-2}],$$

for all intervals $I \subset (1/2, 1)$ with $|I| = 2\gamma^{-1}$, and $t_0 \geq \gamma^{-2}$.

Proof. Let us rescale (8.6) setting $y = \gamma r$, $\tau = \gamma^2(t - t_0)$. In the new coordinates,

$$u_\tau - u_{yy} + b(y)u_y + c(y,\tau)u = 0,$$

where $|b(y)| \leq C$ and $|c(y,\tau)| \leq C$ for all $y \in (\gamma/2, \gamma)$, $\tau \geq 0$. The bounds on b and c follow from the facts that $r \in (1/2, 1)$, $|\partial_r \tilde{H}| \leq C\gamma$, $|\partial_r^2 \tilde{H}| \leq C\gamma$, $\rho_2 \leq \|\rho_2(\cdot,0)\|_\infty \leq \theta$, and

$$\frac{\epsilon\theta}{\gamma^2} = \frac{\epsilon}{\chi\gamma} \leq \tilde{c}^{-1} \tag{8.7}$$

(where $\tilde{c}$ is from Theorem 1.1).

By the parabolic Harnack inequality (e.g. [49, Theorem 6.27 or Corollary 7.42]), for any interval $I' \subset (\gamma/2, \gamma)$ with length 2 we have

$$u(y,\tau) \geq C \int_{I'} \int_0^1 u(y,t)\,dt\,dy \quad \text{for all } y \in I', \tau \in [1,2];$$

here the constant C depends on $\tilde{c}$ in (8.7). Translating this back into the original coordinates finishes the proof. ∎

Consider the time intervals $J_k := [2k\gamma^{-2}, 2(k+1)\gamma^{-2}]$, $k \in \mathbb{N}$, and let n be the smallest integer such that $2(n+1)\gamma^{-2} \geq t_3$. Then for any interval $I \subset (1/2, 1)$ with $|I| = 2\gamma^{-1}$, we can rewrite (8.5) as

$$\sum_{k=0}^{n} \int_{I \times J_k} \partial_r M(r,t)\,dr\,dt \geq \frac{cM_0}{\gamma^2}, \tag{8.8}$$

while Lemma 8.2 gives, for each $k \geq 0$,

$$\partial_r M(r,t) \geq C\gamma^3 \int_{I \times J_k} \partial_r M(s,t)\,ds\,dt \quad \text{for all } r \in I \text{ and } t \in J_{k+1}, \tag{8.9}$$

so that

$$\int_{J_{k+1}} \partial_r M(r,t)\,dt \geq C\gamma \int_{I \times J_k} \partial_r M(s,t)\,ds\,dt.$$

It follows that for each $r \in I$ we have

$$\begin{aligned} \int_0^{t_3+1} \partial_r M(r,t)\,dt &\geq \int_0^{(n+2)\gamma^{-2}} \partial_r M(r,t)\,dt = \sum_{k=0}^{n} \int_{J_{k+1}} \partial_r M(r,t)\,dt \\ &\geq c\gamma \sum_{k=0}^{n} \int_{I \times J_k} \partial_r M(s,t)\,ds\,dt \geq \frac{cM_0}{\gamma}. \end{aligned} \tag{8.10}$$

This finishes the proof of Proposition 8.1.

9. Discussion

In this section, we briefly discuss the nature of the constraints in our main nonlinear application. The arguments here are purely heuristic, though some of the statements can be made rigorous. Observe that for $H(x) = \gamma(-\Delta)^{-1}\chi_{B_1}(x)$, the ground state is

$$e^H = \begin{cases} e^{\gamma(1-r^2)/4}, & r < 1, \\ r^{-\gamma/2}, & r \geq 1. \end{cases} \tag{9.1}$$

A simple calculation shows that for $r < 1$ we have

$$\int_{B_r} e^{H(x)}\,dx = \frac{4\pi}{\gamma} e^{\gamma/4}(1 - e^{-\gamma r^2/4}),$$
$$\int_{(B_r)^c} e^{H(x)}\,dx = \frac{4\pi}{\gamma} e^{\gamma/4}(e^{-\gamma r^2/4} - e^{-\gamma/4}) + \frac{4\pi}{\gamma - 4}.$$

Therefore, most of the mass of e^H is concentrated in a ball of radius $\sim \gamma^{-1/2}$ centered at the origin.

This explains why the radial constraint on the initial conditions is needed to make touch with the heuristics. Indeed, consider ρ_1 that is concentrated initially at a distance L from the support of ρ_2, in a region of size ~ 1 (as opposed to radial). If γ is large, as this mass gets transported towards the origin, it will enter the support of ρ_2 – the unit ball centered at the origin – through a narrow sector and then concentrate overwhelmingly in a tiny region near the origin. After a time $\sim L^2/\gamma$, the density ρ_1 will approximate e^H given by (9.1) since not much reaction has happened during the passage through a narrow sector. Thus, even after the transport phase has taken place, the reaction rate is going to be penalized since ρ_1 is smaller than M_0 by a factor that is exponential in γ on most of the support of ρ_2. As ρ_2 gets depleted near the origin, the potential and so the configuration of ρ_1 will adjust, but this process is not straightforward to control. It seems clear that some essential extra time will be lost.

A similar issue applies in the "risky" regime $\epsilon M_0 \ll 1$, even in the radial case. Then, little reaction happens on the pass through, while the reaction after the transport stage incurs the same penalty due to the aforementioned excessive concentration.

Both of these constraints are due to an artifact of the specific form of the Keller–Segel chemotaxis term. The extreme concentration of e^H can be seen as a consequence of the scaling $\chi\nabla(-\Delta)^{-1}\rho_2 \sim -\gamma$ near and on the support of ρ_2, which is very large when γ is large. But in reality, there is always a speed limit on how fast biological agents can move. A variation of the classical Keller–Segel model is the so-called flux limited chemotaxis system given by

$$\partial_t\rho_1 + \chi\nabla\cdot\left(\rho_1\frac{\nabla c}{|\nabla c|}\psi(|\nabla c|)\right) - \Delta\rho_1 = -\epsilon\rho_1\rho_2, \quad c = (-\Delta)^{-1}\rho_2, \quad \partial_t\rho_2 = -\epsilon\rho_1\rho_2. \tag{9.2}$$

The function ψ appearing in (9.2) satisfies $\psi(0) = 0$, is increasing, and saturates at some level that we can take equal to 1 (given that we have an explicit coupling constant χ). The system (9.2) is more complex to analyze due to the strongly nonlinear flux, but is more realistic. A variety of flux limited Keller–Segel systems have been considered recently in many works (see e.g. [3, 34] for more references); in particular, papers [22, 62, 71] provided derivation of the flux limited Keller–Segel system from kinetic models built on biologically reasonable assumptions about the behavior of the modeled organisms.

In future work, we plan to adapt the techniques developed in this paper to analyze (9.2). The adaptation is not straightforward, but preliminary computations show that in this case the radial assumption is not necessary, and the case of the "risky" reaction can be handled.

Appendix A

A.1. Proof of Theorem 3.1

Let us first assume that ρ_0 is nonnegative. The proof is almost identical to that of [10, Theorem 5], but we include it for the sake of completeness. Let $r(t)$ be a C^1 increasing function to be specified later. We compute the time evolution of $\|\rho(t)\|_{r(t)}$ as follows, where we omit the t, x dependence on the right hand side for notational simplicity:

$$\begin{aligned}
&\frac{d}{dt}\|\rho(\cdot,t)\|_{r(t)} \\
&\quad = -\frac{r'}{r^2}\|\rho\|_r \log(\|\rho\|_r^r) + \frac{r'}{r}\|\rho\|_r^{1-r}\int \rho^r \log\rho\,dx + \|\rho\|_r^{1-r}\int \rho^{r-1}\partial_t\rho\,dx \\
&\quad = \frac{r'}{r^2}\|\rho\|_r^{1-r}\int \rho^r \log\left(\frac{\rho^r}{\|\rho\|_r^r}\right)dx + \|\rho\|_r^{1-r}\int \rho^{r-1}(\Delta\rho - \nabla\cdot(\rho\nabla\Phi) - h\rho)\,dx \\
&\quad \le \frac{r'}{r^2}\|\rho\|_r^{1-r}\left(\int \rho^r \log\left(\frac{\rho^r}{\|\rho\|_r^r}\right)dx - \frac{4(r-1)}{r'}\int |\nabla\rho^{r/2}|^2\,dx\right) + \frac{r-1}{r}\gamma\|\rho\|_r,
\end{aligned} \tag{A.1}$$

where in the last inequality we use the assumptions $h(\cdot,t) \ge 0$ and $\Delta\Phi(\cdot,t) \ge -\gamma$ for all t, as well as the fact that ρ remains nonnegative for all $t \ge 0$. Next we use a sharp form of the logarithm Sobolev inequality in $\mathbb{R}^n$. It is [10, (7.17)], and it is equivalent to Gross's logarithmic Sobolev inequality in [28] after a scale transformation. For all $f \in H^1(\mathbb{R}^d)$ and all $a > 0$,

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\|f\|_2^2}\right)dx + \left(d + \frac{d}{2}\log a\right)\int_{\mathbb{R}^d} f^2\,dx \le \frac{a}{\pi}\int_{\mathbb{R}^d}|\nabla f|^2\,dx. \tag{A.2}$$

Choosing $f = \rho^{r/2}$ and $a = 4\pi(r-1)/r'$, we see that (A.2) becomes

$$\int_{\mathbb{R}^d} \rho^r \log\left(\frac{\rho^r}{\|\rho\|_r^r}\right)dx + \left(d + \frac{d}{2}\log\left(\frac{4\pi(r-1)}{r'}\right)\right)\|\rho\|_r^r \le \frac{4(r-1)}{r'}\int_{\mathbb{R}^d}|\nabla\rho^{r/2}|^2\,dx.$$

Applying this to (A.1) gives

$$\frac{d}{dt}\|\rho(t)\|_{r(t)} \le \frac{r'}{r^2}\|\rho\|_r\left(-d - \frac{d}{2}\log\left(\frac{4\pi(r-1)}{r'}\right)\right) + \frac{r-1}{r}\gamma\|\rho\|_r.$$

Let $G(t) := \log \|\rho(t)\|_{r(t)}$. Then the above differential inequality becomes

$$\frac{dG}{dt} \le \frac{r'}{r^2}\left(-d - \frac{d}{2}\log\left(\frac{4\pi(r-1)}{r'}\right)\right) + \frac{r-1}{r}\gamma. \tag{A.3}$$

Since our goal is to estimate $\|\rho(T)\|_\infty$ using $\|\rho(0)\|_1$ (where $T > 0$ is an arbitrary time at which we want to obtain our estimate), let us set $r(0) = 1$ and $r(T) = p$, where $p > 1$ will be sent to infinity at the end. Integrating (A.3) in $[0, T]$ yields

$$\begin{aligned}
\log\left(\frac{\|\rho(T)\|_p}{\|\rho(0)\|_1}\right) &= G(T) - G(0) \le \int_0^T \left(\frac{r'}{r^2}\left(-d - \frac{d}{2}\log\left(\frac{4\pi(r-1)}{r'}\right)\right) + \frac{r-1}{r}\gamma\right) dt \\
&\le -\int_0^T s'\left(-d - \frac{d}{2}\log(4\pi(s-s^2)) + \frac{d}{2}\log(-s')\right) dt + \gamma T \quad (\text{let } s(t) := 1/r(t)) \\
&\le \int_1^{1/p}\left(d + \frac{d}{2}\log(4\pi(s-s^2))\right) ds + \frac{d}{2}\int_0^T (-s')\log(-s')\,dt + \gamma T.
\end{aligned}$$

The first integral on the right hand side can be explicitly computed, and it is uniformly bounded by some constant $C(d)$ as $p \to \infty$. For the second integral, since $\int_0^T (-s')\,dt$ is fixed as $s(0) - s(T) = 1 - 1/p$, Jensen's inequality gives that the integral is minimized when $-s'$ is a constant. We thus set $-s' = \frac{1-1/p}{T}$, which yields

$$\log\left(\frac{\|\rho(T)\|_p}{\|\rho(0)\|_1}\right) \le C(d) + \frac{d}{2}\left(1 - \frac{1}{p}\right)\log\left(\frac{1-1/p}{T}\right) + \gamma T,$$

hence in the limit $p \to \infty$ we obtain

$$\|\rho(T)\|_\infty \le C(d) T^{-d/2} e^{\gamma T} \|\rho(0)\|_1 \quad \text{for all } T > 0. \tag{A.4}$$

Note that $t^{-d/2}e^{\gamma t}$ reaches its minimum value $(2\gamma/d)^{d/2}e^{d/2}$ at $t = \frac{d}{2\gamma}$. For $t \ge \frac{d}{2\gamma}$, by applying the estimate (A.4) with $t - \frac{d}{2\gamma}$ as the initial time (and using the fact that $\|\rho(t - \frac{d}{2\gamma})\|_1 = \|\rho(0)\|_1$), we obtain $\|\rho(t)\|_\infty \le C(d)\gamma^{d/2}\|\rho_0\|_1$ for all $t \ge \frac{d}{2\gamma}$. Combining this with (A.4) gives

$$\|\rho(t)\|_\infty \le C(d) \max\{t^{-d/2}, \gamma^{d/2}\} \|\rho(0)\|_1 \quad \text{for all } t > 0.$$

To establish the theorem for sign changing ρ_0, notice that equation (3.1) is linear. Thus we can run the evolution separately for the positive and negative parts of the initial data, and both solutions will satisfy (3.2). By linearity, the actual solution of (3.1) is just the difference of these two solutions and (3.2) clearly holds for it as well. ∎

A.2. *Proof of Theorem* 5.1

Existence and uniqueness of a weak solution $\rho(x, t)$ under the assumptions of Theorem 5.1 is well known (see e.g. [47, Theorem 2.3.1]). However, the regularity characterization of the solution available in the standard literature is weaker than what is convenient for us here (though stated for a more general class of coefficients). Here we provide a sketch of a simple proof that applies to the particular case of time independent H.

First, let us introduce the mollified potential $H_\epsilon = \eta_\epsilon * H$, where $\eta_\epsilon(x) = \epsilon^{-2}\eta(x/\epsilon)$ is a standard mollifier. Let us denote by ρ_ϵ the smooth solution of (5.1) with H replaced by H_ϵ. It suffices to prove the uniform-in-ϵ bounds for ρ_ϵ for all $0 < \epsilon < 1$; one can show in a standard way that they are inherited in the limit by ρ. Multiply (5.1) by $\rho_\epsilon|\rho_\epsilon|^{p-2}$, $p > 1$, and integrate by parts in space to obtain

$$\frac{1}{p}\partial_t \int_{\mathbb{R}^2} |\rho_\epsilon|^p \, dx + (p-1)\int_{\mathbb{R}^2} |\nabla \rho_\epsilon|^2 |\rho_\epsilon|^{p-2} \, dx = (p-1)\int_{\mathbb{R}^2} \rho_\epsilon \nabla H_\epsilon \cdot \nabla \rho_\epsilon |\rho_\epsilon|^{p-2} \, dx.$$

The integral on the right hand side can be estimated by

$$\begin{aligned}(p-1)\Big(\int_{\mathbb{R}^2} |\nabla \rho_\epsilon|^2 |\rho_\epsilon|^{p-2} \, dx\Big)^{1/2} \Big(\int_{\mathbb{R}^2} |\nabla H_\epsilon|^2 |\rho_\epsilon|^p\Big)^{1/2} \\ \le \frac{p-1}{2}\int_{\mathbb{R}^2} |\nabla \rho_\epsilon|^2 |\rho_\epsilon|^{p-2} \, dx + C(p-1)\|\nabla H\|_{L^\infty}^2 \|\rho_\epsilon\|_{L^p}^p,\end{aligned}$$

where we have used $\|\nabla H_\epsilon\|_{L^\infty} \le \|\nabla H\|_{L^\infty}$. It follows that given $T \ge 0$, we have $\|\rho_\epsilon\|_p \le C(p, \rho_0, H, T)$ for every $t \le T$.

Now define $v = \rho_\epsilon - \rho_0$. Then

$$\partial_t v - \Delta v + \nabla \cdot (v \nabla H_\epsilon) = f, \quad v(\cdot, 0) = 0,$$

where $f = \Delta \rho_0 - \nabla \cdot (\rho_0 \nabla H_\epsilon)$ satisfies

$$\|f\|_p \le \|\rho_0\|_{W^{2,p}} (\|\nabla H\|_\infty + \|\Delta H\|_\infty).$$

Since H does not depend on t, $w = \partial_t v = \partial_t \rho_\epsilon$ satisfies

$$\partial_t w - \Delta w + \nabla \cdot (w \nabla H) = 0, \quad w(\cdot, 0) = f.$$

By the argument identical to the above estimate for $\|\rho_\epsilon\|_p$, we obtain $\|\partial_t \rho_\epsilon\|_p \le C(p, \rho_0, H, T)$ for every $t \le T$. Finally, the $W^{2,p}$ estimate of v (and thus of ρ_ϵ) follows from standard elliptic regularity estimates. ■

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