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Rationality and fusion rules of exceptional \mathcal{W} -algebras

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Abstract. First, we prove the Kac–Wakimoto conjecture on modular invariance of characters of exceptional affine \mathcal{W} -algebras. In fact more generally we prove modular invariance of characters of all lisse \mathcal{W} -algebras obtained through Hamiltonian reduction of admissible affine vertex algebras. Second, we prove the rationality of a large subclass of these \mathcal{W} -algebras, which includes all exceptional \mathcal{W} -algebras of type A and lisse subregular \mathcal{W} -algebras in simply laced types. Third, for the latter cases we compute S -matrices and fusion rules. Our results provide the first examples of rational \mathcal{W} -algebras associated with nonprincipal distinguished nilpotent elements, and the corresponding fusion rules are rather mysterious.

Keywords. Vertex algebras, \mathcal{W} -algebras, fusion rules, rational conformal field theory

1. Introduction

Let \mathfrak{g} be a finite-dimensional simple Lie algebra, $f \in \mathfrak{g}$ a nilpotent element, and $k \in \mathbb{C}$ a complex number. The universal affine \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{g}, f)$ of level k is obtained from the universal affine vertex algebra $V^k(\mathfrak{g})$ through the process of quantised Drinfeld–Sokolov reduction. This construction was introduced in [39] for f a principal nilpotent element, and in [62] for f a general nilpotent element.

Affine \mathcal{W} -algebras arise as algebras of symmetries of integrable models [61], in the geometric Langlands program [12, 43, 48], the 4d/2d duality [9, 18, 19, 74], $\mathcal{N} = 4$ super Yang–Mills gauge theories [25, 47], and as invariants of 4-manifolds [40].

It is believed that for appropriate choices of nilpotent element $f \in \mathfrak{g}$ and level k the simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ of $\mathcal{W}^k(\mathfrak{g}, f)$ is a rational and lisse vertex algebra, and as such gives rise to a rational conformal field theory. Indeed, let $k = -h^\vee + p/q$ be an admissible level for the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ associated with \mathfrak{g} . (Recall this means that the simple quotient $V_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is admissible as a representation of $\widehat{\mathfrak{g}}$ [65].) Then for f a principal nilpotent element the rationality of $\mathcal{W}_k(\mathfrak{g}, f)$ was conjectured by Frenkel,

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Kac and Wakimoto [44] and proved by the first named author [7]. For general nilpotent elements, the notion of an *exceptional pair* (f, q) , where $f \in \mathfrak{g}$ is nilpotent and $q \geq 1$ is an integer, was introduced in [66] (and later extended in [38]). Kac and Wakimoto conjectured that $\mathcal{W}_k(\mathfrak{g}, f)$ is rational whenever $k = -h^\vee + p/q$ is admissible and (f, q) forms an exceptional pair.

In [6] it was shown that the associated variety [4] of the simple affine vertex algebra $V_k(\mathfrak{g})$ equals the closure of a certain nilpotent orbit $\mathbb{O}_q \subset \mathfrak{g}$ and that $\mathcal{W}_k(\mathfrak{g}, f)$ is nonzero and lisse if $f \in \mathbb{O}_q$. At the risk of ambiguity we should like to refer to (f, q) as an *exceptional pair* whenever $f \in \mathbb{O}_q$. Restriction to those pairs (f, q) for which f is of standard Levi type recovers the notion of exceptional pair of [38], and further restriction to those pairs for which q is coprime to the lacity r^\vee recovers the original notion of exceptional pair of [66]. For \mathfrak{g} of type A all these notions coincide.

It was conjectured in [6] that all exceptional \mathcal{W} -algebras (now in the broader sense of exceptional) are rational. Our first main result gives strong evidence for this conjecture, and thus for the conjecture of Kac and Wakimoto.

Main Theorem 1 (Theorems 4.3 and 5.1). *Let $k = -h^\vee + p/q$ be an admissible level for $\widehat{\mathfrak{g}}$, and let $f \in \mathbb{O}_q$ be a nilpotent element. Then the Ramond twisted Zhu algebra $A(\mathcal{W})$ of $\mathcal{W} = \mathcal{W}_k(\mathfrak{g}, f)$ is semisimple. Let $\{\mathbf{L}_1, \dots, \mathbf{L}_r\}$ be a complete set of representatives of isomorphism classes of simple Ramond twisted \mathcal{W} -modules, and $S_{\mathbf{L}_i}(\tau | u) = \text{Tr}_{\mathbf{L}_i}(u_0 q^{L_0 - c/24})$ for $u \in \mathcal{W}$ the associated trace function. Then $S_{\mathbf{L}_i}(\tau | u)$ converges to a holomorphic function on the upper half-plane for all $u \in \mathcal{W}$, $i = 1, \dots, r$. Moreover, there is a representation $\rho : SL_2(\mathbb{Z}) \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}^r)$ such that*

$$S_{\mathbf{L}_i} \left(\frac{a\tau + b}{c\tau + d} \mid (c\tau + d)^{-L_{[0]}u} \right) = \sum_j \rho(A)_{ij} S_{\mathbf{L}_j}(\tau | u) \quad \text{for all } u \in \mathcal{W}.$$

We note that if f admits a good even grading, as is always the case for \mathfrak{g} of type A , a Ramond twisted module is the same thing as an untwisted module in the usual sense.

Our next result establishes rationality of those \mathcal{W} -algebras appearing in Main Theorem 1 for which the set of principal (or coprincipal) admissible weights of level k satisfies a certain integrability condition relative to f . Let k be an admissible level for $\widehat{\mathfrak{g}}$. The irreducible highest weight representation $L(\widehat{\lambda})$ of $\widehat{\mathfrak{g}}$ with highest weight $\widehat{\lambda} = k\Lambda_0 + \lambda$ is a $V_k(\mathfrak{g})$ -module if and only if λ belongs to the set Pr^k of level k principal (or coprincipal) admissible weights [8]. For such $\lambda \in \text{Pr}^k$ we consider the annihilating ideal $J_\lambda \subset U(\mathfrak{g})$ of $L(\lambda)$ and for a nilpotent orbit \mathbb{O} we denote by $\text{Pr}_{\mathbb{O}}^k$ the subset of Pr^k consisting of those λ for which $\text{Var}(J_\lambda) = \overline{\mathbb{O}}$. The annihilator J_λ depends only on the orbit of λ under the dot action of the finite Weyl group W , and we set $[\text{Pr}_{\mathbb{O}}^k] = \text{Pr}_{\mathbb{O}}^k / W \circ (-)$.

Main Theorem 2 (Theorem 7.9). *Let $k = -h^\vee + p/q$ be an admissible level for $\widehat{\mathfrak{g}}$, and let $f \in \mathbb{O}_q$ be a nilpotent element. Suppose f admits a good even grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ such that every element of $[\text{Pr}_{\mathbb{O}_q}^k]$ has a representative integrable with respect to \mathfrak{g}_0 . Then the vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is rational and lisse, and all irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -modules*

are obtained via quantised Drinfeld–Sokolov “–”-reduction of level k admissible $\hat{\mathfrak{g}}$ -modules.

Main Theorem 2 proves the Kac–Wakimoto rationality conjecture for all exceptional \mathcal{W} -algebras of type A (Theorem 8.5). It also proves the rationality of all exceptional subregular \mathcal{W} -algebras in simply laced types (see Theorem 9.4). The latter algebras actually lie outside the class of Kac–Wakimoto exceptional \mathcal{W} -algebras. The values of q for which $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$ are listed in Table 1.

Some special cases of Main Theorem 2 have already been proved; for f a principal nilpotent element [7], for the Bershadsky–Polyakov algebras $\mathcal{W}_k(\mathfrak{sl}_3, f_{\min})$ [5], and for $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ [27].

We note that subregular \mathcal{W} -algebras of types D and E are distinguished \mathcal{W} -algebras, that is, \mathcal{W} -algebras associated with distinguished nilpotent elements, or equivalently, \mathcal{W} -algebras that have zero weight 1 subspaces. Distinguished \mathcal{W} -algebras play a fundamental role among \mathcal{W} -algebras. However, the representation theory of distinguished \mathcal{W} -algebras that are not of principal type is mysterious even at the level of finite W -algebras, since there are no canonical standard modules. Main Theorem 2 provides the first examples of rational distinguished \mathcal{W} -algebras that are not of principal type.

We say a few words about the proofs of the theorems. The first step is to compute the Zhu algebra of $\mathcal{W}_k(\mathfrak{g}, f)$, which is a quotient of the finite \mathcal{W} -algebra $U(\mathfrak{g}, f)$ [72]. To do this we compute the Zhu algebra of the admissible affine vertex algebra (Theorem 3.4), and then apply the commutativity [7] of the Zhu algebra functor and the Drinfeld–Sokolov reduction functor. The irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -modules are in bijection with those over the Zhu algebra. The role of the \mathfrak{g}_0 -integrality condition is to ensure invariance of irreducible modules under the canonical action of the component group $C(f)$, allowing us to use results of Losev [68] on the representation theory of finite \mathcal{W} -algebras, and thereby characterise the irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -modules. It remains to rule out nontrivial extensions between irreducible modules, which is done by following the same approach as in [7]. Here a result of Gorelik–Kac [51] on complete reducibility of admissible representations of $\hat{\mathfrak{g}}$ is used.

By Huang’s result [53], the module category of a rational, lisse, self-dual vertex algebra is a modular tensor category. Therefore Main Theorem 2 provides a huge supply of modular tensor categories. Following the approach of [44] and [17] we compute the modular S -matrix and fusion rules of $\mathcal{W}_k(\mathfrak{g}, f)$ whenever \mathfrak{g} is simply laced and f is subregular. We now explain the general features in simplified form.

We recall that the irreducible modules of the simple affine vertex algebra $V_{p-h^\vee}(\mathfrak{g})$ are parametrised by regular dominant integral weights of level p . The S -matrix of this vertex algebra is, up to normalisation, given by

$$K_p^{\lambda, \mu} = \sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{p}(w(\lambda), \mu)},$$

where the indices λ, μ run over the set of regular dominant integral weights of $\hat{\mathfrak{g}}$ of level p . These coefficients lie in the cyclotomic field $\mathbb{Q}(\zeta_N)$ where N equals p times the order

of the centre of the adjoint group of \mathfrak{g} . For a coprime to N we let $\varphi_a \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ denote the automorphism defined by $\varphi_a(\zeta_N) = \zeta_N^a$.

The S -matrix of $\mathcal{W} = \mathcal{W}_{-h^\vee + p/q}(\mathfrak{g}, f_{\text{subreg}})$ is, up to normalisation, the Kronecker product matrix

$$\varphi_p(C_q) \otimes \varphi_q(K_p),$$

where C_q is a sort of degenerate analogue of K_p given explicitly by (Theorem 12.2)

$$C_q^{\lambda, \mu} = \sum_{w(\alpha_*) \in \Delta_+} \epsilon(w) \frac{\langle w(\alpha_*), x \rangle}{\langle \alpha_*, x \rangle} e^{-\frac{2\pi i}{q} \langle w(\lambda), \mu \rangle}. \tag{1.1}$$

Here α_* is the unique positive root of the Lie subalgebra \mathfrak{g}_0 and x is an arbitrary element of \mathfrak{h} not orthogonal to α_* . The indices λ, μ run over the set of weights $\gamma \in Q$ of level q satisfying $\langle \gamma, \alpha_i \rangle \in \mathbb{Z}_+$ for $i = 1, \dots, \ell$ and $\langle \gamma, \alpha_i \rangle = 0$ for exactly one i .

Since the fusion product multiplicities are integers determined from the S -matrix via the Verlinde formula, hence Galois invariant, we deduce that the fusion algebra of \mathcal{W} is the tensor product of the fusion algebra of $V_{p-h^\vee}(\mathfrak{g})$ with the fusion algebra whose S -matrix is C_q . This factorisation is quite parallel to the result discovered in [44], that the fusion algebra of the principal \mathcal{W} -algebra $\mathcal{W}_{-h^\vee + p/q}(\mathfrak{g}, f_{\text{prin}})$ is more or less the tensor product of the fusion algebras of two simple affine vertex algebras.

For type A the matrix C_q is the 1×1 identity matrix and therefore the fusion rules of \mathcal{W} depend only on the numerator p and coincide with those of $V_{p-h^\vee}(\mathfrak{g})$. For types D and E the matrix C_q is nontrivial and the fusion rules of \mathcal{W} are more interesting. In most cases (but not quite all, see Conjecture 14.2 below) C_q is itself naturally identified with the S -matrix of a subregular \mathcal{W} -algebra. In those cases for which this \mathcal{W} -algebra has asymptotic growth less than 1, that is, all cases except $\mathfrak{g} = E_7, q = 16, 17$ and $\mathfrak{g} = E_8, q = 27, 28, 29$, we are able to identify it as a simple current extension of a Virasoro minimal model, thus confirming the S -matrix computed by (1.1). We summarise these results in Table 3. Most of the rational lisse \mathcal{W} -algebras obtained above are not unitary. However, we conjecture that

$$\mathcal{W}_{-117/11}(E_6, f_{\text{subreg}}) \quad \text{and} \quad \mathcal{W}_{-267/16}(E_7, f_{\text{subreg}})$$

are unitary, giving rise to two seemingly new unitary modular tensor categories.

Finally, let us make some comments on the relations of the present work to 4d/2d duality [18]. It has been shown in [23, 74, 77, 79] that the exceptional \mathcal{W} -algebras at boundary admissible levels appear as vertex algebras obtained from 4d $\mathcal{N} = 2$ superconformal field theories (or more precisely from Argyres–Douglas theories), and the corresponding modular tensor categories are expected to coincide with those arising from the Coulomb branches of the corresponding four-dimensional theories, or the wild Hitchin moduli spaces [31, 42]. Furthermore, the modular tensor categories associated with exceptional distinguished \mathcal{W} -algebras at boundary admissible levels are closely connected with the Jacobian rings of certain hypersurface singularities [78]. We hope to come back to these points in our future work.

2. Affine Lie algebras and admissible weights

Let \mathfrak{g} be a complex simple finite-dimensional Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let Δ be the set of roots of \mathfrak{g} , let Δ_+ be the set of positive roots and W the Weyl group. Denote by $Q = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha$ the root lattice and by $\check{Q} = \sum_{\alpha \in \Delta} \mathbb{Z}\alpha^\vee$ the coroot lattice. Here $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ where (\cdot, \cdot) is the invariant bilinear form normalised so that the highest root θ satisfies $(\theta, \theta) = 2$. The highest short root is denoted by θ_s and satisfies $(\theta_s, \theta_s) = 2/r^\vee$, where r^\vee is the lacing number of \mathfrak{g} . Let P be the weight lattice of \mathfrak{g} and \check{P} the coweight lattice. The simple coroots α_i^\vee form a basis of \check{Q} , and the basis $\{\varpi_i\}$ of P dual to $\{\alpha_i^\vee\}$ defines the fundamental weights ϖ_i of \mathfrak{g} . Similarly, ϖ_i^\vee will denote the fundamental coweights, which form a basis of \check{P} dual to the simple roots $\{\alpha_i\}$. Let ρ be the half sum of positive roots of \mathfrak{g} , and ρ^\vee the half sum of positive coroots. Let $P_+ = \sum_i \mathbb{Z}_{\geq 0}\varpi_i$ and $\check{P}_+ = \sum_i \mathbb{Z}_{\geq 0}\varpi_i^\vee$ be the sets of dominant weights and dominant coweights, respectively, and for $n \in \mathbb{Z}_{\geq 0}$ set

$$P_+^n = \{\lambda \in P_+ \mid \langle \lambda, \alpha^\vee \rangle \leq n \text{ for } \alpha \in \Delta_+\}, \quad \check{P}_+^n = \{\lambda \in \check{P}_+ \mid \langle \alpha, \lambda \rangle \leq n \text{ for } \alpha \in \Delta_+\}. \tag{2.1}$$

For $\lambda \in \mathfrak{h}^*$, we denote by $M(\lambda)$ the Verma \mathfrak{g} -module with highest weight λ , and by $L(\lambda)$ the unique simple quotient of $M(\lambda)$. A weight λ is said to be *dominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$ for all $\alpha \in \Delta_+$, and *regular* if it has trivial stabiliser under the ‘‘dot’’ action $y \circ \lambda = y(\lambda + \rho) - \rho$ of W .

A *primitive ideal* in the universal enveloping algebra $U(\mathfrak{g})$ is by definition the annihilator of some irreducible \mathfrak{g} -module. Set

$$J_\lambda = \text{Ann}_{U(\mathfrak{g})} L(\lambda).$$

By Duflo’s theorem [36], any primitive ideal in $U(\mathfrak{g})$ is of the form J_λ for some $\lambda \in \mathfrak{h}^*$.

The centre of $U(\mathfrak{g})$ is denoted $\mathcal{Z}(\mathfrak{g})$, and the character $\gamma_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is defined by $z v_\lambda = \gamma_\lambda(z) v_\lambda$, where v_λ is a highest weight vector of $L(\lambda)$.

Let

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

be the affine Kac–Moody algebra associated with \mathfrak{g} , whose commutation relations are given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m(x, y)\delta_{m+n,0}K, \quad [K, \hat{\mathfrak{g}}] = 0.$$

Let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$ be the standard Cartan subalgebra of $\hat{\mathfrak{g}}$, and $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ the extended Cartan subalgebra of $\hat{\mathfrak{g}}$. The dual of $\tilde{\mathfrak{h}}$ is $\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, where $\Lambda_0(K) = \delta(D) = 1$ and $\Lambda(\mathfrak{h} \oplus \mathbb{C}D) = \delta(\mathfrak{h} \oplus \mathbb{C}K) = 0$. The dual $\hat{\mathfrak{h}}^*$ of $\hat{\mathfrak{h}}$ is identified with the subspace $\mathfrak{h} \oplus \mathbb{C}\Lambda_0 \subset \tilde{\mathfrak{h}}^*$.

Let $\hat{\Delta} \subset \hat{\mathfrak{h}}^*$ be the set of roots of $\hat{\mathfrak{g}}$ and

$$\begin{aligned} \hat{\Delta}^{\text{re}} &= \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\}, \\ \hat{\Delta}_+^{\text{re}} &= \{\alpha + n\delta \mid \alpha \in \Delta_+, n \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\alpha + n\delta \mid \alpha \in \Delta_+, n \in \mathbb{Z}_{\geq 1}\} \end{aligned}$$

the subsets of real roots and positive real roots, respectively.

We denote by \widehat{W} the Weyl group of $\widehat{\mathfrak{g}}$, so $\widehat{W} = W \ltimes \check{Q}$, and we denote by $\widetilde{W} = W \ltimes \check{P}$ the extended affine Weyl group of \mathfrak{g} . We have $\widetilde{W} = \widetilde{W}_+ \ltimes \widehat{W}$, where \widetilde{W}_+ is a finite subgroup of \widetilde{W} which we now describe. Write θ as a sum of simple roots: $\theta = \sum_{i=1}^{\ell} a_i \alpha_i$, and set $J = \{i \in \{1, \dots, \ell\} \mid a_i = 1\}$. Then we have

$$\widetilde{W}_+ = \{\pi_j = t_{w_j} \bar{\pi}_j \mid j \in J\}, \tag{2.2}$$

where $\bar{\pi}_j$ is the unique element of W which fixes the set $\{\alpha_1, \dots, \alpha_{\ell}, -\theta\}$ and which satisfies $\bar{\pi}_j(-\theta) = \alpha_j$.

Lemma 2.1 ([44, Lemma 4.1.1]). *Let \mathfrak{g} be a simply laced Lie algebra, and $\widehat{\mathfrak{g}}$ its affinisation. If μ is a dominant integral weight of level coprime to $|J|$ then the elements of the \widetilde{W}_+ -orbit of μ represent a complete set of classes of P/Q . In particular, there exists a unique $\pi \in \widetilde{W}_+$ such that $\pi(\mu) \in Q$.*

For $\lambda \in \widehat{\mathfrak{h}}^*$, let $\widehat{\Delta}(\lambda)$ denote its integral root system and $\widehat{W}(\lambda)$ its integral Weyl group, that is,

$$\widehat{\Delta}(\lambda) = \{\alpha \in \widehat{\Delta}^{\text{re}} \mid \langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \in \mathbb{Z}\}, \quad \widehat{W}(\lambda) = \langle s_\alpha \mid \alpha \in \widehat{\Delta}(\lambda) \rangle \subset \widehat{W},$$

where s_α is the reflection in the root α and $\widehat{\rho} = \rho + h^\vee \Lambda_0$. Let $\widehat{\Delta}(\lambda)_+ = \widehat{\Delta}(\lambda) \cap \widehat{\Delta}_+^{\text{re}}$ be the set of positive roots of $\widehat{\Delta}(\lambda)$ and $\widehat{\Pi}(\lambda) \subset \widehat{\Delta}(\lambda)_+$ its set of simple roots.

A weight $\lambda \in \widehat{\mathfrak{h}}^*$ is called *admissible* [65] if

- (1) λ is *regular dominant*, i.e., $\langle \lambda + \widehat{\rho}, \alpha^\vee \rangle \notin \{0, -1, -2, \dots\}$ for all $\alpha \in \widehat{\Delta}_+^{\text{re}}$, and
- (2) $\mathbb{Q}\widehat{\Delta}(\lambda) = \mathbb{Q}\widehat{\Delta}^{\text{re}}$.

For $\lambda \in \widehat{\mathfrak{h}}^*$ and $k \in \mathbb{C}$, we denote by $\widehat{M}_k(\lambda)$ the Verma module of $\widehat{\mathfrak{g}}$ with highest weight $\lambda + k\Lambda_0 \in \widehat{\mathfrak{h}}^*$, and by $\widehat{L}_k(\lambda)$ the unique simple quotient of $\widehat{M}_k(\lambda)$. The simple highest weight representation $\widehat{L}_k(\lambda)$ is called *admissible* if $\widehat{\lambda}$ is admissible. A complex number k is called admissible if $k\Lambda_0$ is admissible.

When clear from context, we shall write, as above, $\widehat{\lambda}$ for $\lambda + k\Lambda_0$. Occasionally we shall use the notation $\widetilde{\widehat{\lambda}}$ for the image of $\widehat{\lambda} \in \widehat{\mathfrak{h}}^*$ under the natural projection $\widehat{\mathfrak{h}}^* \rightarrow \mathfrak{h}^*$.

Proposition 2.2 ([65, 66]). *The number k is admissible if and only if*

$$k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{Z}_{\geq 1}, (p, q) = 1, p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where h and h^\vee are the Coxeter number and the dual Coxeter number of \mathfrak{g} , respectively. Furthermore,

$$\widehat{\Pi}(k\Lambda_0) = \{\alpha_0, \alpha_1, \dots, \alpha_\ell\},$$

where

$$\alpha_0 = \begin{cases} -\theta + q\delta & \text{if } (r^\vee, q) = 1, \\ -\theta_s + \frac{q}{r^\vee}\delta & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$

Let k be an admissible number. Then

$$\widehat{\Delta}(k\Lambda_0) \cong \begin{cases} \widehat{\Delta}^{\text{re}} & \text{if } (q, r^\vee) = 1, \\ {}^L\widehat{\Delta}^{\text{re}} & \text{if } (q, r^\vee) \neq 1, \end{cases}$$

where ${}^L\widehat{\Delta}^{\text{re}}$ is the real roots system of the Langlands dual ${}^L\widehat{\mathfrak{g}}$ of $\widehat{\mathfrak{g}}$. Let $\text{Pr}^k \subset \mathfrak{h}^*$ be the set of weights λ such that $\widehat{\lambda} = \lambda + k\Lambda_0$ is admissible and there exists $y \in \widetilde{W}$ such that $\widehat{\Delta}(\widehat{\lambda}) = y(\widehat{\Delta}(k\Lambda_0))$. Such admissible weights are said to be *principal admissible* [65] if $(q, r^\vee) = 1$ or *coprincipal admissible* [17] if $(q, r^\vee) \neq 1$. Let $\text{Pr}_{\mathbb{Z}}^k = \text{Pr}^k \cap P$. Then Pr^k decomposes as

$$\text{Pr}^k = \bigcup_{\substack{y \in \widetilde{W} \\ y(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{\text{re}}}} \text{Pr}_y^k, \quad \text{where } \text{Pr}_y^k = \{\lambda \mid \widehat{\lambda} \in y \circ (\text{Pr}_{\mathbb{Z}}^k + k\Lambda_0)\}. \quad (2.3)$$

For later purposes let us assume that the denominator q of k is coprime to r^\vee for the rest of this section and describe the set Pr^k of principal admissible weights in more detail. We have [65, (3.52)]

$$\text{Pr}_y^k \cap \text{Pr}_{y'}^k \neq \emptyset \iff \text{Pr}_y^k = \text{Pr}_{y'}^k \iff y' = yt_{q\varpi_j} \bar{\pi}_j \text{ for some } j \in J, \quad (2.4)$$

where q is the denominator of k . If p is the numerator of k as in Proposition 2.2, we have

$$\text{Pr}_{\mathbb{Z}}^k = P_+^{p-h^\vee}. \quad (2.5)$$

The cardinality of Pr^k is $|\text{Pr}^k| = q^\ell |P_+^{p-h^\vee}|$ ([44, Section 1.5]; see also [8, Proposition 3.2]).

Let ϕ denote the isometry of $\widehat{\mathfrak{h}}$ defined by $\phi|_{\mathfrak{h}} = 1$ and $\phi(\Lambda_0) = (1/q)\Lambda_0$. For $\alpha^\vee \in \mathfrak{h}$ the translate [44, Section 1.2] by $\eta \in \mathfrak{h}$ is

$$t_\eta(\alpha^\vee + sK) = \alpha^\vee + (s - \eta(\alpha^\vee))K.$$

By (2.3) any element of Pr^k is of the form λ where

$$\lambda + k\Lambda_0 = \widehat{\lambda} = \widehat{y}\phi(\widehat{v}) - \widehat{\rho} \quad (2.6)$$

with $\widehat{v} = p\Lambda_0 + v$, $v \in P_{+, \text{reg}}^p = \{\lambda \in P \mid 0 < \langle \lambda, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Delta_+\}$, and $\widehat{y} \in \widetilde{W}$. We may write $\widehat{y} = yt_{-\eta} = t_\beta y$ where $y \in W$ and $\eta, \beta \in P$, and put $\widehat{\eta} = q\Lambda_0 + \eta$. In this way we associate triples (y, η, v) and $(y, \widehat{\eta}, \widehat{v})$ to $\lambda \in \text{Pr}^k$. The associated triple is unique up to the action of the group \widetilde{W}_+ by

$$\pi : (y, \widehat{\eta}, \widehat{v}) \mapsto (y\bar{\pi}^{-1}, \pi(\widehat{\eta}), \pi(\widehat{v})). \quad (2.7)$$

3. Zhu algebras of admissible affine vertex algebras

For $k \in \mathbb{C}$ let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with \mathfrak{g} at level k . By definition

$$V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}k$$

as a $\widehat{\mathfrak{g}}$ -module, where \mathbb{C}_k denotes the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as multiplication by k . In this paper we assume that $k \neq -h^\vee$, so the vertex algebra $V^k(\mathfrak{g})$ is conformal with Virasoro element $L \in V^k(\mathfrak{g})$ and corresponding field

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

given by the Sugawara construction. The central charge of $V^k(\mathfrak{g})$ is $\frac{k \dim \mathfrak{g}}{k+h^\vee}$. The unique simple quotient $V_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is called the *simple affine vertex algebra* associated with \mathfrak{g} at level k . Note that $V_k(\mathfrak{g}) \cong \widehat{L}_k(0)$ as $\widehat{\mathfrak{g}}$ -modules.

The *Zhu algebra* (more precisely the *Ramond twisted Zhu algebra*) $A(V)$ of the vertex algebra V is the quotient of V by its vector subspace $\{a *_2 b \mid a, b \in V\}$, where by definition

$$a *_n b = \sum_{j \in \mathbb{Z}_+} \binom{\Delta(a)}{j} a_{(n+j)} b,$$

equipped with the associative product $a \otimes b \mapsto a *_1 b$. We denote in general the component of lowest L_0 -eigenvalue in a graded V -module M by M_{top} . This graded piece acquires the natural structure of $A(V)$ -module, and it is known that the correspondence $M \mapsto M_{\text{top}}$ is a bijection [30, 81] from the set of isomorphism classes of irreducible Ramond twisted V -modules to the set of simple $A(V)$ -modules.

For all k there exists a natural isomorphism $A(V^k(\mathfrak{g})) \cong U(\mathfrak{g})$ and hence

$$A(V_k(\mathfrak{g})) \cong U(\mathfrak{g})/I_k \tag{3.1}$$

for some two-sided ideal I_k of $U(\mathfrak{g})$. Thus $\widehat{L}_k(\lambda)$ is a $V_k(\mathfrak{g})$ -module if and only if $I_k \cdot L(\lambda) = 0$, that is, $I_k \subset J_\lambda$. The following assertion was conjectured in [1] and proved by the first named author.

Theorem 3.1 ([8]). *Let k be an admissible number for $\widehat{\mathfrak{g}}$ and $\lambda \in \mathfrak{h}^*$. Then $\widehat{L}_k(\lambda)$ is a $V_k(\mathfrak{g})$ -module if and only if $\lambda \in \text{Pr}^k$.*

Corollary 3.2. *Any $A(V_k(\mathfrak{g}))$ -module on which \mathfrak{n}_+ acts locally nilpotently and \mathfrak{h} acts locally finitely is a direct sum of $L(\lambda)$ with $\lambda \in \text{Pr}^k$.*

Proof. It is sufficient to show that $\text{Ext}_{\mathfrak{g}}^1(L(\lambda), L(\mu)) = 0$ for $\lambda, \mu \in \text{Pr}^k$. If $\lambda \neq \mu$ this is obvious since any weight in Pr^k is dominant. So suppose there exists a nonsplit exact sequence $0 \rightarrow L(\lambda) \rightarrow M \rightarrow L(\lambda) \rightarrow 0$. Applying the Zhu induction functor to this sequence gives rise to a nontrivial self-extension of $\widehat{L}_k(\lambda)$ [7, proof of Theorem 10.5]. But this contradicts the fact [51] that admissible representations of $\widehat{\mathfrak{g}}$ do not admit nontrivial self-extensions. ■

Lemma 3.3 ([13, Proposition 2.4]). *Let $\lambda, \mu \in \text{Pr}^k$.*

- (1) J_λ is the unique maximal two-sided ideal containing $U(\mathfrak{g}) \ker \gamma_\lambda$. In particular, $U(\mathfrak{g})/J_\lambda$ is a simple algebra.
- (2) $J_\lambda = J_\mu$ if and only if there exists $w \in W$ such that $\mu = w \circ \lambda$.

Set

$$[\text{Pr}^k] = \text{Pr}^k / \sim, \quad \text{where } \lambda \sim \mu \iff \mu \in W \circ \lambda.$$

By Lemma 3.3, the primitive ideal J_λ depends only on the class of $\lambda \in \text{Pr}^k$ in $[\text{Pr}^k]$.

We are now in a position to state the main result of this section.

Theorem 3.4. *Let k be an admissible number for $\widehat{\mathfrak{g}}$. The Zhu algebra $A(V_k(\mathfrak{g}))$ is isomorphic to the product of the simple algebras $U(\mathfrak{g})/J_\lambda$ as λ runs over $[\text{Pr}^k]$:*

$$A(V_k(\mathfrak{g})) \cong \prod_{\lambda \in [\text{Pr}^k]} U(\mathfrak{g})/J_\lambda.$$

Remark 3.5. (1) Theorem 3.4 implies, in particular, that any $A(V_k(\mathfrak{g}))$ -module admits a central character. Thus the image $[L]$ of the conformal vector L is semisimple in $A(V_k(\mathfrak{g}))$.

(2) For $\lambda \in \text{Pr}^k$, the algebra $U(\mathfrak{g})/J_\lambda$ is finite-dimensional if and only if $\lambda \in P$. (Note that the latter condition implies that λ is a dominant integral weight, since any element of Pr^k is regular dominant.) In this case we have $U(\mathfrak{g})/J_\lambda \cong L(\lambda) \otimes L(\lambda)^*$.

(3) In the special case of $k \in \mathbb{Z}_+$, or equivalently $V_k(\mathfrak{g})$ integrable, Pr^k is the projection P_+^k to \mathfrak{h}^* of the set \widehat{P}_+^k of dominant integral weights of $\widehat{\mathfrak{g}}$ of level k and we have

$$A(V_k(\mathfrak{g})) \cong \prod_{\lambda \in P_+^k} L(\lambda) \otimes L(\lambda)^*,$$

which is well-known [46].

Proof of Theorem 3.4. Fix an admissible number k and put $A = A(V_k(\mathfrak{g})) = U(\mathfrak{g})/I_k$. For any $\lambda \in \text{Pr}^k$ we have

$$A \otimes_{U(\mathfrak{g})} M(\lambda) \cong L(\lambda) \tag{3.2}$$

as A -modules. Indeed, $A \otimes_{U(\mathfrak{g})} M(\lambda)$ is a quotient of $M(\lambda)$, and by Corollary 3.2 the only quotient A -module of $M(\lambda)$ is $L(\lambda)$.

Let $\mathcal{Z}(\mathfrak{g})$ denote the centre of $U(\mathfrak{g})$ and put

$$Z = \mathcal{Z}(\mathfrak{g})/\mathcal{Z}(\mathfrak{g}) \cap I_k.$$

Since the associated variety of $V_k(\mathfrak{g})$ is contained in the nilpotent cone \mathcal{N} of \mathfrak{g} [6], it follows that Z is finite-dimensional (see [16, proof of Corollary 5.3]). Hence

$$Z = \prod_{\gamma \in \text{Specm}(Z)} Z_\gamma, \quad \text{where } Z_\gamma = \{z \in Z \mid (z - \gamma(z))^r = 0 \text{ for } r \gg 0\}.$$

This gives a decomposition

$$A = \prod_{\gamma \in \text{Specm}(Z)} A_\gamma, \quad \text{where } A_\gamma = A \otimes_Z Z_\gamma.$$

Moreover, by Theorem 3.1 and Lemma 3.3, we have

$$\text{Specm}(Z) = \{\gamma_\lambda \mid \lambda \in [\text{Pr}^k]\}.$$

Let $\lambda \in \text{Pr}^k$ and denote by $\mathbb{C}_{\gamma_\lambda}$ the one-dimensional representation of Z corresponding to γ_λ . Then $A \otimes_Z \mathbb{C}_{\gamma_\lambda}$ is a quotient algebra of $U(\mathfrak{g})$, and we claim that

$$A \otimes_Z \mathbb{C}_{\gamma_\lambda} \cong U(\mathfrak{g})/J_\lambda. \tag{3.3}$$

Indeed, $A \otimes_Z \mathbb{C}_{\gamma_\lambda} \cong U(\mathfrak{g})/I$ for some two-sided ideal I , and clearly I contains $U(\mathfrak{g}) \ker \gamma_\lambda$. Since $M(\lambda)/IM(\lambda)$ is an A -module, we have $M(\lambda)/IM(\lambda) \cong L(\lambda)$ by Corollary 3.2. On the other hand, λ is dominant, and by [59] the correspondence $I \mapsto IM(\lambda)$ is an order preserving bijection from the set of two-sided ideals of $U(\mathfrak{g})$ containing $U(\mathfrak{g}) \ker \gamma_\lambda$ to the set of submodules of $M(\lambda)$ for a dominant λ . It follows that $I = J_\lambda$.

There exists a finite filtration

$$0 = Z_0 \subset Z_1 \subset \cdots \subset Z_r = Z_{\gamma_\lambda}$$

of Z_{γ_λ} as a Z_{γ_λ} -module such that each successive quotient Z_i/Z_{i-1} is isomorphic to the one-dimensional representation $\mathbb{C}_{\gamma_\lambda}$ of Z_{γ_λ} . We put $A_i = A \otimes_Z Z_i$ and obtain

$$0 = A_0 \subset A_1 \subset \cdots \subset A_r = A_{\gamma_\lambda}$$

and

$$A_i/A_{i-1} = A \otimes_Z (Z_i/Z_{i-1}) = A \otimes_Z \mathbb{C}_{\gamma_\lambda} \cong U(\mathfrak{g})/J_\lambda.$$

as A -bimodules. If the filtration is trivial, that is, if $r = 1$, then $A_{\gamma_\lambda} \cong U(\mathfrak{g})/J_\lambda$ and we are done. So we now assume that $r > 1$ and deduce a contradiction.

Note that each exact sequence $0 \rightarrow Z_i \rightarrow Z_{i+1} \rightarrow Z_{i+1}/Z_i \rightarrow 0$ is nonsplit since otherwise we obtain a contradiction to (3.3). So we consider the nonsplit exact sequence of Z -modules

$$0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow Z_2/Z_1 \rightarrow 0. \tag{3.4}$$

Since $Z_1 = Z_2/Z_1 = \mathbb{C}_{\gamma_\lambda}$ and λ is regular, it follows from [7, proof of Lemma 10.6] that (3.4) is obtained from an exact sequence $0 \rightarrow \mathbb{C}_\lambda \rightarrow E \rightarrow \mathbb{C}_\lambda \rightarrow 0$ of \mathfrak{h} -modules via the Harish-Chandra homomorphism $\mathcal{Z}(\mathfrak{g}) \rightarrow S(\mathfrak{h})$. We apply induction to obtain an exact sequence

$$0 \rightarrow M(\lambda) \rightarrow N \rightarrow M(\lambda) \rightarrow 0$$

of \mathfrak{g} -modules. Here $N = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n})} E$ where \mathfrak{n} acts trivially on E . Applying the functor $A \otimes_{U(\mathfrak{g})} (-)$ yields the exact sequence

$$L(\lambda) = A \otimes_{U(\mathfrak{g})} M(\lambda) \xrightarrow{\varphi_1} A \otimes_{U(\mathfrak{g})} N \rightarrow A \otimes_{U(\mathfrak{g})} M(\lambda) = L(\lambda) \rightarrow 0. \tag{3.5}$$

Let v_λ be a highest weight vector of $M(\lambda)$. Then $1 \otimes v_\lambda$ is a highest weight vector of $A \otimes_{U(\mathfrak{g})} M(\lambda) = L(\lambda)$. By construction, this vector is mapped to a nonzero vector of $A \otimes_{U(\mathfrak{g})} N$. Thus, the map $L(\lambda) \rightarrow A \otimes_{U(\mathfrak{g})} N$ is an injection and (3.5) is a nontrivial self-extension of $L(\lambda)$. But this contradicts Corollary 3.2. ■

For a two-sided ideal I of $U(\mathfrak{g})$ we denote by $\text{Var}(I)$ the zero locus of $\text{gr } I$ in \mathfrak{g}^* (or rather its image in \mathfrak{g} under the isomorphism induced by the Killing form), where $\text{gr } I$ is the associated graded of I with respect to the filtration induced from the PBW filtration of $U(\mathfrak{g})$. Joseph’s Theorem [60] asserts that $\text{Var}(I) = \overline{\mathbb{O}}$ for some nilpotent orbit \mathbb{O} of \mathfrak{g} . Thus for each nilpotent orbit \mathbb{O} , we denote by $\text{Prim}_{\mathbb{O}}$ the set of those primitive ideals $I \subset U(\mathfrak{g})$ such that $\text{Var}(I) = \overline{\mathbb{O}}$.

Recall the ideal I_k such that $A(V_k(\mathfrak{g})) = U(\mathfrak{g})/I_k$. In [7, Theorem 9.5] the first author has shown that, for an admissible number k , $\text{Var}(I_k)$ coincides with the associated variety [4] $X_{V_k(\mathfrak{g})}$ of $V_k(\mathfrak{g})$, and hence by [6],

$$\text{Var}(I_k) = \overline{\mathbb{O}}_q,$$

where \mathbb{O}_q is some nilpotent orbit of \mathfrak{g} that depends only on the denominator q of k . More precisely, we have

$$\overline{\mathbb{O}}_q = \begin{cases} \{x \in \mathfrak{g} \mid (\text{ad } x)^{2q} = 0\} & \text{if } (r^\vee, q) = 1, \\ \{x \in \mathfrak{g} \mid \pi_{\theta_s}(x)^{2q/r^\vee} = 0\} & \text{if } (r^\vee, q) = r^\vee, \end{cases}$$

where π_{θ_s} is the simple finite-dimensional representation of \mathfrak{g} with highest weight θ_s .

For a nilpotent orbit \mathbb{O} we write

$$\text{Pr}_{\mathbb{O}}^k = \{\lambda \in \text{Pr}^k \mid \text{Var}(J_\lambda) = \overline{\mathbb{O}}\},$$

and we write $[\text{Pr}_{\mathbb{O}}^k]$ for the image of $\text{Pr}_{\mathbb{O}}^k$ in $[\text{Pr}^k]$. We have $\text{Var}(J_\lambda) \subset \text{Var}(I_k) = \overline{\mathbb{O}}_q$ for all $\lambda \in \text{Pr}^k$, so we may write

$$\text{Pr}^k = \bigsqcup_{\mathbb{O} \subset \overline{\mathbb{O}}_q} \text{Pr}_{\mathbb{O}}^k \quad \text{and} \quad [\text{Pr}^k] = \bigsqcup_{\mathbb{O} \subset \overline{\mathbb{O}}_q} [\text{Pr}_{\mathbb{O}}^k]. \tag{3.6}$$

Finally, we put

$$\text{Pr}_{\mathfrak{o}}^k = \text{Pr}_{\mathbb{O}_q}^k \quad \text{and} \quad [\text{Pr}_{\mathfrak{o}}^k] = [\text{Pr}_{\mathbb{O}_q}^k]. \tag{3.7}$$

Theorem 3.6. *For an admissible number we have*

$$\text{Pr}_{\mathfrak{o}}^k = \{\lambda \in \text{Pr}^k \mid |\Delta(\lambda)| = \dim \mathcal{N} - \dim \overline{\mathbb{O}}_q\},$$

where $\Delta(\lambda) = \{\alpha \in \Delta \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$.

Proof. Let $\lambda \in \text{Pr}^k$. Since $\text{Var}(J_\lambda) \subset \text{Var}(I_k) = \overline{\mathbb{O}}_q$, it follows that $\lambda \in \text{Pr}_{\mathfrak{o}}^k$ if and only if

$$\dim \text{Var}(J_\lambda) = \dim \mathbb{O}_q, \tag{3.8}$$

where q is the dominator of k . Since all the elements in Pr^k are regular dominant, we have

$$\dim \text{Var}(J_\lambda) = \dim \mathcal{N} - |\Delta(\lambda)|$$

for such λ by [58, Corollary 3.5]. The result follows. ■

4. Semisimplicity of Zhu algebras of \mathcal{W} -algebras

Let f be a nilpotent element of \mathfrak{g} . Recall [37] that a $\frac{1}{2}\mathbb{Z}$ -grading $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ is called *good* for f if $f \in \mathfrak{g}_{-1}$, $\text{ad}(f) : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j-1}$ is injective for $j \geq 1/2$ and surjective for $j \leq 1/2$. The grading is *even* if $\mathfrak{g}_j = 0$ for $j \notin \mathbb{Z}$. Any nilpotent f can be embedded into an \mathfrak{sl}_2 -triple $\{e, h, f\}$, and \mathfrak{g} thereby acquires a $\frac{1}{2}\mathbb{Z}$ -grading by eigenvalues of $\text{ad}(h/2)$. Such a grading is called a *Dynkin grading*. All Dynkin gradings are good, but not all good gradings are Dynkin.

We denote by $\mathcal{W}^k(\mathfrak{g}, f)$ the affine \mathcal{W} -algebra associated with \mathfrak{g}, f at level k and a good grading $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ [62]. It is defined by the generalised *quantised Drinfeld–Sokolov reduction*:

$$\mathcal{W}^k(\mathfrak{g}, f) = H_f^0(V^k(\mathfrak{g})),$$

where $H_f^*(M)$ denotes the cohomology of the BRST complex associated with \mathfrak{g}, f and k . This vertex algebra carries a conformal structure with central charge [62, Theorem 2.2]

$$\dim(\mathfrak{g}_0) - \frac{1}{2} \dim(\mathfrak{g}_{1/2}) - \frac{12}{k + h^\vee} |\rho - (k + h^\vee)x_0|^2, \tag{4.1}$$

where x_0 is the semisimple element of \mathfrak{g} that defines the grading, that is, $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [x_0, x] = jx\}$. We note that the vertex algebra structure of $\mathcal{W}^k(\mathfrak{g}, f)$ does not depend on the choice of the good grading of \mathfrak{g} [14, Section 3.2.5] but the conformal structure does.

Recall [2, 30] that

$$A(\mathcal{W}^k(\mathfrak{g}, f)) \cong U(\mathfrak{g}, f),$$

where $U(\mathfrak{g}, f)$ is the *finite W -algebra* associated with (\mathfrak{g}, f) [72], whose construction we now briefly recall.

Let \mathcal{HC} be the category of *Harish-Chandra $U(\mathfrak{g})$ -bimodules*, that is, the full subcategory of the category of $U(\mathfrak{g})$ -bimodules consisting of objects that are finitely generated as $U(\mathfrak{g})$ -bimodules and on which the adjoint action of \mathfrak{g} is locally finite. For an object M of \mathcal{HC} one defines a finite-dimensional analogue of the quantised Drinfeld–Sokolov reduction which, by abuse of notation, we write as $H_f^0(M)$ [7, Section 3]. In particular,

$$U(\mathfrak{g}, f) = H_f^0(\overline{U(\mathfrak{g})}),$$

and in general the space $H_f^0(M)$ is a bimodule over the finite W -algebra $U(\mathfrak{g}, f)$.

Let $\mathcal{C}\ell$ be the Clifford algebra associated with the vector space $\mathfrak{g}_{>0} \oplus \mathfrak{g}_{>0}^*$ equipped with the canonical symmetric bilinear form. Let $\chi : \mathfrak{g}_{\geq 1} \rightarrow \mathbb{C}$ be defined by $\chi(x) = (f, x)$, and let $\mathcal{D} = U(\mathfrak{g}_{>0})/I_{>0, \chi}$ where $I_{>0, \chi}$ is the two-sided ideal $U(\mathfrak{g}_{>0})\langle x - \chi(x) \mid x \in \mathfrak{g}_{\geq 1} \rangle$. Note that if the good grading is even then \mathcal{D} is one-dimensional. Now for $M \in \mathcal{HC}$ we put

$$C(M) = M \otimes \mathcal{D} \otimes \mathcal{C}\ell.$$

This inherits a \mathbb{Z} -grading from $\mathcal{C}\ell$ by assigning $\text{deg}(\mathfrak{g}_{>0}) = -1$ and $\text{deg}(\mathfrak{g}_{>0}^*) = +1$. Let $\{x_i\}$ be a homogeneous basis of $\mathfrak{g}_{>0}$, denote by \bar{x}_i the canonical image of x_i in \mathcal{D} , and

let $\{x_i^*\}$ be the dual basis of $\mathfrak{g}_{>0}^*$. Put

$$d = \sum_i (x_i \otimes 1 + 1 \otimes \bar{x}_i) \otimes x_i^* - 1 \otimes 1 \otimes \frac{1}{2} \sum_{i,j,k} c_{ij}^k x_i^* x_j^* x_k \in C^1(U(\mathfrak{g})),$$

where c_{ij}^k are the structure constants, $[x_i, x_j] = \sum_k c_{ij}^k x_k$. Then $\text{ad}(d)$ is a differential on $C(M)$, and one defines

$$H_f^\bullet(M) = H^\bullet(C(M), \text{ad}(d)).$$

The functor $H_f^0(-)$ is identical to $(-)_\dagger$ in Losev’s notation [68] (see [7, Remark 3.5]).

The space $\mathfrak{g}_{1/2}$ is a symplectic vector space with respect to the form $(x, y) \mapsto \chi([x, y])$. Let I be a Lagrangian subspace of $\mathfrak{g}_{1/2}$, and let $\mathfrak{m} = I \oplus \bigoplus_{j \geq 1} \mathfrak{g}_j$. Then \mathfrak{m} is a nilpotent subalgebra of \mathfrak{g} and $\chi : \mathfrak{m} \rightarrow \mathbb{C}$ defines a character. Put $Y = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi$. We have the algebra isomorphism [29]

$$U(\mathfrak{g}, f) \cong \text{End}_{U(\mathfrak{g})}(Y)^{\text{op}}.$$

Let E be a left $U(\mathfrak{g}, f)$ -module. Then $E \mapsto Y \otimes_{U(\mathfrak{g}, f)} E$ is a $U(\mathfrak{g})$ -module on which $x - \chi(x)$ acts locally nilpotently for all $x \in \mathfrak{m}$; moreover, this assignment defines an equivalence of categories known as the *Skryabin equivalence* [72].

By [6, Theorem 4.15], $H_f^0(V_k(\mathfrak{g}))$ is a quotient vertex algebra of $\mathcal{W}^k(\mathfrak{g}, f)$, provided it is nonzero. By [7, Theorem 8.1] we have

$$A(H_f^0(V_k(\mathfrak{g}))) \cong H_f^0(A(V_k(\mathfrak{g}))),$$

and thus by Theorem 3.4 we have, for any admissible number k ,

$$A(H_f^0(V_k(\mathfrak{g}))) \cong \prod_{[\lambda] \in [\text{Pr}^k]} H_f^0(U(\mathfrak{g})/J_\lambda). \tag{4.2}$$

We recall the construction of the component group $C(f)$. Let us embed f into an \mathfrak{sl}_2 triple $\{e, h, f\} \subset \mathfrak{g}$, and let G^\natural be the centraliser of the corresponding copy of \mathfrak{sl}_2 in the simply connected algebraic group G with Lie algebra \mathfrak{g} . The restriction of the adjoint action of G defines actions of G^\natural on $U(\mathfrak{g}, f)$ and $\mathcal{W}^k(\mathfrak{g}, f)$. The Lie algebra \mathfrak{g}^\natural of G^\natural embeds naturally into $U(\mathfrak{g}, f)$ and therefore the action of the unit component $(G^\natural)^\circ$ of G^\natural is trivial. Hence the action of G^\natural descends to the component group $C(f) = G^\natural / (G^\natural)^\circ$.

Let $\lambda \in \mathfrak{h}^*$. Then $H_f^0(U(\mathfrak{g})/J_\lambda)$ is naturally an algebra and the exact sequence $0 \rightarrow J_\lambda \rightarrow U(\mathfrak{g}) \rightarrow U(\mathfrak{g})/J_\lambda \rightarrow 0$ induces an exact sequence

$$0 \rightarrow H_f^0(J_\lambda) \rightarrow H_f^0(U(\mathfrak{g})) \rightarrow H_f^0(U(\mathfrak{g})/J_\lambda) \rightarrow 0$$

([50, 68]; see also [7, Section 3]). Thus, $H_f^0(U(\mathfrak{g})/J_\lambda)$ is a quotient algebra of $U(\mathfrak{g}, f) = H_f^0(U(\mathfrak{g}))$.

Let $I \in \text{Prim}_{G, f}$, that is, $I \subset U(\mathfrak{g})$ is a primitive ideal and $\text{Var}(I) = \overline{G \cdot f}$. For such I we denote by $\text{Fin}_I(U(\mathfrak{g}, f))$ the set of isomorphism classes of finite-dimensional simple $U(\mathfrak{g}, f)$ -modules E such that $\text{Ann}_{U(\mathfrak{g})}(Y \otimes_{U(\mathfrak{g}, f)} E) = I$.

Theorem 4.1 ([68]). *Let $\lambda \in \mathfrak{h}^*$ and let $f \in \mathfrak{g}$ be a nilpotent element.*

(1) *If $\text{Var}(J_\lambda) \subsetneq \overline{G \cdot f}$ then $H_f^0(U(\mathfrak{g})/J_\lambda) = 0$.*

(2) *If $\text{Var}(J_\lambda) = \overline{G \cdot f}$ then*

$$H_f^0(U(\mathfrak{g})/J_\lambda) \cong \prod_{E \in \text{Fin}_{J_\lambda}(U(\mathfrak{g}, f))} E \otimes E^*.$$

In particular, $H_f^0(U(\mathfrak{g})/J_\lambda)$ is a semisimple algebra.

(3) *The natural action of $C(f)$ on $\text{Fin}_{J_\lambda}(U(\mathfrak{g}, f))$ is transitive.*

Theorem 4.2. *Let k be an admissible number with denominator $q \in \mathbb{Z}_{\geq 1}$, and let $f \in \mathbb{O}_q$. Then*

$$A(H_f^0(V_k(\mathfrak{g}))) \cong \prod_{[\lambda] \in [\text{Pr}_\circ^k]} \left(\prod_{E \in \text{Fin}_{J_\lambda}(U(\mathfrak{g}, f))} E \otimes E^* \right).$$

In particular, $A(H_f^0(V_k(\mathfrak{g})))$ is semisimple. Moreover, if $S[\lambda]$ is a complete set of isomorphism classes of $\text{Fin}_{J_\lambda}(U(\mathfrak{g}, f))$, then $\bigsqcup_{[\lambda] \in [\text{Pr}_\circ^k]} S[\lambda]$ is a complete set of isomorphism classes of simple $A(H_f^0(V_k(\mathfrak{g})))$ -modules.

Proof. The first statement follows immediately from Theorems 3.4 and 4.1. To see the second statement, we recall that the centre of $U(\mathfrak{g}, f)$ is isomorphic to $\mathcal{Z}(\mathfrak{g})$ [73], and hence $E \in \text{Fin}_{J_\lambda}(U(\mathfrak{g}, f))$ and $E' \in \text{Fin}_{J_\mu}(U(\mathfrak{g}, f))$ have distinct central characters if $[\lambda], [\mu]$ are distinct in $[\text{Pr}_\circ^k]$. ■

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$.

Theorem 4.3. *Let k be an admissible number with denominator $q \in \mathbb{Z}_{\geq 1}$, and let $f \in \mathbb{O}_q$. Then the Zhu algebra $A(\mathcal{W}_k(\mathfrak{g}, f))$ is semisimple.*

Proof. Since $H_f^0(V_k(\mathfrak{g}))$ is a quotient of $\mathcal{W}^k(\mathfrak{g}, f)$, we infer that $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of $H_f^0(V_k(\mathfrak{g}))$. Hence $A(\mathcal{W}_k(\mathfrak{g}, f))$ is a quotient of $A(H_f^0(V_k(\mathfrak{g})))$. Since the latter is semisimple by Theorem 4.2, so is the former. ■

Remark 4.4. Conjecturally [66], $H_f^0(V_k(\mathfrak{g}))$ is either zero or isomorphic to $\mathcal{W}_k(\mathfrak{g}, f)$. This will be proved in Theorem 7.8 for the exceptional \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, f)$ when f admits a good even grading.

5. Modular invariance of trace functions

For a vertex algebra V let $R_V = V/C_2(V)$ denote Zhu’s C_2 algebra [81], which is a Poisson algebra. The associated variety X_V of V is by definition the Poisson variety $\text{Specm } R_V$ [4]. By [6] we have $X_{H_f^0(V_k(\mathfrak{g}))} \cong X_{V_k(\mathfrak{g})} \cap \mathcal{S}_f$, where \mathcal{S}_f is the Slodowy slice $f + \mathfrak{g}^e$ at f .

Let k be an admissible number with denominator q and let $f \in \mathbb{O}_q$. Since $X_{V_k(\mathfrak{g})} = \overline{\mathbb{O}}_q$, $X_{H_f^0(V_k(\mathfrak{g}))} = \{f\}$ by the transversality of \mathcal{S}_f to G -orbits. Therefore $H_f^0(V_k(\mathfrak{g}))$ is lisse, and so is $\mathcal{W}_k(\mathfrak{g}, f)$ [6]. The vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded, as are untwisted irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -modules. On the other hand, Ramond twisted irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -modules (see [3, 66]) are $\mathbb{Z}_{\geq 0}$ -graded. Note that if $\mathcal{W}_k(\mathfrak{g}, f)$ is \mathbb{Z}_+ -graded then a Ramond twisted module is the same thing as an untwisted module.

For a simple $A(\mathcal{W}_k(\mathfrak{g}, f))$ -module E , we denote by $\mathbf{L}(E)$ the corresponding irreducible Ramond twisted $\mathcal{W}_k(\mathfrak{g}, f)$ -module. The module $\mathbf{L}(E)$ is the unique simple quotient of the Verma module

$$\mathbf{M}(E) = \mathcal{U}(\mathcal{W}^k(\mathfrak{g}, f)) \otimes_{\mathcal{U}(\mathcal{W}^k(\mathfrak{g}, f))_{\geq 0}} E,$$

where $\mathcal{U}(\mathcal{W}^k(\mathfrak{g}, f)) = \bigoplus_{d \in \mathbb{Z}} \mathcal{U}(\mathcal{W}^k(\mathfrak{g}, f))_d$ is the Ramond twisted current algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ [46, 69] and $\mathcal{U}(\mathcal{W}^k(\mathfrak{g}, f))_{\geq 0} = \bigoplus_{d \geq 0} \mathcal{U}(\mathcal{W}^k(\mathfrak{g}, f))_d$.

Let M be an irreducible Ramond twisted representation of $\mathcal{W}_k(\mathfrak{g}, f)$ and for $u \in \mathcal{W}^k(\mathfrak{g}, f)$, set

$$S_M(\tau \mid u) = \text{Tr}_M(u_0 q^{L_0 - c/24}),$$

where c is the central charge, $q = e^{2\pi i \tau}$, and u_0 is the degree preserving Fourier mode of the field $u(z)$. Set

$$L_{[0]} = L_0 - \sum_{j=1}^{\infty} \frac{(-1)^j}{j(j+1)} L_j.$$

Theorem 5.1. *Let k be an admissible number with denominator $q \in \mathbb{Z}_{\geq 1}$, and let $f \in \mathbb{O}_q$. Let $\{E_1, \dots, E_r\}$ be a complete set of representatives of isomorphism classes of simple modules over the Zhu algebra $A(\mathcal{W})$ of $\mathcal{W} = \mathcal{W}_k(\mathfrak{g}, f)$. Then $S_{\mathbf{L}(E_i)}(\tau \mid u)$ converges to a holomorphic function on the upper half-plane for all $u \in \mathcal{W}$, $i = 1, \dots, r$. Moreover, there is a representation $\rho : SL_2(\mathbb{Z}) \rightarrow \text{End}_{\mathbb{C}}(\mathbb{C}^r)$ such that*

$$S_{\mathbf{L}(E_i)}\left(\frac{a\tau + b}{c\tau + d} \mid (c\tau + d)^{-L_{[0]}u}\right) = \sum_{j=1}^r \rho(A)_{ij} S_{\mathbf{L}(E_j)}(\tau \mid u) \quad \text{for all } u \in \mathcal{W}.$$

Proof. As remarked by the second author [76], in Zhu’s result [81] on the modular invariance of the trace function and its generalisations [34, 76], the assumption of rationality of the vertex algebra can be replaced by semisimplicity of the Zhu algebra. Therefore the theorem follows immediately from Theorem 4.3 and the fact that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse [6]. ■

Since $S_M(|0\rangle, \tau)$ is just the normalised character

$$\chi_M(\tau) = \text{Tr}_M(q^{L_0 - c_V/24}),$$

we obtain

Corollary 5.2. *Let k be an admissible number with denominator $q \in \mathbb{Z}_{\geq 1}$, and let $f \in \mathbb{O}_q$. Let $\{E_1, \dots, E_r\}$ be a complete set of representatives of isomorphism classes of simple modules over the Zhu algebra $A(\mathcal{W})$ of $\mathcal{W} = \mathcal{W}_k(\mathfrak{g}, f)$. Then the character $\chi_{\mathbf{L}(E_i)}(\tau)$*

converges to a holomorphic function on the complex upper half-plane, and the span of the $\chi_{L(E_i)}(\tau)$, as i runs over $\{1, \dots, r\}$, is invariant under the action of $SL_2(\mathbb{Z})$.

The exceptional \mathcal{W} -algebras introduced by Kac and Wakimoto [66] are exactly the \mathcal{W} -algebras $\mathcal{W}_k(\mathfrak{g}, f)$ with f of standard Levi type and an admissible number k whose denominator q is coprime to r^\vee [6]. Hence Corollary 5.2 in particular proves the modular invariance of the characters of modules over the exceptional \mathcal{W} -algebras, which was conjectured by Kac and Wakimoto [66].

We now restrict to the $\mathbb{Z}_{\geq 0}$ -graded case and discuss fusion products. The category of modules of a rational lisse self-dual simple vertex algebra $V = \bigoplus_{n \in \mathbb{Z}_+} V_n$ of CFT-type carries the structure of a modular tensor category (MTC) under the fusion product $X \boxtimes Y$ of modules [54]. The fusion product is the one defined, at this level of generality, in [55]. Duals are given by the usual contragredient construction, and twists are given in terms of conformal dimensions. The S -matrix of the MTC coincides with the matrix $S = \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of Zhu’s theorem. In particular, the Verlinde formula asserts that the decomposition multiplicities or *fusion rules* N in $X \boxtimes Y \cong \bigoplus_Z N_{X,Y}^Z \cdot Z$ are given by

$$N_{X,Y}^Z = \sum_{W \in \text{Irr}(V)} \frac{S_{X,W} S_{Y,W} S_{Z',W}}{S_{V,W}}.$$

Furthermore, the *charge conjugation matrix* S^2 is the permutation matrix which exchanges each module with its contragredient.

Let \mathcal{C} be an MTC. The integral Grothendieck group $\mathcal{F}(\mathcal{C})$ acquires a commutative ring structure corresponding to the tensor product, and a distinguished \mathbb{Z} -basis corresponding to simple objects, which comes equipped with an involution corresponding to duality. We refer to this structure as the fusion ring of \mathcal{C} , and we also write $\mathcal{F}(V)$ for the fusion ring of the category of representations of a rational lisse self-dual vertex algebra V . Since the structure of $\mathcal{F}(\mathcal{C})$ is completely encoded in its S -matrix, we shall sometimes abuse notation and write $\mathcal{F}(S)$.

6. Self-duality of \mathcal{W} -algebras

Let V be a vertex algebra of CFT-type, that is, V is $\mathbb{Z}_{\geq 0}$ -graded where $V_0 = \mathbb{C}$. Then V is called *self-dual* if $V \cong V'$ as V -modules, where M' denotes the contragredient dual [45] of the V -module M . Equivalently, V is self-dual if and only if it admits a nondegenerate symmetric invariant bilinear form. According to Li [67], the space of symmetric invariant bilinear forms on V is naturally isomorphic to the linear dual of $V_0/L_1 V_1$.

The condition of self-duality depends on the choice of the conformal vector and is necessary to apply Huang’s result [54] on the Verlinde formula. In this section we consider the question of self-duality of affine \mathcal{W} -algebras.

Proposition 6.1. *Suppose that f admits a good even grading. The simple \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is self-dual if and only if*

$$(k + h^\vee)(x_0, v) - \frac{1}{2} \text{tr}_{\mathfrak{g}_{>0}}(\text{ad } v) = 0 \quad \text{for all } v \in \mathfrak{g}_0^f.$$

Proof. By Li’s result $\mathcal{W}^k(\mathfrak{g}, f)$ admits a nonzero symmetric invariant bilinear form if and only if $L_1 \mathcal{W}^k(\mathfrak{g}, f)_1 = 0$. By [62], $\mathcal{W}^k(\mathfrak{g}, f)_1$ is spanned by the vectors $J^{\{v\}}$, $v \in \mathfrak{g}_0^f$, defined in [62, p. 320]. Hence from [62, Theorem 2.4 (b)] it follows that $\mathcal{W}^k(\mathfrak{g}, f)$ admits a symmetric invariant bilinear form such that $(\mathbf{1}, \mathbf{1}) = 1$ if and only if $(k + h^\vee)(x_0, v) - \frac{1}{2} \operatorname{tr}_{\mathfrak{g}_{>0}}(\operatorname{ad} v) = 0$ for all $v \in \mathfrak{g}_0^f$. If this is the case, the form induces a nondegenerate symmetric invariant bilinear form on the simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$ and thus the latter is self-dual. Conversely, suppose that $(k + h^\vee)(x_0, v) - \frac{1}{2} \operatorname{tr}_{\mathfrak{g}_{>0}}(\operatorname{ad} v) \neq 0$ for some $v \in \mathfrak{g}_0^f$. Then the image of $J^{\{v\}}$ in $\mathcal{W}_k(\mathfrak{g}, f)_1$ is nonzero, and $L_1 J^{\{v\}} = \mathbf{1}$ up to nonzero constant multiplication. Therefore $\mathcal{W}_k(\mathfrak{g}, f)$ is not self-dual according to Li’s criterion. ■

Remark 6.2. The notion of the contragredient dual naturally extends to modules over $\frac{1}{2}\mathbb{Z}$ -graded vertex algebras and the proof of Li’s criterion applies without any change. See [80, Proposition 2.4] for example. Hence Proposition 6.1 is valid without the assumption that f admits a good even grading.

Recall that a nilpotent element f is called *distinguished* if $\mathfrak{g}_0^f = 0$ for the Dynkin grading. For example all principal nilpotent elements are distinguished, and so are subregular nilpotent elements in types D and E . All distinguished nilpotent elements are even. For a distinguished nilpotent element the only good grading is the Dynkin grading. The following assertion is a direct consequence of Proposition 6.1.

Proposition 6.3. *Let f be a distinguished nilpotent element. Then the simple \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is self-dual.*

Remark 6.4. More generally, one can show that $\mathcal{W}_k(\mathfrak{g}, f)$ is self-dual for the Dynkin grading for any nilpotent element [11].

For $\mathfrak{g} = \mathfrak{sl}_n$, only principal nilpotent elements are distinguished. A subregular nilpotent element $f_{\text{subreg}} \in \mathfrak{sl}_n$ is even if and only if n is even.

Proposition 6.5. *Let $\mathfrak{g} = \mathfrak{sl}_n$. The simple subregular \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ is self-dual if and only if either (1) n is even and the grading is Dynkin, or (2) $k + n = n/(n - 1)$.*

Proof. By [37] the good even gradings of f_{subreg} are classified by the pyramids corresponding to the partition $(n - 1, 1)$. So we may take

$$\begin{aligned}
 f &= \sum_{i=1}^{m-2} E_{i+1,i} + E_{m+1,m-1} + \sum_{i=m+1}^{n-1} E_{i+1,i}, \\
 x_0 &= \sum_{i=1}^{m-1} (m - i)E_{i,i} - \sum_{i=m+2}^n (i - m - 1)E_{i,i},
 \end{aligned}
 \tag{6.1}$$

for some $m = 1, \dots, n$, to obtain $\mathfrak{g}_0^f = \mathbb{C}v_m$, where $v_m = E_{m,m} - \frac{1}{n} \sum_{i=1}^n E_{i,i}$. It follows that

$$(k + h^\vee)(x_0, v) - \frac{1}{2} \operatorname{tr}_{\mathfrak{g}_{>0}}(\operatorname{ad} v) = \frac{(n - 1)(n - 2k)}{2n} \left(k + h^\vee - \frac{n}{n - 1} \right).$$

The assertion follows from Proposition 6.1 on noting that $n = 2m$ if and only if n is even and the corresponding pyramid is symmetric, or the grading is Dynkin. ■

Remark 6.6. For $k + n = n/(n - 1)$ we have $W_k(\mathfrak{sl}_n, f_{\text{subreg}}) = \mathbb{C}$.

7. The “-”-reduction functor revisited

From now on we assume that the nilpotent element f admits a good even grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, so that $W^k(\mathfrak{g}, f)$ is $\mathbb{Z}_{\geq 0}$ -graded. Note that this condition is satisfied by all nilpotent elements in type A and subregular nilpotent elements in simply laced types. Without loss of generality we assume that $\mathfrak{h} \subset \mathfrak{g}_0$ and the root system Δ is compatible with the grading, that is, $\Delta_+ = \Delta_{0,+} \sqcup \bigsqcup_{j \geq 1} \Delta_j$, where $\Delta_j = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}$ and $\Delta_{0,+} \subset \Delta_+$ is a set of positive roots of \mathfrak{g}_0 . We write $\mathfrak{g}_{\geq 0} = \bigoplus_{j \geq 0} \mathfrak{g}_j$ and $\mathfrak{g}_{<0} = \bigoplus_{j < 0} \mathfrak{g}_j$.

Let $\chi_- : \mathfrak{g}_{<0} \rightarrow \mathbb{C}$ be the character defined by $\chi_-(e_{-\alpha}) = \chi(e_\alpha)$. As in [3, Section 5] we write

$$H_\bullet^{\text{Lie}}(M) = H_\bullet(\mathfrak{g}_{<0}, M \otimes \mathbb{C}_{\chi_-})$$

for the Whittaker coinvariants functor, where $H_\bullet(-)$ denotes the usual Lie algebra homology functor.

Let \mathcal{O} be the Bernstein–Gelfand–Gelfand category of \mathfrak{g} , and let $\mathcal{O}^{\mathfrak{g}_0}$ be the full subcategory of \mathcal{O} consisting of those objects that are integrable as \mathfrak{g}_0 -modules. We put

$$P_{0,+} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta_{0,+}\}. \tag{7.1}$$

Then $\{L(\lambda) \mid \lambda \in P_{0,+}\}$ is a complete set of isomorphism classes of simple objects in $\mathcal{O}^{\mathfrak{g}_0}$.

Let $\text{Dim } M$ be the Gelfand–Kirillov dimension of the \mathfrak{g} -module M . For $M \in \mathcal{O}^{\mathfrak{g}_0}$ one has

$$\text{Dim } M \leq \dim \mathfrak{g}_{<0} = \frac{1}{2} \dim \overline{G \cdot f},$$

and in the case of equality we shall say that $\text{Dim } M$ is maximal. We recall that M is said to be holonomic if $\text{Dim } M = \frac{1}{2} \dim \text{Var Ann}(M)$.

Theorem 7.1 ([70, 71]; see also [3, Theorem 5.1.1]). (1) *If $M \in \mathcal{O}^{\mathfrak{g}_0}$ then $H_0^{\text{Lie}}(M)$ is finite-dimensional.*

(2) *If $M \in \mathcal{O}^{\mathfrak{g}_0}$ then $H_i^{\text{Lie}}(M) = 0$ for $i > 0$.*

(3) *If $\lambda \in P_{0,+}$ then $H_0^{\text{Lie}}(L(\lambda)) \neq 0$ if and only if $\text{Dim } L(\lambda)$ is maximal.*

Proposition 7.2. *For $\lambda \in P_{0,+}$ we have $\text{Var}(J_\lambda) = \overline{G \cdot f}$ if and only if $\text{Dim } L(\lambda)$ is maximal.*

Proof. We begin by recalling that $L(\lambda)$ is holonomic, by a result of Joseph [58]. Now suppose $\text{Dim } L(\lambda)$ is maximal, so that $\text{Dim } L(\lambda) = \frac{1}{2} \dim \overline{G \cdot f}$. Then $H_0^{\text{Lie}}(L(\lambda)) \neq 0$ and thus there exists a vector $v \in L(\lambda)$ such that its image $[v]$ in $H_0^{\text{Lie}}(L(\lambda))$ is nonzero. As $H_0^{\text{Lie}}(L(\lambda))$ is an $H_f^0(U(\mathfrak{g})/J_\lambda)$ -module and $[v] = [1] \cdot [v]$, the image $[1]$ of 1 in $H_f^0(U(\mathfrak{g})/J_\lambda)$ is nonzero, and hence $H_f^0(U(\mathfrak{g})/J_\lambda) \neq 0$. But this implies $\text{Var}(J_\lambda) \supset \overline{G \cdot f}$.

By the holonomicity of $L(\lambda)$ we have $\dim \text{Var}(J_\lambda) = \dim \overline{G \cdot f}$ and thus $\text{Var}(J_\lambda) = \overline{G \cdot f}$ as required.

On the other hand, if $\text{Dim } L(\lambda) < \frac{1}{2} \dim \overline{G \cdot f}$, then $\dim \text{Var}(J_\lambda) < \dim \overline{G \cdot f}$. This completes the proof. ■

Let $\mathbb{O} = G \cdot f$ and for $I \in \text{Prim}_{\mathbb{O}}$ let $\{E_I[i] \mid i = 1, \dots, N_I\}$ denote the complete set of isomorphism classes of irreducible finite-dimensional representations of $H_f^0(U(\mathfrak{g})/I)$. Recall that the group $C(f)$ acts transitively on this set.

Theorem 7.3.

- Let $\lambda \in \mathfrak{h}^*$, $\mathbb{O} = G \cdot f$ and suppose that $J_\lambda \in \text{Prim}_{\mathbb{O}}$. As $U(\mathfrak{g}, f)$ -modules,

$$H_p^{\text{Lie}}(L(\lambda)) \cong \bigoplus_{i=1}^{N_{J_\lambda}} E_{J_\lambda}[i]^{\oplus n_{p,i}}$$

for some collection of $n_{p,i} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ for each $p \geq 0$.

- Let $\lambda \in P_{0,+}$ and suppose $\text{Dim } L(\lambda)$ is maximal. As $U(\mathfrak{g}, f)$ -modules,

$$H_p^{\text{Lie}}(L(\lambda)) \cong \begin{cases} \bigoplus_{i=1}^{N_{J_\lambda}} E_{J_\lambda}[i]^{\oplus n_\lambda} & \text{for } p = 0, \\ 0 & \text{otherwise} \end{cases}$$

for some $n_\lambda \in \mathbb{Z}_{>0}$.

Proof. By Theorem 4.1,

$$H_f^0(U(\mathfrak{g})/J_\lambda) \cong \bigoplus_{i=1}^{N_{J_\lambda}} E_{J_\lambda}[i] \otimes E_{J_\lambda}[i]^*$$

is a finite-dimensional semisimple algebra. We note that $H_p^{\text{Lie}}(L(\lambda))$ is a module over this algebra, and therefore is a direct sum of $E_{J_\lambda}[i]$ with $i = 1, \dots, N_{J_\lambda}$. This proves the first part. For the second part, there is a natural inclusion of G^\natural in G^0 , the simply connected algebraic group with Lie algebra \mathfrak{g}^0 . Since $L(\lambda)$ is integrable with respect to \mathfrak{g}^0 , we have $H_0^{\text{Lie}}(L(\lambda))$ invariant under the action of G^\natural and hence of $C(f)$. ■

We also need the following result of Matumoto.

Theorem 7.4 ([70]). *Suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \Delta \setminus \Delta_0$. Then $H_0^{\text{Lie}}(L(\lambda))$ is a (nonzero) simple $U(\mathfrak{g}, f)$ -module.*

Theorem 7.5. *Let $\lambda \in P_{0,+}$ be such that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \Delta_{>0}$. Then $\text{Dim } L(\lambda)$ is maximal, $J_\lambda \in \text{Prim}_{G \cdot f}$, and $H_f^0(U(\mathfrak{g})/J_\lambda)$ has a unique simple module E_{J_λ} . Furthermore, $H_0^{\text{Lie}}(L(\lambda)) \cong E_{J_\lambda}$.*

Proof. By Theorem 7.4, $H_0^{\text{Lie}}(L(\lambda))$ is a nonzero simple $U(\mathfrak{g}, f)$ -module. Hence by Theorem 7.1, $\text{Dim } L(\lambda)$ is maximal, and so $J_\lambda \in \text{Prim}_{G \cdot f}$ by Proposition 7.2. Finally, it follows from Theorems 7.3 and 7.4 that $H_0^{\text{Lie}}(L(\lambda))$ is the unique element of $\text{Fin}_{J_\lambda}(U(\mathfrak{g}, f))$. ■

We recall the definition of (the “−” variant of) the quantised Drinfeld–Sokolov reduction functor $H_{f,-}^0(-)$ [3, 62]. For a vector space α we denote by $L\alpha$ the superalgebra $(\alpha \oplus \alpha^*)[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ whose even part is $\mathbb{C}\mathbf{1}$ and odd part is $(\alpha \oplus \alpha^*)[t, t^{-1}]$, and the commutation relation is $[at^m, bt^n] = (a, b)\delta_{m,-n}\mathbf{1}$, where (\cdot, \cdot) is the canonical symmetric bilinear form $(a, \phi) = (\phi, a) = \phi(a)$ for $a \in \alpha, \phi \in \alpha^*$.

Let $\bigwedge^{\infty+\bullet}$ denote the Fock $L\mathfrak{g}_{<0}$ -module with highest weight vector $|0\rangle$, subject to the relations $\varphi_{\alpha,n \geq 1}|0\rangle = 0, \varphi_{\alpha,n \geq 0}^*|0\rangle = 0$. Here $\varphi_\alpha \equiv e_{-\alpha}$ for $\alpha \in \Delta_{>0}$ and $\{\varphi_\alpha^*\}$ is the dual basis of $\mathfrak{g}_{<0}^*$. Assigning $\deg(\varphi) = -1$ and $\deg(\varphi^*) = +1$ makes $\bigwedge^{\infty+\bullet}$ into a \mathbb{Z} -graded vertex superalgebra, with generating fields $\varphi_\alpha(z) = \sum_n \varphi_{\alpha,n}z^{-n}$ and $\varphi_\alpha^*(z) = \sum_n \varphi_{\alpha,n}^*z^{-n-1}$. For any $V^k(\mathfrak{g})$ -module we put

$$C_-^\bullet(M) = M \otimes \bigwedge^{\infty+\bullet}, \tag{7.2}$$

and we introduce the operator Q_- on $C_-^\bullet(M)$ by

$$\begin{aligned} Q_- = & \sum_{\alpha \in \Delta_{>0}, n \in \mathbb{Z}} e_{-\alpha}t^{-n} \otimes \varphi_{\alpha,n}^* - \frac{1}{2} \sum_{\substack{\alpha, \beta, \gamma \in \Delta_{>0} \\ m, n \in \mathbb{Z}}} c_{\beta\gamma}^\alpha : \varphi_{\alpha,-m} \varphi_{\beta,-n}^* \varphi_{\gamma,m+n}^* \\ & + \sum_{\alpha \in \Delta_{>0}} \chi(e_{-\alpha}) \varphi_{\alpha,0}^*. \end{aligned}$$

Then $(Q_-)^2 = 0$, and we define

$$H_{f,-}^\bullet(M) = H^\bullet(C_-^\bullet(M), Q_-).$$

If M is any $V^k(\mathfrak{g})$ -module, then by [3, Section 4.3] (see also [44, Section 2.2]) the space $H_{f,-}^0(M)$ carries the structure of a Ramond twisted $\mathcal{W}^k(\mathfrak{g}, f)$ -module (which, under our assumption that f admits a good even grading, is nothing but the usual untwisted module structure).

Let $\widehat{\mathcal{O}}_k^{\mathfrak{g}_0}$ be the full subcategory of the category of left $\widehat{\mathfrak{g}}$ -modules consisting of objects M such that

- K acts as multiplication by k on M ,
- M admits a weight space decomposition with respect to the action of $\widehat{\mathfrak{h}}$,
- there exists a finite subset $\{\mu_1, \dots, \mu_n\}$ of \mathfrak{h}_k^* such that $M = \bigoplus_{\mu \in \cup_i \mu_i - \widehat{\mathcal{Q}}_+} M^\mu$,
- for each $d \in \mathbb{C}$, M_d is a direct sum of finite-dimensional \mathfrak{g}_0 -modules.

For $\lambda \in P_{0,+}$, put $\widehat{M}_{k,0}(\lambda) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} L(\lambda) \in \widehat{\mathcal{O}}_k^{\mathfrak{g}_0}$, where $L(\lambda)$ is considered as a $\mathfrak{g}[t] \oplus \mathbb{C}K$ -module on which $\mathfrak{g}[t]$ acts trivially and K acts as multiplication by k . The modules $\widehat{L}_k(\lambda)$, as λ ranges over $P_{0,+}$, form a complete set of simple objects of $\widehat{\mathcal{O}}_k^{\mathfrak{g}_0}$.

Theorem 7.6. *Let k be any complex number.*

- (1) ([3, Theorem 5.5.4]) *Let $M \in \widehat{\mathcal{O}}_k^{\mathfrak{g}_0}$. Then $H_{f,-}^i(M) = 0$ for all $i \neq 0$. In particular, the functor $\widehat{\mathcal{O}}_k^{\mathfrak{g}_0} \rightarrow \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, M \mapsto H_{f,-}^0(M)$, is exact.*

- (2) ([3, Theorem 5.5.4]) Let $\lambda \in P_{0,+}$. Then $H_{f,-}^0(\widehat{L}_k(\lambda)) \neq 0$ if and only if $\text{Dim } L(\lambda)$ is maximal.
- (3) Let $\lambda \in P_{0,+}$ and suppose $\text{Dim } L(\lambda)$ is maximal. Then

$$H_{f,-}^0(\widehat{M}_{k,0}(\lambda)) \cong \bigoplus_{i=1}^{N_{J_\lambda}} \mathbf{M}(E_{J_\lambda}[i])^{\oplus n_\lambda} \quad \text{and} \quad H_{f,-}^0(\widehat{L}_k(\lambda)) \cong \bigoplus_{i=1}^{N_{J_\lambda}} \mathbf{L}(E_{J_\lambda}[i])^{\oplus n_\lambda},$$

where n_λ is the multiplicity of $E_{J_\lambda}[i]$ in $H_0^{\text{Lie}}(L(\lambda))$ as in Theorem 7.3.

Proof of (3). We have $H_{f,-}^0(\widehat{M}_{k,0}(\lambda))_{\text{top}} \cong H_{f,-}^0(\widehat{L}_k(\lambda))_{\text{top}} \cong H_0^{\text{Lie}}(L(\lambda))$ (see [3]), and the latter space is isomorphic to $\bigoplus_{i=1}^{N_{J_\lambda}} E_{J_\lambda}[i]^{\oplus n_\lambda}$ by Theorem 7.3. On the other hand, it was shown in [3] that $H_{f,-}^0(\widehat{M}_{k,0}(\lambda))$ is almost highest weight, that is, $H_{f,-}^0(\widehat{M}_{k,0}(\lambda))$ is generated by $H_{f,-}^0(\widehat{M}_{k,0}(\lambda))_{\text{top}}$. Therefore, there is a surjective homomorphism

$$\bigoplus_{i=1}^{N_{J_\lambda}} \mathbf{M}(E_{J_\lambda}[i])^{\oplus n_\lambda} \twoheadrightarrow H_{f,-}^0(\widehat{M}_{k,0}(\lambda)).$$

of $\mathcal{W}^k(\mathfrak{g}, f)$ -modules. But this must be an isomorphism since their characters coincide (see [3]). The exactness result of part (1) now implies that there is a surjective homomorphism

$$\bigoplus_{i=1}^{N_{J_\lambda}} \mathbf{M}(E_{J_\lambda}[i])^{\oplus n_\lambda} \twoheadrightarrow H_{f,-}^0(\widehat{L}_k(\lambda)). \tag{7.3}$$

On the other hand, it was also shown in [3] that $H_{f,-}^0(\widehat{L}_k(\lambda))$ is almost irreducible, that is, any nontrivial submodule of $H_{f,-}^0(\widehat{L}_k(\lambda))$ intersects $H_{f,-}^0(\widehat{L}_k(\lambda))_{\text{top}}$ nontrivially. It follows that (7.3) factors through the isomorphism $\bigoplus_{i=1}^{N_{J_\lambda}} \mathbf{L}(E_{J_\lambda}[i])^{\oplus n_\lambda} \xrightarrow{\sim} H_{f,-}^0(\widehat{L}_k(\lambda))$. This completes the proof. \blacksquare

By the definition of the “-”-reduction [3], it follows that the conformal dimension of the $\mathcal{W}^k(\mathfrak{g}, f)$ -module $\mathbf{L}(E_{J_\lambda}[i])$ is

$$h_\lambda := \frac{|\lambda + \rho|^2 - |\rho|^2}{2(k + h^\vee)} - \frac{k + h^\vee}{2} |x_0|^2 + (x_0, \rho) \tag{7.4}$$

(see [44, (3.1.6)]). A simple but useful observation is that this expression is invariant under the dot action of W on λ .

Theorem 7.7. *Let k be an admissible number for $\widehat{\mathfrak{g}}$. Let $\lambda \in \text{Pr}^k \cap P_{0,+}$ be such that $\text{Dim } L(\lambda)$ is maximal. Then $H_f^0(U(\mathfrak{g})/J_\lambda)$ has a unique simple module, which we denote E_{J_λ} , and*

$$H_{f,-}^0(\widehat{M}_{k,0}(\lambda)) \cong \mathbf{M}(E_{J_\lambda}) \quad \text{and} \quad H_{f,-}^0(\widehat{L}_k(\lambda)) \cong \mathbf{L}(E_{J_\lambda}).$$

Proof. By Theorem 7.5, $H_f^0(U(\mathfrak{g})/J_\lambda)$ has a unique simple module E_{J_λ} . The other statements follow from Theorem 7.6. ■

Theorem 7.8. *Let k be an admissible number for $\widehat{\mathfrak{g}}$ with denominator q and let $\lambda \in \text{Pr}_{\mathbb{Z}}^k$. For $f \in \overline{\mathbb{O}}_q$, we have $\lambda - \frac{p}{q}x_0 \in \text{Pr}^k \cap P_{0,+}$ and*

$$H_f^0(\widehat{L}_k(\lambda)) \cong \mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}}) \cong H_{f,-}^0\left(\widehat{L}_k\left(\lambda - \frac{p}{q}x_0\right)\right).$$

In particular,

$$\mathcal{W}_k(\mathfrak{g}, f) \cong H_f^0(V_k(\mathfrak{g})) \cong \mathbf{L}(E_{J_{-\frac{p}{q}x_0}}).$$

Proof. First, we have $\lambda - \frac{p}{q}x_0 \in \text{Pr}^k \cap P_{0,+}$. Indeed, it is clear that $\lambda - \frac{p}{q}x_0 \in P_{0,+}$. Also, $x_0 \in P_+^\vee$ since we have assumed that f admits a good even grading. We have

$$\begin{cases} \theta(x_0) < q & \text{if } (q, r^\vee) = 1, \\ \theta_s(x_0) < q/r^\vee & \text{if } (q, r^\vee) \neq 1; \end{cases}$$

see [6, Section 5.7]. It follows that $t_{-x_0}(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{\text{re}}$, and so $\lambda - \frac{p}{q}x_0 \in \text{Pr}_{t_{-x_0}}^k \subset \text{Pr}^k$. Moreover by Janzten’s criterion [57] we have $(\lambda - \frac{p}{q}x_0) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} L_{\mathfrak{g}_0}(\lambda - \frac{p}{q}x_0)$, where $\mathfrak{g}_{\geq 0} = \bigoplus_{j \geq 0} \mathfrak{g}_j$ and $L_{\mathfrak{g}_0}(\lambda)$ is the irreducible highest weight representation of \mathfrak{g}_0 with highest weight λ . Hence $\text{Dim } L(\lambda - \frac{p}{q}x_0) = \text{dim } \mathfrak{g}_{\geq 0}$ is maximal. Therefore, by Theorem 7.7, $H_f^0(U(\mathfrak{g})/J_\lambda)$ has a unique simple module $E_{J_{\lambda - \frac{p}{q}x_0}}$ and $H_{f,-}^0(\widehat{L}_k(\lambda - \frac{p}{q}x_0)) \cong \mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}})$.

Now recall that the centre $Z(U(\mathfrak{g}, f))$ of $U(\mathfrak{g}, f)$ is isomorphic to the centre $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$ [73]. By definition, for $\lambda, \mu \in \text{Pr}^k$, $\chi_\lambda = \chi_\mu$ if and only if $[\lambda] = [\mu]$ in $[\text{Pr}^k]$, where the central character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$ is the evaluation at $L_{\mathfrak{g}}(\lambda)$. Hence $E_{J_{-\frac{p}{q}x_0}}$ is the unique simple $A(H_f^0(V_k(\mathfrak{g})))$ -module having central character $\chi_{\lambda - \frac{p}{q}x_0}$. On the other hand, $Z(U(\mathfrak{g}, f))$ acts on $H_f^0(\widehat{L}_k(\lambda))_{\text{top}}$ by the central character $\chi_{\lambda - \frac{p}{q}x_0}$ as well (see [7, Section 5]). Because $H_f^0(\widehat{L}_k(\lambda))$ and $\mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}}) \cong H_{f,-}^0(\widehat{L}_k(\lambda - \frac{p}{q}x_0))$ have the same character (see [6, Proposition 5.12]), we find that $H_f^0(\widehat{L}_k(\lambda)) \cong \mathbf{L}(E_{J_{\lambda - \frac{p}{q}x_0}})$ as required. ■

Theorem 7.9. *Let k be an admissible number for $\widehat{\mathfrak{g}}$ with denominator q and let $f \in \mathbb{O}_q$. Suppose that each element of $[\text{Pr}_{\mathfrak{o}}^k]$ can be represented by an element $\lambda \in \text{Pr}^k \cap P_{0,+}$. Then*

- (1) *for each $[\lambda] \in [\text{Pr}_{\mathfrak{o}}^k]$ the algebra $H_f^0(U(\mathfrak{g})/J_\lambda)$ has a unique simple module, which we denote E_{J_λ} ,*
- (2) *a complete set of representatives of isomorphism classes of simple $\mathcal{W}_k(\mathfrak{g}, f)$ -modules is $\{\mathbf{L}(E_{J_\lambda}) \mid [\lambda] \in [\text{Pr}_{\mathfrak{o}}^k]\}$,*
- (3) *$\mathcal{W}_k(\mathfrak{g}, f)$ is rational.*

Proof. Let $\{\lambda_1, \dots, \lambda_r\}$ be a subset of $P_{0,+}$ such that $[\text{Pr}_0^k] = \{[\lambda_1], \dots, [\lambda_r]\}$. By Theorem 7.7 we have $H_{f,-}^0(\widehat{L}_k(\lambda_i)) \cong \mathbf{L}(E_{J_{\lambda_i}})$ for $i = 1, \dots, r$, where $E_{J_{\lambda_i}}$ is the unique simple $H_f^0(\mathfrak{U}(\mathfrak{g})/J_{\lambda_i})$ -module.

By Theorem 4.2, $\{E_{J_{\lambda_i}} \mid i = 1, \dots, r\}$ is a complete set of representatives of isomorphism classes of simple $A(H_f^0(V_k(\mathfrak{g})))$ -modules. Thus by Theorem 7.8, $\{\mathbf{L}(E_{J_{\lambda_i}}) \mid i = 1, \dots, r\}$ is a complete set of representatives of isomorphism classes of simple $\mathcal{W}_k(\mathfrak{g}, f)$ -modules.

We already know that $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse. It thus remains to show that

$$\text{Ext}_{\mathcal{W}_k(\mathfrak{g}, f)}^1(\mathbf{L}(E_{J_{\lambda_i}}), \mathbf{L}(E_{J_{\lambda_j}})) = 0$$

for all i, j . Let

$$0 \rightarrow \mathbf{L}(E_{J_{\lambda_j}}) \rightarrow M \rightarrow \mathbf{L}(E_{J_{\lambda_i}}) \rightarrow 0 \tag{7.5}$$

be an exact sequence of $\mathcal{W}_k(\mathfrak{g}, f)$ -modules.

Let h_{λ_i} denote the conformal dimension, i.e., the lowest L_0 -eigenvalue, of $\mathbf{L}(E_{J_{\lambda_i}})$. If $h_{\lambda_i} = h_{\lambda_j}$, then (7.5) is obtained by applying the induction functor to the sequence

$$0 \rightarrow \mathbf{L}(E_{J_{\lambda_j}})_{\text{top}} \rightarrow M_{\text{top}} \rightarrow \mathbf{L}(E_{J_{\lambda_i}})_{\text{top}} \rightarrow 0$$

of $A(\mathcal{W}_k(\mathfrak{g}, f))$ -modules, and is therefore split because $A(\mathcal{W}_k(\mathfrak{g}, f))$ is semisimple.

Now suppose that $h_{\lambda_i} < h_{\lambda_j}$. There is a $\mathcal{W}^k(\mathfrak{g}, f)$ -module homomorphism $\mathbf{M}(E_{J_{\lambda_i}}) \rightarrow M$ such that the following diagram commutes:

$$\begin{array}{ccc} & \mathbf{M}(E_{J_{\lambda_i}}) & \\ & \swarrow & \downarrow \\ M & \longrightarrow & \mathbf{L}(E_{J_{\lambda_i}}) \end{array}$$

If (7.5) is nonsplit then M must coincide with a homomorphic image of $\mathbf{M}(E_{J_{\lambda_i}})$. In particular, $[\mathbf{M}(E_{J_{\lambda_i}}) : \mathbf{L}(E_{J_{\lambda_j}})] \neq 0$. By Theorem 7.6 this occurs only if there exists $\mu \in P_{0,+}$ such that $[\widehat{M}_k(\lambda_i) : \widehat{L}_k(\mu)] \neq 0$ and $E_{J_{\lambda_j}}$ is a direct summand of $H_0^{\text{Lie}}(L(\mu))$.

The second of these conditions implies that $\mu \in W \circ \lambda_j$. But since $\widehat{\lambda}_i$ and $\widehat{\lambda}_j$ are dominant, the first condition is only satisfied if $\widehat{\lambda}_i = \widehat{\lambda}_j$, which contradicts $h_{\lambda_i} < h_{\lambda_j}$.

Finally, the case $h_{\lambda_i} < h_{\lambda_j}$ follows from the case $h_{\lambda_i} > h_{\lambda_j}$ by applying the duality functor to (7.5). ■

Remark 7.10. Let $k = p/q - h^\vee$ be an admissible number for $\widehat{\mathfrak{g}}$ and let $f \in \mathbb{O}_q$. By Theorem 7.8 and the exactness of the functor $H_f^0(-)$ [6], we have the exact sequence $0 \rightarrow H_f^0(N_k) \rightarrow \mathcal{W}^k(\mathfrak{g}, f) \rightarrow \mathcal{W}_k(\mathfrak{g}, f) \rightarrow 0$, where N_k is the maximal proper submodule of $V^k(\mathfrak{g})$. The submodule N_k is generated by a singular vector σ_k of weight $s_{\check{\alpha}_0} \circ k\Lambda_0$. From the relations $(\rho, \theta) = h^\vee - 1$ and $(\rho, \theta_s) = h - 1$ we compute

$$s_{\check{\alpha}_0} \circ k\Lambda_0 = \begin{cases} (p - h^\vee + 1)(\theta - q\delta) + k\Lambda_0 & \text{if } (q, r^\vee) = 1, \\ (p - h + 1)(\theta_s - \frac{q}{r^\vee}\delta) + k\Lambda_0 & \text{if } (q, r^\vee) \neq 1. \end{cases}$$

From this we compute the action of the conformal vector $L_0 - (x_0)_0$ on the corresponding singular vector $\sigma_k \otimes |0\rangle$ in $\mathcal{W}^k(\mathfrak{g}, f)$ (cf. [62, Remark 2.3]) to be given by

$$\begin{cases} (p - h^\vee + 1)(q - \langle \theta, x_0 \rangle) & \text{if } (q, r^\vee) = 1, \\ (p - h + 1)\left(\frac{q}{r^\vee} - \langle \theta_s, x_0 \rangle\right) & \text{if } (q, r^\vee) \neq 1. \end{cases}$$

8. Rationality of \mathcal{W} -algebras of type A

For \mathfrak{g} of type A all nilpotent elements are standard Levi type, and so $\mathcal{W}_k(\mathfrak{g}, f)$ is exceptional in the sense of Kac and Wakimoto [66] if and only if k is admissible and $f \in \mathbb{O}_q$, where $q \in \mathbb{Z}_{\geq 1}$ is the denominator of k . In this section we prove the rationality of all exceptional \mathcal{W} -algebras of type A . Throughout this section $\mathfrak{g} = \mathfrak{sl}_n = A_{n-1}$. The Coxeter number of \mathfrak{g} is $h^\vee = h = n$. It is known that the component group $C(f)$ is trivial for every nilpotent element $f \in \mathfrak{g}$ (see e.g. [22, Section 6.1]). Therefore for any primitive ideal I of $U(\mathfrak{g})$ such that $\text{Var}(I) = \overline{G \cdot f}$, the set $\text{Fin}_I(U(\mathfrak{g}, f))$ contains a single element, which we denote E_I . Hence

$$H_f^0(U(\mathfrak{g})/I) \cong E_I \otimes E_I^*.$$

Moreover, the correspondence $I \mapsto E_I$ gives a bijection from the set of primitive ideals of $U(\mathfrak{g})$ satisfying $\text{Var}(I) = \overline{G \cdot f}$ to the set of isomorphism classes of irreducible finite-dimensional $U(\mathfrak{g}, f)$ -modules. The module E_I is described as follows [21].

As usual, we write

$$\Delta = \{\alpha_{i,j} \mid 1 \leq i, j \leq n\} \quad \text{and} \quad \Delta_+ = \{\alpha_{i,j} \mid 1 \leq i < j \leq n\}.$$

The nilpotent orbits are indexed by partitions of n . Indeed, let $Y = (p_1 \leq \dots \leq p_r)$ be a partition of n ; then as in [21] we identify Y with the Young diagram having p_i boxes in the i th row, and we number the boxes of Y by $1, \dots, n$ down columns from left to right. Let $\text{row}(i)$ and $\text{col}(i)$ denote the row and column number of the i th box. Now put

$$f = f_Y = \sum e_{j,i},$$

where the sum runs over (i, j) satisfying $\text{row}(i) = \text{row}(j)$ and $\text{col}(i) = \text{col}(j) - 1$. Here $e_{i,j}$ stands for the i, j -matrix unit. Then f is a nilpotent element of Jordan type Y . Declaring $\text{deg}(e_{i,j}) = \text{col}(j) - \text{col}(i)$ equips \mathfrak{g} with a good even grading for $f_Y \in \mathfrak{g}_{-1}$ [37]. The subsets of roots

$$\Delta_0 = \{\alpha \in \Delta \mid e_\alpha \in \mathfrak{g}_0\}, \tag{8.1}$$

$$\Delta^f = \{\alpha \in \Delta \mid \alpha(h) = 0 \text{ for all } h \in \mathfrak{h}^f\} \tag{8.2}$$

now become

$$\Delta_0 = \{\alpha_{i,j} \in \Delta \mid \text{the } i\text{th and } j\text{th boxes belong to the same column}\},$$

$$\Delta^f = \{\alpha_{i,j} \in \Delta \mid \text{the } i\text{th and } j\text{th boxes belong to the same row}\}.$$

Let $\Delta_+^f = \Delta^f \cap \Delta_+$ and

$$W^f = \{w \in W \mid w(h) = h \text{ for all } h \in \mathfrak{h}^f\}. \tag{8.3}$$

Then W^f is the subgroup of $W \cong \mathfrak{S}_n$ generated by s_α for $\alpha \in \Delta^f$. Finally, we put

$$P_{0,+} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta_{0,+}\}, \tag{8.4}$$

where $\Delta_{0,+} = \Delta_0 \cap \Delta_+$.

Theorem 8.1 (Brundan and Kleshchev [21]). (1) *Let $\lambda \in P_{0,+}$. Then $H_0^{\text{Lie}}(L(\lambda)) \neq 0$ if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1}$ for all $\alpha \in \Delta_+^f$. In this case $H_0^{\text{Lie}}(L(\lambda))$ is an irreducible $U(\mathfrak{g}, f)$ -module. Furthermore, every irreducible finite-dimensional representation of $U(\mathfrak{g}, f)$ arises in this way.*

(2) *Let $\lambda, \mu \in P_{0,+}$ and suppose that $H_0^{\text{Lie}}(L(\lambda))$ and $H_0^{\text{Lie}}(L(\mu))$ are nonzero. Then $H_0^{\text{Lie}}(L(\lambda)) \cong H_0^{\text{Lie}}(L(\mu))$ if and only if $\mu = w \circ \lambda$ for some $w \in W^f$.*

Corollary 8.2. *The assignment $\lambda \mapsto J_\lambda$ sets up a bijection*

$$\{\lambda \in P_{0,+} \mid \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1} \text{ for all } \alpha \in \Delta_+^f\} / \sim \xrightarrow{\sim} \text{Prim}_{G,f},$$

where $\lambda \sim \mu$ if and only if $\mu = w \circ \lambda$ for some $w \in W^f$. Furthermore,

$$E_{J_\lambda} \cong H_0^{\text{Lie}}(L(\lambda)).$$

Proof. By Losev’s result [68] and the fact that $C(f)$ is trivial, there is a bijection between $\text{Prim}_{G,f}$ and the isomorphism classes of simple $U(\mathfrak{g}, f)$ -modules, where the simple $U(\mathfrak{g}, f)$ -module corresponding to $J \in \text{Prim}_{G,f}$ is the unique simple module of the simple algebra $H_f^0(U(\mathfrak{g})/J)$ (see Theorem 4.1). The assertion is obtained by comparing this with Theorem 8.1. ■

In the present case Theorem 7.6 becomes

Theorem 8.3 ([3, Theorem 5.7.1]). *Let k be any complex number and let $\lambda \in P_{0,+}$. Then*

$$H_{f,-}^0(\widehat{L}_k(\lambda)) \cong \begin{cases} \mathbf{L}(E_{J_\lambda}) & \text{if } \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\geq 1} \text{ for all } \alpha \in \Delta_+^f, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8.4. *Any element of $[\text{Pr}_0^k]$ can be represented by an element of $\text{Pr}_0^k \cap P_{0,+}$.*

Proof. By Corollary 8.2, for any $\lambda \in \text{Pr}_0^k$ there exists $\mu \in P_{0,+}$ such that $J_\lambda = J_\mu$. But then $\widehat{L}_k(\mu)$ is a $V_k(\mathfrak{g})$ -module by Theorem 3.4, and hence $\mu \in \text{Pr}^k$ by Theorem 3.1. ■

The following assertion follows immediately from Theorem 7.9 and Lemma 8.4.

Theorem 8.5. *Let k be admissible $f_q \in \mathbb{O}_q$. Then $\mathcal{W}_k(\mathfrak{g}, f_q)$ is rational and the complete set of simple $\mathcal{W}_k(\mathfrak{g}, f)$ -modules is given by $\{\mathbf{L}(E_{J_\lambda}) \mid [\lambda] \in [\text{Pr}_0^k]\}$.*

We now describe the set $[\text{Pr}_\circ^k]$ more precisely. Let $k = -n + p/q$ be an admissible number for $\widehat{\mathfrak{g}}$, so p and q are coprime and $p \geq n$. Since $[\text{Pr}_\circ^k]$ is described in [44] in the cases where $q \geq n$, we assume that $q < n$, so that \mathbb{O}_q is nonprincipal. Let $n = rq + s$ where $r, s \geq 0$ and $0 \leq s < q$. Then \mathbb{O}_q is the nilpotent orbit corresponding to the partition $Y = (s, q, \dots, q)$ of n , in which q appears r times. We have $\mathfrak{g}_j = \{u \in \mathfrak{g} \mid [x_0, u] = ju\}$, where

$$x_0 := \sum_{i=1}^s \varpi_{i(r+1)}^\vee + \sum_{i=s+1}^{q-1} \varpi_{s(r+1)+(i-s)r}^\vee. \tag{8.5}$$

For $\eta \in \check{P}$ we set $\Delta_\eta = \{\alpha \in \Delta \mid \langle \eta, \alpha \rangle = 0\} \subset \Delta$, and define

$$\check{P}_{+,f}^q = \{\eta \in \check{P}_+^q \mid \langle \eta, \theta \rangle \leq q - 1 \text{ and } \Delta_\eta \cong \Delta_0 \text{ as root systems}\}. \tag{8.6}$$

Note that $x_0 \in \check{P}_{+,f}^q$.

Lemma 8.6. *A weight $\eta \in \check{P}_+^q$ belongs to $\check{P}_{+,f}^q$ if and only if there exists a permutation*

(m_1, \dots, m_q) of $\overbrace{(r + 1, \dots, r + 1)}^s, \overbrace{(r, \dots, r)}^{q-s}$ such that

$$\eta = \sum_{j=1}^{q-1} \varpi_{\sum_{a=1}^j m_a}^\vee.$$

Proof. From the description of Δ_0 given at the beginning of this section, applied to the partition Y associated with the orbit \mathbb{O}_q , it follows that Δ_0 is the direct product of q irreducible root systems, s of them of type A_{r+1} and $q - s$ of type A_r (treating the case $r = 0$ as $A_0 = \emptyset$). If Δ_η is to be isomorphic to Δ_0 then we must have $\langle \eta, \alpha \rangle > 0$ for at least $q - 1$ simple roots α , and now the condition $\langle \eta, \theta \rangle \leq q - 1$ implies that $\langle \eta, \alpha \rangle$ equals 1 for these simple roots and 0 for all others. The simple roots $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{q-1}}$ for which $\langle \eta, \alpha_{i_j} \rangle = 1$ are of the form $i_j = \sum_{a=1}^j m_a$ for some permutation (m_1, \dots, m_q) as in the statement of the lemma and so we are done. ■

We define an equivalence relation \sim in the set $P_+^{p-n} \times \check{P}_{+,f}^q$ by declaring $(\lambda, \eta) \sim (\lambda', \eta')$ if and only if

$$\begin{cases} \lambda' = \bar{\pi}_i \circ \lambda + p\varpi_i = \bar{\pi}_i \lambda + (p - n)\varpi_i \\ \eta' = \bar{\pi}_i \eta + q\varpi_i \end{cases} \quad \text{for some } i = 1, \dots, n - 1$$

(cf. (2.4)). We may also describe the equivalence relation in terms of the description of $\check{P}_{+,f}^q$ given in Lemma 8.6. Suppose $(\lambda, \eta) \sim (\lambda', \eta')$ in $P_+^{p-n} \times \check{P}_{+,f}^q$. Writing $\eta = \sum_{j=1}^{q-1} \varpi_{\sum_{a=1}^j m_a}^\vee$ and $\eta' = \sum_{j=1}^{q-1} \varpi_{\sum_{a=1}^j m'_a}^\vee$, setting $m_q = n - \sum_{j=1}^{q-1} m_j$ and $m'_q = n - \sum_{j=1}^{q-1} m'_j$ and, for convenience, considering the index modulo n , we then have $m'_a = m_{a+b}$ for all $a = 1, \dots, q$, for some fixed b .

Theorem 8.7. *Let $k = -n + p/q$ with $q \leq n$. We have a bijection*

$$(P_+^{p-n} \times \check{P}_{+,f}^q) / \sim \xrightarrow{\sim} [\text{Pr}_\circ^k], \quad (\lambda, \eta) \mapsto \left[\lambda - \frac{p}{q} \eta \right].$$

Proof. First we recall that $\dim \mathbb{O}_q$ coincides with the maximal Gelfand–Kirillov dimension of objects of \mathcal{O}^{so} , and thus

$$\dim \mathbb{O}_q = |\Delta| - |\Delta_0|.$$

Hence, by Theorem 3.6, $\lambda \in \text{Pr}^k$ belongs to Pr_\circ^k if and only if $|\Delta(\lambda)| = |\Delta_0|$.

Let $\eta \in \check{P}_{+,f}^q$. Since $\eta \in \check{P}_+$ and $\langle \eta, \theta \rangle \leq q - 1$ we have $\langle \eta, \alpha \rangle < q$ for all $\alpha \in \Delta_+$, and therefore $t_{-\eta}(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{\text{re}}$. Hence for all $\lambda \in P_+^{p-n}$ we have $t_{-\eta} \circ \widehat{\lambda} = \lambda - \frac{p}{q} \eta + k\Lambda_0$ with $\lambda - \frac{p}{q} \eta \in \text{Pr}^k$. Moreover, $\Delta(\lambda - \frac{p}{q} \eta) = \Delta(\eta) \cong \Delta_0$ since $\eta \in \check{P}_{+,f}^q$ and hence $\lambda - \frac{p}{q} \eta \in \text{Pr}_\circ^k$ by the above criterion. On the other hand, let us suppose that $(\lambda', \eta') \sim (\lambda, \eta)$ in $P_+^{p-n} \times \check{P}_{+,f}^q$, so that $\lambda' = \bar{\pi}_i \circ \lambda + p\varpi_i$ and $\eta' = \bar{\pi}_i \eta + q\varpi_i$. Then

$$\bar{\pi}_i \circ \left(\lambda - \frac{p}{q} \eta \right) = \bar{\pi}_i \circ \lambda - \frac{p}{q} \bar{\pi}_i \eta = \lambda' - p\varpi_i - \frac{p}{q} (\eta' + q\varpi_i) = \lambda' - \frac{p}{q} \eta'.$$

We have shown that the map in the theorem statement is well-defined.

Next we show the injectivity of the map. We shall use the relation $\bar{\pi}_{n-i}(\varpi_i) = -\varpi_{n-i}$, which is easily proved. Let $(\lambda, \eta), (\lambda', \eta') \in P_+^{p-n} \times \check{P}_{+,f}^q$ and suppose that $[\lambda' - \frac{p}{q} \eta'] = [\lambda - \frac{p}{q} \eta]$ in $[\text{Pr}_\circ^k]$, i.e., there exists $w \in W$ such that $\lambda' - \frac{p}{q} \eta' = w \circ (\lambda - \frac{p}{q} \eta)$. Since $\lambda' - \frac{p}{q} \eta' \in \text{Pr}^k$, this means $w t_{-\eta}(\widehat{\Delta}(k\Lambda_0)_+) \subset \widehat{\Delta}_+^{\text{re}}$ and $\lambda' - \frac{p}{q} \eta' \in \text{Pr}_{w t_{-\eta}}^k \cap \text{Pr}_{t_{-\eta'}}^k$. By (2.4), we get $w t_{-\eta} = t_{-\eta'} t_{q\varpi_j} \bar{\pi}_j = \bar{\pi}_j t_{-(\bar{\pi}_{n-j} \eta' + q\varpi_{n-j})}$ for some j . Thus

$$w = \bar{\pi}_j \quad \text{and} \quad \eta = \bar{\pi}_{n-j} \eta' + q\varpi_{n-j}.$$

From $\lambda' - \frac{p}{q} \eta' = w \circ (\lambda - \frac{p}{q} \eta)$, and the equalities above, it follows that $\lambda = \bar{\pi}_{n-i} \circ (\lambda' - \frac{p}{q} \eta') + \frac{p}{q} \eta = \bar{\pi}_{n-i} \circ \lambda + p\varpi_{n-i}$. We have shown that $(\lambda', \mu') \sim (\lambda, \mu)$.

Finally, we prove surjectivity. Let $[\lambda] \in [\text{Pr}_\circ^k]$. By Lemma 8.4, we may choose a representative $\lambda \in \text{Pr}_\circ^k \cap P_{0,+}$ of $[\lambda]$. Clearly we have $\Delta(\lambda) = \Delta_0$. By [13, Proposition 2.8] and its proof,

$$\text{Pr}^k = \bigcup \text{Pr}_{y t_{-\eta}}^k,$$

where the union is taken over pairs $(y, \eta) \in W \times \check{P}_+^q$ such that

$$\langle \alpha, \eta \rangle \geq 1 \quad \text{for all } \alpha \in \Delta_+ \cap y^{-1}(\Delta_-). \tag{8.7}$$

Let us therefore take such a pair $(y, \eta) \in W \times \check{P}_+^q$ satisfying (8.7) such that $\lambda \in \text{Pr}_{y t_{-\eta}}^k$. Then we may write $\lambda = y \circ \lambda_1$ and $\lambda_1 = \lambda_0 - \frac{p}{q} \eta$, where $\lambda_1 \in \text{Pr}_{t_{-\eta}}^k$ and $\lambda_0 \in \text{Pr}_{\mathbb{Z}}^k$. We have on the one hand $\Delta(\lambda_1) = y^{-1}(\Delta(\lambda)) = y^{-1}(\Delta_0)$. On the other hand, $\Delta(\lambda_1) = \Delta(\frac{p}{q} \eta)$ and, since $0 \leq \langle \alpha, \eta \rangle \leq \langle \theta, \eta \rangle \leq q - 1 < h^\vee$ for all $\alpha \in \Delta_+$, we have $\langle \frac{p}{q} \eta, \alpha^\vee \rangle \in \mathbb{Z}$ if and only if $\langle \eta, \alpha^\vee \rangle = 0$. Hence $\Delta(\lambda_1) = \Delta_\eta$. Therefore $\Delta_\eta \cong \Delta_0$, and we obtain $[\lambda] = [\lambda_0 - \frac{p}{q} \eta]$ where $(\lambda_0, \mu) \in P_+^{p-n} \times \check{P}_{+,f}^q$. This establishes surjectivity as required. ■

Remark 8.8. Let $f \in \mathfrak{sl}_n$ be a nilpotent element associated with the partition $Y = (p_1 \leq \dots \leq p_r)$ of n . According to [37] the good gradings $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ for f are in natural bijection with combinatorial structures known as *pyramids* associated with Y . A pyramid is an arrangement of n boxes of dimensions 1×1 into r rows, the i th row consisting of a contiguous block of p_i boxes. The projection to the horizontal axis of the i th row must be contained within the projection of the $(i + 1)$ th for each i , and the centre of each box is required to lie directly above the centre or else the edge of the box below.

The symmetric pyramid is the one in which the centres of all rows lie on a single vertical line.

The good grading associated with a pyramid is obtained by labelling its boxes $1, \dots, n$, letting $\text{col}(i)$ and $\text{row}(i)$ denote the horizontal and vertical coordinates of the i th box, respectively, and declaring $\text{deg}(e_{ij}) = \text{col}(j) - \text{col}(i)$. The nilpotent $f = f_Y$ is recovered as $\sum e_{j,i}$ where the sum runs over (i, j) satisfying $\text{row}(i) = \text{row}(j)$ and $\text{col}(i) = \text{col}(j) - 1$.

The grading is even if the centre of each box lies above the centres (and not the edges) of other boxes. Throughout this section we have worked with the even grading associated with the Young diagram, in which all rows are left justified. In particular, the element x_0 given in (8.5) corresponds to this grading. All statements in this section hold upon replacing this grading with any other even grading. In particular, if f is even, so that the Dynkin grading is a good even grading, the results of this section apply to the Dynkin grading. This will be important for applications to the computation of fusion rules in subsequent sections, for instance by Proposition 6.5 the subregular \mathcal{W} -algebra of type A is only self-dual for the Hamiltonian reduction associated with the Dynkin grading.

9. Rationality of simply laced subregular \mathcal{W} -algebras

Let f_{subreg} be a subregular nilpotent element of \mathfrak{g} . We recall that $\mathbb{O}_{\text{subreg}} = G \cdot f_{\text{subreg}}$ is by definition the unique nilpotent orbit of \mathfrak{g} of dimension $\dim \mathfrak{g} - \text{rank } \mathfrak{g} - 2$. The corresponding partition or the Bala–Carter label of f_{subreg} can be found in the third column of Table 1 below.

Lemma 9.1. *Let \mathfrak{g} be a (not necessarily simply laced) simple Lie algebra, and let k be an admissible number with denominator q such that $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$. For $\lambda \in \text{Pr}^k$,*

- (1) $\Delta(\lambda)$ is nonempty,
- (2) $\lambda \in \text{Pr}_0^k = \text{Pr}_{\mathbb{O}_{\text{subreg}}}^k$ if and only if $|\Delta(\lambda)| = 2$.

Proof. If $\Delta(\lambda)$ were empty then we would have $L(\lambda) = M(\lambda)$ and so $\text{Var}(J_\lambda) = \mathcal{N} \supsetneq \overline{\mathbb{O}_q}$, which contradicts (3.6). This proves the first part. The second part follows from Theorem 3.6. ■

By [6] the condition $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$ holds for precisely those values of q listed in the following table.

Type	h^\vee	f_{subreg}	q
A_n	$n + 1$	$[n, 1]$	n
B_n	$2n - 1$	$[2n - 1, 1^2]$	$2n - 1, 2n$
$C_n (n \geq 3)$	$n + 1$	$[2n - 2, 2]$	$2n - 1, 4n - 6, 4n - 4$
D_n	$2n - 2$	$[2n - 3, 3]$	$2n - 4, 2n - 3$
E_6	12	$E_6(a_1)$	9, 10, 11
E_7	18	$E_7(a_1)$	14, 15, 16, 17
E_8	30	$E_8(a_1)$	24, 25, 26, 27, 28, 29
F_4	9	$F_4(a_1)$	9, 11, 12, 14, 16
G_2	4	$G_2(a_1)$	4, 5, 6, 9

Tab. 1. Subregular denominators

Remark 9.2. There are typos in [6, Tables 6, 7]. The central charge of $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ for type F_4 at level $k = -h^\vee + p/q$ should read

$$\frac{6(12p - 13q)(5p - 6q)}{pq}.$$

In the rest of this section we assume \mathfrak{g} is of simply laced type and that $k = -h^\vee + p/q$ is an admissible number with denominator q such that $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$. Let $f_{\text{subreg}} \in \mathbb{O}_{\text{subreg}}$.

We recall that if \mathfrak{g} is of type D or E then f_{subreg} is distinguished, and thus there is exactly one even grading with respect to which f_{subreg} is good, namely its Dynkin grading. Up to conjugacy this grading is given by $\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{g}_{\alpha_*} + \mathfrak{g}_{-\alpha_*}$ and $\mathfrak{g}_1 \supset \bigoplus_{\alpha \in \Pi \setminus \{\alpha_*\}} \mathfrak{g}_\alpha$, where α_* is the simple root corresponding to the trivalent node in the Dynkin diagram of \mathfrak{g} . If \mathfrak{g} is of type A then we fix a simple root α_* arbitrarily and define \mathfrak{g}_0 and \mathfrak{g}_1 as above. This defines n distinct good even gradings on \mathfrak{g} , all of subregular type. If n is odd then one of these gradings is Dynkin; if n is even then none of them are. We have $\Delta_{0,+} = \{\alpha_*\}$. Let

$$x_0 = \sum_{i \neq *} \varpi_i$$

denote the grading element: $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [x_0, x] = jx\}$, and let $h = 2x_0$.

Lemma 9.3. *Let \mathfrak{g} be simply laced. Then every class of $[\text{Pr}_0^k]$ contains a representative $\lambda \in \text{Pr}_0^k$ such that $\Delta(\lambda)_+ = \{\alpha_*\}$.*

Proof. Let $\lambda \in \text{Pr}_0^k$. By Lemma 9.1, $\Delta(\lambda)_+ = \{\alpha\}$ for some $\alpha \in \Delta_+$. Choose $w \in W$ such that $\alpha = w(\alpha_*)$. Then $w \circ \lambda \in \text{Pr}^k$ and $\Delta(w \circ \lambda)_+ = \{\alpha_*\}$ as required. ■

Theorem 9.4. *Let \mathfrak{g} be simply laced and let k be an admissible number for \mathfrak{g} with denominator q such that $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$. Then $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ is rational and a complete set of isomorphism classes of irreducible $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ -modules is*

$$\{\mathbf{L}(E_{J_\lambda}) \mid [\lambda] \in [\text{Pr}_0^k]\},$$

where E_{J_λ} is the unique simple module of $H_f^0(U(\mathfrak{g})/J_\lambda)$.

Proof. By Lemma 9.3 the conditions of Theorem 7.9 are satisfied. Thus the assertion follows immediately from that theorem. ■

We give a more explicit description of the set $[\text{Pr}_\circ^k]$ in the subregular case.

Definition 9.5. Put

$$\check{P}_{+, \text{subreg}}^q = \{\eta \in \check{P}_+^q \mid \langle \alpha_i, \eta + qD \rangle = 0 \text{ for exactly one } i \in \{0, 1, \dots, \ell\}\},$$

where $\alpha_0, \dots, \alpha_\ell \in \hat{\Pi}$ are the simple roots of $\hat{\mathfrak{g}}$.

The finite group \tilde{W}_+ acts on the set $P_+^{p-h^\vee} \times \check{P}_{+, \text{subreg}}^q$ by

$$\pi_j(\lambda, \eta) = (\bar{\pi}_j \lambda + (p - h^\vee)\varpi_j, \bar{\pi}_j \eta + q\varpi_j^\vee) \quad \text{for } j \in J.$$

Theorem 9.6. *Suppose \mathfrak{g} is of simply laced type. For each $\eta \in \check{P}_{+, \text{subreg}}^q$ there exists $y_\eta \in W$ such that $y_\eta t_{-\eta}(\hat{\Delta}(k\Lambda_0)) \subset \hat{\Delta}_+^{\text{re}}$. Moreover, there is a well-defined bijection*

$$\frac{P_+^{p-h^\vee} \times \check{P}_{+, \text{subreg}}^q}{\tilde{W}_+} \xrightarrow{\sim} [\text{Pr}_\circ^k], \quad (\lambda, \eta) \mapsto \left[y_\eta \circ \left(\lambda - \frac{p}{q} \eta \right) \right].$$

Proof. Let $\lambda \in \text{Pr}_\circ^k \cap \text{Pr}_{\hat{y}}^k$ where $\hat{y} = y t_{-\eta} \in \tilde{W}$ with $y \in W$ and $\eta \in P^\vee$. It is straightforward to see that the condition $\hat{y}(\hat{\Delta}(k\Lambda_0)) \subset \hat{\Delta}_+^{\text{re}}$ is equivalent to

$$\begin{cases} 0 \leq \alpha(\eta) \leq q - 1 & \text{for all } \alpha \in \Delta_+ \text{ such that } y(\alpha) \in \Delta_+, \\ 1 \leq \alpha(\eta) \leq q & \text{for all } \alpha \in \Delta_+ \text{ such that } -y(\alpha) \in \Delta_+. \end{cases} \tag{9.1}$$

It is also clear that

$$\Delta(\lambda)_+ = \{y(\alpha) \mid \alpha \in \Delta_+, \alpha(\eta) = 0\} \sqcup \{-y(\alpha) \mid \alpha \in \Delta_+, \alpha(\eta) = q\}.$$

Therefore the condition $|\Delta(\lambda)_+| = 1$ implies that η satisfies one of the following two conditions:

- (i) $0 \leq \alpha(\eta) < q$ for all $\alpha \in \Delta_+$ and there exists a unique simple root α_i of \mathfrak{g} such that $\alpha_i(\eta) = 0$,
- (ii) $0 < \alpha(\eta) \leq q$ for all $\alpha \in \Delta_+$, and $\alpha(\eta) = q$ if and only if $\alpha = \theta$.

But this is equivalent to the statement $\eta \in \check{P}_{+, \text{subreg}}^q$.

Now let $\eta \in \check{P}_{+, \text{subreg}}^q$. For $y \in W$, we have $y t_{-\eta} \circ P_+^{p-h^\vee} \subset \text{Pr}_\circ^k$ if and only if

$$y t_{-\eta}(\hat{\Delta}(k\Lambda_0)) \subset \hat{\Delta}_+^{\text{re}}. \tag{9.2}$$

If η satisfies (i) above then we may take $y = 1$ and condition (9.2) is satisfied. If η satisfies (ii) above then we may take $y = w^\circ$ the longest element of W , and condition (9.2) is again satisfied.

Finally, let $\eta, \eta' \in \check{P}_{+, \text{subreg}}^q$ and suppose that $\eta = \pi_j(\eta')$ for some $j \in J$. Then

$$(y t_{-\eta})(t_{q\varpi_j} \bar{\pi}_j) = y \bar{\pi}_j t_{-\pi_j^{-1}(\eta)} = y \bar{\pi}_j t_{-\eta'}.$$

The assertion now follows from (2.4). ■

We now record some properties of subregular rational \mathcal{W} -algebras.

Type	Conformal weights of generators	$c(p/q)$
A_n (n odd)	$1, 2, \dots, \frac{n-1}{2}, (\frac{n+1}{2})^3, \frac{n+3}{2}, \dots, n$	
D_4	$2^3, 3, 4^2$	$-\frac{6(4p-7q)(3p-4q)}{pq}$
D_n (n odd)	$2^2, 4, 6, 8, \dots, n-3, n-2, (n-1)^2,$ $n, n+1, n+3, \dots, 2n-4$	
D_n (n even, $n \geq 6$)	$2^2, 4, 6, 8, \dots, n-4, (n-2)^2,$ $n^2, n+2, n+4, \dots, 2n-4$	
E_6	$2, 3, 4, 5, 6^2, 8, 9$	$-\frac{8(9p-13q)(7p-9q)}{pq}$
E_7	$2, 4, 6^2, 8, 9, 10, 12, 14$	$-\frac{9(14p-19q)(11p-14q)}{pq}$
E_8	$2, 6, 8, 10, 12, 14, 15, 18, 20, 24$	$-\frac{10(24p-31q)(19p-24q)}{pq}$

Tab. 2. Subregular \mathcal{W} -algebras in simply laced types

The conformal structure, and in particular the central charge, of the subregular \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{g}, f_{\text{subreg}})$ of type $\mathfrak{g} = A_n$ depends on the choice of good grading. In Table 2 we list conformal weights of generators relative to the Dynkin grading, which corresponds to $x_0 = \rho^\vee - \varpi_m^\vee$, where $n = 2m + 1$. The central charge at level $k = -h^\vee + p/q$ is

$$c = -n(n^2 - 1)\frac{p}{q} - (1 + n + 3n^2 + 2n^3) - n(2 + 3n + n^2)\frac{q}{p}.$$

The central charge of the subregular \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{g}, f_{\text{subreg}})$ of type $\mathfrak{g} = D_n$ at $k = -h^\vee + p/q$ is given by

$$c = -2n(13 - 9n + 2n^2)\frac{p}{q} + (26 + 17n - 24n^2 + 8n^3) - 2n(1 - 3n + 2n^2)\frac{q}{p}.$$

Remark 9.7. If \mathfrak{g} is of type A and $q = h^\vee - 1$ then $\check{P}_{+, \text{subreg}}^q = \check{P}_{+, f_{\text{subreg}}}^q \sqcup \{\rho^\vee\}$ and Theorem 9.6 agrees with Theorem 8.7. Furthermore, the action of \tilde{W}_+ on $\check{P}_{+, \text{subreg}}^q$ is simply transitive and the bijection becomes

$$P_+^{p-h^\vee} \xrightarrow{\sim} [\text{Pr}_\circ^k], \quad \lambda \mapsto \lambda - \frac{p}{q}x_0.$$

See also Lemma 13.1 below.

10. Characters of admissible highest weight modules

For a weight $\hat{\lambda} = \lambda + k\Lambda_0 \in \hat{\mathfrak{h}}^*$ of level k we write χ_λ for the formal character $\sum_{\mu \in \hat{\mathfrak{h}}^*} \dim \hat{L}_k(\lambda)_\mu e^\mu$ of the irreducible $\hat{\mathfrak{g}}$ -module $\hat{L}_k(\lambda) = \bigoplus_{\mu \in \hat{\mathfrak{h}}^*} \hat{L}_k(\lambda)_\mu$. For $\lambda \in \text{Pr}^k$ the formula

$$\chi_\lambda = \frac{1}{R} \sum_{w \in \hat{W}(\hat{\lambda})} \epsilon(w)e^{w\hat{\lambda}} \tag{10.1}$$

was proved by Kac and Wakimoto [64] and used to deduce modular properties of the set of characters of modules of admissible highest weight. Here R is the Weyl–Kac denominator for $\widehat{\mathfrak{g}}$, and $\widehat{W}(\lambda) = \langle r_\alpha \mid \alpha \in \Pi^\vee(\lambda) \rangle$ is the integral Weyl group of λ . We may consider χ_λ as a meromorphic function of $(\tau, z) \in \mathcal{H} \times \mathfrak{h}$, more precisely $\chi_\lambda(\tau, z) = \langle \chi_\lambda, -2\pi i(\tau\Lambda_0 - z) \rangle$.

Let us now take $k = -h^\vee + p/q$ and assume that \mathfrak{g} is simply laced for convenience. In particular, all admissible weights are principal. For $\mu \in P \pmod{pqQ}$ we write

$$\Theta_\mu(\tau, z) = \sum_{\alpha \in \frac{\mu}{pq} + Q} e^{\pi i pq \tau(\alpha, \alpha)} e^{2\pi i pq(\alpha, z)}.$$

Up to a change of variable these are exactly the theta functions associated with the discriminant form $L^\vee/L = P/pqQ$ of the integral lattice $L = Q(pq)$.

Let $\lambda \in \text{Pr}^k$ and let (y, η, ν) be a triple associated with λ as in (2.6) and subsequent remarks, and let $\beta = -y(\eta)$ so that $yt_{-\eta} = t_\beta y$. The function

$$B_\lambda(\tau, z) = \epsilon(y) \sum_{w \in W} \epsilon(w) \Theta_{qw(\nu) + p\beta}(\tau, z/q)$$

depends on λ and not on the choice of triple (y, η, ν) . A linear change of coordinates identifies the sum over $\widehat{W}(\widehat{\lambda})$ in (10.1) with a sum over W of theta functions, and in this way one obtains $\chi_\lambda = B_\lambda/R$. Modular properties of theta functions now yield the following result.

Proposition 10.1 ([64]). *The set of functions B_λ , as λ ranges over Pr^k , is $SL_2(\mathbb{Z})$ -invariant. Furthermore,*

$$B_\lambda(-1/\tau, z/\tau) = (-i\tau)^{\ell/2} e^{\pi i(k+h^\vee)(z, z)/\tau} \sum_{\lambda' \in \text{Pr}^k} a^B(\lambda, \lambda') \chi_{\lambda'}(\tau, z), \tag{10.2}$$

where

$$a^B(\lambda, \lambda') = \frac{\epsilon(y)\epsilon(y')}{|P/pqQ|^{1/2}} e^{-2\pi i[(\nu, \beta') + (\nu', \beta)]} e^{-2\pi i \frac{p}{q}(\beta, \beta')} \sum_{w \in W} \epsilon(w) e^{-2\pi i \frac{q}{p}(w(\nu), \nu')}, \tag{10.3}$$

where (y, η, ν) is a triple associated with λ as above, $\beta = -y(\eta)$, and (y', η', ν') and β' are defined similarly.

In [64] the modular transformations of the χ_λ are given by coefficients denoted a . These are related to the a^B of (10.2) by $a = i^{|\overline{\Delta}|+1} a^B$.

Lemma 10.2. *Let $\lambda \in \text{Pr}^k$ and $w \in W$ be such that $w(\Pi^\vee(\lambda)) = \Pi^\vee(w \circ \lambda)$. Then*

$$B_{w \circ \lambda}(\tau, z) = B_\lambda(\tau, w^{-1}(z)). \tag{10.4}$$

In particular, if λ is regular then (10.4) holds for all $w \in W$, and if $\Delta_+(\lambda) = \{\alpha_\}$ then (10.4) holds for all $w \in W$ such that $w(\alpha_*) \in \Delta_+^\vee$.*

Proof. We write $\lambda = \hat{y}\phi(v) - \rho$ where $\Pi^\vee(\lambda) = \hat{y}(\Pi^\vee_{(q)}) \subset \Delta_+^{\vee, \text{re}}$, and $\hat{y} = t_\beta y$. By assumption $\Pi^\vee(w \circ \lambda) = w\hat{y}(\Pi^\vee_{(q)}) \subset \Delta_+^{\vee, \text{re}}$, and therefore

$$t_{w(\beta)}(wy)\phi(v) - \hat{\rho} = w\hat{y}\phi(v) - \hat{\rho} = w \circ \lambda,$$

so $(wy, v, w(\beta))$ is an admissible triple. In general $\Theta_{w(\mu)}(\tau, z) = \Theta(\tau, w^{-1}(z))$, and one immediately deduces (10.4). ■

Let us write

$$\Theta(\tau, v) = \frac{\theta_{1,1}(\tau, -v)}{\eta(\tau)} = q^{1/12} e^{-\pi i v} \prod_{n=1}^\infty (1 - e^{-2\pi i v} q^{n-1})(1 - e^{+2\pi i v} q^n).$$

Then Θ obeys

$$\Theta(-1/\tau, v/\tau) = -i e^{\pi i v^2/\tau} \Theta(\tau, v).$$

Now let f be a subregular nilpotent element in \mathfrak{g} . We fix a good even grading for f , and we consider the characters of Hamiltonian reductions $H_{f,-}^0(L(\lambda))$, i.e., of the cohomology of (7.2). By the Euler–Poincaré principle the supercharacter of $H_{f,-}^\bullet(M)$ equals that of $C_-^\bullet(M) = M \otimes \bigwedge^{\frac{\infty}{2}+\bullet}$. For $\lambda \in \text{Pr}^k$ we write

$$\Psi_\lambda(\tau, z \mid u) = \text{STr}_{C_-^\bullet(\hat{L}_k(\lambda))} u_0 e^{2\pi i(z_0 - (x_0, z))} q^{L_0 - c/24},$$

where u is a d -closed element of $C_-^\bullet(V_k(\mathfrak{g}))$. The central charge of $\bigwedge = \bigwedge^{\frac{\infty}{2}+\bullet}$ is $-2|\Delta_{>0}|$ and

$$\text{STr}_{\bigwedge^{\frac{\infty}{2}+\bullet}} e^{2\pi i(F_0^z + (x_0, z))} q^{L_0 + |\Delta_{>0}|/12} = \prod_{\alpha \in \Delta_{>0}} \Theta(\tau, \alpha(x)).$$

The modular transformations of the Ψ_λ may be derived from [17, Theorem 6.4].

Proposition 10.3. *For all $\lambda \in \text{Pr}^k$ we have*

$$\begin{aligned} \Psi_\lambda\left(\frac{-1}{\tau}, \frac{z}{\tau} \mid \tau^{-L_{[0]}} \exp\left[\frac{1}{\tau} \sum_{n>0} \frac{(-1)^n}{n} z^{(n)}\right] u\right) \\ = e^{\pi i(k+h^\vee)|z^2|/\tau} \sum_{\lambda' \in \text{Pr}^k} i \cdot a^B(\lambda, \lambda') \Psi_{\lambda'}(\tau, z \mid u). \end{aligned}$$

In fact, the proof of this proposition is the same as that of [17, Theorem 8.1], which deals with the case of f being a principal nilpotent element. The only difference is that $\dim(\mathfrak{g}_{>0})$ changes from $|\Delta_+|$ to $|\Delta_+| - 1$ which causes the factor of i to appear in Proposition 10.3 above.

For $\lambda \in \text{Pr}^k \cap P_{0,+}$ the Hamiltonian reduction $H_{f,-}^0(\hat{L}_k(\lambda))$ is an irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -module. Let us define

$$\psi_\lambda(\tau \mid u) = \text{Tr}_{H_{f,-}^0(\hat{L}_k(\lambda))} u_0 q^{L_0 - c/24}.$$

By Theorem 7.6(1) we have

$$\psi_\lambda(\tau \mid u) = \lim_{z \rightarrow 0} \Psi_\lambda(\tau, z \mid u). \tag{10.5}$$

From now on we assume that $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$, i.e., q is one of the denominators listed in Table 1. In particular, $|\Delta_0| = 2$ and we write α_* for the unique element of $\Delta_{0,+}$. Now we define

$$W^{\text{sr}} = \{w \in W \mid w(\alpha_*) \in \Delta_+\}.$$

Since \mathfrak{g} is simply laced, we may fix a set $\mathbb{X} \subset P_{0,+}$ of representatives of $[\text{Pr}^k]$, so that $\text{Pr}^k = \{y \circ \lambda \mid \lambda \in \mathbb{X} \text{ and } y \in W^{\text{sr}}\}$. Since $\mathcal{W}_k(\mathfrak{g}, f)$ is rational and lisse, by Theorem 7.9, the set of trace functions (10.5), as λ ranges over \mathbb{X} , is modular invariant.

We now compute the S -matrix of $\mathcal{W}_k(\mathfrak{g}, f)$ using Proposition 10.3. Since the restrictions $\Psi_\lambda(\tau, z \mid \mathbf{1})$ are linearly independent, it suffices to work with them. The Weyl denominator $R(\tau, x)$ is essentially $\prod_{\alpha \in \Delta_+} \Theta(\tau, \alpha(x))$ and so, since $\Delta_{>0} = \Delta_+ \setminus \{\alpha_*\}$, we have

$$\Psi_\lambda(\tau, z \mid \mathbf{1}) = \chi_\lambda(\tau, z) \cdot \prod_{\alpha \in \Delta_+ \setminus \alpha_*} \Theta(\tau, \alpha(z)) = \frac{1}{\eta(\tau)^\ell} \cdot \frac{B_\lambda(\tau, z)}{\Theta(\tau, \alpha_*(z))}.$$

Note that $\Theta(\tau, \alpha_*(z))$ has a zero along the hyperplane $\alpha_*(z) = 0$. If $\Delta_+(\lambda) = \{\gamma\}$ then $B_\lambda(\tau, z)$ has a zero along the hyperplane $\gamma(z) = 0$. So unless $\lambda \in P_{0,+}$ the function $\Psi_\lambda(\tau, z \mid \mathbf{1})$ has an indeterminate value at $z = 0$. We make an arbitrary choice of $x \in \mathfrak{h}^*$ not orthogonal to α_* and we put $z = tx$. The limit in (10.5) becomes a limit as $t \rightarrow 0$. We then apply l'Hôpital's rule to the formula of Theorem 10.3. Since for $\lambda \in P_{0,+}$ we have $\gamma = \alpha_*$ and hence $\Psi_\lambda(\tau, z \mid \mathbf{1})$ is regular, the final result does not depend on the auxiliary parameter x .

Fix $x \in \mathfrak{h}$ not orthogonal to α_* , and let $\lambda \in \mathbb{X}$. We have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{\eta(-1/\tau)^\ell} \cdot \frac{B_\lambda(-1/\tau, tx/\tau)}{\Theta(-1/\tau, t\alpha_*(x)/\tau)} &= \frac{1}{(-i\tau)^{\ell/2} \eta(\tau)^\ell} \cdot \lim_{t \rightarrow 0} \frac{(-i\tau)^{\ell/2} e^{\frac{\pi i t^2}{\tau} (k+h^\vee)|x|^2} \sum_{\xi \in \text{Pr}^k} a^B(\lambda, \xi) B_\xi(\tau, tx)}{(-i) e^{\frac{\pi i t^2}{\tau} |\alpha_*(x)|^2} \Theta(\tau, t\alpha_*(x))} \\ &= \frac{i}{\eta(\tau)^\ell} \cdot \lim_{t \rightarrow 0} e^{\frac{\pi i t^2}{\tau} [(k+h^\vee)|x|^2 - |\alpha_*(x)|^2]} \sum_{\xi \in \text{Pr}^k} a^B(\lambda, \xi) \frac{B_\xi(\tau, tx)}{\Theta(\tau, t\alpha_*(x))}. \end{aligned}$$

The exponential factor tends to 1. It follows from Lemma 10.2 that

$$\sum_{\xi \in \text{Pr}^k} a^B(\lambda, \xi) \frac{B_\xi(\tau, tx)}{\Theta(\tau, t\alpha_*(x))} = \sum_{\lambda' \in \mathbb{X}} \left[\sum_{y \in W^{\text{sr}}} a^B(\lambda, y \circ \lambda') \frac{B_{\lambda'}(\tau, ty^{-1}(x))}{\Theta(\tau, t\alpha_*(x))} \right].$$

Using the product formula for Θ , l'Hôpital's rule, and the fact that $\lambda' \in P_{0,+}$, we compute

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{B_{\lambda'}(\tau, ty^{-1}(x))}{\Theta(\tau, t\alpha_*(x))} &= \lim_{t \rightarrow 0} \frac{1 - e^{-t\alpha_*(y^{-1}(x))}}{1 - e^{-t\alpha_*(x)}} \cdot \lim_{t \rightarrow 0} \frac{B_{\lambda'}(\tau, ty^{-1}(x))}{\Theta(\tau, t\alpha_*(y^{-1}(x)))} \\ &= \frac{\alpha_*(y^{-1}(x))}{\alpha_*(x)} \cdot \lim_{t \rightarrow 0} \frac{B_{\lambda'}(\tau, tx)}{\Theta(\tau, t\alpha_*(x))}. \end{aligned}$$

Thus we have proved the following theorem.

Theorem 10.4. *Let f be a subregular nilpotent element of the simply laced simple Lie algebra \mathfrak{g} , and let $k = -h^\vee + p/q$ with denominator q such that $\mathbb{O}_q = \mathbb{O}_{\text{subreg}}$. Let \mathbb{X} be a set of representatives of $[\text{Pr}_\circ^k]$ in $P_{0,+}$. For all $\lambda \in \mathbb{X}$ we have*

$$\psi_\lambda(-1/\tau \mid \tau^{-L_{[0]}}u) = \sum_{\lambda' \in \mathbb{X}} S_{\lambda,\lambda'} \psi_{\lambda'}(\tau \mid u),$$

where

$$S_{\lambda,\lambda'} = i \sum_{y \in W^{\text{sr}}} \frac{\langle y(\alpha_*), x \rangle}{\langle \alpha_*, x \rangle} a^B(\lambda, y \circ \lambda'). \tag{10.6}$$

A remarkable feature of the sum appearing in (10.6) is its independence of x . Of course this follows *a posteriori* but clearly an elementary proof would be desirable. Having fixed β, β' let us put

$$r(x) = \sum_{y(\alpha_*) \in \Delta_+^\vee} \epsilon(y) \frac{\langle y(\alpha_*), x \rangle}{\langle \alpha_*, x \rangle} e^{-\frac{2\pi i}{q}(\beta, y(\beta'))}.$$

Let κ_x denote the 1-form of pairing with x , i.e., $\langle x, - \rangle$. We compute the gradient of f to be

$$\nabla r(x) = \frac{1}{\langle \alpha_*, x \rangle^2} \iota_{\kappa_x}(\alpha_* \wedge \xi), \quad \text{where} \quad \xi = \sum_{y(\alpha_*) \in \Delta_+^\vee} \epsilon(y) e^{-\frac{2\pi i}{q}(\beta, y(\beta'))} y(\alpha_*).$$

To show that r is independent of x , it therefore suffices to show that ξ is proportional to α_* . This may be established by a direct computation for any fixed root system, though we do not know a uniform proof.

11. Modular transformations of simple affine vertex algebras

In this section we collect some results on S -matrices and fusion rules of the rational vertex algebra $V_{p-h^\vee}(\mathfrak{g})$ where $p \geq h^\vee$.

Proposition 11.1 ([63]). *Let \mathfrak{g} be a simple Lie algebra and $p \geq h^\vee$ an integer. The irreducible $V_{p-h^\vee}(\mathfrak{g})$ -modules are precisely the irreducible highest weight $\widehat{\mathfrak{g}}$ -modules $L(\widehat{\kappa})$ where $\widehat{\kappa} = \kappa + (p - h^\vee)\Lambda_0$ and κ runs over $P_+^{p-h^\vee}$. The span of the characters $\chi_\kappa(\tau, z)$ is modular invariant. In particular,*

$$\chi_\kappa(-1/\tau, z/\tau) = \sum_{\kappa' \in P_+^{p-h^\vee}} K^{\kappa,\kappa'} \chi_{\kappa'}(\tau, z),$$

where

$$K^{\kappa,\kappa'} = \frac{i^{|\Delta_+|}}{|P/pQ|^{1/2}} \sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{p}(w(\kappa+\rho), \kappa+\rho)}. \tag{11.1}$$

We recall from [44, (4.2.6)] the relation

$$K^{\overline{w(\widehat{\kappa}), \kappa'}} = e^{-2\pi i \overline{w(\Lambda_0), \kappa'}} K^{\kappa, \kappa'} \tag{11.2}$$

for all $w \in \widetilde{W}_+$.

We denote by K_p the matrix of coefficients $K^{\kappa, \kappa'}$ given by (11.1) as κ, κ' run over $P_+^{p-h^\vee}$. We also denote by K_p^{int} the submatrix obtained by restricting $\kappa, \kappa' \in Q$ and by $K_p^{\mathbb{Z}}$ the submatrix obtained by restricting $\kappa, \kappa' \in -\rho + Q$.

Lemma 11.2. *The subgroup $\mathcal{F}(V_{p-h}(\mathfrak{g}))^{\text{int}} \subset \mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))$ spanned by $[\mu]$, where μ runs over Q , is a subring closed under duality. Furthermore, $\mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))^{\text{int}} \cong \mathcal{F}(K_p^{\mathbb{Z}})$.*

Proof. Combining (11.2) with the Verlinde formula yields

$$N_{\lambda, \mu}^\nu = e^{-2\pi i \langle w(\Lambda_0), \lambda + \mu - \nu \rangle} N_{\lambda, \mu}^\nu \quad \text{for all } w \in \widetilde{W}_+.$$

Since $w_j(\Lambda_0) = \varpi_j + \Lambda_0$ for $j \in J$, it follows that $N_{\lambda, \mu}^\nu = 0$ unless $(\lambda + \mu - \nu, \varpi_j) \in \mathbb{Z}$ for all $j \in J$. But the set $\{\varpi_j\}_{j \in J}$ of fundamental weights forms a system of representatives of the quotient P/Q . Therefore

$$N_{\lambda, \mu}^\nu = 0 \quad \text{unless } \lambda + \mu - \nu \in Q. \tag{11.3}$$

This shows that $\mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))^{\text{int}}$ is closed under the fusion product. For duality it is well known that $L(\kappa)^\vee \cong L(-w^\circ(\kappa))$, and if $\kappa \in Q$ then so is $-w^\circ(\kappa)$. This proves the first part.

Let $s \in J$ be defined by the condition $\varpi_s - \rho \in Q$. Then for $\mu \in Q$ we have $w_s(\mu) \in -\rho + Q$. (Here we have used the fact that $2\rho \in Q$.) For $\mu, \mu' \in Q$ we have, by (11.2),

$$K^{w_s(\mu), w_s(\mu')} = e^{-2\pi i \langle \rho, \rho \rangle} K^{\mu, \mu'}.$$

This says that $K_p^{\mathbb{Z}}$ becomes proportional to K_p^{int} upon conjugation by w_s . It therefore suffices to show that the fusion algebra determined by K_p^{int} is isomorphic to $\mathcal{F}(V_{p-h}(\mathfrak{g}))^{\text{int}}$. But this follows easily from (11.2) and the Verlinde formula. ■

12. S-matrices of subregular \mathcal{W} -algebras

Let \mathfrak{g} be a finite-dimensional simple Lie algebra of simply laced type. In this section we compute the fusion rules of rational subregular \mathcal{W} -algebras obtained via Hamiltonian reduction of \mathfrak{g} . Let $k = -h^\vee + p/q$ where q is one of the denominators listed in Table 1 and $p \geq h^\vee$ is coprime to q . The irreducible $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ -modules are parametrised by $[\text{Pr}_\circ^k]$, and by Theorem 9.6 we have a bijection

$$\frac{P_{+, \text{reg}}^p \times \check{P}_{+, \text{subreg}}^q}{\widetilde{W}_+} \rightarrow [\text{Pr}_\circ^k],$$

taking (ν, η) to the $W \circ (-)$ -orbit of

$$t_{-\eta}\phi(\widehat{\nu}) - \widehat{\rho} = (\nu - (p/q)\eta - \rho) + k\Lambda_0.$$

The fusion rules are computed by applying the Verlinde formula to the S -matrix of $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$. Let $\widehat{\lambda} = y t_{-\eta}\phi(\widehat{\nu}) - \widehat{\rho}$ and $\widehat{\lambda}' = y' t_{-\eta'}\phi(\widehat{\nu}') - \widehat{\rho}$. By Proposition 10.1 and Theorem 10.4 the S -matrix is given by

$$S_{\widehat{\lambda}, \widehat{\lambda}'} = i \frac{\epsilon(y)\epsilon(y')}{|P/pqQ|^{1/2}} \sum_{w \in W^{\text{sr}}} \left(\epsilon(w) \frac{\langle w(\alpha_*^\vee), x \rangle}{\langle \alpha_*^\vee, x \rangle} e^{-2\pi i[(\nu, w(\beta')) + (\nu', \beta)]} e^{-2\pi i \frac{p}{q}(\beta, \beta')} \right. \\ \left. \times \sum_{u \in W} \epsilon(u) e^{-2\pi i \frac{q}{p}(u(\nu), \nu')} \right),$$

where $\beta = -y(\eta)$ and $\beta' = -y'(\eta')$. We wish to reduce this to a product of sums over $w \in W^{\text{sr}}$ and $u \in W$, and to do this it is desirable to eliminate the cross terms

$$e^{2\pi i[(\nu, w(\beta')) + (\nu', \beta)]}.$$

If q is coprime to $|J|$ then by Lemma 2.1 we are able to choose, for each $\lambda \in [\text{Pr}_\circ^k]$, a representative (ν, η) for which $\eta \in Q$. Indeed, in this case the action of \widetilde{W}_+ on $\check{P}_{+, \text{subreg}}^q$ is transitive and we obtain a bijection

$$P_{+, \text{reg}}^p \times (\check{P}_{+, \text{subreg}}^q \cap Q) \rightarrow [\text{Pr}_\circ^k]. \tag{12.1}$$

If q is not coprime to $|J|$ then p is coprime to $|J|$ and we have a bijection

$$(P_{+, \text{reg}}^p \cap Q) \times \check{P}_{+, \text{subreg}}^q \rightarrow [\text{Pr}_\circ^k]. \tag{12.2}$$

In either case, for each η , we fix an element $y^{(\eta)} \in W$ such that

$$\widehat{\lambda} = y^{(\eta)} t_{-\eta}\phi(\widehat{\nu}) - \widehat{\rho} \in P_{0,+} \quad \text{for all } \widehat{\nu} \in P_{+, \text{reg}}^p. \tag{12.3}$$

The existence of such elements $y^{(\eta)} \in W$ is ensured by the following lemma.

Lemma 12.1. *Let $\eta \in \check{P}_{+, \text{subreg}}^q$. Then there exists $y = y^{(\eta)} \in W$ such that*

$$\langle \alpha_*^\vee, y(\nu - (p/q)\eta) \rangle \in \mathbb{Z}_+ \quad \text{for all } p \in \mathbb{Z}_+ \text{ and } \nu \in P_+^p.$$

Proof. Let $\eta = \sum_{i=1}^\ell c_i \varpi_i \in \check{P}_{+, \text{subreg}}^q$. Then either $c_k = 0$ for some $k \in \{1, \dots, \ell\}$ or else $\langle \eta, \theta^\vee \rangle \geq \langle \rho, \theta^\vee \rangle = h^\vee - 1$ and hence $q \geq h^\vee - 1$. Since in fact $q \leq h^\vee - 1$ we deduce that either $c_k = 0$ for some $k \in \{1, \dots, \ell\}$ or else $\eta = \rho$.

It clearly suffices to show that there exists $y \in W$ such that

$$\left\langle \alpha_*^\vee, y \left(\varpi_i - \frac{a_i^\vee}{q} \eta \right) \right\rangle \in \mathbb{Z}_+, \quad i = 1, \dots, \ell. \tag{12.4}$$

Here $a_i^\vee = \langle \theta^\vee, \varpi_i \rangle$. If $c_k = 0$ then we have $\langle \eta, \alpha_k^\vee \rangle = 0$ and $\langle \varpi_i - \frac{a_i^\vee}{q} \eta, \alpha_k^\vee \rangle = \delta_{ik}$. On the other hand, if $\eta = \rho$ then $q = h^\vee - 1$ and $\langle \varpi_i - \frac{a_i^\vee}{q} \eta, \theta^\vee \rangle = a_i^\vee - a_i^\vee = 0$. Let us now take $y \in W$ such that $y(\alpha_k^\vee) = \alpha_*^\vee$, respectively $y(\theta^\vee) = \alpha_*^\vee$. We obtain (12.4) as required. ■

Theorem 12.2. *Let \mathfrak{g} be a simple Lie algebra of simply laced type and let $k = -h^\vee + p/q$ where q is a subregular denominator for \mathfrak{g} and $p \geq h^\vee$ is coprime to q . Let $\hat{\lambda} = y^{(\eta)} t_{-\eta} \phi(\hat{v}) - \hat{\rho}$ as in the preceding remarks, and similarly $\hat{\lambda}' = y^{(\eta')} t_{-\eta'} \phi(\hat{v}') - \hat{\rho}$. Put $\beta = -y^{(\eta)}(\eta)$ and $\beta' = -y^{(\eta')}(\eta')$. Then the S -matrix of $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ is given by*

$$S_{\lambda, \lambda'} = i \frac{\epsilon(y^{(\eta)})\epsilon(y^{(\eta')})}{|P/pqQ|^{1/2}} \left(\sum_{w \in W^{\text{sr}}} \epsilon(w) \frac{\langle w(\alpha_*^\vee), x \rangle}{\langle \alpha_*^\vee, x \rangle} e^{-2\pi i \frac{p}{q}(\beta, w(\beta'))} \right) \times \left(\sum_{u \in W} \epsilon(u) e^{-2\pi i \frac{q}{p}(u(v), v')} \right). \tag{12.5}$$

Proof. Having made the choices prescribed above, the result follows immediately from Theorem 10.4. ■

Let q be a subregular denominator for \mathfrak{g} , i.e., one of the denominators listed in Table 1, and $p \geq h^\vee$ coprime to q . We denote by C_q the matrix of coefficients

$$C^{\eta, \eta'} = \sum_{w \in W^{\text{sr}}} \epsilon(w) \frac{\langle w(\alpha_*^\vee), x \rangle}{\langle \alpha_*^\vee, x \rangle} e^{-\frac{2\pi i}{q}(\beta, w(\beta'))}, \tag{12.6}$$

where $\hat{\eta}, \hat{\eta}'$ run over $\check{P}_{+, \text{subreg}}^q$, and by $C_q^{\mathbb{Z}}$ the submatrix with $\eta, \eta' \in Q$. We denote by $S_{\text{sr}}^{p, q}$ the S -matrix of the rational vertex algebra $\mathcal{W}_{-h^\vee + p/q}(\mathfrak{g}, f_{\text{subreg}})$.

Lemma 12.3. *The weight lattice P of \mathfrak{g} satisfies $(P, P) \subset \frac{1}{|J|}\mathbb{Z}$, where (\cdot, \cdot) is the invariant bilinear form on \mathfrak{g} so normalised that $(\theta, \theta) = 2$. If $\mathfrak{g} = D_n$ and n is even then $(P, P) \subset \frac{1}{2}\mathbb{Z}$ in fact.*

Let $\mathbb{Q}(\zeta_N)$ denote the cyclotomic field obtained by adjoining to \mathbb{Q} a primitive N th root of unity ζ_N . For an integer a coprime to N we denote by $\varphi_a \in \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ the automorphism defined by $\varphi_a(\zeta_N) = \zeta_N^a$. The Galois group $\text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ is naturally isomorphic to $(\mathbb{Z}/N)^\times$.

We denote the Kronecker product of matrices by \otimes , and proportionality of matrices with a nonzero scalar by \sim . Now suppose $(q, |J|) = 1$. Then by Lemma 12.3 the entries of K_p lie in $\mathbb{Q}(\zeta_{p|J|})$, on which φ_q acts as an automorphism. Meanwhile the entries of $C_q^{\mathbb{Z}}$ lie in $\mathbb{Q}(\zeta_q)$, on which φ_p acts as an automorphism. Similar considerations apply in case $(p, |J|) = 1$ and Theorem 12.2 can now be summarised as

$$S_{\text{sr}}^{p, q} \sim \varphi_p(C_q^{\mathbb{Z}}) \otimes \varphi_q(K_p) \quad \text{if } (q, |J|) = 1, \tag{12.7}$$

$$S_{\text{sr}}^{p, q} \sim \varphi_p(C_q) \otimes \varphi_q(K_p^{\mathbb{Z}}) \quad \text{if } (p, |J|) = 1. \tag{12.8}$$

In both cases we obtain a presentation of the fusion ring of $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ as the tensor product of the fusion ring of $V_{p-h^\vee}(\mathfrak{g})$ (or its integral weight subalgebra) and the fusion ring associated with the S -matrix $C_q^{\mathbb{Z}}$ (or C_q).

13. Fusion rules of subregular \mathcal{W} -algebras of type A

We now specialise the discussion of the preceding section to the type A case. We note that subregular \mathcal{W} -algebras of type A were previously studied by Feigin and Semikhatov [41] as $\mathcal{W}_n^{(2)}$ -algebras and the isomorphism between them was established by Genra [49].

Lemma 13.1. *Let \mathfrak{g} be the simple Lie algebra of type A_n and let $k = -h^\vee + p/q$ where $q = h^\vee - 1 = n$ and $p \geq h^\vee$ coprime to q . There exists a bijection between the sets of irreducible modules of $\mathcal{W}_{-h^\vee + p/q}(\mathfrak{g}, f_{\text{subreg}})$ and $V_{p-h^\vee}(\mathfrak{g})$ which induces an equality $S_{\text{sr}}^{p,q} = \varphi_q(K_p)$.*

Proof. The set $\check{P}_{+, \text{subreg}}^q$ consists of ρ and the weights $\rho - \varpi_i$ for $i = 1, \dots, \ell$. Let us put $\eta = \rho$ if n is even, and $\eta = \rho - \varpi_m$ if $n = 2m + 1$ is odd. Then $\check{P}_{+, \text{subreg}}^q \cap Q = \{\eta\}$. Let us fix a good even grading on \mathfrak{g} with associated simple root $\alpha_* \in \Delta_{0,+}$ and let ϖ_* be the fundamental weight corresponding to α_* . Finally, we fix $y = y^{(\eta)} \in W$ such that condition (12.3) is satisfied, so that $\hat{\lambda} = (yt_{-\eta}) \circ \hat{\mu}$ defines a bijection $\mu \mapsto \lambda$ from $P_+^{p-h^\vee}$ to $[\text{Pr}_\circ^k]$. By Theorem 12.2 we now have

$$(S_{\text{sr}}^{p,q})_{\lambda, \lambda'} = C \cdot \frac{i}{|P/pqQ|^{1/2}} \sum_{u \in W} \epsilon(u) e^{-2\pi i \frac{q}{p}(u(\mu+\rho), \mu'+\rho)}, \tag{13.1}$$

where the factor $C = C^{\eta, \eta}$, given by formula (12.6), is independent of μ, μ' . By Lemma 12.3 the exponential sums of (11.1) and (13.1) lie in $\mathbb{Q}(\zeta_{ph^\vee})$. Since q is coprime to p and h^\vee , the latter is the conjugate of the former by $\varphi_q \in \text{Gal}(\mathbb{Q}(\zeta_{ph^\vee})/\mathbb{Q})$. ■

Theorem 13.2. *The fusion rules of the exceptional subregular \mathcal{W} -algebras of type A_n , for $q = n$ odd, coincide with those of simple affine vertex algebras at positive integer level. More precisely, the assignment*

$$\hat{L}_{p-h^\vee}(\lambda) \mapsto \mathbf{L}(E_{\lambda - \frac{p}{q}x_0}) \cong H_{DS, f}^0(\hat{L}_k(\lambda)),$$

induces an isomorphism of fusion rings

$$\mathcal{F}(\mathcal{W}_{-h^\vee + p/q}(\mathfrak{g}, f_{\text{subreg}})) \cong \mathcal{F}(V_{p-h^\vee}(\mathfrak{g})).$$

Proof. We let $n = 2m + 1$ and take $\alpha_* = \alpha_m$. The subregular denominator is $q = n$. In the proof of Lemma 13.1 we have $\eta = x_0$ and since $\langle \alpha_*, x_0 \rangle = 0$ we may take $y = 1$. This gives the bijection between irreducible modules stated in the theorem and gives the S -matrix of $\mathcal{W} = \mathcal{W}_{-h^\vee + p/q}(\mathfrak{g}, f_{\text{subreg}})$ as $\varphi_q(K_p)$. Since \mathcal{W} is self-contragredient by Proposition 6.5 (as well as rational and lisse), the fusion rules can be computed via the Verlinde formula. The fusion rules and coefficients of the charge conjugation matrix (which specify duality) are integers and therefore invariant under the Galois group $\text{Gal}(\mathbb{Q}(\zeta_{ph^\vee})/\mathbb{Q})$. It follows immediately that the bijection $\hat{L}_{p-h^\vee}(\lambda) \mapsto \mathbf{L}(E_{\lambda - \frac{p}{q}x_0})$ induces an isomorphism from the fusion ring of $V_{p-h^\vee}(\mathfrak{g})$ to that of \mathcal{W} . ■

Remark 13.3. The rationality of subregular \mathcal{W} -algebras of type A has also been proven in another way by Creutzig and Linshaw [28] after the first version of the present article was submitted. There it was shown that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ at level $k = -n + (n + m)/(n - 1)$ is isomorphic to a simple current extension of the rational vertex algebra $V_{\sqrt{mn}\mathbb{Z}} \otimes \mathcal{W}_\ell(\mathfrak{sl}_m, f_{\text{prin}})$, where $V_{\sqrt{mn}\mathbb{Z}}$ is the lattice vertex algebra associated with the lattice $\sqrt{mn}\mathbb{Z}$ and $\ell = -m + (m + n)/(n + 1)$.

Remark 13.4. Using a result in [10], which appeared after the first version of the present article was submitted, one can strengthen the statement of Theorem 13.2.

Let k be an admissible level for $\mathfrak{g} = \mathfrak{sl}_n$, let KL_k be the full subcategory of $\widehat{\mathcal{O}}_k$ consisting of modules isomorphic to a direct sum of finite-dimensional \mathfrak{g} -modules, and let $\text{KL}(V_k(\mathfrak{g}))$ be the full subcategory of the category of $V_k(\mathfrak{g})$ -modules consisting of those objects which belong to KL_k as $\widehat{\mathfrak{g}}$ -modules. Then $\text{KL}(V_k(\mathfrak{g}))$ is naturally a fusion category [24, 26]. By [10, Theorem 10.4, Remark 10.7], the functor

$$\text{KL}(V_k(\mathfrak{g})) \rightarrow \mathcal{W}_k(\mathfrak{g}, f_q)\text{-mod}, \quad M \mapsto H_f^0(M), \tag{13.2}$$

is an equivalence of fusion categories for $q = n - 1$ and n odd since it induces a bijection between simple objects. Theorem 13.2 then follows from the fact [24] that $\mathcal{F}(\text{KL}(V_k(\mathfrak{g})))$ is isomorphic to $\mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))$.

Remark 13.5. The argument of Remark 13.4 can also be used to describe the fusion categories of exceptional rectangular \mathcal{W} -algebras [15]. Namely, let $\mathfrak{g} = \mathfrak{sl}_n$, let $k = -n + p/q$ be an admissible level and suppose that $n = qr$ for some $r \in \mathbb{Z}_{\geq 1}$, and let $f_q \in \mathcal{O}_q$. Note that f_q is necessarily an even nilpotent. We have $\check{P}_{+,f}^q = \{x_0\}$. It follows from Theorem 8.7 that we have a bijection

$$P_+^{p-n}/\mathbb{Z}_q \xrightarrow{\sim} [\text{Pr}_\circ^k], \quad [\lambda] \mapsto \left[\lambda - \frac{p}{q}x_0 \right],$$

where the cyclic group \mathbb{Z}_q acts on P_+^{p-n} as follows: $i + q\mathbb{Z} \in \mathbb{Z}_q$ sends $\lambda \mapsto \bar{\pi}_{ir}\lambda + (p - n)\varpi_{ir}$. Therefore, (13.2) gives a quotient functor of fusion categories such that

$$H_{DS,f}^0(\widehat{L}_k(\lambda)) \cong H_{DS,f}^0(\widehat{L}_k(\mu))$$

if and only if $\mu \in \mathbb{Z}_q\lambda$. In particular, $\mathcal{F}(\mathcal{W}_k(\mathfrak{g}, f_q)) \cong \mathcal{F}(V_{p-n}(\mathfrak{g}))/\mathbb{Z}_q$.

In a companion paper [11] with Anne Moreau, we will show that in the special case where $p = h^\vee + 1$, we have $\mathcal{W}_{-n+(n+1)/q}(\mathfrak{g}, f_q) \cong V_1(\mathfrak{sl}_r)$. This is compatible with the fact that $\mathcal{F}(V_1(\mathfrak{sl}_{qr}))/\mathbb{Z}_q \cong \mathcal{F}(V_1(\mathfrak{sl}_r))$.

We also note that Ueda [75] has recently constructed a surjective homomorphism from the affine Yangian [20, 52] of type A_r to the current algebra of the rectangular \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{gl}_n, f_q)$. It would be of interest to understand the modular category of exceptional rectangular \mathcal{W} -algebras in terms of the representation theory of the affine Yangian.

14. Fusion rules of subregular \mathcal{W} -algebras of types D and E

Now suppose \mathfrak{g} is of type D or E . In this section we reduce the description of fusion rings of the associated rational subregular \mathcal{W} -algebras to a finite number of cases which can be computed explicitly. We denote by α_* the root associated with the trivalent node of the Dynkin diagram of \mathfrak{g} , and by ϖ_* the corresponding fundamental weight.

Proposition 14.1. (a) *Let q be a subregular denominator for \mathfrak{g} coprime to $h^\vee + 1$ and let $k = -h^\vee + p/q$ where $p \geq h^\vee$ is coprime to q and to $|J|$. Then there exists an isomorphism of fusion rings*

$$\mathcal{F}(\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})) \cong \mathcal{F}(\mathcal{W}_{-h^\vee+(h^\vee+1)/q}(\mathfrak{g}, f_{\text{subreg}})) \otimes \mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))^{\text{int}}.$$

(b) *Let $q = h^\vee - 1$ and let $k = -h^\vee + p/q$ where $p \geq h^\vee$ is coprime to q . Then there exists an isomorphism of fusion rings*

$$\mathcal{F}(\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})) \cong \mathcal{F}(\mathcal{W}_{-h^\vee+h^\vee/(h^\vee-1)}(\mathfrak{g}, f_{\text{subreg}})) \otimes \mathcal{F}(V_{p-h^\vee}(\mathfrak{g})).$$

We remark that the only subregular denominator q excluded by the condition $(q, h^\vee + 1) = 1$ is \mathfrak{g} of type D_n for $n \equiv 2 \pmod{3}$ and $q = 2n - 4$.

Proof of Proposition 14.1. (a) Since $(p, |J|) = 1$ we have the relation (12.8). By Lemma 11.2 the fusion ring of $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ is the tensor product of $\mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))^{\text{int}}$ and the fusion ring associated with the S -matrix C_q . Now in general P_+^1 is in bijection with the finite set J and carries a transitive action of \tilde{W}_+ , hence $|P_+^1 \cap Q| = 1$ and $K_{h^\vee+1}^{\mathbb{Z}}$ is a 1×1 matrix. Since we assume $(q, h^\vee + 1) = 1$, we have

$$S_{\text{sr}}^{h^\vee+1,q} \sim \varphi_{h^\vee+1}(C_q) \otimes \varphi_q(K_{h^\vee+1}^{\mathbb{Z}}) \sim \varphi_{h^\vee+1}(C_q),$$

so C_q is a cyclotomic conjugate of the S -matrix of $\mathcal{W}_{-h+(h+1)/q}(\mathfrak{g}, f_{\text{subreg}})$, and the result follows.

(b) Since $(q, |J|) = 1$ we have the relation (12.7). The fusion ring of $\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ is therefore the tensor product of the fusion ring of $V_{p-h}(\mathfrak{g})$ and the fusion ring associated with the S -matrix $C_q^{\mathbb{Z}}$. Evidently $|P_+^0| = 1$ and so K_{h^\vee} is a 1×1 matrix. Since $(q, h^\vee) = 1$ we have

$$S_{\text{sr}}^{h^\vee,q} \sim \varphi_{h^\vee}(C_q) \otimes \varphi_q(K_{h^\vee}) = \varphi_{h^\vee}(C_q),$$

so C_q is a cyclotomic conjugate of the S -matrix of $\mathcal{W}_{-h+h/(h-1)}(\mathfrak{g}, f_{\text{subreg}})$, and the result follows. ■

In the following table we record some data on the \mathcal{W} -algebras of minimal numerator. The isomorphisms of the final column will be proved in Section 15. See that section also for background on the effective central charge.

\mathfrak{g}	p/q	c	c^{eff}	#(Irr. rep.)	Isom. type
A_n	$\frac{n+1}{n}$	0	0	1	1
D_n	$\frac{2n-2}{2n-3}$	$-\frac{12n^2-62n+78}{2n-3}$	$1 - \frac{3}{2n-3}$	$n-2$	$\text{Vir}_{2,2n-3}$
D_n ($3 \nmid n-2$)	$\frac{2n-1}{2n-4}$	$-\frac{2n^2-21n+52}{n-2}$	$1 - \frac{2}{n-2}$	$n-3$	$\text{Vir}_{3,n-2}$
E_6	13/9	0	0	1	1
E_6	13/10	4/5	4/5	6	$\mathcal{W}_{-7/4}(A_2, f_{\text{prin}})$
E_6	12/11	-350/11	10/11	7	$\text{Vir}_{3,22} \oplus L(21, 1)$
E_7	19/14	0	0	1	1
E_7	19/15	-3/5	3/5	4	$\text{Vir}_{3,5}$
E_7	19/16	-135/8	9/8	13	
E_7	18/17	-1420/17	20/17	16	
E_8	31/24	0	0	1	1
E_8	31/25	-22/5	2/5	2	$\text{Vir}_{2,5}$
E_8	31/26	-350/13	10/13	6	$\text{Vir}_{2,13}$
E_8	31/27	-590/9	10/9	12	
E_8	31/28	-830/7	10/7	25	
E_8	30/29	-7518/29	42/29	44	

Tab. 3. Subregular \mathcal{W} -algebras $\mathcal{W}_{-h+p/q}(\mathfrak{g}, f_{\text{subreg}})$

We observe that for \mathfrak{g} of type E_6, E_7, E_8 of rank ℓ the cardinality of $\check{P}_{+, \text{subreg}}^q$ coincides with that of the set of regular dominant integral weights of level $q - \ell + 2$ for the root system $\Delta^\perp = \{\alpha \in \Delta \mid (\alpha, \alpha_*) = 0\}$ (which is, respectively, A_5, D_6, E_7). A similar pattern extends to types A_n and D_n but is more complicated since, for Δ of type D_n for example, Δ^\perp has type $D_{n-2} \times A_1$. These bijections do not correspond to isomorphisms of fusion rings however. Indeed, the fusion ring of $V_1(E_7)$ is isomorphic to the group ring of $\mathbb{Z}/2$, while the fusion ring of $\mathcal{W}_{-719/25}(E_8, f_{\text{subreg}}) \cong \text{Vir}_{2,5}$ is different.

We close this section with a remark on the case $\mathfrak{g} = D_n$ where $n \equiv 2 \pmod 3$, and $k = -h + p/q$ where $q = 2n - 4$, and $p \geq h^\vee$ is coprime to q . By the proof of Proposition 14.1 we have an isomorphism of fusion rings

$$\mathcal{F}(\mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})) \cong \mathcal{F}(C_q) \otimes \mathcal{F}(V_{p-h^\vee}(\mathfrak{g}))^{\text{int}}.$$

Since $h^\vee + 1$ is not coprime to q we cannot set $p = h^\vee + 1$ and thereby identify the fusion ring $\mathcal{F}(C_q)$ as the fusion ring of a subregular \mathcal{W} -algebra. Nevertheless, we may compute the fusion rules of $\mathcal{F} = \mathcal{F}(C_q)$ explicitly for small n . For $\mathfrak{g} = D_5$ for instance, \mathcal{F} is the group ring of $\mathbb{Z}/2$ with its canonical basis. In general \mathcal{F} contains $n - 3$ simple objects (naturally indexed by $\eta = \rho - \varpi_i$, where ϖ_i are those fundamental weights of D_n with Kac label $a_i^\vee = 2$). Based on explicit computation of \mathcal{F} for low ranks, we propose the following conjecture.

Conjecture 14.2. *For each positive integer m there exists a rational lisse vertex algebra of central charge $c = 13 - 6(m + 1/m)$ whose $r = 3m - 1$ irreducible modules, denoted $[i]$ for $i = 0, 1, \dots, r$, have the following fusion rules:*

$$[i] \boxtimes [j] \cong \bigoplus_{\substack{|i-j| \leq k \leq \min\{i+j, r-i-j\} \\ k \equiv i+j \pmod 2}} [k].$$

Curiously, the central charges that appear here are the central charges of the triplet vertex algebras.

15. Sporadic isomorphisms

In this section we explain the final column of Table 3. First we fix some notation regarding the Virasoro minimal models. Recall that

$$\text{Vir}_{p,q} = H_{f_{\text{prin}}}^0(V_{-2+p/q}(A_1))$$

is a rational vertex algebra of central charge $c_{p,q} = 1 - 6(p - q)^2/pq$. It has $(p - 1)(q - 1)/2$ irreducible modules, all of the form

$$L(r, s) = H_{f_{\text{prin},-}}^0(L(\lambda)) \quad \text{where} \quad \lambda = k\Lambda_0 + [s - 1 - r(p/q)]\varpi_1.$$

Here $1 \leq r \leq q - 1, 1 \leq s \leq p - 1$ and $L(r, s) \cong L(q - r, p - s)$. The conformal dimensions of the irreducible modules are given by

$$\Delta(L(r, s)) = h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}. \tag{15.1}$$

The *effective central charge* of a rational vertex algebra V is by definition

$$c^{\text{eff}} = c - 24h_{\text{min}},$$

where h_{min} is the minimal conformal dimension of the irreducible V -modules.

Next let $M = \bigoplus_{n \in \mathbb{Z}_+} M_n$ be a graded vector space for which the series $\chi_M(\tau) = \sum_{n=0}^\infty \dim(M_n)q^{h+n}$ is convergent. If for some constants A, β and g , the character χ_M has asymptotic behaviour

$$\chi(it) \sim At^\beta e^{\pi g/12t}, \tag{15.2}$$

then one says that M has *asymptotic growth* g and *asymptotic dimension* A .

The following proposition is a well known consequence of the modular invariance of characters of rational vertex algebras [81].

Proposition 15.1. *Let V be a rational lisse vertex algebra of CFT type. The asymptotic growth of an irreducible V -module does not exceed the effective central charge of V , and equality occurs if all entries of the S -matrix of V are nonzero.*

The asymptotic growth g and asymptotic dimension A of $\mathcal{W}_{-h^\vee+p/q}(\mathfrak{g}, f)$ are given in [66, Theorem 2.16] as

$$g = \dim(\mathfrak{g}^f) - \frac{h^\vee}{pq} \dim(\mathfrak{g}), \tag{15.3}$$

and, if f possesses a good even grading,

$$A = \frac{1}{(pq)^{\ell/2} |P/Q^\vee|^{1/2}} \cdot \frac{1}{q^{|\Delta_{>0}|}} \prod_{\alpha \in \Delta_{>0}} 2 \sin\left(\frac{\pi}{q}(\alpha, x_0)\right) \cdot \prod_{\alpha \in \Delta_+} 2 \sin\left(\frac{\pi}{p}(\alpha, \rho)\right). \tag{15.4}$$

The asymptotic dimension of the $\text{Vir}_{p,q}$ -module $L(r, s)$ is

$$A_{p,q}^{L(r,s)} = (8/pq)^{1/2} (-1)^{(r+s)(r_0+s_0)} \sin\left(\pi \frac{p-q}{q} r r_0\right) \cdot \sin\left(\pi \frac{p-q}{p} s s_0\right), \tag{15.5}$$

where r_0 and s_0 are positive integers characterised by $r_0 < q$, $s_0 < p$ and $r_0 p - s_0 q = 1$ (see [64]).

Proposition 15.2. *For each entry in Table 3 for which $c^{\text{eff}} < 1$, the isomorphism type of $\mathcal{W}_{-h^\vee+p/q}(\mathfrak{g}, f_{\text{subreg}})$ is as listed in the final column.*

Proof. First we consider the case $\mathfrak{g} = D_n$, $q = 2n - 4$ (for $n \not\equiv 2 \pmod{3}$). By (4.1) and (15.3) the central charge c and the asymptotic growth g of $\mathcal{W} = \mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ are

$$c = -\frac{2n^2 - 21n + 52}{n - 2} \quad \text{and} \quad g = 1 - \frac{2}{n - 2}.$$

These coincide with the values for $\text{Vir}_{3,n-2}$.

Let $N \subset \mathcal{W}$ be the vertex subalgebra generated by the Virasoro vector. By the representation theory of the Virasoro algebra, N is either the universal Virasoro vertex algebra $\text{Vir}^{2,2n-3}$ of central charge c , or its simple quotient $\text{Vir}_{3,n-2}$. However, by [64, Proposition 4 (b)] the asymptotic growth of $\text{Vir}^{3,n-2}$ is 1, which is greater than the asymptotic growth of \mathcal{W} . Thus we cannot have $N \cong \text{Vir}^{3,n-2}$ and so in fact $N \cong \text{Vir}_{3,n-2}$. Since $\text{Vir}_{3,n-2}$ is a rational vertex algebra, its embedding into \mathcal{W} induces a decomposition of the latter into a direct sum of irreducible $\text{Vir}_{3,n-2}$ -modules. The direct sum is finite since \mathcal{W} has finite-dimensional graded pieces.

Each irreducible $\text{Vir}_{3,n-2}$ -module has $\beta = 0$ in (15.2) and nonzero asymptotic dimension, and all have the same asymptotic growth g . It follows that the asymptotic dimension of \mathcal{W} equals the sum of the asymptotic dimensions of the modules in its decomposition, and so to prove $\mathcal{W} \cong \text{Vir}_{3,n-2}$ it suffices to prove equality of their asymptotic dimensions. By (15.5) the asymptotic dimension of $\text{Vir}_{3,n-2}$ is

$$A_{3,n-2}^{L(1,1)} = \frac{2}{q^{1/2}} \sin\left(\frac{2\pi}{q}\right).$$

To compute the asymptotic dimension A of \mathcal{W} we recall that $\Delta_{>0} = \Delta_+ \setminus \{\alpha_*\}$ and $x_0 = \rho - \varpi_*$. We count, for each $m \in \mathbb{Z}_+$, the number of $\alpha \in \Delta_{>0}$ satisfying $(\alpha, x_0) = m$

and we deduce

$$\prod_{\alpha \in \Delta_{>0}} 2 \sin\left(\frac{\pi}{q}(\alpha, x_0)\right) = 4 \sin\left(\frac{\pi}{q}\right) \sin\left(\frac{(q/2 - 1)\pi}{q}\right) \\ \times \left(\prod_{k=1}^{q-1} 2 \sin\left(\frac{\pi k}{q}\right)\right)^{n/2+1} \cdot \left(\prod_{k=1}^{q/2-1} 2 \sin\left(\frac{2\pi k}{q}\right)\right)^{-1/2}.$$

Using the identity $\prod_{k=1}^{r-1} 2 \sin(\pi k/r) = r$, the product reduces to

$$4q^{n/2+1}(q/2)^{-1/2} \sin\left(\frac{2\pi}{q}\right).$$

Similarly the product over Δ_+ reduces to $p^{n/2}$. Substituting into (15.4) yields $A = A_{3,n-2}^{L(1,1)}$ as required.

Next we consider the case $\mathfrak{g} = D_n, q = 2n - 3$. As above we conclude that $\mathcal{W} = \mathcal{W}_k(\mathfrak{g}, f_{\text{subreg}})$ is an extension of $\text{Vir}_{2,2n-3}$, and computation of asymptotic dimensions reveals $\mathcal{W} \cong \text{Vir}_{2,2n-3}$. For (\mathfrak{g}, q) one of the pairs $(E_7, 15), (E_8, 25)$ or $(E_8, 26)$ the proof is again the same.

Now we consider the case $\mathcal{W} = \mathcal{W}_{-h^\vee+12/11}(E_6, f_{\text{subreg}})$, which, by the same arguments as above, is an extension of $\text{Vir}_{3,22}$. The unique irreducible $\text{Vir}_{3,22}$ -module with conformal dimension $\Delta \in \mathbb{Z}_{\geq 1}$ is $L(21, 1)$ with $\Delta = 5$. Thus $\mathcal{W} \cong \text{Vir}_{3,22} \oplus L(21, 1)^{\oplus n}$ for some $n \in \mathbb{Z}_+$. Comparison of asymptotic dimensions reveals $n = 1$.

The final case $\mathfrak{g} = E_6, q = 10$ is quite similar to the last case. Analysis of central charges, asymptotic growths and asymptotic dimensions reveals that $\mathcal{W}_{-h^\vee+13/10}(E_6, f_{\text{subreg}})$ decomposes as $\text{Vir}_{5,6} \oplus L(5, 1)$. The same arguments imply that $\mathcal{W}_{-7/4}(A_2, f_{\text{prin}}) = \text{Vir}_{5,6} \oplus L(5, 1)$ (a fact well known in the physics literature [32, p. 227]). Now the module $L(5, 1)$ is a simple current, so by uniqueness of simple current extensions [35, Proposition 5.3] we obtain the claimed isomorphism. ■

16. New modular tensor categories

We have computed the fusion rules of $\mathcal{W}_{-h^\vee+p/q}(\mathfrak{g}, f_{\text{subreg}})$ in terms of the fusion rules of the affine vertex algebra $V_{p-h^\vee}(\mathfrak{g})$ and those of the vertex algebras listed in Table 3. We have identified most of these vertex algebras, the remaining cases being $E_6, p/q = 12/11$ and the five algebras with asymptotic growth greater than 1. We compute the fusion rings of these vertex algebras from their S -matrices. In this section we present in detail the cases $E_6, p/q = 12/11$ and $E_7, p/q = 19/16$.

A vertex algebra V is said to be *positive* if every irreducible V -module besides V itself has positive conformal dimension. We observe that the vertex algebras

$$U_6 = \mathcal{W}_{-h^\vee+15/11}(E_6, f_{\text{subreg}}) \quad \text{and} \quad U_7 = \mathcal{W}_{-h^\vee+21/16}(E_7, f_{\text{subreg}})$$

are positive. It is natural to expect the following.

Conjecture 16.1. *The vertex operator algebras U_6 and U_7 are unitary.*

We remark that the analogue $U_8 = \mathcal{W}_{-h^\vee + 32/25}(E_8, f_{\text{subreg}})$ is isomorphic to the unitary theory $\text{Vir}_{4,5}$. The MTCs associated with U_6 and U_7 , which we also believe to be unitary, appear to be new and interesting. We recall that the quantum dimension of an irreducible V -module L is $S_{V,L}/S_{V,V}$ where S is the S -matrix of V . By results of [33] the quantum dimensions of all irreducible U_6 - and U_7 -modules L must satisfy $\text{qdim}(L) \geq 1$. By Theorem 14.1 the S -matrices are

$$\begin{aligned} S^{(U_6)} &= S_{\text{sr}}^{15,11} = \varphi_3(S_{\text{sr}}^{12,11}) \otimes \varphi_{11}(K_{15}(E_6)), \\ S^{(U_7)} &= S_{\text{sr}}^{21,16} = \varphi_3(S_{\text{sr}}^{19,16}) \otimes \varphi_{16}(K_{21}(E_7)^{\text{int}}), \end{aligned} \tag{16.1}$$

and the relation $\text{qdim}(L) \geq 1$ is verified by direct computation of quantum dimensions from (16.1).

To compute the fusion rules of U_6 and U_7 it suffices to compute the fusion rules of $V_6 = \mathcal{W}_{-h^\vee + 12/11}(E_6, f_{\text{subreg}})$ and $V_7 = \mathcal{W}_{-h^\vee + 19/16}(E_7, f_{\text{subreg}})$.

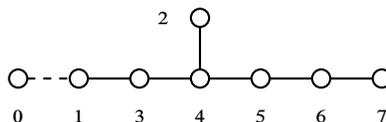
The vertex algebra V_6 is an extension of $V^0 = \text{Vir}_{3,22}$ by its simple current $M = L(21, 1)$. We note that $M \boxtimes L(r, 1) \cong L(22 - r, 1)$. The seven irreducible V_6 -modules are obtained as follows: the fusion product $V_6 \boxtimes_{V^0} L(r, 1)$ is an irreducible V_6 -module for $1 \leq r \leq 9$ odd, while for $r = 11$ it decomposes into two irreducible V_6 -modules consisting of symmetric resp. antisymmetric tensors. We also note that V_6 carries a $\mathbb{Z}/2$ -action by virtue of the decomposition $V_6 = V^0 \oplus M$, and for r even $V_6 \boxtimes_{V^0} L(r, 1)$ is an irreducible $\mathbb{Z}/2$ -twisted V -module.

The fusion rules of V_6 can be compactly described as follows. We write $[i] = V \boxtimes_{V^0} L(2i + 1, 1)$ for $0 \leq i \leq 5$ and $[5] = [5_+] \oplus [5_-]$. Then $[5_+] = [5_-]$ and all the other $[i]$ are self-dual. The conformal dimensions are given by $\Delta([i]) = i(3i - 19)/22$. For $0 \leq i, j \leq 4$ we have $[i] \boxtimes [j] = \bigoplus_{|i-j| \leq k \leq i+j} [k]$, where we identify $[5 + \epsilon]$ with $[5 - \epsilon]$. Also $[5_\pm] \boxtimes [i] = [5_{\pm(-1)^i}] + \bigoplus_{5-i \leq k \leq 4} [k]$. Finally, $[5_\pm] \boxtimes [5_\pm] = [1] \oplus [3] \oplus [5_\mp]$ and $[5_+] \boxtimes [5_-] = [0] \oplus [2] \oplus [4]$.

The quantum dimensions of the irreducible V_6 -modules lie in the cyclotomic field $\mathbb{Q}(\zeta)$ of degree 11 and are given explicitly as

$$\begin{aligned} \text{qdim}([0]) &= 1, & \text{qdim}([3]) &= \zeta^9 + 2\zeta^8 + \zeta^6 + \zeta^5 + 2\zeta^3 + \zeta^2 + 2, \\ \text{qdim}([1]) &= -\zeta^7 - \zeta^4 + 1, & \text{qdim}([4]) &= \zeta^9 + 2\zeta^8 + 2\zeta^6 + 2\zeta^5 + 2\zeta^3 + \zeta^2 + 2, \\ \text{qdim}([2]) &= \zeta^8 - \zeta^7 - \zeta^4 + \zeta^3 + 1, & \text{qdim}([5_\pm]) &= \zeta^8 + \zeta^6 + \zeta^5 + \zeta^3 + 1. \end{aligned}$$

We now examine the vertex algebra V_7 . Since the asymptotic growth of V_7 is $9/8 > 1$, it is not a finite extension of a Virasoro minimal model. The central charge is $c = -135/8$ and it has 13 irreducible modules, parametrised by $\check{P}_{+, \text{subreg}}^{16}$. We denote by σ the nontrivial diagram automorphism of the Dynkin diagram of E_7 .



$$F_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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