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Miaofen Chen

Fargues–Rapoport conjecture for p-adic period domains in the non-basic case

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Abstract. We prove the Fargues–Rapoport conjecture for p-adic period domains in the non-basic case with minuscule cocharacter. More precisely, we give a group-theoretical criterion for the cases when the admissible locus and weakly admissible locus coincide. This generalizes the result of Hartl (2013) for the group GL_n. In the last section, we also give a conjecture about the intersection of weakly admissible locus and the Newton strata in the flag variety.

Keywords. p-adic period domain, Fargues–Fontaine curve, Newton stratification

Introduction

Let F be a finite extension of $\mathbb{Q}_p$, and let $\breve{F}$ be the p-adic completion of the maximal unramified extension of F with Frobenius σ. We consider the flag variety $\mathcal{F}(G,\mu)$ associated to a pair (G,μ), where G is a reductive group over F, and μ a minuscule cocharacter of G. It is a projective variety over the local reflex field $E = E(G,\{\mu\})$, the field of definition of the geometric conjugacy class $\{\mu\}$ of μ. We still denote by $\mathcal{F}(G,\mu)$ the associated adic space over $\breve{E}$. For any $b \in G(\breve{F})$ satisfying certain conditions with respect to μ (cf. Section 2.2), we are interested in two open adic subspaces

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}} \subseteq \mathcal{F}(G,\mu,b)^{\mathrm{wa}}$$

of $\mathcal{F}(G,\mu)$, where $\mathcal{F}(G,\mu,b)^{\mathrm{wa}}$ is the weakly admissible locus, defined by Rapoport and Zink [33] by removing a profinite number of Schubert varieties from $\mathcal{F}(G,\mu)$ which contradict the weak admissibility condition defined by Fontaine (cf. Section 2.2), and where $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ is the admissible locus, also called the p-adic period domain. The latter is much more mysterious. The existence of the admissible locus has been conjectured by Rapoport and Zink. It is characterized by the properties that it has the same classical points

Miaofen Chen: School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, No. 500, Dong Chuan Road, Shanghai 200241, China; mfchen@math.ecnu.edu.cn

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as the weakly admissible locus (see [9], in which the weakly admissible locus is also considered as its algebraic approximation) and there exists a local system with G-structures on it which interpret the crystalline representations on all classical points. When the triple (G, μ, b) is of PEL type, the admissible locus $\mathcal{F}(G, \mu, b)^{\mathrm{a}}$ is the image of the p-adic period mapping from the Rapoport–Zink space associated to (G, μ, b) to the flag variety $\mathcal{F}(G, \mu)$. There is also a direct construction of $\mathcal{F}(G, \mu, b)^{\mathrm{a}}$ in special cases by Hartl [21] and Faltings [12]. In the most general case, the existence of the admissible locus equipped with the étale local system is known due to the work of Fargues–Fontaine [18], Kedlaya–Liu [23] and Scholze [35]. In fact, the admissible locus is defined by semistable conditions on the modification of the G-bundle associated to b of type μ on the Fargues–Fontaine curve. Moreover, it can also be considered as the image of the p-adic period mapping from the local Shimura variety associated to (G, μ, b) to the flag variety $\mathcal{F}(G, \mu)$ [30, 36].

We want to understand the structure of the p-adic period domain. As the structure of its approximation, the weakly admissible locus, is well known, it is natural to ask when the p-adic period domain coincides with the weakly admissible locus. Hartl [21] classified all such cases for $G = \mathrm{GL}_n$. For a general group G, Rapoport and Fargues have conjectured a group-theoretic criterion when b is basic, which has now become the following theorem.

Theorem (Fargues–Rapoport conjecture, [6, Theorem 6.1]). *Suppose b is basic. The equality $\mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(G, \mu, b)^{\mathrm{a}}$ holds if and only if (G, μ) is fully Hodge–Newton-decomposable.*

Recently, Shen [40] also generalized the Fargues–Rapoport conjecture to non-minuscule cocharacters. Here the full Hodge–Newton-decomposability condition is purely group-theoretic. This notion is first introduced and systematically studied by Görtz, He and Nie [19], who classified all the fully Hodge–Newton-decomposable pairs and gave more conditions equivalent to full Hodge–Newton-decomposability. When G is a general linear group, the triple (G, μ, b) is *Hodge–Newton-indecomposable* if no breakpoints of the Newton polygon defined by b touch the Hodge polygon defined by μ. Otherwise, the triple is called *Hodge–Newton-decomposable*. Let $B(G)$ be the set of σ-conjugacy classes of $G(\breve{F})$. For $b \in G(\breve{F})$, let $[b] \in B(G)$ be the σ-conjugacy class of b. An element $[b] \in B(G)$ is called *basic* if the Newton polygon of b is central. Kottwitz defined a subset $B(G, \mu)$ in $B(G)$. When G is the general linear group, $[b] \in B(G, \mu)$ if and only if the Newton polygon of $[b]$ lies on or above the Hodge polygon of μ and they have the same endpoint. The pair (G, μ) is *fully Hodge–Newton-decomposable* if for any non-basic $[b']$ in the Kottwitz set $B(G, \mu)$, the triple (G, μ, b') is Hodge–Newton-decomposable. We refer to Section 3 for the details of these notions.

The main result of this article is a generalized version of the Fargues–Rapoport conjecture which works for any b. It is inspired by the Fargues–Rapoport conjecture for basic elements and Hartl's result for GL_n. For simplicity, we assume that G is quasi-split in this introduction. There is also a similar description for the non-quasi-split case in Section 4 (Theorem 4.3).

Theorem (Theorem 4.1). *Suppose M is the standard Levi subgroup of G such that $[b] \in B(M, \mu)$ and (M, b, μ) is Hodge–Newton-indecomposable. Then the equality $\mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(G, \mu, b)^{\mathrm{a}}$ holds if and only if $(M, \{\mu\})$ is fully Hodge–Newton-decomposable and $[b]$ is basic in $B(M)$.*

The key ingredients of the proof of the main theorem are Proposition 3.7 which describes the relation of the (weakly) admissible locus for different groups G and its Levi subgroup M and the following proposition.

Proposition (Proposition 4.5). *Suppose (G, μ, b) is Hodge–Newton-indecomposable. If b is not basic, then $\mathcal{F}(G, \mu, b)^{\mathrm{a}} \neq \mathcal{F}(G, \mu, b)^{\mathrm{wa}}$.*

Indeed, we know that on the flag variety there is a group action of $\tilde{J}_b$. We prove that the group action preserves the admissible locus but not the weakly admissible locus by producing a point which is weakly admissible but not admissible. Such a point exists in the $\tilde{J}_b$-orbit of a non-weakly-admissible point.

We briefly describe the structure of this article. In Section 1, we review the basic facts about the Kottwitz set and G-bundles on the Fargues–Fontaine curve. In Section 2, we review the reduction of G-bundles and introduce the weakly admissible locus and admissible locus in the flag variety in terms of (weakly) semistable condition on the modification of G-bundles. In Section 3, we review the Hodge–Newton-decomposability condition and prove Proposition 3.7, which is one of the main ingredients for the proof of the main theorems. In Section 4, we prove Proposition 4.5 and the main theorems 4.1 and 4.3 by using the $\tilde{J}_b$-action on the flag variety. In Section 5, we discuss the relation between the Newton strata and the weakly admissible locus. We introduce a conjecture predicting which Newton strata contain weakly admissible points in the basic case. We prove this conjecture in a very special case in Proposition 5.4. The full conjecture has been proved by Viehmann [41] very recently.

Notations

We use the following notations:

- F is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and a uniformizer π_F.
- $\overline{F}$ is an algebraic closure of F and $\Gamma = \mathrm{Gal}(\overline{F}|F)$.
- $\breve{F} = \widehat{F^{\mathrm{un}}}$ is the completion of the maximal unramified extension with Frobenius σ.
- G is a connected reductive group over F, and H is a quasi-split inner form of G equipped with an inner twisting $G_{\breve{F}} \xrightarrow{\sim} H_{\breve{F}}$.
- $A \subseteq T \subseteq B$, where A is a maximal split torus, $T = Z_H(A)$ is the centralizer of A in T, and B is a Borel subgroup in H.
- $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is the absolute root datum with positive roots Φ^+ and simple roots Δ with respect to the choice of B.

- $W = N_H(T)/T$ is the absolute Weyl group of T in H, and w_0 is the maximal length element in W.
- $(X^*(A), \Phi_0, X_*(A), \Phi_0^\vee)$ is the relative root datum with positive roots Φ_0^+ and simple (reduced) roots Δ_0.
- If M is a standard Levi subgroup in H we denote by Φ_M the corresponding roots or coroots showing up in Lie M, and by W_M the Weyl group of M. If P is the standard parabolic subgroup of H with Levi component M, sometimes we also write W_P for W_M.

1. Kottwitz set and G-bundles on the Fargues–Fontaine curve

In this section, we will recall some basic facts about G-bundles on the Fargues–Fontaine curve which will be the main tool for our study of p-adic period domains.

1.1. The Fargues–Fontaine curve

Let K be a perfectoid field over $\mathbb{F}_q$ with $\omega_K \in K$ satisfying $0 < |\omega_K| < 1$. Let

$$\mathcal{Y}_K = \mathrm{Spa}(W_{\mathcal{O}_F}(\mathcal{O}_K)) \backslash V(\pi_F[\omega_K])$$

be an adic space over F equipped with an automorphism φ induced from the Frobenius $K|\mathbb{F}_q$. The *Fargues–Fontaine curve* over F associated to K is the scheme

$$X = X_K := \mathrm{Proj}\Big(\bigoplus_{d \geq 0} B_K^{\varphi = \pi_F^d}\Big)$$

where $B_K = H^0(\mathcal{Y}_K, \mathcal{O}_{\mathcal{Y}_K})$. The scheme X is a curve (which means it is a one-dimensional noetherian regular scheme) over F [18, Theorems 6.5.2, 7.3.3].

If we replace K by an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $\mathbb{F}_q$, we can similarly construct $\mathcal{Y}_S$ and X_S over F; the latter is called the *relative Fargues–Fontaine curve* (cf. [23]).

1.2. G-bundles

From now on, suppose $K = C^\flat$ is the tilt of a complete algebraically closed field C over F. Then the curve X is equipped with a closed point ∞ with residue field $k(\infty) = C$. Let Bun_X be the category of vector bundles on X. The classification of vector bundles on X is well known due to the work of Fargues–Fontaine.

Theorem 1.1 ([18, Theorem 8.2.10]). *Every vector bundle on X is a direct sum of stable vector subbundles, and the isomorphism classes of stable vector bundles on X are parametrized by the slope in $\mathbb{Q}$.*

For $\lambda \in \mathbb{Q}$, let $\mathcal{O}(\lambda)$ be a stable vector bundle on X of slope $\lambda \in \mathbb{Q}$.

By [18], $\{\infty\} = V^+(t)$ with $t \in H^0(X, \mathcal{O}(1))$. Then

$$X \setminus \{\infty\} = \mathrm{Spec}(B_e) \quad \text{and} \quad \widehat{X}_\infty = \mathrm{Spec}(B_{\mathrm{dR}}^+)$$

where $B_e = B_K[1/t]^{\varphi=1}$ is a principal ideal domain, and B_{dR}^+ is a complete discrete valuation ring with residue field C. Let B_{dR} be the fraction field of B_{dR}^+. The following proposition tells us that a vector bundle on X is determined by its restrictions to $X \setminus \{\infty\}$ and $\widehat{X}_\infty$ with a gluing datum.

Proposition 1.2 ([18, Corollary 5.3.2]). *Let $\mathcal{C}$ be the category of triples $(M_e, M_{\mathrm{dR}}, u)$ where*

- *M_e is a free B_e-module of finite rank;*
- *M_{dR} is a free B_{dR}^+-module of finite rank;*
- *$u : M_e \otimes_{B_e} B_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$ is an isomorphism of B_{dR}-modules.*

Then there is an equivalence of categories

$$\mathrm{Bun}_X \xrightarrow{\sim} \mathcal{C}, \quad \mathcal{E} \mapsto (\Gamma(X \setminus \{\infty\}, \mathcal{E}), \widehat{\mathcal{E}}_\infty, \mathrm{Id}).$$

Let $\mathrm{Isoc}_{\breve{F}|F}$ be the category of isocrystals relative to $\breve{F}|F$. By [18, Theorem 8.2.10], there is an essentially surjective functor

$$\mathcal{E}(-) : \mathrm{Isoc}_{\breve{F}|F} \to \mathrm{Bun}_X$$

where for any $(D, \varphi) \in \mathrm{Isoc}_{\breve{F}|F}$, the vector bundle $\mathcal{E}(D, \varphi)$ on X is associated to the graded $\bigoplus_{d \geq 0} B_K^{\varphi = \pi_F^d}$-module

$$\bigoplus_{d \geq 0} (D \otimes B_K)^{\varphi \otimes \varphi = \pi_F^d}.$$

In fact, it maps a simple isocrystal of slope λ to $\mathcal{O}(-\lambda)$.

To any $b \in G(\breve{F})$, we can associate an isocrystal with G-structures:

$$\mathcal{F}_b : \mathrm{Rep}\, G \to \mathrm{Isoc}_{\breve{F}|F}, \quad V \mapsto (V_{\breve{F}}, b\sigma).$$

Its isomorphism class only depends on the σ-conjugacy class $[b] \in B(G)$ of b, where $B(G)$ is the set of σ-conjugacy classes in $G(\breve{F})$. In this way, $B(G)$ parametrizes the set of isomorphism classes of F-isocrystals with G-structure (cf. [31, Remarks 3.4 (i)]).

Recall that a G-*bundle* on X is a G-torsor on X which is locally trivial for the étale topology. Equivalently, a G-bundle on X can also be viewed as an exact functor $\mathrm{Rep}\, G \to \mathrm{Bun}_X$ where $\mathrm{Rep}\, G$ is the category of rational algebraic representations of G. The étale cohomology set $H^1_{\text{ét}}(X, G)$ classifies the isomorphism classes of G-bundles on X.

For $b \in G(\breve{F})$, we can associate to b a G-bundle on X,

$$\mathcal{E}_b = \mathcal{E}(-) \circ \mathcal{F}_b : \mathrm{Rep}\, G \xrightarrow{\mathcal{F}_b} \mathrm{Isoc}_{\breve{F}|F} \xrightarrow{\mathcal{E}(-)} \mathrm{Bun}_X.$$

By [15, Theorem 5.1], there is a bijection of sets

$$B(G) \xrightarrow{\sim} H^1_{\text{ét}}(X, G), \quad [b] \mapsto [\mathcal{E}_b].$$

In this way, the set $B(G)$ also classifies G-bundles on X.

1.3. Kottwitz set

There are two invariants on the set $B(G)$, the Newton map and the Kottwitz map. For any $b \in G(\breve{F})$, there is a composed functor

$$\mathcal{F} : \operatorname{Rep} G \xrightarrow{\mathcal{F}_b} \operatorname{Isoc}_{\breve{F}|F} \to \mathbb{Q}\text{-grVect}_{\breve{F}},$$

where $\mathbb{Q}\text{-grVect}_{\breve{F}}$ is the category of $\mathbb{Q}$-graded vector spaces over $\breve{F}$ and the second functor is given by the Dieudonné–Manin classification of isocrystals which decomposes an isocrystal into isocline subisocrystals parametrized by $\mathbb{Q}$. We can attach to $\mathcal{F}$ a slope morphism

$$\nu_b : \mathbb{D}_{\breve{F}} = \operatorname{Aut}^{\otimes}(\omega) \to \operatorname{Aut}^{\otimes}(\omega \circ \mathcal{F}) = G_{\breve{F}},$$

where $\omega : \mathbb{Q}\text{-grVect}_{\breve{F}} \to \operatorname{Vect}_{\breve{F}}$ is the natural forgetful functor and $\mathbb{D}$ is the pro-algebraic torus over $\breve{F}$ with $X^*(\mathbb{D}) = \mathbb{Q}$.

The conjugacy class of the slope morphism ν_b is defined over F and it only depends on the σ-conjugacy class of b. We thus obtain the *Newton map*

$$\nu : B(G) \to \mathcal{N}(G), \quad [b] \mapsto [\nu_b],$$

where $\mathcal{N}(G) = \mathcal{N}(H) = X_*(A)^+_{\mathbb{Q}}$ is the Newton chamber. The σ-conjugacy class $[b] \in B(G)$ is called *basic* if ν_b is central. Denote by $B(G)_{\text{basic}}$ the subset of basic σ-conjugacy classes in $B(G)$.

The other invariant is the *Kottwitz map* ([25, §4.9, §7.5], [31, Theorem 1.15]):

$$\kappa_G : B(G) \to \pi_1(G)_\Gamma, \quad [b] \mapsto \kappa_G([b]),$$

where $\pi_1(G) = \pi_1(H) = X_*(T)/\langle \Phi^\vee \rangle$ is the algebraic fundamental group of G, and $\pi_1(G)_\Gamma$ is the Galois coinvariants. For $G = \mathrm{GL}_n$, $\kappa_{\mathrm{GL}_n}([b]) = v_p(\det(b)) \in \mathbb{Z} = \pi_1(\mathrm{GL}_n)_\Gamma$ where v_p denotes the p-adic valuation on $\breve{F}$. For general G, the Kottwitz map is characterized by the unique natural transformation $\kappa_{(-)} : B(-) \to \pi_1(-)_\Gamma$ of set valued functors on the category of connected reductive groups over F such that κ_{GL_n} is defined as above. Let $B(G)_{\text{basic}}$ be the subset of basic elements in $B(G)$. Then the restriction of κ_G to $B(G)_{\text{basic}}$ induces a bijection

$$\kappa_G : B(G)_{\text{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

Moreover, if $[b] \in B(G)_{\text{basic}}$ with $\kappa_G([b]) = \mu^\sharp \in \pi_1(G)_\Gamma$ with $\mu \in X_*(T)^+$, then $[\nu_b] = \mathrm{Av}_\Gamma \mathrm{Av}_W(\mu)$ where Av_W (resp. Av_Γ) denotes the W-average (resp. Γ-average).

The elements in $B(G)$ are determined by the Newton map and the Kottwitz map. Namely, the map

$$(\nu, \kappa_G) : B(G) \to \mathcal{N}(G) \times \pi_1(G)_\Gamma \tag{1.3.1}$$

is injective [25, §4.13].

Definition 1.3. (1) Let $[b] \in B(G)$ and $\mu \in X_*(T)^+$. We define the *Kottwitz sets*

$$B(G,\mu) := \{[b] \in B(G) \mid [\nu_b] \leq \mu^\diamond,\ \kappa_G(b) = \mu^\sharp\},$$
$$A(G,\mu) := \{[b] \in B(G) \mid [\nu_b] \leq \mu^\diamond\},$$

which are finite subsets in $B(G)$, where

- $\mu^\diamond := \mathrm{Av}_\Gamma(\mu) \in \mathcal{N}(G)$ is the Galois average of μ,
- $\mu^\sharp \in \pi_1(G)_\Gamma$ is the image of μ via the natural quotient map $X_*(T) \to \pi_1(G)_\Gamma$,
- the order $\leq$ on $\mathcal{N}(G)$ is the usual order: $\nu_1 \leq \nu_2$ if and only if $\nu_2 - \nu_1 \in \mathbb{Q}_{\geq 0}\Phi_0^+$.

(2) We define a partial order on $A(G,\mu)$: for $[b_1], [b_2] \in A(G,\mu)$, we write $[b_1] \leq [b_2]$ if $[\nu_{b_1}] \leq [\nu_{b_2}]$.

We will also need the following generalized Kottwitz set defined in [6].

Definition 1.4. For $\varepsilon \in \pi_1(G)_\Gamma$ and $\delta \in X_*(A)^+_{\mathbb{Q}}$ we set

$$B(G,\varepsilon,\delta) = \{[b] \in B(G) \mid \kappa_G(b) = \varepsilon \text{ and } [\nu_b] \leq \delta\}.$$

Remark 1.5. (1) We follow the notations in [6]. For the generalized Kottwitz set $B(G,\varepsilon,\delta)$, the term ε is written in an additive way while the term δ is written in a multiplicative way.

(2) We have

$$B(G,\mu) = B(G,\mu^\sharp,\mu^\diamond),$$
$$A(G,\mu) = \coprod_{\varepsilon \in \pi_1(G)_{\Gamma,\mathrm{tor}}} B(G,\mu^\sharp + \varepsilon,\mu^\diamond),$$

where $\pi_1(G)_{\Gamma,\mathrm{tor}}$ denotes subgroup of torsion elements in $\pi_1(G)_\Gamma$.

Definition 1.6. Suppose G is quasi-split. Let M be a standard Levi subgroup of G. We define a partial order $\preceq_M$ in $\pi_1(M)_\Gamma$ and in $\pi_1(M)_{\Gamma,\mathbb{Q}}$ as follows: for y_1, y_2 in $\pi_1(M)_\Gamma$ (resp. $\pi_1(M)_{\Gamma,\mathbb{Q}}$), we write $y_1 \preceq_M y_2$ if and only if $y_2 - y_1$ is a non-negative integral linear combination of images in $\pi_1(M)_\Gamma$ (resp. $\pi_1(M)_{\Gamma,\mathbb{Q}}$) of coroots corresponding to the simple roots of T in N with N the unipotent radical of the standard parabolic subgroup of G with Levi component M.

We have the following characterization of the generalized Kottwitz set.

Lemma 1.7. *Suppose G is quasi-split. Let M be a standard Levi subgroup of G. Let $b \in M(\breve{F})$ be such that $[b]_M \in B(M)$ is basic. Then $[b] \in B(G, \varepsilon, \delta)$ if and only if*

$$\kappa_G(b) = \varepsilon \quad \text{and} \quad \kappa_M(b) \preceq_M \delta^\sharp \quad \text{in } \pi_1(M)_{\Gamma,\mathbb{Q}},$$

where $\delta^\sharp$ denotes the image of δ via the natural map $X_*(A)_\mathbb{Q} \to \pi_1(M)_{\Gamma,\mathbb{Q}}$ by abuse of notation.

Moreover, for elements in the Kottwitz set $B(G, \mu)$, we can have a simpler characterization: $[b] \in B(G, \mu)$ if and only if $\kappa_M(b) \preceq_M \mu^\sharp$ in $\pi_1(M)_\Gamma$ where $\mu^\sharp$ denotes the image of μ in $\pi_1(M)_\Gamma$ by abuse of notation.

Proof. The proof is the same as that of [26, Proposition 4.10]. We repeat the proof for the generalized Kottwitz set here, while the proof for $B(G, \mu)$ is similar. The necessity is obvious. For the sufficiency, the inequality $\kappa_M(b) \preceq_M \delta^\sharp$ in $\pi_1(M)_{\Gamma,\mathbb{Q}}$ implies

$$[\nu_b] = \mathrm{Av}_\Gamma \mathrm{Av}_{W_M}(\widetilde{\kappa_M(b)}) \le \mathrm{Av}_\Gamma \mathrm{Av}_{W_M}(\delta) \le \delta,$$

where $\widetilde{\kappa_M(b)}$ denotes a preimage of $\kappa_M(b)$ via the natural map $X_*(T) \to \pi_1(M)_\Gamma$ and the second inequality is due to the fact that δ is dominant. ■

The following lemma will be used in the proof of the main theorem.

Lemma 1.8. *Suppose G is quasi-split and $\mu \in X_*(T)^+$ is minuscule. Let M be a standard Levi subgroup of G and $b \in M(\breve{F})$.*

(1) *Then the natural map $\pi_1(M)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}}$ is injective.*
(2) *Suppose $[b] \in B(G, \mu)$. Then there exists $\mu' \in X_*(T)$ such that $[b]_M \in B(M, \mu')$ and μ' is conjugate to μ in G.*
(3) *Suppose $[b] \in B(G, \mu^\sharp + \varepsilon, \mu^\diamond)$ with $\varepsilon \in \pi_1(G)_{\Gamma,\mathrm{tor}}$. Then*

$$\varepsilon \in \mathrm{Im}(\pi_1(M)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}})$$

and there exists $w \in W$ such that $[b] \in B(M, (w\mu)^{\sharp,M} + \varepsilon, (w\mu)^{\diamond,M})$ where we view ε as an element in $\pi_1(M)_{\Gamma,\mathrm{tor}}$.

Proof. For (1), the map $\pi_1(M)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}}$ can be identified with the map $H^1(F, M) \to H^1(F, G)$, which is injective (cf. [38, Exercise 1 in III §2.1]).

Part (2) of the lemma is proved in [32, Lemma 8.1 (2)] when G is unramified. For the general case, we want to reduce to the unramified case. Without loss of generality, we may assume that G is adjoint and simple by [25, §6.5]. Moreover, after replacing M by a smaller Levi subgroup, we may assume that $[b]_M \in B(M)_{\mathrm{basic}}$. Then by Lemma 1.7,

$$[b] \in B(G, \mu) \iff \kappa_M(b) \preceq_M \mu^\sharp \text{ in } \pi_1(M)_\Gamma.$$

We want to show that there exists $\mu' \in X_*(T)$ which is conjugate to μ with $\kappa_M(b) = (\mu')^\sharp$ in $\pi_1(M)_\Gamma$.

Let $Q := \mathrm{Ker}(\pi_1(M) \to \pi_1(G))$.

Claim. $Q_\Gamma = \mathrm{Ker}(\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma)$.

Let $A := \mathrm{Ker}(\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma)$, which is torsion free by [6, proof of Lemma 4.11]. As the functor $(-)_\Gamma$ is right exact, there exists a natural surjection $Q_\Gamma \to A$. We need to show it is injective. Therefore it suffices to show $\mathrm{rank}_{\mathbb{Q}} A_{\mathbb{Q}} = \mathrm{rank}_{\mathbb{Q}} Q_{\Gamma,\mathbb{Q}}$. Since the functor $(-)_{\Gamma,\mathbb{Q}}$ is canonically isomorphic to $(-)^{\Gamma}_{\mathbb{Q}}$, we have

$$A_{\mathbb{Q}} \simeq \mathrm{Ker}(\pi_1(M)^{\Gamma}_{\mathbb{Q}} \to \pi_1(G)^{\Gamma}_{\mathbb{Q}}) = Q^{\Gamma}_{\mathbb{Q}}.$$

The Claim follows.

By the Claim, $\mu - \kappa_M(b) \in Q_\Gamma$. We need to show there exists $\mu' \in X_*(T)$ which is conjugate to μ such that

$$\mu - \kappa_M(b) = \mu - \mu' \quad \text{in } Q_\Gamma.$$

This is a question only about a root system with Galois action. Indeed, by the classification of k-forms of G, we can construct an unramified group $\tilde{G}$ over F which is a form of G and they both have the same Tits indices. More precisely, we can find $\tilde{T} \subset \tilde{B} \subset \tilde{G}$ over F where $\tilde{T}$ is a maximal torus and $\tilde{B}$ is a Borel subgroup such that

- $X^*(T) \simeq X^*(\tilde{T})$ and via this identification $\Delta_G = \Delta_{\tilde{G}}$;
- Δ_G and $\Delta_{\tilde{G}}$ have the same Galois orbits.

Then the absolute Weyl groups of (G, T) and $(\tilde{G}, \tilde{T})$ are isomorphic. Let $\tilde{M}$ be the standard Levi subgroup of $\tilde{G}$ such that $\Delta_{\tilde{M}} = \Delta_M$. The isomorphism between the character groups induces an identification $\pi_1(\tilde{M})_\Gamma = \pi_1(M)_\Gamma$. Let $\tilde{\mu} \in X_*(\tilde{T})$ be the cocharacter corresponding to μ via the identification $X_*(T) \simeq X_*(\tilde{T})$. Let $[\tilde{b}] \in B(\tilde{M})_{\mathrm{basic}}$ be such that $\kappa_{\tilde{M}}(\tilde{b}) \in \pi_1(\tilde{M})_\Gamma$ maps to $\kappa_M(b) \in \pi_1(M)_\Gamma$ via the identification $\pi_1(\tilde{M})_\Gamma = \pi_1(M)_\Gamma$. As $\tilde{G}$ is unramified, we can find $\tilde{\mu}'$ for $[\tilde{b}] \in B(\tilde{G}, \tilde{\mu})$. Then μ' is the cocharacter of G corresponding to $\tilde{\mu}'$.

For (3), as before, we may assume $[b] \in B(M)_{\mathrm{basic}}$ after replacing M by a smaller group. Then $[\nu_b] \le \mu^\diamond$ implies that

$$\kappa_M(b) \preceq_M \mu^{\sharp,M} \quad \text{in } \pi_1(M)_{\Gamma,\mathbb{Q}}.$$

Hence there exists $\varepsilon' \in \pi_1(M)_{\Gamma,\mathrm{tor}}$ such that

$$\kappa_M(b) - \varepsilon' \preceq_M \mu^{\sharp,M} \quad \text{in } \pi_1(M)_\Gamma.$$

Let $[b'] \in B(M)_{\mathrm{basic}}$ be such that $\kappa_M(b') = \kappa_M(b) - \varepsilon'$. Then $[b'] \in B(G, \mu)$. By (2), there exists $w \in W$ such that $[b'] \in B(M, w\mu)$. Therefore $[\nu_b]_M = [\nu_{b'}]_M \le (w\mu)^{\diamond,M}$ and

$$\mu^\sharp = \kappa_G(b') = \kappa_G(b) - \varepsilon' = \mu^\sharp + \varepsilon - \varepsilon' \quad \text{in } \pi_1(G)_\Gamma,$$

where ε' is considered to be an element in $\pi_1(G)_{\Gamma,\mathrm{tor}}$ via the natural map in (1). Hence $\varepsilon = \varepsilon'$ and

$$\kappa_M(b) = \kappa_M(b') + \varepsilon = (w\mu)^\sharp + \varepsilon \quad \text{in } \pi_1(M)_\Gamma. \qquad \blacksquare$$

1.4. Classification of G-bundles in terms of φ-modules over $\bar{B}$

Let

$$B^{b,+} := W_{\mathcal{O}_F}(\mathcal{O}_K)[1/\pi_F], \quad \bar{B} := (B^{b,+}/[\omega_K])_{\mathrm{red}}.$$

Here $\bar{B}$ is a local F-algebra with residue field $W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}$.

The Frobenius on $\mathcal{O}_K$ induces an automorphism φ on $B^{b,+}$ and on $\bar{B}$.

Let $\varphi\text{-Mod}_{\bar{B}}$ (resp. $\varphi\text{-Mod}_{W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}}$) be the category of free $\bar{B}$-modules (resp. $W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}$-vector spaces) of finite rank equipped with a semilinear isomorphism.

Theorem 1.9 ([18, Theorems 11.1.7 and 11.1.9]). *There is an equivalence of additive tensor categories*

$$\mathrm{Bun}_X \xrightarrow{\sim} \varphi\text{-Mod}_{\bar{B}}.$$

For $(M, \varphi) \in \varphi\text{-Mod}_{\bar{B}}$, the Harder–Narasimham filtration of the corresponding vector bundle gives a $\mathbb{Q}$-filtration $(M^{\geq\lambda})_{\lambda\in\mathbb{Q}}$ of M which is called the Harder–Narasimham filtration of M (cf. [15, §5.4.1]).

For any $\beta \in G(\bar{B})$, we define

$$\mathcal{E}_\beta : \mathrm{Rep}\, G \to \varphi\text{-Mod}_{\bar{B}} \xrightarrow{\sim} \mathrm{Bun}_X, \quad (V, \rho) \mapsto (V \otimes_F \bar{B}, \rho(\beta)\varphi).$$

Proposition 1.10 ([15, Proposition 5.11]). *The functor $\beta \mapsto \mathcal{E}_\beta$ induces a bijection between the set of φ-conjugacy classes in $G(\bar{B})$ and the set of isomorphism classes of G-bundles on X.*

We also define a functor $\mathrm{red}_{\bar{B},\breve{F}}$ as composition of two functors:

$$\mathrm{red}_{\bar{B},\breve{F}} : \varphi\text{-Mod}_{\bar{B}} \xrightarrow{\otimes_{\bar{B}} W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}} \varphi\text{-Mod}_{W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}} \xrightarrow{\sim} \mathrm{Isoc}_{\breve{F}|F},$$

where the second functor is a quasi-inverse of the functor

$$(-) \otimes_{\breve{F}} W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}} : \mathrm{Isoc}_{\breve{F}|F} \xrightarrow{\sim} \varphi\text{-Mod}_{W_{\mathcal{O}_F}(k_K)_{\mathbb{Q}}}$$

which is an equivalence of categories due to Dieudonné–Manin's theorem of classification of isocrystals.

1.5. The automorphism group $\tilde{J}_b$

For $[b] \in B(G)$, let $\tilde{J}_b = \underline{\mathrm{Aut}}(\mathcal{E}_b)$ be the pro-étale sheaf of automorphisms of $\mathcal{E}_b$ on the category of affinoid perfectoid spaces $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ over $\overline{\mathbb{F}}_q$. More precisely, for any affinoid perfectoid space S over $\overline{\mathbb{F}}_q$, one has $\tilde{J}_b(S) = \mathrm{Aut}(\mathcal{E}_{b|X_S})$.

In this subsection, we review the structure of the group $\tilde{J}_b(K)$ studied in [15, §5.4.2]. Suppose $\mathcal{E}_b$ corresponds to the φ-conjugacy class of $\beta \in G(\bar{B})$ as in Proposition 1.10. Then

$$\tilde{J}_b(K) \simeq \{g \in G(\bar{B}) \mid g\beta = \beta\varphi(g)\}.$$

We will identify these two groups via this isomorphism. In order to study the structure of $\tilde{J}_b(K)$, we need to use a parabolic subgroup of $G \otimes \bar{B}$ that contains $\tilde{J}_b(K)$.

Consider the functor

$$\begin{aligned} \operatorname{Rep} G &\to \varphi\text{-}\mathrm{Mod}_{\bar{B}} \to \mathbb{Q}\text{-filtered } \bar{B}\text{-modules}, \\ (V, \rho) &\mapsto (V \otimes_F \bar{B}, \rho(\beta)\varphi), \end{aligned}$$

where the second functor is given by the Harder–Narasimham filtration. By [43, Theorem 4.40], this functor corresponds to a parabolic subgroup $\mathscr{P} \subset G \otimes_F \bar{B}$ satisfying $\tilde{J}_b(K) \subset \mathscr{P}(\bar{B})$. The structure of $\mathscr{P}$ is well understood. Let

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

be the adjoint representation with $\mathfrak{g} := \operatorname{Lie} G$. Then $(\mathfrak{g} \otimes_F \bar{B}, \mathrm{Ad}(\beta)\varphi)$ is the φ-module over $\bar{B}$ corresponding to the vector bundle $\mathrm{Ad}(\mathscr{E}_\beta) := \mathscr{E}_\beta \times^{G,\mathrm{Ad}} \mathfrak{g}$. Hence it has the Harder–Narasimham filtration $(\mathfrak{g}_{\bar{B}}^{\geq \lambda})_{\lambda \in \mathbb{Q}}$. In particular, for $\lambda \neq 0$, the dimension of $\mathrm{gr}^\lambda \mathfrak{g}_{\bar{B}}$ equals the number of roots $\alpha \in \Phi$ such that $\langle \alpha, \nu_b \rangle = \lambda$. Then

$$\mathscr{P} = \{g \in G_{\bar{B}} \mid \mathrm{Ad}(g)(\mathfrak{g}_{\bar{B}}^{\geq \bullet}) = \mathfrak{g}_{\bar{B}}^{\geq \bullet}\}, \quad \operatorname{Lie} \mathscr{P} = \mathfrak{g}_{\bar{B}}^{\geq 0}.$$

Moreover, the parabolic subgroup $\mathscr{P}$ is filtered by $(\mathscr{P}^{\geq \lambda})_{\lambda \in \mathbb{Q}_{\geq 0}}$ such that

$$\begin{aligned} &\mathscr{P}^{>0} = R_u \mathscr{P}, \\ &\forall \lambda > 0, \quad \mathscr{P}^{\geq \lambda} / \mathscr{P}^{> \lambda} \xrightarrow{\sim} \mathrm{gr}^\lambda \mathfrak{g}_{\bar{B}} \otimes \mathbb{G}_\mathrm{a}, \\ &\mathscr{P}^{\geq \lambda} = \{g \in G_{\bar{B}} \mid (\mathrm{Ad}(g) - \mathrm{Id})(\mathfrak{g}_{\bar{B}}^{\geq \bullet}) = \mathfrak{g}_{\bar{B}}^{\geq \bullet + \lambda}\}. \end{aligned}$$

Let $\tilde{J}_b^{\geq \lambda}(K) = \tilde{J}_b(K) \cap \mathscr{P}^{\geq \lambda}(\bar{B})$ for all $\lambda \in \mathbb{Q}_{\geq 0}$. Then we can understand the graded pieces:

$$\begin{aligned} &\tilde{J}_b(K) / \tilde{J}_b^{>0}(K) \simeq J_b = \{g \in G(\breve{F}) \mid b\sigma(g) = gb\}, \\ &\forall \lambda > 0, \quad \tilde{J}_b^{\geq \lambda}(K) / \tilde{J}_b^{> \lambda}(K) \simeq (\mathrm{gr}^\lambda \mathfrak{g}_{\bar{B}})^{\mathrm{Ad}(\beta)\varphi = \mathrm{Id}}, \end{aligned}$$

where $(\mathrm{gr}^\lambda \mathfrak{g}_{\bar{B}})^{\mathrm{Ad}(\beta)\varphi = \mathrm{Id}}$ is $(\dim \mathrm{gr}^\lambda \mathfrak{g}_{\bar{B}})/h$ copies of $H^0(X, \mathscr{O}_X(\lambda))$ if $\lambda = d/h$ with $(d, h) = 1$. In particular $\tilde{J}_b^{\geq \lambda}(K) \supsetneq \tilde{J}_b^{> \lambda}(K)$ if there exists $\alpha \in \Phi$ such that $\langle \alpha, \nu_b \rangle = \lambda > 0$.

1.6. Modifications of a G-bundle on X

Definition 1.11. Let $\mathscr{E}$ be a G-bundle on X. A *modification* of G-bundles of $\mathscr{E}$ (on ∞) is a pair $(\mathscr{E}', u)$, where $\mathscr{E}'$ is a G-bundle on X and

$$u : \mathscr{E}_{|X \setminus \{\infty\}} \xrightarrow{\sim} \mathscr{E}'_{|X \setminus \{\infty\}}$$

is an isomorphism of G-bundles on $X \setminus \{\infty\}$. Two modifications $(\mathscr{E}', u)$ and $(\tilde{\mathscr{E}}', \tilde{u})$ of $\mathscr{E}$ are said to be *equivalent* if there exists an isomorphism $f : \mathscr{E}' \xrightarrow{\sim} \tilde{\mathscr{E}}'$ such that $\tilde{u} = f_{|X \setminus \{\infty\}} \circ u$.

Consider the B_{dR}-affine Grassmannian $\mathrm{Gr}_G^{B_{\mathrm{dR}}}$ attached to G (cf. [36]). We only need its C-points,

$$\mathrm{Gr}_G^{\mathrm{dR}}(C) := G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+).$$

For any $b \in B(G)$, let $\mathcal{E}_b$ be the associated G-bundle on X. For any $x \in \mathrm{Gr}_G^{\mathrm{dR}}(C)$, we can construct a modification $\mathcal{E}_{b,x}$ of $\mathcal{E}_b$ à la Beauville–Laszlo given by gluing $\mathcal{E}_{b|X\backslash\{\infty\}}$ and the trivial bundle on $\mathrm{Spec}(B_{\mathrm{dR}}^+)$ via the gluing datum given by x (cf. [5, Theorem 3.4.5] and [13, §4.2], [14, Proposition 3.20]). Moreover, by [14, Proposition 3.20], there is a bijection

$$\begin{aligned} \mathrm{Gr}_G^{\mathrm{dR}}(C) &\xrightarrow{\sim} \{\text{equivalence classes of modifications of } \mathcal{E}_b\}, \\ x &\mapsto \text{equivalence class of } (\mathcal{E}_{b,x}, \mathrm{Id}). \end{aligned} \tag{1.6.1}$$

For $\mu \in X_*(T)^+$, the corresponding affine Schubert cell is

$$\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+) \subset \mathrm{Gr}_G^{B_{\mathrm{dR}}}(C).$$

Here we use the non-standard notion of affine Schubert cell associated to anti-dominant μ^{-1}. This affine Schubert cell is closely related to the modification of G-bundles of type μ as in the following definition.

Definition 1.12. A modification of $\mathcal{E}_b$ is of *type* μ if its equivalence class falls in the affine Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}(C)$ via (1.6.1).

The natural action of $\tilde{J}_b(K) = \mathrm{Aut}(\mathcal{E}_b)$ on the set of modifications of $\mathcal{E}_b$ induces via (1.6.1) an action of $\tilde{J}_b(K)$ on $\mathrm{Gr}_G^{B_{\mathrm{dR}}}(C)$.

Let $\widehat{\mathcal{E}}_{b\infty}$ be the local completion of $\mathcal{E}_b$ at ∞. It is canonically trivialized. Hence there is a natural morphism

$$\alpha_{b,G} : \tilde{J}_b(K) = \mathrm{Aut}(\mathcal{E}_b) \to \mathrm{Aut}(\widehat{\mathcal{E}}_{b\infty}) = G(B_{\mathrm{dR}}^+).$$

The action of $\tilde{J}_b(K)$ on $\mathrm{Gr}_G^{B_{\mathrm{dR}}}(C)$ is given by left multiplication via $\alpha_{b,G}$.

Lemma 1.13. *Let $\gamma \in \tilde{J}_b(K)$. For any $x \in \mathrm{Gr}_G^{B_{\mathrm{dR}}}(C)$, the automorphism $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ induces an automorphism*

$$\tilde{\gamma} : \mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b,\gamma(x)}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E}_{b|X\backslash\{\infty\}} & \xrightarrow{\gamma} & \mathcal{E}_{b|X\backslash\{\infty\}} \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathcal{E}_{b,x|X\backslash\{\infty\}} & \xrightarrow{\tilde{\gamma}} & \mathcal{E}_{b,\gamma(x)|X\backslash\{\infty\}} \end{array}$$

Proof. By Tannakian formalism, it suffices to deal with the case when $G = \mathrm{GL}_n$. Then G-bundles on X are the same thing as vector bundles of rank n. Suppose $\mathcal{E}_b$ corresponds to the triple $(M_e, M_{\mathrm{dR}}, u)$ as in Proposition 1.2. Then $x \in \mathrm{Gr}_G^{B_{\mathrm{dR}}}(C)$ corresponds to a

B_{dR}^+-lattice M_x in $M_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$ and $\mathcal{E}_{b,x}$ corresponds to the triple (M_e, M_x, u). The automorphism $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ corresponds to a pair $(\gamma_e, \gamma_{\mathrm{dR}})$ of automorphisms compatible with u, where

$$\gamma_e : M_e \xrightarrow{\sim} M_e, \quad \gamma_{\mathrm{dR}} : M_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}}.$$

Then $\mathcal{E}_{b,\gamma(x)}$ corresponds to the triple $(M_e, \gamma_{\mathrm{dR}}(M_x), u)$ where $\gamma_{\mathrm{dR}}(M_x)$ is the image of M_x via $\gamma_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} : M_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} B_{\mathrm{dR}}$. We define $\tilde{\gamma} : \mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b,\gamma(x)}$ to be the automorphism corresponding to

$$\gamma_e : M_e \xrightarrow{\sim} M_e, \quad \gamma_{\mathrm{dR}} : M_x \xrightarrow{\sim} \gamma_{\mathrm{dR}}(M_x).$$

The commutativity of the diagram can be verified directly. ■

Recall that we have the *Białynicki-Birula map* (cf. [5, Proposition 3.4.3])

$$\pi_{G,\mu} : \mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}(C) \to \mathcal{F}(G,\mu)(C).$$

By Tannakian formalism, we may reduce the construction to the case when $G = \mathrm{GL}_n$. In this case, $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}(C)$ parametrizes the lattices in $\mathrm{B}_{\mathrm{dR}}^n$ that has relative position μ^{-1} with the standard lattice $B_{\mathrm{dR}}^{+,n}$. Write $\mu = (k_1, \dots, k_n)$ with $k_1 \geq \cdots \geq k_n$. Suppose $\Lambda \in \mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}(C)$. We define an increasing filtration $\mathrm{Fil}_\Lambda^\bullet$ of C^n as follows: for any $m \in \mathbb{Z}$,

$$\mathrm{Fil}_\Lambda^m C^n := (B_{\mathrm{dR}}^+)^n \cap t^{-m}\Lambda/((tB_{\mathrm{dR}}^+)^n \cap t^{-m}\Lambda) \subseteq (B_{\mathrm{dR}}^+)^n/(tB_{\mathrm{dR}}^+)^n = C^n.$$

It is easy to check that

$$\dim_C \mathrm{Fil}_\Lambda^m C^n = \max\{1 \leq i \leq n \mid k_i \geq -m\}.$$

Therefore $\pi_{G,\mu}(\Lambda) := \mathrm{Fil}_\Lambda^\bullet \in \mathcal{F}(G,\mu)(C)$.

From now on, suppose μ is minuscule. Then the Białynicki-Birula map $\pi_{G,\mu}$ is an isomorphism by [5, Lemma 3.4.4]. For $x \in \mathcal{F}(G,\mu)(C)$, we denote by $\mathcal{E}_{b,x}$ the modification $\mathcal{E}_{b,\pi_{G,\mu}^{-1}(x)}$ of $\mathcal{E}_b$ of type μ.

When $[b] \in B(G)$ is basic, the isomorphism classes of the modifications of $\mathcal{E}_b$ can be classified as follows.

Proposition 1.14 ([30, Corollary A.10], [6, Proposition 5.2]). *Let $[b] \in B(G)$ be basic. Let*

$$B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1}) := B(G, \kappa_G(b) - \mu^\sharp, \nu_b(w_0\mu^{-1})^\diamond) \subseteq B(G).$$

(Here we write the element $\nu_b(w_0\mu^{-1})^\diamond$ in $X_*(A)_{\mathbb{Q}}^+$ in multiplicative form and not in the usual additive form.) The map $[b'] \mapsto [\mathcal{E}_{b'}]$ gives a bijection

$$B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1}) \simeq \{\mathcal{E}_{b,x} \mid x \in \mathcal{F}(G,\mu)(C)\}/\sim.$$

The action of $\tilde{J}_b(K)$ on $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}(C)$ defined by multiplication on the left via the morphism $\alpha_{b,G}$ induces an action of $\tilde{J}_b(K)$ on $\mathcal{F}(G,\mu)(C)$ via the Białynicki-Birula map $\pi_{G,\mu}$. For any $\gamma \in \tilde{J}_b(K)$, we have an automorphism (still denoted by)

$$\tilde{\gamma} : \mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b,\gamma(x)}$$

of G-bundles for any $x \in \mathcal{F}(G,\mu)(C)$.

2. Admissible locus and weakly admissible locus

2.1. Reductions of G-bundles

Definition 2.1. (1) Let $H \subseteq G$ be a closed subgroup of G. Suppose $\mathcal{E}$ is a G-bundle on X. A *reduction* of $\mathcal{E}$ to H is a pair $(\mathcal{E}_H, \iota)$ where $\mathcal{E}_H$ is an H-bundle and $\iota : \mathcal{E}_H \times^H G \xrightarrow{\sim} \mathcal{E}$ is an isomorphism of G-bundles. We will also write $\mathcal{E}_H$ for such a reduction if we do not need to emphasize ι.

(2) Two reductions $(\mathcal{E}_H, \iota)$ and $(\mathcal{E}'_H, \iota')$ of $\mathcal{E}$ to H are called *equivalent* if there exists an isomorphism $u : \mathcal{E}_H \xrightarrow{\sim} \mathcal{E}'_H$ such that $\iota = \iota' \circ (u \times^H G)$.

Remark 2.2. The equivalence classes of reductions of $\mathcal{E}$ to H are in bijection with the sections of the fibration $H\backslash\mathcal{E} \to X$.

We will assume G is quasi-split in the rest of this subsection.

Definition 2.3. Let $b \in G(\breve{F})$. For a Levi subgroup M of G, a *reduction* of b to M is a pair (b_M, g) with $b_M \in M(\breve{F})$ and $g \in G(\breve{F})$ such that $b = g b_M \sigma(g)^{-1}$. We also write b_M for such a reduction if we do not need to emphasize g. Two reductions (b_M, g) and (b'_M, g') of b to M are *equivalent* if there exists $h \in M(\breve{F})$ such that $(b'_M, g') = (h b_M \sigma(h)^{-1}, g h^{-1})$. Similarly, we can define the same notion for parabolic subgroups.

There is a natural injective map

$$\{\text{equivalence classes of reductions of } b \text{ to } M\} \to \{\text{equivalence classes of reductions of } \mathcal{E}_b \text{ to } M\}.$$

This map is in general not surjective.

Example 2.4. Let $G = \mathrm{GL}_5$ with Levi subgroup $M = \mathrm{GL}_3 \times \mathrm{GL}_2$. Let $b \in G(\breve{F})$ with Newton slopes $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$. Then there exists a unique equivalence class of reductions of b to M. However, as the decomposition of $\mathcal{E}_b = \mathcal{O}(-\frac{1}{3}) \oplus \mathcal{O}(-\frac{1}{2})$ into semistable vector bundles is not canonical because of the existence of morphisms $\mathcal{O}(-\frac{1}{2}) \to \mathcal{O}(-\frac{1}{3})$, there exist infinitely many equivalence classes of reductions of $\mathcal{E}_b$ to M.

The following lemma will be used frequently.

Lemma 2.5 ([6, Lemma 2.4]). *Let $\mathcal{E}$ and $\mathcal{E}'$ be two G-bundles on X with a modification $\mathcal{E}_{|X\backslash\{\infty\}} \xrightarrow{\sim} \mathcal{E}'_{|X\backslash\{\infty\}}$. Then for any parabolic subgroup P of G, we have a bijection*

$$\{\textit{reductions of } \mathcal{E} \textit{ to } P\} \to \{\textit{reductions of } \mathcal{E}' \textit{ to } P\}.$$

Let $\mathcal{E}$ be a G-bundle on X. By [15, §5.1], there exists the canonical reduction $\mathcal{E}_P$ of $\mathcal{E}$ to a unique standard parabolic subgroup P of G such that

- the associated M-bundle $\mathcal{E}_P \times^P M$ is semistable, where M is the Levi component of P,
- for any $\chi \in X^*(P/Z_G) \setminus \{0\} \cap \mathbb{N}\Delta_G$, we have $\deg \chi_*\mathcal{E}_P > 0$.

Using the Harder–Narasimham reduction $\mathcal{E}_P$, we can define the slope

$$\nu_{\mathcal{E}} \in X_*(A)_{\mathbb{Q}}$$

of $\mathcal{E}$ by the Galois invariant morphism $X^*(P) \to \mathbb{Z}$ which maps $\chi \in X^*(P)$ to $\deg \chi_* \mathcal{E}_P$ combined with the inclusion

$$\mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{Z})^{\Gamma} = X_*(M^{\mathrm{ab}})^{\Gamma} \subset X_*(A_M)_{\mathbb{Q}} \subseteq X_*(A)_{\mathbb{Q}},$$

where M^{ab} is the cocenter of M and $A_M \subseteq A$ is a maximal split central torus of M.

Proposition 2.6 ([18, Proposition 6.6], [5, Lemma 3.5.5]). *Let $[b] \in B(G)$ and let $x \in \mathrm{Gr}^{B_{\mathrm{dR}}}_{G,\mu}(C)$. Then*

(1) *in the positive Weyl chamber we have*

$$\nu_{\mathcal{E}_b} = -w_0[\nu_b],$$

where w_0 is the element of maximal length in the Weyl group W;

(2) $c_1^G(\mathcal{E}_{b,x}) = \mu^{\sharp} - \kappa_G(b) \in \pi_1(G)_{\Gamma}$, *where c_1^G denotes the G-equivariant first Chern class. In particular, $c_1^G(\mathcal{E}_b) = -\kappa_G(b)$.*

Recall the following fact:

Theorem 2.7 ([34, Theorem 4.5.1]). *Let $\mathcal{E}$ be a G-bundle on X.*

(1) *Suppose $\mathcal{E}_Q$ is a reduction of $\mathcal{E}$ to the standard parabolic subgroup Q. Consider the vector*

$$v : X^*(Q) \to \mathbb{Z}, \quad \chi \mapsto \deg\, \chi_* \mathcal{E}_Q,$$

seen as an element of $X_*(A)_{\mathbb{Q}}$. Then $v \le \nu_{\mathcal{E}}$. Moreover, if this inequality is an equality, then $Q \subset P$ and $\mathcal{E}_P \simeq \mathcal{E}_Q \times^Q P$, where $\mathcal{E}_P$ is the canonical reduction of $\mathcal{E}$.

(2) *The vector $\nu_{\mathcal{E}}$ can be defined as the supremum of all such vectors v associated to all possible reductions $\mathcal{E}_Q$ in the poset $X_*(A)_{\mathbb{Q}}$.*

Remark 2.8. If we view v as an element in $X_*(A)_{\mathbb{Q}}$, then $v = \mathrm{Av}_{W_{M_Q}}(\nu_{\mathcal{E}_{M_Q}})$, where M_Q is the Levi component of Q and $\mathcal{E}_{M_Q} = \mathcal{E}_Q \times^Q M$.

Corollary 2.9. *Suppose $\mathcal{E}$ is a G-bundle with $\mathcal{E}_P$ a reduction to P. Let $\mathcal{E}' = (\mathcal{E}_P \times^P M) \times^M G$. Then $\nu_{\mathcal{E}} \preceq \nu_{\mathcal{E}'}$. In particular, if $\mathcal{E}_P \times^P M$ is a trivial M-bundle, then $\mathcal{E}$ is a trivial G-bundle.*

Remark 2.10. When $G = \mathrm{GL}_n$, this corollary is shown in [22, Lemma 3.4.17].

Proof of Corollary 2.9. Suppose $\mathcal{E}_Q$ is a reduction of $\mathcal{E}$ to a standard parabolic subgroup Q. Suppose

$$s_P : X \to P\backslash\mathcal{E}, \quad s_Q : X \to Q\backslash\mathcal{E}$$

are the corresponding sections for $\mathcal{E}_P$ and $\mathcal{E}_Q$ respectively. Then the relative position map

$$Q\backslash G \times P\backslash G \to Q\backslash G/P = W_Q\backslash W/W_P, \quad (Qg_1, Pg_2) \mapsto Qg_1g_2^{-1}P,$$

gives

$$X \xrightarrow{(s_Q, s_P)} Q\backslash\mathcal{E} \times P\backslash\mathcal{E} \to W_Q\backslash W/W_P.$$

Let $W_Q \dot{w} W_P$ be the image of the generic point of X, where $\dot{w}$ is the minimal length representative of the double coset $W_Q \dot{w} W_P$. Let $X' \subseteq X$ be the preimage of $W_Q \dot{w} W_P$. It is an open subscheme of X.

By Theorem 2.7 (2), the corollary follows from the following Claim.

Claim. *$\mathcal{E}_Q$ induces a reduction $\mathcal{E}'_Q$ of $\mathcal{E}'$ to the parabolic subgroup Q such that $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ have the same vector v as defined in Theorem 2.7.*

Now it remains to prove the Claim. The composition

$$\mathcal{E}_P \to \mathcal{E} \xrightarrow{\times \dot{w}} \mathcal{E}$$

induces a monomorphism

$$P \cap \dot{w}^{-1} Q \dot{w} \backslash \mathcal{E}_P \to Q\backslash\mathcal{E}.$$

The pullback by the section $s_Q : X \to Q\backslash\mathcal{E}$ of this morphism gives a section of

$$P \cap \dot{w}^{-1} Q \dot{w} \backslash (\mathcal{E}_P)_{|X'} \to X'.$$

Combining it with the natural morphism

$$P \cap \dot{w}^{-1} Q \dot{w} \backslash \mathcal{E}_P \to P'_M \backslash \mathcal{E}'_M$$

induced by the projection to the Levi quotient, where $\mathcal{E}'_M = \mathcal{E}_P \times^P M$ and $P'_M = M \cap \dot{w}^{-1} Q \dot{w}$, we get a reduction $(\mathcal{E}'_{M,P'_M})_{|X'}$ of $(\mathcal{E}'_M)_{|X'}$ to its standard parabolic subgroup P'_M. The composition

$$\mathcal{E}'_M \to \mathcal{E}' \xrightarrow{\times \dot{w}} \mathcal{E}'$$

induces a morphism

$$(P'_M \backslash \mathcal{E}'_M)_{|X'} \to (Q\backslash\mathcal{E}')_{|X'},$$

and we get a reduction $\mathcal{E}'_{|X',Q}$ of $\mathcal{E}'_{|X'}$ to Q induced from $\mathcal{E}'_{M,P'_M}$ via this morphism. Then the desired reduction $\mathcal{E}'_Q$ of the Claim is obtained by applying the valuative criterion of properness to $Q\backslash\mathcal{E}' \to X$, as $Q\backslash G$ is proper and X is a Dedekind scheme.

Now it remains to show that $\mathcal{E}_Q$ and $\mathcal{E}'_Q$ have the same vector v as defined in Theorem 2.7. By the construction of $\mathcal{E}'_Q$, this results from the commutativity of the following diagram for any $\chi \in X^*(Q)$:

$$\begin{array}{ccccc} P \cap \dot{w}^{-1} Q \dot{w} & \xrightarrow{\operatorname{ad} \dot{w}} & Q & \xrightarrow{\chi} & \mathbb{G}_m \\ \downarrow{\scriptstyle \mathrm{pr}_M} & & & & \| \\ P'_M & \xrightarrow{\operatorname{ad} \dot{w}} & Q & \xrightarrow{\chi} & \mathbb{G}_m \end{array} \tag{2.1.1}$$

where pr_M is restriction to $P \cap w^{-1} Q w$ of the projection of P to its Levi component M.

■

Remark 2.11. In the notations of the above corollary, when $G = \mathrm{GL}_n$ and $M = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$ with $n_1 + n_2 = n$, then $\mathcal{E}$ corresponds to a vector bundle of rank n over X, and $\mathcal{E} \times^P M$ corresponds to a pair of a vector bundle of rank n_1 and a vector bundle of rank n_2 over X which admits an extension by $\mathcal{E}$. Therefore the above corollary gives a necessary condition for a vector bundle over the Fargues–Fontaine curve X to be an extension of two given vector bundles over X.

The converse of the corollary is the following conjecture.

Conjecture 2.12. *Let P be a standard parabolic subgroup of G with Levi component M. Let $\mathcal{E}$ be a G-bundle and $\mathcal{E}'_M$ be a semistable M-bundle with $\nu_{\mathcal{E}} \preceq \nu_{\mathcal{E}'}$ where $\mathcal{E}' = \mathcal{E}'_M \times^M G$. Suppose $\nu_{\mathcal{E}'_M}$ is G-anti-dominant. Then $\mathcal{E}$ has a reduction $\mathcal{E}_P$ to P such that $\mathcal{E}_P \times^P M \simeq \mathcal{E}'_M$.*

Remark 2.13. When $G = \mathrm{GL}_n$, the conjecture is proved in [1]. Recently, this conjecture has been proved by Viehmann [41]. Note that in the conjecture the G-anti-dominant assumption for $\nu_{\mathcal{E}'_M}$ is necessary since $H^1(X, \mathcal{O}(\lambda)) = 0$ for $\lambda \geq 0$. The assumption that $\mathcal{E}'_M$ is semistable (i.e. corresponds to a basic element in $B(M)$) is also necessary, as explained in [8].

Let b_M be a reduction of b to M, where M is a standard Levi subgroup of G. Let P be the standard parabolic subgroup with Levi component M. Recall that for any $w \in W$, there is an affine fibration

$$\mathrm{pr}_w : \mathcal{F}(G,\mu)(C)^w := P(C)wP_\mu(C)/P_\mu(C) \to \mathcal{F}(M, w\mu)(C)$$

by projection to the Levi quotient. We have the following fact.

Lemma 2.14 ([6, Lemma 2.6]). *For $x \in P(C)wP_\mu(C)/P_\mu(C)$ there is an isomorphism*

$$(\mathcal{E}_{b,x})_P \times^P M \simeq \mathcal{E}_{b_M,\mathrm{pr}_w(x)},$$

where b_M is a reduction of b to M and $(\mathcal{E}_{b,x})_P$ is the reduction of $\mathcal{E}_{b,x}$ to P induced by the reduction $\mathcal{E}_{b_M} \times^M P$ of $\mathcal{E}_b$ to P.

2.2. *Weakly admissible locus*

Recall that $\{\mu\}$ is a geometric conjugacy class of a minuscule cocharacter $\mu : \mathbb{G}_{\mathrm{m}} \to G_{\bar{F}}$. After choosing a suitable representative in $\{\mu\}$, we may assume $\mu \in X_*(T)^+$ via inner twisting, where $^+$ stands for the dominant cocharacters. We consider the adic space $\mathcal{F}(G,\mu)$ associated to the flag variety over $\mathrm{Spa}(\breve{E})$. For $b \in G(\breve{F})$, Rapoport and Zink have defined a weakly admissible locus

$$\mathcal{F}(G,\mu,b)^{\mathrm{wa}} \subseteq \mathcal{F}(G,\mu)$$

associated to (G,μ,b). Now we recall its definition.

Let $L|\breve{F}$ be a complete field extension. To any $x \in \mathcal{F}(G,\mu)(L)$, we can associate a cocharacter $\mu_x \in \{\mu\}$ defined over L. Let φ-$\mathrm{FilMod}_{L|\breve{F}}$ be the category of filtered isocrystals over $L|\breve{F}$. There is a functor

$$\mathcal{I}_{b,x} : \mathrm{Rep}\, G \to \varphi\text{-}\mathrm{FilMod}_{L|\breve{F}}, \quad (V,\rho) \mapsto (V_{\breve{F}}, \rho(b)\sigma, \mathrm{Fil}^{\bullet}_{\rho\circ\mu_x} V_L).$$

The pair (b,x) is called *weakly admissible* if for any $(V,\rho) \in \mathrm{Rep}\, G$, the filtered isocrystal $\mathcal{I}_{b,x}(V,\rho)$ is weakly admissible in the sense of Fontaine. More precisely, a filtered isocrystal $\mathcal{V} = (V, \varphi, \mathrm{Fil}^{\bullet} V_L)$ over $L|\breve{F}$ is called *weakly admissible* if for any subobject $\mathcal{V}'$ of $\mathcal{V}$ with filtration induced from $\mathcal{V}$, we have

$$t_H(\mathcal{V}) = t_N(\mathcal{V}) \quad \text{and} \quad t_H(\mathcal{V}') \leq t_N(\mathcal{V}'),$$

where $t_N(\mathcal{V})$ is the π_F-adic valuation of $\det \varphi$ and

$$t_H(\mathcal{V}) := \sum_{i\in\mathbb{Z}} i \cdot \dim_L \mathrm{gr}^i_{\mathrm{Fil}^\bullet}(V_L).$$

Let

$$\mathcal{F}(G,\mu,b)^{\mathrm{wa}}(L) := \{x \in \mathcal{F}(G,\mu)(L) \mid (b,x) \text{ is weakly admissible}\}.$$

This defines the weakly admissible locus $\mathcal{F}(G,\mu,b)^{\mathrm{wa}}$, which is a partially proper open subspace inside $\mathcal{F}(G,\mu)$ by [33, Proposition 1.36].

Remark 2.15. (1) Let $b, b' \in G(\breve{F})$ with $[b] = [b'] \in B(G)$, then $\mathcal{F}(G,\mu,b)^{\mathrm{wa}} \simeq \mathcal{F}(G,\mu,b)^{\mathrm{wa}}$.

(2) By [32, Proposition 3.1], $\mathcal{F}(G,\mu,b)^{\mathrm{wa}}$ is non-empty if and only if $[b] \in A(G,\mu)$.

(3) Suppose the Frobenius maps on $H(\breve{F})$ maps to $g^{-1}\sigma(-)g$ via the inner twisting $H_{\breve{F}} \xrightarrow{\sim} G_{\breve{F}}$ with $g \in G(\breve{F})$. We have a bijection $B(G) \xrightarrow{\sim} B(H)$ which maps $[b]$ to $[b^H]$, where b^H maps to $bg \in G(\breve{F})$ via the inner twisting. By [10, Proposition 9.5.3], there is an identification

$$\mathcal{F}(G,\mu,b)^{\mathrm{wa}} = \mathcal{F}(H,\mu,b^H)^{\mathrm{wa}}.$$

Therefore for the study of weakly admissible locus, it suffices to reduce to the quasi-split case.

In the following proposition, we will use the modification of G-bundles on the curve X to give an equivalent definition of weak admissibility of a pair (b,x) when G is quasi-split.

Proposition 2.16 ([6, Proposition 2.7]). *Assume that G is quasi-split. Let $[b] \in A(G,\mu)$ and $x \in \mathcal{F}(G,\mu)(L)$. Then the pair (b,x) is weakly admissible if and only if for any standard parabolic subgroup P with Levi component M, any reduction b_M of b to M, and any $\chi \in X^*(P/Z_G)^+$ where Z_G is the center of G, we have*

$$\deg \chi_*(\mathcal{E}_{b,x})_P \leq 0,$$

where $(\mathcal{E}_{b,x})_P$ is the reduction of $\mathcal{E}_{b,x}$ to P induced by the reduction $\mathcal{E}_{b_M} \times^M P$ of $\mathcal{E}_b$ by Lemma 2.5.

2.3. Admissible locus

Rapoport and Zink [33] have conjectured the existence of an open subspace

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}} \subseteq \mathcal{F}(G,\mu,b)^{\mathrm{wa}}$$

with an étale-G-local system $\mathcal{L}$ on $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ such that these two spaces have the same classical points and the G-local system $\mathcal{L}$ interpolates a family of crystalline representations with value in $G(F)$.

When the local Shimura datum (G,μ,b) corresponds to a Rapoport–Zink space $\mathcal{M}(G,\mu,b)$ over $\breve{E}$, the admissible locus $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ is the image of the p-adic period mapping [33, Chapter 5]

$$\breve{\pi} : \mathcal{M}(G,\mu,b) \to \mathcal{F}(G,\mu),$$

and the G-local system $\mathcal{L}$ corresponds to the Tate module of the universal p-divisible group with G-structures by descent via the p-adic period mapping.

For the general local Shimura datum (G,μ,b), the existence of the admissible locus is due to the work of Fargues–Fontaine [18], Kedlaya–Liu [23] and Scholze [35].

Definition 2.17. Let $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ be a subspace of $\mathcal{F}(G,\mu)$ stable under generalization with C-points, defined as follows:

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}}(C) = \{x \in \mathcal{F}(G,\mu)(C) \mid \nu_{\mathcal{E}_{b,x}} \text{ is trivial}\}$$

for any complete algebraically closed field C over F.

Remark 2.18. (1) $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ is an open subset of $\mathcal{F}(G,\mu)$ (see [23]), and by definition

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}} \subset \mathcal{F}(G,\mu,b)^{\mathrm{wa}}.$$

Moreover,

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}}(K) = \mathcal{F}(G,\mu,b)^{\mathrm{wa}}(K)$$

for any finite extension K over $\breve{E}$ [30, Remarks A.5], [9]. In particular, $\mathcal{F}(G,\mu,b)^{\mathrm{a}} \neq \emptyset$ if and only if $[b] \in A(G,\mu)$.

(2) For $[b] \in A(G,\mu)$, the admissible locus $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ coincides with the image of the p-adic period mapping from the local Shimura variety attached to (G,μ,b) to the flag variety $\mathcal{F}(G,\mu)$ [36, 37]. It also coincides with the construction of admissible locus of Hartl [21] and Faltings [12] when (G,μ,b) is a Hodge type local Shimura datum.

(3) Via the bijection $B(G) \xrightarrow{\sim} B(H)$ by inner twisting which maps $[b]$ to $[b^H]$, there is an identification

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}} = \mathcal{F}(H,\mu,b^H)^{\mathrm{a}}.$$

Therefore for the study of admissible locus, we can also reduce to the quasi-split case.

For any $[b] \in B(G)$, consider the Newton stratification

$$\mathscr{F}(G,\mu) = \coprod_{[b'] \in B(G)} \mathscr{F}(G,\mu,b)^{[b']},$$

where $\mathscr{F}(G,\mu,b)^{[b']}$ is a subspace of $\mathscr{F}(G,\mu)$ stable under generalization with C-points defined by

$$\mathscr{F}^{[b']}(C) = \{x \in \mathscr{F}(C) \mid \mathscr{E}_{b,x} \simeq \mathscr{E}_{b'}\}$$

for any complete algebraically closed field C over F. Note that when $[b]$ is basic, we know precisely which strata show up in the Newton stratification by Proposition 1.14. But this is unknown for non-basic $[b]$. Each stratum in the Newton stratification is locally closed by Kedlaya–Liu [23]. And it is clear that

$$\mathscr{F}(G,\mu,b)^{[1]} = \mathscr{F}(G,\mu,b)^{\mathrm{a}}.$$

Remark 2.19. The $\tilde{J}_b(K)$-action on $\mathscr{F}(G,\mu)(C)$ induces an action on each stratum $\mathscr{F}(G,\mu,b)^{[b']}(C)$. In particular, $\tilde{J}_b(K)$ acts on $\mathscr{F}(G,\mu,b)^{\mathrm{a}}(C)$.

3. Hodge–Newton-decomposability

Let $M_b = \mathrm{Cent}_H([\nu_b])$ be the centralizer of $[\nu_b]$.

Definition 3.1. (1) A triple (G,μ,b) (resp. (G,δ,b) with $\delta \in X_*(A)^+_{\mathbb{Q}}$) is called *Hodge–Newton-decomposable* (or HN-decomposable for short) if $[b] \in A(G,\mu)$ (resp. $[b] \in B(G,\varepsilon,\delta)$ with $\varepsilon = \kappa_G(b) \in \pi_1(G)_\Gamma$) and there exists a strict standard Levi subgroup M of the quasi-split inner form H of G containing M_b, such that $\mu^\diamond - [\nu_b] \in \langle \Phi^\vee_{0,M} \rangle_{\mathbb{Q}}$ (resp. $\delta - [\nu_b] \in \langle \Phi^\vee_{0,M} \rangle_{\mathbb{Q}}$). Otherwise, the triple (G,μ,b) (resp. (G,δ,b)) is called *Hodge–Newton-indecomposable* (or HN-indecomposable for short).

(2) A pair (G,μ) is called *fully Hodge–Newton-decomposable* (or fully HN-decomposable for short) if for any non-basic $[b] \in B(G,\mu)$, the triple (G,μ,b) is HN-decomposable. In this case, we also say the Kottwitz set $B(G,\mu)$ is fully Hodge–Newton-decomposable.

(3) The generalized Kottwitz set $B(G,\varepsilon,\delta)$ is called *fully HN-decomposable* if for any non-basic $[b] \in B(G,\varepsilon,\delta)$, the triple (G,δ,b) is HN-decomposable.

Remark 3.2. The notion of full Hodge–Newton-decomposability has first been introduced and systematically studied by Görtz, He and Nie [19]. They gave equivalent conditions for a pair (G,μ) to be fully HN-decomposable and classified all such pairs.

In the quasi-split case, we have the following equivalent definition for HN-decomposability.

Lemma 3.3 (cf. [6, Lemma 4.11]). *Suppose G is quasi-split. Let $[b] \in B(G,\mu^\sharp + \varepsilon,\mu^\diamond)$ for some $\varepsilon \in \pi_1(G)_{\Gamma,\mathrm{tor}}$. Then the following three conditions are equivalent:*

(1) *the triple (G, μ, b) is HN-decomposable,*

(2) *there exist a strict standard Levi subgroup M containing M_b and a unique element $\varepsilon_M \in \pi_1(M)_{\Gamma,\mathrm{tor}}$ such that $[b_M] \in B(M, \mu^\sharp + \varepsilon_M, \mu^\diamond)$ and ε_M maps to ε via the natural map $\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma$, where b_M is the reduction of b to M deduced from its canonical reduction to M_b combined with the inclusion $M_b \subseteq M$,*

(3) *there exist a strict standard Levi subgroup M containing $w_0 M_b w_0^{-1}$ and a unique element $\tilde{\varepsilon}_M \in \pi_1(M)_{\Gamma,\mathrm{tor}}$ such that $[\tilde{b}_M] \in B(M, (\tilde{w}_0\mu)^\sharp + \tilde{\varepsilon}_M, (\tilde{w}_0\mu)^\diamond)$ and $w_0\tilde{\varepsilon}_M$ maps to ε via the natural map $\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma$, where $\tilde{b}_M$ is the reduction of b to M deduced from its canonical reduction $w_0 b_{M_b} w_0^{-1}$ to $w_0 M_b w_0^{-1}$ combined with the inclusion $w_0 M_b w_0^{-1} \subseteq M$ and $\tilde{w}_0\mu = (w_0\mu)_{M\text{-dom}}$ is the M-dominant representative in $W_M w_0 \mu$.*

Proof. The proof of the equivalence of (1) and (2) is similar to that of [6, Lemma 4.11]. Indeed, by definition, (G, μ, b) is HN-decomposable if and only if $[b_M] \in A(M, \mu)$ for some strict standard Levi subgroup of G. By Remark 1.5, we may assume $[b_M] \in B(M, \mu^\sharp + \varepsilon_M, \mu^\diamond)$ for some $\varepsilon_M \in \pi_1(M)_{\Gamma,\mathrm{tor}}$ which maps to $\varepsilon \in \pi_1(G)_\Gamma$. The uniqueness of ε_M follows from Lemma 1.8 (1).

The equivalence between (2) and (3) is due to the fact that there is a bijection between $B(M, \mu^\sharp + \varepsilon_M, \mu^\diamond)$ and $B(w_0 M w_0^{-1}, (\tilde{w}_0\mu)^\sharp + w_0\varepsilon_M, (\tilde{w}_0\mu)^\diamond)$ induced by the conjugation by w_0. ∎

Corollary 3.4. *Suppose G is quasi-split. Let $[b] \in A(G, \mu)$. Then the following three conditions are equivalent:*

(1) *the triple (G, μ, b) is HN-indecomposable,*

(2) *$\mu^\sharp - \kappa_{M_b}(b_{M_b}) \in \pi_1(M_b)_\Gamma \otimes \mathbb{Q}$ is a linear combination of elements of*

$$\{\alpha^{\vee,\sharp} \in \pi_1(M_b)_\Gamma \otimes \mathbb{Q} \mid \alpha \in \Delta_0,\ \langle \alpha, [\nu_b] \rangle > 0\}$$

with positive integer coefficients,

(3) *for all $\alpha \in \Delta_0$, $\langle \alpha, [\nu_b] \rangle > 0$ implies the coefficient of $\alpha^\vee$ in $\mu^\diamond - [\nu_b]$ is positive.*

Corollary 3.5. *Suppose G is quasi-split. Suppose $[b] \in B(G, \mu^\sharp + \varepsilon, \mu^\diamond)$ for some ε in $\pi_1(G)_{\Gamma,\mathrm{tor}}$ and (G, μ, b) is HN-decomposable. Then there exists a unique strict standard Levi subgroup M, a unique element $\varepsilon_M \in \pi_1(M)_{\Gamma,\mathrm{tor}}$ and a reduction $\tilde{b}_M$ such that*

(1) *ε is the image of $w_0\varepsilon_M$ under the natural map $\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma$,*

(2) *$[\tilde{b}_M] \in B(M, (\tilde{w}_0\mu)^\sharp + \varepsilon_M, (\tilde{w}_0\mu)^\diamond)$,*

(3) *$(M, \tilde{w}_0\mu, \tilde{b}_M)$ is HN-indecomposable.*

Proof. Let M_1 be the standard Levi subgroup of G such that its simple roots are described as follows:

$$\Delta_{M_1,0} = \Delta_{M_b,0} \cup \{\alpha \in \Delta_0 \mid n_\alpha > 0\},$$

where $\mu^\diamond - [\nu_b] = \sum_{\alpha\in\Delta_0} n_\alpha \alpha^\vee$ with $n_\alpha \geq 0$. Let b_{M_1} be the reduction of b to M_1 deduced from its canonical reduction to M_b combined with the inclusion $M_b \subseteq M_1$.

Then it is easy to see (M_1, μ, b_{M_1}) is HN-indecomposable and M_1 is the unique standard Levi subgroup with this property. Let $M = w_0 M_1 w_0^{-1}$. The rest of the assertions can be proved in the same way as Lemma 3.3. ■

Remark 3.6. With notations as in Corollary 3.5, as $w_0\varepsilon$ is the image of ε_M under the natural injective map $\pi_1(M)_\Gamma \to \pi_1(G)_\Gamma$, we may identify ε_M and $w_0\varepsilon$.

The following proposition is a key ingredient in the proof of the main result.

Proposition 3.7. *Suppose G is quasi-split. Let $[b] \in A(G, \mu)$ be such that (G, μ, b) is HN-decomposable. Let $M \subseteq G$ be the strict standard Levi subgroup such that $(M, w_0\mu, \tilde{b}_M)$ is HN-indecomposable as in Corollary 3.5. Then*

$$\mathrm{pr}_{\tilde{w}_0}^{-1}(\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{a}}(C)) = \mathcal{F}(G, \mu, b)^{\mathrm{a}}(C),$$
$$\mathrm{pr}_{\tilde{w}_0}^{-1}(\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{wa}}(C)) = \mathcal{F}(G, \mu, b)^{\mathrm{wa}}(C).$$

Remark 3.8. The appearance of $\tilde{w}_0$ in the statement is due to the fact that $\nu_{\mathcal{E}_b} = -w_0[\nu_b]$ by Proposition 2.6.

The remainder of this section is devoted to the proof of the above proposition. Suppose (G, μ, b) is HN-decomposable with G quasi-split. Suppose M is the Levi subgroup of G such that $(M, \tilde{w}_0\mu, \tilde{b}_M)$ is HN-indecomposable as in Corollary 3.5. Let P be the standard parabolic subgroup of G with M as Levi component.

Lemma 3.9. $\mathcal{F}(G, \mu, b)^{\mathrm{wa}}(C) \subseteq P(C)w_0P_\mu(C)/P_\mu(C)$.

Proof. Suppose $\mathcal{F}(G, \mu, b)^{\mathrm{wa}}(C) \cap P(C)wP_\mu(C)/P_\mu(C) \neq \emptyset$ for some $w \in W$. We want to show $w_0 \in W_P w W_{P_\mu}$. For any $x \in \mathcal{F}(G, \mu, b)^{\mathrm{wa}}(C) \cap P(C)wP_\mu(C)/P_\mu(C)$, we have

$$(\mathcal{E}_{b,x})_P \simeq \mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}$$

by Lemma 2.14. The weak semistability of $\mathcal{E}_{b,x}$ implies that $\deg \chi_*(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}) \leq 0$ for any $\chi \in X^*(M/Z_G)^+$. On the other hand, using the fact that $[\tilde{b}_M] \in A(M, \tilde{w}_0\mu)$, we have

$$c_1^M(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_w(x)}) = (w\mu)^\sharp - \kappa_M(\tilde{b}_M) = (w\mu)^\sharp - (w_0\mu)^\sharp \quad \text{in } \pi_1(M)_\Gamma \otimes \mathbb{Q},$$

and $\deg \chi_*(\mathcal{E}_{b_M, \mathrm{pr}_w(x)}) = \langle w\mu - w_0\mu, \chi\rangle \leq 0$ for any $\chi \in X^*(M/Z_G)^+$ by Proposition 2.6. Therefore equality holds for any χ and $(w\mu)^\sharp = (w_0\mu)^\sharp$ in $\pi_1(M)_\Gamma$. On the other hand, as $w\mu \geq w_0\mu$, $w\mu$ and $w_0\mu$ have the same image in $\pi_1(M)$. The result follows. ■

With notations as in Corollary 3.5, let $\mathcal{E}_\varepsilon^G$ be the G-bundle such that $\nu_{\mathcal{E}_\varepsilon^G}$ is trivial and $c_1^G(\mathcal{E}_\varepsilon^G) = -\varepsilon$. Similarly, let $\mathcal{E}_{w_0\varepsilon}^M$ be the M-bundle such that $\nu_{\mathcal{E}_{w_0\varepsilon}^M}$ is trivial and $c_1^M(\mathcal{E}_{w_0\varepsilon}^M) = -w_0\varepsilon$ (cf. Remark 3.6). Let $\mathcal{E}_{w_0\varepsilon}^P := \mathcal{E}_{w_0\varepsilon}^M \times^M P$. When $[b] \in B(G, \mu)$ (i.e., $\varepsilon = 0$), the bundles $\mathcal{E}_\varepsilon^G = \mathcal{E}_1^G$, $\mathcal{E}_{w_0\varepsilon}^M = \mathcal{E}_1^M$ and $\mathcal{E}_{w_0\varepsilon}^P = \mathcal{E}_1^P$ are trivial.

Consider the following commutative diagram (cf. [28, 39]) of de Rham period maps for different groups from local Shimura varieties at infinite level to flag varieties:

$$
\begin{array}{ccccc}
 & & \mathcal{M}(P, \tilde{w}_0\mu, \tilde{b}_P)_\infty & & \\
 & \xi_M \swarrow & & \searrow \xi_G & \\
\mathcal{M}(M, \tilde{w}_0\mu, \tilde{b}_M)_\infty & & \Big\downarrow \pi_{\mathrm{dR}}^P & & \mathcal{M}(G, \mu, b)_\infty \\
\Big\downarrow \pi_{\mathrm{dR}}^M & & \mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)^{\mathrm{a}} & & \Big\downarrow \pi_{\mathrm{dR}}^G \\
 & \tilde{\xi}_M \swarrow & & \searrow \tilde{\xi}_G & \\
\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{a}} & & & & \mathcal{F}(G, \mu, b)^{\mathrm{a}}
\end{array}
$$

where:

- $\tilde{b}_P$ is the reduction of b to P induced by the reduction $\tilde{b}_M$ of b to M combined with the inclusion $M \subseteq P$.
- $\mathcal{M}(G, \mu, b)_\infty$ (resp. $\mathcal{M}(M, \tilde{w}_0\mu, \tilde{b}_M)$, resp. $\mathcal{M}(P, \tilde{w}_0\mu, \tilde{b}_P)$) classifies modifications of type μ (resp. $\tilde{w}_0\mu$, resp. $\tilde{w}_0\mu$) between $\mathcal{E}_b^G$ (resp. $\mathcal{E}_b^M$, resp. $\mathcal{E}_b^P$) and $\mathcal{E}_\varepsilon^G$ (resp. $\mathcal{E}_{w_0\varepsilon}^M$, resp. $\mathcal{E}_{w_0\varepsilon}^P$).
- π_{dR}^G, π_{dR}^M, π_{dR}^P are the de Rham period maps. More precisely, for a modification in $\mathcal{M}(G, \mu, b)_\infty$ its image by π_{dR}^G is x if $\mathcal{E}_{b,x}^G = \mathcal{E}_\varepsilon^G$, and similarly for π_{dR}^M and π_{dR}^P. We define $\mathcal{F}(P, \tilde{w}_0\mu, \tilde{b}_P)^{\mathrm{a}}$ to be the image of π_{dR}^P.
- ξ_G (resp. ξ_M) is the induced modification via the natural morphism $P \to G$ (resp. the projection to the Levi quotient $P \to M$).
- $\tilde{\xi}_G$ is induced from

$$
\begin{aligned}
\mathcal{F}(P, \tilde{w}_0\mu) = P/P_{\tilde{w}_0\mu} \cap P &\to P\tilde{w}_0 P_\mu/P_\mu \subseteq G/P_\mu = \mathcal{F}(G, \mu), \\
aP_{\tilde{w}_0\mu} \cap P &\mapsto a(P_{\tilde{w}_0\mu} \cap P)\tilde{w}_0 P_\mu = a\tilde{w}_0 P_\mu.
\end{aligned}
\tag{3.0.1}
$$

- $\tilde{\xi}_M$ is induced from the natural projection $P \to M$ to the Levi component

$$
\mathcal{F}(P, \tilde{w}_0\mu) = P/P_{\tilde{w}_0\mu} \to M/M_{\tilde{w}_0\mu} = \mathcal{F}(M, \tilde{w}_0\mu).
$$

By [6, proof of Lemma 6.3], we have the following fact.

Lemma 3.10. *Let $\mathcal{E}$ be a G-bundle and let $P' \subseteq P$ be a standard parabolic subgroup of G contained in P. There is a bijection between*

- *reductions $\mathcal{E}_{P'}$ of $\mathcal{E}$ to P',*
- *reductions $\mathcal{E}_P$ to P together with a reduction $(\mathcal{E}_P \times^P M)_{M\cap P'}$ of $\mathcal{E}_P \times^P M$ of $M \cap P'$.*

Moreover, this bijection identifies $\mathcal{E}_{P'} \times^{P'} M'$ and $(\mathcal{E}_P \times^P M)_{M\cap P'} \times^{M\cap P'} M'$, where M' is the Levi component of P'.

Now we can prove the following result.

Lemma 3.11. *In the above diagram, $\tilde{\xi}_G$ is an isomorphism of adic spaces and*

$$\mathcal{M}(G,\mu,b)_\infty \simeq \coprod_{\mathrm{Aut}(\mathcal{E}^G_\varepsilon)/\mathrm{Aut}(\mathcal{E}^P_{\tilde{w}_0\varepsilon})} \mathcal{M}(P,\tilde{w}_0\mu,b)_\infty.$$

Proof. We only deal with the case $[b]\in B(G,\mu)$; the proof for $[b]\in A(G,\mu)$ is similar. For the first assertion, as $\mathcal{F}(P,\tilde{w}_0\mu)\to\mathcal{F}(G,\mu)$ is an open immersion, it suffices to show that

$$\mathcal{F}(P,\tilde{w}_0\mu,\tilde{b}_P)^{\mathrm{a}}(C)\to\mathcal{F}(G,\mu,b)^{\mathrm{a}}(C)$$

is surjective for any complete algebraically closed field C. For any $x\in\mathcal{F}(G,\mu,b)^{\mathrm{a}}(C)$, by Lemma 3.9, $x\in P(C)\omega_0 P_\mu(C)/P_\mu(C)$. Let $(\mathcal{E}_{b,x})_P$ be the reduction of $\mathcal{E}_{b,x}$ to P induced by the reduction $\mathcal{E}_{\tilde{b}_P}$ of $\mathcal{E}_b$ to P. Write $(\mathcal{E}_{b,x})_P=\mathcal{E}_{\tilde{b}_P,y}$ where y is the preimage of x via (3.0.1). We want to show $y\in\mathcal{F}(P,\tilde{w}_0\mu,\tilde{b}_P)^{\mathrm{a}}(C)$. This is equivalent to showing that $(\mathcal{E}_{b,x})_P=\mathcal{E}_{\tilde{b}_P,y}$ is a trivial G-bundle. The isomorphism of G-bundles $\mathcal{E}_{b,x}\simeq\mathcal{E}_1$ induces a reduction $(\mathcal{E}_1)_P$ of $\mathcal{E}_1$ to P and an isomorphism $(\mathcal{E}_{b,x})_P\simeq(\mathcal{E}_1)_P$ of P-bundles. We want to show that $(\mathcal{E}_1)_P$ is a trivial P-bundle. By Corollary 2.9, it suffices to show that $\mathcal{E}_M:=(\mathcal{E}_1)_P\times^P M$ is a trivial M-bundle. We first show that

$$c_1^M(\mathcal{E}_M)=0. \tag{3.0.2}$$

Indeed,

$$\begin{aligned} c_1^M((\mathcal{E}_M) &= c_1^M((\mathcal{E}_{b,x})_P\times^P M)\\ &= c_1^M(\mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{w_0}(x)}) \quad \text{(by Lemma 2.14)}\\ &= (w_0\mu)^\sharp-\kappa_M(\tilde{b}_M)=0\in\pi_1(M)_\Gamma, \end{aligned}$$

where the last equality holds because $[\tilde{b}_M]\in B(M,\tilde{w}_0\mu)$.

Now it remains to show the slope $\nu_{\mathcal{E}_M}=0$. Let $(\mathcal{E}_M)_{P'_M}$ be the canonical reduction of $\mathcal{E}_M$ to a standard parabolic subgroup P'_M of M. Write $P'=P'_M\cdot R_u(P)$, where $R_u(P)$ denotes the unipotent radical of P. Note that P' is a standard parabolic subgroup of G with $P'\cap M=P'_M$. Let M' be the Levi component of P'_M. By [15, Proposition 5.16],

$$\nu_{(\mathcal{E}_M)_{P'_M}\times^{P'_M}M'}=\nu_{\mathcal{E}_M}\in X_*(A)_{\mathbb{Q}}.$$

By Lemma 3.10, $(\mathcal{E}_M)_{P'_M}$ corresponds to a reduction of $\mathcal{E}_1$ to P'. Hence the semistability of $\mathcal{E}_1$ implies that

$$\langle\chi,\nu_{\mathcal{E}_M}\rangle\le 0,\quad \forall\chi\in X^*(M'/Z_G)^+.$$

On the other hand, (3.0.2) implies that $\nu_{\mathcal{E}_M}$ is a non-negative linear combination of simple coroots in M. Hence

$$\langle\chi,\nu_{\mathcal{E}_M}\rangle\ge 0,\quad \forall\chi\in X^*(M'/Z_G)^+.$$

It follows that $\nu_{\mathcal{E}_M}=0$.

The second assertion is obtained from the first assertion and the fact that $\mathcal{M}(G,\mu,b)_\infty$ (resp. $\mathcal{M}(P,\tilde{w}_0\mu,\tilde{b}_P)_\infty$) is the $\underline{\mathrm{Aut}(\mathcal{E}_\varepsilon^G)}$-torsor (resp. $\underline{\mathrm{Aut}(\mathcal{E}_{w_0\varepsilon}^P)}$-torsor) over $\mathcal{F}(G,\mu,b)^{\mathrm{a}}$ (resp. $\mathcal{F}(P,\tilde{w}_0\mu,\tilde{b}_P)$). ■

Remark 3.12. This lemma was proved for an unramified local Shimura datum of PEL type by Mantovan [28, §8.2] and Shen [39, Corollary 6.4].

In order to prove Proposition 3.7, we also need the following lemmas.

Lemma 3.13. *Suppose G is quasi-split. Consider $\mathcal{F}(M,w_0\mu)$ as a subspace of $\mathcal{F}(G,\mu)$ via the natural injective morphism $\mathcal{F}(M,\tilde{w}_0\mu)\to\mathcal{F}(G,\mu)$. Then*

$$\mathcal{F}(M,\tilde{w}_0\mu,\tilde{b}_M)^{\mathrm{a}}\subseteq\mathcal{F}(G,\mu,b)^{\mathrm{a}},\quad \mathcal{F}(M,\tilde{w}_0\mu,\tilde{b}_M)^{\mathrm{wa}}\subseteq\mathcal{F}(G,\mu,b)^{\mathrm{wa}}.$$

Proof. The first assertion is clear. For the second one, let $x\in\mathcal{F}(M,\tilde{w}_0\mu,\tilde{b}_M)^{\mathrm{wa}}(C)$ and let x_G be its image via the natural morphism $\mathcal{F}(M,\tilde{w}_0\mu)\to\mathcal{F}(G,\mu)$. Suppose Q is a standard parabolic subgroup with Levi component M_Q. Let b_{M_Q} be a reduction of b to M_Q. Let $w\in W^\Gamma$ and $b_1\in M_1(\breve{F})$ with $M_1=M\cap w^{-1}M_Q w$ as in Lemma 3.14 below. Let Q_1 be the standard parabolic subgroup of M with Levi component M_1. Since

$$(\mathcal{E}_{b_1}\times^{M_1}Q_1)\times^{Q_1,\mathrm{ad}\,w}Q\simeq\mathcal{E}_{b_{M_Q}}\times^{M_Q}Q$$

by Lemma 3.14, we have

$$(\mathcal{E}_{b,x_G})_Q\simeq(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}\times^{Q_1}Q,$$

where $Q_1\to Q$ is induced by $\mathrm{ad}\,\dot{w}$ and where $(\mathcal{E}_{b,x_G})_Q$ (resp.$(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}$) is the reduction of $\mathcal{E}_{b,x_G}$ (resp. $\mathcal{E}_{\tilde{b}_M,x}$) to Q (resp. Q_1) induced by the reduction $\mathcal{E}_{b_{M_Q}}\times^{M_Q}Q$ (resp. $\mathcal{E}_{b_1}\times^{M_1}Q_1$) of $\mathcal{E}_b$ (resp. $\mathcal{E}_{\tilde{b}_M}$) to Q (resp. Q_1). For any $\chi\in X^*(Q/Z_G)^+$, we have

$$\deg\chi_*(\mathcal{E}_{b,x_G})_Q=\deg\chi'_*(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}$$

with $\chi'=\chi\circ\mathrm{ad}\,\dot{w}\in X^*(Q_1)$. Write $\chi'=\chi'_1+\chi'_2$ with $\chi'_1=\mathrm{Av}_{W_M}(\chi')$ the W_M-average of χ'. Up to replacing χ by a multiple $m\chi$ for $m\in\mathbb{N}$ large enough, we may assume $\chi'_1\in(\Phi_M^\vee)^\perp=X^*(M)$ and $\chi'_2\in X^*(M_1/Z_M)$. Moreover, the choice of w implies $w\beta\in\Phi_G^+$ for any $\beta\in\Phi_M^+$. Hence $\chi'_2\in X^*(M_1/Z_M)^+$, where $^+$ stands for M-dominant. Then

$$\deg\chi_*(\mathcal{E}_{b,x})_Q=\deg\chi'_*(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}=\underbrace{\deg\chi'_{1*}(\mathcal{E}_{\tilde{b}_M,x})}_{=0}+\underbrace{\deg\chi'_{2*}(\mathcal{E}_{\tilde{b}_M,x})_{Q_1}}_{\leq 0}\leq 0$$

since $[\tilde{b}_M]\in B(M,\tilde{w}_0\mu)$ combined with the weak admissibility of x. ■

Recall that M and $\tilde{b}_M$ are as in Corollary 3.5. The following lemma reflects the fact that the category of isocrystals is semisimple. If we can decompose an isocrystal in two different ways, then we can decompose it in a way that is finer than the previous two decompositions.

Lemma 3.14. *Suppose G is quasi-split. Let b_{M_Q} be a reduction of b to M_Q, where M_Q is the Levi component of a standard parabolic subgroup Q. Then there exist $w \in W^\Gamma$ (where W^Γ can be identified with the relative Weyl group of G), $g_1 \in M(\breve{F})$, $g_1' \in (w^{-1}M_Q w)(\breve{F})$ and $b_1 \in M_1(\breve{F})$ with $M_1 = w^{-1}M_Q w \cap M$ such that*

(1) *w is the minimal length element in $W^\Gamma_{M_Q} w W^\Gamma_M$;*

(2) *(b_1, g_1) is a reduction of $\tilde{b}_M$ to M_1;*

(3) *(b_1, g_1') is a reduction of $\dot{w}^{-1} b_{M_Q} \sigma(\dot{w})$ to M_1 with $\dot{w} \in N(T)(\breve{F})$ a representative of w.*

Moreover, $W_{M_Q} w W_M$ is the generic relative position between the reduction $\mathcal{E}_{\tilde{b}_M} \times^M P$ of $\mathcal{E}_b$ to P and the reduction $\mathcal{E}_{b_{M_Q}} \times^{M_Q} Q$ of $\mathcal{E}_b$ to Q.

Proof. The last assertion is implied by conditions (2) and (3). We first show that there exist elements satisfying (2) and (3). More precisely, we claim that there exist $\tilde{w} \in W^\Gamma$, $\tilde{g}_1 \in M(\breve{F})$, $\tilde{g}_1' \in \tilde{w}^{-1}M_Q\tilde{w}(\breve{F})$ and $\tilde{b}_1 \in \tilde{M}_1(L)$ with $\tilde{M}_1 = \tilde{w}^{-1}M_Q\tilde{w} \cap M$ such that

(2) $(\tilde{b}_1, \tilde{g}_1)$ is a reduction of $\tilde{b}_M$ to $\tilde{M}_1$;

(3) $(\tilde{b}_1, \tilde{g}_1')$ is a reduction of $\dot{\tilde{w}}^{-1} b_{M_Q} \sigma(\dot{\tilde{w}})$ to $\tilde{M}_1$ with $\dot{\tilde{w}} \in N(T)(\breve{F})$ a representative of $\tilde{w}$.

Note that a representative $\dot{\tilde{w}}$ of $\tilde{w}$ can be chosen in $N(T)(\breve{F})$ by Steinberg's theorem [38, III, §2.3].

By definition, $\tilde{b}_M$ is induced from $w_0 b_{M_b} w_0^{-1}$ via the natural inclusion $w_0 M_b w_0^{-1} \subseteq M$. Therefore, without loss of generality, we may assume $M = w_0 M_b w_0^{-1}$. Since b_{M_Q} is a reduction of b to M_Q, up to σ-conjugation, we may assume $\nu_{b_{M_Q}} : \mathbb{D} \to M_Q$ is defined over F and has image in the split maximal torus A which is M_Q-anti-dominant. Choose $\tilde{w} \in W^\Gamma$ such that $\tilde{w}^{-1}\nu_{b_{M_Q}} = w_0[\nu_b]$ is G-anti-dominant. Then $\mathrm{Cent}_{\tilde{w}^{-1}M_Q\tilde{w}}(\nu_{\dot{\tilde{w}}^{-1}b_{M_Q}\sigma(\dot{\tilde{w}})}) = \tilde{w}^{-1}M_Q\tilde{w} \cap M = \tilde{M}_1$ and $\dot{\tilde{w}}^{-1} b_{M_Q} \sigma(\dot{\tilde{w}})$ has a canonical reduction $(\tilde{b}_1, \tilde{g}_1')$ to $\tilde{M}_1$. Then (3) follows.

Let $\tilde{b}_1'$ be the image of $\tilde{b}_1$ via the inclusion $\tilde{M}_1 \subseteq M$. For (2), it suffices to show $[\tilde{b}_1'] = [\tilde{b}_M]$ in $B(M)$. Clearly,

$$[\nu_{\tilde{b}_1'}]_M = [\nu_{\tilde{b}_M}]_M = (w_0[\nu_b])_{M\text{-dom}} \in \mathcal{N}(M).$$

By the injectivity of the map (see (1.3.1))

$$(\nu, \kappa_M) : B(M) \to \mathcal{N}(M) \times \pi_1(M)_\Gamma,$$

it suffices to show $\kappa_M(\tilde{b}_1') = \kappa_M(\tilde{b}_M) \in \pi_1(M)_\Gamma$. Since

$$\kappa_M(a) = \nu_a \quad \text{in } \pi_1(M)_\Gamma \otimes_{\mathbb{Z}} \mathbb{Q},\ \forall a \in M(\breve{F}),$$

we have $\kappa_M(\tilde{b}_1') - \kappa_M(\tilde{b}_M) \in \pi_1(M)_{\Gamma,\mathrm{tor}}$. As $\tilde{b}_1$ and $\tilde{b}_M$ are both reductions of b, we have

$$\kappa_G(\tilde{b}_1') = \kappa_G(\tilde{b}_1) = \kappa_G(\tilde{b}_M) \quad \text{in } \pi_1(G)_\Gamma.$$

The result follows by the injectivity of the map $\pi_1(M)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}}$ (cf. proof of Lemma 3.3).

Let $w = w_1\tilde{w}w_2$ be the unique minimal length element in $W^{\Gamma}_{M_Q}\tilde{w}W^{\Gamma}_{M}$ with $w_1 \in W^{\Gamma}_{M_Q}$ and $w_2 \in W^{\Gamma}_{M}$. Then $w \in W^{\Gamma}$ and we can choose representatives $\dot{w}_2 \in M(\breve{F})$, $\dot{w} \in G(\breve{F})$ of w_2 and w respectively again by Steinberg's theorem. Let

$$\begin{aligned} b_1 &:= \dot{w}_2^{-1}\tilde{b}_1\sigma(\dot{w}_2) \in M_1(\breve{F}),\\ g_1 &:= \tilde{g}_1\dot{w}_2 \in M(\breve{F}),\\ g_1' &:= \dot{w}^{-1}\dot{\tilde{w}}\tilde{g}_1'\dot{w}_2 \in w^{-1}M_Q w(\breve{F}). \end{aligned}$$

They satisfy the desired properties (1)–(3). ■

Now we are ready to prove the main result of this section.

Proof of Proposition 3.7. We only deal with the case $[b] \in B(G,\mu)$; the proof for $[b] \in A(G,\mu)$ is similar. By Lemmas 3.9 and 3.11, we have $\mathrm{pr}_{\tilde{w}_0} = \xi_M \circ \tilde{\xi}_G^{-1}$ and

$$\mathcal{F}(G,\mu,b)^{\mathrm{a}} \subseteq \mathrm{pr}_{w_0}^{-1}(\mathcal{F}(M, w_0\mu, \tilde{b}_M)^{\mathrm{a}}).$$

It remains to show that $\mathcal{E}_{b,x}$ is a trivial G-bundle for any $x \in \mathrm{pr}_{\tilde{w}_0}^{-1}(\mathcal{F}(M,\tilde{w}_0\mu,b)^{\mathrm{a}}(C))$. By Lemma 2.14, $(\mathcal{E}_{b,x})_P \times^P M \simeq \mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{\tilde{w}_0}(x)}$, which is a trivial M-bundle. The result then follows by Corollary 2.9.

The proof for weak admissibility is similar. Suppose $x \in \mathcal{F}(G,\mu,b)^{\mathrm{wa}}(C)$. Then $x \in P(C)w_0P_{\mu}(C)/P_{\mu}(C)$ by Lemma 3.9. We show that $\mathrm{pr}_{\tilde{w}_0}(x) \in \mathcal{F}(M,\tilde{w}_0\mu,\tilde{b}_M)^{\mathrm{wa}}(C)$. Suppose $M' \subset M$ is a standard Levi subgroup of M with $b' \in M'(L)$ a reduction of $\tilde{b}_M$ to M'. Let $P'_M \subseteq M$ (resp. $P' \subseteq G$) be the standard parabolic subgroup of M (resp. G) with Levi component M'. We need to show that

$$\deg \chi_*(\mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{\tilde{w}_0}(x)})_{P'_M} \leq 0, \quad \forall \chi \in X^*(P'_M/Z_M)^+,$$

where $(\mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{\tilde{w}_0}(x)})_{P'_M}$ is the reduction of $\mathcal{E}^M_{b'_M,\mathrm{pr}_{\tilde{w}_0}(x)}$ to P'_M induced by the reduction $\mathcal{E}_{b'} \times^{M'} P'_M$ of $\mathcal{E}_{\tilde{b}_M}$. By Remark 2.8, this is equivalent to showing

$$\mathrm{Av}_{W_{M'}}(\nu_{\mathcal{E}_{M'}}) \leq_M 0 \quad \text{in } X_*(A)_{\mathbb{Q}}, \tag{3.0.3}$$

where $\mathcal{E}_{M'} := (\mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{w_0}(x)})_{P'_M} \times^{P'_M} M'$. By Theorem 2.7,

$$\mathrm{Av}_{W_{M'}}(\nu_{\mathcal{E}_{M'}}) \leq_M \nu_{\mathcal{E}_M} \quad \text{in } X_*(A)_{\mathbb{Q}}, \tag{3.0.4}$$

where $\mathcal{E}_M := \mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{w_0}(x)}$. On the other hand, by Lemmas 2.14 and 3.10, we have

$$\begin{aligned} \mathcal{E}_{M'} &= (\mathcal{E}_{\tilde{b}_M,\mathrm{pr}_{w_0}(x)})_{P'_M} \times^{P'_M} M' \simeq ((\mathcal{E}_{b,x})_P \times^P M)_{P'_M} \times^{P'_M} M'\\ &\simeq (\mathcal{E}_{b,x})_{P'} \times^{P'} M', \end{aligned}$$

where $(\mathcal{E}_{b,x})_{P'}$ is the reduction of $\mathcal{E}_{b,x}$ to P' induced by the reduction $\mathcal{E}_{b'} \times^{M'} P'$ of $\mathcal{E}_b$. The weak semistability of $\mathcal{E}_{b,x}$ implies that

$$\mathrm{Av}_{W_{M'}}(\nu_{\mathcal{E}_{M'}}) \leq_G 0 \quad \text{in } X_*(A)_{\mathbb{Q}}. \tag{3.0.5}$$

Hence inequality (3.0.3) follows from (3.0.4) and (3.0.5) combined with the fact that

$$\mathrm{Av}_{W_M}(\nu_{\mathcal{E}_M}) = 0$$

since $[\tilde{b}_M] \in B(M, \tilde{w}_0\mu)$.

Conversely, suppose

$$x \in P(C)w_0P_\mu(C)/P_\mu(C) \quad \text{with} \quad \mathrm{pr}_{\tilde{w}_0}(x) \in \mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{wa}}(C).$$

We want to show $x \in \mathcal{F}(G, \mu, b)^{\mathrm{wa}}(C)$. Suppose Q is a standard parabolic subgroup of G with Levi component M_Q. Let b_{M_Q} be a reduction of b to M_Q. We need to show

$$\deg \chi_*(\mathcal{E}_{b,x})_Q \leq 0, \quad \forall \chi \in X^*(Q/Z_G)^+, \tag{3.0.6}$$

where $(\mathcal{E}_{b,x})_Q$ is the reduction of $\mathcal{E}_{b,x}$ to Q induced by the reduction $\mathcal{E}_{b_{M_Q}} \times^{M_Q} Q$ of $\mathcal{E}_b$ to Q. By the proof of Corollary 2.9, $(\mathcal{E}_{b,x})_Q$ induces a reduction

$$(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)} \times^M G)_Q$$

of $\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)} \times^M G$ to Q. Moreover, we have

$$\deg \chi_*(\mathcal{E}_{b,x})_Q = \deg \chi_*(\mathcal{E}_{\tilde{b}_M, \mathrm{pr}_{w_0}(x)} \times^M G)_Q.$$

Hence (3.0.6) follows by the weak admissibility of $\mathrm{pr}_{w_0}(x)$ and Lemma 3.13. ∎

4. The action of $\tilde{J}_b$ on modifications of G-bundles

Now we state the main results of this article. We first state the main result in the quasi-split case.

Suppose G is quasi-split and $[b] \in A(G, \mu)$. Recall that by Remark 1.5, we may assume $[b] \in B(G, \mu^\sharp + \varepsilon, \mu^\diamond)$ for some $\varepsilon \in \pi_1(G)_{\Gamma,\mathrm{tor}}$. By Lemma 3.3 (cf. also Corollary 3.5), there exists a unique standard Levi subgroup M of G with a reduction b_M of b to M such that (M, μ, b) is HN-indecomposable and ε is in the image of the injective map $\pi_1(M)_{\Gamma,\mathrm{tor}} \to \pi_1(G)_{\Gamma,\mathrm{tor}}$. So we may consider ε as an element in $\pi_1(M)_{\Gamma,\mathrm{tor}}$. In the following, sometimes we also identify b_M with b.

Theorem 4.1. *Suppose G is quasi-split and μ is minuscule. Let $[b] \in B(G, \mu^\sharp + \varepsilon, \mu^\diamond)$. Suppose M is the standard Levi subgroup of G such that (M, μ, b) is Hodge–Newton-indecomposable. Then the equality $\mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(G, \mu, b)^{\mathrm{a}}$ holds if and only if $B(M, \mu^\sharp + \varepsilon, \mu^\diamond)$ is fully Hodge–Newton-decomposable and $[b]$ is basic in $B(M)$.*

In particular, for $[b] \in B(G, \mu)$, the equality $\mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(G, \mu, b)^{\mathrm{a}}$ holds if and only if the pair (M, μ) is fully Hodge–Newton-decomposable and $[b]$ is basic in $B(M)$.

Remark 4.2. For GL_n and $[b] \in B(G, \mu)$, this theorem was proved by Hartl [21, Theorem 9.3].

We have a similar description when the group G is non-quasi-split.

Theorem 4.3. *Suppose μ is minuscule and $[b] \in B(G, \mu)$. Let $[b^H] \in B(H)$ be the image of $[b]$ via the natural bijection $B(G) \simeq B(H)$. Let M^H be the standard Levi subgroup of H with a reduction b^H_M of b^H to M^H such that $[b^H_M] \in A(M^H, \mu)$ and (M^H, μ, b^H_M) is HN-indecomposable. Then*

(1) *the Levi subgroup M^H of H corresponds to a Levi subgroup M of G via the inner twisting,*
(2) *the equality $\mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(G, \mu, b)^{\mathrm{a}}$ holds if and only if the pair (M, μ) is fully HN-indecomposable and $[b_M]$ is basic in $B(M)$, where $[b_M]$ corresponds to $[b^H_M]$ via $B(M) \simeq B(M^H)$.*

In order to prove the main theorems, we need some preparations.

Lemma 4.4. *Suppose M is a standard Levi subgroup of H defined over F. Let $\dot{w}_0 \in G(\breve{F})$ be a representative of w_0. Then the map*

$$M(\breve{F}) \simeq (w_0 M w_0^{-1})(\breve{F}), \quad g \mapsto \dot{w}_0 g \sigma(\dot{w}_0)^{-1},$$

induces the bijections

$$\begin{aligned} B(M) &\simeq B(w_0 M w_0^{-1}), \\ B(M)_{\mathrm{basic}} &\simeq B(w_0 M w_0^{-1})_{\mathrm{basic}}, \\ B(M, \mu) &\simeq B(w_0 M w_0^{-1}, \tilde{w}_0 \mu), \\ B(M, \mu^\sharp + \varepsilon, \mu^\diamond) &\simeq B(w_0 M w_0^{-1}, (\tilde{w}_0\mu)^\sharp + w_0\varepsilon, (\tilde{w}_0\mu)^\diamond), \end{aligned}$$

where $\tilde{w}_0\mu := (w_0\mu)_{w_0 M w_0^{-1}\text{-dom}}$ is the $w_0 M w_0^{-1}$-domominant representative of $w_0\mu$ in its $W_{w_0 M w_0^{-1}}$-orbit. Moreover, $B(M, \mu^\sharp + \varepsilon, \mu^\diamond)$ is fully HN-decomposable if and only if so is $B(w_0 M w_0^{-1}, (\tilde{w}_0\mu)^\sharp + w_0\varepsilon, (\tilde{w}_0\mu)^\diamond)$.

Proof. We may assume that $\tilde{w}_0$ is the minimal length representative in $W_{w_0 M w_0^{-1}} w_0$. Note that for any $g \in M(\breve{F})$,

$$[\nu_{\dot{w}_0 g \sigma(\dot{w}_0)^{-1}}] = \tilde{w}_0[\nu_g] \in \mathcal{N}(w_0 M w_0^{-1}).$$

The other assertions can be checked easily. ∎

We also need the following proposition which is a key ingredient in the proof of the main result.

Proposition 4.5. *Suppose G is quasi-split. Suppose $[b] \in A(G, \mu)$ and (G, μ, b) is HN-indecomposable. If b is not basic, then $\mathcal{F}(G, \mu, b)^{\mathrm{a}} \neq \mathcal{F}(G, \mu, b)^{\mathrm{wa}}$.*

Before proving this proposition, we first show how to use it combined with Proposition 3.7 to prove the main theorem.

Proof of Theorem 4.1. For the sufficiency, suppose $B(M, \mu^\sharp + \varepsilon, \mu^\diamond)$ is fully HN-decomposable and $[b_M]$ is basic in $B(M)$. Then $B(\tilde{M}, (\tilde{w}_0\mu)^\sharp + w_0\varepsilon, (\tilde{w}_0\mu)^\diamond)$ is fully HN-decomposable and $[\tilde{b}_M]$ is basic in $B(\tilde{M})$ with $\tilde{M} = w_0 M w_0^{-1}$ and $\tilde{b}_M = w_0 b_M \sigma(w_0)^{-1}$ by Lemma 4.4. After applying [6, Theorem 6.1] and its proof for the non-quasi-split case to the triple $(\tilde{M}, \tilde{w}_0\mu, \tilde{b}_M)$, we get $\mathcal{F}(\tilde{M}, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{a}} = \mathcal{F}(\tilde{M}, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{wa}}$. The result follows by Proposition 3.7.

For the necessity, if b_M is not basic in M or $B(M, \mu^\sharp + \varepsilon, \mu^\diamond)$ is not fully HN-indecomposable, then equivalently $\tilde{b}_M$ is not basic in $\tilde{M}$ or $B(\tilde{M}, (\tilde{w}_0\mu)^\sharp + w_0\varepsilon, (\tilde{w}_0\mu)^\diamond)$ is not fully HN-indecomposable. Hence after applying Proposition 4.5 (in the first case) or [6, Theorem 6.1 and its proof] for the non-quasi-split case to the triple $(\tilde{M}, \tilde{w}_0\mu, \tilde{b}_M)$, we get $\mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{a}} \neq \mathcal{F}(M, \tilde{w}_0\mu, \tilde{b}_M)^{\mathrm{wa}}$. And $\mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(G, \mu, b)^{\mathrm{a}}$ follows again by Proposition 3.7. ∎

Proof of Theorem 4.3. The reduction to the quasi-split case is similar to the proof of [6, Theorem 6.1]. We may assume that G is adjoint. Then $G = J_{b^*}$ is an extended pure inner form of H, where $[b^*] \in B(H)_{\mathrm{basic}}$. We have

$$\mathcal{F}(G, \mu, b)^{\mathrm{a}} = \mathcal{F}(H, \mu, b^H)^{\mathrm{a}}, \quad \mathcal{F}(G, \mu, b)^{\mathrm{wa}} = \mathcal{F}(H, \mu, b^H)^{\mathrm{wa}},$$

and $[b^H] \in B(H, \mu^\sharp + \kappa(b^*), \mu^\diamond)$. By Lemma 1.8,

$$\kappa(b^*) \in \mathrm{Im}(\pi_1(M^H)_{\Gamma,\mathrm{tor}} \to \pi_1(H)_{\Gamma,\mathrm{tor}}),$$

and if we view $\kappa(b^*)$ as an element in $\pi_1(M^H)_{\Gamma,\mathrm{tor}}$, we have

$$[b_M^H] \in B(M^H, \mu^\sharp + \kappa(b^*), \mu^\diamond).$$

Let $[b_M^*]_{M^H} \in B(M^H)_{\mathrm{basic}}$ with $\kappa_{M^H}(b_M^*) = \kappa(b^*) \in \pi_1(M^H)_\Gamma$. Then $[b^*] = [b_M^*] \in B(H)$ and we may assume $b^* = b_M^* \in M^H(\breve{F})$. It follows that $M = J_{b_M^*}$ is a pure inner form of M^H, which is also a Levi subgroup of G. Therefore (1) follows.

For (2), by Theorem 4.1, $\mathcal{F}(H, \mu, b^H)^{\mathrm{a}} = \mathcal{F}(H, \mu, b^H)^{\mathrm{wa}}$ if and only if $[b_M^H]$ is basic in H and $B(M^H, \mu^\sharp + \kappa(b^*), \mu^\diamond)$ is fully HN-indecomposable. The latter condition is equivalent to (M, μ) being fully HN-indecomposable. ∎

The rest of the section is devoted to proving Proposition 4.5. Suppose G is quasi-split. In order to distinguish the roots for different groups, we will write Δ_G and $\Delta_{G,0}$ for Δ and Δ_0 respectively. For any $\beta \in \Delta_G$, let M_β be the standard Levi subgroup of G such that $\Delta_{M_\beta} = \Delta_G \backslash \Gamma\beta$. For any $\alpha \in \Delta_{G,0}$, let $M_\alpha := M_\beta$ for any $\beta \in \Delta_G$ such that $\beta_{|A} = \alpha$.

Lemma 4.6. *Suppose G is quasi-split. Let $[b] \in B(G, \mu^\sharp + \varepsilon, \mu^\diamond)$ be a non-basic element, where $\varepsilon \in \pi_1(G)_{\Gamma,\mathrm{tor}}$. Suppose (G, μ, b) is HN-indecomposable. Then there exist $\tilde{\alpha} \in \Delta_0$, $w \in W$ and $x \in \mathcal{F}(M_{\tilde{\alpha}}, w\mu)(C)$ satisfying the following conditions:*

(1) $\langle \tilde{\alpha}, [\nu_b] \rangle > 0$,

(2) *$w\mu$ is $M_{\tilde\alpha}$-dominant,*

(3) $\varepsilon \in \mathrm{Im}(\pi_1(M_{\tilde\alpha})_{\Gamma,\mathrm{tor}} \hookrightarrow \pi_1(G)_{\Gamma,\mathrm{tor}})$ *(hence we may view $\varepsilon \in \pi_1(M_{\tilde\alpha})_{\Gamma,\mathrm{tor}}$),*

(4) $\mathcal{E}_{b_{M_{\tilde\alpha}},x} \simeq \mathcal{E}_{b'_{M_{\tilde\alpha}}}$*, where $b_{M_{\tilde\alpha}}$ is the reduction of b to $M_{\tilde\alpha}$ deduced from its canonical reduction b_{M_b} to M_b combined with the inclusion $M_b \subseteq M_{\tilde\alpha}$, and $b'_{M_{\tilde\alpha}} \in B(M_{\tilde\alpha})_{\mathrm{basic}}$ is such that $\kappa_{M_{\tilde\alpha}}(b'_{M_{\tilde\alpha}}) = -(\tilde\beta^\vee)^\sharp + \varepsilon$ with $\tilde\beta \in \Delta$ and $\tilde\beta_{|A} = \tilde\alpha$.*

Proof. By Lemma 1.8, $[b_{M_b}] \in B(M_b, (w_1\mu)^\sharp + \varepsilon, (w_1\mu)^\diamond)$ for some $w_1 \in W$. Moreover, $\varepsilon \in \mathrm{Im}(\pi_1(M_b)_{\Gamma,\mathrm{tor}} \hookrightarrow \pi_1(G)_{\Gamma,\mathrm{tor}})$. Hence (3) holds for any $M_{\tilde\alpha}$ containing M_b. For any $\alpha \in \Delta_{G,0}$ such that $\langle\alpha, [\nu_b]\rangle > 0$, let $[b_{M_\alpha}]$ be the image of $[b_{M_b}]$ via the natural map $B(M_b) \to B(M_\alpha)$. Then $[b_{M_\alpha}] \in A(M_\alpha, (w_1\mu)_{M_\alpha\text{-dom}})$. As (b, μ) is not HN-decomposable, we have $(w_1\mu)_{M_\alpha\text{-dom}} \neq \mu$. Therefore there exists $\beta \in \Delta_G$ such that $\beta_{|A} = \alpha$ and $\langle\beta, (w_1\mu)_{M_\alpha\text{-dom}}\rangle < 0$. Let

$$R_\alpha := \{(s_\beta(w_1\mu)_{M_\alpha\text{-dom}})_{M_\alpha\text{-dom}} \in W\mu \mid \beta \in \Delta_G, \beta_{|A} = \alpha, \langle\beta, (w_1\mu)_{M_\alpha\text{-dom}}\rangle < 0\},$$

where $s_\beta \in W$ is the reflection corresponding to β. Let $w\mu$ be a maximal element in the subset

$$\bigcup_{\alpha\in\Delta_{G,0},\, \langle\alpha,[\nu_b]\rangle>0} R_\alpha \subset W\mu.$$

Suppose $w\mu \in R_\alpha$ for any $\alpha \in \Delta_{G,0}$. Then $(\alpha, w\mu)$ satisfies (1) and (2). It remains to find $x \in \mathcal{F}(M_{\tilde\alpha}, w\mu)(C)$ satisfying condition (4) for some $\tilde\alpha$ with $w\mu \in R_{\tilde\alpha}$.

This is equivalent to finding $\tilde\alpha \in \Delta_{G,0}$ and $y \in \mathcal{F}(M_{\tilde\alpha}, (w\mu)^{-1})(C)$ such that $\mathcal{E}_{b_{M_{\tilde\alpha}}} \simeq \mathcal{E}_{b'_{M_{\tilde\alpha}},y}$ and $w\mu \in R_{\tilde\alpha}$.

By Proposition 1.14, we need to find $\tilde\alpha \in \Delta_{G,0}$ such that $w\mu \in R_{\tilde\alpha}$ and

$$[b_{M_{\tilde\alpha}}] \in B(M_{\tilde\alpha}, \kappa_{M_{\tilde\alpha}}(b'_{M_{\tilde\alpha}}) + (w\mu)^\sharp, \nu_{b'_{M_{\tilde\alpha}}}(w\mu)^\diamond).$$

By Lemma 1.7, the latter is equivalent to the conditions

$$\kappa_{M_{\tilde\alpha}}(b_{M_{\tilde\alpha}}) = (w\mu - \tilde\beta^\vee)^\sharp + \varepsilon \quad \text{in } \pi_1(M_{\tilde\alpha})_\Gamma, \tag{4.0.1}$$

$$\kappa_{M_b}(b_{M_b}) \preceq_{M_b} -\mathrm{Av}_\Gamma \mathrm{Av}_{W_{M_{\tilde\alpha}}}(\tilde\beta^\vee) + \mathrm{Av}_\Gamma(w\mu) \quad \text{in } \pi_1(M_b)_{\Gamma,\mathbb{Q}}. \tag{4.0.2}$$

Let $(\beta_j)_{j\in J}$ be a set of representatives of Galois orbits in $\Delta_G\backslash\Delta_{M_b}$. Let

$$w\mu - w_1\mu = \sum_{j\in J} n_j\beta_j^\vee \quad \text{in } \pi_1(M_b)_\Gamma$$

with $n_j \in \mathbb{N}$ for all $j \in J$.

Claim 1. *$n_j \geq 1$ for all $j \in J$.*

Suppose $n_{j_0} = 0$ for some $j_0 \in J$. Let $\alpha_0 = \beta_{j_0|A}$. Then $(w\mu)_{M_{\alpha_0}\text{-dom}} = (w_1\mu)_{M_{\alpha_0}\text{-dom}}$. Again by HN-indecomposability, $(w\mu)_{M_{\alpha_0}\text{-dom}} \neq \mu$. Then there exists $\beta \in \Gamma\beta_{j_0}$ such that $\langle\beta, (w\mu)_{M_{\alpha_0}\text{-dom}}\rangle < 0$. It follows that

$$(s_\beta(w_1\mu)_{M_{\alpha_0}\text{-dom}})_{M_{\alpha_0}\text{-dom}} \not\preceq w\mu.$$

This contradicts the maximality of $w\mu$. Hence Claim 1 follows.

By definition, suppose $w\mu \in R_\alpha$ for some α and $w\mu = (s_\beta(w_1\mu)_{M_\alpha\text{-dom}})_{M_\alpha\text{-dom}}$, where $\alpha \in \Delta_{G,0}$ with $\langle \alpha, [\nu_b] \rangle > 0$ and $\beta \in \Delta_G$ with $\beta_{|A} = \alpha$. Suppose $\beta \in \Gamma\beta_{j_0}$ for some $j_0 \in J$. Then $n_{j_0} = 1$ by the definition of $w\mu$.

The subset
$$J_1 := \{j \in J \mid n_j = 1\}$$
of J consists of a single element j_0. Let $\tilde{\alpha} := \alpha$ and $\tilde{\beta} := \beta$. We will verify that $\tilde{\alpha}$ has the desired properties. Since $[b_{M_b}] \in B(M_b, (w_1\mu)^\sharp + \varepsilon, (w_1\mu)^\diamond)$, we have
$$\kappa_{M_{\tilde{\alpha}}}(b_{M_{\tilde{\alpha}}}) = (w_1\mu)^\sharp + \varepsilon = (w\mu - \tilde{\beta}^\vee)^\sharp + \varepsilon \quad \text{in } \pi_1(M_{\tilde{\alpha}})_\Gamma,$$
which is (4.0.1). For (4.0.2), we have
$$\mathrm{Av}_\Gamma(w\mu) - \kappa_{M_b}(b_{M_b}) - \mathrm{Av}_\Gamma \mathrm{Av}_{W_{M_{\tilde{\alpha}}}}(\tilde{\beta}^\vee) = (w\mu - w_1\mu) - \mathrm{Av}_{W_{M_{\tilde{\alpha}}}}(\tilde{\beta}^\vee) \succcurlyeq_{M_b} 0$$
in $\pi_1(M_b)_{\Gamma,\mathbb{Q}}$, where the last inequality follows from Lemma 4.7 (1) (because the E_8 case does not occur as there are only trivial minuscule cocharacters in that case) combined with the fact that $n_j \geq 2$ for any $j \in J \setminus \{j_0\}$.

It remains to deal with the case when J_1 has at least two elements. By Claim 1, for any $j \in J$, up to replacing β_j by some other representative in the same Galois orbit, we may assume β_j appears in the linear combination of $w\mu - w_1\mu$.

Claim 2. *$w\mu = \mu = (s_{\beta_{j'}}(w_1\mu)_{M_{\beta_{j'}}\text{-dom}})_{M_{\beta_{j'}}\text{-dom}}$ for any $j' \in J_1$.*

We want to show $w\mu$ is G-dominant. Suppose $j_0' \in J_1$ with $j_0 \neq j_0'$. Let $w\mu = w_3 s_{\beta_{j_0'}} w_2 w_1 \mu$ with $w_2, w_3 \in W_{M_{\beta_{j_0'}}}$. Then
$$(s_{\beta_{j_0'}}(w_1\mu)_{M_{\beta_{j_0'}}\text{-dom}})_{M_{\beta_{j_0'}}\text{-dom}} \succcurlyeq s_{\beta_{j_0'}} w_2 w_1 \mu.$$
Since both sides are in $W\mu$ with difference a linear combination of coroots in $M_{\beta_{j_0'}}$, we have
$$\bigcup_\alpha R_\alpha \ni (s_{\beta_{j_0'}}(w_1\mu)_{M_{\beta_{j_0'}}\text{-dom}})_{M_{\beta_{j_0'}}\text{-dom}} = (s_{\beta_{j_0'}} w_2 w_1 \mu)_{M_{\beta_{j_0'}}\text{-dom}} \succcurlyeq w\mu.$$
By the maximality of $w\mu$, we deduce that $w\mu$ is both $M_{\beta_{j_0}}$-dominant and $M_{\beta_{j_0'}}$-dominant. Therefore
$$w\mu = (s_{\beta_{j_0'}} w_2 w_1 \mu)_{M_{\beta_{j_0'}}\text{-dom}}$$
is G-dominant and Claim 2 follows.

Let $(I_i)_{0 \leq i \leq r}$ be the increasing sequence of subsets in Δ_G as in Lemma 4.7. Suppose i_0 is the smallest integer such that $\{\beta_j \mid j \in J_1\} \cap I_{i_0}$ is not an empty set. Choose $\tilde{\beta} \in \{\beta_j | j \in J_1\} \cap I_{i_0}$. Let $\tilde{\alpha} := \tilde{\beta}_{|A}$. By the same arguments as before we can verify that $\tilde{\alpha}$ satisfies the condition (4.0.1). For (4.0.2), let
$$\mathrm{Av}_{W_{M_{\tilde{\alpha}}}}(\tilde{\beta}^\vee) := \sum_{j \in J} m_j \beta_j^\vee \quad \text{in } \pi_1(M_b)_{\Gamma,\mathbb{Q}},$$

with $m_j \in \mathbb{Q}$ for all $j \in J$. By Lemma 4.7, $0 \le m_j \le 2$ (as the E_8 case will not occur) for all $j \in J$, and $0 \le m_j \le 1$ for all $j \in J_1$. Then

$$\begin{aligned}\mathrm{Av}_\Gamma(w\mu) - \kappa_{M_b}(b_{M_b}) - \mathrm{Av}_\Gamma \mathrm{Av}_{W_{M_{\tilde\alpha}}}(\tilde\beta^\vee) &= (w\mu - w_1\mu) - \mathrm{Av}_{W_{M_{\tilde\alpha}}}(\tilde\beta^\vee)\\ &= \sum_{j\in J}(n_j - m_j)\beta_j^\vee \succcurlyeq_{M_b} \sum_{j\in J_1}(1-m_j)\beta_j^\vee + \sum_{j\in J\setminus J_1}(2-m_j)\beta_j^\vee \succcurlyeq_{M_b} 0. \quad \blacksquare\end{aligned}$$

Lemma 4.7. *Suppose G is quasi-split. Let $\beta \in \Delta_G$. Suppose*

$$\mathrm{Av}_{M_\beta}(\beta^\vee) = \beta^\vee + \sum_{\gamma\in(\Delta_{M_\beta})_\Gamma} n_{\beta,\gamma}\gamma^\vee \quad \text{in } X_*(T)_{\Gamma,\mathbb{Q}}. \tag{4.0.3}$$

Then:

(1) $0 \le n_{\beta,\gamma} \le 3$ *for all* $\gamma \in (\Delta_G)_\Gamma$. *Moreover, if no connected component of the Dynkin diagram of G is of type E_8, then* $0 \le n_{\beta,\gamma} \le 2$ *for all* $\gamma \in (\Delta_G)_\Gamma$.

(2) *There exist $r \in \mathbb{N}_{\ge 1}$ and an increasing sequence*

$$\emptyset = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = \Delta_G$$

of Γ-invariant subsets such that if $\beta \in I_i$ for some i, then $n_{\beta,\gamma} \le 1$ for all $\gamma \notin (I_{i-1})_\Gamma$. In particular, if $r = 1$, then $n_{\beta,\gamma} \le 1$ for all $\beta \in \Delta_G$ and $\gamma \in (\Delta_G)_\Gamma$.

Proof. This lemma only depends on the absolute root system of G with Galois action. After considering separately each connected component of the Dynkin diagram of G, we may assume the Dynkin diagram of G is connected. The first assertion can be checked directly case by case. Indeed, it suffices to compute explicitly all the $n_{\beta,\gamma}$ in (4.0.3). As $\langle \mathrm{Av}_{M_\beta}(\beta^\vee), \alpha\rangle = 0$ for any $\alpha \in \Delta_{M_\beta}$, it follows that

$$\langle \beta^\vee, \alpha\rangle + \sum_{\gamma\in(\Delta_{M_\beta})_\Gamma} n_{\beta,\gamma}\langle\gamma^\vee,\alpha\rangle = 0$$

for any $\alpha \in \Delta_{M_\beta}$. Then $\{n_{\beta,\gamma}\}$ is the unique solution of this system of linear equations.

For the second assertion, we consider the increasing sequence of Γ-invariant subsets in Δ_G case by case according to the type of the Dynkin diagram of G. We can check directly that this increasing sequence of subsets has the desired property. We leave the details of the verification to the readers.

Case A_n: In the ${}^1A_n$ case or ${}^2A_n$ case with n even, take $r = 1$. Otherwise, we are in the ${}^2A_n$ case with n odd; then take $r = 2$ and $I_1 = \Delta_G \setminus \{\beta\}$ where β is the unique Γ-invariant root in Δ_G.

Case B_n: Take $r = 2$ and I_1 the subset of long roots in Δ_G.

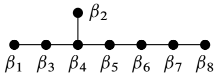
Case C_n: Take $r = 1$.

Case D_n: In the ${}^1D_n$ case with $n > 4$, take $r = 2$ and let $\Delta_G \setminus I_1$ consist of two roots which are the end points of the Dynkin diagram and are neighbors to the same simple root.

In the ${}^2D_n$ case or ${}^1D_4$ case, take $r = 1$.

In the ${}^3D_4$ case, take $r = 2$ and $I_1 = \Delta_G \setminus \{\beta\}$ where β is the unique Γ-invariant root in Δ_G.

Case E_n: Suppose the Dynkin diagram of E_8 is



In the E_7 (resp. E_6) case, we remove β_8 (resp. β_7 and β_8) in the diagram.

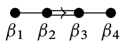
In the ${}^1E_6$ case, take $r = 3$, $I_1 = \{\beta_4\}$, $I_2 = \{\beta_2, \beta_3, \beta_4, \beta_5\}$.

In the ${}^2E_6$ case, take $r = 4$, $I_1 = \{\beta_3, \beta_5\}$, $I_2 = I_1 \cup \{\beta_1, \beta_6\}$, $I_3 = I_2 \cup \{\beta_4\}$.

In the E_7 case, take $r = 4$, $I_1 = \{\beta_4\}$, $I_2 = \{\beta_3, \beta_4, \beta_5\}$, $I_3 = \Delta_G \setminus \{\beta_7\}$.

In the E_8 case, take $r = 5$, $I_1 = \{\beta_4\}$, $I_2 = \{\beta_4, \beta_5\}$, $I_3 := I_2 \cup \{\beta_3, \beta_6\}$, $I_4 = \Delta_G \setminus \{\beta_8\}$.

Case F_4: Suppose the Dynkin diagram is



Take $r = 3$, $I_1 = \{\beta_2\}$, $I_2 = \{\beta_1, \beta_2, \beta_3\}$.

Case G_2: Take $r = 2$, I_1 the set consisting of the unique long root in Δ_G. ■

Now we can prove Proposition 4.5.

Proof of Proposition 4.5. As b is not basic, there exist $\alpha \in \Delta_0$, $w \in W$ and $x \in \mathcal{F}(M_{\tilde{\alpha}}, w\mu)(C)$ satisfying properties (1)–(3) in Lemma 4.6. Let $b_{M_{\tilde{\alpha}}}$ and $b'_{M_{\tilde{\alpha}}}$ be as in that lemma.

Let $x_G \in \mathcal{F}(G, \mu)(C)$ be the image of x via the natural morphism

$$\mathcal{F}(M_{\tilde{\alpha}}, w\mu)(C) \to \mathcal{F}(G, \mu)(C), \quad a(M_{\tilde{\alpha}} \cap wP_\mu w^{-1}) \mapsto awP_\mu.$$

Then $\mathcal{E}_{b,x_G} \simeq \mathcal{E}_{b'_{M_{\tilde{\alpha}}}} \times^{M_{\tilde{\alpha}}} G \simeq \mathcal{E}_{b'}$ where $[b'] \in B(G)$ is the image of $[b'_{M_{\tilde{\alpha}}}] \in B(M_{\tilde{\alpha}})$ via the natural map $B(M_{\tilde{\alpha}}) \to B(G)$. Hence $x_G \notin \mathcal{F}^{\mathrm{a}}(C)$. Moreover, $x_G \notin \mathcal{F}^{\mathrm{wa}}(C)$. Indeed, the canonical reduction $(\mathcal{E}_{b'})_{P_{\tilde{\alpha}}}$ of $\mathcal{E}_{b'}$ to the standard parabolic subgroup $P_{\tilde{\alpha}}$ corresponding to $M_{\tilde{\alpha}}$ induces the reduction $\mathcal{E}_{b_{M_{\tilde{\alpha}}}} \times^{M_{\tilde{\alpha}}} P_{\tilde{\alpha}}$ of $\mathcal{E}_b$ to $P_{\tilde{\alpha}}$. Take $\chi \in X^*(P_{\tilde{\alpha}}/Z_G)^+$. Then $\deg \chi_*(\mathcal{E}_{b'})_{P_{\tilde{\alpha}}} > 0$.

For any element $\gamma \in \tilde{J}_b(K)$, we have $\gamma(x_G) \notin \mathcal{F}^{\mathrm{a}}(C)$. Choose $\gamma \in \tilde{J}_b^{\geq \lambda_{\max}}(K) \setminus \{1\}$ with $\lambda_{\max} = \max_{\gamma \in \Phi} \langle \nu_b, \gamma \rangle$; it remains to show that $\gamma(x_G) \in \mathcal{F}^{\mathrm{wa}}(C)$.

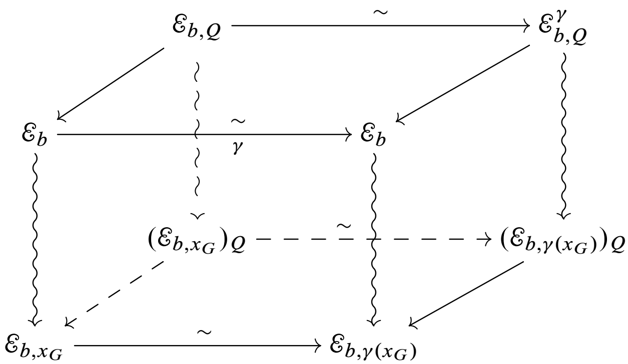
Suppose the pair $(b, \gamma(x_G))$ is not weakly admissible. There exist a standard maximal parabolic subgroup Q, a reduction b_{M_Q} of b to the Levi component M_Q of Q and $\chi \in X^*(Q/Z_G)^+$ such that

$$\deg \chi_*(\mathcal{E}_{b,\gamma(x_G)})_Q > 0,$$

where $(\mathcal{E}_{b,\gamma(x_G)})_Q$ is the reduction of $\mathcal{E}_{b,\gamma(x_G)}$ to Q induced by a reduction $\mathcal{E}^{\gamma}_{b,Q}$ of $\mathcal{E}_b$ to Q, where $\mathcal{E}^{\gamma}_{b,Q} := \mathcal{E}_{\tilde{b}_{M_Q}} \times^{M_Q} Q$ is induced by a reduction $\tilde{b}_{M_Q}$ of b to M_Q (and hence to Q). The isomorphism $\gamma : \mathcal{E}_b \xrightarrow{\sim} \mathcal{E}_b$ induces an isomorphism $\mathcal{E}_b/Q \xrightarrow{\sim} \mathcal{E}_b/Q$, hence by Remark 2.2, $\mathcal{E}_{b,Q}$ induces a reduction $\mathcal{E}^{\gamma}_{b,Q}$ of $\mathcal{E}_b$ to Q and an isomorphism $\mathcal{E}_{b,Q} \xrightarrow{\sim} \mathcal{E}^{\gamma}_{b,Q}$ yielding the commutative diagram

$$\begin{array}{ccc} \mathcal{E}_b & \xrightarrow[\sim]{\gamma} & \mathcal{E}_b \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathcal{E}_{b,Q} \times^Q G & \xrightarrow{\sim} & \mathcal{E}^{\gamma}_{b,Q} \times^Q G \end{array}$$

Suppose the reduction $(\mathcal{E}_{b,x_G})_Q$ of $\mathcal{E}_{b,x_G}$ to Q is induced by the reduction $(\mathcal{E}_{b,\gamma(x_G)})_Q$ of $\mathcal{E}_{b,\gamma(x_G)}$ to Q by Lemma 1.13 and Remark 2.2. We get a cubic commutative diagram



where the vertices of the front face are G-bundles and the vertices of the back face are the corresponding reductions of the G-bundles to Q; the vertical waved arrows denote modifications of G- or Q-bundles. It follows that $\deg \chi_*(\mathcal{E}_{b,x_G})_Q = \deg \chi_*(\mathcal{E}_{b,\gamma(x_G)})_Q > 0$.

According to Theorem 2.7 the vector

$$v : X^*(Q/Z_G) \to \mathbb{Z}, \quad \chi \mapsto \deg \chi_*(\mathcal{E}_{b,x_G})_Q,$$

seen as an element of $X_*(A)_{\mathbb{Q}}$, satisfies $v \le \nu_{\mathcal{E}_{b,x_G}} = \nu_{\mathcal{E}_{b'}}$ with $v \neq 0$. As $\nu_{\mathcal{E}_{b'}} \in \mathcal{N}_G \setminus \{0\}$ is minimal, one deduces that $v = \nu_{b,x_G}$, $Q = P_{\tilde{\alpha}}$ and $(\mathcal{E}_{b,x_G})_Q$ is the HN canonical reduction of $\mathcal{E}_{b,x_G}$. Therefore $\mathcal{E}_{b,Q} = \mathcal{E}_{b_{M_{\tilde{\alpha}}}} \times^{M_{\tilde{\alpha}}} P_{\tilde{\alpha}}$ is a reduction of $\mathcal{E}_b$ to Q. Pushing forward via the natural projection $P_{\tilde{\alpha}} \to M_{\tilde{\alpha}}$, the isomorphism of $P_{\tilde{\alpha}}$-bundles $\mathcal{E}_{b,P_{\tilde{\alpha}}} \simeq \mathcal{E}^{\gamma}_{b,P_{\tilde{\alpha}}}$ induces an isomorphism of $M_{\tilde{\alpha}}$-bundles, hence we have $[b_{M_{\tilde{\alpha}}}] = [\tilde{b}_{M_Q}] \in B(M_{\tilde{\alpha}})$. According to Lemma 4.8 below, the two reductions $b_{M_{\tilde{\alpha}}}$ and $\tilde{b}_{M_{\tilde{\alpha}}} := \tilde{b}_{M_Q}$ of b to M_Q are equivalent. In particular, the reductions $\mathcal{E}_{b,Q}$ and $\mathcal{E}^{\gamma}_{b,Q}$ of $\mathcal{E}_b$ to Q are equivalent. Hence they give the same vector subbundle

$$\mathcal{E}_{b,Q} \times^{Q,\mathrm{Ad}} \mathrm{Lie}\, Q = \mathcal{E}^{\gamma}_{b,Q} \times^{Q,\mathrm{Ad}} \mathrm{Lie}\, Q$$

over X of $\mathrm{Ad}(\mathcal{E}_b) = \mathcal{E}_b \times^{G,\mathrm{Ad}} \mathfrak{g}$. By Theorem 1.9, this subbundle corresponds to a $\bar{B}$-submodule $\mathfrak{q}$ in $\mathfrak{g}_{\bar{B}}$ which is stable under the action of $\mathrm{Ad}(\gamma)$, where we identify $\tilde{J}_b(K)$ with a subgroup of $G(\bar{B})$ (cf. Section 1.5). Recall that in loc. cit., we have also defined a filtration $(\mathfrak{g}_{\bar{B}}^{\geq\lambda})_{\lambda\in\mathbb{Q}}$ on $\mathfrak{g}_{\bar{B}}$. As $\nu_{b_{M_{\tilde{\alpha}}}} = \nu_b$, the non-zero elements of $\mathfrak{q}$ are in the $-\langle\gamma, \nu_b\rangle$ graded pieces of $\mathfrak{g}_{\bar{B}}$ for some absolute root γ in $Q = P_{\tilde{\alpha}}$. In particular, $\mathfrak{q} \cap \mathfrak{g}_{\bar{B}}^{\geq\lambda_{\max}} = 0$. On the other hand, since $\gamma \neq 1$, we can always choose an element $y \in \mathfrak{q} \cap \mathfrak{g}_{\bar{B}}^{\geq 0}$ such that $\mathrm{Ad}(\gamma)(y) \neq y$. Note that

$$0 \neq \mathrm{Ad}(\gamma)(y) - y \in \mathfrak{g}_{\bar{B}}^{\geq\lambda_{\max}},$$

which implies $\mathfrak{q} \cap \mathfrak{g}_{\bar{B}}^{\geq\lambda_{\max}} \neq 0$. We get a contradiction. ∎

Lemma 4.8. *Suppose $b \in G(\breve{F})$. Suppose $\tilde{\alpha} \in \Delta_0$ is such that $\langle\tilde{\alpha}, \nu_b\rangle > 0$. Let $(b_{M_{\tilde{\alpha}}}, g)$ and $(\tilde{b}_{M_{\tilde{\alpha}}}, \tilde{g})$ be two reductions of b to $M_{\tilde{\alpha}}$ with $[b_{M_{\tilde{\alpha}}}] = [\tilde{b}_{M_{\tilde{\alpha}}}] \in B(M_{\tilde{\alpha}})$ and $\nu_{b_{M_{\tilde{\alpha}}}} = \nu_b$. Then these reductions are equivalent.*

Proof. As $[b_{M_{\tilde{\alpha}}}] = [\tilde{b}_{M_{\tilde{\alpha}}}] \in B(M_{\tilde{\alpha}})$, we may assume $b_{M_{\tilde{\alpha}}} = \tilde{b}_{M_{\tilde{\alpha}}}$. Then

$$\tilde{g}g^{-1} \in J_b = \{h \in G(\breve{F}) \mid b\sigma(h) = hb\}.$$

Since $\langle\tilde{\alpha}, \nu_b\rangle > 0$, $J_b \subseteq M_{\tilde{\alpha}}(\breve{F})$. It follows that $(b_{M_{\tilde{\alpha}}}, g)$ and $(\tilde{b}_{M_{\tilde{\alpha}}}, \tilde{g})$ are equivalent. ∎

5. Newton stratification and weakly admissible locus

In this section, we suppose G is quasi-split and $[b] \in A(G, \mu)$ is basic. Under this condition, the proof of [6, Theorem 6.1] in fact shows the following finer result.

Theorem 5.1 ([6]). *Let $[b'] \in B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1})$.*

(1) *If $(G, \nu_b(w_0\mu^{-1})^\diamond, b')$ is HN-decomposable, then $\mathcal{F}(G, b, \mu)^{[b']} \cap \mathcal{F}(G, b, \mu)^{\mathrm{wa}} = \emptyset$.*
(2) *If $(G, \nu_b(w_0\mu^{-1})^\diamond, b')$ is HN-indecomposable and $[b']$ a minimal element in the set $B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1}) \setminus [1]$ for the dominance order, then $\mathcal{F}(G, b, \mu)^{[b']} \cap \mathcal{F}(G, b, \mu)^{\mathrm{wa}} \neq \emptyset$.*

Inspired by this theorem, we make the following conjecture.

Conjecture 5.2. *Suppose $[b'] \in B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1})$ with $(G, \nu_b(w_0\mu^{-1})^\diamond, b')$ HN-indecomposable. Then*

$$\mathcal{F}(G, b, \mu)^{[b']} \cap \mathcal{F}(G, b, \mu)^{\mathrm{wa}} \neq \emptyset.$$

Remark 5.3. This conjecture has been completely proved by Viehmann [41] very recently.

In the rest of the section, we will prove this conjecture for the linear algebraic groups for special μ.

For $r, s \in \mathbb{Z}$ with $r > 0$, let $\mathcal{O}([\frac{s}{r}]) := \mathcal{O}(\frac{s}{r})^d$ if $d = (s, r)$. Then $\deg \mathcal{O}([\frac{s}{r}]) = s$ and $\mathrm{rank}\, \mathcal{O}([\frac{s}{r}]) = r$.

Proposition 5.4. *Let $G = \mathrm{GL}_n$. Suppose $[b'] \in B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1})$ with $(G, \nu_b(w_0\mu^{-1})^\diamond, b')$ HN-indecomposable. If $\mathcal{E}_{b'} \simeq \mathcal{O}([\frac{2}{r_1}]) \oplus \mathcal{O}([\frac{-2}{r_2}])$ or $\mathcal{O}([\frac{1}{r_1}]) \oplus \mathcal{O}([\frac{0}{r_2}]) \oplus \mathcal{O}([\frac{-1}{r_3}])$ for some $r_1, r_2, r_3 > 0$, then*

$$\mathcal{F}(G,b,\mu)^{[b']} \cap \mathcal{F}(G,b,\mu)^{\mathrm{wa}} \neq \emptyset.$$

In particular, Conjecture 5.2 holds when $\mu = (1^{(r)}, 0^{(n-r)})$ with $r(n-r) \leq 2n$.

Proof. For the last assertion, if $\mu = (1^{(r)}, 0^{(n-r)})$ with $r(n-r) \leq 2n$, then by Proposition 1.14, $[b'] \in B(G, \kappa_G(b) - \mu^\sharp, \nu_b\mu^{-1}) \setminus [1]$ implies $\mathcal{E}_{b'} \simeq \mathcal{O}([\frac{2}{r_1}]) \oplus \mathcal{O}([\frac{-2}{r_2}])$ or $\mathcal{O}([\frac{1}{r_1}]) \oplus \mathcal{O}([\frac{0}{r_2}]) \oplus \mathcal{O}([\frac{-1}{r_3}])$ for some $r_1, r_3 > 0$ and $r_2 \geq 0$. So the last assertion follows from the first one combined with Theorem 5.1 (when $r_2 = 0$). Now it remains to prove the first assertion.

We claim that there exists an exact sequence of vector bundles

$$0 \to \mathcal{E}' \to \mathcal{E}_b \to \mathcal{E}'' \to 0 \tag{5.0.1}$$

satisfying a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E}_b & \longrightarrow & \mathcal{E}'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \tilde{\mathcal{E}}' & \longrightarrow & \mathcal{E}_{b'} & \longrightarrow & \tilde{\mathcal{E}}'' & \longrightarrow & 0 \end{array}$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are semistable vector bundles and the vertical arrows are the modifications of minuscule type. Indeed, suppose $\mathcal{E}_b = \mathcal{O}([\frac{s}{r}])$. If $\mathcal{E}_{b'} \simeq \mathcal{O}([\frac{2}{r_1}]) \oplus \mathcal{O}([\frac{-2}{r_2}])$, then $r = r_1 + r_2$. Let $\tilde{\mathcal{E}}' = \mathcal{O}([\frac{2}{r_1}])$ and $\tilde{\mathcal{E}}'' = \mathcal{O}([\frac{-2}{r_2}])$. If $s \leq r_2$, then let $\mathcal{E}' = \mathcal{O}([\frac{2}{r_1}])$ and $\mathcal{E}'' = \mathcal{O}([\frac{s-2}{r_2}])$. Otherwise, let $\mathcal{E}' = \mathcal{O}([\frac{s+2-r_2}{r_1}])$ and $\mathcal{E}'' = \mathcal{O}([\frac{r_2-2}{r_2}])$. If $\mathcal{E}'_b \simeq \mathcal{O}([\frac{1}{r_1}]) \oplus \mathcal{O}([\frac{0}{r_2}]) \oplus \mathcal{O}([\frac{-1}{r_3}])$, then $r_1 + r_2 + r_3 = r$. We can easily check that one of the following two inequalities holds:

$$\frac{s-1}{r-r_1} \leq \frac{r_3-1}{r_3}, \quad \frac{s-r_3+1}{r-r_3} \geq \frac{1}{r_1}.$$

(Otherwise, the inequalities give upper and lower bounds for s. The comparison of the two bounds leads to a contradiction $r < r_1 + r_3$). In the former case, let

$$\mathcal{E}' = \mathcal{O}\big(\big[\tfrac{1}{r_1}\big]\big), \quad \mathcal{E}'' = \mathcal{O}\big(\big[\tfrac{s-1}{r-r_1}\big]\big),$$
$$\tilde{\mathcal{E}}' = \mathcal{O}\big(\big[\tfrac{1}{r_1}\big]\big), \quad \tilde{\mathcal{E}}'' = \mathcal{O}\big(\big[\tfrac{0}{r_2}\big]\big) \oplus \mathcal{O}\big(\big[\tfrac{-1}{r_3}\big]\big).$$

In the latter case, let

$$\mathcal{E}' = \mathcal{O}\big(\big[\tfrac{s-r_3+1}{r-r_3}\big]\big), \qquad \mathcal{E}'' = \mathcal{O}\big(\big[\tfrac{r_3-1}{r_3}\big]\big),$$
$$\tilde{\mathcal{E}}' = \mathcal{O}\big(\big[\tfrac{1}{r_1}\big]\big) \oplus \mathcal{O}\big(\big[\tfrac{0}{r_2}\big]\big), \quad \tilde{\mathcal{E}}'' = \mathcal{O}\big(\big[\tfrac{-1}{r_3}\big]\big).$$

The existence of the extension (5.0.1) is due to [1], and the existence of the modifications given by the left and right vertical arrows is given by Proposition 1.14. The vertical arrow in the middle of the commutative diagram gives a point $x \in \mathcal{F}(G,b,\mu)^{[b']}(C)$. It suffices to show $x \in \mathcal{F}^{\mathrm{wa}}(C)$. Suppose $\mathcal{G} \subseteq \mathcal{E}_{b'}$ is any vector subbundle of $\mathcal{E}_{b'}$ corresponding to a reduction $(\mathcal{E}_{b,x})_P$ of $\mathcal{E}_{b'}$ to a maximal parabolic subgroup P such that $\deg \mathcal{G} > 0$. Let $\mathcal{G}' \subseteq \mathcal{E}_b$ be the corresponding vector subbundle of $\mathcal{E}_b$ corresponding to the reduction $(\mathcal{E}_b)_P$ induced by $(\mathcal{E}_{b,x})_P$. We want to show that $\mathcal{G}'$ does not come from subisocrystals.

Suppose $\mathcal{G}'$ comes from subisocrystals. The fact that $\deg \mathcal{G} > 0$ combined with the particular choice of b' implies that $\mathcal{G} \supseteq \tilde{\mathcal{E}}'$ or $\mathcal{G} \subseteq \tilde{\mathcal{E}}'$. Then $\mathcal{G}' \supseteq \mathcal{E}'$ or $\mathcal{G}' \subseteq \mathcal{E}'$. The latter case is obviously impossible. For the former, $\mathcal{E}'' = \mathcal{E}_b/\mathcal{E}'$ must have a direct summand of slope s/r, which is also impossible as $\mathcal{E}''$ is semistable. ■

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