J. Eur. Math. Soc. 25, 2919–2925 (2023) DOI 10.4171/JEMS/1364

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Corrigendum to "The Brauer group and the Brauer–Manin set of products of varieties"

Received August 30, 2021

Abstract. In his Zentralblatt review of our paper [J. Eur. Math. Soc. 16, 749–768 (2014)], Faltings pointed out that he could not follow the proof of Proposition 2.2. In this corrigendum we rectify this and other mistakes in that paper.

Keywords. Künneth formula, Brauer group

The main results of [6], Theorems A, B and C, are correct as stated. However, the version of the Künneth formula in degree 2 with coefficients in an arbitrary ring mentioned on p. 750 of [6], with reference to Proposition 2.2, is not true in this generality (see Remark 1.2 for a counterexample). A similar correction needs to be made to Theorem 2.6.

1. Correction to Proposition 2.2

Proposition 1.1. *Let X and Y be non-empty path-connected CW-complexes such that $\mathrm{H}_1(X,\mathbb{Z})$ and $\mathrm{H}_1(Y,\mathbb{Z})$ are finitely generated abelian groups (which holds when X and Y are finite CW-complexes). For any abelian group G we have a canonical isomorphism*

$$\mathrm{H}^1(X\times Y,G)\cong \mathrm{H}^1(X,G)\oplus \mathrm{H}^1(Y,G).$$

If $G=\mathbb{Z}$ or $G=\mathbb{Z}/n$, where n is a positive integer, then there is a canonical isomorphism

$$\mathrm{H}^2(X\times Y,G)\cong \mathrm{H}^2(X,G)\oplus \mathrm{H}^2(Y,G)\oplus \mathrm{Hom}\big(\mathrm{H}^1(X,G)^\vee,\mathrm{H}^1(Y,G)\big),$$

where for a G-module M we write $M^\vee=\mathrm{Hom}(M,G)$.

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Mathematics Subject Classification (2020): Primary 14F22; Secondary 14G25

Proof. We write $\mathrm{H}_n(X) = \mathrm{H}_n(X, \mathbb{Z})$. Since X is non-empty and path-connected, we have $\mathrm{H}_0(X) = \mathbb{Z}$ (see [3, Prop. 2.7]). The Künneth formula for homology [3, Thm. 3.B.6] gives a split exact sequence of abelian groups

$$0 \to \bigoplus_{i=0}^{n} \big(\mathrm{H}_i(X) \otimes \mathrm{H}_{n-i}(Y)\big) \to \mathrm{H}_n(X \times Y) \to \bigoplus_{i=0}^{n-1} \mathrm{Tor}(\mathrm{H}_i(X), \mathrm{H}_{n-1-i}(Y)) \to 0.$$

Since $\mathrm{H}_0(X) = \mathbb{Z}$, in degrees 1 and 2 this gives canonical isomorphisms

$$\mathrm{H}_1(X \times Y) \cong \mathrm{H}_1(X) \oplus \mathrm{H}_1(Y), \tag{1}$$

$$\mathrm{H}_2(X \times Y) \cong \mathrm{H}_2(X) \oplus \mathrm{H}_2(Y) \oplus \big(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y)\big). \tag{2}$$

For any abelian group G, the universal coefficients theorem [3, Thm. 3.2] gives the following (split) exact sequence of abelian groups:

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(X), G) \to \mathrm{H}^n(X, G) \to \mathrm{Hom}(\mathrm{H}_n(X), G) \to 0, \tag{3}$$

where the third map evaluates a cocycle on a cycle. This gives a canonical isomorphism

$$\mathrm{H}^1(X, G) \cong \mathrm{Hom}(\mathrm{H}_1(X), G). \tag{4}$$

The desired isomorphism for H^1 now follows from (1).

Using the functoriality of the universal coefficients formula (3) with respect to the projections of $X \times Y$ to X and Y, together with the isomorphisms (1) and (2), we obtain a split short exact sequence

$$0 \to \mathrm{H}^2(X, G) \oplus \mathrm{H}^2(Y, G) \to \mathrm{H}^2(X \times Y, G) \to \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), G) \to 0. \tag{5}$$

The second map here has a retraction induced by the embedding of $X \times y_0$ and $x_0 \times Y$, for some base points x_0 and y_0. The third map in (5) is given by evaluating a cocycle on $X \times Y$ on the product of a cycle on X and a cycle on Y. A similar map with $G = G_1 \otimes G_2$ fits into the following commutative diagram with the natural right-hand vertical map:

$$\begin{array}{ccc} \mathrm{H}^2(X \times Y, G_1 \otimes G_2) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), G_1 \otimes G_2) \\ \cup \uparrow & & \uparrow \\ \mathrm{H}^1(X, G_1) \otimes \mathrm{H}^1(Y, G_2) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{H}_1(X), G_1) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), G_2) \end{array} \tag{6}$$

Let $G = \mathbb{Z}$. By assumption, $\mathrm{H}_1(X)$ and $\mathrm{H}_1(Y)$ are finitely generated abelian groups. Let M and N be their respective quotients by the torsion subgroups. The map induced by multiplication in $\mathbb{Z}$,

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}),$$

coincides with $\mathrm{Hom}(M, \mathbb{Z}) \otimes \mathrm{Hom}(N, \mathbb{Z}) \to \mathrm{Hom}(M \otimes N, \mathbb{Z})$, which is clearly an isomorphism, so the displayed map is also an isomorphism. Using (4) we rewrite it as

$$\mathrm{H}^1(X, \mathbb{Z}) \otimes \mathrm{H}^1(Y, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}).$$

Now (5) gives a canonical isomorphism

$$\mathrm{H}^2(X \times Y, \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathrm{H}^2(Y, \mathbb{Z}) \oplus \big(\mathrm{H}^1(X, \mathbb{Z}) \otimes \mathrm{H}^1(Y, \mathbb{Z})\big). \tag{7}$$

In view of diagram (6) the last summand is embedded into $\mathrm{H}^2(X \times Y, \mathbb{Z})$ via the cup-product map. Since $\mathrm{H}^1(X, \mathbb{Z})$ is a free abelian group of finite rank, we can rewrite (7) and obtain the desired isomorphism for $\mathrm{H}^2(X \times Y, \mathbb{Z})$.

Now let $G = \mathbb{Z}/n$. Then $\mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}/n)$ is canonically isomorphic to

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}/n)) \cong \mathrm{Hom}(\mathrm{H}_1(X)/n, \mathrm{H}^1(Y, \mathbb{Z}/n)).$$

Since $\mathrm{Hom}(\mathrm{H}_1(X)/n, \mathbb{Z}/n) \cong \mathrm{H}^1(X, \mathbb{Z}/n)$, we have $\mathrm{H}^1(X, \mathbb{Z}/n)^\vee \cong \mathrm{H}_1(X)/n$. Now (5) produces the required isomorphism for $\mathrm{H}^2(X \times Y, \mathbb{Z}/n)$. ■

Remark 1.2. For $X = Y = \mathbb{RP}^2$ we have $\mathrm{H}_1(X) = \mathbb{Z}/2$, so in this case the map induced by multiplication in $\mathbb{Z}/n$ with $n = 4$

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}/n) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}/n) \to \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}/n)$$

is zero. From diagram (6) we see that in this case the cup-product map

$$\mathrm{H}^1(X, \mathbb{Z}/n) \otimes \mathrm{H}^1(Y, \mathbb{Z}/n) \to \mathrm{H}^2(X \times Y, \mathbb{Z}/n)$$

is zero.

2. Correction to Theorem 2.6

Let k be a separably closed field. Let G be a finite commutative group k-scheme of order not divisible by $\mathrm{char}(k)$. The Cartier dual of G is defined as $\widehat{G} = \mathrm{Hom}(G, \mathbb{G}_{\mathrm{m},k})$ in the category of commutative group k-schemes.

For a proper and geometrically integral variety X over k, the natural pairing

$$\mathrm{H}^1_{\text{ét}}(X, G) \times \widehat{G} \to \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{\mathrm{m},X}) = \mathrm{Pic}(X)$$

gives rise to a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, G) \xrightarrow{\sim} \mathrm{Hom}(\widehat{G}, \mathrm{Pic}(X)). \tag{8}$$

The map in (8) associates to a class of a G-torsor $\mathcal{T} \to X$ its 'type' (see [5, Thm. 2.3.6]).

Let n be a positive integer not divisible by $\mathrm{char}(k)$. Define S_X as the finite commutative group k-scheme whose Cartier dual is

$$\widehat{S}_X = \mathrm{H}^1_{\text{ét}}(X, \mu_n) \cong \mathrm{Pic}(X)[n]. \tag{9}$$

We shall often consider the Tate twist $\widehat{S}_X(-1)$. So for a finite commutative group k-scheme G such that $nG = 0$ we introduce the notation

$$G^\vee = \mathrm{Hom}(G, \mathbb{Z}/n).$$

In particular, we have $S_X^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$. The pairing $G \times G^\vee \to \mathbb{Z}/n$ gives rise to a canonical isomorphism $G \xrightarrow{\sim} (G^\vee)^\vee$.

Let $T_X \to X$ be an S_X-torsor whose type is the natural inclusion

$$\widehat{S}_X = \mathrm{Pic}(X)[n] \hookrightarrow \mathrm{Pic}(X);$$

it is unique up to isomorphism. The natural pairing

$$\mathrm{H}^1_{\text{ét}}(X, S_X) \times S_X^\vee \to \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$$

with the class $[T_X] \in \mathrm{H}^1_{\text{ét}}(X, S_X)$ induces the identity map on $S_X^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$. In other words, the image of $[T_X]$ with respect to the map induced by $a \colon S_X \to \mathbb{Z}/n$ equals $a \in S_X^\vee$.

Suppose that Y is also a proper and geometrically integral variety over k. The image of $[T_X] \otimes [T_Y]$ under the map

$$\mathrm{H}^1_{\text{ét}}(X, S_X) \otimes \mathrm{H}^1_{\text{ét}}(Y, S_Y) \to \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)$$

induced by $a \colon S_X \to \mathbb{Z}/n$ and $b \colon S_Y \to \mathbb{Z}/n$ equals $a \otimes b \in S_X^\vee \otimes S_Y^\vee$.

We refer to [4, Prop. V.1.16] for the existence and properties of the cup-product. Thus we can consider $[T_X] \cup [T_Y] \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y)$ and

$$a \cup b \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n \otimes \mathbb{Z}/n) \cong \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

The cup-product is functorial, so the image of $[T_X] \cup [T_Y]$ under the map induced by $a \otimes b$ is $a \cup b$. This can be rephrased by saying that the natural pairing

$$\mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y) \ \times \ S_X^\vee \otimes S_Y^\vee \to \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \tag{10}$$

with $[T_X] \cup [T_Y]$ gives rise to the cup-product map

$$S_X^\vee \otimes S_Y^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n).$$

It is important to note that (10) factors through the pairing

$$\mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y) \ \times \ \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n). \tag{11}$$

The pairing (11) with $[T_X] \cup [T_Y]$ induces a map

$$\varepsilon \colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

We thus have a commutative diagram, where ξ is induced by multiplication in $\mathbb{Z}/n$:

$$\begin{array}{ccc} S_X^\vee \otimes S_Y^\vee & \xrightarrow{\ \ \xi\ \ } & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) & \xrightarrow{\ \ \cup\ \ } & \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \end{array} \tag{12}$$

The canonical isomorphism $\mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \cong \mathrm{Hom}(S_X, S_Y^\vee)$ allows us to rewrite ε as the map sending $\varphi \in \mathrm{Hom}(S_X, S_Y^\vee)$ to $\varepsilon(\varphi) = \varphi_*[T_X] \cup [T_Y]$, where $\cup$ stands for the cup-product pairing

$$\mathrm{H}^1_{\text{ét}}(X, S_Y^\vee) \times \mathrm{H}^1_{\text{ét}}(Y, S_Y) \to \mathrm{H}^2_{\text{ét}}(X \times Y, S_Y^\vee \otimes S_Y) \to \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

We write $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$ for the natural projections. Since X and Y are geometrically integral over the separably closed field k, we can choose base points $x_0 \in X(k)$ and $y_0 \in Y(k)$. We have the induced map

$$(\mathrm{id}_X, y_0)^* : \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \to \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n)$$

and a similar map for Y. Using these maps we see that

$$(p_X^*, p_Y^*) : \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \to \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \tag{13}$$

is split injective, so we have an isomorphism

$$\mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \cong \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}, \tag{14}$$

where $\mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}$ is the intersection of the kernels of $(\mathrm{id}_X, y_0)^*$ and $(x_0, \mathrm{id}_Y)^*$. Since k is separably closed, we have $\mathrm{H}^i(k, M) = 0$ for any abelian group M and any $i \geq 1$. Thus $[T_X] \cup [T_Y]$ goes to zero under the maps induced by the restrictions to $x_0 \times Y$ and to $X \times y_0$. This implies that $\mathrm{Im}(\varepsilon) \subset \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}}$.

The following is a corrected version of [6, Thm. 2.6].

Theorem 2.1. *Let X and Y be proper and geometrically integral varieties over a separably closed field k. Let n be a positive integer not divisible by* $\mathrm{char}(k)$. *Then we have the following statements:*

(i) *Write* $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)^\vee = \mathrm{Hom}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n), \mathbb{Z}/n)$ *and similarly for Y. The maps ε and ξ defined above fit into the commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) & \xrightarrow{\ \xi\ } & \mathrm{Hom}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)^\vee, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)) \\ \cup \downarrow & & \cong \downarrow \varepsilon \\ \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n) & \hookleftarrow & \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)_{\text{prim}} \end{array} \tag{15}$$

where ε is an isomorphism.

(ii) *If $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module* (*which holds if* $\mathrm{NS}(X)[n] = 0$)*, then ξ is an isomorphism, so we have*

$$\mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n) \cong \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \big(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)\big).$$

Proof. Part (ii) is the degree 2 case of [4, Cor. VI.8.13].

Let us prove (i). Diagram (15) is obtained from diagram (12) since $\mathrm{Im}(\varepsilon)$ is a subset of $\mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)_{\text{prim}}$, as explained above. It remains to show that ε is an isomorphism. From the spectral sequence

$$E_2^{p,q}=\mathrm{H}^p_{\text{ét}}(X,\mathrm{H}^q_{\text{ét}}(Y,\mathbb{Z}/n))\Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)$$

we get an isomorphism $\mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)_{\text{prim}}\cong \mathrm{H}^1_{\text{ét}}(X,\mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n))$. As a particular case of (8) we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(X,\mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n))\cong \mathrm{Hom}(S_Y,S_X^\vee)\cong \mathrm{Hom}(S_X,S_Y^\vee).$$

Thus the source and the target of ε are isomorphic finite abelian groups. One can finish the proof following the original arguments in [6] with small adjustments; see [1, pp. 161–162] for this revised proof.

Here we give a short proof communicated to us by Yang Cao. Since the source and the target of ε have the same cardinality, it is enough to show that

$$\varepsilon\colon \mathrm{Hom}(S_X\otimes S_Y,\mathbb{Z}/n)\to \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)_{\text{prim}}$$

is *injective*. More generally, for an integer $m\,|\,n$ consider the map

$$\varepsilon_m\colon \mathrm{Hom}(S_X\otimes S_Y,\mathbb{Z}/m)\to \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/m)_{\text{prim}}$$

defined via pairing with $[T_X]\cup[T_Y]$. We prove that ε_m is injective by induction on $m\,|\,n$. If p is a prime, the usual Künneth formula [4, Cor. VI.8.13] for the field $\mathbb{F}_p$ implies that the cup-product map

$$\cup\colon \mathrm{H}^1_{\text{ét}}(X,\mathbb{F}_p)\otimes \mathrm{H}^1_{\text{ét}}(Y,\mathbb{F}_p)\to \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{F}_p)_{\text{prim}}$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(S_X,\mathbb{F}_p)\otimes \mathrm{Hom}(S_Y,\mathbb{F}_p) & \xrightarrow{\ \xi\ } & \mathrm{Hom}(S_X\otimes S_Y,\mathbb{F}_p)\\
\cong\downarrow & & \downarrow\varepsilon_p\\
\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/p)\otimes \mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/p) & \xrightarrow{\ \cup\ } & \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/p)_{\text{prim}}
\end{array}\qquad (16)$$

In this case ξ is an isomorphism, hence ε_p is also an isomorphism.

Now for a positive integer $m\,|\,n$ assume that ε_a is injective for all $a\,|\,m$, $a\neq m$. Write $m=ab$. The exact sequence of abelian groups

$$0\to\mathbb{Z}/a\to\mathbb{Z}/m\to\mathbb{Z}/b\to 0$$

gives rise to the long exact sequences of étale cohomology groups of X, Y and $X\times Y$, which are linked by the split injective maps (13). Using (14) and the well-known fact that $\mathrm{H}^1_{\text{ét}}(X\times Y,\mathbb{Z}/b)_{\text{prim}}=0$ (see [6, Cor. 1.8] or [1, Thm. 5.7.7 (i)]) we find that the top row of the following commutative diagram is exact (see [1, p. 160] for an alternative argument):

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/a)_{\text{prim}} & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/m)_{\text{prim}} & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/b)_{\text{prim}} \\ & \uparrow \varepsilon_a & & \uparrow \varepsilon_m & & \uparrow \varepsilon_b \\ 0 \longrightarrow & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/a) & \longrightarrow & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/m) & \longrightarrow & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/b) \end{array}$$

The bottom row is obviously exact. The diagram implies that the middle map is injective too. We conclude that $\varepsilon = \varepsilon_n$ is injective, hence an isomorphism.

References

[1] Colliot-Thélène, J.-L., Skorobogatov, A. N.: The Brauer–Grothendieck group. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 71, Springer, Cham (2021) Zbl 1490.14001 MR 4304038

[2] Faltings, G.: Zentralblatt review of [6]. Zbl 1295.14021

[3] Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002) Zbl 1044.55001 MR 1867354

[4] Milne, J. S.: Étale cohomology. Princeton Mathematical Series 33, Princeton University Press, Princeton, NJ (1980) Zbl 0433.14012 MR 559531

[5] Skorobogatov, A.: Torsors and rational points. Cambridge Tracts in Mathematics 144, Cambridge University Press, Cambridge (2001) Zbl 0972.14015 MR 1845760

[6] Skorobogatov, A. N., Zarhin, Y. G.: The Brauer group and the Brauer–Manin set of products of varieties. J. Eur. Math. Soc. **16**, 749–768 (2014) Zbl 1295.14021 MR 3191975