

# Boolean Valued Decomposition Theory of States

By

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## Abstract

Boolean valued analysis (i.e., analysis based on Boolean valued set theory) enables us to introduce a natural framework for decompositions of states over  $C^*$ -algebras with identity. The development apparently runs parallel with the traditional approach originating in von Neumann's reduction theory, but our logical approach succeeds in getting rid of separability restrictions that have haunted the traditional approach.

## §1. Introduction

For all its simple and intuitive nature, direct integral theory invented by von Neumann, one of the most revered mathematical figures of this century, is cumbersome in semblance and has been obsessed with separability restrictions inherited from measure-theoretic techniques it exploits. There has long been a strong tendency to make this theory trimmer and more widely applicable by using Boolean algebras and the like (notable Segal [7] and Tomita [10, 11] as pioneering work in this vein), but it is very recently that we realized the significance of Boolean valued set theory, which has been applied successfully to independence problems of axiomatic set theory, but on which this tendency is finally to be based. Indeed Ozawa [6] and Takeuti [8, 9] have applied this new technique, called *Boolean valued analysis*, to some rudiments of Hilbert space theory and the theory of von Neumann algebras.

The main purpose of this paper is to approach to the rudiments of decomposition theory of states over  $C^*$ -algebras with identity by rolling our Boolean valued juggernaut. Our approach, like its predecessor, is based on two principal facts, say, the one-to-one correspondence between states and cyclic representations on the one hand and that between orthogonal measures and their commutative von Neumann algebras on the other. These results as well as the fundamentals of Boolean valued analysis are reviewed in Section 2. The main results are presented in Section 3, followed by their ergodic counterparts in Section 4.

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## § 2. Preliminaries

### 2.1. States

Let  $\mathfrak{A}$  be a  $C^*$ -algebra with identity 1. A linear functional  $\omega$  over  $\mathfrak{A}$  is called *positive* if

$$\omega(A^*A) \geq 0$$

for any  $A \in \mathfrak{A}$ . A positive linear functional  $\omega$  over  $\mathfrak{A}$  with  $\omega(1)=1$  is called a *state*. We denote by  $E_{\mathfrak{A}}$  the set of all states over  $\mathfrak{A}$ , which is convex and weakly\* compact. Given two positive linear functionals  $\omega_1, \omega_2$  over  $\mathfrak{A}$ , we write  $\omega_1 \geq \omega_2$  and say that  $\omega_1$  *majorizes*  $\omega_2$ , provided  $\omega_1 - \omega_2$  is positive. A state  $\omega$  over  $\mathfrak{A}$  is called *pure* if every positive linear functional majorized by  $\omega$  is of the form  $\lambda\omega$  with  $0 \leq \lambda \leq 1$ .

Every nondegenerate representation  $\pi$  of  $\mathfrak{A}$  on a Hilbert space  $\mathfrak{H}$  and every vector  $\Omega$  in  $\mathfrak{H}$  with  $\|\Omega\|=1$  give rise to a linear functional

$$\omega_{\Omega}(A) = (\Omega, \pi(A)\Omega),$$

which is easily seen to be a state. It is known that this construction gives a one-to-one correspondence between states on  $\mathfrak{A}$  and cyclic representations of  $\mathfrak{A}$  up to unitary equivalence. Indeed, given a state  $\omega$  on  $\mathfrak{A}$ , there is a canonical construction of a cyclic representation  $\pi_{\omega}$  of  $\mathfrak{A}$  on a Hilbert space  $\mathfrak{H}_{\omega}$  with a unit vector  $\Omega_{\omega} \in \mathfrak{H}_{\omega}$  such that

$$\omega(A) = (\Omega_{\omega}, \pi_{\omega}(A)\Omega_{\omega})$$

for any  $A \in \mathfrak{A}$ . This cyclic representation  $(\mathfrak{H}_{\omega}, \pi_{\omega}, \Omega_{\omega})$  is called the *canonical cyclic representation of  $\mathfrak{A}$  associated with  $\omega$* , for which we have the following two fundamental theorems.

**Theorem 2.1.1.** *The following three conditions are equivalent :*

- (1)  $(\mathfrak{H}_{\omega}, \pi_{\omega})$  is irreducible ;
- (2)  $\omega$  is pure ;
- (3)  $\omega$  is an extremal point of  $E_{\mathfrak{A}}$ .

**Theorem 2.1.2.** *The correspondence*

$$\omega_T(A) = (\Omega_{\omega}, T\pi_{\omega}(A)\Omega_{\omega})$$

*from positive operators  $T$  in the commutant  $\pi_{\omega}(\mathfrak{A})'$  with  $\|T\| \leq 1$  to positive functionals  $\omega_T$  over  $\mathfrak{A}$  majorized by  $\omega$  is bijective.*

A state  $\omega$  is called a *factor state* if  $\pi_{\omega}(\mathfrak{A})''$  is a factor.

**2.2. Orthogonal Measures**

Two positive linear functionals  $\omega_1, \omega_2$  over  $\mathfrak{A}$  are called *orthogonal* and written  $\omega_1 \perp \omega_2$  provided for any positive linear functional  $\omega$  over  $\mathfrak{A}$ ,  $\omega \leq \omega_1$  and  $\omega \leq \omega_2$  imply  $\omega = 0$ . A regular Borel measure  $\mu$  over  $E_{\mathfrak{A}}$  is called an *orthogonal measure* if for any Borel set  $S \subseteq E_{\mathfrak{A}}$  one has

$$\left(\int_S \omega d\mu(\omega)\right) \perp \left(\int_{E_{\mathfrak{A}}-S} \omega d\mu(\omega)\right).$$

For any  $\omega \in E_{\mathfrak{A}}$ , the set of orthogonal probability measures  $\mu$  on  $E_{\mathfrak{A}}$  with

$$\omega = \int \omega' d\mu(\omega')$$

is denoted by  $\mathcal{O}_{\omega}(E_{\mathfrak{A}})$ .

**Theorem 2.2.1.** *If  $\mu \in \mathcal{O}_{\omega}(E_{\mathfrak{A}})$ , then there exists a \*-isomorphism  $\kappa_{\mu}$  from  $L_{\infty}(\mu)$  to  $\pi_{\omega}(\mathfrak{A})'$  such that for any  $A \in \mathfrak{A}$ ,*

$$\langle \Omega_{\omega}, \kappa_{\mu}(f) \pi_{\omega}(A) \Omega_{\omega} \rangle = \int f(\omega') \omega'(A) d\mu(\omega').$$

**Theorem 2.2.2.** *By assigning to each  $\mu \in \mathcal{O}_{\omega}(E_{\mathfrak{A}})$  the commutative von Neumann subalgebra  $\kappa_{\mu}(L_{\infty}(\mu))$  of  $\pi_{\omega}(\mathfrak{A})'$ , we obtain a bijective correspondence between  $\mathcal{O}_{\omega}(E_{\mathfrak{A}})$  and the commutative von Neumann subalgebras of  $\pi_{\omega}(\mathfrak{A})'$ .*

**2.3. Boolean Valued Analysis**

Let  $\mathcal{B}$  be a complete Boolean algebra. We define  $V_{\alpha}^{(\mathcal{B})}$  by transfinite induction on ordinal  $\alpha$  as follows:

- (1)  $V_0^{(\mathcal{B})} = \phi$ ;
- (2)  $V_{\alpha}^{(\mathcal{B})} = \{u \mid u : \mathcal{D}(u) \rightarrow \mathcal{B} \text{ and } \mathcal{D}(u) \subseteq \bigcup_{\xi < \alpha} V_{\xi}^{(\mathcal{B})}\}$ .

Then the Boolean valued universe  $V^{(\mathcal{B})}$  of Scott and Solovay is defined as follows:

$$V^{(\mathcal{B})} = \bigcup_{\alpha \in \text{On}} V_{\alpha}^{(\mathcal{B})}, \text{ where On is the class of all ordinal numbers.}$$

$V^{(\mathcal{B})}$  can be considered to be a Boolean valued model of set theory by defining  $\llbracket u \in v \rrbracket$  and  $\llbracket u = v \rrbracket$  for  $u, v \in V^{(\mathcal{B})}$  as

- (1)  $\llbracket u \in v \rrbracket = \sup_{y \in \mathcal{D}(v)} (v(y) \wedge \llbracket u = y \rrbracket)$ , and
- (2)  $\llbracket u = v \rrbracket = \inf_{x \in \mathcal{D}(u)} (u(x) \Rightarrow \llbracket x \in v \rrbracket) \wedge \inf_{y \in \mathcal{D}(v)} (v(y) \Rightarrow \llbracket y \in u \rrbracket)$ ,

and by assigning a Boolean value  $\llbracket \varphi \rrbracket$  to each formula  $\varphi$  without free variables inductively as

- (1)  $\llbracket \neg \varphi \rrbracket = \neg \llbracket \varphi \rrbracket$ ,

- (2)  $\llbracket \varphi_1 \vee \varphi_2 \rrbracket = \llbracket \varphi_1 \rrbracket \vee \llbracket \varphi_2 \rrbracket$ ,
- (3)  $\llbracket \varphi_1 \wedge \varphi_2 \rrbracket = \llbracket \varphi_1 \rrbracket \wedge \llbracket \varphi_2 \rrbracket$ ,
- (4)  $\llbracket \forall x \varphi(x) \rrbracket = \inf_{u \in V^{(\mathcal{B})}} \llbracket \varphi(u) \rrbracket$ , and
- (5)  $\llbracket \exists x \varphi(x) \rrbracket = \sup_{u \in V^{(\mathcal{B})}} \llbracket \varphi(u) \rrbracket$ .

The following theorem is of fundamental importance to Boolean valued analysis.

**Theorem 2.3.1.** *If  $\varphi$  is a theorem of ZFC, then so is  $\llbracket \varphi \rrbracket = 1$ .*

The class  $V$  of all sets can be embedded into  $V^{(\mathcal{B})}$  by transfinite induction as follows:

$$\check{y} = \{ \langle \check{x}, 1 \rangle \mid x \in y \} \quad \text{for } y \in V.$$

Then we have

**Proposition 2.2.2.** *For  $x, y \in V$ ,*

- (1)  $\llbracket \check{x} \in \check{y} \rrbracket = \begin{cases} 1 & \text{if } x \in y \\ 0 & \text{otherwise;} \end{cases}$
- (2)  $\llbracket \check{x} = \check{y} \rrbracket = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$

A subset  $\{b_\alpha\}$  of  $\mathcal{B}$  is called a *partition of unity* if  $\sup_\alpha b_\alpha = 1$  and  $b_\alpha \wedge b_\beta = 0$  whenever  $\alpha \neq \beta$ . Given a partition of unity  $\{b_\alpha\}$  and a subset  $\{u_\alpha\}$  of  $V^{(\mathcal{B})}$ , it can be proved easily that

**Proposition 2.3.3.** *There exists an element  $u$  of  $V^{(\mathcal{B})}$  such that  $\llbracket u = u_\alpha \rrbracket \geq b_\alpha$  for any  $\alpha$ . Furthermore this  $u$  is determined uniquely in the sense that if  $v$  is another element of  $V^{(\mathcal{B})}$  with the above property, then  $\llbracket u = v \rrbracket = 1$ .*

The above  $u$  is denoted by  $\sum_\alpha u_\alpha b_\alpha$  or  $u_{\alpha_1} b_{\alpha_1} + \dots + u_{\alpha_n} b_{\alpha_n}$  if  $\{b_\alpha\}$  is a finite set.

We define the interpretation  $X^{(\mathcal{B})}$  of  $X = \{x \mid \varphi(x)\}$  to be  $\{u \in V^{(\mathcal{B})} \mid \llbracket \varphi(u) \rrbracket = 1\}$ , assuming that it is not empty. For technical convenience, if  $x$  is a set, then  $X^{(\mathcal{B})}$  is usually considered to be a set by choosing a representative from an equivalence class  $\{v \in V^{(\mathcal{B})} \mid \llbracket u = v \rrbracket = 1\}$ . Such a convention is implicit in the sequel of the paper.

Let  $D \subseteq V^{(\mathcal{B})}$ . A function  $g: D \rightarrow V^{(\mathcal{B})}$  is called *extensional* if  $\llbracket d = d' \rrbracket \leq \llbracket g(d) = g(d') \rrbracket$  for any  $d, d' \in D$ . A  $B$ -valued set  $u \in V^{(\mathcal{B})}$  is said to be *definite* if  $u(d) = 1$  for any  $d \in \mathcal{D}(u)$ . Then we have the following characterization theorem of extensional maps.

**Theorem 2.3.4.** *Let  $u, v \in V^{(\mathcal{B})}$  be definite and  $D = \mathcal{D}(u)$ . Then there is a bijective correspondence between  $f \in V^{(\mathcal{B})}$  satisfying  $\llbracket f : u \rightarrow v \rrbracket = 1$  and extensional maps  $\phi : D \rightarrow v^{(\mathcal{B})}$ , where  $v^{(\mathcal{B})} = \{u \mid \llbracket u \in v \rrbracket = 1\}$ . The correspondence is given by the relation  $\llbracket f(d) = \phi(d) \rrbracket = 1$  for any  $d \in D$ .*

Let  $(X, \mathcal{S}, \mu)$  be a  $\sigma$ -finite measure space. Then its measure algebra (i.e., measurable sets modulo null sets) forms a complete Boolean algebra  $B$  and  $\mathcal{C}^{(\mathcal{B})}$  is represented by the set  $L(\mu)$  of all measurable complex valued functions (modulo  $\mu$ ). The set  $L_\infty(\mu)$  of all essentially bounded measurable functions on  $X$  can be considered to be a commutative von Neumann algebra acting on the Hilbert space  $L_2(\mu)$  of square  $\mu$ -integrable functions as multiplication operators.

Now let  $\mathfrak{H}$  be a Hilbert space and  $\mathcal{Z}$  be a commutative von Neumann algebra on  $\mathfrak{H}$ . Let us suppose also that  $\mathcal{Z}$  is  $*$ -isomorphic to  $L_\infty(\mu)$  for some  $\sigma$ -finite measure space  $(X, \mathcal{S}, \mu)$ . Then obviously the projection lattice  $\mathcal{B}$  of  $\mathcal{Z}$  is isomorphic to the measure algebra of  $(X, \mathcal{S}, \mu)$ , so that they can be identified. The quotient set  $\tilde{\mathfrak{H}}$  of  $\mathfrak{H}$  with respect to the equivalence relation

$$\begin{aligned} & \{ \langle \langle a, b \rangle^v, e(a, b) \rangle \mid a, b \in \mathfrak{H} \} \\ & (e(a, b) = \sup \{ P \in \mathcal{B} \mid Pa = Pb \}) \end{aligned}$$

is a Hilbert space in  $V^{(\mathcal{B})}$ , where the inner product  $(\tilde{a}, \tilde{b})$  of the quotient classes of  $\tilde{a}$  and  $\tilde{b}$  is defined, using the Radon-Nikodym theorem, to be the unique function  $f \in L(\mu)$  such that

$$\int_P f d\mu = (a, Pb)$$

for all  $P \in \mathcal{B}$ . Each bounded linear operator  $T$  on  $H$  belonging to the commutant  $\mathcal{Z}'$  of  $\mathcal{Z}$  induces a bounded linear operator  $\tilde{T}$  in  $V^{(\mathcal{B})}$  by

$$\tilde{T}(\tilde{a}) = (T(a))^\sim.$$

And every bounded linear operator on  $\tilde{\mathfrak{H}}$  is of the form  $\sum_i \tilde{T}_i P_i$  for some countable partition  $\{P_i\}$  of  $\mathcal{B}$  and  $T_i \in \mathcal{Z}'$  for any  $i$ . In particular, every projection in  $V^{(\mathcal{B})}$  is of the form  $\tilde{P}$  for some  $P \in \mathcal{Z}'$ . We have

$$\llbracket \tilde{P}_1 = \tilde{P}_2 \rrbracket = \sup \{ P \in \mathcal{B} \mid PP_1 = PP_2 \}$$

for any projections  $P_1, P_2 \in \mathcal{Z}'$ .

### § 3. Making States Boolean Valued

Let  $\omega$  be a state over a  $C^*$ -algebra  $\mathfrak{A}$  with identity 1. Let  $\mathcal{Z}$  be a commutative von Neumann subalgebra of  $\pi_\omega(\mathfrak{A})'$  with a complete Boolean algebra  $\mathcal{B}$  as its projection lattice and let  $\mu$  be the orthogonal probability measure on  $E_{\mathfrak{A}}$  corresponding to  $\mathcal{Z}$  under Theorem 2.2.2. Then, for each  $A \in \mathfrak{A}$  the function defined on the Borel field of  $E_{\mathfrak{A}}$

$$S \longmapsto (\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(A)\mathcal{Q}_\omega)$$

is easily seen to be absolutely continuous with respect to  $\mu$ . Therefore, by making an appeal to the Radon-Nikodym theorem, there should be a unique  $\mu$ -integrable complex valued function  $f$  on  $E_{\mathfrak{X}}$  modulo  $\mu$  such that

$$\int_S f(\omega')d\mu(\omega') = (\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(A)\mathcal{Q}_\omega).$$

We denote this  $f$  by  $\bar{\omega}(A)$ . By Theorem 2.3.3  $\bar{\omega}$  can be regarded as a function from  $\mathfrak{Y}$  to  $C^{(\mathfrak{P})}$  in  $V^{(\mathfrak{P})}$ . Since

$$\begin{aligned} |(\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(A)\mathcal{Q}_\omega)| &= |(\kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega, \pi_\omega(A)\kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega)| \\ &\leq \|\kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega\| \|\pi_\omega(A)\kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega\| \\ &\leq \|\pi_\omega(A)\| \|\kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega\|^2 \\ &\leq \|A\| \|\kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega\|^2 \\ &= \|A\| \mu(S), \end{aligned}$$

$\bar{\omega}$  is a bounded linear functional on  $\mathfrak{Y}$ , which is to be extended uniquely to a bounded linear functional  $\tilde{\omega}$  on the completion  $\tilde{\mathfrak{Y}}$  of  $\mathfrak{Y}$  in  $V^{(\mathfrak{P})}$ . Then we have

**Theorem 3.1.**  $\tilde{\omega}$  is a state on  $\tilde{\mathfrak{Y}}$  in  $V^{(\mathfrak{P})}$ .

*Proof.* Now that  $\tilde{\omega}$  is known to be a linear functional, this follows readily from the following:

- (1)  $(\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(A^*A)\mathcal{Q}_\omega) = (\pi_\omega(A)\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(A)\mathcal{Q}_\omega) \geq 0$ ;
- (2)  $(\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(1)\mathcal{Q}_\omega) = (\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\mathcal{Q}_\omega) = \mu(S)$ . Q. E. D.

Now we would like to determine the cyclic representation of  $\tilde{\mathfrak{Y}}$  associated with  $\tilde{\omega}$ . For each  $A \in \mathfrak{Y}$ ,  $\pi_\omega(A)$  induces a bounded linear operator  $(\pi_\omega(A))^\sim$  and the function  $A \mapsto (\pi_\omega(A))^\sim$  on  $\mathfrak{Y}$  is bounded linear, so that it can be extended uniquely to a bounded linear operator  $\tilde{\pi}_\omega$  on  $\tilde{\mathfrak{Y}}$ .

**Theorem 3.2.**  $(\tilde{\mathfrak{H}}_\omega, \tilde{\pi}_\omega, \tilde{\mathcal{Q}}_\omega)$  is a cyclic representation for  $\tilde{\omega}$ .

*Proof.* Obviously  $(\tilde{\mathfrak{H}}_\omega, \tilde{\pi}_\omega, \tilde{\mathcal{Q}}_\omega)$  is a cyclic representation of  $\tilde{\mathfrak{Y}}$ , and so it is sufficient to recall that both  $\tilde{\omega}(A)$  and  $(\tilde{\mathcal{Q}}_\omega, \tilde{\pi}_\omega(A)\tilde{\mathcal{Q}}_\omega)$  are the a. e. unique measurable function such that

$$\int_S f(\omega')d\mu(\omega') = (\mathcal{Q}_\omega, \kappa_\mu(\mathcal{X}_S)\pi_\omega(A)\mathcal{Q}_\omega)$$

for any Borel subset  $S$  of  $E_{\mathfrak{X}}$ . Q. E. D.

**Theorem 3.3.**  $\tilde{\omega}$  is pure in  $V^{(\mathfrak{A})}$  iff  $\mathcal{Z}$  is maximal among commutative von Neumann subalgebras of  $\pi_\omega(\mathfrak{A})'$ .

*Proof.* If  $\tilde{\omega}$  is pure, then  $(\tilde{\mathfrak{H}}_\omega, \tilde{\pi}_\omega)$  is irreducible by Theorems 2.1.1 and 3.2. Thus if  $P$  is a projection of  $(\mathcal{Z} \cup \pi_\omega(\mathfrak{A}))'$ ,  $[[\tilde{P} = \tilde{I} \vee \tilde{P} = \tilde{O}]]$  should be  $I$ , which implies  $P \in (\mathcal{Z} \cup \pi_\omega(\mathfrak{A}))'$ . Hence  $\mathcal{Z}$  is maximal among commutative von Neumann subalgebras of  $\pi_\omega(\mathfrak{A})'$ . Conversely, if  $\mathcal{Z}$  is maximal among commutative von Neumann subalgebras of  $\pi_\omega(\mathfrak{A})'$ , then every projection commuting with  $\pi_\omega(\mathfrak{A})$  in  $V^{(\mathfrak{A})}$  is of the form  $\tilde{P}$  for some projection  $P$  in  $\mathcal{Z}$ , so that

$$\begin{aligned} [[\tilde{P} = \tilde{I} \vee \tilde{P} = \tilde{O}]] &= [[\tilde{P} = \tilde{I}]] \vee [[\tilde{P} = \tilde{O}]] \\ &= P \vee (I - P) \\ &= I. \end{aligned}$$

Thus  $\tilde{\omega}$  is pure by Theorems 2.1.1 and 3.2.

Q. E. D.

A similar argument gives

**Theorem 3.4.**  $\tilde{\omega}$  is a factor state in  $V^{(\mathfrak{A})}$  iff  $\mathcal{Z}$  contains the center of  $\pi_\omega(\mathfrak{A})'$ .

#### § 4. Boolean Valued Ergodic Decomposition

Let  $\mathfrak{A}$  be a  $C^*$ -algebra with identity and  $G$  be a group of  $*$ -automorphisms of  $\mathfrak{A}$ . We denote the action of  $G$  by

$$A \in \mathfrak{A} \longmapsto \tau_g(A) \in \mathfrak{A}$$

with  $g \in G$ . A positive linear functional over  $\mathfrak{A}$  is called  $G$ -invariant of  $\omega(A) = \omega(\tau_g(A))$ . The set of all  $G$ -invariant states over  $A$ , denoted by  $E_{\mathfrak{A}}^G$ , is apparently a compact convex subset of  $E_{\mathfrak{A}}$ . A  $G$ -invariant state  $\omega$ , which is an extremal point of  $E_{\mathfrak{A}}^G$ , is called  $G$ -ergodic. A  $G$ -invariant state  $\omega$  is said to be  $G$ -pure if every  $G$ -invariant positive linear function majorized by  $\omega$  is of the form  $\lambda\omega$  with  $0 \leq \lambda \leq 1$ . Each action  $g \in G$  induces a unitary operator  $U_\omega(g)$  on the Hilbert space  $\tilde{\mathfrak{H}}_\omega$  characterized by the following requirements:

- (1)  $U_\omega(g)\pi_\omega(A)U_\omega(g)^* = \pi_\omega(\tau_g(A))$  for all  $A \in \mathfrak{A}$ ;
- (2)  $U_\omega(g)\Omega_\omega = \Omega_\omega$ .

Then the following  $G$ -invariant counterpart of Theorem 2.1.1 obtains for any  $\omega \in E_{\mathfrak{A}}^G$ .

**Theorem 4.1.** The following three conditions are equivalent:

- (1)  $\pi_\omega(\mathfrak{A}) \cup U_\omega(G)$  is irreducible on  $\tilde{\mathfrak{H}}_\omega$ ;
- (2)  $\omega$  is  $G$ -pure;
- (3)  $\omega$  is  $G$ -ergodic.

The discussion of the previous section holds mutatis mutandis with due regard to  $G$ -invariance. In particular, we have the following counterparts of Theorems 3.3 and 3.4 for any  $G$ -invariant state  $\omega$  and any commutative von Neumann subalgebra  $\mathfrak{Z}$  of  $(\pi_\omega(\mathfrak{A}) \cup U_\omega(G))'$  with a complete Boolean algebra  $\mathfrak{B}$  as its projection lattice.

**Theorem 4.2.** *The state  $\bar{\omega}$  is  $G$ -ergodic in  $V^{(\mathfrak{B})}$  iff  $\mathfrak{Z}$  is maximal among commutative von Neumann subalgebras of  $(\pi_\omega(\mathfrak{A}) \cup U_\omega(G))'$ .*

**Theorem 4.3.** *The state  $\bar{\omega}$  is centrally  $G$ -ergodic in  $V^{(\mathfrak{B})}$  iff  $\mathfrak{Z}$  contains  $\mathfrak{B}(\pi_\omega(\mathfrak{A})'') \cap U_\omega(G)'$ , where a  $G$ -invariant state  $\omega'$  is called centrally  $G$ -ergodic if  $\mathfrak{B}(\pi_{\omega'}(\mathfrak{A})'') \cap U_{\omega'}(G)'$  consists only of scalar multiples of the identity operator.*

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