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Eleonora Di Nezza · Vincent Guedj · Henri Guenancia

Families of singular Kähler–Einstein metrics

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Abstract. Refining Yau's and Kołodziej's techniques, we establish very precise uniform a priori estimates for degenerate complex Monge–Ampère equations on compact Kähler manifolds, that allow us to control the blow up of the solutions as the cohomology class and the complex structure both vary.

We apply these estimates to the study of various families of possibly singular Kähler varieties endowed with twisted Kähler–Einstein metrics, by analyzing the behavior of canonical densities, establishing uniform integrability properties, and developing the first steps of a pluripotential theory in families. This provides interesting information on the moduli space of stable varieties, extending works by Berman–Guenancia and Song, as well as on the behavior of singular Ricci-flat metrics on (log) Calabi–Yau varieties, generalizing works by Rong–Ruan–Zhang, Gross–Tosatti–Zhang, Collins–Tosatti and Tosatti–Weinkove–Yang.

Keywords. Singular Kähler–Einstein metrics, families of complex spaces, stable families, log Calabi–Yau manifolds

Contents

Eleonora Di Nezza: Institut Mathématique de Jussieu, Sorbonne Université, Paris, France; eleonora.dinezza@imj-prg.fr

Vincent Guedj: Institut de Mathématiques de Toulouse; UMR 5219, Université de Toulouse; CNRS, UPS, 118 route de Narbonne, 31062 Toulouse Cedex 9, France; vincent.guedj@math.univ-toulouse.fr

Henri Guenancia: Institut de Mathématiques de Toulouse; UMR 5219, Université de Toulouse; CNRS, UPS, 118 route de Narbonne, 31062 Toulouse Cedex 9, France; henri.guenancia@math.cnrs.fr

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Introduction

Let $p : X \to Y$ be a proper, surjective holomorphic map with connected fibers between Kähler varieties. It is a central question in complex geometry to relate the geometry of X to that of Y and the fibers X_y of p. An important instance of such a situation is when one can endow X_y with a Kähler–Einstein metric and study the geometry of X induced by the properties of the resulting family of metrics. This is the main theme of this article.

Einstein metrics are a central object of study in differential geometry. A Kähler–Einstein metric on a complex manifold is a Kähler metric whose Ricci tensor is proportional to the metric tensor. This notion still makes sense on mildly singular varieties, as was observed in [28, Section 7]. The solution of the (singular) Calabi Conjecture [28,71] provides a very powerful existence theorem for Kähler–Einstein metrics with negative or zero Ricci curvature. It is important to study the ways in which these canonical metrics behave when they are moving in families. In this paper we consider the case when both the complex structure and the Kähler class vary and we try to understand how the corresponding metrics can degenerate.

Constructing singular Kähler–Einstein metrics on a mildly singular variety V boils down to solving degenerate complex Monge–Ampère equations of the form

$$(\omega + i\partial\bar{\partial}\varphi)^n = f e^{\lambda\varphi} dV_X,$$

where

- $\pi : X \to V$ is a resolution of singularities, dV_X is a volume form on X,
- $\omega = \pi^*\omega_V$ is the pull-back of a Kähler form on V,
- the sign of $\lambda \in \mathbb{R}$ depends on that of $c_1(V)$,
- $f \in L^p(X)$ with $p > 1$ if the singularities of V are mild (klt singularities),

and φ is the unknown. The latter should be ω-plurisubharmonic (ω-psh for short), i.e. it is locally the sum of a psh and a smooth function, and satisfies $\omega + i\partial\bar{\partial}\varphi \geq 0$ in the weak sense of currents. We let $\mathrm{PSH}(X, \omega)$ denote the set of all such functions.

The uniform estimate

A crucial step in order to prove the existence of a solution to the above equation is to establish a uniform a priori estimate. In order to understand the behavior of the solution φ as the cohomology class $\{\omega_V\}$ and the complex structure of V vary, we revisit the proof by Yau [71], as well as its recent generalizations [28,48], and establish the following (see Theorem 1.1):

Theorem A. *Let X be a compact Kähler manifold of complex dimension $n \in \mathbb{N}^*$ and let ω be a semipositive form such that $V := \int_X \omega^n > 0$. Let ν and $\mu = f\nu$ be probability measures, with $0 \leq f \in L^p(\nu)$ for some $p > 1$. Assume the following assumptions are satisfied:*

(H1) *there exist* $\alpha > 0$ *and* $A_\alpha > 0$ *such that for all* $\psi \in \mathrm{PSH}(X, \omega)$,

$$\int_X e^{-\alpha(\psi - \sup_X \psi)} d\nu \leq A_\alpha;$$

(H2) *there exists* $C > 0$ *such that* $(\int_X |f|^p \, d\nu)^{1/p} \leq C$.

Let φ *be the unique* ω*-psh solution* φ *to the complex Monge–Ampère equation*

$$V^{-1}(\omega + i\partial\bar{\partial}\varphi)^n = \mu,$$

normalized by $\sup_X \varphi = 0$. *Then* $-M \leq \varphi \leq 0$ *where*

$$M = 1 + C^{1/n} A_\alpha^{1/nq} e^{\alpha/nq} b_n [5 + e\alpha^{-1} C (q!)^{1/q} A_\alpha^{1/q}],$$

$1/p + 1/q = 1$ *and* b_n *is a constant such that* $\exp(-1/x) \leq b_n^n x^{2n}$ *for all* $x > 0$.

Remark 0.1. Let us observe that condition (H1) in Theorem A above guarantees that the measure ν does not charge pluripolar sets, since any such set can be included in the polar locus of a *global* ω-psh function by [33, Thm. 7.2]. The existence (and uniqueness) of the solution φ in Theorem A follows from [11, Thm. A].

We also establish slightly more general versions of Theorem A valid for less regular densities (Theorem 1.5) or big cohomology classes (Theorem 1.9). We then move on to checking hypotheses (H1) and (H2) in various geometrical contexts.

• *Hypothesis* (H1). If $\pi : \mathcal{X} \to \mathbb{D}$ is a projective family whose fibers $X_t = \pi^{-1}(t)$ have degree d with respect to a given projective embedding $\mathcal{X} \subset \mathbb{P}^N \times \mathbb{D}$, and $\omega = \omega_t$ is the restriction of the Fubini–Study metric, we observe in Proposition 2.5 that

$$V = \int_{X_t} \omega_t^n = \int_{\mathbb{P}^N} \omega_{\mathrm{FS}}^n \wedge [X_t] = d$$

is independent of t and the following uniform integrability holds.

Proposition B. *For all* $\psi \in \mathrm{PSH}(X_t, \omega_t)$,

$$\int_{X_t} e^{-\frac{1}{nd}(\psi - \sup_{X_t} \psi)} \omega_t^n \leq (4n)^n \cdot d \cdot \exp\left\{ -\frac{1}{nd} \int_{X_t} \left(\psi - \sup_{X_t} \psi \right) \omega_t^n \right\}.$$

Hypothesis (H1) is thus satisfied in this projective setting, with $\alpha = 1/nd$, as soon as we can uniformly control the L^1-norm of ψ. We take care of this in Section 3. This non-trivial control requires the varieties X_t to be irreducible (see Example 3.5).

Bypassing the projectivity assumption, we show that (H1) is actually satisfied for many Kähler families of interest, by generalizing a uniform integrability result of Skoda–Zeriahi [61, 72] (see Theorem 2.9). This is the content of Theorem 3.4.

• *Hypothesis* (H2). We analyze (H2) in Section 4. We show that, up to shrinking the base, it is always satisfied if the f_t's are canonical densities associated to a proper, holomorphic surjective map $\pi : \mathcal{X} \to \mathbb{D}$ from a normal, $\mathbb{Q}$-Gorenstein Kähler space $\mathcal{X}$ to the unit disk such that the central fiber has only canonical singularities; see Lemma 4.4 and its application to families of Calabi–Yau varieties, Theorem F.

While previous works tend to use sophisticated arguments from Variations of Hodge Structures (see e.g. the Appendix by Gross in [56]), we use here direct elementary computations in adapted coordinates, in the spirit of [28, Sect. 6].

In the context of families of varieties with negative curvature though, it is essential to allow worse singularities than the ones described above; see Setting 4.1 for the precise context. The trade-off is that the canonical densities do not satisfy condition (H2) anymore, reflecting the fact that the local potentials of the Kähler–Einstein metrics at stake need not be bounded anymore. This legitimizes the introduction of a weaker condition (H2′) (see Theorem 1.5 and Lemma 4.6). This allows us to derive almost optimal control of the potentials of Kähler–Einstein metrics along a stable family; see Theorem E below.

Let us end this subsection by emphasizing that our approach enables us to work with singular families (i.e. families where the generic fiber is singular; see Theorems E and F) as opposed to all previously known results on that topic, requiring one to approximate a singular variety by smooth ones using either a smoothing or a crepant resolution.

We now describe more precisely four independent geometric settings to which we apply the uniform estimate provided by Theorem A.

Families of manifolds of general type

Let $\mathcal{X}$ be an irreducible and reduced complex space endowed with a Kähler form ω and a proper, holomorphic map $\pi : \mathcal{X} \to \mathbb{D}$. We assume that for each $t \in \mathbb{D}$, the (schematic) fiber X_t is an n-dimensional Kähler manifold X_t of general type, i.e. such that its canonical bundle K_{X_t} is big. In particular, $\mathcal{X}$ is automatically non-singular and the map π is smooth.

We fix a closed differential $(1,1)$-form Θ on $\mathcal{X}$ which represents $c_1(K_{\mathcal{X}/\mathbb{D}})$ and set $\theta_t = \Theta|_{X_t}$.

It follows from [11], a generalization of the Aubin–Yau theorem [3, 71], that there exists a unique Kähler–Einstein current on X_t. This is a positive closed current T_t in $c_1(K_{X_t})$ which is a smooth Kähler form in the ample locus $\mathrm{Amp}(K_{X_t})$, where it satisfies the Kähler–Einstein equation

$$\mathrm{Ric}(T_t) = -T_t.$$

It can be written $T_t = \theta_t + dd^c\varphi_t$, where φ_t is the unique θ_t-psh function with minimal singularities that satisfies the complex Monge–Ampère equation

$$(\theta_t + dd^c\varphi_t)^n = e^{\varphi_t + h_t}\omega_t^n \quad \text{on } \mathrm{Amp}(K_{X_t}),$$

where h_t is such that $\mathrm{Ric}(\omega_t) - dd^c h_t = -\theta_t$ and $\int_{X_t} e^{h_t}\omega_t^n = \mathrm{vol}(K_{X_t})$. For $x \in \mathcal{X}$, set

$$\phi(x) := \varphi_{\pi(x)}(x) \tag{0.1}$$

and consider

$$V_\Theta = \sup\{u \in \mathrm{PSH}(\mathcal{X}, \Theta)\,;\, u \le 0\}. \tag{0.2}$$

We prove that conditions (H1) and (H2) are satisfied in this setting. It then follows from Theorem A and the plurisubharmonic variation of the T_t's [14, Thm. A] that $\phi - V_\Theta$ is uniformly bounded on compact subsets of $\mathcal{X}$ (cf. Theorem 5.5 and Remark 5.6):

Theorem C. *Let $\pi : \mathcal{X} \to \mathbb{D}$ be a smooth Kähler family of manifolds of general type, let $\Theta \in c_1(K_{\mathcal{X}/\mathbb{D}})$ be a smooth representative and let ϕ be the Kähler–Einstein potential as in (0.1). Given any compact subset $\mathcal{K} \Subset \mathcal{X}$, there exists a constant $M_{\mathcal{K}}$ such that*

$$-M_{\mathcal{K}} \le \phi - V_\Theta \le M_{\mathcal{K}}$$

on $\mathcal{K}$, where V_Θ is defined by (0.2).

The same results can be proved if the family $\pi : \mathcal{X} \to \mathbb{D}$ is replaced by a smooth family $\pi : (\mathcal{X}, B) \to \mathbb{D}$ of pairs (X_t, B_t) of log-general type, i.e. such that (X_t, B_t) is klt and $K_{X_t} + B_t$ is big for all $t \in \mathbb{D}$.

Stable families

A *stable variety* is a projective variety X such that X has semi-log-canonical singularities and the $\mathbb{Q}$-line bundle K_X is ample. We refer to [43, 50] for a detailed account of such varieties and their connection to moduli theory.

In [7], it was proved that a stable variety admits a unique Kähler–Einstein metric ω, i.e. a smooth Kähler metric on X_{reg} such that, if $n = \dim_{\mathbb{C}} X$,

$$\mathrm{Ric}(\omega) = -\omega \quad \text{and} \quad \int_{X_{\mathrm{reg}}} \omega^n = (K_X^n).$$

The metric ω extends canonically across X_{sing} to a closed, positive current in the class $c_1(K_X)$. It is desirable to understand the singularities of ω near X_{sing}. In [38, Thm. B], it is proved that ω has cusp singularities near the double crossings of X. Moreover, it is proved in [62] that the potential φ of ω with respect to a given Kähler form $\omega_X \in c_1(K_X)$, i.e. $\omega = \omega_X + dd^c\varphi$, is locally bounded on the klt locus of X. We make this assertion more precise by establishing the following (see Proposition 5.9).

Proposition D. *For any $\varepsilon > 0$, there is a constant C_ε such that*

$$C_1 \ge \varphi \ge -(n + 1 + \varepsilon)\log(-\log|s|) - C_\varepsilon, \tag{0.3}$$

where $(s = 0)$ is any reduced divisor containing the non-klt locus of X.

This estimate is almost optimal. Indeed, if X is the Satake–Baily–Borel compactification of a ball quotient, it is a normal stable variety and it admits a resolution $(\overline{X}, D)$ which is a toroidal compactification of the ball quotient obtained by adding disjoint abelian varieties. Then the potential φ of the Kähler–Einstein metric on $(\overline{X}, D)$ with respect to a smooth form in $c_1(K_{\overline{X}} + D)$ satisfies

$$\varphi = -(n+1)\log(-\log|s_D|) + O(1)$$

if $(s_D = 0) = D$.

A slight refinement of Theorem A (see Theorem 1.5) allows us to establish a uniform family version of estimate (0.3). In order to state it, let $\mathcal{X}$ be a normal Kähler space and let $\pi : \mathcal{X} \to \mathbb{D}$ be a proper, surjective, holomorphic map such that each fiber X_t has slc singularities and $K_{\mathcal{X}/\mathbb{D}}$ is an ample $\mathbb{Q}$-line bundle. If $\omega_{\mathcal{X}} \in c_1(K_{\mathcal{X}/\mathbb{D}})$ is a relative Kähler form and $\omega_{X_t} := \omega_{\mathcal{X}}|_{X_t}$, then the Kähler–Einstein metric of X_t can be written as $\omega_{X_t} + dd^c\varphi_t$ where φ_t is uniquely determined by equation (5.7). The behavior of φ_t is then described by the following (see Theorem 5.11)

Theorem E. *Let $\mathcal{X}$ be a normal Kähler space and let $\pi : \mathcal{X} \to \mathbb{D}$ be a proper, surjective, holomorphic map such that:*

- *Each schematic fiber X_t has semi-log-canonical singularities.*
- *$K_{\mathcal{X}/\mathbb{D}}$ is an ample $\mathbb{Q}$-line bundle.*

In particular, X_t is a stable variety for any $t \in \mathbb{D}$. Assume additionally that the central fiber X_0 is irreducible.

Let $\omega_{X_t} + dd^c\varphi_t$ be the Kähler–Einstein metric of X_t and let $D = (s = 0) \subset \mathcal{X}$ be a divisor which contains $\mathrm{Nklt}(\mathcal{X}, X_0)$ *(see* (4.4)*). Fix a smooth hermitian metric $|\cdot|$ on $\mathcal{O}_{\mathcal{X}}(D)$. Up to shrinking $\mathbb{D}$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$C_1 \geq \varphi_t \geq -(n+1+\varepsilon)\log(-\log|s|) - C_\varepsilon$$

on X_t for any $t \in \mathbb{D}$.

Let us finally mention the very recent results of Song, Sturm and Wang [63, Proposition 3.1] where similar bounds are derived in the context of smoothings of stable varieties over higher-dimensional bases, with application towards Weil–Petersson geometry of the KSBA compactification of canonically polarized manifolds.

Families of $\mathbb{Q}$-Calabi–Yau varieties

A $\mathbb{Q}$-*Calabi–Yau variety* is a compact, normal Kähler space X with canonical singularities such that the $\mathbb{Q}$- line bundle K_X is torsion. Up to taking a finite, quasi-étale cover referred to as the index 1 cover (see e.g. [46, Def. 5.19]), one can assume that $K_X \sim_{\mathbb{Z}} \mathcal{O}_X$. Given any Kähler class α on X, it follows from [28] and [55] that there exists a unique singular Ricci-flat Kähler metric $\omega_{\mathrm{KE}} \in \alpha$, i.e. a closed, positive current $\omega_{\mathrm{KE}} \in \alpha$ with globally bounded potentials inducing a smooth, Ricci-flat Kähler metric on X_{reg}.

Now, we can consider families of such varieties and ask how the bounds on the potentials vary. This is the content of the following (see Theorem 6.1 and Remark 6.2).

Theorem F. *Let $\mathcal{X}$ be a normal, $\mathbb{Q}$-Gorenstein Kähler space and let $\pi : \mathcal{X} \to \mathbb{D}$ be a proper, surjective, holomorphic map. Let α be a relative Kähler cohomology class on $\mathcal{X}$ represented by a relative Kähler form ω. Assume additionaly that:*

- *The relative canonical bundle $K_{\mathcal{X}/\mathbb{D}}$ is trivial.*
- *The central fiber X_0 has canonical singularities.*
- *Assumption 3.2 is satisfied.*

Up to shrinking $\mathbb{D}$, each fiber X_t is a $\mathbb{Q}$-Calabi–Yau variety. Let $\omega_{\mathrm{KE},t} = \omega_t + dd^c\varphi_t$ be the singular Ricci-flat Kähler metric in α_t, normalized by $\int_{X_t} \varphi_t \omega_t^n = 0$. Then, given any compact subset $K \Subset \mathbb{D}$, there exists $C = C(K) > 0$ such that

$$\mathrm{osc}_{X_t} \varphi_t \le C$$

for any $t \in K$, where $\mathrm{osc}_{X_t} \varphi_t = \sup_{X_t} \varphi_t - \inf_{X_t} \varphi_t$.

In the case of a projective smoothing (i.e. when $\mathcal{X}$ admits a π-ample line bundle and X_t is smooth for $t \neq 0$), the result above has been obtained previously by Rong–Zhang [56] by using the Moser iteration process.

Log Calabi–Yau families

Let X be a compact Kähler manifold and let $B = \sum b_i B_i$ be an effective $\mathbb{R}$-divisor such that the pair (X, B) has klt singularities and $c_1(K_X + B) = 0$.

It follows from [11,28,71] that one can find a unique Ricci-flat metric in each Kähler class α_t. A basic problem is to understand the asymptotic behavior of these metrics as α_t approaches the boundary of the Kähler cone. Despite motivations coming from mirror symmetry, not much is known when the norm of α_t converges to $+\infty$ (this case is expected to be the mirror of a large complex structure limit; see [49]). We thus only consider the case when $\alpha_t \to \alpha_0 \in \partial\mathcal{K}_X$.

The non-collapsing case ($\mathrm{vol}(\alpha_0) > 0$) can be easily understood by using Theorem A (see Theorem 6.5). We describe here a particular instance of the more delicate collapsing case $\mathrm{vol}(\alpha_0) = 0$. Let $f : X \to Z$ be a surjective holomorphic map with connected fibers, where Z is a normal Kähler space. Let ω_X (resp. ω_Z) be a Kähler form on X (resp. Z). Set $\omega_t := f^*\omega_Z + t\omega_X$. There exists a unique singular Ricci-flat current $\omega_{\varphi_t} := \omega_t + dd^c\varphi_t$ in $\{f^*\omega_Z + t\omega_X\}$ for $t > 0$, where $\int_X \varphi_t \omega_X^n = 0$. It satisfies

$$\omega_{\varphi_t}^n = V_t \cdot \mu_{(X,B)}, \quad \text{where} \quad \mu_{(X,B)} = (s \wedge \bar{s})^{1/m} e^{-\phi_B}.$$

Here, $s \in H^0(X, m(K_X + B))$ is any non-zero section (for some $m \ge 1$) and ϕ_B is the unique singular psh weight on $\mathcal{O}_X(B)$ solving $dd^c\phi_B = [B]$ and normalized by

$$\int_X (s \wedge \bar{s})^{1/m} e^{-\phi_B} = 1.$$

The probability measure $f_*\mu_{(X,B)}$ has $L^{1+\varepsilon}$ density with respect to ω_Z^m thanks to [29, Lem. 2.3]. It follows therefore from [28] that there exists a unique current $\omega_\infty \in \{\omega_Z\}$

that solves the Monge–Ampère equation

$$\omega_\infty^m = f_*\mu_{(X,B)}.$$

In the case where X is smooth, $B = 0$ and $c_1(X) = 0$, the Ricci curvature of ω_∞ coincides with the Weil–Petersson form of the fibration f of Calabi–Yau manifolds.

Understanding the asymptotic behavior of the ω_{φ_t}'s as $t \to 0$ is an important problem with a long history; we refer the reader to the thorough survey [68] for references. We prove here the following:

Theorem G. *Let (X, B) be a log smooth klt pair such that $c_1(K_X + B) = 0$ and such that X admits a fibration $f : X \to Z$. With the notations above, the conic Ricci-flat metrics $\omega_{\varphi_t} \in \{f^*\omega_Z + t\omega_X\}$ converge to $f^*\omega_\infty$ as currents on X when t goes to* 0.

When $B = 0$ is empty, it has been shown in [32, 39, 67, 69] that the metrics ω_{φ_t} converge to $f^*\omega_\infty$ in the $\mathscr{C}^\alpha$ sense on compact subsets of $X \setminus S_X$ for some $\alpha > 0$, where $S_X = f^{-1}(S_Z)$ and S_Z denotes the smallest proper analytic subset $\Sigma \subset Z$ such that Σ contains the singular locus Z_{sing} of Z and the map f is smooth on $f^{-1}(Z \setminus \Sigma)$.

The proof of Theorem G follows the strategy developed by the above papers with several twists that notably require the extensive use of Theorem A and conical metrics.

1. Chasing the constants

Our goal in this section is to establish the following a priori estimate which is a refinement of the main result of Kołodziej [48] (see also [24, 27, 28]):

Theorem 1.1. *Let (X, ω_X) be a compact Kähler manifold of complex dimension $n \in \mathbb{N}^*$ and let ω be a semipositive form which is big, i.e. such that*

$$V := \mathrm{vol}_\omega(X) = \int_X \omega^n > 0.$$

Let ν and $\mu = f\nu$ be probability measures, with $0 \le f \in L^p(\nu)$ for some $p > 1$. Assume the following two assumptions are satisfied:

(H1) *there exist $\alpha > 0$ and $A_\alpha > 0$ such that for all $\psi \in \mathrm{PSH}(X, \omega)$,*

$$\int_X e^{-\alpha(\psi - \sup_X \psi)}\, d\nu \le A_\alpha;$$

(H2) *there exists $C > 0$ such that $(\int_X |f|^p\, d\nu)^{1/p} \le C$.*

Let φ be the unique ω-psh solution φ to the complex Monge–Ampère equation

$$V^{-1}(\omega + dd^c\varphi)^n = \mu,$$

normalized by $\sup_X \varphi = 0$. *Then $-M \le \varphi \le 0$ where*

$$M = 1 + C^{1/n} A_\alpha^{1/nq}\, e^{\alpha/nq} b_n [5 + e\alpha^{-1} C (q!)^{1/q} A_\alpha^{1/q}],$$

$1/p + 1/q = 1$ and b_n is a constant such that $\exp(-1/x) \le b_n^n x^{2n}$ *for all* $x > 0$.

Here $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{2}(\partial - \overline{\partial})$ so that $dd^c = i\partial\overline{\partial}$. Recall that a function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is *ω-plurisubharmonic* (ω-psh for short) if it is locally given as the sum of a smooth and a psh function, and such that $\omega + dd^c\varphi \geq 0$ in the weak sense of currents. We let $\mathrm{PSH}(X, \omega)$ denote the set of all ω-psh functions.

The non-pluripolar Monge–Ampère measure for arbitrary ω-psh functions has been defined in [11]. It follows from assumption (H1) that the measure μ does not charge pluripolar sets, since the latter can be defined by ω-psh functions (as follows easily from [33, Thm. 7.2] since a big class contains a Kähler current). The existence of a unique normalized ω-psh solution to $V^{-1}(\omega + dd^c\varphi)^n = \mu$ follows from [11, Thm. A] (the case of Kähler forms had been earlier treated in [26, 34]).

We will use this result to obtain uniform a priori estimates on normalized solutions φ_t to families of complex Monge–Ampère equations

$$V_t^{-1}(\omega_t + dd^c\varphi_t)^n = \mu_t,$$

when hypotheses (H1)–(H2) are satisfied, i.e. the constants $1/\alpha_t, A_{\alpha_t}, q_t, C_t$ in the theorem are actually bounded from above by uniform constants $1/\alpha, A, q, C$ independent of t. Here q denotes the conjugate exponent of $p > 1$, $1/p + 1/q = 1$. The assumption on this exponent is thus that $p > 1$ stays bounded away from 1.

The reader should keep in mind that assumption (H1) is the strongest of all. In some applications one can assume $f \equiv 1$, hence (H2) is trivially satisfied.

We are going to eventually obtain a version of Theorem 1.1 that applies to big cohomology classes, extending [11, Thm. B]. The proof is almost identical but explaining the statement requires introducing various notions and technical notations, so we first treat the case of semipositive classes and deal with the general case in Section 1.4.

1.1. Preliminaries on capacities

Let $K \subset X$ be a Borel set and consider

$$V_{K,\omega} := (\sup\{\psi\,;\, \psi \in \mathrm{PSH}(X, \omega) \text{ and } \psi \leq 0 \text{ on } K\})^*,$$

where * denotes upper semicontinuous regularization.

The *Alexander–Taylor capacity* is

$$T_\omega(K) := \exp\Big(-\sup_X V_{K,\omega}\Big).$$

It is shown in [35, Lem. 9.17] that if K is pluripolar then $V_{K,\omega} \equiv +\infty$ and $T_\omega(K) = 0$. If K is not pluripolar then

- $0 \leq V_{K.\omega} \in \mathrm{PSH}(X, \omega)$ and $V_{K,\omega} = 0$ on K off a pluripolar set;
- the Monge–Ampère measure $\mathrm{MA}(V_{K,\omega})$ is concentrated on E.

Here and in what follows, we denote by

$$\mathrm{MA}(u) = \frac{1}{V}(\omega + dd^c u)^n$$

the normalized Monge–Ampère measure of a ω-psh function u, where $V = \int_X \omega^n = \{\omega\}^n$ is the volume of the cohomology class $\{\omega\}$. It is defined for any ω-psh function u (see e.g. [34, §1.1]). For a Borel set $K \subset X$, the *Monge–Ampère capacity* is

$$\mathrm{Cap}_\omega(K) := \sup\left\{\int_K \mathrm{MA}(u)\,;\, u \in \mathrm{PSH}(X,\omega) \text{ and } 0 \le u \le 1\right\}.$$

This capacity also characterizes pluripolar sets, i.e.

$$\mathrm{Cap}^*_\omega(P) = 0 \iff P \text{ is pluripolar.}$$

Here Cap^*_ω is the outer capacity associated to Cap_ω, defined for any set $E \subset X$ as

$$\mathrm{Cap}^*_\omega(E) := \inf\{\mathrm{Cap}_\omega(G)\,;\, G \text{ open},\ E \subset G\}.$$

Moreover, if $K \subset X$ is a compact set then $\mathrm{Cap}^*_\omega(K) = \mathrm{Cap}_\omega(K)$.

The Monge–Ampère and the Alexander–Taylor capacities compare as follows:

Lemma 1.2.

$$T_\omega(K) \le \exp\left[1 - \frac{1}{\mathrm{Cap}_\omega(K)^{1/n}}\right].$$

We refer the reader to [33, Prop. 7.1] for a proof, which also provides a reverse inequality that is not needed in the following.

1.2. Proof of Theorem 1.1

1.2.1. Domination by capacity. It follows from Hölder's inequality and (H2) that

$$\mu \le C\nu^{1/q},$$

where q is the conjugate exponent, $1/p + 1/q = 1$.

Let $K \subset X$ be a non-pluripolar Borel set. Recall that $V_{K,\omega}(x) = 0$ for ν-almost every point $x \in K$. Hypothesis (H1) therefore implies that

$$\nu(K) \le \int_X e^{-\alpha V_{K,\omega}}\, d\nu \le A_\alpha T_\omega(K)^\alpha.$$

Combining previous information we obtain

$$\mu(K) \le C A_\alpha^{1/q} e^{\alpha/q} \exp\left[-\frac{\alpha/q}{\mathrm{Cap}_\omega(K)^{1/n}}\right] \le D\,\mathrm{Cap}_\omega(K)^2,$$

where

$$D = b_n^n C A_\alpha^{1/q} e^{\alpha/q},$$

with b_n a numerical constant such that $\exp(-1/x) \le b_n^n x^{2n}$ for all $x > 0$.

We now need to relate the Monge–Ampère capacity of sublevel sets of a ω-psh function to the Monge–Ampère measure of similar sublevel sets:

Lemma 1.3. *Let φ be a bounded ω-psh function. For all $s > 0$ and $0 < \delta < 1$,*

$$\delta^n\,\mathrm{Cap}_\omega(\{\varphi < -s - \delta\}) \le \mathrm{MA}(\varphi)(\{\varphi < -s\})$$

We refer to [28, Lem. 2.2] for a proof.

1.2.2. Bounding the solution from below. Under our assumptions (H1)–(H2), it follows from general arguments that there is a unique bounded ω-psh solution φ of $\mathrm{MA}(\varphi) = \mu$ normalized by $\sup_X \varphi = 0$ (see Remark 0.1). The non-expert reader could even think that φ is smooth; the point here is to establish a uniform a priori bound from below.

We let $f : \mathbb{R}^+ \to \mathbb{R}^+$ denote the function defined by

$$f(s) := -\frac{1}{n} \log \mathrm{Cap}_\omega(\{\varphi < -s\}).$$

Observe that f is non-decreasing and such that $f(+\infty) = +\infty$. It follows from our previous estimates that for all $s > 0$ and $0 < \delta < 1$,

$$f(s+\delta) \geq 2f(s) + \log \delta - \frac{\log D}{n}.$$

Our next lemma guarantees that such a function reaches $+\infty$ in finite time:

Lemma 1.4. *$f(s) = +\infty$ for all $s \geq 5D^{1/n} + s_0$, where*

$$s_0 = \inf\{s > 0 \, ; eD^{1/n} \, \mathrm{Cap}_\omega(\{\varphi < -s\}) < 1\}.$$

Proof. We define a sequence (s_j) of positive reals by induction as follows:

$$s_{j+1} = s_j + \delta_j \quad \text{with} \quad \delta_j = eD^{1/n} \exp(-f(s_j)).$$

We fix s_0 large enough (as in the statement of the lemma) so that $\delta_0 < 1$. It is straightforward to check, by induction, that the sequence (s_j) is increasing, while (δ_j) is decreasing. Thus $0 < \delta_j < 1$ and

$$f(s_{j+1}) \geq f(s_j) + 1, \quad \text{hence} \quad f(s_j) \geq j.$$

We infer $\delta_j \leq eD^{1/n} \exp(-j)$ and

$$s_\infty = s_0 + \sum_{j \geq 0} (s_{j+1} - s_j) \leq s_0 + \sum_{j \geq 0} eD^{1/n} \exp(-j) \leq s_0 + 5D^{1/n}. \qquad \blacksquare$$

It remains to obtain a uniform bound on s_0. It follows from the Chebyshev inequality and Lemma 1.3 (used with $\delta = 1$) that for all $s > 0$,

$$\mathrm{Cap}_\omega(\{\varphi < -s-1\}) \leq \frac{1}{s} \int_X (-\varphi)\, d\mu,$$

since $\mathrm{MA}(\varphi) = \mu$. The Hölder inequality and (H2) yield

$$\int_X (-\varphi)\, d\mu \leq C \left(\int_X (-\varphi)^q \, d\nu \right)^{1/q}.$$

Observe that for all $t \geq 0$,

$$t^q \leq \frac{q!}{\alpha^q} \exp(\alpha t)$$

and use (H1) to conclude that

$$\mathrm{Cap}_\omega(\{\varphi < -s-1\}) \leq \frac{C(q!)^{1/q} A_\alpha^{1/q}}{\alpha s}.$$

Thus

$$s_0 = 1 + eD^{1/n} \frac{C(q!)^{1/q} A_\alpha^{1/q}}{\alpha}$$

is a convenient choice. This yields the desired a priori estimate and concludes the proof.

1.3. More general densities

The setting of Theorem 1.1 is the most commonly used in geometric applications, as it allows one e.g. to construct Kähler–Einstein currents on varieties with log-terminal singularities (see Section 6). For varieties of general type with semi-log-canonical singularities (see Section 5.2), one has to deal with slightly more general densities. The following result is a refinement of [48, Thm. 2.5.2] and [28, Thm. A].

Theorem 1.5. *Let (X, ω_X) be a compact Kähler manifold of complex dimension $n \in \mathbb{N}^*$ and let ω be a semipositive form with $V := \mathrm{vol}_\omega(X) = \int_X \omega^n > 0$. Let ν and $\mu = f\nu$ be probability measures, with $0 \leq f \in L^1(\nu)$. Assume the following assumptions are satisfied:*

(H1) *there exists $\alpha > 0$ and $A_\alpha > 0$ such that for all $\psi \in \mathrm{PSH}(X, \omega)$,*

$$\int_X e^{-\alpha(\psi - \sup_X \psi)} d\nu \leq A_\alpha;$$

(H2′) *there exist $C, \varepsilon > 0$ such that*

$$\int_X |f|\, |\log f|^{n+\varepsilon}\, d\nu \leq C.$$

Let φ be the unique ω-psh solution φ to the complex Monge–Ampère equation

$$V^{-1}(\omega + dd^c\varphi)^n = \mu,$$

normalized by $\sup_X \varphi = 0$. Then $-M \leq \varphi \leq 0$, where $M = M(C, \varepsilon, n, A_\alpha)$.

Proof. The proof follows the same lines as that of Theorem 1.1, so we only emphasize the main technical differences and focus on the case $\varepsilon = 1$. Set, for $t \geq 0$,

$$\chi(t) = (t+1)\sum_{j=0}^{n+1} (-1)^{n+1-j} \frac{(n+1)!}{j!} (\log(t+1))^j.$$

Observe that χ is a convex function such that $\chi(0) = 0$ and $\chi'(t) = (\log(t+1))^{n+1}$. Its Legendre transform is

$$\chi^*(s) = \sup_{t>0}\{s \cdot t - \chi(t)\} = st(s) - \chi(t(s)),$$

where $1 + t(s) = \exp(s^{\frac{1}{n+1}})$ satisfies $s = \chi'(t(s))$, thus

$$\chi^*(s) = P(s^{\frac{1}{n+1}})\exp(s^{\frac{1}{n+1}}) - s,$$

where P is the following polynomial of degree n:

$$P(X) = \sum_{j=0}^{n}(-1)^{n-j}\frac{(n+1)!}{j!}X^j.$$

We let the reader check that (H2′) is equivalent to $\|f\|_\chi \le C'$, where $\|f\|_\chi$ denotes the Luxemburg norm of f,

$$\|f\|_\chi := \inf\left\{r > 0\,;\int_X \chi(|f|/r)\,d\nu \le 1\right\}.$$

Let $K \subset X$ be a non-pluripolar Borel set. It follows from the Hölder–Young inequality [6, Prop. 2.15] that

$$\mu(K) \le 2C'\|\mathbf{1}_K\|_{\chi^*},$$

where $\|\mathbf{1}_K\|_{\chi^*} = \inf\{r > 0\,;\nu(K)\chi^*(1/r) \le 1\} =: r_K$ with

$$\chi^*(1/r_K) = 1/\nu(K).$$

We are interested in the behavior of this function as $\nu(K)$ approaches zero, i.e. for small values of r_K. Observe that $\chi^*(s) \le \exp(2s^{\frac{1}{n+1}})$ for $s \ge 1/r_n$, hence

$$\nu(K) \le \delta_n \implies \mu(K) \le 2C'r_K \le \frac{2^{n+2}C'}{(-\log\nu(K))^{n+1}}.$$

Recall that (H1) and Lemma 1.2 yield

$$\nu(K) \le A_\alpha e^\alpha \exp\left(-\frac{\alpha}{\mathrm{Cap}_\omega(K)^{1/n}}\right).$$

It follows that for $\nu(K) \le \delta_n$,

$$\mu(K) \le C''\,\mathrm{Cap}_\omega(K)^{1+1/n},$$

and we can then conclude by reasoning as in Lemma 1.4. This completes the proof when $\varepsilon = 1$. The proof for arbitrary $\varepsilon > 0$ is similar, the crucial point being the domination of μ by a multiple of $\mathrm{Cap}_\omega^{1+\varepsilon/n}$, with an exponent $1 + \varepsilon/n > 1$. ∎

1.4. Big cohomology classes

We now consider a similar situation where the reference cohomology class α is still big but no longer semipositive. We assume for convenience that the ambient manifold (X, ω_X) is again compact Kähler, but one could equally well develop this material when X belongs to the Fujiki class (i.e. when X is merely bimeromorphic to a Kähler manifold).

By definition α is big if it contains a *Kähler current*, i.e. there is a positive current $T \in \alpha$ and $\varepsilon > 0$ such that $T \geq \varepsilon\omega_X$. It follows from [23] that one can further assume that T has *analytic singularities*, i.e. it can be locally written as $T = dd^c u$ with

$$u = \frac{c}{2}\log\Big[\sum_{j=1}^{s}|f_j|^2\Big] + v,$$

where $c > 0$, v is smooth and the f_j's are holomorphic functions.

Definition 1.6. We let $\mathrm{Amp}(\alpha)$ denote the *ample locus* of α, i.e. the Zariski open subset of all points $x \in X$ for which there exists a Kähler current in α with analytic singularities which is smooth in a neighborhood of x.

It follows from the work of Boucksom [10] that one can find a single Kähler current T_0 with analytic singularities in α such that

$$\mathrm{Amp}(\alpha) = X \setminus \mathrm{Sing}\, T_0.$$

We fix a smooth closed differential $(1,1)$-form θ representing α. Following Demailly, one defines the following θ-psh function with *minimal singularities*:

$$V_\theta := \sup\{u\,;u \in \mathrm{PSH}(X,\theta) \text{ and } u \leq 0\}.$$

Definition 1.7. A θ-psh function φ has *minimal singularities* if for any other θ-psh function u, there exists $C \in \mathbb{R}$ such that $u \leq \varphi + C$.

There are plenty of such functions, which play here the role of bounded functions when α is semipositive. Demailly's regularization result [23] ensures that α contains many θ-psh functions which are smooth in $\mathrm{Amp}(\alpha)$. In particular, a θ-psh function φ with minimal singularities is locally bounded in $\mathrm{Amp}(\alpha)$. The Monge–Ampère measure $(\theta + dd^c\varphi)^n$ is thus well defined in $\mathrm{Amp}(\alpha)$ in the sense of Bedford and Taylor [4].

Definition 1.8. It follows from the work of Boucksom [9] that

$$\int_{\mathrm{Amp}(\alpha)}(\theta + dd^c\varphi)^n =: V_\alpha > 0$$

is independent of φ; it is the *volume* of the cohomology class α.

One can therefore develop a pluripotential theory in the Zariski open set $\mathrm{Amp}(\alpha)$. This was done in [11], where the following properties have been established:

- the class $\mathrm{PSH}(X,\theta)$ enjoys several compactness properties;
- the operator $\mathrm{MA}(\varphi) = V_\alpha^{-1}(\theta + dd^c\varphi)^n$ is a well defined probability measure on the set of θ-psh functions with minimal singularities;
- the extremal functions $V_{K,\theta} = \sup\{u\,;u \in \mathrm{PSH}(X,\theta) \text{ and } u \leq 0 \text{ on } K\}$ and the Alexander–Taylor capacity $T_\theta(K) = \exp(-\sup_X V_{K,\theta})$ enjoy similar properties to those in the semipositive case;

– in particular $T_\theta(K)$ compares similarly to the Monge–Ampère capacity

$$\mathrm{Cap}_\theta(K) := \sup\left\{\int_K \mathrm{MA}(u)\,;u \in \mathrm{PSH}(X,\theta) \text{ and } 0 \le u - V_\theta \le 1\right\};$$

– the comparison principle holds so Lemma 1.3 holds here as well.

The same proof as above therefore produces the following uniform a priori estimate, which is a refinement of [11, Thm. 4.1]:

Theorem 1.9. *Let (X,ω_X) be a compact Kähler manifold of complex dimension $n \in \mathbb{N}^*$. Let α be a big cohomology class of volume $V_\alpha > 0$ and fix a smooth closed differential $(1,1)$-form θ representing α.*

Let ν and $\mu = f\nu$ be probability measures with $0 \le f \in L^p(\nu)$ for some $p > 1$. Assume the following assumptions are satisfied:

(H1) *there exist $\alpha > 0$, $A_\alpha > 0$ such that for all $\psi \in \mathrm{PSH}(X,\theta)$, $\int_X e^{-\alpha(\psi - \sup_X \psi)}\,d\nu \le A_\alpha$;*

(H2) *there exists $C > 0$ such that $(\int_X |f|^p\,d\nu)^{1/p} \le C$.*

Let φ be the unique θ-psh function with minimal singularities such that

$$V_\alpha{}^{-1}(\theta + dd^c\varphi)^n = \mu$$

and $\sup_X \varphi = 0$. Then $-M \le \varphi - V_\theta \le 0$ where

$$M = 1 + C^{1/n} A_\alpha^{1/nq}\, e^{\alpha/nq} b_n [5 + e\alpha^{-1} C (q!)^{1/q} A_\alpha^{1/q}],$$

where b_n is a uniform constant such that $\exp(-1/x) \le b_n^n x^{2n}$ for all $x > 0$.

Remark 1.10. We also have an analogue of Theorem 1.5 in the big setting.

2. Uniform integrability

We wish to apply the previous uniform estimates when the complex structure of the underlying manifold varies. In this section we pay a special attention to assumption (H1), by generalizing an integrability result of Skoda–Zeriahi [61,72].

2.1. Notations

In all what follows, given a positive real number r, we denote by $\mathbb{D}_r := \{z \in \mathbb{C}\,; |z| < r\}$ the open disk of radius r in the complex plane. If $r = 1$, we simply write $\mathbb{D}$ for $\mathbb{D}_1$.

Setting 2.1. *Let $\mathcal{X}$ be an irreducible and reduced complex Kähler space. We let $\pi : \mathcal{X} \to \mathbb{D}$ denote a proper, surjective holomorphic map such that each fiber $X_t = \pi^{-1}(t)$ is an n-dimensional, reduced, irreducible, compact Kähler space, for any $t \in \mathbb{D}$.*

For later purposes, we pick a covering $\{U_\alpha\}_\alpha$ of $\mathcal{X}$ by open sets admitting an embedding $j_\alpha : U_\alpha \hookrightarrow \mathbb{C}^N$ for some $N \ge n+1$. Moreover, we fix a Kähler form ω on $\mathcal{X}$. Up to refining the covering, the datum of ω is equivalent to the datum of Kähler metrics on open neighborhoods of $j_\alpha(U_\alpha) \subset \mathbb{C}^N$ that agree on each intersection $U_\alpha^{\mathrm{reg}} \cap U_\beta^{\mathrm{reg}}$. Equivalently, ω is a genuine Kähler metric on $\mathcal{X}_{\mathrm{reg}}$ such that $(j_\alpha)_*(\omega|_{U_\alpha^{\mathrm{reg}}})$ is the restriction of a Kähler metric defined on an open neighborhood of $j_\alpha(U_\alpha) \subset \mathbb{C}^N$.

Let us point out that this definition of a Kähler metric on a singular space X is much more restrictive than merely asking for a Kähler metric on X_{reg} (even if one requires, say, that the latter has local potentials near X_{sing}, and that those are bounded). One important property that Kähler metrics satisfy is that their pull-back under a modification is a smooth form (i.e. locally the restriction of a smooth form under a local embedding in $\mathbb{C}^N$); in particular, it is dominated by a Kähler form.

For each $t \in \mathbb{D}$, we set

$$\omega_t := \omega|_{X_t}.$$

An easy yet important observation is the following.

Lemma 2.2. *In Setting 2.1 and using the notation above, the quantity $\int_{X_t} \omega_t^n$ is independent of $t \in \mathbb{D}$. We will denote it by V in the following.*

Proof. The function $\mathbb{D} \ni t \mapsto \int_{X_t} \omega_t^n$ coincides with the push-forward current $\pi_*\omega^n$ of bidimension $(1,1)$. Its distributional differential is zero as d commutes with π_* and ω is closed. ■

We fix a smooth, closed differential $(1,1)$-form Θ on $\mathcal{X}$ and set $\theta_t = \Theta|_{X_t}$. Up to shrinking $\mathbb{D}$, one will always assume that there exists a constant $C_\Theta > 0$ such that

$$-C_\Theta \omega \le \Theta \le C_\Theta \omega. \tag{2.1}$$

In particular, $\mathrm{PSH}(X_t, \theta_t) \subseteq \mathrm{PSH}(X_t, C_\Theta \omega_t)$. We assume that the cohomology classes $\{\theta_t\} \in H^{1,1}(X_t, \mathbb{R})$ are psef, i.e. the sets $\mathrm{PSH}(X_t, \theta_t)$ are non-empty for all t. The notions of (quasi-)plurisubharmonic functions, positive currents and Monge–Ampère measure are well defined on singular spaces [22].

2.2. *Uniform integrability index*

Recall from [21, Déf. 3] that if T is a closed, positive current of bi-dimension (p,p) on a complex space X and if $x \in X$ is a closed point, then the *Lelong number* of T at x is defined as the limit

$$\nu(T,x) := \lim_{r\to 0}{}^{\downarrow} \frac{1}{r^{2p}} \int_{\{\psi<r\}} T \wedge (dd^c\psi)^p, \tag{2.2}$$

where $\psi := \sum_{i\in I} |g_i|^2$ and $(g_i)_{i\in I}$ is a (finite) system of generators of the maximal ideal $\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$. It is proved in *loc. cit.* that the limit above is a decreasing limit, independent of the choice of the generators. Moreover, one has the formula

$$\nu(T,x) = \int_{\{x\}} T \wedge (dd^c \log\psi)^p \tag{2.3}$$

(see [21, bottom of p. 45]). Finally, if φ is a θ-psh function on X for some smooth, closed $(1,1)$-form θ, then the Lelong number of φ at a given point $x \in X$ is defined to be the quantity $\nu(\theta + dd^c\varphi, x)$.

Proposition 2.3. *In Setting 2.1, let $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$ be a collection of θ_t-psh functions on X_t. Then*

$$\sup_{t\in\overline{\mathbb{D}}_{1/2}} \sup_{x\in X_t} \nu(\varphi_t, x) < +\infty.$$

Proof. Let $U'_\alpha \Subset U_\alpha$ be a relatively compact open subset such that the U'_α are still a covering of $\mathcal{X}$. Up to adding more elements to the initial covering, one can always assume that one can find such a refinement. One picks cut-off functions χ_α such that $\chi_\alpha \equiv 1$ on U'_α and $\mathrm{Supp}(\chi_\alpha) \subset U_\alpha$. Now, let $x \in \mathcal{X}$; there exists $\alpha = \alpha(x)$ such that $x \in U'_\alpha$. Recall that we have an embedding $j_\alpha : U_\alpha \to \mathbb{C}^N$; we set $x' := j_\alpha(x)$ and $G_{x'} : \mathbb{C}^N \ni z \mapsto \log(\sum_{i=1}^N |z_i - x'_i|^2)$. One can easily check that there exists a constant $A > 0$, independent of the point x now ranging over the compact set $\pi^{-1}(\overline{\mathbb{D}}_{1/2})$, such that the function

$$H_x := \chi_\alpha \cdot j_\alpha^* G_{x'}$$

defines an $A\omega$-psh function on the whole $\mathcal{X}$. By (2.3), one has

$$\begin{aligned}
\nu(\varphi_t, x) &= \int_{\{x\}} (\theta_t + dd^c\varphi_t) \wedge (dd^c(j_\alpha^* G_{x'})|_{X_t})^{n-1} \\
&\leq \int_{U'_\alpha \cap X_t} (\theta_t + dd^c\varphi_t) \wedge (dd^c H_x)^{n-1} \\
&\leq \int_{U'_\alpha \cap X_t} (\theta_t + dd^c\varphi_t) \wedge (A\omega_t + dd^c H_x)^{n-1} \\
&\leq \int_{X_t} (C_\Theta\omega_t + dd^c\varphi_t) \wedge (A\omega_t + dd^c H_x)^{n-1} \\
&= C_\Theta A^{n-1} \cdot V.
\end{aligned}$$

The conclusion follows. ∎

It follows from Skoda's integrability theorem [61] that the Lelong number $\nu(\varphi_t, x)$ controls the local integrability index $\alpha(\varphi_t, x)$ of a θ_t-psh function φ_t,

$$\alpha(\varphi_t, x) := \sup\{c > 0\,;\, e^{-c\varphi_t} \in L^2_{\mathrm{loc}}(X_t, x)\},$$

via

$$\frac{1}{\nu(\varphi_t, x)} \leq \alpha(\varphi_t, x) \leq \frac{n}{\nu(\varphi_t, x)}.$$

Proposition 2.3 thus yields:

Corollary 2.4. *In Setting 2.1,*

$$\alpha(\Theta) := \inf\{\alpha(\varphi_t, x)\,;\, t \in \overline{\mathbb{D}}_{1/2},\ x \in X_t,\ \varphi_t \in \mathrm{PSH}(X_t, \theta_t)\} > 0.$$

2.3. Skoda's integrability theorem in families: the projective case

Zeriahi [72] has established a uniform version of Skoda's integrability theorem. We now further generalize Zeriahi's result by establishing its family version.

We first provide a very explicit result in the projective case which does not rely on Corollary 2.4 unlike its general Kähler analogue that will be given later (see Theorem 2.9). This should also help the reader in following the somehow tricky computations in the general Kähler case.

Proposition 2.5. *Let $V \subseteq \mathbb{P}^N$ be a projective variety of complex dimension n and degree d. Let $\omega = \omega_{\mathrm{FS}}|_V$ and $\varphi \in \mathrm{PSH}(V, \omega)$ be such that $\sup_V \varphi = 0$. Then*

$$\int_V e^{-\frac{1}{nd}\varphi}\omega^n \leq (4n)^n \cdot d \cdot \exp\left\{-\frac{1}{nd}\int_V \varphi\omega^n\right\}.$$

To our knowledge, the inequality in Proposition 2.5 is new.

Remark 2.6. When $\pi : \mathcal{X} \to \mathbb{D}$ is a projective family whose fibers have degree d with respect to a given projective embedding, the above result gives the integrability of $e^{-\frac{1}{nd}\varphi_t}$ on $V_t := \pi^{-1}(t)$. In particular, $\alpha(\omega_{\mathrm{FS}}) \geq \frac{1}{2nd}$.

Proof of Proposition 2.5. Embedding $\mathbb{P}^1$ in $\mathbb{P}^2$ if necessary, we assume without loss of generality that $N \geq 2$. We first claim that it is enough to prove the proposition when φ is *smooth*. Indeed, thanks to [19, Cor. C], there exists a sequence of smooth functions $\varphi_n \in \mathrm{PSH}(V, \omega_{\mathrm{FS}})$ decreasing pointwise to φ. Let $\varepsilon_n := \sup_V \varphi_n$; by Hartogs' theorem, we have $\varepsilon_n \to 0$. If the proposition holds for smooth functions, we will have

$$\int_V e^{-\frac{1}{nd}\varphi_n}\omega^n \leq e^{\frac{\varepsilon_n\cdot(d-1)}{nd}}(4n)^n \cdot d \cdot \exp\left\{-\frac{1}{nd}\int_V \varphi_n\omega^n\right\}.$$

Using the Fatou lemma and the monotone convergence theorem, we deduce the expected inequality for φ. From now on, we assume that φ is smooth.

The projective logarithmic kernel on $\mathbb{P}^N \times \mathbb{P}^N$ is defined by

$$G(x,y) := \log\left(\frac{\|x \wedge y\|}{\|x\| \cdot \|y\|}\right), \quad x, y \in \mathbb{P}^N,$$

writing x, y in homogeneous coordinates. By [2, Lem. 4.1], for any fixed y, $x \mapsto G(x,y)$ is a non-positive ω_{FS}-psh function in $\mathbb{P}^N$ such that $(\omega_{\mathrm{FS}} + dd^c_x G(\cdot, y))^N = \delta_y$. We set $g = G|_V$ and $g_y = g(\cdot, y)$. By definition, g_y has Lelong number 1 at y. Therefore, it follows from [22, Cor. 4.8] that $\omega^n_{g_y} := (\omega + dd^c g(\cdot, y))^n \geq \delta_y$. From the Stokes formula (cf. Lemma 2.11 below) it follows that

$$\begin{aligned}
\varphi(y) &\geq \int_V \varphi\omega^n_{g_y} = \int_V \varphi(\omega + dd^c g_y) \wedge \omega^{n-1}_{g_y} \\
&= \int_V \varphi\omega \wedge \omega^{n-1}_{g_y} + \int_V g_y(\omega + dd^c\varphi) \wedge \omega^{n-1}_{g_y} - \int_V g_y\omega \wedge \omega^{n-1}_{g_y} \\
&\geq \int_V \varphi\omega \wedge \omega^{n-1}_{g_y} + \int_V g_y\omega_\varphi \wedge \omega^{n-1}_{g_y},
\end{aligned}$$

using the fact that $g_y \le 0$. One obtains similarly

$$\int_V \varphi\omega \wedge \omega_{g_y}^{n-1} \ge \int_V \varphi\omega^2 \wedge \omega_{g_y}^{n-2} + \int_V g_y \omega \wedge \omega_\varphi \wedge \omega_{g_y}^{n-2}$$
$$\ge \int_V \varphi\omega^2 \wedge \omega_{g_y}^{n-2} + \int_V g_y \omega_\varphi \wedge \omega_{g_y}^{n-1},$$

where the second inequality follows from

$$\int_V g_y \omega \wedge \omega_\varphi \wedge \omega_{g_y}^{n-2} = \int_V g_y \omega_\varphi \wedge \omega_{g_y}^{n-1} + \int_V dg_y \wedge d^c g_y \wedge \omega_\varphi \wedge \omega_{g_y}^{n-2}$$
$$\ge \int_V g_y \omega_\varphi \wedge \omega_{g_y}^{n-1}.$$

Iterating the process n times we end up with

$$\varphi(y) \ge \int_V \varphi\omega^n + n\int_V g_y \omega_\varphi \wedge \omega_{g_y}^{n-1}.$$

Hence

$$\int_V e^{-\frac{1}{nd}\varphi}\omega^n \le \exp\left\{-\frac{1}{nd}\int_V \varphi\omega^n\right\} \cdot I,$$

where

$$I := \int_{y\in V} \exp\left\{-\frac{1}{d}\int_{x\in V} g_y(x)\omega_\varphi(x) \wedge \omega_{g_y}(x)^{n-1}\right\}\omega(y)^n.$$

The (n,n)-form $\frac{1}{d} \cdot \omega_\varphi \wedge \omega_{g_y}^{n-1}$ induces a probability measure on V given that

$$\int_V \omega_\varphi \wedge \omega_{g_y}^{n-1} = \int_{\mathbb{P}^N} \omega_\varphi \wedge \omega_{g_y}^{n-1} \wedge [V] = \{\omega_{\mathrm{FS}}\}^n \cdot \{V\} = d.$$

From Jensen's inequality, one can then derive

$$I \le \frac{1}{d}\int_{y\in V}\int_{x\in V} e^{-g(x,y)}\, \omega_\varphi(x) \wedge (\omega(x) + dd_x^c g(x,y))^{n-1} \wedge \omega(y)^n.$$

Lemma 2.8 (i) below yields

$$\omega_\varphi(x) \wedge (\omega(x) + dd_x^c g(x,y))^{n-1} \le e^{-2(n-1)g(x,y)}\omega_\varphi(x) \wedge \omega(x)^{n-1}.$$

Lemma 2.8 (ii) (for $\delta = 1/2n$) now yields

$$I \le \frac{1}{d}\int_{y\in V}\int_{x\in V} e^{(-2n+1)g(x,y)}\omega_\varphi(x) \wedge \omega(x)^{n-1} \wedge \omega(y)^n$$
$$= \frac{1}{d}\int_{x\in V}\left(\int_{y\in V} [e^{-2(1-\frac{1}{2n})g(x,y)}\omega(y)]^n\right)\omega_\varphi(x) \wedge \omega(x)^{n-1}$$
$$\le (4n)^n \int_{x\in V}\left(\frac{1}{d}\int_{y\in V} (\omega + dd^c \chi_{\frac{1}{2n}} \circ g_x)^n\right)\omega_\varphi(x) \wedge \omega(x)^{n-1}$$
$$= (4n)^n \int_{x\in V} \omega_\varphi(x) \wedge \omega(x)^{n-1} = (4n)^n \cdot d.$$

■

Remark 2.7. The same arguments as above show that for any $\gamma \in (0,2)$,

$$\int_V e^{-\frac{\gamma}{nd}\varphi}\omega^n \leq C_\gamma \cdot d \exp\left\{-\frac{\gamma}{nd}\int_V \varphi\omega^n\right\},$$

where $C_\gamma > 0$ depends on n and γ. We have fixed $\gamma = 1$ in the above proposition to simplify the statement.

Lemma 2.8. *With the notations of the proof of Proposition 2.5, fix a point $y \in V$ and set $g := g_y$. Moreover, let $\delta \in (0,1)$. Then the following inequalities hold as currents on V:*

(i) $\omega_g \leq e^{-2g}\omega$,

(ii) $\frac{\delta}{2}e^{-2(1-\delta)g}\omega \leq \omega + dd^c\chi_\delta \circ g$.

Here, χ_δ is the function defined on $\mathbb{R}$ by $\chi_\delta(t) := \frac{e^{2\delta t}}{4\delta}$.

It is understood here that we take derivatives with respect to x and the estimates are uniform both in x and y.

Proof. We proceed in three steps.

Step 1. *Reduction to a computation on $\mathbb{C}^N$.* First of all we observe that the function g as well as the (1, 1)-currents ω and ω_g are the restrictions to V of a function or (1, 1)-currents on $\mathbb{P}^N$. As positivity is preserved by restriction to a subvariety, it is enough to prove the inequalities of currents above on the whole $\mathbb{P}^N$ where they make sense as well.

Now, recall that $\mathrm{PU}(N,\mathbb{C})$ acts transitively on $\mathbb{P}^N$ by transformations preserving ω_{FS} and an isometry u sends G_y to $G_{u(y)}$. Therefore it suffices to prove all the inequalities above on $\mathbb{P}^N$, for the special point $y = [1:0:\cdots:0]$. We work in the affine chart (U_1, z) where $U_1 := \{x \in \mathbb{P}^N : x_1 \neq 0\}$ and $z := (z_j)_j$, $z_j = x_j/x_1$. In these coordinates $\omega_{\mathrm{FS}}|_{U_1} = \frac{1}{2}dd^c\log(1+\|z\|^2)$. Note that U_1 is dense in $\mathbb{P}^N$ and both $\omega_{\mathrm{FS}}, \omega_G$ are smooth on the complement $\mathbb{P}^N \setminus U_1$; thus it is sufficient to prove the inequalities on $U_1 \simeq \mathbb{C}^N$.

We actually claim that it is sufficient to prove the inequalities on $U_1 \setminus \{y\}$, where all the currents involved are smooth differential forms. This is because neither of the positive currents $e^{-2G}\omega_{\mathrm{FS}}$ and $\omega_{\mathrm{FS}} + dd^c\chi_\delta \circ G$ on $\mathbb{P}^N$ puts any mass on $\{y\}$. This follows from the integrability of e^{-2G} for the first one (recall that $N \geq 2$) and the boundedness of $\chi_\delta \circ G$ for the second one.

As observed in [2, Lem. 4.1], for $(x, y := [1:0:\cdots:0]) \in U_1 \times U_1$ we have

$$G(x,y) = N(z,0) - \tfrac{1}{2}\log(1+\|z\|^2),$$

where $z = z(x)$ and $N(z,0) := \frac{1}{2}\log\|z\|^2$. Thus in U_1 we have $e^{-2G} = 1 + \frac{1}{\|z\|^2}$ and

$$\omega(x) + dd^c_x G_y(x) = dd^c_z N(z,0) = \tfrac{1}{2}dd^c_z \log\|z\|^2.$$

Let

$$\beta := dd^c\|z\|^2 = i\sum_{k=1}^N dz_k \wedge d\bar{z}_k \quad \text{and} \quad \alpha_1 := \sum_{k=1}^N \bar{z}_k dz_k.$$

Step 2. *Proof of (i).* Standard computations give

$$(\omega_{\mathrm{FS}})_{j\bar{k}} = \frac{(1+\|z\|^2)\delta_{j\bar{k}} - \bar{z}_j z_k}{2(1+\|z\|^2)^2} \quad \text{and} \quad N_{j\bar{k}} = \frac{1}{2}\cdot\frac{\|z\|^2\delta_{j\bar{k}} - \bar{z}_j z_k}{\|z\|^4},$$

or equivalently

$$\omega_{\mathrm{FS}} = \frac{1}{2}\left(\frac{1}{1+\|z\|^2}\beta - \frac{1}{(1+\|z\|^2)^2} i\alpha_1\wedge\bar{\alpha}_1\right) \quad \text{and} \quad \omega_G = \frac{1}{2}\left(\frac{1}{\|z\|^2}\beta - \frac{1}{\|z\|^4} i\alpha_1\wedge\bar{\alpha}_1\right).$$

The matrix $A(z) := (z_i\bar{z}_j)_{ij}$ is semipositive with rank at most 1 and trace $\|z\|^2$. Therefore, if $\lambda, \mu \in \mathbb{R}$ (they can depend on z), the matrix $\lambda\,\mathrm{Id} + \mu A$ is hermitian with eigenvalues λ (with multiplicity $N-1$) and $\lambda + \|z\|^2\cdot\mu$ (with multiplicity 1). In particular, it is semipositive if and only if $\lambda \geq \max(0, -\|z\|^2\cdot\mu)$.

The computations above show that the eigenvalues of the $(1,1)$-form $\lambda\beta + \mu i\alpha_1\wedge\bar{\alpha}_1$ with respect to β are λ and $\lambda + \|z\|^2\cdot\mu$. Now, if C is some non-negative constant, the $(1,1)$-form $Ce^{-2g}\omega_{\mathrm{FS}} - \omega_G$ can be rewritten as follows:

$$\frac{1}{2(1+\|z\|^2)\|z\|^4}\cdot\left[(C-1)\|z\|^2(1+\|z\|^2)\cdot\beta + [(1+\|z\|^2) - C\|z\|^2]\cdot i\alpha_1\wedge\bar{\alpha}_1\right].$$

The latter form is semipositive if and only if $C \geq 1$. This proves (i).

Step 3. *Proof of (ii).* Observe that χ_δ is convex increasing with $0 \leq \chi'_\delta \leq 1/2$ for $t \leq 0$. Standard computations give $dd^c\chi_\delta\circ G = \chi'_\delta\circ G\, dd^c G + \chi''_\delta\circ G\, dG\wedge d^c G$. Next, we have

$$dd^c G = \frac{1}{2\|z\|^2(1+\|z\|^2)}\left[\beta - \frac{1+2\|z\|^2}{\|z\|^2(1+\|z\|^2)}\cdot i\alpha_1\wedge\bar{\alpha}_1\right]$$

with the notation introduced in Step 1. Similarly,

$$dG\wedge d^c G = \frac{1}{4\|z\|^4(1+\|z\|^2)^2} i\alpha_1\wedge\bar{\alpha}_1.$$

To lighten notation, we will from now on write χ' (resp. χ'') to denote $\chi'_\delta\circ G$ (resp. $\chi''_\delta\circ G$). One has

$$\omega_{\mathrm{FS}} + dd^c\chi_\delta\circ G = \frac{1}{2(1+\|z\|^2)}\left[\left(1+\frac{\chi'}{\|z\|^2}\right)\beta + \frac{\frac{1}{2}\chi'' - \chi'(1+2\|z\|^2)}{\|z\|^4(1+\|z\|^2)} i\alpha_1\wedge\bar{\alpha}_1\right].$$

As a result, the two eigenvalues λ, μ of $\omega_{\mathrm{FS}} + dd^c\chi_\delta\circ G$ with respect to ω_{FS} are given by

$$\lambda = 1 + \frac{\chi'}{\|z\|^2},$$
$$\mu = (1+\|z\|^2)\cdot\left(1 + \frac{\chi'}{\|z\|^2} + \frac{\frac{1}{2}\chi'' - \chi'(1+2\|z\|^2)}{\|z\|^2(1+\|z\|^2)}\right) = (1+\|z\|^2 - \chi') + \frac{\chi''}{2\|z\|^2}.$$

Using the definition of χ and the fact that $e^{-2G} = 1 + \frac{1}{\|z\|^2}$, one easily sees that $\lambda \geq \frac{1}{2}e^{-2(1-\delta)G}$ and $\mu \geq \frac{\delta}{2}e^{-2(1-\delta)G}$. The conclusion follows. ∎

2.4. *Skoda's integrability theorem in families: the general case*

In this section, we bypass the projectivity assumption and establish a quite general family version of Skoda's integrability theorem, valid for families of compact Kähler varieties:

Theorem 2.9. *In Setting 2.1, choose $\alpha \in (0, \alpha(\Theta))$, which is possible thanks to Corollary 2.4. Then there exist constants $A_\alpha, C > 0$ such that for all $t \in \overline{\mathbb{D}}_{1/2}$ and for all $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$ with $\sup_{X_t} \varphi_t = 0$,*

$$\int_{X_t} e^{-\alpha\varphi_t}\omega_t^n \le C \exp\left\{-A_\alpha \int_{X_t} \varphi_t \omega_t^n\right\}. \tag{2.4}$$

Proof. The proof follows the strategy of [72], as presented in [35, Thm. 2.50]. There exist a finite number of trivializing charts $\{U_\tau\}$ of $\mathcal{X}$ such that $\pi^{-1}(\overline{\mathbb{D}}_{1/2}) \subset \bigcup_\tau U_\tau$. The statement will then follow if we prove the bound for the integral on the left-hand side replacing X_t by $X_t \cap U_\tau$. Moreover, we can assume that we have an immersion $j_\tau : U_\tau \hookrightarrow \mathbb{B}$, where $\mathbb{B}$ is the unit ball in $\mathbb{C}^N$. Up to shrinking U_τ, one can also assume that there exists a smooth function ρ on $\mathbb{B}$ such that $\sup_{\mathbb{B}} \rho = -2$ and $\Theta|_{U_\tau} = dd^c j_\tau^*\rho$. We define $\rho_t := (j_\tau^*\rho)|_{U_\tau \cap X_t}$; this is a potential of $\theta_t|_{U_\tau\cap X_t}$. Note that $\psi_t := \varphi_t + \rho_t$ is a non-positive psh function in $U_\tau \cap X_t$ such that

$$\varphi_t - 2 \ge \psi_t \ge \varphi_t - C_\tau \tag{2.5}$$

for some constant $C_\tau > 0$ depending only on U_τ. It is also clear that proving (2.4) is equivalent to showing that

$$\int_{U_\tau \cap X_t} e^{-\alpha\psi_t}\omega_t^n \le C_\tau \exp\left\{-A_{\alpha,\tau} \int_{U_\tau\cap X_t} \psi_t \omega_t^n\right\} \tag{2.6}$$

for some constants $C_\tau, A_{\alpha,\tau}$ that do not depend on t.

Claim 2.10. *It is sufficient to prove (2.6) for smooth, non-positive psh functions ψ_t on $U_\tau \cap X_t$ such that*

$$dd^c\psi_t \ge (j_\tau^* dd^c\|z\|^2)|_{X_t}. \tag{2.7}$$

Proof of Claim 2.10. Indeed, as

$$\int_{U_\tau \cap X_t} e^{-\alpha\psi_t}\omega_t^n \le e^\alpha \int_{U_\tau\cap X_t} e^{-\alpha(\psi_t + j_\tau^*\|z\|^2)}\omega_t^n,$$

we can replace ψ_t by the function $\psi_t + j_\tau^*\|z\|^2$, bounded above by -1. Next, thanks to a result of Fornæss–Narasimhan [30, Thm. 5.5], one can write ψ_t as a decreasing limit of non-positive, smooth psh functions on $U_\tau \cap X_t$ (up to possibly shrinking U_τ). The combination of the monotone convergence theorem and the integrability of $e^{-\alpha\varphi_t}$ on X_t provided by Corollary 2.4 settles the claim. ∎

From now on, we assume that ψ_t is smooth, and we work exclusively on U_τ, which we view inside the unit ball $\mathbb{B}$ of $\mathbb{C}^N$. By abuse of notation, we will denote by $\mathbb{B} \cap X_t$ the set $U_\tau \cap X_t$. In the same vein, we will identify the coordinate functions $z = (z_1, \dots, z_N)$ on $\mathbb{B} \subset \mathbb{C}^N$ with their pull-backs by j_τ on U_τ.

Let us pick some number $t \in \overline{\mathbb{D}}_{1/2}$ and some point $x \in \mathbb{B} \cap X_t$. We denote by Φ_x the automorphism of the unit ball $\mathbb{B}$ that sends x to the origin and consider

$$G_x(z) := \log \|\Phi_x(z)\|,$$

the pluricomplex Green function of the unit ball $\mathbb{B}$. Recall that G_x is the unique plurisubharmonic function in $\mathbb{B}$ such that $(dd^c G_x)^N = \delta_x$ in the weak sense of currents, $G_x \le 0$ and G_x is identically zero on $\partial\mathbb{B}$. Standard computations yield

$$dd^c G_x \le \frac{C_0}{\|\Phi_x(z)\|^2}\, dd^c \|z\|^2 \quad \text{on } \mathbb{B} \tag{2.8}$$

for some dimensional constant $C_0 = C_0(N) > 0$.

Since $[X_t|_{\mathbb{B}}]$ is a positive $(N-n, N-n)$-current on $\mathbb{B}$ and the singular set of the restriction of the Green function $G_x|_{X_t}$ is compact (it is indeed equal to $\{x\}$), the mixed Monge–Ampère measure $(dd^c G_x)^n \wedge [X_t]$ is well defined [35, Prop. 3.15] and it has a Dirac mass with coefficient ≥ 1 at the point x. Since $\psi_t \le 0$ we then have

$$\psi_t(x) \ge \int_{\mathbb{B}} \psi_t (dd^c G_x)^n \wedge [X_t] = \int_{\mathbb{B}\cap X_t} \psi_t (dd^c G_x)^n.$$

Now, we have the following result, which is the Stokes formula in the context of isolated singularities.

Lemma 2.11. *Let $X \subset B_{\mathbb{C}^N}(0,2)$ be a proper, n-dimensional complex subspace of the ball of radius 2 in $\mathbb{C}^N$, centered at the origin. Let u, v, w be psh functions on $B_{\mathbb{C}^N}(0,2)$ with isolated singularities, i.e. they are smooth outside a discrete set of points in $B_{\mathbb{C}^N}(0,2)$ which we assume does not meet $\partial B_{\mathbb{C}^N}(0,1)$. Finally, let $\mathbb{B} := B_{\mathbb{C}^N}(0,1) \cap X$. Then*

$$\int_{\partial\mathbb{B}} (u d^c v - v d^c u) \wedge (dd^c w)^{n-1} = \int_{\mathbb{B}} (u dd^c v - v dd^c u) \wedge (dd^c w)^{n-1}. \tag{2.9}$$

We include a proof for the reader's convenience.

Proof. By using a (regularized) maximum operation, we can find a family of smooth psh functions u_ε (resp. $v_\varepsilon, w_\varepsilon$) decreasing to u (resp. v, w) and which coincide with their limit outside a compact set $K_\varepsilon \Subset \mathbb{B}$ which collapses to a finite set $S \Subset \mathbb{B}$. By the usual Stokes formula, one has

$$\int_{\partial\mathbb{B}} (u_\varepsilon d^c v_\varepsilon - v_\varepsilon d^c u_\varepsilon) \wedge (dd^c w_\varepsilon)^{n-1} = \int_{\mathbb{B}} (u_\varepsilon dd^c v_\varepsilon - v_\varepsilon dd^c u_\varepsilon) \wedge (dd^c w_\varepsilon)^{n-1}.$$

The left-hand side above is identical to the left-hand side of (2.9). To prove that the right-hand side above converges to the right-hand side of (2.9), we prove that the current $(u_\varepsilon dd^c v_\varepsilon - v_\varepsilon dd^c u_\varepsilon) \wedge (dd^c w_\varepsilon)^{n-1}$ on $\mathbb{B}$ converges to $(u dd^c v - v dd^c u) \wedge (dd^c w)^{n-1}$

weakly, globally on $\mathbb{B}$ and locally smoothly on $\overline{\mathbb{B}} \setminus S$. The local smooth convergence outside S is obvious. As for the global weak convergence, it follows from the convergence $u_\varepsilon dd^c v_\varepsilon \wedge (dd^c w_\varepsilon)^{n-1} \rightharpoonup u dd^c v \wedge (dd^c w)^{n-1}$ (and its symmetrical version swapping u and v), proved by Demailly (see e.g. [22, Thm. 2.6 and Rem. 2.10]). ■

Applying Lemma 2.11 to $X = X_t$, $u = \psi_t$, $v = w = G_x$ (recall that $G_x|_{\partial\mathbb{B}} \equiv 0$), we get

$$\int_{\mathbb{B}\cap X_t} \psi_t\,(dd^c G_x)^n = \underbrace{\int_{\mathbb{B}\cap X_t} G_x\, dd^c\psi_t \wedge (dd^c G_x)^{n-1}}_{=:I_t} + \underbrace{\int_{\partial\mathbb{B}\cap X_t} \psi_t\, d^c G_x \wedge (dd^c G_x)^{n-1}}_{=:J_t}.$$

By Lemma 2.12, in order to get a lower bound for J_t, it is enough to bound from above the quantity $\int_{\partial\mathbb{B}\cap X_t}(-\psi_t)\, d^c\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1}$. Applying (2.9) to $u = -\psi_t$, $v = w = \|z\|^2 - 1$, we find

$$\begin{aligned}\int_{\partial\mathbb{B}\cap X_t} (-\psi_t)\, d^c\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1} &= \int_{\mathbb{B}\cap X_t} (-\psi_t)\,(dd^c\|z\|^2)^n \\ &\quad + \int_{\mathbb{B}\cap X_t} (\|z\|^2 - 1)\, dd^c\psi_t \wedge (dd^c\|z\|^2)^{n-1} \\ &\le \int_{\mathbb{B}\cap X_t} (-\psi_t)\,(dd^c\|z\|^2)^n \\ &\le C_1^n \left[\int_{X_t} (-\varphi_t)\,\omega_t^n + C_\tau \cdot V\right],\end{aligned}$$

where C_1 is such that $dd^c\|z\|^2 \le C_1\omega$ on $\mathbb{B}$ and C_τ is given in (2.5).

We now take care of the most singular term I_t. Set

$$\gamma_t(x) := \int_{\mathbb{B}} dd^c\psi_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]$$

so that $\mu := \gamma_t^{-1} dd^c\psi_t \wedge (dd^c G_x)^{n-1} \wedge [X_t]$ is a probability measure on $\mathbb{B}$ (depending on x). We claim that for any $x \in \mathbb{B}$ there exists a constant $\nu > 0$ independent of t and x such that $1 \le \gamma_t < \nu$. The uniform upper bound follows from the same computations as in the proof of Proposition 2.3. By (2.7) we can infer that

$$\begin{aligned}\int_{\mathbb{B}} dd^c\psi_t \wedge (dd^c G_x)^{n-1} \wedge [X_t] &\ge \int_{\mathbb{B}} dd^c\|z\|^2 \wedge (dd^c G_x)^{n-1} \wedge [X_t] \\ &\ge \nu((dd^c G_x)^{n-1} \wedge [X_t], x) \ge \nu([X_t], x) = m(X_t, x) \ge 1.\end{aligned}$$

In the second inequality we have used the fact that $r \mapsto \frac{1}{r^2}\int_{\mathbb{B}_r} dd^c\|z\|^2 \wedge T$ is decreasing to $\nu(T, x)$ when $r \downarrow 0$ (see (2.2)). The first equality follows from (2.3), while the second one comes from Thie's theorem. Recall that the origin of $\mathbb{B}$ is identified with the point x.

We now use Jensen's formula and (2.8) to obtain

$$\begin{aligned}\exp(-\alpha I_t(x)) &= \exp\Big(\int_{z\in\mathbb{B}} -\alpha\gamma_t G_x\, d\mu\Big)\\ &\le \frac{1}{\gamma_t}\int_{z\in\mathbb{B}} e^{-\alpha\gamma_t G_x}\, dd^c\psi_t\wedge(dd^c G_x)^{n-1}\wedge[X_t]\\ &= \frac{1}{\gamma_t}\int_{z\in\mathbb{B}} \frac{dd^c\psi_t\wedge(dd^c G_x)^{n-1}\wedge[X_t]}{\|\Phi_x(z)\|^{\alpha\gamma_t}}\\ &\le C_0\int_{z\in\mathbb{B}} \frac{dd^c\psi_t\wedge(dd^c\|z\|^2)^{n-1}\wedge[X_t]}{\|\Phi_x(z)\|^{\alpha\nu+2n-2}},\end{aligned}$$

where we can assume that $\alpha\nu < 2$. By Fubini's theorem, we have

$$\begin{aligned}\int_{x\in\mathbb{B}_{1/2}} e^{-\alpha\psi_t}\omega^n\wedge[X_t] &\le \int_{x\in\mathbb{B}_{1/2}} e^{-\alpha(I_t+J_t)}\omega^n\wedge[X_t] \le K\int_{x\in\mathbb{B}_{1/2}} e^{-\alpha I_t}\omega^n\wedge[X_t]\\ &\le C_0 K\int_{x\in\mathbb{B}_{1/2}}\Big(\int_{z\in\mathbb{B}}\frac{dd^c\psi_t\wedge(dd^c\|z\|^2)^{n-1}\wedge[X_t]}{\|\Phi_x(z)\|^{\alpha\nu+2n-2}}\Big)\omega^n\wedge[X_t]\\ &\le C_0 K\int_{z\in\mathbb{B}}\Big(\int_{x\in\mathbb{B}_{1/2}}\frac{(dd^c\|x\|^2)^n\wedge[X_t]}{\|\Phi_x(z)\|^{\alpha\nu+2n-2}}\Big)dd^c\psi_t\wedge(dd^c\|z\|^2)^{n-1}\wedge[X_t],\end{aligned}$$

where $K := \exp\{-\alpha\, C_1^n \int_{X_t}\psi_t\,\omega_t^n\}$. Moreover, using the same computation as in the proof of Lemma 2.13 below, one can check that if $\beta := \frac{2-\alpha\nu}{2n} > 0$, then there exists a constant $C_\beta > 0$ such that the inequality between (n,n)-currents below holds on $\mathbb{B}$:

$$C_\beta^{-1}\,(dd^c_x\|\Phi_x(z)\|^{2\beta})^n \le \frac{1}{\|\Phi_x(z)\|^{\alpha\nu+2n-2}}(dd^c\|x\|^2)^n \le C_\beta\,(dd^c_x\|\Phi_x(z)\|^{2\beta})^n. \tag{2.10}$$

Fix $z\in\mathbb{B}$ and for any $x\in\mathbb{B}$ let $f_x(z) := \|\Phi_x(z)\|$. We define an extension of f_x to $\mathcal{X}$ by

$$F_x(z) := \begin{cases}\chi\cdot f_x(z) & \text{if } x\in\mathbb{B},\\ 0 & \text{else.}\end{cases}$$

Here, χ is a smooth cut-off function such that $\mathrm{Supp}(\chi)\subset\mathbb{B}$ and $\chi\equiv 1$ on $\mathbb{B}_{1/2}$. It is easy to check that F_x is an $A\omega$-psh function on $\mathcal{X}$ for some $A = A_\tau$ large enough (which a priori depends on U_τ but can be chosen independently of $x\in\mathbb{B}_{1/2}$). Thus

$$\begin{aligned}\int_{x\in\mathbb{B}_{1/2}}\frac{1}{\|\Phi_x(z)\|^{\alpha\nu+2n-2}}(dd^c\|x\|^2)^n\wedge[X_t] &\le C_\beta\int_{x\in\mathbb{B}_{1/2}}(dd^c_x\|\Phi_x(z)\|^{2\beta})^n\wedge[X_t]\\ &\le C_\beta\int_{x\in\mathcal{X}}(A\omega+dd^c_x F_x(z)^{2\beta})^n\wedge[X_t] \le C_\beta A^n V =: C_2.\end{aligned}$$

It then follows that

$$\int_{x\in\mathbb{B}_{1/2}} e^{-\alpha\psi_t}\omega^n\wedge[X_t] \le C_0 C_2 K\int_{z\in\mathbb{B}} dd^c\psi_t\wedge(dd^c\|z\|^2)^{n-1}\wedge[X_t] \le C_3 K,$$

where $C_3 := C_0 C_2 C_\Theta C_1^{n-1} \cdot V$. The last inequality follows from the fact that on $\mathbb{B}_t$, we have $dd^c\psi_t \wedge (dd^c\|z\|^2)^{n-1} \le (\theta_t + dd^c\varphi_t) \wedge (C_1\omega)^{n-1}$, and one can dominate the integral of the right-hand side on $\mathbb{B}_t$ by its integral on X_t and use (2.1). This is the conclusion. ■

Lemma 2.12. *With the notations introduced at the beginning of the proof of Theorem 2.9, there is a constant $C = C(n) > 0$ such that for all $x \in \mathbb{B}_{1/2} \subset \mathbb{C}^N$ and $z \in X_t \cap \mathbb{S}^{2N-1}$,*

$$\frac{1}{C} d^c\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1} \le d^c G_x \wedge (dd^c G_x)^{n-1} \le C d^c\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1}. \tag{2.11}$$

Proof. There exists a neighborhood U of $\mathbb{S}^{2N-1} \subset \mathbb{C}^N$ not containing x such that $dd^c\|\Phi_x\|^2$ defines a Kähler form ω_x on U; this follows for instance from the fact that Φ_x can be extended as a holomorphic map to an open neighborhood of the closed ball – and that neighborhood can be chosen to be independent of $x \in \mathbb{B}_{1/2}$. On U, ω_x is comparable to the euclidean metric on $\mathbb{C}^N$, and therefore ω_x and ω_{eucl} induce uniformly equivalent Riemannian metrics g_x and g_{eucl} on $U \cap X_t$, and then as well on the real hypersurface $X_t \cap \mathbb{S}^{2N-1}$; we denote them by g_x' and g_{eucl}' respectively. In particular, their volume forms $dV_{g_x'}, dV_{g_{\mathrm{eucl}}'}$ are equivalent too. One has $dV_{g_{\mathrm{eucl}}'} = \iota_v dV_{g_{\mathrm{eucl}}}$ where v is the restriction to X_t of the unit outward radial vector

$$\sum_{j=1}^{n+k} \left(z_j \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

Hence, on $X_t \cap \mathbb{S}^{2N-1}$ one has

$$dV_{g_{\mathrm{eucl}}'} = \iota_v (dd^c\|z\|^2)^n = 2\left(\frac{i}{\pi}\right)^{n-1} d^c\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1}.$$

In the same way, $dV_{g_x'} = \iota_{v_x} dV_{g_x}$, where v_x is the restriction to X_t of the unit outward vector with respect to $dd^c\|\Phi_x\|^2$, hence $v_x = \Phi_x^* v$. Therefore, on $X_t \cap \mathbb{S}^{2N-1}$,

$$\begin{aligned} dV_{g_x'} &= \iota_{v_x}(dd^c\|\Phi_x\|^2)^n = \Phi_x^*(\iota_v(dd^c\|z\|^2)^n) \\ &= 2\left(\frac{i}{\pi}\right)^{n-1} d^c\|\Phi_x\|^2 \wedge (dd^c\|\Phi_x\|^2)^{n-1} = 2^{n+1}\left(\frac{i}{\pi}\right)^{n-1} d^c G_x \wedge (dd^c G_x)^{n-1}, \end{aligned}$$

since $G_x = \frac{1}{2}\log\|\Phi_x\|^2$ vanishes on the sphere and $d^c \log u \wedge (dd^c \log u)^{n-1} = \frac{1}{u^n} d^c u \wedge (dd^c u)^{n-1}$ for any smooth function u. This shows that the above two volume forms on $X_t \cap \mathbb{S}^{2N-1}$ are uniformly equivalent on $X_t \cap \mathbb{S}^{2N-1}$, which ends the proof. ■

Lemma 2.13. *Let $\beta > 0$ and $\mathbb{B} \subset \mathbb{C}^n$ be the unit ball. Then $\|z\|^{2\beta}$ is psh on $\mathbb{B}$ and there exists a constant $C_\beta > 0$ (that depends only on β) such that*

$$\frac{C_\beta^{-1}}{\|z\|^{2(1-\beta)}} \cdot dd^c\|z\|^2 \le dd^c\|z\|^{2\beta} \le \frac{C_\beta}{\|z\|^{2(1-\beta)}} \cdot dd^c\|z\|^2.$$

Proof. Let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\chi(t) := t^\beta$ and $u := \|z\|^2$. One has

$$dd^c \chi \circ u = \beta u^{\beta-1}(dd^c u - (1-\beta)u^{-1} du \wedge d^c u).$$

Note that $\min(1,\beta) \cdot dd^c u \leq dd^c u - (1-\beta)u^{-1} du \wedge d^c u \leq \max(1,\beta) \cdot dd^c u$. Observe that the hermitian matrix associated to the $(1,1)$-form $du \wedge d^c u$ is $(\bar{z}_i z_j)_{i\bar{j}}$. The latter has rank 1 and its non-zero eigenvalue coincides with its trace, i.e. u. Therefore the eigenvalues of the hermitian matrix $A := \mathrm{I}_n - (1-\beta)u^{-1}(\bar{z}_i z_j)_{i\bar{j}}$ are 1 (with multiplicity $n-1$) and β (with multiplicity 1). This ends the proof. ■

3. Normalization in families

The previous section allows us to check hypothesis (H1), as soon as the mean value of sup-normalized θ_t-psh functions is uniformly controlled. It is classical that one can compare the supremum and the mean value of θ-psh functions on a fixed compact Kähler variety (see [35, Prop. 8.5]). We conjecture that the following result holds:

Conjecture 3.1. In Setting 2.1, there exists a constant $C > 0$ such that

$$\sup_{X_t} \varphi_t - C \leq \frac{1}{V} \int_{X_t} \varphi_t \, {\omega_t}^n \leq \sup_{X_t} \varphi_t$$

for all $t \in \overline{\mathbb{D}}_{1/2}$ and for every function $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$.

In a preprint version of this paper, we claimed a proof of the conjecture above but a referee, whom we thank, pointed out a gap. In this section, we propose a large class of families for which the conjecture holds. More precisely, let us consider the following

Assumption 3.2. *In Setting 2.1, we assume additionally that one of the following conditions is satisfied by the family $\pi : \mathcal{X} \to \mathbb{D}$:*

(1) *The map π is projective.*

(2) *The map π is locally trivial.*

(3) *The fibers X_t are smooth for $t \neq 0$.*

(4) *The fibers X_t have isolated singularities for every $t \in \mathbb{D}$.*

Recall that π is said to be

– *projective* if we have a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \overset{\iota}{\hookrightarrow} & \mathbb{P}^N \times \mathbb{D} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & \mathbb{D} & \end{array}$$

– *locally trivial* if, up to shrinking $\mathbb{D}$, there exists a euclidean open cover $(U_\alpha)_\alpha$ of $\mathcal{X}$ and a collection of isomorphisms

$$F_\alpha : \mathcal{X}|_{U_\alpha} \xrightarrow{\simeq} (U_\alpha \cap X_0) \times \mathbb{D}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{X}|_{U_\alpha} & \xrightarrow{\ F_\alpha\ } & (U_\alpha \cap X_0) \times \mathbb{D} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & \mathbb{D} & \end{array} \tag{3.1}$$

For instance, if $\mathcal{X}$ is smooth and if the map π is a holomorphic submersion, then π is automatically locally trivial.

The main result in this section is

Proposition 3.3. *In Setting 2.1 and if Assumption 3.2 is satisfied, then Conjecture 3.1 holds. That is, there exists a constant* $C > 0$ *such that*

$$\sup_{X_t} \varphi_t - C \leq \frac{1}{V} \int_{X_t} \varphi_t \, \omega_t^n \leq \sup_{X_t} \varphi_t$$

for all $t \in \overline{\mathbb{D}}_{1/2}$ *and for every function* $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$.

We will prove Proposition 3.3 in several independent steps.

- In §3.2, we prove the locally trivial case.
- In §3.3, we treat the case of isolated singularities.
- In §§3.4–3.6 we introduce the material (Sobolev and Poincaré inequalities, heat kernels and Green functions) that we will use in the final section.
- In §3.7, we establish at the same time the projective case and the case of a smoothing, thereby completing the proof of Proposition 3.3.

By combining the above result with Theorem 2.9, we get the following

Theorem 3.4. *In Setting 2.1, choose* $\alpha \in (0, \alpha(\Theta))$, *which is possible thanks to Corollary 2.4. If Assumption 3.2 is satisfied, then there exists a constant* $C_\alpha > 0$ *such that for all* $t \in \overline{\mathbb{D}}_{1/2}$ *and all* $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$,

$$\int_{X_t} e^{-\alpha(\varphi_t - \sup_{X_t} \varphi_t)} \omega_t^n \leq C_\alpha.$$

3.1. Irreducibility of the fibers

The irreducibility of *all* the fibers is a necessary assumption for the left-hand inequality in Conjecture 3.1 to hold, as the following example shows:

Example 3.5. Consider $\mathcal{X} \subset \mathbb{P}^2 \times \mathbb{C}$ where

$$\mathcal{X} := \{([x:y:z], t)\,;\, xy - tz^2 = 0\}.$$

The variety $\mathcal{X}$ is smooth and comes equipped with the proper morphism $\pi : \mathcal{X} \to \mathbb{C}$ induced by the second projection $\mathbb{P}^2 \times \mathbb{C} \to \mathbb{C}$. Set $X_t = \{[x:y:z] \in \mathbb{P}^2\,;\, xy = tz^2\}$.

Note that X_t is a smooth conic for $t \neq 0$ while $X_0 = \{[x:y:z] \in \mathbb{P}^2\,;\, xy = 0\}$ is the union of two lines. The quasi-psh function φ on $\mathbb{P}^2$ defined by

$$\varphi([x:y:z]) = \frac{1}{4}\big(\log(|x|^2+|z|^2) + \log|y|^2\big) - \frac{1}{2}\log(|x|^2+|y|^2+|z|^2) + \frac{\log 2}{2}$$

clearly induces a ω-psh function Φ on $\mathcal{X}$, where $\omega = \omega_{\mathrm{FS}} + dd^c|t|^2$,

$$\Phi([x:y:z],t) = \varphi([x:y:z]).$$

We set $\varphi_t := \Phi|_{X_t}$ and $\omega_t := \omega|_{X_t}$. A simple computation shows that $\sup_{\mathcal{X}} \Phi = 0$ and it is attained at points $([x:y:z],t)$ such that $|y|^2 = |x|^2+|z|^2$. We also find that $\sup_{X_t} \varphi_t = 0$ and the supremum is attained on the set

$$S_t := \left\{[x:1:z]\,;\, |x| = \frac{1}{2|t|}\cdot\big(\sqrt{4|t|^2+1}-1\big),\ z^2 = xt^{-1}\right\}.$$

As $t \to 0$, S_t becomes the circle $\mathcal{C} := \{[0:1:e^{i\theta}]\,;\,\theta \in \mathbb{R}\} \subset X_0$. Note also that $X_0 = \ell \cup \ell'$, where $\ell := \{[0:y:z]\}$, $\ell' := \{[x:0:z]\}$ and $\mathcal{C} \subset \ell$. The open annulus $U_t := \{[z^2:t:z]\,;\, 1 < |z|^2 < 2\} \subset X_t$ satisfies

$$\int_{U_t} \omega_t \geq \delta$$

for some $\delta > 0$ independent of t, as well as

$$\varphi_t|_{U_t} \leq \tfrac{1}{2}(\log|t|+1),$$

from which it follows that

$$\lim_{t\to 0}\int_{X_t} \varphi_t\,\omega_t = -\infty.$$

3.2. *The locally trivial case*

In this section, we prove Proposition 3.3 under the assumption that π is locally trivial; we borrow the notations from diagram (3.1).

One can reduce the problem to showing that there exists a constant $C > 0$ depending only on π such that given any sequence of complex numbers $t_k \to 0$ and any functions $\varphi_k \in \mathrm{PSH}(X_{t_k}, \theta_{t_k})$ such that $\sup_{X_{t_k}} \varphi_k = 0$, one has

$$\int_{X_{t_k}} \varphi_k \omega_{t_k}^n \geq -C.$$

By compactness of $\pi^{-1}(\overline{\mathbb{D}}_{1/2})$, one can assume that α ranges over the finite set $\{1,\ldots,r\}$ and without loss of generality, one can assume that $U_{\alpha+1} \cap U_\alpha \neq \emptyset$ for any $\alpha \in \{1,\ldots,r-1\}$. Up to splitting the sequence (φ_k) into (at most) r subsequences, we can assume that for every k, φ_k attains its maximum in the same set U_{α_0} for some fixed $\alpha_0 \in \{1,\ldots,r\}$. For simplicity, we assume that $\alpha_0 = 1$.

Let $G_{\alpha,k} : U_\alpha \cap X_0 \to U_\alpha \cap X_{t_k}$ be the biholomorphism defined as the inverse of the restriction of F_α to $U_\alpha \cap X_{t_k}$ and let us analyze the sequence of functions $\psi_{\alpha,k} := G^*_{\alpha,k}\varphi_k$. As $F^*_\alpha(\omega_0 + i dt \wedge d\bar{t})$ is commensurable to ω, there exists $C > 0$ depending only on π such that

$$C^{-1}\omega_0 \le G^*_{\alpha,k}\omega_{t_k} \le C\omega_0. \tag{3.2}$$

In particular, up to increasing C, one can assume that $G^*_{\alpha,k}\theta_{t_k} \le C\omega_0$. As a result, one has $\psi_{\alpha,k} \in \mathrm{PSH}(U_\alpha \cap X_0, C\omega_0)$.

The family $(\psi_{1,k})_k$ is a family of non-positive $C\omega_0$-psh functions on the complex space $U_1 \cap X_0$ attaining the value zero there, so it is relatively compact for the L^1_{loc} topology (see e.g. [35, Prop. 8.5]). In particular, given any compact subset $U_1' \Subset U_1$, the integral $\int_{U_1'} \psi_{1,k}\omega_0^n$ admits a lower bound depending only on U_1' but not on k.

Next, the family $(\psi_{2,k})_k$ is a family of non-positive $C\omega_0$-psh functions on $U_2 \cap X_0$. Therefore, either it converges locally uniformly to $-\infty$ or it is relatively compact on each compact subset. From (3.2), it follows that the family of automorphisms $H_k := (G^{-1}_{2,k})|_{U_1\cap U_2 \cap X_{t_k}} \circ G_1|_{U_1 \cap U_2 \cap X_0}$ of $U_1 \cap U_2 \cap X_0$ satisfies

$$C^{-1}\omega_0 \le H_k^*\omega_0 \le C\omega_0 \quad \text{and} \quad \psi_{2,k} = H_k^*\psi_{1,k}.$$

One then deduces easily that for any compact subset $U_{12}' \Subset U_1 \cap U_2$, the integral $\int_{U_{12}'} \psi_{2,k}\omega_0^n$ admits a lower bound independent of k. In turn, this implies that $(\psi_{2,k})_k$ is relatively compact for the L^1_{loc} topology on the whole $U_2 \cap X_0$.

By iterating the argument, one finds that for any α, the family $(\psi_{\alpha,k})_k$ is relatively compact for the $L^1_{\mathrm{loc}}(U_\alpha \cap X_0)$ topology and using estimate (3.2), one concludes easily that $\int_{X_{t_k}} \varphi_k \omega_{t_k}^n$ admits a uniform lower bound as claimed.

This shows that Proposition 3.3 holds whenever π is locally trivial. An easy consequence is

Corollary 3.6. *In Setting 2.1, there exists a discrete set $Z \subset \mathbb{D}$ such that for every compact subset $K \Subset \mathbb{D} \setminus Z$, there exists a constant C_K such that*

$$\int_{X_t} \varphi_t \omega_t^n \ge -C_K$$

for any collection of functions $\varphi_t \in \mathrm{PSH}(X_t, \theta_t)$ such that $\sup_{X_t} \varphi_t = 0$. Moreover, one can take $Z = \emptyset$ provided that the family $\pi : \mathcal{X} \to \mathbb{D}$ admits a simultaneous resolution of singularities, i.e. a proper, surjective holomorphic map $f : \mathcal{Y} \to \mathcal{X}$ from a Kähler manifold $\mathcal{Y}$ such that for any $t \in \mathbb{D}$, the induced morphism $f|_{Y_t} : Y_t \to X_t$ is a resolution of singularities, where $Y_t := f^{-1}(X_t)$.

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a resolution of singularities of $\mathcal{X}$. One can assume that $\mathcal{Y}$ is a Kähler manifold; let us pick a Kähler form ω_Y on $\mathcal{Y}$. The induced map $\rho := \pi \circ f : \mathcal{Y} \to \mathbb{D}$ is surjective, hence by generic smoothness, it is smooth over the complement of a proper analytic subset Z of $\mathbb{D}$. In particular, Z is discrete. Note that over Z, the fibers of ρ may have several irreducible components.

We denote by f_t the restriction $f|_{Y_t} : Y_t \to X_t$ of f to the fiber $Y_t := \rho^{-1}(t)$. For any $t \in \mathbb{D} \setminus Z$, the map f_t is bimeromorphic, i.e. it is a resolution of singularities of X_t. Let us choose a compact subset $K \Subset \mathbb{D}$. There exists a constant C_K such that $f^*\omega \leq C_K \omega_Y$ on $\rho^{-1}(K)$. In particular, for any $t \in K$, one has $f_t^*\varphi_t \in \mathrm{PSH}(Y_t, C_K C_\Theta \, \omega_Y)$ and $\sup_{Y_t} f_t^*\varphi_t = 0$. Now, if we additionally assume that $K \Subset \mathbb{D} \setminus Z$, we can apply the result above to the *smooth* family $\rho|_{\rho^{-1}(K)} : \rho^{-1}(K) \to K$ to find another constant $C'_K > 0$ satisfying

$$\int_{Y_t} (f_t^*\varphi_t) \, \omega_Y^n \geq -C'_K$$

for any $t \in K$. As $\omega_Y^n \geq C_K^{-n} \, f_t^*\omega_t^n$, we deduce that

$$\int_{X_t} \varphi_t \, \omega_t^n \geq -C'_K \cdot C_K^n,$$

which concludes the first part of the proof. The second statement is an immediate consequence of the proof of the first one. Indeed, if Y_t is smooth (as an analytic space), then $\pi \circ f$ is smooth in a neighborhood of Y_t and the argument above can be run over a neighborhood of t. ■

3.3. *The case of isolated singularities*

In this section, we prove Proposition 3.3 in the case where all fibers X_t, $t \in \mathbb{D}$, have isolated singularities.

Remark 3.7. We start with two observations.

- This case includes the case where $n = \dim X_t = 1$.
- If one only assumes that X_0 has isolated singularities, then it is easy to check that there exists $\varepsilon > 0$ such that X_t has isolated singularities for any t satisfying $|t| < \varepsilon$. This is because the locus $Z \subset \mathcal{X}$ where π is not smooth is an analytic set such that $\dim(Z \cap X_0) = 0$ and by upper semicontinuity, Z has relative dimension 0 over a neighborhood of $0 \in \mathbb{D}$.

We now proceed to prove Proposition 3.3 in several steps.

Step 1. *Localization of the problem at $t = 0$.* Let $f : \mathcal{Y} \to \mathcal{X}$ be a resolution of singularities $\mathcal{X}$. The induced family $\pi \circ f : \mathcal{Y} \to \mathbb{D}$ is generically smooth over $\mathbb{D}$, so for $r > 0$ small enough, the restriction of $\pi \circ f$ to the inverse image of $\mathbb{D}_r$ has at most one singular fiber, corresponding to $t = 0$. In particular, the family $\mathcal{Y} \to \mathbb{D}_r$ is locally trivial away from Y_0. Applying the result in the locally trivial case (see §3.2) to the collection of $f^*\theta_t$-psh functions $f^*\varphi_t$, we see that for every compact subset $K \Subset \mathbb{D}_r^*$, there exists a constant C_K independent of the chosen family such that

$$\sup_{t \in K} \int_{X_t} (-\varphi_t) \omega_t^n \leq C_K$$

(cf. also Corollary 3.6). This shows that it is enough to prove that for any sequence $t_k \to 0$ and any collection of sup-normalized θ_{t_k}-psh functions φ_{t_k}, one has

$$\sup_{k\geq 1} \int_{X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n < +\infty.$$

Step 2. *Choice of a good covering.* As the fibers are reduced, it follows from the jacobian criterion for smoothness that the smooth locus of π coincides with the union of the smooth loci of X_t when t ranges over $\mathbb{D}$. Recall that Z, the singular locus of π, is an analytic space of relative dimension at most zero. It has finitely many irreducible components (say when restricted to $\pi^{-1}(\mathbb{D}_{1/2})$) and we can assume without loss of generality that this number is equal to the cardinality of $Z \cap X_0$. Let $(V_\alpha)_\alpha$ be a finite collection of (small) open balls in $\mathcal{X}$ centered at the (finitely many) singular points of X_0. Up to adding a finite number of balls to the collection, one can assume that:

(i) The union $V := \bigcup_\alpha V_\alpha$ is an open neighborhood of $X_0 \subset \mathcal{X}$.
(ii) Each point of $Z \cap X_0$ belongs to exactly one element V_α of the covering.
(iii) For all α, there exists $\rho_\alpha \in \mathcal{C}^\infty(V_\alpha, \mathbb{R})$ such that $\omega|_{V_\alpha} = dd^c\rho_\alpha$.
(iv) There exists $r > 0$ such that for all α,

$$Z \cap \partial V_\alpha \cap \pi^{-1}(\mathbb{D}_r) = \emptyset.$$

Up to subtracting a constant from ρ_α, one can assume that ρ_α is non-negative. Moreover, there exists a constant $C_1 > 0$ such that $\rho_\alpha \leq C_1$ on V_α for any α. Let $(\chi_\alpha)_\alpha$ be a partition of unity associated to the covering $(V_\alpha)_\alpha$. That means that $\sum_\alpha \chi_\alpha \equiv 1$ and $\mathrm{Supp}(\chi_\alpha) \subset V_\alpha$. Finally, let $\rho := \sum \chi_\alpha\rho_\alpha$. It follows from (ii) that $\omega = dd^c\rho$ in some neighborhood W'_α of each point of $Z \cap X_0$. We pick a relatively compact open subset $W_\alpha \Subset W'_\alpha$ and set $W := \bigcup W_\alpha$. Up to decreasing r a little, one can assume that $Z \cap \partial W \cap \pi^{-1}(\overline{\mathbb{D}}_r) = \emptyset$. In particular, there exists $\delta > 0$ such that for any $t \in \overline{\mathbb{D}}_r$, one has $d_\omega(\partial W \cap X_t, Z) \geq \delta$. In summary,

$$0 \leq \rho \leq C_1, \quad \omega = dd^c\rho \text{ on } W, \quad d_\omega(\partial W \cap X_t, Z) \geq \delta \text{ for all } t \in \overline{\mathbb{D}}_r. \tag{3.3}$$

Step 3. *Weak compactness locally outside Z.* Let t_k be a sequence of numbers converging to zero, and let $\varphi_{t_k} \in \mathrm{PSH}(X_{t_k}, \theta_{t_k})$ be such that $\sup_{X_{t_k}} \varphi_{t_k} = 0$. We claim that there exists a sequence of points $x_k \in X_{t_k}$ and a constant $C_2 > 0$ such that:

(i) $\varphi_{t_k}(x_k) \geq -C_2$.
(ii) $d_\omega(x_k, Z) \geq \delta/2$.

Indeed, let $y_k \in X_{t_k}$ be such that $\varphi_{t_k}(y_k) = 0$. If $d_\omega(y_k, Z) \geq \delta/2$, then we are done. Otherwise, $y_k \in W$ by the third statement of (3.3). Now, the function $C_\Theta\rho + \varphi_{t_k}$ is psh on $\overline{W}$ so by the maximum principle, there exists $x_k \in \partial W$ such that $(C_\Theta\rho + \varphi_{t_k})(x_k) \geq (C_\Theta\rho + \varphi_{t_k})(y_k) \geq 0$. By the first statement of (3.3), we deduce $\varphi_{t_k}(x_k) \geq -C_2$ where $C_2 := C_1C_\Theta$.

Let $U := \{x \in \pi^{-1}(\mathbb{D}_r)\,; d(x, Z) > \delta/2\}$. The map π is smooth on U and one can cover U by *finitely* many open subsets $(U_j)_{1\leq j\leq p}$ isomorphic to $(U_j \cap X_0) \times \mathbb{D}_r$ over $\mathbb{D}_r$.

Because of (i), we can argue as in the locally trivial case (cf. §3.2) by exporting the functions $\varphi_{t_k}|_{U_j \cap X_{t_k}}$ to the fixed space $U_j \cap X_0$ and get relative compactness there. In particular, one can find a constant $C_3 > 0$ independent of k such that

$$\int_{U \cap X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n \leq C_3. \tag{3.4}$$

Step 4. *The integral bound.* On W, one has $\omega = dd^c\rho$. This implies that $\omega^n = (dd^c\rho)^n + T$ for some smooth, closed (n,n)-form T on $\pi^{-1}(\mathbb{D}_r)$ such that $T|_W \equiv 0$. Let us introduce constants C_4, C_5 such that $-C_4\omega^{n-1} \leq (dd^c\rho)^{n-1} \leq C_4\omega^{n-1}$ and $T \leq C_5\omega^n$. As the complement of W in $\pi^{-1}(\mathbb{D}_r)$ is included in U, it follows from (3.4) that

$$\int_{X_{t_k}} (-\varphi_{t_k})\, T \leq C_5 C_3. \tag{3.5}$$

Moreover,

$$\begin{aligned}
\int_{X_{t_k}} (-\varphi_{t_k})(dd^c\rho)^n &= \int_{X_{t_k}} -\rho\, dd^c\varphi_{t_k} \wedge (dd^c\rho)^{n-1} \\
&= -\int_{X_{t_k}} \rho(\theta_{t_k} + dd^c\varphi_{t_k}) \wedge (dd^c\rho)^{n-1} + \int_{X_{t_k}} \rho\theta_{t_k} \wedge (dd^c\rho)^{n-1} \\
&\leq C_4 C_1 \int_{X_{t_k}} (\theta_{t_k} + dd^c\varphi_{t_k}) \wedge \omega^{n-1} + C_\Theta C_4 C_1 \cdot V \\
&\leq 2C_1 C_4 C_\Theta \cdot V.
\end{aligned}$$

All in all, one finds that

$$\int_{X_{t_k}} (-\varphi_{t_k})\omega_{t_k}^n \leq C_6,$$

where $C_6 = C_3C_5 + 2C_1C_4C_\Theta \cdot V$.

3.4. Sobolev and Poincaré inequalities

In this section, we work in Setting 2.1 and we assume from now on that the relative dimension $n = \dim_{\mathbb{C}} X_t$ satisfies $n > 1$, since the case $n = 1$ has already been dealt with in §3.3 (cf. Remark 3.7).

For $t \in \mathbb{D}$, we set $X_t := \pi^{-1}(t)$ and denote by X_t^{reg} the regular locus of X_t. We fix a Kähler form ω on $\mathcal{X}$ and set

$$\omega_t := \omega|_{X_t}.$$

Proposition 3.8. *Let $K \Subset \mathbb{D}$. There exists $C_S = C_S(K)$ such that*

$$\forall t \in K,\ \forall f \in \mathcal{C}_0^\infty(X_t^{\mathrm{reg}}), \quad \left(\int_{X_t} |f|^{\frac{2n}{n-1}}\omega_t^n\right)^{\frac{n-1}{n}} \leq C_S \int_{X_t} (|f|^2 + |df|^2_{\omega_t})\omega_t^n.$$

Remark 3.9. The inequality above extends immediately to the functions $f \in W^{1,2}(X_t^{\mathrm{reg}})$, i.e. such that $f, df \in L^2(X_t^{\mathrm{reg}}, \omega_t)$.

Proof of Proposition 3.8. Because of the existence of a partition of unity, the statement above is local. This means that it is enough to show the above inequality for any $t \in K$ and any $f \in \mathscr{C}_0^\infty(U_i \cap X_t^{\mathrm{reg}})$ where $U_i \subset \mathcal{X}$ are open sets such that $\bigcup U_i = \mathcal{X}$.

We fix such an open set U_i and we drop the index i in what follows. Without loss of generality, one can assume that there exists an embedding $U_i \hookrightarrow \mathbb{C}^N$ and that $\omega|_U$ and $\omega_{\mathbb{C}^N}|_U$ are quasi-isometric. Because the Sobolev inequality is essentially insensitive to quasi-isometry, it is enough to show the inequality replacing ω_t by $\omega_{\mathbb{C}^N}|_{U_t}$ where $U_t := U \cap X_t$.

Now, the isometric embeddings $(U_t^{\mathrm{reg}}, \omega_{\mathbb{C}^N}|_{U_t}) \hookrightarrow (\mathbb{C}^N, \omega_{\mathbb{C}^N})$ provide a family of minimal submanifolds (i.e. with zero mean curvature vector) of the euclidean space by virtue of the Wirtinger inequality. The expected inequality is now a direct application of Michael–Simon's Sobolev inequality [53, Thm. 2.1]. ■

Proposition 3.10. *Let $K \Subset \mathbb{D}$. There exists $C_P = C_P(K)$ such that*

$$\forall t \in K,\ \forall f \in W_0^{1,2}(X_t^{\mathrm{reg}}), \quad \int_{X_t} |f|^2 \omega_t^n \le C_P \int_{X_t} |df|^2_{\omega_t} \omega_t^n.$$

In the statement above, the space $W_0^{1,2}(X_t^{\mathrm{reg}})$ is defined as the space of functions f on X_t^{reg} such that $f, df \in L^2(X_t^{\mathrm{reg}}, \omega_t)$ and $\int_{X_t} f\omega_t^n = 0$.

Proof. First, we claim that for each $t \in \mathbb{D}$, there exists such a Poincaré constant $C_{P,t}$. Indeed, thanks to [5, Thm. 0.2], the Laplacian Δ_{ω_t} is positive, self-adjoint and its spectrum is discrete. It remains to show that its kernel is one-dimensional. Now, if $f \in W^{1,2}(X_t^{\mathrm{reg}})$ is such that $\Delta_t f = 0$, it means that for every $u \in W^{1,2}(X_t^{\mathrm{reg}})$, we have $\langle \nabla u, \nabla f \rangle = 0$. In particular, taking $u = f$ shows that f is locally constant on X_t^{reg}. As X_t is irreducible, X_t^{reg} is connected and the result follows.

Given the absolute case explained above, the family version of the Poincaré inequality follows from Proposition 3.8 and the irreducibility of the fibers: we refer the reader to [57, Prop. 3.2] for a detailed argument (the projectivity assumption made by Ruan–Zhang being unnecessary for this part of the argument). ■

3.5. *Heat kernels and Green's functions*

In this section and the following one, we go back to the absolute case and consider an *irreducible* and reduced Kähler space (X, ω) of dimension $n = \dim_{\mathbb{C}} X$ satisfying $n > 1$.

When X is smooth, it is well known (see e.g. [15, §VI]) that there exists a smooth, positive function H on $X \times X \times (0, +\infty)$, symmetric in its space variable and such that if $\Delta := \mathrm{tr}_\omega dd^c$, then

- $(-\Delta_y + \partial_t)H(x, y, t) = 0$.
- For every $x \in X$, $H(x, \cdot, t)\omega^n \to \delta_x$ weakly as $t \to 0$.

In the general case where X may have singularities, one can introduce $X_\varepsilon = X \setminus V_\varepsilon$ where V_ε is a closed ε-neighborhood of X_{sing} with smooth boundary. Then there exists a unique smooth, positive function H_ε on $X_\varepsilon \times X_\varepsilon \times (0, +\infty)$ such that:

- $(-\Delta_y + \partial_t)H_\varepsilon(x, y, t) = 0$;
- $H_\varepsilon(x, y, t) \to 0$ whenever x or y approaches ∂X_ε.
- For every $x \in X_\varepsilon$, $H_\varepsilon(x, \cdot, t)\omega^n \to \delta_x$ weakly as $t \to 0$.

Moreover, given $(x, y, t) \in X_{\varepsilon_0} \times X_{\varepsilon_0} \times (0, +\infty)$, the function $(0, \varepsilon_0) \ni \varepsilon \mapsto H_\varepsilon(x, y, t)$ is decreasing. Using [15, VIII.2 Thm. 4] and its proof, we additionally see that the limit $H := \lim_\varepsilon H_\varepsilon$ is everywhere finite and satisfies:

- H is positive and smooth on $X_{\text{reg}} \times X_{\text{reg}} \times (0, +\infty)$.
- $(-\Delta_y + \partial_t)H(x, y, t) = 0$.
- For all $x, y \in X_{\text{reg}}$ and $t, s > 0$,

$$H(x, y, t + s) = \int_X H(x, \cdot, t)H(\cdot, y, s)\omega^n. \tag{3.6}$$

- For any $x \in X_{\text{reg}}$, $H(x, \cdot, t)\omega^n \to \delta_x$ weakly as $t \to 0$.

When $X \subset \mathbb{P}^N$ is *projective* and $\omega = \omega_{\text{FS}}|_X$, Li and Tian [51] have showed that there is an absolute inequality

$$H(x, y, t) \le H_{\mathbb{P}^n}(d_{\mathbb{P}^N}(x, y), t) \tag{3.7}$$

for any $x, y \in X_{\text{reg}}$ and $t \in (0, +\infty)$, where $H_{\mathbb{P}^n}$ is the heat kernel of $(\mathbb{P}^n, \omega_{\text{FS}})$, whose dependence on the space variables x, y is known to reduce to a single real variable, namely the distance between those two points.

In particular, $H(x, \cdot, t)$ is *bounded* on X_{reg} for any $x \in X_{\text{reg}}$ and $t > 0$. Since X_{sing} has real codimension at least 2, it admits cut-off functions whose gradient converges to zero in L^2, and this allows one to perform integration by parts as in the compact case for *bounded* functions in $W^{1,2}$. We refer to [51, Lem. 3.1] for more details; we will also rely on the latter result which states that $H(x, \cdot, t)$ is in $W^{1,2}$ and that it satisfies the conservation property

$$\forall t > 0, \quad \int_X H(x, \cdot, t)\omega^n = 1.$$

Below are a few more properties that will be useful later, which are certainly standard in the smooth case. For this purpose, one introduces the function

$$G(x, y, t) := H(x, y, t) - 1/V, \quad \text{where} \quad V := \int_X \omega^n.$$

The key information for us will be given by the fourth item, for which the arguments are borrowed from [16]; see also [59, App. A].

Lemma 3.11. *Assume either that X is smooth or that $X \subset \mathbb{P}^N$ is projective and $\omega = \omega_{\text{FS}}|_X$. Let $x, y \in X_{\text{reg}}$. Then:*

(1) *$G(x,y,t) > -1/V$, $\int_X G(x,\cdot,t)\omega^n = 0$ and $\int_X |G(x,\cdot,t)|\omega^n \le 2$.*

(2) *$|G(x,y,t)|^2 \le G(x,x,t)G(y,y,t)$.*

(3) *$H(x,x,t) \to +\infty$ as $t \to 0$.*

(4) *There exists a constant C_0 depending only on the Sobolev and Poincaré constants of (X_{reg},ω) such that*
$$|G(x,y,t)| \le C_0 t^{-n}$$
for any $x,y \in X_{\text{reg}}$ and any $t > 0$.

Proof. Under the assumptions on X, we know that $H(x,\cdot,t)$ is bounded in $W^{1,2}$ on X_{reg} and satisfies the conservation property. We will only rely on these non-quantitative properties to establish the items below, and not on the more precise inequality (3.7) which certainly does not hold if X is not projective.

(1) is a trivial consequence of the positivity of H and the fact that $\int_X H(x,\cdot,t)\omega^n = 1$.

(2) is classical when X is smooth, so we assume for the time being that X is projective. Let K_ε be the *Neumann* heat kernel on X_ε, let $V_\varepsilon := \int_{X_\varepsilon} \omega^n$ and let $\widetilde{G}_\varepsilon := K_\varepsilon - 1/V_\varepsilon$. Then
$$K_\varepsilon(x,y,t) = \sum_{i\ge 0} e^{-\lambda_{i,\varepsilon} t}\phi_{i,\varepsilon}(x)\phi_{i,\varepsilon}(y),$$
where $(\phi_{i,\varepsilon})$ is an orthonormal basis of $L^2(X_\varepsilon)$ consisting of Neumann eigenfunctions of $-\Delta$ with eigenvalues $\lambda_{i,\varepsilon}$. Note that $\phi_{0,\varepsilon} = 1/\sqrt{V_\varepsilon}$. By Cauchy–Schwarz, we find that
$$|\widetilde{G}_\varepsilon(x,y,t)|^2 \le \widetilde{G}_\varepsilon(x,x,t)\cdot \widetilde{G}_\varepsilon(y,y,t).$$
Thanks to [51, Lem. 3.2], K_ε converges to H locally smoothly on $X^2_{\text{reg}} \times (0,+\infty)$ when $\varepsilon \to 0$, hence $\widetilde{G}_\varepsilon \to G$ in the same way and we get the second item.

(3) Since $H \ge H_\varepsilon$, it is enough to show the third claim for H_ε. We consider a Sturm–Liouville decomposition as before,
$$H_\varepsilon(x,y,t) = \sum_{i\ge 0} e^{-\mu_{i,\varepsilon} t}\psi_{i,\varepsilon}(x)\psi_{i,\varepsilon}(y),$$
but now $(\psi_{i,\varepsilon})$ is an orthonormal basis of $L^2(X_\varepsilon)$ consisting of *Dirichlet* eigenfunctions of $-\Delta$ with eigenvalues $\mu_{i,\varepsilon}$ (see [15, VII (31)]). The sought property now follows since $\sum \psi_{i,\varepsilon}(x)^2$ is the norm of the unbounded functional $L^2 \cap \mathscr{C}^\infty(X_\varepsilon) \ni f \mapsto f(x)$.

(4) We start from the identity (3.6), which holds for G as well as one checks easily. Taking $y = x$ and differentiating with respect to s and eventually setting $s := t$, one finds
$$-G'(x,x,2t) = \|dG(x,\cdot,t)\|^2_{L^2} \ge (C_S(C_P+1))^{-1}\|G(x,\cdot,t)\|^2_{L^{\frac{2n}{n-1}}}$$
since integration by parts is legitimate as we explained above and $\int_X G(x,\cdot,t)\omega^n = 0$. Moreover, the interpolation inequality gives
$$G(x,x,2t) = \|G(x,\cdot,t)\|^2_{L^2} \le \|G(x,\cdot,t)\|^{\frac{2}{n+1}}_{L^1}\cdot \|G(x,\cdot,t)\|^{\frac{2n}{n+1}}_{L^{\frac{2n}{n-1}}},$$

hence

$$\|G(x,\cdot,t)\|^2_{L^{\frac{2n}{n-1}}} \geq 2^{-\frac{2}{n}} G(x,x,2t)^{\frac{n+1}{n}}$$

and

$$-\frac{1}{n} G'(x,x,t) G(x,x,t)^{-1-\frac{1}{n}} \geq C^{-1} \quad \text{for } C = n4^{1/n} \cdot C_S(C_P+1).$$

Integrating this inequality with respect to t and using the second item, we get the fourth item – recall that $G(x,x,t) > 0$ for any $x \in X_{\mathrm{reg}}$ given its expansion as power series (cf. (2)). ∎

Under the assumptions of Lemma 3.11 above, the integral

$$G(x,y) := \int_0^{+\infty} G(x,y,t)\,dt$$

is convergent whenever $x \neq y$ and defines a function G on $X_{\mathrm{reg}} \times X_{\mathrm{reg}}$ such that $G(x,\cdot)$ is in $L^1(X_{\mathrm{reg}})$. Moreover, since $(-\Delta + \partial_t)G(x,\cdot,t) = 0$ and

$$G(x,\cdot,t) \xrightarrow{t\to+\infty} 0, \quad G(x,\cdot,t)\omega^n \xrightarrow{t\to 0} \delta_x - 1/V,$$

we have

$$dd^c G(x,\cdot) \wedge \omega^n = \omega^n/V - \delta_x,$$

i.e. for all $f \in \mathscr{C}_0^\infty(X_{\mathrm{reg}})$, we have

$$\int_X \Delta f \cdot G(x,\cdot)\,\omega^n = \frac{1}{V}\int_X f\,\omega^n - f(x). \tag{3.8}$$

Finally, the first and fourth items of Lemma 3.11 enable us to find a lower bound of the Green function as follows

$$G(x,y) = \int_0^1 G(x,y,t)\,dt + \int_1^{+\infty} G(x,y,t)\,dt \geq -\frac{1}{V} - \frac{C}{n-1}, \tag{3.9}$$

where C only depends on the Sobolev and Poincaré constants of $(X_{\mathrm{reg}},\omega)$.

3.6. *Green's inequality for general psh functions*

In this section, we assume that the assumptions of Lemma 3.11 are satisfied.

Let us first generalize formula (3.8) to some functions $f \in \mathscr{C}^\infty(X_{\mathrm{reg}})$ that are not necessarily compactly supported. For that purpose, let $p: Y \to X$ be a log resolution of singularities, let D be the exceptional divisor of p and let $Y^\circ := p^{-1}(X_{\mathrm{reg}}) = Y \setminus D$. We claim that for any $f \in \mathscr{C}^\infty(X_{\mathrm{reg}})$ such that p^*f extends smoothly across D,

$$\int_{X_{\mathrm{reg}}} \Delta f \cdot G(x,\cdot)\omega^n = \frac{1}{V}\int_{X_{\mathrm{reg}}} f\,\omega^n - f(x) \tag{3.10}$$

for all $x \in X_{\mathrm{reg}}$. First observe that all the terms are well defined as one sees by pulling back by p, which is an isomorphism over X_{reg}. Indeed, recall that $x \in X_{\mathrm{reg}}$ and that $G(x,\cdot)$

is locally bounded near X_{sing} so that $p^*G(x,\cdot)$ is in $L^1(Y^\circ,\omega_Y)$ for any Kähler form ω_Y on Y.

Next, we choose a family $(\chi_\delta)_\delta$ of cut-off functions for D. As they are identically 0 on D, they come from X under p and one can see them as functions on X or Y interchangeably. It is classical (see e.g. [12, Sect. 9]) that one can choose χ_δ such that both $d\chi_\delta\wedge d^c\chi_\delta$ and $\pm dd^c\chi_\delta$ are dominated by some fixed Poincaré metric ω_P (independently of δ). In particular, using Cauchy–Schwarz and the dominated convergence theorem, one finds

$$\lim_{\delta\to 0}\int_{X_{\mathrm{reg}}} G(x,\cdot)[f dd^c\chi_\delta + df\wedge d^c\chi_\delta + d\chi_\delta\wedge d^c f]\wedge\omega^{n-1} = 0. \tag{3.11}$$

Formula (3.10) is now a direct application of (3.8).

The next result is the key for the proof of Proposition 3.3.

Claim 3.12. *Under the assumptions of Lemma 3.11, let $\varphi\in\mathrm{PSH}(X,\omega)$, $V=\int_X\omega^n$ and let $x\in X_{\mathrm{reg}}$. Then*

$$\frac{1}{V}\int_X \varphi\omega^n - \varphi(x) \geq nV\cdot \inf_{X_{\mathrm{reg}}} G(x,\cdot).$$

Proof. Replacing φ by $\max(\varphi,-j)$ and letting $j\to+\infty$, one sees that it is enough to prove the claim for bounded functions φ. Next, thanks to Demailly's regularization theorem, one can write $p^*\varphi$ as a pointwise decreasing limit of smooth function ψ_ε satisfying $p^*\omega+\varepsilon\omega_Y+dd^c\psi_\varepsilon\geq 0$ for some fixed Kähler metric ω_Y on Y. Using (3.10) and setting $G_x:=G(x,\cdot)$, one finds

$$\frac{1}{V}\int_X \varphi\omega^n - \varphi(x) = \lim_{\varepsilon\to 0}\int_{Y^\circ} np^*G_x dd^c\psi_\varepsilon\wedge p^*\omega^{n-1}.$$

Moreover, as G_x have zero mean value, one has

$$\begin{aligned}
\int_{Y^\circ} p^*G_x dd^c\psi_\varepsilon\wedge p^*\omega^{n-1} &= \int_{Y^\circ}\Big(p^*G_x - \inf_{X_{\mathrm{reg}}} G_x\Big) dd^c\psi_\varepsilon\wedge p^*\omega^{n-1}\\
&= \int_{Y^\circ}\Big(p^*G_x - \inf_{X_{\mathrm{reg}}} G_x\Big)(p^*\omega+\varepsilon\omega_Y+dd^c\psi_\varepsilon)\wedge p^*\omega^{n-1}\\
&\quad - \int_{Y^\circ} p^*G_x\wedge(p^*\omega+\varepsilon\omega_Y)\wedge p^*\omega^{n-1} + \inf_{X_{\mathrm{reg}}} G_x\cdot\Big(V+\varepsilon\int_Y\omega_Y\wedge p^*\omega^{n-1}\Big)\\
&\geq \inf_{X_{\mathrm{reg}}} G_x\cdot V + \varepsilon\cdot\Big(\inf_{X_{\mathrm{reg}}} G_x\cdot\int_Y \omega_Y\wedge p^*\omega^{n-1} - \int_{Y^\circ} p^*G_x\omega_Y\wedge p^*\omega^{n-1}\Big).
\end{aligned}$$

Taking the limit as $\varepsilon\to 0$, we get the expected result. ∎

3.7. Proof of Proposition 3.3

We can now finish the proof of Proposition 3.3. It remains to treat the cases where π is projective or X_t is smooth for $t\neq 0$. Moreover, we can assume that $n=\dim X_t\geq 2$, since otherwise X_t would have at most isolated singularities and we could then appeal to §3.3 (see Remark 3.7).

Moreover, the content of Proposition 3.3 is insensitive to replacing ω by another Kähler metric on $\mathcal{X}$. In the case where π is projective, i.e. if $\mathcal{X} \subset \mathbb{P}^N \times \mathbb{D}$ is such that π commutes with the second projection, then we will assume that $\omega = \omega_{\mathrm{FS}}|_{\mathcal{X}}$.

Finally, in the case where X_t is smooth for $t \neq 0$, it is sufficient to prove Proposition 3.3 for $t \neq 0$ since it is already well known that the L^1-sup comparison holds on the *fixed* irreducible complex space X_0.

We know from §3.4 that the Kähler manifolds $(X_t^{\mathrm{reg}}, \omega_t)$ admit uniform Poincaré and Sobolev constants. As the volume V of (X_t, ω_t) is constant, it follows from (3.9) that there exists $C_G > 0$ independent of t such that

$$\forall x, y \in X_t^{\mathrm{reg}}, \quad G_t(x, y) \geq -C_G,$$

where $G_t(\cdot,\cdot)$ is the Green function of (X_t, ω_t). As φ_t is sup-normalized and upper semicontinuous, there exists $x_t \in X_t^{\mathrm{reg}}$ such that $\varphi_t(x_t) \geq -1$. Applying Claim 3.12 to $\varphi := \varphi_t$ and $x := x_t$, we find

$$\frac{1}{V}\int_{X_t} (-\varphi_t)\omega_t^n \leq nVC_G + 1.$$

The proposition is proved.

4. Densities along a log-canonical map

We now pay attention to hypotheses (H2) and (H2′). We focus on the integrability properties of some canonical densities.

4.1. Semistable model

Setting 4.1. *Let $\pi : \mathcal{X} \to \mathbb{D}$ be a proper, holomorphic surjective map from a Kähler space $\mathcal{X}$ with connected fibers to the unit disk of relative dimension n. We make the following assumption:*

$$\textit{For each } t \in \mathbb{D}, \textit{ the pair } (\mathcal{X}, X_t) \textit{ has log-canonical singularities.} \tag{4.1}$$

Here $X_t = \pi^{-1}(t)$ is the schematic fiber at $t \in \mathbb{D}$ [46, Def. 7.1].

About the singularities. In Setting 4.1, the following properties hold:

(1) Every fiber is reduced, $K_{\mathcal{X}/\mathbb{D}}$ is $\mathbb{Q}$-Cartier and $\mathcal{X}$ has log-canonical singularities.

(2) The space $\mathcal{X}$ has canonical singularities if and only if the general fiber X_t has canonical singularities [46, Lem. 7.2].

(3) The condition (4.1) is preserved by finite base change from a smooth curve [46, Lem. 7.6].

(4) If $(\mathcal{X}, X_0)$ has lc singularities, then $(\mathcal{X}, X_t)$ has lc singularities for $|t| \ll 1$ ([44, Cor. 4.10 (2)] and [45, Thm. 2.3]).

(5) By *loc. cit.*, the condition (4.1) is equivalent to asking $\mathcal{X}$ to be normal, $\mathbb{Q}$-Gorenstein, and that each fiber X_t has semi-log-canonical singularities.

By [42], one can find a semistable model of π.[1] More precisely, up to shrinking $\mathbb{D}$, there exists a finite cover $\varphi : t \mapsto t^k$ of the disk for some integer $k \geq 1$ and a proper, surjective birational morphism $f : \mathcal{X}' \to \mathcal{X} \times_\varphi \mathbb{D}$

$$\begin{array}{ccccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \times_\varphi \mathbb{D} & \xrightarrow{g} & \mathcal{X} \\ & \searrow^{\pi'} & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{D} & \xrightarrow{\varphi} & \mathbb{D} \end{array} \tag{4.2}$$

such that $\mathcal{X}'$ is smooth, f is isomorphic over the smooth locus of π and such that around any point $x' \in X'_0$, there exists an integer $p \leq n+1$ and a system of coordinates $(z_0, \dots, z_n)$ centered at x' and such that $\pi'(z_0, \dots, z_n) = z_0 \cdots z_p$.

Additional assumption. Up to shrinking $\mathbb{D}$, we will assume that π' is smooth away from 0 so that for any $t \neq 0$, the induced morphism $(g \circ f)|_{X'_t} : X'_t \to X_t$ is a resolution of singularities. Note that X'_t need not be connected.

Let $m \geq 1$ be an integer such that $mK_{\mathcal{X}/\mathbb{D}}$ is a Cartier divisor. We can cover $\mathcal{X}$ with open sets U_i such that $U_i \cap \mathcal{X}^{\mathrm{reg}}$ admits a nowhere vanishing section $\Omega_{U_i} \in H^0(U_i \cap \mathcal{X}^{\mathrm{reg}}, mK_{\mathcal{X}/\mathbb{D}})$. For any $t \in \mathbb{D}$, the restriction $\Omega_{U_i}|_{X_t^{\mathrm{reg}}}$ defines a nowhere vanishing section $\Omega_{U_i}|_{X_t^{\mathrm{reg}}} \in H^0(U_i \cap X_t^{\mathrm{reg}}, mK_{X_t})$. In particular, mK_{X_t} is a Cartier divisor for all t. We want to understand the behavior of the volume forms $(\Omega_{U_i} \wedge \overline{\Omega_{U_i}})|_{X_t^{\mathrm{reg}}}^{1/m}$ when $t \to 0$. In order to do so, it is enough to work on $\mathcal{X} \times_\varphi \mathbb{D}$ directly as explained below.

Reduction step. The finite map g induces an isomorphism of $\mathbb{Q}$-line bundles $K_{\mathcal{X}\times_\varphi \mathbb{D}/\mathbb{D}} \simeq g^* K_{\mathcal{X}/\mathbb{D}}$. In particular, one can replace $\mathcal{X}$ by $\mathcal{X} \times_\varphi \mathbb{D}$ in the following, or equivalently assume that $\varphi = \mathrm{Id}_{\mathbb{D}}$, i.e. $k = 1$. By what was said above, the "new" family still satisfies condition (4.1).

4.2. *Analytic expression of the densities in a semistable model*

Let us start with some notation. Once and for all, we fix an open set $U := U_{i_0}$ for some i_0. We set $\Omega := \Omega_U$ and $\Omega_t := \Omega|_{X_t^{\mathrm{reg}}}$. One can cover $f^{-1}(U)$ by a finite number of open subsets $V_j \subset \mathcal{X}'$ isomorphic to the unit polydisk of $\mathbb{C}^{n+1}$ and endowed with a system of coordinates as above. We let $V := V_{j_0}$ be one of them. The goal is to understand $f^*\Omega$ when restricted to V, using our preferred set of coordinates. Finally, we set $U_t := U \cap X_t$ and $V_t := V \cap X'_t$.

[1] The reference [42] deals with the case of a proper morphism between algebraic varieties but the construction extends to the analytic case *mutatis mutandis*, as stated e.g. in [46, Thm. 7.17].

Next, we write

$$K_{\mathcal{X}'} + Y_0 = f^*(K_{\mathcal{X}} + X_0) + \sum_i a_i E_i \tag{4.3}$$

where the E_i's are f-exceptional divisors with $a_i \geq -1$ for all i and Y_0 is the strict transform of X_0. Note that some of the divisors E_i may be irreducible components of X_0'. The others surject onto $\mathbb{D}$ thanks to the additional assumption made in the previous section. The divisor $E := \sum_i E_i$ is the exceptional locus of f and $E + Y_0$ has simple normal crossing support. Under our assumptions, the analytic set

$$\mathrm{Nklt}(\mathcal{X}, X_0) := f\Big(\bigcup_{a_i=-1} E_i\Big) \tag{4.4}$$

contains the non-klt locus of every fiber X_t, $t \in \mathbb{D}$. This is an easy consequence of the adjunction formula, at least when the X_t's are normal.

We now let $x' \in Y_0$ and we assume that the coordinates mentioned above are chosen such that $Y_0 = (z_0 \cdots z_r = 0)$ locally, for $0 \leq r \leq p$ being the number of irreducible components of Y_0 minus 1 on that chosen open set.

On V_t, $t \neq 0$, the functions $(z_1, \dots, z_n)$ induce a system of coordinates and the form $f^*\Omega$ on V can be seen as a collection of m-th powers of holomorphic n-forms

$$f^*\Omega_t = g_t(z_1, \dots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m}$$

for some holomorphic function g_t on $V_t \setminus E$, with poles of order at most $(-ma_i)_+$ along $E_i \cap X_t$. The form $\Omega \wedge \pi^*\big(\frac{dt}{t}\big)^{\otimes m}$ is a trivialization of $m(K_{\mathcal{X}} + X_0)$ over U^{reg}. The pull-back $f^*(\Omega \wedge \pi^*\big(\frac{dt}{t}\big)^{\otimes m})$ is a well defined m-th power of an $(n+1)$-form on $f^{-1}(U^{\mathrm{reg}})$ with logarithmic poles along Y_0 that extends meromorphically over $f^{-1}(U)$ with poles of order at most $(-ma_i)_+$ along E_i. As

$$f^*\pi^*\left(\frac{dt}{t}\right) = (\pi')^*\left(\frac{dt}{t}\right) = \sum_{i=0}^{p} \frac{dz_i}{z_i}$$

on V, the form $f^*(\Omega \wedge \pi^*(\frac{dt}{t})^{\otimes m})$ is equal on that set to

$$(-1)^{mn}(z_1 \cdots z_r)^m g_{\pi'(z)}(z_1, \dots, z_n)\left(\frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_r}{z_r} \wedge dz_{r+1} \wedge \cdots \wedge dz_n\right)^{\otimes m}$$

so that the function $(V \setminus E \cup Y_0) \ni z \mapsto (-1)^{nm}(z_1 \cdots z_r)^m g_{\pi'(z)}(z_1, \dots, z_n)$ extends to a meromorphic function h on V, holomorphic along Y_0 and with poles of order at most $(-ma_i)_+$ along E_i and satisfying

$$f^*\Omega_t = (-1)^{mn} \frac{h(z)}{(z_1 \cdots z_r)^m}(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m} \tag{4.5}$$

on V_t, for $t \neq 0$. When $t = 0$, one can also obtain a formula as above for $f^*\Omega_0$ but it requires to first choose a component $Y_0^{(k)}$ of Y_0. Let $0 \leq i \leq r$ be such that $Y_0^{(k)} \cap V_0 = (z_i = 0)$. On that set (say after removing E), one has

$$f^*\Omega_0 = (-1)^{i+mn} \frac{h(z)}{(z_1 \cdots \widehat{z_i} \cdots z_r)^m}(dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n)^{\otimes m}. \tag{4.6}$$

Note that if X_0 (or equivalently Y_0) is irreducible, then $r = 0$ in the formula above.

Claim 4.2. *If X_0 has canonical singularities, then $r = 0$ and the meromorphic function $V \ni z \mapsto h(z)$ is holomorphic on V.*

Proof. As X_0 is normal, it is irreducible, hence Y_0 is smooth and irreducible. In particular, the map $f|_{Y_0} : Y_0 \to X_0$ induces a resolution of singularities.

As X_0 has canonical singularities, the pull-back $f^*\Omega_0$ of the form Ω_0 on $X_0^{\mathrm{reg}} \cap U$ extends holomorphically across $Y_0 \cap E$. Given (4.6), it means that $h|_{V\cap Y_0}$ extends holomorphically along each $E_i \cap Y_0$. As h is holomorphic on V and does not vanish outside V_0, its divisor is an n-dimensional variety supported on $V \cap E$, we have $\mathrm{div}(h) = \sum b_i E_i$ for some integers b_i. As $E + Y_0$ is snc, the decomposition $\mathrm{div}(h|_{Y_0}) = \sum b_i (E_i \cap Y_0)$ is the decomposition into irreducible components. As $h|_{Y_0}$ is holomorphic along the non-empty set $Y_0 \cap E_i$, we necessarily have $b_i \geq 0$ for any i. The claim is proved. ∎

4.3. Integrability properties of the canonical densities

Definition 4.3. In Setting 4.1, let ω be a Kähler form on $\mathcal{X}$. We define the function γ on $U \cap \mathcal{X}_{\mathrm{reg}}$ by

$$(\Omega \wedge \overline{\Omega})^{1/m} = e^{-\gamma}\omega^n.$$

We want to analyze the integrability properties of $e^{-\gamma}$. Arguing as in [56, proof of Thm. B.1 (i)] (see also [28, Lem. 6.4]), it is easy to infer from the normality of $\mathcal{X}$ that given any small open set $U' \subset U$, there exist bounded holomorphic functions $(f_1, \dots, f_\ell)$ on U' such that $V(f_1, \dots, f_\ell) \subset U'_{\mathrm{sing}}$ and

$$\gamma|_{U'_{\mathrm{reg}}} = \frac{1}{m} \log \sum_i |f_i|^2. \tag{4.7}$$

Let us pick a section $s_E \in H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}(E))$ cutting out the exceptional divisor E and let us choose a smooth hermitian metric $|\cdot|$ on $\mathcal{O}_{\mathcal{X}'}(E)$. Given (4.7), there exists a constant $A > 0$ such that

$$f^*\gamma \geq A \log |s_E|^2. \tag{4.8}$$

Lemma 4.4. *Assume that X_0 has canonical singularities and set $\omega_t := \omega|_{X_t}$. Then up to shrinking $\mathbb{D}$, there exists $p > 1$ and a constant $C > 0$ such that for any $t \in \mathbb{D}$, one has*

$$\int_{U_t} e^{-p\gamma}\omega_t^n \leq C.$$

Proof. We set $p := 1 + \delta$ for some $\delta > 0$ small enough to be chosen later. Given (4.8), we have

$$\int_{U_t} e^{-p\gamma}\omega_t^n = \int_{f^{-1}(U_t)} e^{-\delta f^*\gamma} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m} \leq \int_{f^{-1}(U_t)} |s_E|^{-2\delta A} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m}.$$

Now, one can cover $f^{-1}(U_t)$ by finitely many open sets $V_t = V \cap X'_t$ as above. On V, the system of coordinates $(z_0, \dots, z_n)$ induces a system of coordinates $(z_1, \dots, z_n)$ such that

$$|s_E|^{-2\delta A} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m} \le C \prod_{i=1}^{p} |z_i|^{-2\delta A} i dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge i dz_n \wedge d\bar{z}_n$$

for some uniform constant C thanks to (4.5) and Claim 4.2. Recall that $V = \prod_{i=0}^{n} \{|z_i| < 1\} \subset \mathbb{C}^{n+1}$ and

$$V_t = V \cap \{z_0 \cdots z_p = t\} \hookrightarrow \{(z_1, \dots, z_n) \in \mathbb{C}^n \, ; t \le |z_i| < 1\} \subset \mathbb{D}^n,$$

where the injective map is given by $\mathrm{pr}_{z_1,\dots,z_n}|_{V_t}$, i.e. the restriction to V_t of the projection map onto the last n coordinates in $\mathbb{C}^{n+1}$. For δ small enough, the function $\mathbb{D} \ni z \mapsto |z|^{-\delta A}$ is integrable with respect to the area measure; this concludes the proof. ■

For the next lemma, we come back to the general case. We start by choosing a component $Y_0^{(k_0)}$ of Y_0, and we denote by $X_0^{(k_0)}$ the irreducible component of X_0 birational to $Y_0^{(k_0)}$ via f. Next, we consider the reduced divisor F on $\mathcal{X}'$ whose support consists of the union of the other components $Y_0^{(k)}$, $k \ne k_0$, along with the divisors E_i whose discrepancy a_i is equal to -1 (see (4.3)).

Let h_F be a smooth hermitian metric on $\mathcal{O}_{\mathcal{X}'}(F)$ and let $s_F \in H^0(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}(F))$ such that $\mathrm{div}(s_F) = F$. We let

$$\psi_F := -\log(-\log |s_F|^2_{h_F}). \tag{4.9}$$

Similarly, let $F_{\mathrm{klt}} := E - F \cap E$ and $\psi_{\mathrm{klt}} := \log |s_{F_{\mathrm{klt}}}|^2$.

Claim 4.5. *There exists $\delta > 0$ small enough such that for any $\varepsilon > 0$, there exists a constant C_ε such that for any $t \in \mathbb{D}$,*

$$\int_{f^{-1}(U_t)} e^{(1+\varepsilon)\psi_F - \delta \psi_{\mathrm{klt}}} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m} \le C_\varepsilon.$$

Proof. The statement is local on $\mathcal{X}'$, so it is enough to control the integrals over V_t. Up to relabeling, one can assume that $Y_0^{(k_0)} \cap V = (z_0 = 0)$, $F \cap V = (z_1 \cdots z_s = 0)$ so that for $s + 1 \le i \le p$, $f^*\Omega_t$ has a pole of order at most $m - 1$ along $(z_i = 0)$. We have implicitly assumed that V meets $Y_0^{(k_0)}$; actually, the computation is insensitive to whether that condition is fulfilled or not. Using (4.5), our integral is bounded by

$$\int_{V_t} \prod_{i=1}^{s} \frac{1}{|z_i|^2(-\log |z_i|)^{1+\varepsilon}} \cdot \prod_{i=s+1}^{p} \frac{1}{|z_i|^{2(\delta - a_i)}} \, d\lambda_{\mathbb{C}^n},$$

where $-1 < a_i < 0$, $V = \prod_{i=0}^{n} \{|z_i| < 1\} \subset \mathbb{C}^{n+1}$ and $V_t = V \cap \{z_0 \cdots z_p = t\}$. By the Fubini theorem, one can reduce the integral to $V_t^p := V_t \cap \mathbb{C}^{p+1}$ (i.e. fixing $z_{p+1}, \dots, z_n$). There is no harm in assuming that $\delta < \min_i (1 + a_i)/2$ so that the integral is bounded by

$$\int_{V_t^p} \prod_{i=1}^{s} \frac{1}{|z_i|^2(-\log |z_i|^2)^{1+\varepsilon}} \cdot \prod_{i=s+1}^{p} \frac{1}{|z_i|^{2(1-\delta/2)}} \, d\lambda_{\mathbb{C}^p}.$$

Using polar coordinates, one can assume that t is real (in $(0,1)$) and the integral becomes over $W_t := \{(r_i)_{1\le i\le p} \in [0,1/2]^p\,;\, r_1\cdots r_p \ge t\}$

$$\int_{W_t} \prod_{i=1}^{s} \frac{1}{r_i(-\log r_i)^{1+\varepsilon}} \cdot \prod_{i=s+1}^{p} \frac{1}{r_i^{1-\delta}}\, d\lambda_{\mathbb{R}^p}.$$

As $W_t \subset \prod_{i=1}^{p}\{t \le r_i \le 1/2\}$ and the functions $r \mapsto \frac{1}{r(-\log r)^{1+\varepsilon}}$ and $r \mapsto \frac{1}{r^{1-\delta}}$ are integrable on $[0,1/2]$, the conclusion follows from Fubini's theorem. ∎

The result above allows us to generalize Lemma 4.4 when no assumption on the central fiber is made. To do so, we first need some notation. The function ψ_F is well defined on $\mathcal{X}'$ but it does not necessarily come from $\mathcal{X}$. Given that $\mathrm{Nklt}(\mathcal{X}, X_0)$ is an analytic set in $\mathcal{X}$ and up to shrinking $\mathbb{D}$ a little, one can construct a function ρ such that:

- $\rho \le -1$ on $\mathcal{X}$.
- ρ is quasi-psh and has analytic singularities along $\mathrm{Nklt}(\mathcal{X}, X_0)$; in particular, it is identically $-\infty$ on that set.

We set

$$\psi := -\log(-\rho) \quad \text{on } \mathcal{X}.$$

Up to scaling ρ, one can assume that

$$f^*\psi \le \psi_F. \tag{4.10}$$

Next, we introduce for $\varepsilon > 0$ the function $\gamma_\varepsilon := \gamma - (n+1+2\varepsilon)\psi$ defined on U. In other words,

$$e^{(n+1+2\varepsilon)\psi}(\Omega \wedge \overline{\Omega})^{1/m} = e^{-\gamma_\varepsilon}\omega^n. \tag{4.11}$$

Lemma 4.6. *With the notation above, there exists a constant $\widetilde{C}_\varepsilon$ such that*

$$\int_{U_t} |\gamma_\varepsilon|^{n+\varepsilon} e^{-\gamma_\varepsilon}\omega_t^n \le \widetilde{C}_\varepsilon \quad \textit{for any } t \in \mathbb{D}.$$

Proof. In order to compute the integral, we pull it back by f and work on V_t. We have successively

$$|f^*\gamma_\varepsilon| \lesssim -\log|s_E| + \log(-\log|s_F|) \lesssim -\log|s_F| - \log|s_{F_{\mathrm{klt}}}|.$$

The first inequality is a combination of (4.8) and (4.10). To obtain the second inequality, we use the fact that $E = F \cup F_{\mathrm{klt}}$ to split the term $\log|s_E|$ while $\log(-\log|s_F|)$ can be absorbed by the more singular $-\log|s_F|$. The integral to bound becomes

$$\int_{V_t} [(-\log|s_F|)^{n+\varepsilon} + (-\log|s_{F_{\mathrm{klt}}}|)^{n+\varepsilon}] e^{(n+1+2\varepsilon)\psi_F} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m},$$

which itself is controlled by

$$\int_{V_t} e^{(1+\varepsilon)\psi_F} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m} + \int_{V_t} e^{2\psi_F - \delta\psi_{\mathrm{klt}}} f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m}$$

for any given $\delta > 0$. The lemma now follows from Claim 4.5. ∎

5. Negative curvature

In this section we apply our previous results to the study of families of varieties with "negative canonical bundle": we consider families of manifolds of general type, as well as families of "stable varieties".

5.1. *Families of manifolds of general type*

Setting 5.1. *Let $\mathcal{X}$ be an irreducible and reduced complex space endowed with a Kähler form ω and a proper, holomorphic map $\pi : \mathcal{X} \to \mathbb{D}$. We assume that for each $t \in \mathbb{D}$, the (schematic) fiber X_t is an n-dimensional Kähler manifold X_t of general type, i.e. its canonical bundle K_{X_t} is big. In particular, $\mathcal{X}$ is automatically non-singular and the map π is smooth. One can view the fibers X_t as deformations of X_0.*

We fix a closed differential $(1,1)$-form Θ on $\mathcal{X}$ which represents $c_1(K_{\mathcal{X}/\mathbb{D}}) \in H^{1,1}_{\partial\bar\partial}(\mathcal{X})$ and set $\theta_t = \Theta|_{X_t}$. Shrinking $\mathbb{D}$ if necessary and rescaling, we can assume that

$$-\omega \le \Theta \le \omega.$$

Lemma 5.2. *In Setting 5.1, the quantity* $\mathrm{vol}(K_{X_t})$ *is independent of $t \in \mathbb{D}$.*

Proof. We work in two steps. First, we assume that the family $\pi : \mathcal{X} \to \mathbb{D}$ is projective, i.e. there exists a positive line bundle L over $\mathcal{X}$. In that case, we know that the invariance of plurigenera holds [54, 60], meaning that the function $t \mapsto h^0(X_t, mK_{X_t})$ is constant on $\mathbb{D}$, without even assuming that X_t is of general type for all t. In particular, it would be enough to assume that only X_0 is of general type, from which it results that X_t is of general type for all t and $\mathrm{vol}(K_{X_t})$ is independent of t.

Coming back to the general case, we know that $K_{\mathcal{X}/\mathbb{D}}$ is big. Thanks to Demailly's regularization theorem, there exists a Kähler current $T \in c_1(K_{\mathcal{X}/\mathbb{D}})$ with analytic singularities along $V(\mathcal{I})$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$. Let $f : \mathcal{X}' \to \mathcal{X}$ be a log resolution of $(\mathcal{X}, \mathcal{I})$. By Hironaka's theorem, one can construct such a morphism f by a sequence of blow-ups along smooth centers only. We write $f^*T = T' + [F]$ for some smooth semipositive form T' on $\mathcal{X}'$ and some effective divisor F. This sequence may be infinite; however, the centers project onto a locally finite family of subsets of $\mathcal{X}$. Up to co-restricting f to $\pi^{-1}(K)$ for some compact subset $K \Subset \mathbb{D}$, one can assume that f is a finite composition of blow-ups and that $T' \ge \delta\pi^*\omega$ for some $\delta > 0$ small enough.

Let E be the exceptional divisor of f, with irreducible components $E = \sum_{k=1}^N E_k$. A classical argument (see e.g. [25, Lem. 3.5]) allows one to find smooth $(1,1)$-forms $\theta_{E_k} \in c_1(E_k)$ with support in an arbitrarily small neighborhood of E_k along with positive numbers (a_k) such that the sum $\theta = \sum_k a_k\theta_k$ defines a $(1,1)$-form on $\mathcal{X}'$ which is negative definite along the fibers of f. It follows that for $\varepsilon > 0$ small enough, the smooth form $\pi^*\omega - \varepsilon\theta_E$ is Kähler. In particular, $T' - \delta\varepsilon\theta$ is a Kähler form whose cohomology class belongs to $\mathrm{NS}_{\mathbb{R}}(\mathcal{X}')$. This implies that the Kähler cone of $\mathcal{X}'$ meets $\mathrm{NS}_{\mathbb{Z}}(\mathcal{X}')$, i.e. $\pi \circ f$ is projective.

Let $X_t' := f^{-1}(X_t)$ and let $K^\circ \subset K$ be the set of regular values of $\pi \circ f$. For any $t \in K^\circ$, the map $f|_{X_t'} : X_t' \to X_t$ is birational, hence $\mathrm{vol}(K_{X_t'}) = \mathrm{vol}(K_{X_t})$. By the first step, $\mathrm{vol}(K_{X_t'})$ is independent of $t \in K^\circ$, hence the same holds for $\mathrm{vol}(K_{X_t})$. The set $K \setminus K^\circ$ is finite and without loss of generality, one can assume that it consists of the single element $\{0\}$. The fiber X_0' can be decomposed as $X_0' = Y_0 + \sum E_i$ where $f|_{Y_0} : Y_0 \to X_0$ is birational and E_i is contracted by $f|_{X_0'}$. Let $Y_0' \to Y_0$ be a resolution of singularities. By [64, Thm. 1.2], we have $\mathrm{vol}(K_{Y_0'}) \le \mathrm{vol}(K_{X_t'})$ for $t \neq 0$. As X_0 and Y_0' are smooth and birational, we have $\mathrm{vol}(K_{X_0}) = \mathrm{vol}(K_{Y_0'}) \le \mathrm{vol}(K_{X_t})$. Finally, as $t \mapsto \mathrm{vol}(K_{X_t})$ is upper semicontinous, we have $\mathrm{vol}(K_{X_0}) = \mathrm{vol}(K_{X_t})$ for any $t \in K$. The lemma is proved. ■

Remark 5.3. In the last step of the proof of Lemma 5.2, we could also use the existence of relative minimal models, provided $\mathbb{D}$ is replaced by a quasi-projective smooth curve C. The general fiber of the projective morphism $\mathcal{X}' \to C$ is a projective variety of general type, hence it admits a good minimal model over $\mathbb{C}$ by [8]. By [31, Thm. 3.3] and [65, Cor. 1.2], it follows that $\mathcal{X}' \to C$ admits a birational model $\phi : \mathcal{X}' \dashrightarrow \mathcal{X}''$ over C such that ϕ^{-1} does not contract any divisor, every fiber X_t'' of $\mathcal{X}'' \to C$ has canonical singularities and $K_{X_t''}$ is semiample and big. For any $t \in C$, one has $\mathrm{vol}(K_{X_t''}) = (K^n_{X_t''})$. By flatness, this quantity does not depend on t.

Finally, we claim that X_0'' is birational to X_0. This is a combination of the following two facts. First, the variety X_0'' has canonical singularities and $K_{X_0''}$ is big, hence it is of general type and in particular it is not uniruled. Next, X_0'' is birational to a component of X_0' and all of them but the strict transform of X_0 by f are covered by rational curves as f is a composition of blow-ups of smooth centers from a smooth manifold.

The positive (n,n)-forms $(\omega_t^n)_{t\in\mathbb{D}}$ induce a smooth hermitian metric on $-K_{\mathcal{X}/\mathbb{D}}$. Since $[\Theta] = c_1(K_{\mathcal{X}/\mathbb{D}}) \in H^{1,1}_{\partial\bar\partial}(\mathcal{X})$ there exists a smooth function $\widetilde{h}$ on $\mathcal{X}$ such that

$$-dd^c_{\mathcal{X}} \log \omega_t^n = -\Theta + dd^c_{\mathcal{X}} \widetilde{h}.$$

We will denote by $\widetilde{h}_t := \widetilde{h}|_{X_t}$ the restriction to the fiber X_t. The function $\widetilde{h}$ becomes unique (and remains smooth) after imposing the normalization

$$\int_{X_t} \widetilde{h}_t \omega_t^n = 0.$$

We define a function h on $\mathcal{X}$ by imposing that $h_t := h|_{X_t}$ satisfies

$$h_t = \widetilde{h}_t - \log\left(\frac{1}{V_t}\int_{X_t} e^{\widetilde{h}_t}\omega_t^n\right).$$

In particular,

$$\int_{X_t} e^{h_t}\omega_t^n = V_t := \mathrm{vol}(K_{X_t}). \tag{5.1}$$

As $\widetilde{h}$ is smooth on $\mathcal{X}$, one has the following obvious consequence.

Lemma 5.4. *Given any compact subset $K \Subset \mathbb{D}$, one has*

$$\sup_{t\in K} \|h_t\|_{L^\infty(X_t)} < +\infty.$$

It follows from [11], a generalization of the Aubin–Yau theorem [3, 71], that there exists a unique Kähler–Einstein current on X_t. This is a positive closed current T_t in $c_1(K_{X_t})$ which, by [8, 28], is a smooth Kähler form in the ample locus $\mathrm{Amp}(K_{X_t})$, where it satisfies the Kähler–Einstein equation

$$\mathrm{Ric}(T_t) = -T_t.$$

It can be written as $T_t = \theta_t + dd^c\varphi_t$, where φ_t is the unique θ_t-psh function with minimal singularities that satisfies the complex Monge–Ampère equation

$$(\theta_t + dd^c\varphi_t)^n = e^{\varphi_t + h_t}\omega_t^n \quad \text{on } \mathrm{Amp}(K_{X_t}).$$

The minimal singularity assertion is equivalent to the following uniform bound: for all $x \in X_t$,

$$-M_t \le \Big(\varphi_t(x) - \sup_{X_t}\varphi_t\Big) - V_{\theta_t}(x) \le 0,$$

where

$$V_{\theta_t}(x) = \sup\{u_t(x)\,;\, u_t \in \mathrm{PSH}(X_t, \theta_t) \text{ and } u_t \le 0\}.$$

We can choose M_t independent of t by using Theorem 1.9:

Theorem 5.5. *In Setting 5.1, let $K \Subset \mathbb{D}$ be a compact subset. There exists a constant M_K such that for all $x \in \pi^{-1}(K)$, one has*

$$-M_K \le \varphi_t(x) - V_{\theta_t}(x) \le M_K \quad \textit{where } t = \pi(x).$$

Proof. From Lemma 5.2, it follows that the volume V_t of K_{X_t} is independent of t. We denote it by V.

Set $\mu_t = e^{h_t}\omega_t^n/V$ and recall that this is a probability measure, by our choice of normalization. We first observe that

$$0 \le \sup_{X_t}\varphi_t \le -\inf_{\pi^{-1}(K)} h \le C_K. \tag{5.2}$$

Let us first prove the left-hand inequality. As the measures

$$\frac{1}{V}(\theta_t + dd^c\varphi_t)^n = e^{\varphi_t}\mu_t$$

have mass 1, one has

$$1 \le \int_{X_t} e^{\sup_{X_t}\varphi_t}\, d\mu_t = e^{\sup_{X_t}\varphi_t},$$

and therefore $\sup_{X_t}\varphi_t \ge 0$.

To prove the middle inequality in (5.2), we observe that, since $\theta_t \le \omega_t$, φ_t is a subsolution of the equation

$$(\omega_t + dd^c \varphi_t)^n \ge (\theta_t + dd^c \varphi_t)^n = e^{\varphi_t + h_t} \omega_t^n,$$

while the constant function $u_t(x) = -\inf_{\pi^{-1}(K)} h$ is a supersolution of the same equation,

$$(\omega_t + dd^c u_t)^n = \omega_t^n \le e^{u_t + h_t} \omega_t^n.$$

It follows from the comparison principle [35, Prop. 10.6] that $\varphi_t \le -\inf_{\pi^{-1}(K)} h$. The rightmost inequality in (5.2) follows from Lemma 5.4 above.

We can thus rewrite the complex Monge–Ampère equation as

$$\frac{1}{V}(\theta_t + dd^c \psi_t)^n = e^{\psi_t + \sup_{X_t} \varphi_t} \mu_t = f_t \mu_t,$$

where $\psi_t = \varphi_t - \sup_{X_t} \varphi_t$ and $f_t = \exp(\psi_t + \sup_{X_t} \varphi_t)$. Combining the inequalities $\psi_t \le 0$ and (5.2), it follows that the densities f_t are uniformly bounded.

Recall that π is smooth, so in particular it is locally trivial. Therefore, Theorem 3.4 applies and we can now appeal to Theorem 1.9 with $p = +\infty$ and $0 < \alpha < \alpha(\Theta, \mathcal{X})$ and obtain

$$-M_K \le \psi_t - V_{\theta_t} \le 0.$$

Note that one uses here the fact that the volumes V_t stay away from zero. The conclusion follows since $\psi_t - \varphi_t$ is uniformly bounded by (5.2). ∎

Remark 5.6. Set

$$V_\Theta(x) := V_{\theta_{\pi(x)}}(x) \quad \text{and} \quad \phi(x) := \varphi_{\pi(x)}(x).$$

It is tempting to compare ϕ to

$$\hat{V}_\Theta = \sup\{u \in \mathrm{PSH}(\mathcal{X}, \Theta)\,;\, u \le 0\}.$$

Clearly $\hat{V}_\Theta \le V_\Theta$, hence $\hat{V}_\Theta - M_K \le \phi$. It follows from [14, Thm. A] that ϕ is Θ-psh on $\mathcal{X}$, thus $\phi - \sup_{\pi^{-1}(K)} \phi \le \hat{V}_\Theta$ and

$$-M_K \le \phi - \hat{V}_\Theta \le M_K.$$

Remark 5.7. The same results can be proved if the family $\pi : \mathcal{X} \to \mathbb{D}$ is replaced by a smooth family $\pi : (\mathcal{X}, B) \to \mathbb{D}$ of pairs (X_t, B_t) of log-general type, i.e. such that (X_t, B_t) is klt and $K_{X_t} + B_t$ is big for all $t \in \mathbb{D}$.

5.2. *Stable varieties*

A *stable variety* is a projective variety X such that:

(1) X has semi-log-canonical singularities.

(2) The $\mathbb{Q}$-line bundle K_X is ample.

We refer to [1,41,43,47,50] for a detailed account of these varieties and their connection to moduli theory.

In [7], it was proved that a stable variety admits a unique Kähler–Einstein metric ω. There are several equivalent definitions for such an object, but the simplest is probably the following:

Definition 5.8. A *Kähler–Einstein metric* ω on a stable variety is a smooth Kähler metric on X_{reg} such that

$$\mathrm{Ric}(\omega) = -\omega \quad \text{and} \quad \int_{X_{\mathrm{reg}}} \omega^n = (K_X^n) \quad \text{where } n = \dim_{\mathbb{C}} X.$$

It is proved in *loc. cit.* that ω extends canonically across X_{sing} to a closed, positive current in the class $c_1(K_X)$. It is desirable to understand the singularities of ω near X_{sing}. In [38, Thm. B], it is proved that ω has cusp singularities near the double crossings of X. Moreover, it is proved in [62] that the potential φ of ω with respect to a given Kähler form $\omega_X \in c_1(K_X)$, i.e. $\omega = \omega_X + dd^c\varphi$, is locally bounded on the klt locus of X. More precisely, given any divisor $D = (s = 0) \sim_{\mathbb{Q}} K_X$ containing the non-klt locus of X and given any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\varphi \geq \varepsilon \log |s|^2 - C_\varepsilon, \tag{5.3}$$

where $|\cdot|$ is some smooth hermitian metric on $\mathcal{O}_X(D)$. We wish to refine that estimate and obtain a version for families of canonically polarized manifolds degenerating to a stable variety.

Proposition 5.9. *Let X be a stable variety of dimension n and let $\omega_X \in c_1(K_X)$ be a Kähler metric. Next, let $\omega = \omega_X + dd^c\varphi$ be the Kähler–Einstein metric of X. Let $D = (s = 0)$ be a divisor containing the non-klt locus of X and let $|\cdot|$ be some smooth hermitian metric on $\mathcal{O}_X(D)$. For any $\varepsilon > 0$, there is a constant C_ε such that*

$$\varphi \geq -(n + 1 + \varepsilon) \log(-\log |s|) - C_\varepsilon. \tag{5.4}$$

Remark 5.10. Estimate (5.4) is an important refinement of (5.3), as it ensures that φ belongs to the finite energy class $\mathcal{E}^1(X, \omega_X)$; see [34] or [11, Sect. 2] for the definitions and main properties of these classes.

This estimate is almost optimal. Indeed, if X is the Satake–Baily–Borel compactification of a ball quotient, it is a normal stable variety and it admits a resolution $(\overline{X}, D)$ which is a toroidal compactification of the ball quotient obtained by adding disjoint abelian varieties. The potential φ of the Kähler–Einstein metric on $(\overline{X}, D)$ with respect to a smooth form in $c_1(K_{\overline{X}} + D)$ satisfies

$$\varphi = -(n + 1) \log(-\log |s_D|) + O(1) \quad \text{if } (s_D = 0) = D.$$

Proof of Proposition 5.9. Let $f : Y \to X$ be a resolution of singularities of X such that f induces an isomorphism over X_{reg}. The complex Monge–Ampère equation satisfied by φ pulls back to Y and reads

$$(f^*\omega_X + dd^c f^*\varphi)^n = e^{f^*\varphi} d\mu_Y, \tag{5.5}$$

where $d\mu_Y := \prod_{i=1}^r |t_i|^{2a_i} \omega_Y^n$ is a positive measure with possibly infinite mass. Here, ω_Y is a Kähler form on Y, and $(t_i = 0)$ are divisors sitting over X_{sing} (they need not be exceptional though, as X may have singularities in codimension 1). Finally, $a_i \geq -1$ for all i, and any divisor $(t_i = 0)$ such that $a_i = -1$ sits above the non-klt locus of X.

Now, let F be an effective divisor on X and let $\sigma_X \in H^0(X, \mathcal{O}_X(F))$ be a section cutting out F. Let h be a smooth hermitian metric on $\mathcal{O}_X(F)$; there exists a constant C_F such that $\Theta_h(F) \leq C_F \omega_X$. One can scale h so that $|\sigma_X|_h^2 < e^{-2(n+2)C_F}$ on X. Finally, let $\sigma_Y := f^*\sigma_X$ and $\psi := -\log(-\log|\sigma_Y|^2)$. We have

$$dd^c\psi = \frac{\langle D\sigma_Y, D\sigma_Y\rangle}{|\sigma_Y|^2(-\log|\sigma_Y|^2)} - \frac{1}{-\log|\sigma_Y|^2} \cdot f^*\Theta_h(F).$$

By our choice of scaling, the function $A\psi$ is $f^*\omega_X$-psh for any $0 \leq A \leq 2(n+2)$. Moreover, it belongs to the class $\mathcal{E}(Y, f^*\omega_X)$ thanks to e.g. [36, Prop. 2.3] and [20, Thm. 1.1 (ii)].

We apply this construction to F some (very ample, say) divisor containing the non-klt locus of X. This yields a section σ_Y of f^*F that vanishes to order at least 1 along the $(t_i = 0)$ for which $a_i = -1$. As a result, the measure

$$e^{(n+1+2\varepsilon)\psi} d\mu_Y \lesssim \prod_{a_i=-1} \frac{1}{|t_i|^2(-\log|t_i|^2)^{n+1+2\varepsilon}} \prod_{a_i>-1} |t_i|^{2a_i} \cdot \omega_Y^n$$

has a density g_ε with respect to ω_Y^n that satisfies

$$\int_Y g_\varepsilon |\log g_\varepsilon|^{n+\varepsilon} \omega_Y^n < +\infty$$

for any $\varepsilon > 0$. By Theorem 1.5, this implies that the unique solution $u_\varepsilon \in \mathcal{E}(Y, \frac{1}{2}f^*\omega_X)$ of the Monge–Ampère equation

$$\left(\tfrac{1}{2}f^*\omega_X + dd^c u_\varepsilon\right)^n = e^{u_\varepsilon + (n+1+2\varepsilon)\psi} d\mu_Y$$

is bounded, i.e. there exists a constant $C_\varepsilon > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(Y)} \leq C_\varepsilon. \tag{5.6}$$

Now, the function $v_\varepsilon := u_\varepsilon + (n+1+2\varepsilon)\psi \in \mathcal{E}(Y, f^*\omega_X)$ satisfies the inequality

$$(f^*\omega_X + dd^c v_\varepsilon)^n \geq \left(\tfrac{1}{2}f^*\omega_X + dd^c u_\varepsilon\right)^n = e^{v_\varepsilon} d\mu_Y,$$

i.e. v_ε is a subsolution of (5.5). By the comparison principle, we obtain $f^*\varphi \geq v_\varepsilon$ and it follows from (5.6) that

$$f^*\varphi \geq (n+1+2\varepsilon)\psi - C_\varepsilon,$$

from which the conclusion follows. ∎

5.3. Stable families

Now one can establish a family version of the previous estimate, i.e. of Proposition 5.9. In Setting 4.1, assume additionally that $K_{\mathcal{X}/\mathbb{D}}$ is ample. We let h be a smooth hermitian metric on $K_{\mathcal{X}/\mathbb{D}}$ whose curvature is a Kähler form $\omega_{\mathcal{X}} := \Theta_h(K_{\mathcal{X}/\mathbb{D}})$; we set

$$\omega_{X_t} := \omega_{\mathcal{X}}|_{X_t}.$$

If Ω is a local trivialization of $mK_{\mathcal{X}/\mathbb{D}}$, then the quantity

$$\mu_{\mathcal{X}/\mathbb{D},h} := \frac{i^{n^2}(\Omega \wedge \overline{\Omega})^{1/m}}{|\Omega|_h^{2/m}}$$

is independent of Ω and m (yet it depends on h), and for any $t \in \mathbb{D}$, it restricts to X_t^{reg} as a positive measure

$$\mu_{X_t,h} := \mu_{\mathcal{X}/\mathbb{D}}|_{X_t^{\mathrm{reg}}},$$

which we extend by zero across X_t^{sing}. For each $t \in \mathbb{D}$, there exists a unique Kähler–Einstein metric $\omega_{\mathrm{KE},t} \in c_1(K_{X_t})$ which solves the Monge–Ampère equation

$$(\omega_{X_t} + dd^c\varphi_t)^n = e^{\varphi_t}\mu_{X_t,h} \tag{5.7}$$

on X_t. This is due to [3, 71] when X_t is smooth and to [7] in general.

Theorem 5.11. *In Setting 4.1, assume that:*

- *The relative canonical bundle $K_{\mathcal{X}/\mathbb{D}}$ is ample.*
- *The central fiber X_0 is irreducible.*

Let $\omega_{X_t} + dd^c\varphi_t$ be the Kähler–Einstein metric of X_t that solves (5.7) and let $D = (s = 0) \subset \mathcal{X}$ be a divisor which contains $\mathrm{Nklt}(\mathcal{X}, X_0)$ *(see (4.4)). Fix a smooth hermitian metric $|\cdot|$ on $\mathcal{O}_{\mathcal{X}}(D)$. Up to shrinking $\mathbb{D}$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\varphi_t \geq -(n + 1 + \varepsilon)\log(-\log|s|) - C_\varepsilon \tag{5.8}$$

on X_t for any $t \in \mathbb{D}$.

This estimate improves an interesting control obtained previously by J. Song (see [62, Lem. 4.2]).

Proof of Theorem 5.11. Let $f : \mathcal{X}' \to \mathcal{X}$ be a semistable model as in (4.2). The first observation is that the behavior of $f^*(\Omega_t \wedge \overline{\Omega}_t)^{1/m}$ and $f^*\mu_{X_t,h}$ on X_t is the same, uniformly in t, because there exists a constant $C > 0$ such that for any trivializing open set, one has $C \geq |\Omega|_h^2 \geq C^{-1}$, where Ω ranges among the finitely many trivializations of $mK_{\mathcal{X}/\mathbb{D}}$. This follows from the fact h is a smooth hermitian metric on $mK_{\mathcal{X}/\mathbb{D}}$.

We set $\psi := f^*(-\log(-\log|s|^2))$; it is a quasi-psh function on $\mathcal{X}'$ satisfying

$$\psi \leq \psi_F + O(1),$$

where ψ_F is defined in (4.9).

By scaling the metric $|\cdot|$ on $\mathcal{O}_{\mathcal{X}}(D)$, one can assume that $A\psi$ is $f^*\omega_X$-psh for any $0 \le A \le 2(n+2)$. For any $t \in \mathbb{D}^*$, the function $\psi_t := \psi|_{X'_t}$ belongs to $\mathcal{E}(X'_t, f^*\omega_{X_t})$ by the same argument as in the proof of Proposition 5.9.

Let $u_{\varepsilon,t} \in \mathcal{E}(X'_t, \frac{1}{2} f^*\omega_{X_t})$ be the unique solution of the Monge–Ampère equation

$$\left(\tfrac{1}{2} f^*\omega_{X_t} + dd^c u_{\varepsilon,t}\right)^n = e^{u_{\varepsilon,t}+(n+1+2\varepsilon)\psi_t} f^*\mu_{X_t,h}. \tag{5.9}$$

One can write $e^{(n+1+2\varepsilon)\psi_t} f^*\mu_{X_t,h} = e^{\rho_t} f^*\omega^n_{X_t}$, where ρ_t is the restriction to X'_t of the difference of quasi-psh functions on $\mathcal{X}'$ with uniformly bounded L^1 norm on X'_t. Set $V := \int_{X_t} \omega^n_{X_t}$. Integrating both sides of (5.9) and using the Jensen inequality we obtain

$$\begin{aligned}\frac{V}{2^n} &= \int_{X'_t} e^{u_{\varepsilon,t}+(n+1+2\varepsilon)\psi_t} f^*\mu_{X_t,h} = V \int_{X'_t} e^{u_{\varepsilon,t}+\rho_t} \frac{f^*\omega^n_{X_t}}{V} \\ &\ge V \cdot e^{\frac{1}{V}\int_{X'_t}(u_{\varepsilon,t}+\rho_t) f^*\omega^n_{X_t}}.\end{aligned}$$

Since $\int_{X'_t} |\rho_t| f^*\omega^n_{X_t}$ is uniformly bounded, we get $\int_{X'_t} u_{\varepsilon,t} f^*\omega^n_{X_t} \le C$ for some $C > 0$ independent of ε, t. Since $u_{\varepsilon,t}$ is $f^*(\frac{1}{2}\omega_{X_t})$-psh, it is the pull-back of a $\frac{1}{2}\omega_{X_t}$-psh function on X_t to which one can apply Proposition 3.3 since π is projective. To summarize, we get an upper bound

$$u_{\varepsilon,t} \le C. \tag{5.10}$$

Next, we wish to apply Theorem 1.5; in order to do so, one has to check that hypotheses (H1) and (H2′) are satisfied in our situation. For (H1), it is a consequence of Theorem 3.4 – recall that up to shrinking $\mathbb{D}$, all fibers X_t are irreducible since so is X_0. As for (H2′), it follows from Lemma 4.6 that we pull back via f to the smooth Kähler manifold X'_t. All in all, we can find $C_\varepsilon > 0$ independent of $t \in \mathbb{D}$ such that

$$\|u_{\varepsilon,t}\|_{L^\infty(X'_t)} \le C_\varepsilon. \tag{5.11}$$

Now, the function $v_{\varepsilon,t} := u_{\varepsilon,t} + (n+1+2\varepsilon)\psi_t \in \mathcal{E}(X'_t, f^*\omega_{X_t})$ satisfies

$$(f^*\omega_{X_t} + dd^c v_{\varepsilon,t})^n \ge \left(\tfrac{1}{2} f^*\omega_{X_t} + dd^c u_{\varepsilon,t}\right)^n = e^{v_{\varepsilon,t}} f^*\mu_{X_t,h},$$

i.e. $v_{\varepsilon,t}$ is a subsolution of (5.7). By the comparison principle, we obtain $f^*\varphi_t \ge v_{\varepsilon,t}$ and it follows from (5.11) that

$$f^*\varphi_t \ge (n+1+2\varepsilon)\psi_t - C_\varepsilon,$$

from which the conclusion follows. ∎

6. Log Calabi–Yau families

6.1. Families of Calabi–Yau varieties

In Setting 4.1, let us assume additionally that $K_{\mathcal{X}/\mathbb{D}}$ is relatively trivial and that X_0 has canonical singularities. For t small enough, X_t has canonical singularities as well and K_{X_t} is linearly trivial.

Let α be a relative Kähler cohomology class on $\mathcal{X}$ represented by a relative Kähler form ω. We set $\alpha_t := \alpha|_{X_t}$, $\omega_{X_t} := \omega|_{X_t}$ and $V := \int_{X_t} \omega_t^n$; V does not depend on t (see Lemma 2.2). Let Ω be a trivialization of $K_{\mathcal{X}/\mathbb{D}}$, so that the quantity

$$\mu_{\mathcal{X}/\mathbb{D}} := i^{n^2} \Omega \wedge \overline{\Omega}$$

restricts to X_t^{reg} as a positive measure

$$\mu_{X_t} := \mu_{\mathcal{X}/\mathbb{D}}|_{X_t^{\mathrm{reg}}},$$

which we extend by zero across X_t^{sing}. We set $c_t := \log \int_{X_t} d\mu_{X_t}$. For each $t \in \mathbb{D}$, there exists a unique Kähler–Einstein metric $\omega_{\mathrm{KE},t} = \omega_t + dd^c \varphi_t \in \alpha_t$ which solves the Monge–Ampère equation

$$\frac{1}{V}(\omega_t + dd^c \varphi_t)^n = e^{-c_t} \mu_{X_t} \tag{6.1}$$

on X_t and that we normalize by $\sup_{X_t} \varphi_t = 0$. This is due to [71] when X_t is smooth and to [28] in general.

Theorem 6.1. *In Setting 4.1, assume that:*

- *The relative canonical bundle $K_{\mathcal{X}/\mathbb{D}}$ is trivial.*
- *The central fiber X_0 has canonical singularities.*
- *Assumption 3.2 is satisfied.*

Let $\omega_t + dd^c \varphi_t$ be the Kähler–Einstein metric of X_t solving (6.1). Up to shrinking $\mathbb{D}$, there exists $C > 0$ such that

$$\operatorname{osc}_{X_t} \varphi_t \leq C \tag{6.2}$$

for any $t \in \mathbb{D}$, where $\operatorname{osc}_{X_t} \varphi_t = \sup_{X_t} \varphi_t - \inf_{X_t} \varphi_t$.

A particular case of this result has been obtained previously by Rong–Zhang (see [56, Lemma 3.1]) by using the Moser iteration process.

Remark 6.2. One can replace the first two assumptions in Theorem 6.1 above by the following weaker ones: $\mathcal{X}$ is normal, $\mathbb{Q}$-Gorenstein, $K_{\mathcal{X}/\mathbb{D}}$ is trivial and X_0 has canonical singularities. Indeed, it follows from the inversion of adjunction [45, Thm. 2.3] that (X, X_t) is lc for t close enough to 0. Moreover, an easy computation relying on the adjunction formula shows that X_t has canonical singularities for t close to 0.

Proof of Theorem 6.1. A first observation is that the quantities c_t remain bounded when t varies thanks to Lemma 4.2. The result now follows from Theorem 1.1. Indeed, (H1) is satisfied thanks to Theorem 3.4, while (H2) holds thanks to Lemma 4.4 that we pull back to X'_t via f, with the notation of that lemma. ∎

6.2. The log Calabi–Yau setting

We will use the following setting.

Setting 6.3. *Let X be an n-dimensional compact Kähler space and let $B = \sum b_i B_i$ be an effective $\mathbb{R}$-divisor such that the pair (X, B) has klt singularities. We assume furthermore that the log Kodaira dimension of the pair (X, B) vanishes, i.e.*

$$\kappa(K_X + B) = 0.$$

In what follows, we denote by E the (unique) effective $\mathbb{R}$-divisor in $c_1(K_X + B)$. Thanks to log abundance in numerical dimension zero (see [13, Cor. 1.18]), a particular instance of such pairs is provided by klt pairs (X, B) with rational boundary such that the Chern class $c_1(K_X + B) \in H^2(X, \mathbb{Q})$ vanishes.

Definition 6.4. In Setting 6.3, given a cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ that is nef and big, it follows from [11] that there exists a unique *singular Ricci-flat current* $T \in \alpha$, i.e. a closed, positive current of bidegree $(1, 1)$ representing α with the following properties:

(i) T has minimal singularities in α.

(ii) T is a Kähler form on the analytic open set $\Omega_\alpha := (X_{\text{reg}} \setminus \text{Supp}(B + E)) \cap \text{Amp}(\alpha)$.

(iii) $\text{Ric}(T) = [B] - [E]$ on X_{reg}.

The current T can be found by solving the Monge–Ampère equation

$$\text{vol}(\alpha)^{-1}(\theta + dd^c\varphi)^n = \mu_{(X,B)}, \tag{6.3}$$

where $\theta \in \alpha$ is a smooth representative, $\varphi \in \text{PSH}(X, \theta)$ is the unknown function and

$$\mu_{(X,B)} = (s \wedge \bar{s})^{1/m} e^{-\phi_B}.$$

Here, $s \in H^0(X, m(K_X + B))$ is any non-zero section (for some $m \geq 1$) and ϕ_B is the unique singular psh weight on $\mathcal{O}_X(B)$ solving $dd^c\phi_B = [B]$ and normalized by

$$\int_X (s \wedge \bar{s})^{1/m} e^{-\phi_B} = 1.$$

We let $\mathcal{K}_X$ denote the *Kähler cone*, i.e. the set of cohomology classes $\alpha \in H^{1,1}(X, \mathbb{R})$ which can be represented by a Kähler form. We fix a path $(\alpha_t)_{0<t\leq 1} \subset \mathcal{K}_X$ of Kähler classes and assume that $\alpha_t \to \partial\mathcal{K}_X$ as $t \to 0$.

When X is smooth and $B = 0$, the existence of a unique Ricci-flat Kähler metric ω_t in α_t for each $0 < t \leq 1$ dates back to the celebrated work of Yau [71]. A basic problem is to understand the asymptotic behavior of the ω_t's as $t \to 0$. This problem has a long history; we refer the reader to [32] for references.

Despite motivations coming from mirror symmetry, not much is known when the norm of α_t converges to $+\infty$ (this case is expected to be the mirror of a large complex structure limit; see [49] or the recent survey [68]). We thus only consider the case when $\alpha_t \to \alpha_0 \in \partial\mathcal{K}_X$. There are two rather different settings, depending on whether α_0 is big ($\text{vol}(\alpha_0) > 0$), or merely nef with $\text{vol}(\alpha_0) = 0$.

6.3. *The non-collapsing case*

We first consider the case when the volumes of the α_t's are non-collapsing, i.e. $\mathrm{vol}(\alpha_0) > 0$. Then we have the following result, generalizing theorems of Tosatti [66] and Collins–Tosatti [18].

Theorem 6.5. *Let (X, B) be a pair as in Setting 6.3 and let $(\alpha_t)_{0<t\leq 1} \subset \mathcal{K}_X$ be a smooth path of Kähler classes such that $\alpha_t \to \alpha_0 \in \partial\mathcal{K}_X$ as $t \to 0$, with $\mathrm{vol}(\alpha_0) > 0$. Then the singular Ricci-flat currents $T_t \in \alpha_t$ converge to T_0 as $t \to 0$ weakly on X, and locally smoothly on Ω_α.*

Proof. One can work in a desingularization $p : Y \to X$ of X. The path α_t induces a path $\beta_t = p^*\alpha_t$ of semipositive and big classes. The currents T_t can be decomposed as $T_t = \theta_t + dd^c\varphi_t$ where $\theta_t \in \beta_t$ is a smooth representative and φ_t are normalized by $\sup_{X_t} \varphi_t = 0$ and solve the complex Monge–Ampère equation

$$\frac{1}{V_t}(\theta_t + dd^c\varphi_t)^n = \mu_Y = f\, dV_Y,$$

where the volumes $V_t = \alpha_t^n$ are bounded away from zero and infinity, $C^{-1} \leq V_t \leq C$, and $\mu_Y = f\, dV_Y$ is a fixed volume form, with $f \in L^p(Y)$ for some $p > 1$ (because (X, B) has klt singularities [28, Lem. 6.4]).

Hypothesis (H2) of Theorem 1.1 is thus trivially satisfied, while (H1) follows if we initially bound from above $\alpha_t \leq \gamma_X$ by a fixed Kähler class. The most delicate $\mathcal{C}^0$-estimate follows thus here from Theorem 1.9. When X is smooth, the $\mathcal{C}^0$-estimate in [66] is obtained by using a Moser iteration argument as in Yau's celebrated paper [71], but this argument can no longer be applied in the present more singular setting.

The rest of the proof is then roughly the same as in the case of smooth manifolds. It consists in adapting Yau's Laplacian estimate by using Tsuji's trick (first used in [70]), the remaining higher order estimates being local ones. ∎

6.4. *The collapsing case*

We now consider the case when the volumes of the α_t's are collapsing, i.e. $\mathrm{vol}(\alpha_0) = 0$. This case is more involved and only special cases are fully understood.

Suppose there is a surjective, holomorphic map $f : X \to Z$ with connected fibers, where Z is a compact, normal Kähler space of positive dimension m. We denote by $k := n - m = \dim X - \dim Z$ the relative dimension of the fiber space f. We let S_Z denote the smallest proper analytic subset $\Sigma \subset Z$ such that:

- Σ contains the singular locus Z_{sing} of Z.
- The map f is smooth on $f^{-1}(Z \setminus \Sigma)$.
- For any $z \in Z \setminus \Sigma$, $\mathrm{Supp}(B)$ intersects X_z transversally.

We set $S_X = f^{-1}(S_Z)$. Finally, we set $Z^\circ := Z \setminus S_Z$ and $X^\circ := X \setminus S_X = f^{-1}(Z^\circ)$. By the last item, each component of $B|_{X^\circ}$ dominates Z°.

A general fiber X_z satisfies $\kappa(K_{X_z} + B_z) \geq 0$, but the inequality may be strict. If $c_1(K_X + B) = 0$, then log abundance implies that $K_{X_z} + B_z \sim_{\mathbb{Q}} \mathcal{O}_{X_z}$ for z general. Moreover, Iitaka's conjecture predicts that $\kappa(K_{X_z} + B_z)$ vanishes as long as $\kappa(Z) \geq 0$, which in turn should be equivalent to Z not being uniruled.

Fix a Kähler form ω_Z on Z. For simplicity, we assume that $\int_Z \omega_Z^m = 1$. The form $f^*\omega_Z$ is a semipositive form such that $f^*\omega_Z^p = 0$ for any $p > m$. We also choose a Kähler form ω_X on X. The quantity $\int_{X_z} \omega_X^k = f_*\omega_X^k$ is constant in $z \in Z$; up to renormalizing ω_X, we may assume that the constant is 1.

We assume that our path $(\alpha_t)_{t\geq 0}$ in $H^{1,1}(X, \mathbb{R})$ is given by $\alpha_0 = \{f^*\omega_Z\}$ and $\alpha_t = \alpha_0 + t\{\omega_X\}$. As a result,

$$V_t := \mathrm{vol}(\alpha_t) = \binom{n}{k} t^k \int_X f^*\omega_Z^m \wedge \omega_X^k + o(t^k) = \binom{n}{k} t^k + o(t^k). \tag{6.4}$$

We set $\omega_t := f^*\omega_Z + t\omega_X$ and let $\omega_{\varphi_t} := \omega_t + dd^c\varphi_t$ denote the singular Ricci-flat current in α_t, normalized by $\int_X \varphi_t \omega_X^n = 0$. It satisfies

$$\omega_{\varphi_t}^n = V_t \cdot \mu_{(X,B)},$$

by (6.3). The probability measure $f_*\mu_{(X,B)}$ has $L^{1+\varepsilon}$ density with respect to ω_Z^m thanks to [29, Lem. 2.3]. Therefore, there exists a unique positive current $\omega_\infty \in \{\omega_Z\}$ with bounded potentials that solves the Monge–Ampère equation

$$\omega_\infty^m = f_*\mu_{(X,B)}$$

(see [28]). In the case where X is smooth, $B = 0$ and $c_1(X) = 0$, the Ricci curvature of $f_*\mu_X$ (or equivalently ω_∞) coincides with the Weil–Petersson form of the fibration f of Calabi–Yau manifolds. We propose the following problem.

Problem 1. *Let $f : X \to Z$ be a surjective holomorphic map with connected fibers between compact, normal Kähler spaces. Assume that there exists an effective divisor B on X such that (X, B) is klt and $\kappa(K_X + B) = 0$. Let ω_X (resp. ω_Z) be a Kähler form on X (resp. Z) and let ω_{φ_t} be the unique singular Ricci-flat current in $\{f^*\omega_Z + t\omega_X\}$ for $t > 0$. Then the currents ω_{φ_t} converge weakly to $f^*\omega_\infty$ as $t \to 0$, where $\omega_\infty \in \{\omega_Z\}$ solves $\omega_\infty^{\dim Z} = f_*\mu_{(X,B)}$.*

The problem above is motivated by a string of papers (see below) where the expected result is proved along with some additional information on the convergence.

Theorem 6.6 ([32,39,67,69]). *Assume that X is smooth, $B = 0$ and $c_1(K_X) = 0$. Then the metrics ω_{φ_t} converge to $f^*\omega_\infty$ in the $\mathcal{C}^\alpha_{\mathrm{loc}}$ sense on $X \setminus S_X$, for some $\alpha > 0$.*

In this section, we aim at providing a positive answer to Problem 1 whenever X is smooth, B has simple normal crossings support and $c_1(K_X + B) = 0$. We will follow the strategy of Tosatti [67] rather closely. However, some adjustments need to be made, requiring the use of conical metrics and the results of the present paper.

Theorem 6.7. *In the setting of Problem 1, assume furthermore that X is smooth, B has snc support and $c_1(K_X + B) = 0$. Then $\omega_{\varphi_t} \to f^*\omega_\infty$ as currents on X when $t \to 0$.*

Proof. We will proceed in several steps, similarly to [67]. In order to simplify some computations to follow, we will assume that S_Z is contained in a divisor D_Z, cut out by a section $\sigma_Z \in H^0(Z, \mathcal{O}_Z(D_Z))$. If Z is projective, this is not a restriction. The general case requires following Tosatti's computations more closely but does not present significant additional difficulties.

Step 1. *Choice of some suitable conical metrics.* In the proposition below we list the properties of the conical metric that will be important for the following. It is mostly a recollection of well known results (see e.g. [37]). By abuse of notation, we will not distinguish between B and $\mathrm{Supp}(B)$.

Proposition 6.8. *There exists a Kähler current $\omega_B \in \{\omega_X\}$ on X such that:*

(1) *ω_B is a smooth Kähler form on $X \setminus B$ and has conical singularities along B.*

(2) *There exists a constant $C > 0$ and a quasi-psh function $\Psi \in \mathcal{C}^\infty(X \setminus B) \cap L^\infty(X)$ such that the following inequalities between tensors hold in the sense of Griffiths on $X \setminus B$:*
$$-(C\omega_B + dd^c\Psi) \otimes \mathrm{Id}_{T_X} \le \Theta_{\omega_B}(T_X) \le C\omega_B \otimes \mathrm{Id}_{T_X}.$$

(3) *Let $h := \omega_B^n/\omega_X^n$. There exists $p > 1$ such that for any $K \Subset Z^\circ$,*
$$\sup_{z\in K} \|h|_{X_z}\|_{L^p(\omega_{X_z}^k)} < +\infty.$$

Sketch of proof of Proposition 6.8. To construct such a metric ω_B, one first chooses smooth metrics h_i on B_i and sections $s_i \in H^0(X, \mathcal{O}_X(B_i))$ cutting out B_i, and one sets $\omega_B := \omega_X + dd^c \sum_i |s_i|^{2(1-b_i)}$. Up to scaling down the metrics h_i, one can easily achieve the first condition. The third condition also follows easily.

The left-hand inequality of (2) ("lower bound" on the holomorphic bisectional curvature) follows from [37, (4.3)] with $\varepsilon = 0$. As for the right-hand inequality (upper bound on the holomorphic bisectional curvature), a proof has been given in [40, App. A] in the case where B is smooth but a very simple argument has been found by Sturm [58, Lem. 3.14]. ∎

Step 2. *Estimates.* In the proposition below we list various estimates on ω_{φ_t} that will be useful for the last step. First, we define for $z \in Z^\circ$ the quantity $\underline{\varphi_t}(z) := \int_{X_z} \varphi_t \omega_{X_z}^k$. In the following, we will not distinguish between $\underline{\varphi_t}$ and $f^*\underline{\varphi_t}$.

Proposition 6.9. *There exist a constant $C > 0$ as well as a positive function $g \in \mathcal{C}^\infty(X^\circ)$, both independent of t, such that:*

(1) $\|\varphi_t\|_{L^\infty(X)} \le C$.

(2) $\omega_{\varphi_t} \ge C^{-1} f^*\omega_Z$.

(3) $|\varphi_t - \underline{\varphi_t}| \le g \cdot t$.

(4) $g^{-1}t \cdot \omega_B \le \omega_{\varphi_t} \le g \cdot \omega_B$.

(5) $g^{-1}t \cdot \omega_{B_z} \le \omega_{\varphi_t}|_{X_z} \le gt \cdot \omega_{B_z}$ *for all* $z \in Z^\circ$.

Proof of Proposition 6.9. In this proof, C will denote a constant that may change from line to line but is independent of t. In the same way, g will be a smooth, positive function on X° that should be thought of as blowing up to $+\infty$ near S_X; it can be assumed to come from Z° via f.

(1) This is a consequence of [27, Thm. A] or [24, p. 401].

(2) Let us consider the holomorphic map $f : (X \setminus B, \omega_{\varphi_t}) \to (Z, \omega_Z)$. Since $\mathrm{Ric}(\omega_{\varphi_t}) = 0$ and ω_Z is a smooth Kähler metric on the compact space Z, Chern–Lu's formula [17,52] provides a constant $C > 0$ such that the non-negative function $u := \mathrm{tr}_{\omega_{\varphi_t}} f^*\omega_Z$ satisfies

$$\Delta_{\omega_{\varphi_t}} \log u \ge -C(1+u)$$

on $X \setminus B$. Now,

$$\Delta_{\omega_{\varphi_t}}(-\varphi_t) = \mathrm{tr}_{\omega_{\varphi_t}}(-\omega_{\varphi_t} + f^*\omega_Z + t\omega_X) \ge u - n,$$

so that setting $A = C + 1$, one finds

$$\Delta_{\omega_{\varphi_t}}(\log u - A\varphi_t) \ge u - C.$$

Let τ be a section of $\mathcal{O}_X(\lceil B \rceil)$ cutting out B and let h_B be a smooth hermitian metric on that line bundle. We set

$$\chi := \log |\tau|^2_{h_B}.$$

As ω_{φ_t} is a Kähler current and χ is quasi-psh, there exists a constant $C_t > 0$ such that $dd^c\chi \ge -C_t\omega_{\varphi_t}$. Therefore, for any $\delta \in (0, C_t^{-1})$,

$$\Delta_{\omega_{\varphi_t}}(\log u - A\varphi_t + \delta\chi) \ge u - C.$$

As ω_{φ_t} is a conical metric for $t > 0$, the function u is bounded above on $X \setminus B$, and therefore $H_{t,\delta} := \log u - A\varphi_t + \delta\chi$ attains its maximum at a point $x_{t,\delta} \in X \setminus B$ such that $u(x_{t,\delta}) \le C$. As a result, the estimate in (1) allows one to show that for any $x \in X \setminus B$,

$$\log u(x) = H_{t,\delta}(x) + A\varphi_t(x) - \delta\chi(x) \le H_{t,\delta}(x_{t,\delta}) + C - \delta\chi(x) \le C - \delta\chi.$$

As this holds for any $\delta > 0$ small enough, we can pass to the limit and conclude that $u \le e^C$ on $X \setminus B$, hence everywhere.

(3) The equation solved by φ_t can be rewritten as

$$(f^*\omega_Z + t\omega_X + dd^c\varphi_t)^n = t^k e^{F_t}\omega_B^n, \tag{6.5}$$

where F_t is uniformly bounded independently of t. Next, on X_z ($z \in Z^\circ$),

$$\frac{(\omega_{\varphi_t}|_{X_z})^k}{\omega_{X_z}^k} = \frac{\omega_{\varphi_t}^k \wedge f^*\omega_Z^m}{\omega_X^k \wedge f^*\omega_Z^m} \le Cg \cdot \frac{\omega_{\varphi_t}^n}{\omega_X^n} \tag{6.6}$$

thanks to (2). Observing that $\omega_{\varphi_t}|_{X_z} = (\omega_{\varphi_t} - dd^c \underline{\varphi_t})|_{X_z}$, one sees from (6.5) that $(\varphi_t - \underline{\varphi_t})|_{X_z}$ satisfies

$$\left(\omega_{X_z} + dd^c\left(\frac{1}{t}(\varphi_t - \underline{\varphi_t})|_{X_z}\right)\right)^k \le gh|_{X_z} \cdot \omega_{X_z}^k,$$

where $h = \omega_B^n/\omega_X^n$. Thanks to the third item of Proposition 6.8, Theorem 3.4 and Theorem 1.1, we can derive (3). Actually, we have used a version of Theorem 3.4 for higher-dimensional bases, but only for smooth morphisms, in which case the proofs in the one-dimensional case go through without any change.

(4.a) We first prove the right-hand inequality. Let us start by writing $\omega_B = \omega_X + dd^c\psi_B$, where $\psi_B \in L^\infty(X) \cap \mathcal{C}^\infty(X \setminus B)$. From Proposition 6.8 (2) and Siu's Laplacian inequality [37, (2.2)], one concludes that

$$\Delta_{\omega_{\varphi_t}}(\log \operatorname{tr}_{\omega_B} \omega_{\varphi_t} + \Psi) \ge -C(1 + \operatorname{tr}_{\omega_{\varphi_t}} \omega_B).$$

Next,

$$\Delta_{\omega_{\varphi_t}}(-\varphi_t + t\psi_B) = \operatorname{tr}_{\omega_{\varphi_t}}(-\omega_{\varphi_t} + f^*\omega_Z + t\omega_B) \ge t \operatorname{tr}_{\omega_{\varphi_t}} \omega_B - n, \tag{6.7}$$

so that

$$\Delta_{\omega_{\varphi_t}}\left(\log \operatorname{tr}_{\omega_B} \omega_{\varphi_t} + \Psi - \frac{A}{t}\varphi_t + A\psi_B\right) \ge \operatorname{tr}_{\omega_{\varphi_t}} \omega_B - \frac{C}{t}. \tag{6.8}$$

We want to bound the term $dd^c\underline{\varphi_t}$ from below. In order to achieve this, we write

$$\begin{aligned} dd^c\underline{\varphi_t} &= dd^c f_*(\varphi_t\omega_X^k) = f_*(dd^c\varphi_t \wedge \omega_X^k) \ge -f_*(f^*\omega_Z \wedge \omega_X^k + t\omega_X^{k+1}) \\ &\ge -\omega_Z - tf_*\omega_X^{k+1} \ge -g \cdot \omega_Z \end{aligned} \tag{6.9}$$

because $f_*\omega_X^k = 1$. In particular,

$$\Delta_{\omega_{\varphi_t}} \underline{\varphi_t} \ge -g \tag{6.10}$$

thanks to (2). Combining that estimate with (6.8), one finds

$$\Delta_{\omega_{\varphi_t}}\left(\log \operatorname{tr}_{\omega_B} \omega_{\varphi_t} + \Psi - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) + A\psi_B\right) \ge \operatorname{tr}_{\omega_{\varphi_t}} \omega_B - \frac{g}{t}. \tag{6.11}$$

We now set $F := \Psi - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) + A\psi_B$; it is a bounded function on X, smooth on $X^\circ \setminus B$, such that

$$|F| \le g \tag{6.12}$$

thanks to (3). Next, we set $\rho := \chi + f^* \log |\sigma_Z|^2_{h_{D_Z}}$ where χ is defined in the proof of (2) and h_{D_Z} is a smooth hermitian metric on the divisor D_Z (containing S_Z). As ρ is quasi-psh on X, there exists $C_t > 0$ such that

$$dd^c\rho \ge -C_t\omega_{\varphi_t}. \tag{6.13}$$

In particular,

$$\Delta_{\omega_{\varphi_t}}(\log \operatorname{tr}_{\omega_B} \omega_{\varphi_t} + F + \delta\rho) \geq \operatorname{tr}_{\omega_{\varphi_t}} \omega_B - g/t \tag{6.14}$$

as long as $\delta \in (0, C_t^{-1})$. We choose such a δ for the following. As the quantity $\log \operatorname{tr}_{\omega_B} \omega_{\varphi_t} + F$ is globally bounded on X and smooth on $X^\circ \setminus B$, the function $\log \operatorname{tr}_{\omega_B} \omega_{\varphi_t} + F + \delta\rho$ attains its maximum at a point $y_{t,\delta} \in X^\circ \setminus B$ such that

$$\operatorname{tr}_{\omega_{\varphi_t}} \omega_B(y_{t,\delta}) \leq g/t$$

thanks to the maximum principle. Combining this with (2), one finds

$$\operatorname{tr}_{\omega_{\varphi_t}}(f^*\omega_Z + t\omega_B)(y_{t,\delta}) \leq g. \tag{6.15}$$

Using the standard inequality

$$\operatorname{tr}_{\omega'} \omega \leq \frac{\omega^n}{\omega'^n}(\operatorname{tr}_\omega \omega')^{n-1},$$

valid for any two positive $(1,1)$-forms, one gets from (6.15)

$$\operatorname{tr}_{f^*\omega_Z + t\omega_B}(\omega_{\varphi_t})(y_{t,\delta}) \leq g$$

since $\omega_{\varphi_t}^n \simeq t^k \omega_B^n$ is uniformly comparable to $(f^*\omega_Z + t\omega_B)^n$ by Claim 6.10 below. As ω_B dominates $f^*\omega_Z + t\omega_B$, we infer from the inequality above that

$$\operatorname{tr}_{\omega_B} \omega_{\varphi_t}(y_{t,\delta}) \leq g. \tag{6.16}$$

Given the definition of $y_{t,\delta}$, the boundedness of F and that $\delta > 0$ is arbitrary, we find as in the proof of (2) above that (6.16) actually implies

$$\operatorname{tr}_{\omega_B} \omega_{\varphi_t} \leq g \quad \text{on } X^\circ \setminus B,$$

hence on the whole X°.

To conclude the proof of the right-hand inequality in (4), it remains to prove

Claim 6.10. *We have*

$$g^{-1} t^k \cdot \omega_B^n \leq (f^*\omega_Z + t\omega_B)^n \leq g t^k \cdot \omega_B^n. \tag{6.17}$$

Proof of Claim 6.10. The statement is local, so one can assume that $f : \mathbb{C}^n \to \mathbb{C}^m$ is given by the projection onto the last m factors and that $B = \sum_{i=1}^r b_i(z_i = 0)$ for some $r \leq k$. As the inequality is invariant under quasi-isometry, one can choose $\omega_Z = \sum_{j=k+1}^n i dz_j \wedge d\bar{z}_j$ to be the euclidean metric on $\mathbb{C}^m$ while

$$\omega_B = \sum_{j=1}^r \frac{i dz_j \wedge d\bar{z}_j}{|z_j|^{2b_j}} + \sum_{j=r+1}^n i dz_j \wedge d\bar{z}_j$$

is the standard cone metric. Setting $K := \prod_{j=1}^r |z_j|^{-2b_j}$ and $\omega_{\mathbb{C}^n} := \sum_{j=1}^n i dz_j \wedge d\bar{z}_j$, one finds

$$\omega_B^n = K \cdot \omega_{\mathbb{C}^n}^n \quad \text{and} \quad (f^*\omega_Z + t\omega_B)^n = t^k(1+t)^m K \cdot \omega_{\mathbb{C}^n}^n,$$

which gives the expected result. ∎

(4.b) We now move on to the left-hand inequality of (4). Let us set $v := \mathrm{tr}_{\omega_{\varphi_t}}(t\omega_B)$. Recall from Proposition 6.8 (2) that ω_B has holomorphic bisectional curvature bounded from above on $X \setminus B$. By Chern–Lu's inequality, we get

$$\Delta_{\omega_{\varphi_t}} \log v \geq -Ct^{-1}v \quad \text{on } X \setminus B.$$

Combining that inequality with (6.7)–(6.10) and (6.13), one finds, for $A = C + 1$,

$$\Delta_{\omega_{\varphi_t}} \left(\log v - \frac{A}{t}(\varphi_t - \underline{\varphi_t}) + A\psi_B + \delta\rho \right) \geq \frac{1}{t}(v - g).$$

Applying the maximum principle and arguing as before, we eventually find $v \leq g$ on $X^\circ \setminus B$, hence on X°.

(5) The left-hand inequality is a direct consequence of (4), by restriction. As for the right-hand inequality, it follows easily from the left-hand one since

$$\mathrm{tr}_{\omega_{B_z}} \omega_{\varphi_t}|_{X_z} \leq \frac{(\omega_{\varphi_t}|_{X_z})^k}{\omega_{B_z}^k} \cdot (\mathrm{tr}_{\omega_{\varphi_t}|_{X_z}} \omega_{B_z})^{k-1} \leq g t^{k-(k-1)}$$

thanks to (6.6). This ends the proof of Proposition 6.9. ∎

Step 3. *Convergence.* Thanks to Proposition 6.9 (1), the family $(\varphi_t)_{0<t\leq 1}$ is relatively compact for the L^1-topology. All we have to do is to show that all its cluster values coincide. Let φ_∞ be such a cluster value; it is an $f^*\omega_Z$-psh function but f has connected fibers so that φ_∞ is necessarily constant on the fibers. Equivalently, one has $\varphi_\infty = f^*\underline{\varphi_\infty}$ for the (unique) ω_Z-psh function $\underline{\varphi_\infty}$ satisfying $\underline{\varphi_\infty}(z) := \int_{X_z} \varphi_\infty \omega_X^k$ for $z \in Z^\circ$. We want to show that the equality of measures

$$(\omega_Z + dd^c \underline{\varphi_\infty})^m = f_*\mu_{(X,B)} \tag{6.18}$$

holds on Z. Since (6.18) has a unique normalized bounded solution, this will prove the theorem. As $\underline{\varphi_\infty}$ is globally bounded on X thanks to Proposition 6.9 (1), and $f_*\mu_{(X,B)}$ does not charge any pluripolar set, it is actually enough to show that the equality (6.18) of measures holds on Z°. In order to prove (6.18) on Z°, since $f_*\omega_X^k = 1$, it is enough to prove instead that for any $u \in \mathscr{C}_0^\infty(Z^\circ)$,

$$\int_X f^*u \cdot (f^*\omega_Z + dd^c\varphi_\infty)^m \wedge \omega_X^k = \int_X f^*u \cdot d\mu_{(X,B)}. \tag{6.19}$$

We start from the identity

$$\omega_{\varphi_t}^n = (f^*\omega_Z + t\omega_X + dd^c\varphi_t)^n = V_t \cdot \mu_{(X,B)}, \tag{6.20}$$

where $V_t = \binom{n}{k} t^k + o(t^k)$ when $t \to 0$ (see (6.4)). Set $\psi_t := \varphi_t - \underline{\varphi_t}$ and decompose ω_{φ_t} as

$$\omega_{\varphi_t} = f^*(\omega_Z + dd^c \underline{\varphi_t}) + (t\omega_X + dd^c\psi_t).$$

By expanding, one obtains

$$\omega_{\varphi_t}^n = \sum_{i=0}^{m} \binom{n}{i} \underbrace{f^*(\omega_Z + dd^c \underline{\varphi_t})^i \wedge (t\omega_X + dd^c \psi_t)^{n-i}}_{=:\alpha_i}.$$

• *Case* $i = m$. We expand again

$$\alpha_m = \sum_{j=0}^{k-1} \binom{k}{j} t^j \underbrace{f^*(\omega_Z + dd^c \underline{\varphi_t})^m \wedge \omega_X^j \wedge (dd^c \psi_t)^{k-j}}_{=:\beta_j} + t^k f^*(\omega_Z + dd^c \underline{\varphi_t})^m \wedge \omega_X^k.$$

Integrating by parts, one gets

$$\int_X f^* u \cdot \beta_j = \int_X \psi_t \cdot f^* \underbrace{\left(dd^c u \wedge (\omega_Z + dd^c \underline{\varphi_t})^m\right)}_{=0} \wedge \omega_X^j \wedge (dd^c \psi_t)^{k-j-1} = 0$$

for degree reasons.

By the dominated convergence theorem, $\underline{\varphi_t} \to \underline{\varphi_\infty}$ in the $L^1_{\mathrm{loc}}(Z^\circ)$ topology. Moreover, as B intersects the fibers of f transversally over Z°, an easy argument relying on a partition of unity shows that $f_*(\omega_B \wedge \omega_X^k)$ is a smooth $(1,1)$-form on Z°. Combining this with Proposition 6.9 (4), we find $dd^c \underline{\varphi_t} = f_*(dd^c \varphi_t \wedge \omega_X^k) \le f_*(g\omega_B \wedge \omega_X^k) \le (f_* g) \cdot \omega_Z$. Together with (6.9), this implies

$$\pm dd^c \underline{\varphi_t} \le (f_* g) \cdot \omega_Z. \tag{6.21}$$

By a standard result, this shows that $\underline{\varphi_t} \to \underline{\varphi_\infty}$ in $C^{1,\alpha}_{\mathrm{loc}}(Z^\circ)$ for any $\alpha < 1$. In particular, the quasi-psh functions $\underline{\varphi_t}$ converge uniformly on $\mathrm{Supp}(u)$. By Bedford–Taylor theory, one deduces that

$$\int_X f^* u \cdot f^*(\omega_Z + dd^c \underline{\varphi_t})^m \wedge \omega_X^k \to \int_X f^* u \cdot f^*(\omega_Z + dd^c \underline{\varphi_\infty})^m \wedge \omega_X^k.$$

In the end, we have showed that

$$\frac{\binom{n}{m}}{V_t} \int_X f^* u \cdot \alpha_m \to \int_X f^* u \cdot f^*(\omega_Z + dd^c \underline{\varphi_\infty})^m \wedge \omega_X^k \tag{6.22}$$

since $V_t \sim \binom{n}{m} t^k$.

• *Case* $i < m$. We expand

$$\alpha_i = \sum_{j=0}^{n-i-1} \binom{n-i}{j} t^j \underbrace{f^*(\omega_Z + dd^c \underline{\varphi_t})^i \wedge \omega_X^j \wedge (dd^c \psi_t)^{n-i-j}}_{=:\gamma_{ij}}$$
$$+ t^{n-i} f^*(\omega_Z + dd^c \underline{\varphi_t})^i \wedge \omega_X^{n-i}.$$

From (6.21), we find

$$\frac{t^{n-i}}{V_t}\int_X f^*u\cdot f^*(\omega_Z+dd^c\underline{\varphi_t})^i\wedge\omega_X^{n-i}=O(t^{m-i})=o(1). \tag{6.23}$$

For the remaining terms, an integration by parts yields

$$\int_X f^*u\cdot\gamma_{ij}=\int_X \psi_t\cdot f^*\big(dd^c u\wedge(\omega_Z+dd^c\underline{\varphi_t})^i\big)\wedge\omega_X^j\wedge(dd^c\psi_t)^{n-i-j-1}.$$

From Proposition 6.9 (3), one has $|\psi_t|\le gt$. Moreover, among the $(n-i-j-1)$ eigenvalues of $dd^c\psi_t$ involved in the integral, at least $(n-i-j-1)-(m-(i+1))=k-j$ must come from the fiber. By Proposition 6.9 (4, 5), the integrand is $O(t^{1+k-j})$. As a result,

$$\frac{t^j}{V_t}\int_X f^*u\cdot\gamma_{ij}=O(t).$$

Combining that result with (6.23), we see that for any $i>m$,

$$\lim_{t\to 0}\frac{1}{V_t}\int_X f^*u\cdot\alpha_i=0. \tag{6.24}$$

Putting together (6.20), (6.22) and (6.24), we obtain

$$\begin{aligned}\int_X f^*u\cdot d\mu_{(X,B)}&=\frac{1}{V_t}\int_X f^*u\cdot\omega_{\varphi_t}^n=\lim_{t\to 0}\sum_{i=0}^m\binom{n}{i}\frac{1}{V_t}\int_X f^*u\cdot\alpha_i\\&=\lim_{t\to 0}\frac{\binom{n}{m}}{V_t}\int_X f^*u\cdot\alpha_m=\int_X f^*u\cdot f^*(\omega_Z+dd^c\underline{\varphi_\infty})^m\wedge\omega_X^k.\end{aligned}$$

In summary, (6.19) is proved, which concludes the proof of the theorem. ∎

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