



Matija Bucić · Benny Sudakov · Tuan Tran

# Erdős–Szekeres theorem for multidimensional arrays

Received November 5, 2019; revised February 2, 2022

**Abstract.** The classical Erdős–Szekeres theorem dating back almost a hundred years states that any sequence of  $(n - 1)^2 + 1$  distinct real numbers contains a monotone subsequence of length  $n$ . This theorem has been generalised to higher dimensions in a variety of ways but perhaps the most natural one was proposed by Fishburn and Graham more than 25 years ago. They defined the concept of a monotone and a lex-monotone array and asked how large an array one needs in order to be able to find a monotone or a lex-monotone subarray of size  $n \times \cdots \times n$ . Fishburn and Graham obtained Ackerman-type bounds in both cases. We significantly improve these results. Regardless of the dimension we obtain at most a triple exponential bound in  $n$  in the monotone case and a quadruple exponential one in the lex-monotone case.

**Keywords.** Erdős–Szekeres theorem, high-dimensional permutations, monotone arrays, Ramsey theory

---

## 1. Introduction

A classical paper of Erdős and Szekeres [14] from 1935 is one of the starting points of a very rich discipline within combinatorics: Ramsey theory. The main result of the paper, which has become known as the Erdős–Szekeres theorem, says that any sequence of  $(n - 1)^2 + 1$  distinct real numbers contains either an increasing or a decreasing subsequence of length  $n$ , and this is tight. Among simple results in combinatorics, only few can compete with this one in terms of beauty and utility. See, for example, Steele [29] for a collection of six proofs and some applications.

A very natural question which arises is how does one generalise the Erdős–Szekeres theorem to higher dimensions? The main concept which does not have an obvious generalisation is that of the monotonicity of a subsequence. Several candidates have been

---

Matija Bucić: School of Mathematics, Institute for Advanced Study and Department of Mathematics, Princeton University, 304 Washington Road, Princeton, NJ 08544, USA; [matija.bucic@ias.edu](mailto:matija.bucic@ias.edu)

Benny Sudakov: Department of Mathematics, ETH, Zürich, Switzerland; [benjamin.sudakov@math.ethz.ch](mailto:benjamin.sudakov@math.ethz.ch)

Tuan Tran: School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China; [trantuan@ustc.edu.cn](mailto:trantuan@ustc.edu.cn)

*Mathematics Subject Classification (2020):* Primary 05D10; Secondary 05C55, 05A05

proposed [8, 9, 20–23, 28, 30] but perhaps the most natural one was introduced more than 25 years ago by Fishburn and Graham [16]. A multidimensional array is said to be *monotone* if for each dimension all the 1-dimensional subarrays along the direction of this dimension are increasing or are all decreasing. To be more formal, a  $d$ -dimensional array  $f$  is an injective function from  $A_1 \times \dots \times A_d$  to  $\mathbb{R}$  where  $A_1, \dots, A_d$  are non-empty subsets of  $\mathbb{Z}$ ; we say  $f$  has size  $|A_1| \times \dots \times |A_d|$ .

**Definition** (Monotone array). A  $d$ -dimensional array  $f: A_1 \times \dots \times A_d \rightarrow \mathbb{R}$  is *monotone* if for each  $i \in [d]$  one of the following alternatives occurs:

- (i)  $f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_d)$  is increasing in  $x$  for all choices of  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d$ ;
- (ii)  $f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_d)$  is decreasing in  $x$  for all choices of  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d$ .

For example, of the following 2-dimensional arrays the first and second are monotone, while the third is not (since some rows contain increasing and some rows decreasing sequences).

7	8	9
4	5	6
1	2	3

1	3	6
2	5	7
4	8	9

7	8	9
6	5	4
1	2	3

The higher-dimensional version of the Erdős–Szekeres problem introduced by Fishburn and Graham [16] now becomes: given positive integers  $d$  and  $n$ , determine the smallest  $N$  such that any  $d$ -dimensional array of size  $N \times \dots \times N$  contains a monotone  $d$ -dimensional subarray of size  $n \times \dots \times n$ ; we denote this  $N$  by  $M_d(n)$ . The Erdős–Szekeres theorem can now be rephrased as  $M_1(n) = (n - 1)^2 + 1$ . Fishburn and Graham [16, Section 3] showed that  $M_2(n) \leq \text{towr}_5(O(n))$ ,<sup>1</sup> that  $M_3(n)$  is bounded by a tower of height at least a tower in  $n$ , and that  $M_d(n)$  is bounded from above by an Ackermann-type<sup>2</sup> function of order at least  $d$  for  $d \geq 4$ . We significantly improve upon these results.

**Theorem 1.1.**

- (i)  $M_2(n) \leq 2^{2^{(2+o(1))n}}$ ,
- (ii)  $M_3(n) \leq 2^{2^{(2+o(1))n^2}}$  and
- (iii)  $M_d(n) \leq 2^{2^{2^{O_d(n^{d-1})}}}$  for  $d \geq 4$ .

Fishburn and Graham introduced another very natural generalisation of the notion of monotonicity of a sequence to higher-dimensional arrays. A multidimensional array

---

<sup>1</sup>We define the tower function  $\text{towr}_k(x)$  by  $\text{towr}_1(x) = x$  and  $\text{towr}_k(x) = 2^{\text{towr}_{k-1}(x)}$  for  $k \geq 2$ .  
<sup>2</sup>The Ackermann function  $A_k$  of order  $k$  is defined recursively by  $A_k(1) = 2$ ,  $A_1(n) = 2n$  and  $A_k(n) = A_{k-1}(A_k(n - 1))$ . It is an incredibly fast growing function, for example  $A_2(n) = 2^n$ ,  $A_3(n) = \text{towr}_n(2)$  and  $A_4(n)$  is a tower of height tower of height tower, iterated  $n$  times, of 2.

is said to be *lexicographic* if for any two entries the one which has the larger position in the first coordinate in which they differ is larger. For example, the following array is lexicographic:

3	6	9
2	5	8
1	4	7

An array is said to be *lex-monotone* if it is possible to permute the coordinates and reflect the array along some dimensions to obtain a lexicographic array. To be more formal, for two vectors  $\mathbf{u} = (u_1, \dots, u_d)$  and  $\mathbf{v} = (v_1, \dots, v_d)$  in  $\mathbb{R}^d$ , we write  $\mathbf{u} <_{\text{lex}} \mathbf{v}$  if  $u_i < v_i$ , where  $i$  is the smallest index such that  $u_i \neq v_i$ .

**Definition** (Lex-monotone array). A  $d$ -dimensional array  $f$  is said to be *lex-monotone* if there exist a permutation  $\sigma: [d] \rightarrow [d]$  and a sign vector  $s \in \{-1, 1\}^d$  such that

$$f(\mathbf{x}) < f(\mathbf{y}) \iff (s_{\sigma(1)}x_{\sigma(1)}, \dots, s_{\sigma(d)}x_{\sigma(d)}) <_{\text{lex}} (s_{\sigma(1)}y_{\sigma(1)}, \dots, s_{\sigma(d)}y_{\sigma(d)}).$$

Note that a 1-dimensional array is lex-monotone if and only if it is a monotone sequence. The following 2-dimensional arrays are lex-monotone since for the first one the above matrix is obtained by swapping the coordinates, for the second one by reflecting along the first dimension and for the third by performing both of these operations.

7	8	9
4	5	6
1	2	3

9	6	3
8	5	2
7	4	1

9	8	7
6	5	4
3	2	1

Given positive integers  $d$  and  $n$ , let  $L_d(n)$  denote the minimum  $N$  such that for any  $d$ -dimensional array of size  $N \times \dots \times N$ , one can find a lex-monotone subarray of size  $n \times \dots \times n$ . Fishburn and Graham [16, Theorem 1] showed that  $L_d(n)$  exists. This result has been used to prove interesting results in poset dimension theory [15] and computational complexity theory [4].

Note that any lex-monotone array is monotone, so a very natural strategy to bound  $L_d(n)$  is to first find a monotone subarray and then within this subarray find a lex-monotone subarray. This motivates the following problem which is of independent interest. For positive integers  $d$  and  $n$ , we define  $F_d(n)$  to be the minimum  $N$  such that any  $d$ -dimensional monotone array of size  $N \times \dots \times N$  contains a lex-monotone subarray of size  $n \times \dots \times n$ . It is easy to see by the above reasoning that  $L_d(n) \leq M_d(F_d(n))$ . Fishburn and Graham [16, Lemma 1] showed  $F_2(n) \leq 2n^2 - 5n + 4$  and  $F_3(n) \leq 2^{2n+o(n)}$ , while for  $d \geq 4$  their argument gives  $F_d(n) \leq \text{towr}_{d-1}(O_d(n))$ . We determine  $F_2(n)$  completely and significantly improve the bound for all  $d \geq 3$ .

**Theorem 1.2.**

- (i)  $F_2(n) = 2n^2 - 5n + 4$  and
- (ii)  $F_d(n) \leq 2^{O_d(n^{d-2})}$  for  $d \geq 3$ .

Part (i) of Theorem 1.2 answers in the negative a question of Fishburn and Graham asking whether  $F_2(n) = (1 + o(1))n^2$ . Combining Theorems 1.1 and 1.2 with the inequality  $L_d(n) \leq M_d(F_d(n))$  gives the following upper bounds on  $L_d(n)$ .

**Theorem 1.3.**

- (i)  $L_2(n) \leq 2^{2^{(4+o(1))n^2}}$ ,
- (ii)  $L_3(n) \leq 2^{2^{2^{(2+o(1))n}}}$  and
- (iii)  $L_d(n) \leq \text{towr}_5(O_d(n^{d-2}))$  for  $d \geq 4$ .

For comparison, the best lower bound on  $L_d(n)$ , due to Fishburn and Graham [16, Theorem 2], is  $L_d(n) \geq n^{(1-1/d)n^{d-1}}$  for all  $d \geq 2, n \geq 3$ , achieved by taking a random array.

**Notation and organisation.** The rest of the paper is organised as follows. We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3. The final section contains some concluding remarks and open problems.

We use standard set-theoretic and asymptotic notation throughout the paper. We write  $[n]$  for  $\{1, \dots, n\}$ . For sequences  $a(n)$  and  $b(n)$  we write  $a(n) = O(b(n))$  to mean there is a constant  $C$  such that  $|a(n)| \leq C|b(n)|$  for all  $n \in \mathbb{N}$ , and we write  $a(n) = o(b(n))$  to mean that  $a(n)/b(n) \rightarrow 0$  as  $n \rightarrow \infty$ . For the purpose of asymptotics we always treat  $d$ , the number of dimensions, as a constant while taking  $n \rightarrow \infty$ . Given  $d \in \mathbb{N}$ , we denote the set of all permutations of  $[d]$  by  $\mathfrak{S}_d$ . For real numbers  $\alpha$  and  $\beta$ , we employ the interval notation

$$[\alpha, \beta] := \{x \in \mathbb{Z} : \alpha \leq x \leq \beta\}.$$

A set of the form  $A_1 \times \dots \times A_d$ , where  $A_i$  is a finite subset of  $\mathbb{Z}$  for each  $i \in [d]$ , is called an  $|A_1| \times \dots \times |A_d|$  grid or a grid of size  $|A_1| \times \dots \times |A_d|$ . Note that a  $d$ -dimensional array  $f: A_1 \times \dots \times A_d \rightarrow \mathbb{R}$  is equivalent to an ordering of the vertices of the  $d$ -dimensional grid  $A_1 \times \dots \times A_d$ ; we switch between these points of view depending on which is more suitable for the topic at hand.

We generally use lowercase boldface letters for vectors and uppercase boldface letters for grids. For a vector  $\mathbf{u}$  we denote by  $u_i$  the value of the  $i$ -th coordinate.

**2. Monotone arrays**

In this section we will prove Theorem 1.1. We begin with a few preliminaries.

*2.1. Preliminaries*

We collect here two well-known Ramsey-type results for grids. We mention first a relation between the grid Ramsey problem and the (hyper)graph Zarankiewicz problem which offers an alternative perspective. Given  $d, n_1, \dots, n_d \in \mathbb{N}$ , let  $K_{n_1, \dots, n_d}^{(d)}$  denote the complete  $d$ -uniform  $d$ -partite hypergraph with parts  $V_1 = [n_1], \dots, V_d = [n_d]$ . Edges

of  $K_{n_1, \dots, n_d}^{(d)}$  correspond in the obvious way to vertices of the  $d$ -dimensional grid  $[n_1] \times \dots \times [n_d]$ . Subhypergraphs of  $K_{n_1, \dots, n_d}^{(d)}$  of the form  $K_{m_1, \dots, m_d}^{(d)}$  then correspond to subgrids of  $[n_1] \times \dots \times [n_d]$  of size  $m_1 \times \dots \times m_d$ . Using this correspondence, the two lemmas below follow, for example, from [18, Theorem 2] and [12, Theorem 4], respectively.

**Lemma 2.1.** *Given  $k, n, t \in \mathbb{N}$  in any vertex  $k$ -colouring of an  $tk \binom{nk}{n} \times kn$  grid there is a monochromatic subgrid of size  $t \times n$ .*

In the higher-dimensional case we will not need an asymmetric variant.

**Lemma 2.2.** *Given integers  $d, k \geq 2$  there exists a positive constant  $C = C(d, k)$  such that for any positive integers  $n$  and  $N$  with  $N \geq 2^{Cn^{d-1}}$ , in any  $k$ -colouring of the  $d$ -dimensional  $N \times \dots \times N$  grid there is a monochromatic subgrid of size  $n \times \dots \times n$ .*

### 2.2. Proofs of the main results on monotonicity

We begin with the 2-dimensional case. We will actually prove a stronger version of Theorem 1.1 (i) as we will need it for the 3-dimensional case.

**Theorem 2.3.** *For every  $n, t \in \mathbb{N}$ , any  $4n^2 \times (2t)^{2^{2n}}$  array contains an  $n \times t$  monotone subarray.*

*Proof.* Let  $f$  be an array indexed by  $[N] \times [M]$ , where  $N = 4n^2$  and  $M = (2t)^{2^{2n}}$ . By the Erdős–Szekeres theorem we know that in each column of  $f$  there is a monotone subsequence of length  $2n$ . The entries of this subsequence can appear in  $\binom{4n^2}{2n}$  different positions so there must be a set  $R \subseteq [N]$  of  $2n$  positions for which at least  $M / \binom{4n^2}{2n}$  columns are monotone when restricted to (rows)  $R$ . We take  $C \subseteq [N]$  to be the subset of these columns for which the restriction is increasing; without loss of generality we may assume that  $C$  consists of at least half of these columns. We obtain a subarray  $f' = f|_{R \times C}$  which is increasing in each column and has size  $2n \times M'$  where  $M' \geq \frac{M}{2 \binom{4n^2}{2n}} \geq t^{2^{2n}}$ .

By applying the Erdős–Szekeres theorem to the sequence given by the first row of  $f'$ , we can find a subset  $C_1 \subseteq C$  of size  $|C_1| \geq \sqrt{|C|}$  such that the first row of  $f'|_{R \times C_1}$  is monotone. Repeating this argument at step  $i$  we find a subset  $C_i \subseteq C_{i-1}$  of size  $|C_i| \geq \sqrt{|C_{i-1}|} \geq |C|^{1/2^i}$  such that the first  $i$  rows of  $f'|_{R \times C_i}$  are monotone. Continuing this process until  $i = 2n$  we obtain a  $2n \times t$  array with each row being either increasing or decreasing. By taking the ones of the type which appears more often we obtain a monotone  $n \times t$  subarray as claimed. ■

One can easily generalise the above proof to give a bound of the form  $M_d(n) \leq \text{towr}_{d+1}(O(n^{d-1}))$  for any  $d \geq 2$ , which would already be a substantial improvement over the Ackermann bound due to Fishburn and Graham [16]. However, to prove the desired significantly better bound  $M_d(n) \leq \text{towr}_4(O(n^{d-1}))$  for  $d \geq 4$ , we need to consider an intermediate problem which we find interesting in its own right.

**Definition** (Inconsistently monotone array). An array  $f: A_1 \times \dots \times A_d \rightarrow \mathbb{R}$  is *inconsistently monotone* if for each  $i \in [d]$ ,  $f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_d)$  is monotone in  $x$  for all choices of  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_d$ .

The main difference compared with the definition of a monotone array is that we do not require the lines along a fixed dimension to be all increasing or all decreasing but allow some to be increasing and some to be decreasing. For positive integers  $d$  and  $n$ , let  $M'_d(n)$  denote the minimum  $N$  such that for any  $d$ -dimensional array of size  $N \times \dots \times N$ , one can find a  $d$ -dimensional inconsistently monotone subarray of size  $n \times \dots \times n$ . We have  $M'_1(n) = (n - 1)^2 + 1$  according to the Erdős–Szekeres theorem. When  $d \geq 2$  we obtain the following version of Theorem 1.1 which gives stronger bounds but only guarantees us an inconsistently monotone array, a weaker notion compared to actual monotone arrays.

**Theorem 2.4.** *For every  $d \geq 2$ , we have  $M'_d(n) \leq 2^{2^{(1+o(1))n^{d-1}}}$ .*

*Proof.* We will prove the following recursive bound:

$$M'_d(n) \leq \binom{M'_{d-1}(n)}{n}^{d-1} n^{2^{n^{d-1}}}. \tag{1}$$

Let  $m = M'_{d-1}(n)$  and  $N = \binom{m}{n}^{d-1} n^{2^{n^{d-1}}}$ . To prove (1), let  $f$  be an array indexed by  $[m]^{d-1} \times [N]$ . For each “height”  $h \in [N]$ , consider the restriction of  $f$  to  $[m]^{d-1} \times \{h\}$ . As  $m = M'_{d-1}(n)$ , there exist an  $n \times \dots \times n$  subgrid  $S_h$  of  $[m]^{d-1}$  such that  $f$  is inconsistently monotone on  $S_h \times \{h\}$ . Given  $h \in [N]$ , there are at most  $\binom{m}{n}^{d-1}$  possibilities for the location of  $S_h$ . Hence, by the pigeonhole principle, we can find an  $n \times \dots \times n$  subgrid  $S$  of  $[m]^{d-1}$  and a subset  $H \subset [N]$  of size

$$|H| \geq \frac{N}{\binom{m}{n}^{d-1}} = n^{2^{n^{d-1}}}$$

such that  $f$  is inconsistently monotone on  $S \times \{h\}$  for every  $h \in H$ . Let us denote the elements of  $S$  by  $s_1, \dots, s_{n^{d-1}}$ . By the Erdős–Szekeres theorem, we can construct a nested sequence  $H_0 := H \supseteq H_1 \supseteq \dots \supseteq H_{n^{d-1}}$  such that  $|H_i| \geq \sqrt{|H_{i-1}|}$  for every  $i \geq 1$ , and  $\{s_j\} \times H_i$  is monotone for  $j = 1, \dots, i$ . In particular,  $|H_{n^{d-1}}| \geq |H|^{1/2^{n^{d-1}}} \geq n$ , and the restriction of  $f$  to  $S \times H_{n^{d-1}}$  is inconsistently monotone. This completes the proof of (1).

It remains to show that (1) implies the desired bound  $M'_d(n) \leq \text{towr}_3((1 + o(1))n^{d-1})$ . We proceed by induction on  $d$ , noting that the case  $d = 2$  follows from (1) and the fact that  $M'_1(n) = (n - 1)^2 + 1$ . For the induction step, in the case  $d \geq 3$ , using (1) and the induction hypothesis we find

$$\begin{aligned} M'_d(n) &\leq M'_{d-1}(n)^{(d-1)n} \cdot n^{2^{n^{d-1}}} \leq \text{towr}_3(O(n^{d-2})) \cdot \text{towr}_3((1 + o(1))n^{d-1}) \\ &= \text{towr}_3((1 + o(1))n^{d-1}), \end{aligned}$$

finishing the proof. ■

The following definition is going to help us find a monotone array inside an inconsistently monotone array.

**Definition 2.5** (Monotonicity pattern). Let  $f: A \rightarrow \mathbb{R}$  be an inconsistently monotone  $d$ -dimensional array. Let  $\mathbf{a} = (a_1, \dots, a_d) \in A$ . For each  $i \in [d]$ , let  $s_i \in \{-1, 1\}$  be such that  $s_i f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_d)$  is increasing in  $x$ . The vector  $\mathbf{s} = (s_1, \dots, s_d)$  is called the *monotonicity pattern* of  $f$  at  $\mathbf{a}$ .

Notice that if  $f$  is a monotone array then  $f$  has the same monotonicity pattern at all points, in which case we just call it the monotonicity pattern of  $f$ . We now use Theorem 2.4 to prove part (iii) of Theorem 1.1.

**Theorem 2.6.** For every  $d \geq 4$ , we have  $M_d(n) \leq \text{towr}_4(O(n^{d-1}))$ .

*Proof.* Let  $N = \text{towr}_4(O(n^{d-1}))$ , and let  $C = C(d, 2^d)$  be the positive constant given by Lemma 2.2. It follows from Theorem 2.4 that in any  $d$ -dimensional array of size  $N \times \dots \times N$ , one can find an inconsistently monotone subarray  $f$  indexed by  $A = A_1 \times \dots \times A_d$  such that  $|A_1| = \dots = |A_d| = 2^{Cn^{d-1}}$ .

Let us colour every point in  $A$  with the monotonicity pattern of  $f$  at this point. This gives us a vertex-colouring of  $A$  with  $2^d$  colours given by  $\{-1, 1\}^d$ . By Lemma 2.2 and the choice of  $C$ , there exists a monochromatic  $n \times \dots \times n$  subgrid  $B$  of  $A$  with colour  $(s_1, \dots, s_d)$ . From the definition of the monotonicity pattern  $(s_1, \dots, s_d)$ , we can see that  $f|_B$  is monotone. ■

In order to prove  $M_3(n) \leq 2^{2^{(2+o(1))n^2}}$ , we devise a different argument, not going through the intermediate problem of bounding  $M'_d(n)$ .

**Theorem 2.7.** We have  $M_3(n) \leq 2^{2^{(2+o(1))n^2}}$ .

*Proof.* Let  $X_1 = [16n^2]$ ,  $X_2 = [2^{2^{6n}}]$  and  $X_3 = [2^{2^{(2+o(1))n^2}}]$ . To prove Theorem 1.1 (ii) it suffices to show that any 3-dimensional array  $f$  indexed by  $X_1 \times X_2 \times X_3$  contains an  $n \times n \times n$  monotone subarray.

For each “height”  $h \in X_3$ , let  $C_h = X_1 \times X_2 \times \{h\}$ . As  $2^{2^{6n}} \geq (2n2^{2n})^{2^{4n}}$ , Theorem 2.3 implies that each  $C_h$  contains a  $2n \times n2^{2n} \times 1$  monotone subarray. There are  $\binom{16n^2}{2n} \binom{2^{2^{6n}}}{n2^{2n}}$  different possibilities for a  $2n \times n2^{2n} \times 1$  monotone subarray, and the monotonicity pattern of each such subarray is a vector  $\mathbf{s} \in \{-1, 1\}^2$ . Since  $4 \binom{16n^2}{2n} \binom{2^{2^{6n}}}{n2^{2n}} = 2^{2^{o(n^2)}}$ , by the pigeonhole principle we can find a vector  $\mathbf{s} \in \{-1, 1\}^2$  and three subsets  $S_1 \subseteq X_1, S_2 \subseteq X_2, S_3 \subseteq X_3$  with  $|S_1| = 2n, |S_2| = n2^{2n}$  and  $|S_3| = 2^{2^{(2+o(1))n^2}}$  such that for any  $h \in S_3$  the array  $f|_{S_1 \times S_2 \times \{h\}}$  is monotone with pattern  $\mathbf{s}$ . Our remaining goal is to find an  $n \times n$  subgrid of  $S_1 \times S_2$  such that for any pair  $(a_1, a_2)$  of this subgrid,  $f(a_1, a_2, \cdot)$  is always increasing or always decreasing on some fixed subset of size  $n$  of  $S_3$ .

For each  $h \in S_3$ , let  $L_h = S_1 \times S_2 \times \{h\}$ . We can think of  $L_h$ ’s as “layers” stacked one on top of each other. Given two layers  $L_h$  and  $L_{h'}$  with  $h < h'$ , we colour an element  $\mathbf{v} \in S_1 \times S_2$  red if  $f(\mathbf{v}, h) > f(\mathbf{v}, h')$ , and blue otherwise. This way we obtain a colouring of  $S_1 \times S_2$  with two colours, so by Lemma 2.1 we can find a monochromatic subgrid  $B_{hh'}$  of size  $n \times n$ . We now consider the following edge-colouring of the complete graph on the vertex set  $S_3$  using  $k$  colours. We colour the edge between  $h$  and  $h'$  by a pair made of  $B_{hh'}$

and its monochromatic colour. Since there are at most  $\binom{2n}{n} \binom{n2^{2n}}{n}$  possibilities for  $\mathbf{B}_{hh'}$ , we must have  $k \leq 2 \binom{2n}{n} \binom{n2^{2n}}{n} = 2^{(2+o(1))n^2}$ , giving  $k^{kn} \leq 2^{2^{(2+o(1))n^2}} = |S_3|$ . From this and a result of Erdős and Rado [13, Theorem 1] on the multicolour Ramsey numbers which states that any  $k$ -edge colouring of the complete graph on  $k^{kn}$  vertices contains a monochromatic  $K_n$ , we deduce that our colouring contains a monochromatic  $K_n$ . Let  $H \subseteq S_3$  correspond to the vertices of this  $K_n$  and let its colour correspond to an  $n \times n$  subgrid  $\mathbf{B}$  and say, without loss of generality, be blue. This means that  $f(a_1, a_2, \cdot) : H \rightarrow \mathbb{R}$  is increasing for all  $(a_1, a_2) \in \mathbf{B}$ . So by our construction of  $S_1, S_2$  we find that  $f$  when restricted to  $\mathbf{B} \times H$  is a monotone array of size  $n \times n \times n$ . ■

**Remark.** In the above proof we used the usual Ramsey theorem on our colouring of the complete graph on  $S_3$ . However, our colouring is not arbitrary and in fact one can instead use the ordered Ramsey number of a path (see [11]). The third alternative is to only colour an edge according to  $\mathbf{B}_{hh'}$ , and record whether the values increase or decrease between  $\mathbf{B}_{hh'} \times \{h\}$  and  $\mathbf{B}_{hh'} \times \{h'\}$  by a directed edge. This gives us a colouring of a tournament in which we want to find a monochromatic directed path (see [10, 19]). Both approaches give slightly better bounds than the one in Theorem 1.1 (ii), but unfortunately still give bounds of the form  $M_3(n) \leq \text{towr}_3(O(n^2))$ .

### 3. Lexicographic arrays

In this section we show our bounds on  $F_d(n)$ , in particular we prove Theorem 1.2.

#### 3.1. Preliminaries

A monotone array  $f$  is said to be *increasing* if restriction of  $f$  to any axis parallel line is an increasing sequence (i.e. case (i) of the definition of monotonicity always occurs). More formally:

**Definition 3.1.** A  $d$ -dimensional array  $f$  is *increasing* if  $f(\mathbf{x}) \leq f(\mathbf{y})$  whenever  $x_i \leq y_i$  for all  $i \in [d]$ .

The following definition generalises the notion of a lexicographic array to allow for a custom priority order of coordinates.

**Definition 3.2.** Given a  $d$ -dimensional array  $f$  and a permutation  $\sigma \in \mathfrak{S}_d$ , we say  $f$  is (*lexicographic*) of *type*  $\sigma$  if  $f(\mathbf{x}) < f(\mathbf{y}) \Leftrightarrow (x_{\sigma(1)}, \dots, x_{\sigma(d)}) <_{\text{lex}} (y_{\sigma(1)}, \dots, y_{\sigma(d)})$  for all possible  $\mathbf{x}$  and  $\mathbf{y}$ .

Recall that an array is said to be *lex-monotone* if it is possible to permute the coordinates and reflect the array along some dimensions to obtain a lexicographic array. The above definition allows us to separate these two actions. In particular, an alternative definition is that an array is *lex-monotone* if one can reflect the array along some dimensions to obtain a lexicographic array of some type.

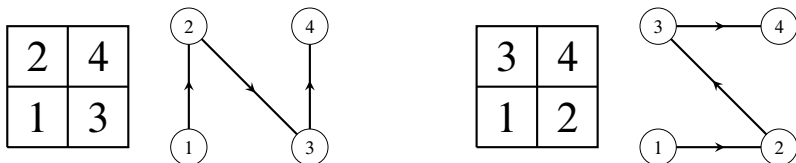


Notice that any subarray of a monotone array is itself monotone and moreover has the same monotonicity pattern. This means that when looking for a lex-monotone subarray within a monotone array we can only find one with the same monotonicity pattern. In other words, we may without loss of generality assume that the starting array is increasing. The following immediate lemma makes this statement formal.

**Lemma 3.3.** *For all  $d, n \in \mathbb{N}$ ,  $F_d(n)$  equals the minimum  $N$  such that any increasing  $d$ -dimensional array of size  $N \times \dots \times N$  contains an  $n \times \dots \times n$  subarray of type  $\sigma$  for some  $\sigma \in \mathfrak{S}_d$ .*

### 3.2. 2-dimensional case

Notice first that in two dimensions there are only two possible types of a (lexicographic) array, namely (1, 2) and (2, 1). See Figure 1 for an illustration of both together with an example of the arrow notation which we found useful when thinking about the problem.



**Fig. 1.** Lexicographic arrays of type (1, 2) and (2, 1); arrows point towards larger points.

We begin with a proof of the upper bound  $F_2(n) \leq 2n^2 - 5n + 4$  as it sheds some light on where our lower bound construction is coming from.

**Theorem 3.4** (Fishburn and Graham [16]). *For  $n \in \mathbb{N}$  we have  $F_2(n) \leq 2n^2 - 5n + 4$ .*

*Proof.* Let  $f$  be an increasing array indexed by  $[N] \times [N]$ , where  $N = (n - 1)(2n - 3) + 1$ . For  $i \in [2n - 2]$ , let  $a_i = (n - 1)(i - 1) + 1$ . Define a red-blue colouring of the grid  $\{a_1, \dots, a_{2n-3}\} \times \{a_1, \dots, a_{2n-3}\}$  as follows. For every  $i, j \in [2n - 3]$ , we colour  $(a_i, a_j)$  red if  $f(a_{i+1}, a_j) < f(a_i, a_{j+1})$ , and blue otherwise. As  $(n - 2)(2n - 3) + (n - 2)(2n - 3) < (2n - 3)^2$ , there exists a row with at least  $n - 1$  red points or a column with at least  $n - 1$  blue points. By symmetry, we can assume  $(a_i, a_{j_1}), \dots, (a_i, a_{j_{n-1}})$  are  $n - 1$  red points in a row  $a_i$  with  $j_1 < \dots < j_{n-1}$ . One can check that the  $n \times n$  subarray of  $f$  indexed by  $[a_i, a_{i+1}] \times \{a_{j_1}, \dots, a_{j_{n-1}}, a_{j_{n-1}+1}\}$  is of type (1, 2). Hence  $F_2(n) \leq N = 2n^2 - 5n + 4$ , as required. ■

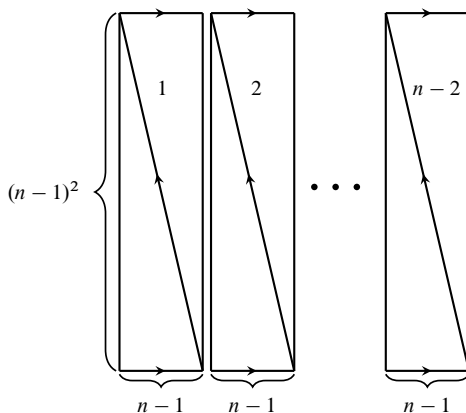
The remainder of this subsection is devoted to the proof of the lower bound  $F_2(d) \geq 2n^2 - 5n + 4$ . We will make ample use of the immediate observation that any subarray of a lexicographic array of type  $\sigma$  which has size at least 2 in each dimension must also be of type  $\sigma$ . We first construct a “building block” for our actual construction showing  $F_2(d) \geq 2n^2 - 5n + 4$ .

**Lemma 3.5.** For  $n \geq 3$ , there exists an increasing array  $g$  of size  $(n - 1)(n - 2) \times (n - 1)^2$  such that

- (G1)  $g$  does not contain an  $(n - 1) \times 2$  subarray of type  $(1, 2)$ ,
- (G2)  $g$  does not contain an  $n \times 2$  subarray of type  $(2, 1)$ .

*Proof.* For  $1 \leq i \leq n - 2$ , let  $C_i = [(n - 1)(i - 1) + 1, (n - 1)i] \times [(n - 1)^2]$ . We choose an array  $g$  (see Figure 2 for an illustration), indexed by  $[(n - 1)(n - 2)] \times [(n - 1)^2]$ , such that

- $g|_{C_1} < \dots < g|_{C_{n-2}}$ ,
- for each  $i \in [n - 2]$ ,  $g|_{C_i}$  is of type  $(2, 1)$ .



**Fig. 2.** An illustration of the array  $g$ . Directed arrows point towards a position with a larger value of  $g$ . Numbers denote relative order of subarrays.

For (G1), if such a subarray exists, then at least one of  $C_i$ 's would need to intersect this subarray in some  $2 \times 2$  subarray. By the second property of  $g$ , the  $2 \times 2$  subarray is of type  $(2, 1)$ , a contradiction.

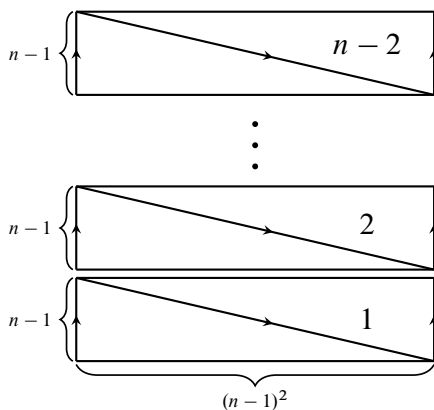
For (G2), if such a subarray exists it would intersect at least two distinct  $C_i$ 's, and so it would contain a  $2 \times 2$  subarray of type  $(1, 2)$ , a contradiction. ■

Another building block of our construction is the following.

**Lemma 3.6.** For  $n \geq 3$ , there is an increasing array  $h$  of size  $(n - 1)^2 \times (n - 1)(n - 2)$  such that

- (H1)  $h$  does not contain a  $2 \times n$  subarray of type  $(1, 2)$ ,
- (H2)  $h$  does not contain a  $2 \times (n - 1)$  subarray of type  $(2, 1)$ .

*Proof.* Let  $R_i = [(n - 1)^2] \times [(n - 1)i - n + 1, (n - 1)i]$  for  $i \in [n - 2]$ . Let us define  $h$  to be an array indexed by  $[(n - 1)^2] \times [(n - 1)(n - 2)]$  so that  $h|_{R_i} < h|_{R_j}$  whenever



**Fig. 3.** An illustration of the array  $h$ . Directed arrows point towards a position with a larger value of  $h$ . Numbers denote relative order of subarrays.

$i < j$  and so that  $h|_{\mathbb{R}_i}$  is of type (1, 2) (see Figure 3 for an illustration). This array satisfies the properties (H1) and (H2) by the same argument as in Lemma 3.5. ■

We are now in a position to prove Theorem 1.2 (i).

**Theorem 3.7.** For  $n \in \mathbb{N}$  we have  $F_2(n) \geq 2n^2 - 5n + 4$ .

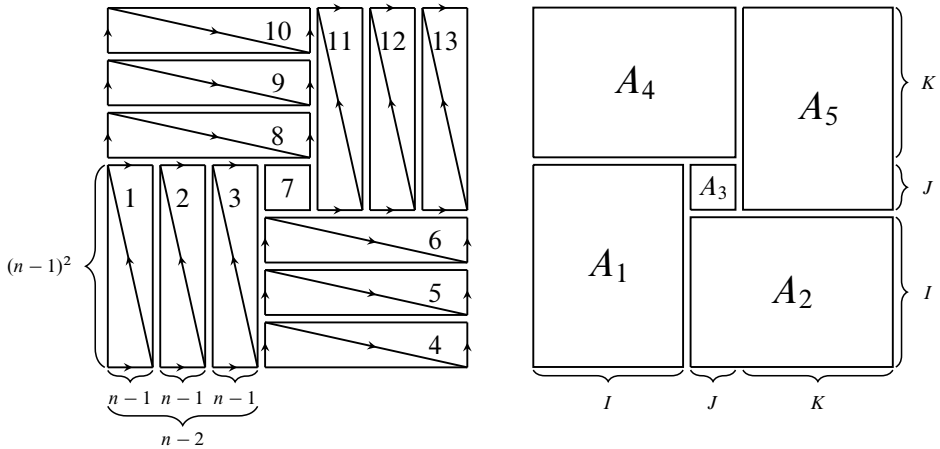
*Proof.* It is immediate that the statement holds for  $n = 1, 2$ . We henceforth assume that  $n \geq 3$ .

Let  $N = 2n^2 - 5n + 3 = (n - 1)^2 + (n - 1)(n - 2)$ . To prove the statement, it suffices to construct an increasing array  $f : [N]^2 \rightarrow \mathbb{R}$  which does not contain an  $n \times n$  subgrid of type (1, 2) or (2, 1).

We first split  $[N]^2$  into five subgrids  $A_1, \dots, A_5$  (see Figure 4) such that both  $A_1$  and  $A_5$  have size  $(n - 1)(n - 2) \times (n - 1)^2$ , both  $A_2$  and  $A_4$  have size  $(n - 1)^2 \times (n - 1)(n - 2)$ , while  $A_3$  has size  $(n - 1) \times (n - 1)$ . Let  $g$  and  $h$  be arrays given by Lemmas 3.5 and 3.6, respectively. The array  $f$  is chosen so that  $f|_{A_1} < \dots < f|_{A_5}$ ,  $f|_{A_1}$  and  $f|_{A_5}$  are copies of  $g$ ,  $f|_{A_2}$  and  $f|_{A_4}$  are copies of  $h$ , and  $f|_{A_3}$  is an arbitrary increasing array. Since  $f|_{A_1} < \dots < f|_{A_5}$  and  $f|_{A_i}$  is increasing for every  $1 \leq i \leq 5$ ,  $f$  is increasing as well. It remains to show that  $f$  does not contain an  $n \times n$  subarray of type (1, 2) or (2, 1).

As  $f|_{A_1} < \dots < f|_{A_5}$  and  $f|_{A_i}$  is increasing for every  $1 \leq i \leq 5$ , we also find that

- (P1) any  $2 \times 2$  subarray with two vertices in  $A_1$  and two to the right of  $A_1$  is of type (1, 2),
- (P2) any  $2 \times 2$  subarray with two vertices in  $A_5$  and two to the left of  $A_5$  is of type (1, 2),
- (P3) any  $2 \times 2$  subarray with two vertices in  $A_2$  and two above  $A_2$  is of type (2, 1),
- (P4) any  $2 \times 2$  subarray with two vertices in  $A_4$  and two below  $A_4$  is of type (2, 1).



**Fig. 4.** Directed arrows point towards a larger value of  $f$  and indicate whether the given subarray is of type (1, 2) or (2, 1) (as in Figure 1). Numbers denote the relative order of subarrays.

We will show that  $f$  has the desired property using properties (P1)–(P4) together with conditions (G1), (G2), (H1), (H2) from Lemmas 3.5 and 3.6.

Suppose towards a contradiction that  $[N]^2$  contains an  $n \times n$  subgrid  $L = L_1 \times L_2$  such that  $f|_L$  is of type (1, 2) or (2, 1). Letting  $I = [(n - 1)(n - 2)]$ ,  $J = [(n - 1)(n - 2) + 1, (n - 1)^2]$ ,  $K = [(n - 1)^2 + 1, N]$ , we define

$$a = |L_1 \cap I|, \quad b = |L_1 \cap J|, \quad c = |L_1 \cap K|,$$

$$x = |L_2 \cap I|, \quad y = |L_2 \cap J|, \quad z = |L_2 \cap K|.$$

We will obtain various inequalities involving  $a, b, c, x, y, z$ , and eventually reach a contradiction. Since  $L$  has size  $n \times n$  and  $|J| = n - 1$  we obtain

$$a + b + c = x + y + z = n, \quad 0 \leq a, c, x, z \leq n \quad \text{and} \quad 0 \leq b, y \leq n - 1. \quad (2)$$

We divide our analysis into two cases.

**Case 1:**  $L$  is of type (2, 1). We have the following series of observations:

$$(G2) \implies a \leq n - 1 \text{ or } x + y \leq 1, \quad (3)$$

$$(G2) \implies c \leq n - 1 \text{ or } y + z \leq 1, \quad (4)$$

$$(H2) \implies a + b \leq 1 \text{ or } z \leq n - 2, \quad (5)$$

$$(H2) \implies b + c \leq 1 \text{ or } x \leq n - 2, \quad (6)$$

$$(P1) \implies a = 0 \text{ or } b + c = 0 \text{ or } x + y \leq 1, \quad (7)$$

$$(P2) \implies a + b = 0 \text{ or } c = 0 \text{ or } y + z \leq 1. \quad (8)$$

Observe that

$$a = 0 \quad \text{or} \quad x + y \leq 1, \quad (9)$$

$$c = 0 \quad \text{or} \quad y + z \leq 1. \quad (10)$$

To see (9), if  $b + c = 0$  then by (2) we have  $a = n$ , which according to (3) implies  $x + y \leq 1$ , so (7) implies (9). Similarly, to see (10), if  $a + b = 0$  then by (2) we have  $c = n$ , which according to (4) implies  $y + z \leq 1$ , so (8) implies (10).

To complete our analysis of Case 1, we show  $a = c = 0$ , giving a contradiction to (2). Suppose to the contrary that  $a \geq 1$ . Then  $x + y \leq 1$  by (9), and so  $z \geq n - 1$  by (2), which according to (5) shows  $a + b \leq 1$ . Hence

$$c = n - (a + b) \geq n - 1 \geq 1 \quad \text{and} \quad y + z \geq z \geq n - 1 \geq 2,$$

giving a contradiction to (10). It remains to show that  $c = 0$ . If instead  $c \geq 1$ , then (10) implies  $y + z \leq 1$ , and so  $x \geq n - 1$  by (2), which by (6) implies  $b + c \leq 1$ . Thus

$$a = n - (b + c) \geq n - 1 \geq 1 \quad \text{and} \quad x + y \geq x \geq n - 1 \geq 2,$$

contradicting (9) and completing the proof in this case.

**Case 2:**  $L$  is of type (1, 2). The analysis of this case is very similar to that of Case 1. We first have the following observations:

$$(G1) \implies a \leq n - 2 \text{ or } x + y \leq 1, \tag{11}$$

$$(G1) \implies c \leq n - 2 \text{ or } y + z \leq 1, \tag{12}$$

$$(H1) \implies a + b \leq 1 \text{ or } z \leq n - 1, \tag{13}$$

$$(H1) \implies b + c \leq 1 \text{ or } x \leq n - 1, \tag{14}$$

$$(P4) \implies a + b \leq 1 \text{ or } x + y = 0 \text{ or } z = 0, \tag{15}$$

$$(P3) \implies b + c \leq 1 \text{ or } x = 0 \text{ or } y + z = 0. \tag{16}$$

We next show

$$a + b \leq 1 \quad \text{or} \quad z = 0, \tag{17}$$

$$b + c \leq 1 \quad \text{or} \quad x = 0. \tag{18}$$

To see (17), if  $x + y = 0$  then by (2) we have  $z = n$ , which according to (13) implies  $a + b \leq 1$ , so (15) implies (17). Similarly, to see (18), if  $y + z = 0$  then by (2) we have  $x = n$ , which according to (14) implies  $b + c \leq 1$ , so (16) implies (18).

Finally, we show  $x = z = 0$ , giving a contradiction to (2). Suppose  $x \geq 1$ . Then  $b + c \leq 1$  by (18), and so (2) gives  $a \geq n - 1$ , which by (11) forces  $x + y \leq 1$ . From this we conclude

$$a + b \geq a \geq n - 1 \geq 2 \quad \text{and} \quad z = n - (x + y) \geq n - 1 \geq 1,$$

giving a contradiction to (17). To show  $z = 0$ , we suppose  $z \geq 1$ . Then (17) gives  $a + b \leq 1$ , and so (2) implies  $c \geq n - 1$ , which by (12) results in  $y + z \leq 1$ . Thus

$$b + c \geq c \geq n - 1 \geq 2 \quad \text{and} \quad x = n - (y + z) \geq n - 1 \geq 1,$$

contradicting (18). This completes our proof of Theorem 1.2 (i). ■

The example used above is partially motivated by certain examples considered in [5]. That paper also considers higher-dimensional examples which may be of some use in

higher-dimensional instances of our problem as well, but only in terms of optimizing the dependence on  $d$ .

### 3.3. High-dimensional case

In this subsection we prove Theorem 1.2 (ii). Our first ingredient in the proof will be the following lemma.

**Lemma 3.8** (Dominant coordinate). *Let  $d, m, t \geq 2$  be integers, and let  $f$  be an increasing array indexed by  $[d^2mt]^d$ . Then there exist a dimension  $i \in [d]$ , sets  $B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_d \subseteq [d^2mt]$  of size  $m + 1$  and  $t$  subgrids  $A_h := B_1 \times \dots \times B_{i-1} \times \{h\} \times B_{i+1} \times \dots \times B_d$  such that  $f|_{A_h} < f|_{A_{h'}}$  whenever  $h < h'$ .*

One should think of this lemma as saying that there is a dimension  $i$  such that one can find a “stack” of subgrids appearing at the same location along the remaining  $d - 1$  dimensions and different positions along dimension  $i$ , which can be thought of as heights of the subgrids. Furthermore, the subgrids are not much smaller than the initial one in the remaining dimensions and our array is always bigger on a higher subgrid. Our proof of this lemma borrows some ideas of [16].

*Proof of Lemma 3.8.* The proof is in some sense a high-dimensional generalisation of the argument used to prove Theorem 3.4. We split the grid  $[d^2mt]^d$  along each coordinate into  $td$  intervals of equal size, obtaining a partition of  $[d^2mt]^d$  into translates of  $[dm]^d$ ,

$$[d^2mt]^d = \bigcup_{\mathbf{u} \in T} (\mathbf{u} + [dm]^d), \tag{19}$$

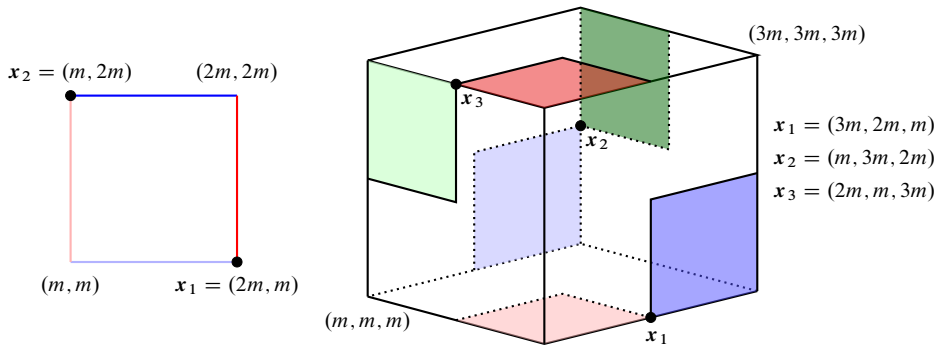
where  $T = \{0, dm, 2dm, \dots, (dt - 1)dm\}^d$ . The reason behind considering this is that we are now going to compare values taken by the array on certain points in  $\mathbf{u} + [dm]^d$  for each  $\mathbf{u} \in T$ , and once we find the one with the largest entry, the fact that points of  $T$  are suitably spaced apart will allow us to get information about the ordering of a relatively large  $(d - 1)$ -dimensional subarray. For each  $i \in [d]$ , the aforementioned points are given by  $\mathbf{x}_i$  and the subarrays by  $C_i$ , where

$$\begin{aligned} \mathbf{x}_i &= ((i - 1)m, (i - 2)m, \dots, m, dm, (d - 1)m, \dots, im) \in [dm]^d, \\ C_i &= [(i - 1)m, im] \times \dots \times [m, 2m] \times \{m\} \times [(d - 1)m, dm] \\ &\quad \times \dots \times [im, (i + 1)m] \subset [dm]^d. \end{aligned}$$

Notice that  $\mathbf{x}_{i+1}$  has every coordinate larger than  $\mathbf{x}_i$ , except the  $i$ -th; here  $\mathbf{x}_{d+1} := \mathbf{x}_1$ . Notice further that  $\mathbf{x}_{i+1} \in C_i$  is larger in every coordinate than any other point of  $C_i$ . Similarly  $\mathbf{x}_i$  with its  $i$ -th coordinate reduced by  $(d - 1)m$  is the point of  $C_i$  which is smaller than any other in every coordinate. In other words, with respect to the component-wise order of  $[dm]^d$ ,

$$\max C_i = \mathbf{x}_{i+1}, \quad \text{and} \quad \min C_i = \mathbf{x}_i - (d - 1)m\mathbf{e}_i, \tag{20}$$

where  $\mathbf{e}_i$  stands for the  $i$ -th standard unit vector.



**Fig. 5.** Lightly shaded  $(d - 1)$ -dimensional regions in the figure denote  $C_i$ 's with their maximum point being  $x_{i+1}$ . Depending on which of the  $x_i$ 's has largest value of  $f$  one of these  $C_i$ 's has value of  $f$  on  $x_i$  smaller than that of  $f$  on the minimal point of a translate of  $C_i$  (strongly shaded region of the same colour) at  $x_i$ .

Now consider a colouring  $\chi: T \rightarrow [d]$  given by

$$\chi(u) = i \quad \text{if and only if} \quad f(u + x_i) = \max \{f(u + x_1), \dots, f(u + x_d)\}.$$

By the pigeonhole principle, there is a colour  $i \in [d]$  which appears at least  $(td)^d/d$  times. This implies that the grid  $T$  contains a column in the direction of the  $i$ -th coordinate for which at least  $t$  vertices of this column have colour  $i$ . We list those vertices of  $T$  from smallest to largest with respect to their  $i$ -th coordinates:  $u_1, \dots, u_t$ .

We show that the grids  $A_1 = u_1 + C_i, \dots, A_t = u_t + C_i$  have the desired properties. Indeed, (19) implies that  $A_1, \dots, A_t$  are subgrids of  $[d^2mt]^t$ . Since we have chosen  $u_j$ 's as in the same column in the direction of the  $i$ -th coordinate, all of them have the same coordinates in all other dimensions. This implies that there are  $d - 1$  sets  $B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_d$  and  $t$  "heights"  $h_1, \dots, h_t$  such that  $A_j$  can be written as  $B_1 \times \dots \times B_{i-1} \times \{h_j\} \times B_{i+1} \times \dots \times B_d$  for every  $1 \leq j \leq t$ . Since  $C_i$  has size  $(m + 1) \times \dots \times (m + 1)$ , each  $B_j$  has size  $m + 1$ . Finally, since  $f$  is increasing, for  $1 \leq j < k \leq t$  we have

$$\begin{aligned} \max f(A_j) &= f(u_j + x_{i+1}) < f(u_j + x_i) \\ &< f(u_k - (d - 1)me_i + x_i) = \min f(A_k). \end{aligned}$$

The first equality follows since (20) implies  $u_j + x_{i+1}$  is the largest point of  $A_j$  so since  $f$  is increasing we conclude that  $f$  is maximised over  $A_j$  at  $u_j + x_{i+1}$ . Similarly, we get the last equality as well. The first inequality follows since  $\chi(u_j) = i$ . The second inequality follows since  $j < k$  implies the  $i$ -th coordinate of  $u_j$  is smaller than that of  $u_k$  (since  $u_j$  and  $u_k$  belong to the same column along the  $i$ -th dimension and since we named them according to their  $i$ -th coordinate) by at least  $dm$  (since  $u_j, u_k \in T$ ). This finishes the proof of Lemma 3.8. ■

This lemma provides us with a stack of subgrids  $A_1, \dots, A_t$  on which  $f$  is increasing in dimension  $i$ . It is natural to try to iterate and now apply the lemma within each  $A_j$ , but notice that we do not only want to find a lexicographic subgrid in some  $n$  different  $A_j$ 's, but they also have to appear at the same positions in each of them. One can now apply a Ramsey result for  $O(2^{F_{d-1}(n)})$  colours to ensure that the subgrids found appear at the same locations. Consequently, this approach gives at best  $F_d(n) \leq \text{towr}_{d-1}(O_d(n))$ . Fishburn and Graham [16, Section 4] follow a similar approach and their arguments hit the same barrier in terms of the bounds they can obtain.

We take a different approach in order to prove Theorem 1.2 (ii).

**Theorem 3.9.** *For  $d \geq 3$ , we have  $F_d(n) \leq 2^{(c_d+o(1))n^{d-2}}$ , where  $c_d = \frac{1}{2}(d - 1)!$ .*

*Proof.* We prove the statement by induction on  $d$ .

**The base case:**  $d = 3$ . Let  $m = 2n^3, t = 2m \binom{m/n}{n} / \binom{m/(2n)}{n} = 2^{n+o(n)}$ , and  $N = d^2mt = 2^{n+o(n)}$ . Consider an increasing array  $f : [N]^3 \rightarrow \mathbb{R}$ . By Lemma 3.8, we may assume without loss of generality that  $[N]^3$  contains a stack of  $t$  subgrids  $A_1 = B_1 \times B_2 \times \{h_1\}, \dots, A_t = B_1 \times B_2 \times \{h_t\}$  of size  $m \times m \times 1$  such that  $h_1 < \dots < h_t$  and  $f|_{A_1} < \dots < f|_{A_t}$ . We drop the third dimension from now on, and think of  $A_i$ 's as 2-dimensional grids.

We split each  $A_i$  into  $(m/n)^2$  smaller subgrids of size  $n \times n$ . Colour each such smaller subgrid red if its top left corner is smaller than the bottom right one, and blue otherwise. As in the proof of Theorem 3.4, any  $n$  red subgrids in the same row of  $A_i$  give rise to an  $n \times n$  subgrid of type (1, 2). If we further manage to find  $n$  layers of the stack, each having such a sequence of the same  $n$  red  $n \times n$  subgrids, we obtain an  $n \times n \times n$  subgrid of type (3, 1, 2) (using the property that  $h_1 < \dots < h_t$  and  $f|_{A_1} < \dots < f|_{A_t}$ ). Similarly, if we find  $n$  layers each having the same sequence of  $n$  blue  $n \times n$  subgrids in the same column, we find an  $n \times n \times n$  subgrid of type (3, 2, 1).

By the pigeonhole principle, each layer  $A_i$  of the stack has a row with  $\frac{m}{2n}$  red subgrids or a column with at least  $\frac{m}{2n}$  blue subgrids. This row or column can be chosen in  $2 \cdot \frac{m}{n}$  ways, so there are  $\frac{nt}{2m}$  layers having in the same row/column  $\frac{m}{2n}$  red/blue subgrids. Let us say without loss of generality that it is the first row. Then the first row of such a layer contains at least  $\binom{m/(2n)}{n}$  tuples of  $n$  red subgrids. By the pigeonhole principle, there are at least

$$\frac{\frac{nt}{2m} \binom{m/(2n)}{n}}{\binom{m/n}{n}} \geq n$$

layers having the same red  $n$ -tuples, as desired.

**The induction step:** Suppose  $d \geq 4$  and that the lemma holds for  $d - 1$ . It is easy to see that the desired estimate  $F_d(n) \leq 2^{(c_d+o(1))n^{d-2}}$  follows from the induction hypothesis and the following recursive bound:

$$F_d(n) \leq d^d F_{d-1}(n)^{(d-1)n+1} \quad \text{for all } d \geq 4 \text{ and } n \geq 2.$$

We now prove this inequality. Let  $m = F_{d-1}(n), t = (d - 1)!n \binom{m}{n}^{d-1}$ , and  $N = d^2mt \leq d^d F_{d-1}(n)^{(d-1)n+1}$ . Consider an increasing array  $f : [N]^d \rightarrow \mathbb{R}$ .



By Lemma 3.8, we may assume without loss of generality that  $[N]^d$  contains a stack of  $t$  subgrids  $A_1 = B_1 \times \dots \times B_{d-1} \times \{h_1\}, \dots, A_t = B_1 \times \dots \times B_{d-1} \times \{h_t\}$  of size  $m \times \dots \times m \times 1$  such that  $h_1 < \dots < h_t$  and  $f|_{A_1} < \dots < f|_{A_t}$ .

Given  $i \in [t]$ , as  $m = F_{d-1}(n)$ , one can find a permutation  $\sigma \in \mathfrak{S}_{d-1}$  and a subgrid  $A'_i \subset A_i$  of size  $n \times \dots \times n \times 1$  such that  $f|_{A'_i}$  is of type  $\sigma$ . Since  $\frac{t}{(d-1)! \binom{m}{n}^{d-1}} = n$ , the pigeonhole principle implies the existence of a permutation  $\sigma \in \mathfrak{S}_{d-1}$ , an  $n \times \dots \times n$  subgrid  $B'_1 \times \dots \times B'_d$  of  $B_1 \times \dots \times B_d$ , and  $n$  layers  $1 \leq i_1 < \dots < i_n \leq t$  such that for every  $k \in [n]$ , we have  $A'_{i_k} = B'_1 \times \dots \times B'_{d-1} \times \{h_{i_k}\}$ , and the restriction of  $f$  to  $A'_{i_k}$  is of type  $\sigma$ . As  $f|_{A_{i_1}} < \dots < f|_{A_{i_n}}$ , the restriction of  $f$  to  $B'_1 \times \dots \times B'_{d-1} \times \{h_{i_1}, \dots, h_{i_n}\}$  is an  $n \times \dots \times n$  array of type  $(d, \sigma)$ . This shows  $F_d(n) \leq N \leq d^d F_{d-1}(n)^{(d-1)n+1}$ , as required. ■

#### 4. Concluding remarks

We obtain a major improvement on best known upper bounds for  $M_d(n)$ . However, our bounds are still off from the best known lower bound of  $M_d(n) \geq n^{(1+o(1))n^{d-1}/d}$  due to Fishburn and Graham [16, Theorem 3]. Perhaps the most interesting open question regarding  $M_d(n)$  is to determine the behaviour in two dimensions.

**Question 4.1.** *What is the behaviour of  $M_2(n)$ ? Is it closer to exponential or to double exponential in  $n$ ?*

It is natural to ask whether our argument used to get a double exponential bound in the monotone case in three dimensions (Theorem 2.7) generalises to higher dimensions. Unfortunately, the natural generalisation of our approach to more dimensions gives a bound of the form  $M_d(n) \leq \text{tower}_{\lfloor d/2 \rfloor + 2}(O_d(n))$  which has a tower of height growing with  $d$ . However, this does still imply a better bound than Theorem 2.6 in four and five dimensions. The main issue preventing us from extending our argument to more dimensions is that it seems hard to obtain asymmetric results which would allow us to find a monotone subarray with exponential size in at least two dimensions. For example, if we could find an  $n \times 2^n \times 2^{n^2}$  monotone subarray within any array of size  $2^{2^{O(n^2)}} \times 2^{2^{O(n^2)}} \times 2^{2^{O(n^2)}}$  we would obtain a double exponential bound  $M_4(n) \leq 2^{2^{O(n^3)}}$ . However, if we knew how to do this then by considering an array which is always increasing in the first dimension and has the same but arbitrary ordering for each 2-dimensional subarray with fixed value in the first dimension, we would also be able to get a better than double exponential bound in the 2-dimensional case, which leads us back to Question 4.1. Our better bounds in three, four and five dimensions make it seem unlikely that a triple exponential is ever needed.

**Question 4.2.** *For  $d \geq 4$  is  $M_d(n)$  bounded from above by a double exponential in  $n^{d-1}$ ?*

For the problem of determining  $F_d(n)$ , we completely settle the 2-dimensional case and give exponential upper bounds for  $d \geq 3$ . The best known lower bound

$F_d(n) \geq (n - 1)^d$ , also due to Fishburn and Graham [16], is still only polynomial. We find the 3-dimensional case particularly interesting since via Lemma 3.8 it reduces to the following nice problem.

**Question 4.3.** *What is the smallest  $N$  such that given  $N$  increasing arrays of size  $N \times N$  one can find an  $n \times n$  lexicographic array of the same type appearing in the same positions in at least  $n$  of the arrays?*

In particular, is this  $N$  bounded by a polynomial in  $n$  or is it exponential in  $n$ ?

The study of  $M_d(n)$ ,  $F_d(n)$  and  $L_d(n)$ , while interesting in its own right, is also closely related to various other interesting problems. We present just a few here.

#### 4.1. Long common monotone subsequence

The problem of estimating  $M_2(n)$  is closely related to the longest common monotone subsequence problem. A *common monotone subsequence* of two permutations  $\pi, \sigma \in \mathfrak{S}_N$  is a set  $I \subseteq [N]$  such that the restrictions of  $\pi$  and  $\sigma$  to  $I$  are either both increasing or both decreasing. A common monotone subsequence of more than two permutations is defined analogously. Given positive integers  $t, k$  and  $N$ , let  $\text{LMS}(t, k, N)$  denote the maximum  $\ell$  such that any size- $k$  multisubset  $P \subseteq \mathfrak{S}_N$  contains a size- $t$  multisubset  $P'$  such that the length of the longest common monotone subsequence of  $P'$  is at least  $\ell$ .

We now describe the connection between  $M_2(n)$  and  $\text{LMS}(t, k, N)$ . Let  $f: [N]^2 \rightarrow \mathbb{R}$  be a 2-dimensional array. Similarly to the first part of the proof of Theorem 2.3, we can show that  $[N]^2$  contains a subgrid  $R \times C$  of size  $(\log N)^{1-o(1)} \times N^{1-o(1)}$  such that either  $f|_{R \times C}$  is increasing in each column, or  $f|_{R \times C}$  is decreasing in each column. For each  $r \in R$ , the restriction of  $f$  to the row  $\{r\} \times C$  induces a permutation  $\pi_r$  of  $C$ , since  $f$  is assumed to be injective. (Note that  $\pi_r$ 's are not necessarily distinct.) It is not hard to see that if among  $(\log N)^{1-o(1)}$  permutations  $\{\pi_r : r \in R\}$  of  $C$  there are  $n$  permutations whose longest common monotone subsequence has length at least  $n$ , then  $f|_{R \times C}$  contains an  $n \times n$  monotone subarray. Therefore every  $N \times N$  array contains a monotone subarray of size  $n \times n$ , where  $n$  is the maximum  $t \in \mathbb{N}$  such that  $\text{LMS}(t, (\log N)^{1-o(1)}, N^{1-o(1)})$  is greater than or equal to  $t$ . By an iterative application of the Erdős–Szekeres theorem, one can take  $n = (1/2 - o(1)) \log_2 \log_2 N$ , or equivalently  $N = 2^{2^{(2+o(1))n}}$ .

The problem of determining the parameter  $\text{LMS}(t, k, N)$  for other ranges of  $t$  and  $k$  is also very appealing. For example, it would be interesting to have a good estimate for  $\text{LMS}(t, k, N)$  when  $t$  is fixed and  $k$  grows to infinity with  $N$ . We refer the reader to [2, 3, 7] for some related results in this direction.

#### 4.2. Ramsey type problems for vertex-ordered graphs

One can place the problems we have studied in this paper under the framework of (vertex-)ordered Ramsey numbers. For simplicity of presentation we choose to illustrate this through the 2-dimensional monotone subarray problem. Let  $K_{N,N}^{(3)}$  be the 3-uniform

hypergraph with vertex set  $A \cup B$ , where  $A$  and  $B$  are two copies of  $[N]$ , and edge set consisting of all those triples which intersect both  $A$  and  $B$ . Given an array  $f: [N]^2 \rightarrow \mathbb{R}$ , we can associate to  $f$  an edge-colouring  $\chi$  of  $K_{N,N}^{(3)}$  with two colours, red and blue. For  $i \in A$  and  $j, j' \in B$  with  $j < j'$ , let  $\chi(i, j, j') = \text{red}$  if and only if  $f(i, j) < f(i, j')$ . Similarly, for  $j \in B$  and  $i, i' \in A$  with  $i < i'$ , we assign colour red to  $(i, i', j)$  if and only if  $f(i, j) < f(i', j)$ . The following simple observation connects the multidimensional Erdős–Szekeres world with the ordered Ramsey world.

**Observation 4.4.** Suppose there are two size- $n$  subsets  $\{a_1 < \dots < a_n\} \subseteq A$  and  $\{b_1 < \dots < b_n\} \subseteq B$  such that

- $\{(a_i, b_j, b_{j+1}) : i \in [n], j \in [n-1]\}$  is monochromatic under  $\chi$ ,
- $\{(a_i, a_{i+1}, b_j) : j \in [n], i \in [n-1]\}$  is monochromatic under  $\chi$ .

Then the restriction of  $f$  to  $\{a_1, \dots, a_n\} \times \{b_1, \dots, b_n\}$  is monotone.

Now define  $\text{OR}(n)$  to be the smallest  $N$  such that in every red-blue colouring of the edges of  $K_{N,N}^{(3)}$  we can always find two size- $n$  subsets  $\{a_1 < \dots < a_n\} \subseteq A$  and  $\{b_1 < \dots < b_n\} \subseteq B$  with the aforementioned properties. From the observation, we know  $M_2(n) \leq \text{OR}(n)$ . A closer inspection of our proof of the inequality  $M_2(n) \leq 2^{2^{(2+o(1))n}}$  reveals that it actually gives  $\text{OR}(n) \leq 2^{2^{(2+o(1))n}}$ . Thus it is natural to ask whether  $M_2(n)$  and  $\text{OR}(n)$  have the same order of magnitude.

### 4.3. Canonical orderings of discrete structures

An ordering of the edges of a (vertex-ordered)  $d$ -graph  $G$  with  $V(G) \subset \mathbb{Z}$  is *lex-monotone* if one can find a permutation  $\sigma \in \mathfrak{S}_d$  and a sign vector  $s \in \{-1, 1\}^d$  such that the edges  $(a_1, \dots, a_d)$  of  $G$  with  $a_1 < \dots < a_d$  are ordered according to the lexicographical order of the tuple  $(s_{\sigma(1)}a_{\sigma(1)}, \dots, s_{\sigma(d)}a_{\sigma(d)})$ . An old result of Leeb and Prömel (see [24, Theorem 2.8]) says that for all  $d, n \in \mathbb{N}$  there is a positive integer  $\text{LP}_d(n)$  such that every edge-ordering of a (vertex-ordered) complete  $d$ -graph on  $\text{LP}_d(n)$  vertices contains a copy of the complete  $d$ -graph on  $n$  vertices whose edges induce a lex-monotone ordering. Theorem 1.3 in our paper can be viewed naturally as a  $d$ -partite version (with a better bound) of this result. It would be interesting to know if our approach can lead to an improvement on the upper bound  $\text{LP}_d(n) \leq \text{towr}_{2d}(O_d(n))$  for  $d \geq 2$ , due to Nešetřil and Rödl [26, Theorem 14]. For other interesting results on edge-ordered Ramsey numbers, we refer the reader to [1, 17].

Theorem 1.3 is also related to the work of Nešetřil, Prömel, Rödl and Voigt [25] on linear orders of the combinatorial cube  $[k]^n$  when  $k$  is fixed and  $n$  is large. For simplified presentations of this work, see [6, 27].

*Funding.* Work of M.B. was supported in part by the NSF grant CCF-1900460.

Research of B.S. was supported in part by SNSF grant 200021\_196965.

Work of T.T. was supported by the Institute for Basic Science (IBS-R029-Y1), and the National Natural Science Foundation of China.

**References**

- [1] Balko, M., Vizer, M.: Edge-ordered Ramsey numbers. *Eur. J. Combin.* **87**, art. 103100, 11 pp. (2020) Zbl [1439.05129](#) MR [4081481](#)
- [2] Beame, P., Blais, E., Huynh-Ngoc, D.-T.: Longest common subsequences in sets of permutations. arXiv:[0904.1615](#) (2009)
- [3] Beame, P., Huynh-Ngoc, D.-T.: On the value of multiple read/write streams for approximating frequency moments. In: *Proc. 49th Annual Symposium on Foundations of Computer Science*, IEEE, 499–508 (2008)
- [4] Bodirsky, M., Kára, J.: The complexity of temporal constraint satisfaction problems. *J. ACM* **57**, art. 9, 41 pp. (2010) Zbl [1327.68125](#) MR [2606084](#)
- [5] Bucic, M., Lidický, B., Long, J., Wagner, A. Z.: Partition problems in high dimensional boxes. *J. Combin. Theory Ser. A* **166**, 315–336 (2019) Zbl [1416.05038](#) MR [3927902](#)
- [6] Bukh, B., Sevekari, A.: Linear orderings of combinatorial cubes. arXiv:[1906.11866](#) (2019)
- [7] Bukh, B., Zhou, L.: Twins in words and long common subsequences in permutations. *Israel J. Math.* **213**, 183–209 (2016) Zbl [1354.68213](#) MR [3509473](#)
- [8] Burkill, H., Mirsky, L.: Combinatorial problems on the existence of large submatrices. I. *Discrete Math.* **6**, 15–28 (1973) Zbl [0357.05011](#) MR [321761](#)
- [9] Burkill, H., Mirsky, L.: Monotonicity. *J. Math. Anal. Appl.* **41**, 391–410 (1973) Zbl [0268.26007](#) MR [335714](#)
- [10] Chvátal, V.: Monochromatic paths in edge-colored graphs. *J. Combin. Theory Ser. B* **13**, 69–70 (1972) Zbl [0221.05064](#) MR [309785](#)
- [11] Conlon, D., Fox, J., Lee, C., Sudakov, B.: Ordered Ramsey numbers. *J. Combin. Theory Ser. B* **122**, 353–383 (2017) Zbl [1350.05085](#) MR [3575208](#)
- [12] Conlon, D., Fox, J., Sudakov, B.: Ramsey numbers of sparse hypergraphs. *Random Structures Algorithms* **35**, 1–14 (2009) Zbl [1203.05099](#) MR [2532871](#)
- [13] Erdős, P., Rado, R.: Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc. (3)* **2**, 417–439 (1952) Zbl [0048.28203](#) MR [65615](#)
- [14] Erdős, P., Szekeres, G.: A combinatorial problem in geometry. *Compos. Math.* **2**, 463–470 (1935) Zbl [61.0651.04](#) MR [1556929](#)
- [15] Felsner, S., Fishburn, P. C., Trotter, W. T.: Finite three-dimensional partial orders which are not sphere orders. *Discrete Math.* **201**, 101–132 (1999) Zbl [0941.06005](#) MR [1687878](#)
- [16] Fishburn, P. C., Graham, R. L.: Lexicographic Ramsey theory. *J. Combin. Theory Ser. A* **62**, 280–298 (1993) Zbl [0773.05097](#) MR [1207738](#)
- [17] Fox, J., Li, R.: On edge-ordered Ramsey numbers. *Random Structures Algorithms* **57**, 1174–1204 (2020) Zbl [1454.05076](#) MR [4170446](#)
- [18] Füredi, Z.: An upper bound on Zarankiewicz’ problem. *Combin. Probab. Comput.* **5**, 29–33 (1996) Zbl [0857.05048](#) MR [1395691](#)
- [19] Gyárfás, A., Lehel, J.: A Ramsey-type problem in directed and bipartite graphs. *Period. Math. Hungar.* **3**, 299–304 (1973) Zbl [0267.05115](#) MR [335354](#)
- [20] Kalmanson, K.: On a theorem of Erdős and Szekeres. *J. Combin. Theory Ser. A* **15**, 343–346 (1973) Zbl [0362.05027](#) MR [329928](#)
- [21] Kruskal, J. B., Jr.: Monotonic subsequences. *Proc. Amer. Math. Soc.* **4**, 264–274 (1953) Zbl [0052.04901](#) MR [53256](#)
- [22] Linial, N., Simkin, M.: Monotone subsequences in high-dimensional permutations. *Combin. Probab. Comput.* **27**, 69–83 (2018) Zbl [1378.05003](#) MR [3734331](#)
- [23] Morse, A. P.: Subfunction structure. *Proc. Amer. Math. Soc.* **21**, 321–323 (1969) Zbl [0179.01702](#) MR [258640](#)
- [24] Nešetřil, J.: Some nonstandard Ramsey-like applications. *Theoret. Computer Sci.* **34**, 3–15 (1984) Zbl [0546.05044](#) MR [774038](#)

- 
- [25] Nešetřil, J., Prömel, H. J., Rödl, V., Voigt, B.: Canonizing ordering theorems for Hales–Jewett structures. *J. Combin. Theory Ser. A* **40**, 394–408 (1985) Zbl [0582.06001](#) MR [814422](#)
- [26] Nešetřil, J., Rödl, V.: Statistics of orderings. *Abh. Math. Sem. Univ. Hamburg* **87**, 421–433 (2017) Zbl [1377.05178](#) MR [3696159](#)
- [27] Prömel, H. J.: Some remarks on natural orders for combinatorial cubes. *Discrete Math.* **73** (1989), 189–198 Zbl [0666.06001](#) MR [0974827](#)
- [28] Siders, R.: Monotone subsequences in any dimension. *J. Combin. Theory Ser. A* **85**, 243–253 (1999) Zbl [0917.05001](#) MR [1673952](#)
- [29] Steele, J. M.: Variations on the monotone subsequence theme of Erdős and Szekeres. In: *Discrete Probability and Algorithms* (Minneapolis, MN, 1993), IMA Vol. Math. Appl. 72, Springer, New York, 111–131 (1995) Zbl [0832.60012](#) MR [1380525](#)
- [30] Szabó, T., Tardos, G.: A multidimensional generalization of the Erdős–Szekeres lemma on monotone subsequences. *Combin. Probab. Comput.* **10**, 557–565 (2001) Zbl [1113.52301](#) MR [1869846](#)