J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1207

Published by EMS Press

Dmitry Gourevitch · Eyal Kaplan
(with an appendix by Avraham Aizenbud and Dmitry Gourevitch)

Multiplicity one theorems for the generalized doubling method

Received November 14, 2019

Abstract. We prove the local multiplicity at most one theorem underlying the definition and theory of local γ-, ϵ- and L-factors, defined by virtue of the generalized doubling method, over any local field of characteristic 0. We also present two applications: one to the existence of local factors for genuine representations of covering groups, the other to the global unfolding argument of the doubling integral.

Keywords. Doubling method, multiplicity one, invariant distributions, covering groups, Schwartz functions

Contents

Dmitry Gourevitch: Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel; dmitry.gourevitch@weizmann.ac.il

Eyal Kaplan: Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel; kaplaney@gmail.com

Avraham Aizenbud: Faculty of Mathematics and Computer Science, Weizmann Institute of Science, POB 26, Rehovot 76100, Israel; aizenr@gmail.com

Mathematics Subject Classification (2020): Primary 11F70; Secondary 11F55, 11F66, 22E50, 22E55

The doubling method of [18, 20, 90] constructs an integral representation for the tensor product of a pair of irreducible cuspidal automorphic representations of $G(\mathbb{A})$ and $\mathrm{GL}_k(\mathbb{A})$, for a range of reductive groups G defined over a number field F_0 with a ring of adeles $\mathbb{A}$. One of the advantages of this method is that it does not rely on the existence of a model (or a nonzero Fourier coefficient) for the representation of $G(\mathbb{A})$; it is applicable to any cuspidal representation. The family of local integrals can be used to define local γ-, ϵ- and L-factors. These factors are defined for arbitrary irreducible admissible representations, and as such, generalize the corresponding (tensor product) factors defined by Shahidi [96] for irreducible generic representations, using his method of local coefficients. We prove the local multiplicity at most one theorem underlying the definition of the local factors.

Let F be a local field of characteristic 0. Let G be one of the split groups Sp_c, SO_c, GSpin_c or GL_c, where in the symplectic case c is even. For an integer k, let H be the split group of the same type as G, which is either Sp_{2kc}, SO_{2kc}, GSpin_{2kc} or GL_{2kc}. There is a unipotent subgroup U of H, and a character ψ_U of U which is generic with respect to the unipotent orbit $((2k-1)^c 1^c)$ associated with H, such that $G \times G$ can be mapped into the normalizer of U and stabilizer of ψ_U. We denote the image of $G \times G$ under this map by (G, G) and let D be the subgroup $U \rtimes (G, G)$ of H.

We identify F-groups with their F-points. The underlying principle of the doubling construction is the multiplicity at most one property of the restriction to D of representations of H parabolically induced from certain degenerate representations of GL_{kc}.

A representation ρ of GL_{kc} is called a (k, c) representation if its wave-front set contains (k^c) as the unique maximal orbit, and its degenerate Whittaker model with respect to this orbit is unique. The simplest examples are the representation det of GL_c or its twist by a quasi-character of F^*, which is a $(1, c)$ representation, or irreducible generic representations of GL_k, which are $(k, 1)$. The generalized Speh representation $\rho_c(\tau)$ of GL_{kc} attached to c copies of an irreducible unitary representation τ of GL_k is (k, c) [19, Theorem 4].

Let P be a maximal parabolic subgroup of H which is a Siegel parabolic subgroup if $G \neq \mathrm{GL}_{kc}$. Denote the Levi part of P by M_P. If $M_P = \mathrm{GL}_{kc}$, let ρ be a (k, c) representation of M_P. For a complex parameter s, consider the space $V(s, \rho)$ of the representation of H parabolically induced from $|\det|^{s-1/2}\rho$ and P to H. In the other cases of subgroups M_P the representation $V(s, \rho)$ slightly varies: for $M_P = \mathrm{GL}_{kc} \times \mathrm{GL}_1$ we induce from $\rho \otimes \eta$ with a quasi-character η of F^*, and for $M_P = \mathrm{GL}_{kc} \times \mathrm{GL}_{kc}$, ρ is the (exterior) tensor product of two (k, c) representations of GL_{kc}.

Theorem A (see Theorem 2.1). *Let π_1 and π_2 be irreducible admissible representations of G, and ρ be an admissible finite length (k, c) representation of* GL_{kc}. *Outside a discrete* *subset $\mathcal{B} \subset \mathbb{C}$ of s,*

$$\dim \mathrm{Hom}_D(V(s,\rho), \psi_U^{-1} \otimes \pi_1 \otimes \pi_2) \leq \dim \mathrm{Hom}_G(\pi_1^\vee, \pi_2^\iota).$$

Over nonarchimedean fields, for supercuspidal representations π_1 and π_2 the result holds for all s, under certain additional conditions.

Here ι is a certain involution of G; when $G = \mathrm{Sp}_c$, $\pi^\iota = \pi^\vee$, and for GL_c, ι is trivial. As usual, if the field is nonarchimedean and its residue field contains q elements, the set $\mathcal{B}$ consists of finitely many values of q^{-s}. For the stronger statement for supercuspidal representations we must exclude minimal rank cases where G does not contain nontrivial unipotent subgroups, and for GL_c there is an additional condition on ρ. In the setup of the doubling method of [18,20], $(\pi_1, \pi_2) = (\pi^\vee, \pi^\iota)$ and the dimension is precisely 1 outside a discrete subset of s. Note that there is no canonical isomorphism between the spaces appearing in the theorem. For the definitions of all objects, notation and for the more precise statement, see §1, in particular §1.5 where we recall the generalized doubling setup, and Theorem 2.1 (e.g., ι is defined in §1.5).

The case $k = 1$ of the theorem for supercuspidal representations was proved by Harris et al. [51, §4], but the general setting of the theorem (even for $k = 1$) has not been studied. In this sense we close a historical gap.

Theorem A is the local counterpart of the global unfolding argument in [18]. We briefly recall the global result, focusing on the parts relevant to us here. For more detail on the global setting see §3.2.

Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$, and $\mathcal{E}_\tau$ be the generalized Speh representation of $\mathrm{GL}_{kc}(\mathbb{A})$ corresponding to c copies of τ, defined by Jacquet [55]. The representation $\mathcal{E}_\tau$ is a global (k,c) representation, in the sense that it does not support any Fourier coefficient along an orbit greater than or incomparable with (k^c), it supports a Fourier coefficient along (k^c), and all of its local components are (k,c) [18–20,41,60]. Let $E(h; s, f)$ denote the Eisenstein series attached to a suitable section f in the space of the representation of $H(\mathbb{A})$ parabolically induced from $|\det|^{s-1/2}\mathcal{E}_\tau$ and $P(\mathbb{A})$. One can consider the Fourier coefficient $E^{U,\psi_U}(h; s, f)$ of $E(h; s, f)$ along (U, ψ_U) (see (3.7)) as an automorphic function on $G(\mathbb{A}) \times G(\mathbb{A})$. The global integral was defined in [18] by integrating $E^{U,\psi_U}(h; s, f)$ against two cusp forms in the space of a unitary irreducible cuspidal automorphic representation π of $G(\mathbb{A})$. In a right half-plane $\mathrm{Re}(s) \gg 0$, one can rewrite the integral as a sum (of integrals) parametrized by representatives of $P(F_0)\backslash H(F_0)/D(F_0)$. All summands but one vanish, and the remaining summand was shown to produce an Eulerian integral.

There are three methods for showing the vanishing of a summand. The first is by finding a subgroup $U' < U$ such that ψ_U is nontrivial on $U'(\mathbb{A})$, and showing that the summand admits an inner integral of ψ_U over $U'(F_0)\backslash U'(\mathbb{A})$, which is then zero. Second, if the summand admits an inner integral which constitutes a Fourier coefficient of $\mathcal{E}_\tau$ that is greater than or incomparable with (k^c). This summand vanishes because $\mathcal{E}_\tau$ is (k,c). Third, if one can obtain an inner integral of one of the cusp forms along a unipotent radical U_R of a parabolic subgroup $R = M_R \ltimes U_R$ of G, then the summand vanishes because π is cuspidal.

Our local result is, in some sense, parallel to the global unfolding. We consider distributions on the orbits of $P\backslash H/D$. The argument involving ψ_U can be applied locally. The second case, where we use the (k, c) representation alone, is not difficult to carry out in the nonarchimedean setting, using the local "exchange of roots" arguments of Ginzburg et al. [44] and the theory of derivatives of Bernstein and Zelevinsky [11, 12]. Results involving the Jacquet functor are in general more difficult and subtle over archimedean fields. Fortunately, we are able to benefit from the recent (partial) extension of the theory of derivatives to archimedean fields by Aizenbud et al. [3, 4]. In fact, our argument in this case is greatly simplified and streamlined using the precise reformulation of Gomez et al. [46] of the connection between the wave-front set and the theory of derivatives, over both archimedean and nonarchimedean fields.

The class of double cosets, where the global vanishing follows using the fact that the representations of G are cuspidal, requires a different approach. The difficulty arises because in the local setting one must consider non-supercuspidal representations of G as well. This is where we lose the subset $\mathscr{B}$.

In more detail, if the global summand was treated using, say, $U_R < R < G$, the corresponding orbit should be handled by analyzing the action of the center C_{M_R} of M_R. Consider the nonarchimedean setting. Following the method of Jacquet et al. [58], if the local representations (which are usually Jacquet modules of π_i and ρ) restrict to finite length representations of M_R, the action of C_{M_R} is filtered by a finite sequence of quasi-characters, combined with a quasi-character determined by $|\det|^s$. This produces a compatibility condition that rules out a discrete subset of s, i.e., there are no distributions on the orbit unless s belongs to a discrete set. This argument was carried out in several works, including [40, 44, 66, 97]. The main difficulty was to show that indeed the representations involved restrict to finite length representations of a Levi subgroup, or more precisely in those works, of the reductive part of the appropriate mirabolic subgroup, and the key tool in the proofs was the theory of derivatives [11, 12]. By contrast, here the Jacquet modules of ρ that occur in the analysis do not afford a representation of the mirabolic subgroup, but we are still able to show that they restrict to finite length representations of M_R. Moreover, while in the previous aforementioned works the number of double cosets was finite, here there are in general uncountably many (unless $k = 1$). Therefore we must be careful to apply this argument to only finitely many representatives.

In fact, treating uncountably many orbits is another difficulty. In the nonarchimedean case, in principle if there are no distributions (satisfying certain equivariance properties) on the orbits, there are no global distributions (see, e.g., [11, §6]). Over archimedean fields this is considerably more complicated. Kolk and Varadarajan [68] extended parts of the archimedean Bruhat theory to this case using transverse symbols. In Appendix A by the first named author and Avraham Aizenbud, we will present a generalization of the main result of [68], which is sufficiently strong for our application and is of independent interest, using tools from functional analysis.

The proof here clarifies several arguments of [18], and is applicable to a wide class of groups, in particular all groups treated in [20] (the unfolding argument in [18] was presented only for symplectic groups). The original doubling method of [90] was stated for a slightly different class of groups, e.g., the full orthogonal groups O_c, and also unitary groups; general spin groups, as well as the double cover of the symplectic group, were mentioned in [90, §4.3] but not treated in any way. The extension of the present proof to other classes of groups would be straightforward.

Note that the aforementioned proof in [51, §4] does not involve ψ_U nor the structure of ρ (at any rate ρ for $k = 1$ is plainly a character), and involves only finitely many orbits. Once the proof here is reduced to the open orbit, it "mirrors" the arguments of *loc. cit.* when the representations are supercuspidal, but again in the general case it is more difficult and relies on the deep properties of ρ.

Our work has two immediate applications. The first concerns covering groups (topological central extensions by finite cyclic groups). The classical doubling method ($k = 1$) was extended by Gan [34] to the double cover of the symplectic group. In the recent work [64], the doubling method was extended to m-fold coverings $\mathrm{Sp}_c^{(m)}(\mathbb{A})$ of $\mathrm{Sp}_c(\mathbb{A})$ (defined by [80]) for all m, and any k, providing an integral representation for the tensor product of a pair of genuine irreducible cuspidal automorphic representations π of $\mathrm{Sp}_c^{(m)}(\mathbb{A})$ and τ of $\widetilde{\mathrm{GL}}_k(\mathbb{A})$, where $\widetilde{\mathrm{GL}}_k(\mathbb{A})$ is a covering group of $\mathrm{GL}_k(\mathbb{A})$ defined by restriction from $\mathrm{Sp}_{2k}^{(m)}(\mathbb{A})$. Alongside, the local doubling construction for $\widetilde{\mathrm{GL}}_c$ was developed as well (for all m and k). The construction of [64] is still subject to local and global conjectures regarding generalized Speh representations, but the local theory for unramified data over nonarchimedean fields does not depend on these conjectures.

Theorem A can be reformulated for covering groups, granted certain conditions hold (see §3.1 for details). In the particular cases of $\mathrm{Sp}_c^{(m)}$ and $\widetilde{\mathrm{GL}}_c$, Theorem A is applicable and as a consequence, we can define local factors using uniqueness, at least when data are unramified. Specifically, define the γ-factor as the proportionality factor between two integrals, then use it to define ϵ- and L-factors; see (0.2) and the explanation below in the linear setting. To the best of our knowledge, at present no other method for an analytic definition of these factors is known (see below; of course, for a formal abstract definition of local factors for unramified data one can use the Satake parametrization). Moreover, granted the conjectures of [64], Theorem A is expected to imply the existence of local γ-, ϵ- and L-factors in general, i.e., also in the ramified case, for $\mathrm{Sp}_c^{(m)} \times \widetilde{\mathrm{GL}}_k$ (and additional covering groups).

As explained above, the doubling method does not rely on the existence of a model for the representation of G. This is advantageous for linear groups, but even more so when considering covering groups. As a rule, Whittaker models are not unique for representations of covering groups (the double cover of Sp_c is an exception), first and foremost, for genuine irreducible unramified principle series representations. This means that Shahidi's theory of local coefficients is no longer applicable, even in the unramified setting. The fact that one can still define local factors using analytic methods and uniqueness, is perhaps a surprise.

We note that the number of Whittaker models is still finite. In recent works, Gao et al. [38, 39] and Szpruch [106] studied generalizations of Shahidi's local coefficients, namely a local coefficients matrix and a scattering matrix, and extracted interesting representation-theoretic invariants.

The second application is global. Since the local components of $\mathcal{E}_\tau$ are (k, c) representations, our local analysis expresses the Fourier coefficient of the Eisenstein series as a sum over a finite number of cosets in (the infinite space) $P(F_0)\backslash H(F_0)/D(F_0)$. Then it is visible that the integral of this coefficient against two cusp forms reduces to a single summand, explicating the unfolding process. In this sense we fill out the gap of the unfolding for the cases of groups considered in [20]. Note that our arguments also readily globalize.

The original doubling method of Piatetski-Shapiro and Rallis [90] produced an integral representation for the standard automorphic L-function of an irreducible cuspidal automorphic representation of a classical group, or its rank-1 twists, which is the case $k = 1$. The local theory for $k = 1$ was fully developed by Lapid and Rallis [73]. The doubling construction was extended to arbitrary k in [18], and the corresponding theory of local factors was developed in [20].

We briefly explain how Theorem A is used for the definition of the local factors. Fix a nontrivial additive character ψ of F. Let π be an irreducible admissible representation of G, and τ be an irreducible admissible and generic representation of GL_k. If τ is unitary, the representation $\rho_c(\tau)$ was introduced above; in general $\rho_c(\tau)$ is defined using Langlands' classification and the tempered case. The local doubling integral $Z(s, \omega, f)$ is defined for a matrix coefficient ω of $\pi^\vee$ and a holomorphic section f of $V(s, \rho_c(\tau))$. In its domain of absolute convergence (a right half-plane), $Z(s, \omega, f)$ defines a morphism in

$$\mathrm{Hom}_D(V(s, \rho_c(\tau)), \psi_U^{-1} \otimes \pi^\vee \otimes \pi^\iota). \tag{0.1}$$

Applying a standard intertwining operator

$$M(s, w) : V(s, \rho_c(\tau)) \to V(1 - s, {}^w\rho_c(\tau)),$$

where

$${}^w\rho_c(\tau) = \rho_c(\tau^\vee) \quad \text{for } G = \mathrm{Sp}_c, \mathrm{SO}_c,$$

we obtain a second integral $Z(1 - s, \omega, M(s, w)f)$, absolutely convergent in a left half-plane, which still defines a morphism in (0.1). In fact, $M(s, w)$ is further normalized using a second functional equation, $M^*(s, w) = C(s, c, \tau, \psi)M(s, w)$, where $C(s, c, \tau, \psi)$ is a meromorphic function of s (see [20, §4]). By Theorem A we can define the γ-factor $\gamma(s, \pi \times \tau, \psi)$ by

$$\begin{aligned} \gamma(s, \pi \times \tau, \psi)Z(s, \omega, f) &= Z^*(s, \omega, f), \\ Z^*(s, \omega, f) &= Z(1 - s, \omega, M^*(s, w)f). \end{aligned} \tag{0.2}$$

The main local result of [20] was the characterization of this factor, according to the prescribed list of properties formulated by Shahidi in the context of generic representations

[96, Theorem 3.5] (see also [73, Theorem 4]). In turn, the γ-factor was used in [20] to define the local ϵ- and L-factors, following Shahidi's method [96, §7] (see also [92–95]). The main motivation of [20] was to find a new proof of global functoriality from $G(\mathbb{A})$ to the appropriate general linear group, via the Converse Theorem of Cogdell and Piatetski-Shapiro [27, 28], thereby extending the global result of [6, 25, 26] from globally generic representations to arbitrary cuspidal ones. Note that the endoscopic functorial transfer for quasi-split orthogonal or symplectic groups was obtained by Arthur [5] using the twisted stable trace formula, and extended to quasi-split unitary groups by Mok [84].

We mention that if π is supercuspidal (under certain additional assumptions), our uniqueness results imply, using Bernstein's continuation principle [7], that the integral is holomorphic (see Corollary 2.4). Using the fact that the integral can always be made constant, it follows that the only poles appearing in $\gamma(s, \pi \times \tau, \psi)$ are poles of $M^*(s, w)$. This observation hints that a "g.c.d. definition" of the L-function using the generalized doubling method must involve "good sections" (see, e.g., [65, pp. 589–590]). Indeed, this was the approach of Yamana [113], who studied this definition of the L-function for $k = 1$.

Similar multiplicity at most one theorems exist in the literature. In the context of Rankin–Selberg integrals for representations of $G \times \mathrm{GL}_k$ admitting unique Whittaker models, where G is a classical group, see [40, 44, 66, 97, 98]. See also [2, 35, 62, 74, 102, 104, 108, 109] where strong general uniqueness results were proved (which in particular imply multiplicity one for the same Rankin–Selberg constructions with irreducible generic representations). Our proof technique resembles Soudry's [97, §8] and [98].

In a more general context, for a representation ρ of an arbitrary group H, a subgroup $D < H$ and a representation ξ of D, one can consider the space $\mathrm{Hom}_D(\rho, \xi)$. Typical questions involve the multiplicity of this space, or the structure of ξ for which $\mathrm{Hom}_D(\rho, \xi) \neq 0$. In certain cases, the nonvanishing is related to special values of L-functions. Globally, one is often interested in a period integral of an automorphic form on $H(\mathbb{A})$ over $D(F_0)\backslash D(\mathbb{A})$ (with $\xi = 1$). There is a vast amount of research on such problems; let us mention [32, 33, 56, 57, 59, 75–79, 86–89, 114, 115].

For other works involving the doubling method see, e.g., [14, 31, 36, 51–53, 72]. The doubling method is not the only integral representation to lift the barrier of globally generic representations: other constructions of similar generality were developed, thus far without complete local theory, in [10, 42, 43, 63, 99, 100]. While the local ramified theory is probably within reach (see, e.g., [99, Theorem 4.2]), it will require an abundance of work. For example, these integrals are far less uniform than the doubling method, and for orthogonal groups one uses the Bessel model of π which involves an auxiliary representation.

The rest of this work is organized as follows. In §1 we provide some general preliminaries, define (k, c) representations and recall the doubling construction of [18, 20]. The proof of our main result is given in §2. Section 3 contains our main applications.

Parts of the nonarchimedean version of Theorem A for supercuspidal representations appear in [17]; Cai and the present authors were working independently.

1. Preliminaries

1.1. The groups

Let $l \geq 1$ be an integer. Let $B_{\mathrm{GL}_l} = T_{\mathrm{GL}_l} \ltimes N_{\mathrm{GL}_l}$ denote the Borel subgroup of upper triangular invertible matrices, where N_{GL_l} is its unipotent radical. The standard parabolic subgroups of GL_l can be identified with the set of compositions $\beta = (\beta_1, \ldots, \beta_a)$ of l ($\beta_i \geq 0$, $a \geq 1$), where $P_\beta = M_\beta \ltimes V_\beta$ denotes the parabolic subgroup with $M_\beta = \mathrm{GL}_{\beta_1} \times \cdots \times \mathrm{GL}_{\beta_a}$ and $V_\beta < N_{\mathrm{GL}_l}$. Let J_l be the permutation matrix with 1 on the anti-diagonal and 0 otherwise. For $g \in \mathrm{GL}_l$, ${}^tg$ denotes the transpose of g, and $g^* = J_l\, {}^tg^{-1} J_l$.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ (resp., $\lceil x \rceil$) denotes the largest (resp., smallest) integer smaller (resp., greater) than or equal to x.

For an even l, define

$$\mathrm{Sp}_l = \left\{ g \in \mathrm{GL}_l : {}^tg \begin{pmatrix} & J_{l/2} \\ -J_{l/2} & \end{pmatrix} g = \begin{pmatrix} & J_{l/2} \\ -J_{l/2} & \end{pmatrix} \right\}.$$

Let $B_{\mathrm{Sp}_l} = \mathrm{Sp}_l \cap B_{\mathrm{GL}_l}$. For any l, let $\mathrm{SO}_l = \{g \in \mathrm{SL}_l : {}^tgJ_lg = J_l\}$ and fix $B_{\mathrm{SO}_l} = \mathrm{SO}_l \cap B_{\mathrm{GL}_l}$. Let Spin_l be the algebraic double cover of SO_l, with the Borel subgroup which is the preimage of B_{SO_l}. This defines the set of simple roots $\alpha_0, \ldots, \alpha_{\lfloor l/2 \rfloor - 1}$ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $0 \leq i < \lfloor l/2 \rfloor - 1$, and GSpin_l can be defined as the Levi subgroup of Spin_{l+2} obtained by removing α_0. For $l = 0, 1$, $\mathrm{GSpin}_l = \mathrm{GL}_1$, and $\mathrm{GSpin}_2 = \mathrm{GL}_1 \times \mathrm{GL}_1$.

Henceforth we fix one of the families of groups GL_l, Sp_l (when l is even), SO_l or GSpin_l, and for a given l denote the member by $\mathscr{G}_l$, e.g., $\mathscr{G}_l = \mathrm{Sp}_l$. Write the Borel subgroup in the form $B_{\mathscr{G}_l} = T_{\mathscr{G}_l} \ltimes N_{\mathscr{G}_l}$, where $N_{\mathscr{G}_l}$ is the unipotent radical. For a parabolic subgroup $R < \mathscr{G}_l$, δ_R denotes its modulus character, and we write $R = M_R \ltimes U_R$ where M_R is the Levi part and U_R is the unipotent radical. If $U < \mathscr{G}_l$ is a unipotent subgroup, U^- denotes the opposite subgroup. The Weyl group of $\mathscr{G}_l$ is denoted $W(\mathscr{G}_l)$, and similar notation is used for any reductive group. The center of an algebraic group X is denoted C_X, and its connected component by C_X°.

The unipotent subgroups of GSpin_l are isomorphic (as algebraic groups) to the unipotent subgroups of SO_l, and $W(\mathrm{GSpin}_l)$ is isomorphic to $W(\mathrm{SO}_l)$. Also $C_{\mathrm{GSpin}_{2l+1}}$ is connected and for $l > 2$, $C^\circ_{\mathrm{GSpin}_l} \cong \mathrm{GL}_1$.

Let F be a local field with characteristic 0. Throughout, we identify F-groups with their F-points, e.g., $\mathscr{G}_l = \mathscr{G}_l(F)$. The additive group of $l \times l'$ matrices (over F) is denoted $\mathrm{Mat}_{l \times l'}$ and $\mathrm{Mat}_l = \mathrm{Mat}_{l \times l}$. The trace map is denoted tr. If F is nonarchimedean, we let q denote the cardinality of its residue field. When we say that a property holds outside a discrete subset of s, over a nonarchimedean field we mean for all but finitely many values of q^{-s}. For any group X, $x, y \in X$ and $Y < X$, ${}^xy = xyx^{-1}$ and ${}^xY = \{{}^xy : y \in Y\}$.

1.2. Representations

We describe the general notation involving representations that appear in this work. In this section $\mathscr{G}_l$ can be replaced with any reductive algebraic group. By a representation of a closed subgroup of $\mathcal{G}_l$ we always mean a smooth representation on a complex vector space. Over archimedean fields, an admissible representation is understood to be admissible Fréchet of moderate growth. If π is a representation of a closed subgroup $Y < \mathcal{G}_l$, $\pi^\vee$ is the representation contragredient to π, and for $x \in \mathcal{G}_l$, ${}^x\pi$ denotes the representation of ${}^xY$ on the same space as π, with the action given by ${}^x\pi(y) = \pi({}^{x^{-1}}y)$. Parabolic induction is normalized. Morphisms are continuous and induction is smooth, and $\otimes$ is the complete tensor product, over archimedean fields.

In this work supercuspidal representations are not automatically irreducible (or unitary). When the field is nonarchimedean, a representation of a group which does not have unipotent subgroups is also (trivially) supercuspidal. By definition, supercuspidal representations only exist over nonarchimedean fields.

For a closed unipotent subgroup $U < \mathcal{G}_l$, denote the set of (unitary) characters of U by $\widehat{U}$. Let π be a representation of U on a space $\mathcal{V}$. For $\psi \in \widehat{U}$, let $\mathcal{V}(U, \psi) \subset \mathcal{V}$ be the subspace spanned by the vectors $\pi(u)\xi - \psi(u)\xi$ for all $u \in U$ and $\xi \in \mathcal{V}$ over nonarchimedean fields, and over archimedean fields $\mathcal{V}(U, \psi)$ is the closure of this subspace. The Jacquet module $J_{U,\psi}(\pi)$ is the quotient $\mathcal{V}(U, \psi)\backslash\mathcal{V}$. Assume $R < \mathcal{G}_l$ is a closed subgroup containing U. Denote the normalizer of U in R by $N_R(U)$. If π is a representation of R, $J_{U,\psi}(\pi)$ is a representation of the subgroup of $N_R(U)$ which stabilizes ψ. We do not twist the action, i.e., we do not multiply by a modulus character. For any $r \in R$, we have an isomorphism ${}^rJ_{U,\psi}(\pi) \cong J_{{}^rU,{}^r\psi}(\pi)$ of representations of ${}^rU$ (use $\xi \mapsto \pi(r)\xi$). In particular, if $r \in N_R(U)$, then ${}^rJ_{U,\psi}(\pi) \cong J_{U,{}^r\psi}(\pi)$.

Over nonarchimedean fields, if U is abelian and $N_R(U)$ acts on $\widehat{U}$ with finitely many orbits, by [11, §§5.9–5.12] if $J_{U,\psi'}(\pi) = 0$ when ψ' varies over a complete set of representatives for the nontrivial orbits, U acts trivially on the space of π, i.e., $\pi = J_{U,1}(\pi)$.

Let $J_{U,\psi}(\pi)^*$ be the algebraic dual of $J_{U,\psi}(\pi)$ over a nonarchimedean field, and the continuous dual over archimedean fields. By definition $\mathrm{Hom}_U(\pi, \psi) = J_{U,\psi}(\pi)^*$.

Over archimedean fields we will also need the notion of generalized Jacquet modules. Let π be a representation of $\mathcal{G}_l$, and $R = M_R \ltimes U_R < \mathcal{G}_l$ be a parabolic subgroup. Denote the Lie algebra of U_R by $\mathfrak{u} = \mathfrak{u}_R$. For any positive integer i, we call $\pi/\overline{\mathfrak{u}^i\pi}$ the i-th *generalized Jacquet module* of π.

Lemma 1.1. *If π is an admissible finite length representation of $\mathcal{G}_l$, the i-th generalized Jacquet module is an admissible finite length representation of M_R.*

This lemma is proven in the same way as the classical case ($i = 1$); see Wallach [110, Lemma 4.3.1].

Lemma 1.2. *Assume π is an admissible finite length representation of $\mathcal{G}_l$. The set of central exponents of $\pi/\mathfrak{u}^i\pi$, i.e., the central characters of the irreducible constituents of $\pi/\mathfrak{u}^i\pi$ as a representation of M_R, where i varies over the positive integers, belong in a discrete set.*

Proof. Let V denote the Harish-Chandra module of π, i.e., the space of K-finite vectors, where $K \subset \mathcal{G}_l$ is a maximal compact subgroup. By [22, Proposition 2.2], V is dense in π, and by [22, Proposition 5.1 and Lemma 5.3], $V/\mathfrak{u}V$ has finitely many central exponents. In other words, for any X in the Lie algebra of the center of $\mathcal{G}_l$ there exists a polynomial p such that $p(X)$ acts by zero on $V/\mathfrak{u}V$.

Now, $V/\mathfrak{u}^iV$ is filtered by the modules $\mathfrak{u}^jV/\mathfrak{u}^{j+1}V$, $0 \leq j < i$, and each of these is a quotient of $\mathfrak{u}^j \otimes V/\mathfrak{u}V$. When i varies, the set of central exponents of $V/\mathfrak{u}^iV$ is contained in the set of central exponents of $\mathfrak{u}^j \otimes V/\mathfrak{u}V$, $j \geq 0$. Regarding $\mathfrak{u}^j$, its central exponents can be computed using the adjoint action, and when j varies they belong in a lattice. Since $V/\mathfrak{u}V$ admits only finitely many central exponents, the central exponents of $V/\mathfrak{u}^iV$ for all i lie in a finite union of lattices.

Finally, note that for any i, the set of central exponents of $\pi/\mathfrak{u}^i\pi$ lies in the set of central exponents of $V/\mathfrak{u}^iV$. Indeed, if $p(X)$ acts by zero on $V/\mathfrak{u}^iV$ then it acts by zero on $\pi/\mathfrak{u}^i\pi$, since V is dense in π. ■

Remark 1.3. In particular, the set of central exponents of the i-th generalized Jacquet modules of π, where i varies over the positive integers, belongs in a discrete set.

Let ψ be a nontrivial additive character of F. For $v \in V_{(c^l)}$, write $v = (v_{i,j})_{1\leq i,j\leq l}$ with $v_{i,j} \in \mathrm{Mat}_c$. Denote

$$\psi_l(v) = \psi\Big(\sum_{i=1}^{l-1} \mathrm{tr}(v_{i,i+1})\Big).$$

For a representation π of GSpin_l which admits a central character, let χ_π be the restriction of the central character of π to $C^\circ_{\mathrm{GSpin}_l}$.

1.3. Distribution vanishing theorem

Let a real algebraic group C act on a real algebraic manifold X. Let E be a smooth representation of C in a Fréchet space. Assume the actions of C on X and on E extend to a Lie group A, which contains C as a closed normal subgroup.

Let $Z \subset X$ be a closed subset which is a union of finitely many locally closed A-orbits. For any $\nu \in \mathbb{Z}_{>0}$ and $z \in Z$ let Λ_z^ν be the symmetric ν-th power of the conormal space at z to the orbit Cz in X. Let C_z denote the stabilizer of z in C, and δ be the ratio of modular functions of C and C_z.

Denote the space of E-valued distributions on X, i.e., functionals on the space of compactly supported smooth E-valued functions on X, by $\mathcal{D}'(X, E)$, and let $\mathcal{D}'_Z(X, E) \subset \mathcal{D}'(X, E)$ denote the subspace of distributions supported on Z. For a smooth character χ of Z, let $\mathcal{D}'_Z(X, E)^{C,\chi} \subset \mathcal{D}'_Z(X, E)$ be the subspace of (C, χ)-equivariant distributions.

The following theorem follows from Theorem A.13 in the appendix:

Theorem 1.4. *Assume that for any $z \in Z$, the set $\{\chi^a|_{C_z} : a \in A\}$ is a union of finitely many locally closed orbits under the action of the stabilizer A_z of z in A. Suppose also that for any $z \in Z$ and any $\nu \geq 0$,*

$$((E \otimes \Lambda_\nu \otimes \delta)^*)^{C_z,\chi} = 0. \tag{1.1}$$

Then $\mathcal{D}'_Z(X, E)^{C,\chi} = 0$.

Remark 1.5. If χ is trivial or $A = C$, the theorem already follows from [68, Theorem 3.15, Cases (i, ii)]. Note that in both cases $\chi^a = \chi$ for any $a \in A$.

Remark 1.6. If A, X and the action of A on X are semialgebraic, the A-orbits in X are automatically locally closed. If in addition C is semialgebraic and C_z is unipotent, the condition (1.1) is equivalent to $(E^*)^{C_z,\chi} = 0$, independently of ν (see [103]).

In order to check the conditions of the theorem we will need the following lemma.

Lemma 1.7. *Let H be a real reductive group and $Q < H$ be a parabolic subgroup with a unipotent radical $U = U_Q$. The set $\widehat{U}$ (the unitary characters of U) is a finite union of locally closed Q-orbits.*

Proof. Let $\mathfrak{u}$ denote the Lie algebra of U. There exists a hyperbolic semisimple element $S \in H$ such that $\mathfrak{u}$ is the sum of positive eigenspaces of the adjoint action $\mathrm{ad}(S)$. The eigenspace $\mathfrak{u}_1$ corresponding to the smallest positive eigenvalue of $\mathrm{ad}(S)$ is called the *first internal Chevalley module* of Q. Clearly, $\mathfrak{u}_1$ projects onto (and in fact identifies with) the space of characters of $\mathfrak{u}$, which in turn identifies with $\widehat{U}$ by multiplying by i and exponentiation. By [91, Theorem E′], Q has finitely many orbits on $\mathfrak{u}_1$, and each orbit is locally closed since the action is algebraic. ∎

1.4. Representations of type (k, c)

Let k and c be positive integers. For a partition σ of kc, let $V(\sigma) < N_{\mathrm{GL}_{kc}}$ denote the corresponding unipotent subgroup, and $\widehat{V}(\sigma)_{\mathrm{gen}}$ denote the set of generic characters. If σ' is another partition of kc, write $\sigma' \gtrsim \sigma$ if σ' is greater than or incomparable with σ, with respect to the natural partial ordering on partitions. See [41], [29, §5] and [21] for details on these notions. For convenience, we provide the definition of $V(\sigma)$. Identify σ with an l-tuple $(a_1, \ldots, a_l)$ of integers such that $a_1 \geq \cdots \geq a_l > 0$. Let p_σ be the kc-tuple of integers obtained by arranging the multi-set $\{a_i - 2j + 1 : 1 \leq i \leq l,\ 1 \leq j \leq a_i\}$ in decreasing order. For any $x \in F^*$, put $x^{p_\sigma} = \mathrm{diag}(x^{p_\sigma(1)}, \ldots, x^{p_\sigma(kc)}) \in T_{\mathrm{GL}_{kc}}$. The one-parameter subgroup $\{x^{p_\sigma} : x \in F^*\}$ acts on the Lie algebra of $N_{\mathrm{GL}_{kc}}$ by conjugation, and $V(\sigma)$ is the subgroup generated by the weight subspaces of weight at least 2.

For the orbit (k^c), $V((k^c)) = V_{(c^k)}$, the group $M_{(c^k)}$ acts transitively on the set $\widehat{V}((k^c))_{\mathrm{gen}}$, and $\psi_k \in \widehat{V}((k^c))_{\mathrm{gen}}$. The stabilizer of ψ_k in $M_{(c^k)}$ is then the diagonal embedding $\mathrm{GL}_c^{\triangle}$ of GL_c in $M_{(c^k)}$.

Let ρ be a representation of GL_{kc}. We say that ρ is a (k, c) *representation* if $\mathrm{Hom}_{V(\sigma)}(\rho, \psi') = 0$ for all $\sigma \gtrsim (k^c)$ and $\psi' \in \widehat{V}(\sigma)_{\mathrm{gen}}$, and $\dim \mathrm{Hom}_{V_{(c^k)}}(\rho, \psi_k) = 1$.

We briefly recall the definition of the wave-front set (see e.g., [48, §4.1] for some more details). When ρ is admissible of finite length, its character defines a distribution on a neighborhood of 0 in the Lie algebra of GL_{kc}. This distribution (in the nonarchimedean case) or the leading term of its asymptotic expansion near 0 (archimedean case) is a combination of Fourier transforms of Haar measures of nilpotent coadjoint orbits ([54], [50, p. 180], [9, Theorems 1.1 and 4.1]). For a nilpotent orbit $\mathcal{O}$, let $c_{\mathcal{O}}$ denote its coefficient in this expansion (for a suitable normalization of the measures). The *wave-front set* $\mathrm{WF}(\rho)$ of ρ is defined to be the set of orbits $\mathcal{O}$ such that $c_{\mathcal{O}} \neq 0$ and for every other orbit $\mathcal{O}'$ containing $\mathcal{O}$ in its closure, $c_{\mathcal{O}'} = 0$.

In this case, an equivalent definition of a (k, c) representation can be given in terms of $\mathrm{WF}(\rho)$: Now ρ is (k, c) if (k^c) is the unique maximal orbit in $\mathrm{WF}(\rho)$ and the dimension of the space of degenerate Whittaker functionals on ρ with respect to $V_{(c^k)}$ and ψ_k is 1 (see [46, Theorem E]).

For $c = 1$, a representation is $(k, 1)$ if and only if it affords a unique Whittaker model. On the other end, a representation is $(1, c)$ if and only if $\dim \mathrm{Hom}_{V_{(c)}}(\rho, 1) = 1$, equivalently ρ is a character ($V_{(c)}$ is the trivial group).

For a (k, c) representation ρ, $\dim J_{V_{(c^k)},\psi_k}(\rho)^* = 1$, hence $\dim J_{V_{(c^k)},\psi_k}(\rho) = 1$ so that $\mathrm{SL}_c^{\triangle}$ acts trivially on $J_{V_{(c^k)},\psi_k}(\rho)$ and $\mathrm{GL}_c^{\triangle}$ acts on $J_{V_{(c^k)},\psi_k}(\rho)$ by a character.

We recall the map ρ_c defined (implicitly) in [19, §2.2] from irreducible admissible generic representations of GL_k to admissible finite length (k, c) representations of GL_{kc}. For an irreducible tempered representation τ of GL_k, $\rho_c(\tau)$ is the generalized Speh representation, i.e., the unique irreducible quotient of $\mathrm{Ind}_{P_{(k^c)}}^{\mathrm{GL}_{kc}}((\tau \otimes \cdots \otimes \tau)\delta_{P_{(k^c)}}^{1/(2k)})$ (see [55, 83]). Then if $\tau = \mathrm{Ind}_{P_\beta}^{\mathrm{GL}_k}(\otimes_{i=1}^{d} |\det|^{a_i} \tau_i)$ where β is a composition of d parts of k, $a_1 > \cdots > a_d$ and each τ_i is tempered, $\rho_c(\tau) = \mathrm{Ind}_{P_{\beta c}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^{d} |\det|^{a_i} \rho_c(\tau_i))$. By [19, Theorem 4] the representation $\rho_c(\tau)$ is (k, c). The definition of $\rho_c(\tau)$ was also extended to unramified principal series $\mathrm{Ind}_{B_{\mathrm{GL}_k}}^{\mathrm{GL}_k}(\otimes_{i=1}^{k} |\det|^{a_i} \tau_i)$, where τ_i are unramified unitary quasi-characters of F^* and $a_1 \geq \cdots \geq a_k$, again by letting $\rho_c(\tau) = \mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^{k} |\det|^{a_i} \rho_c(\tau_i))$ (note that $\rho_c(\tau_i) = \tau \circ \det_{\mathrm{GL}_c}$). While $\rho_c(\tau)$ might be reducible in the general case, it is still admissible, of finite length and admits a central character. Also note that (over any local field) $\mathrm{GL}_c^{\triangle}$ acts on $J_{V_{(c^k)},\psi_k}(\rho_c(\tau))$ by $g \mapsto \tau((\det g)I_k)$ [19, Lemma 12].

We mention that over nonarchimedean fields, certain structural properties of irreducible (k, c) representations follow from [82, §II.2]. For principal series representations, irreducible or not, over any local field, a representation is (k, c) if and only if it takes the form $\mathrm{Ind}_{P_{(c^k)}}^{\mathrm{GL}_{kc}}(\otimes_{i=1}^{k} \chi_i \det_{\mathrm{GL}_c})$ for quasi-characters χ_i of F^*. This follows from [3,4,46] (their focus was archimedean; the nonarchimedean case essentially follows from [11,82]).

1.5. Doubling setup

We define the basic setup for the doubling method: the groups G and H, the image of $G \times G$ in H, and the definition of the local integral. The precise details depend on G.

Let $c, k \geq 1$ be integers, $G = \mathcal{G}_c$ and $H = \mathcal{G}_{2kc}$ (if $G = \mathrm{Sp}_c$, c must be even). Let $n = \lfloor c/2 \rfloor$ if $G \neq \mathrm{GL}_c$, otherwise $n = c$. Also set $\epsilon_0 = -1$ for $G = \mathrm{Sp}_c$ and $\epsilon_0 = 1$ otherwise, and if $G = \mathrm{SO}_c, \mathrm{GSpin}_c$ and c is odd, define $(\epsilon_1, \epsilon_2) = (1, 1/2)$ if k is even and $(\epsilon_1, \epsilon_2) = (1/2, 1)$ if k is odd.

Recall $B_H = T_H \ltimes N_H$ is our fixed Borel subgroup in H (see §1.1). Set $H_0 = \mathcal{G}_{2c}$. Let $Q = M_Q \ltimes U_Q$ be the standard parabolic subgroup of H such that its Levi part M_Q

is isomorphic to $\mathrm{GL}_c \times \cdots \times \mathrm{GL}_c \times H_0$ if $H \neq \mathrm{GL}_{2kc}$, otherwise $Q = P_{(c^{k-1},2c,c^{k-1})}$. Denote $U = U_Q$. We construct the following character ψ_U of U.

For $k > 1$, denote the middle $4c \times 4c$ block of an element in U by

$$\begin{pmatrix} I_c & u & v \\ & I_{2c} & u' \\ & & I_c \end{pmatrix}. \tag{1.2}$$

Let $u^{1,1}$ be the top left $n \times n$ block of u, and if $H \neq \mathrm{GL}_{2kc}$, denote the bottom right $n \times n$ block of u by $u^{2,2}$. For $H = \mathrm{GL}_{2kc}$, $u^{2,2}$ is defined to be the top $c \times c$ block of u'. If $H = \mathrm{SO}_{2kc}, \mathrm{GSpin}_{2kc}$ and c is odd, denote the middle two coordinates of row $n+1$ of u by $(u^3, u^4) \in \mathrm{Mat}_{1\times 2}$.

If $H \neq \mathrm{GL}_{2kc}$ and $k > 1$, the character ψ_U restricts to ψ_{k-1} on the group $V_{(c^{k-1})}$, identified with a subgroup of U via the embedding $v \mapsto \mathrm{diag}(v, I_{2c}, v^*) \in U$. For $H = \mathrm{GL}_{2kc}$ and $k > 1$, ψ_U restricts to ψ_{k-1}^{-1} on each of the two copies of $V_{(c^{k-1})}$, embedded in U via $(v_1, v_2) \mapsto \mathrm{diag}(v_1, I_{2c}, v_2)$ ($v_1, v_2 \in V_{(c^{k-1})}$). The character ψ_U is given on (1.2) by

$$\begin{cases} \psi(\mathrm{tr}(-u^{1,1} + u^{2,2})), & H = \mathrm{GL}_{2kc}, \\ \psi(\mathrm{tr}(u^{1,1} + u^{2,2})), & H = \mathrm{Sp}_{2kc}, \mathrm{SO}_{2kc}, \mathrm{GSpin}_{2kc}, \text{ even } c, \\ \psi(\mathrm{tr}(u^{1,1} + u^{2,2}) + \epsilon_1 u^3 - \epsilon_2 u^4), & H = \mathrm{SO}_{2kc}, \mathrm{GSpin}_{2kc}, \text{ odd } c. \end{cases}$$

For $k = 1$, U and thereby ψ_U are trivial.

Now consider the case $H \neq \mathrm{GSpin}_{2kc}$. In this case $G \times G$ is embedded in the stabilizer of ψ_U in M_Q. Explicitly, assume $k \geq 1$ and $g_1, g_2 \in G$. If $H = \mathrm{Sp}_{2kc}, \mathrm{SO}_{2kc}$ with an even c, write $g_1 = \left(\begin{smallmatrix} g_{1,1} & g_{1,2} \\ g_{1,3} & g_{1,4} \end{smallmatrix}\right)$, $g_{1,i} \in \mathrm{Mat}_n$; then

$$(g_1, g_2) = \mathrm{diag}\left(g_1, \ldots, g_1, \begin{pmatrix} g_{1,1} & & g_{1,2} \\ & g_2 & \\ g_{1,3} & & g_{1,4} \end{pmatrix}, g_1^*, \ldots, g_1^*\right),$$

where g_1^* appears $k-1$ times. For $H = \mathrm{GL}_{2kc}$,

$$(g_1, g_2) = \mathrm{diag}(g_1, \ldots, g_1, g_1, g_2, g_1, \ldots, g_1).$$

Here g_1 appears k times on the left of g_2 and $k-1$ on the right.

For odd c and $H = \mathrm{SO}_{2kc}$, take column vectors $e_{\pm i}$, $1 \leq i \leq c$, whose Gram matrix is J_{2c} (i.e., ${}^t e_i e_{-j} = \delta_{i,j}$). Let

$$\begin{aligned} b &= (e_1, \ldots, e_{c-1}, \epsilon_1 e_c - \epsilon_2 e_{-c}, \epsilon_1 e_c + \epsilon_2 e_{-c}, e_{-c+1}, \ldots, e_{-1}), \\ b_1 &= (e_1, \ldots, e_n, \epsilon_1 e_c - \epsilon_2 e_{-c}, e_{-n}, \ldots, e_{-1}), \\ b_2 &= (e_{n+1}, \ldots, e_{c-1}, \epsilon_1 e_c + \epsilon_2 e_{-c}, e_{-c+1}, \ldots, e_{-n-1}), \\ m &= \mathrm{diag}\left(I_{c-1}, \left(\begin{smallmatrix} \epsilon_1 & \epsilon_1 \\ -\epsilon_2 & \epsilon_2 \end{smallmatrix}\right), I_{c-1}\right). \end{aligned}$$

The Gram matrices of (b, b_1, b_2) are $(J_{2c}, \mathrm{diag}(I_n, -1, I_n)J_c, J_c)$. The left (resp., right) copy of SO_c acts on the subspace spanned by b_1 (resp., b_2); the left copy is defined by

$$\{g_1 \in \mathrm{SL}_c : {}^t g_1 \mathrm{diag}(I_n, -1, I_n) J_c g_1 = \mathrm{diag}(I_n, -1, I_n) J_c\},$$

and the right copy is defined using the convention of §1.1 (the Gram matrix of b_2 is J_c). Extend g_i by letting it fix the vectors of b_{3-i}, then write this extension as a matrix $g_i' \in \mathrm{SO}_{2c}$ with respect to b, $i = 1, 2$. The matrices ${}^m g_1'$ and ${}^m g_2'$ commute and the embedding is given by

$$(g_1, g_2) = \mathrm{diag}(g_1, \ldots, g_1, {}^m g_1' \, {}^m g_2', g_1^*, \ldots, g_1^*).$$

The notation $(1, g)$ or $(g, 1)$ is used for the embedding of one of the copies of G in H, where 1 denotes the identity element of G.

Example 1.8. Here are a few examples for the embedding in the odd orthogonal case, adapted from [20, Example 15]. Consider the standard Siegel parabolic subgroup R of G. For $a, b \in \mathrm{GL}_n \cong M_R$,

$$(a, b) = \mathrm{diag}(\mathrm{diag}(a, 1, a^*)^{\triangle'}, \mathrm{diag}(a, b, I_2, b^*, a^*)),$$

where $\triangle'$ denotes the diagonal embedding of GL_c in $\mathrm{GL}_{(k-1)c}$, and we omitted, here and below, the bottom right $(k-1)c \times (k-1)c$ block of (a, b) (it is uniquely defined by the given blocks and H). The images of $(U_R, 1)$ and $(1, U_R)$ take the form

$$\mathrm{diag}\left(\begin{pmatrix} I_n & x & y \\ & 1 & x' \\ & & I_n \end{pmatrix}^{\triangle'}, \begin{pmatrix} I_n & & \epsilon_2 x & -\epsilon_1 x & & y \\ & I_n & & & & \\ & & 1 & & & \epsilon_1 x' \\ & & & 1 & & -\epsilon_2 x' \\ & & & & I_n & \\ & & & & & I_n \end{pmatrix}\right),$$

$$\mathrm{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_n & & & & & \\ & I_n & \epsilon_2 x & \epsilon_1 x & y & \\ & & 1 & & \epsilon_1 x' & \\ & & & 1 & \epsilon_2 x' & \\ & & & & I_n & \\ & & & & & I_n \end{pmatrix}\right),$$

where x' is uniquely determined given x and H. We also note that

$$\begin{aligned} &\left(\begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}, 1\right) \\ &= \mathrm{diag}\left(\begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}^{\triangle'}, \begin{pmatrix} I_{n-1} & & & & & & & \\ & & & & & & 1 & \\ & & I_n & & & & & \\ & & & & 2\epsilon_1^2 & & & \\ & & & 2\epsilon_2^2 & & & & \\ & & & & & I_n & & \\ & 1 & & & & & & \\ & & & & & & & I_{n-1} \end{pmatrix}\right), \end{aligned}$$

$$\left(1, \begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & -1 & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}\right) = \mathrm{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_{2n-1} & & & & & \\ & & & & 1 & \\ & & & -2\epsilon_1^2 & & \\ & & -2\epsilon_2^2 & & & \\ & 1 & & & & \\ & & & & & I_{2n-1} \end{pmatrix}\right).$$

The case of $H = \mathrm{GSpin}_{2kc}$ is slightly more complicated: we have an embedding of

$$\{(z, z) : z \in C_G^\circ\} \backslash G \times G \tag{1.3}$$

in M_Q, in the stabilizer of ψ_U. Here, with a minor abuse of notation, (z, z) is regarded as an element of $G \times G$. For details see [20, §3.5].

We define the space of the induced representation of H, which is used for the construction of the integral. First assume $H = \mathrm{Sp}_{2kc}, \mathrm{SO}_{2kc}$. Let $P = M_P \ltimes U_P$ be a standard maximal parabolic subgroup of H such that $M_P \cong \mathrm{GL}_{kc}$ and $M_P < M_{(kc,kc)}$. Let ρ be a representation of GL_{kc}. For a complex parameter s, let $V(s, \rho)$ be the space of $\mathrm{Ind}_P^H(|\det|^{s-1/2}\rho)$. For $H = \mathrm{GSpin}_{2kc}$ we take the standard parabolic subgroup P obtained by removing the simple root α_{kc}, then $M_P \cong \mathrm{GL}_{kc} \times \mathrm{GL}_1$, and note that GL_1 is identified with C_H°. Let ρ be as above, and η be a quasi-character of F^*. Then $V(s, \rho \otimes \eta)$ is the space of the induced representation $\mathrm{Ind}_P^H(|\det|^{s-1/2}\rho \otimes \eta)$. For $H = \mathrm{GL}_{2kc}$ we take $P = P_{(kc,kc)}$, $\rho = \rho_1 \otimes \rho_2$ for two representations ρ_1 and ρ_2 of GL_{kc}, and $V(s, \rho)$ denotes the space of $\mathrm{Ind}_P^H(|\det|^{s-1/2}\rho_1 \otimes |\det|^{-s+1/2}\rho_2)$.

Assume $H \neq \mathrm{GSpin}_{2kc}$. Take $\delta_0 \in H$ satisfying ${}^{\delta_0}U_P = U_P^-$ if kc is even or $H = \mathrm{GL}_{2kc}$; otherwise take $\delta_0' \in \mathrm{O}_{2kc}$ with ${}^{\delta_0'}U_P = U_P^-$ and let δ_0 be the product of δ_0' and a representative of the transposition in O_{2kc} which normalizes N_H (to obtain $\det \delta_0 = 1$). Let $\delta_1 \in H_0 \cap U_P$ ($H_0 < M_Q$) be such that its natural identification with a matrix in Mat_c is of rank c, unless $H = \mathrm{SO}_{2kc}$ and c is odd, in which case the rank is $c - 1$ (this is the maximal rank). Put $\delta = \delta_0\delta_1$. Then let ι be an involution of G such that ${}^{\delta}\{(g, {}^{\iota}g) : g \in G\} < M_P$.

One concrete choice of δ_0, δ_1 and ι was given in [20]:

$$\delta_0 = \begin{cases} \begin{pmatrix} & I_{kc} \\ \epsilon_0 I_{kc} & \end{pmatrix}, & H \neq \mathrm{SO}_{2kc} \text{ or } c = 2n, \\ \begin{pmatrix} & I_{kc} \\ I_{kc} & \end{pmatrix} \mathrm{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_n & \\ & (-1)^k \\ & I_n & \end{pmatrix}, \begin{pmatrix} & I_n \\ (-1)^k & \\ & & I_n \end{pmatrix}, I_{(k-1)c}\right) J_{kc}, & \\ & H = \mathrm{SO}_{2kc}, c = 2n + 1, \end{cases}$$

where $J_{kc} = \mathrm{diag}\left(I_{kc-1}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}^{kc}, I_{kc-1}\right)$, $\delta_1 = \mathrm{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_c & A \\ & I_c \end{pmatrix}, I_{(k-1)c}\right)$ with

$$A = \begin{cases} I_c, & H = \mathrm{Sp}_{2kc}, \mathrm{GL}_{2kc}, \\ \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix}, & H = \mathrm{SO}_{2kc}, c = 2n, \\ \begin{pmatrix} & -I_n & \\ & & I_n \\ 0 & & \end{pmatrix}, & H = \mathrm{SO}_{2kc}, c = 2n + 1, \end{cases}$$

and

$$\iota = \begin{cases} \begin{pmatrix} & I_n \\ -\epsilon_0 I_n & \end{pmatrix}, & H = \mathrm{Sp}_{2kc}, \mathrm{SO}_{2kc}, c = 2n, \\ I_c, & H = \mathrm{GL}_{2kc}, \\ \begin{pmatrix} & & I_n \\ & I_2 & \\ I_n & & \end{pmatrix}, & H = \mathrm{SO}_{2kc}, c = 2n + 1, k \text{ is odd}, \\ \begin{pmatrix} & & & I_n \\ & & -2\epsilon_1^2 & \\ & -2\epsilon_2^2 & & \\ I_n & & & \end{pmatrix}, & H = \mathrm{SO}_{2kc}, c = 2n + 1, k \text{ is even}. \end{cases}$$

Also set $U_0 = U \cap {}^{J_{kc}}U_P$; then ψ_U is a character of U_0 by restriction.

For $H = \mathrm{GSpin}_{2kc}$, $\delta_0 \in H$ is defined using the isomorphism $W(H) \cong W(\mathrm{SO}_{2kc})$, δ_1 is the element taken for SO_{2kc}, and ι satisfies the same condition as above (the concrete examples of ι extend to involutions of GSpin_c as well). The important observation for us here concerning GSpin_{2kc} is that the images of unipotent subgroups, and subgroups GL_l occurring as direct factors of standard Levi subgroups, can be read off from the corresponding orthogonal cases.

For any representation π of G, π^ι is the representation on the space of π, with the action defined by $\pi^\iota(g) = \pi({}^\iota g)$. The definitions imply $(\pi^\iota)^\vee = (\pi^\vee)^\iota$.

We define the (local) doubling integral. Let π be an irreducible admissible representation of G. If $H \neq \mathrm{GL}_{2kc}$, let ρ be an admissible finite length (k, c) representation of GL_{kc} which admits a central character. Otherwise $\rho = \rho_1 \otimes \chi^{-1}\rho_2$ where ρ_1 and ρ_2 are admissible finite length (k, c) representations of GL_{kc}, each admitting a central character, and such that the central character of ρ_1 is the inverse of the central character of ρ_2, and χ is a quasi-character of F^*.

Let ω be a matrix coefficient of $\pi^\vee$. Let f be a holomorphic section of $V(s, \rho)$ if $H \neq \mathrm{GSpin}_{2kc}$, and for GSpin_{2kc}, f is a holomorphic section of $V(s, \rho \otimes \chi_\pi)$. The doubling integral for $\pi \times \rho$ is defined by

$$Z(s, \omega, f) = \int_G \int_{U_0} \omega(g) f(s, \delta u_0(1, {}^\iota g))\, \psi_U(u_0)\, du_0\, dg. \tag{1.4}$$

Here if $H = \mathrm{GSpin}_{2kc}$, the domain of integration is $C_G^\circ \backslash G$ instead of G.

Theorem 1.9 ([20, Propositions 17, 20, 21]). *Integral* (1.4) *enjoys the following properties.*

(1) *Formally, it belongs to the space*

$$\mathrm{Hom}_{(G,G)}\big(J_{U,\psi_U^{-1}}(V(s, \rho \otimes \chi_\pi)), \chi^{-k}\pi^\vee \otimes \pi^\iota\big). \tag{1.5}$$

Here χ_π and χ are omitted for the cases where they are undefined.

(2) *It is absolutely convergent for* $\mathrm{Re}(s) \gg 0$*, independent of the data* (ω, f)*.*

(3) *Over nonarchimedean fields there is data (ω, f), where f is a polynomial section in $q^{\mp s}$, such that $Z(s, \omega, f)$ is absolutely convergent in $\mathbb{C}$ and equals a nonzero constant (independent of s). Over archimedean fields for each s there is ω and a smooth section f such that the integral is nonzero at s.*

Proof. The theorem was proved in *loc. cit.*, for the representation $\rho = \rho_c(\tau)$ ($H \neq \mathrm{GL}_{2kc}$) or $\rho = \rho_c(\tau) \otimes \chi^{-1}\rho_c(\tau^\vee)$ ($\rho_c(\tau)$ was defined in §1.4). However, the proofs of these statements remain valid when we take the more general representation ρ as described above. ∎

Over nonarchimedean fields, once we prove that (1.5) is at most one-dimensional outside a discrete subset of s, Theorem 1.9 together with Bernstein's continuation principle (in [7]) imply that for a rational section f, $Z(s, \omega, f)$ admits meromorphic continuation to a rational function in q^{-s}. Over archimedean fields for the choice of ρ in [20], the meromorphic continuation of the integral and continuity of this continuation in the input data were proved in [20, §6.13].

2. Uniqueness results

2.1. Outline of the proof of Theorem A

Let π_1 and π_2 be admissible finite length representations of G. If $H \neq \mathrm{GL}_{2kc}$, let ρ be an admissible finite length (k,c) representation of GL_{kc}. For $H = \mathrm{GL}_{2kc}$ put $\rho = \rho_1 \otimes \rho_2$ where each ρ_i is an admissible finite length (k,c) representation of GL_{kc}, and let χ_0 be the quasi-character of F^* such that the diagonal action of $\mathrm{GL}_c^{\triangle}$ on $J_{V_{(c^k)},\psi_k}(\rho_1) \otimes J_{V_{(c^k)},\psi_k}(\rho_2)$ is given by $g \mapsto \chi_0(\det g)$ $(g \in \mathrm{GL}_c)$. If $H = \mathrm{GSpin}_{2kc}$, assume in addition that $\chi_{\pi_1}, \chi_{\pi_2}$ exist and $\chi_{\pi_1}^{-1} = \chi_{\pi_2}$, and put $\eta = \chi_{\pi_1}^{-1}$. To preserve uniform notation the characters χ_0, χ_{π_i} and η are simply ignored in all other cases.

Let $D = U \rtimes (G,G) < H$. We will prove our main result by analyzing distributions on the orbits of the right action of D on the homogeneous space $P\backslash H$. The space $P\backslash H/D$ is finite if $k=1$ (see [90, Lemma 2.1]) or $c=1$ (then either $n=0$ and $U = N_H$, or $G = \mathrm{GL}_1$ and U contains all the roots of N_H but one, on which (G,G) acts with two orbits). Otherwise it is infinite, even uncountable (e.g., for $H = \mathrm{Sp}_{2kc}$ and $k > 2$), but contains a unique Zariski open orbit which is $P\delta D$. This follows by showing that the dimension of $P\delta D$ is equal to the dimension of PU_P^-. For $h, h' \in H$, write $h \sim h'$ if $PhD = Ph'D$, otherwise $h \nsim h'$.

Regard $\psi_U \otimes \pi_1^\vee \otimes \pi_2^\vee$ as a representation of D. For $H = \mathrm{GSpin}_{2kc}$, (G,G) is a homomorphic image of $G \times G$ (see (1.3)), and the condition $\chi_{\pi_1}^{-1} = \chi_{\pi_2}$ above implies that $\pi_1^\vee \otimes \pi_2^\vee$ is a representation of (G,G).

Consider the space

$$\mathrm{Hom}_{(G,G)}\big(J_{U,\psi_U^{-1}}(V(s,\rho\otimes\eta)), \pi_1 \otimes \pi_2\big) \cong \mathrm{Hom}_D\big(V(s,\rho\otimes\eta) \otimes \psi_U \otimes \pi_1^\vee \otimes \pi_2^\vee, 1\big), \tag{2.1}$$

which is isomorphic to

$$\mathrm{Hom}_D\Big(\mathrm{Ind}_{P\times D}^{H\times D}\big((|\det|^{s-1/2}\rho\otimes\eta)\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee)\big), 1\Big). \tag{2.2}$$

Here the action of D on the space of functions ξ on $H \times D$ is given by $d \cdot \xi(h', d') = \xi(h'd, d'd)$; and if $H = \mathrm{GL}_{2kc}$, $|\det|^{s-1/2}\rho$ is short for $|\det|^{s-1/2}\rho_1 \otimes |\det|^{-s+1/2}\rho_2$.

For any $h \in H$, denote $P_h = {}^{h^{-1}}P \cap D$. We will study (2.2) by considering the following spaces of distributions on the orbits PhD (this is well defined, see below):

$$\mathrm{Hom}_D\Big(\mathrm{ind}_{P_h}^D\big({}^{h^{-1}}((|\det|^{s-1/2}\rho\otimes\eta)\delta_P^{1/2})\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee\otimes\Lambda_\nu)\big), 1\Big). \tag{2.3}$$

Here over nonarchimedean fields, ind denotes the compact nonnormalized induction, while for archimedean fields, ind is the Schwartz induction of [30, §2] (see also [47, §2.3]); and Λ_0 is the trivial character. If the field is nonarchimedean or $h \sim \delta$, we only have $\nu = 0$. Over archimedean fields when $h \nsim \delta$, we further have, for each integer $\nu > 0$, a finite-dimensional algebraic representation Λ_ν which is the algebraic dual of the symmetric ν-th power of the normal bundle to the double coset. Note that when $h \sim \delta$, i.e., for the open orbit $P\delta D$, the tangent space to the double coset coincides with the total tangent space, and thus the normal space is trivial.

By the Frobenius reciprocity (2.3) is isomorphic to

$$\mathcal{H}_\nu(h) = \mathrm{Hom}_{P_h}\big({}^{h^{-1}}(|\det|^{s-1/2}\rho \otimes \eta) \otimes (\psi_U \otimes \pi_1^\vee \otimes \pi_2^\vee) \otimes \Lambda_\nu, \theta_h\big). \tag{2.4}$$

Here $\theta_h(x) = \delta_{P_h}(x)\delta_D^{-1}(x)\delta_P^{-1/2}({}^h x)$ $(x \in P_h)$. We define $\mathcal{H}(h) = \mathcal{H}_0(h)$ if F is nonarchimedean or $h \sim \delta$, otherwise $\mathcal{H}(h) = \bigoplus_\nu \mathcal{H}_\nu(h)$.

Our main result (Theorem 2.1 below) is that (2.1) is at most one-dimensional outside a discrete subset of s. We will prove there is a discrete subset $\mathcal{B} \subset \mathbb{C}$ such that for all $s \notin \mathcal{B}$, $\mathcal{H}(h) = 0$ for all $h \nsim \delta$, and $\dim \mathcal{H}(\delta) \leq 1$.

Over nonarchimedean fields this already implies (2.1) is at most one-dimensional outside $\mathcal{B}$. Indeed, this follows from the theory of distributions on l-sheafs of [11]. In more detail, let $\mathcal{F}$ be the l-sheaf of the induced representation in (2.2). The right action of D on $P\backslash H$ is constructive, by [11, Theorem A] applied to $X(F)$ where X is the algebraic F-variety $P\backslash H$. Each $\mathcal{H}(h)$ (see (2.3)) is the space of distributions on the restriction of $\mathcal{F}$ to the orbit PhD (the orbits are locally closed, hence this restriction is well defined). Fix $s \notin \mathcal{B}$ and let $\mathcal{T}, \mathcal{T}'$ be nonzero distributions in (2.2). Since $P\delta D$ is open, by [11, §1.16] both $\mathcal{T}$ and $\mathcal{T}'$ restrict to distributions on $\mathcal{H}(\delta)$, which is one-dimensional, hence there is $\alpha \in \mathbb{C}$ such that $\alpha\mathcal{T}|_{P\delta D} = \mathcal{T}'|_{P\delta D}$. Then $\alpha\mathcal{T} - \mathcal{T}'$ is well defined on the quotient l-sheaf $\mathcal{F}(P\delta D)\backslash\mathcal{F}$ (see [11, §1.16] for the definition and notation), which is an l-sheaf on the complement of $P\delta D$ in H. Since there are no nonzero distributions on any $\mathcal{H}(h)$ for $h \nsim \delta$, by [11, Theorem 6.9] we deduce $\alpha\mathcal{T} - \mathcal{T}'$ vanishes on $\mathcal{F}$, i.e., $\alpha\mathcal{T} = \mathcal{T}'$.

Over archimedean fields the argument also depends on the precise methods we use in order to handle each $\mathcal{H}(h)$. We describe this below.

2.1.1. Basic properties of $\mathcal{H}(h)$. In general every algebraic representation of a unipotent group is unipotent, i.e., admits a (finite) filtration such that the group acts trivially on each of its quotients. We can hence filter each Λ_ν and consider these quotients. If (2.4) is nonzero, it is nonzero when Λ_ν is replaced by one of these quotients. Since we will prove $\mathcal{H}_\nu(h) = 0$ for all $\nu > 0$, we can consider each of these quotients, re-denoted Λ_ν (at the cost of relabeling the index set of ν), separately, so that we assume Λ_ν is a trivial representation of U for all $\nu \geq 0$.

In general if $Y < {}^hU \cap M_P$, then ${}^{h^{-1}}Y < P_h$ and by definition any morphism in $\mathcal{H}(h)$ factors through $J_{Y,{}^h\psi_U^{-1}}(\rho)$. Indeed, since ${}^{h^{-1}}Y < U$, for $y \in Y$ we have

$$^{h^{-1}}(|\det|^{s-1/2}\rho \otimes \eta)({}^{h^{-1}}y) = \rho(y), \quad (\psi_U \otimes \pi_1^\vee \otimes \pi_2^\vee \otimes \Lambda_\nu)({}^{h^{-1}}y) = \psi_U({}^{h^{-1}}y),$$

so if $\mathcal{T} \in \mathcal{H}_\nu(h)$ for some ν, and $\xi_\rho \otimes \xi$ is a pure tensor in the space of $\rho \otimes (\pi_1^\vee \otimes \pi_2^\vee \otimes \Lambda_\nu)$, then

$$\psi_U({}^{h^{-1}}y)\mathcal{T}(\rho(y)\xi_\rho \otimes \xi) = \mathcal{T}(\psi_U({}^{h^{-1}}y)\rho(y)\xi_\rho \otimes \xi) = \mathcal{T}(\xi_\rho \otimes \xi).$$

Thus

$$\mathcal{T}((\rho(y)\xi_\rho - {}^h\psi_U^{-1}(y)\xi_\rho) \otimes \xi) = 0. \tag{2.5}$$

This means that $\mathcal{T}$ factors through $J_{Y,{}^h\psi_U^{-1}}(\rho)$, where in the archimedean case note that $\mathcal{T}$ is continuous, and because the argument is applicable to all ν, we conclude that any morphism in $\mathcal{H}(h)$ factors through $J_{Y,{}^h\psi_U^{-1}}(\rho)$.

2.1.2. The vanishing of $\mathcal{H}(h)$. One can prove the vanishing of $\mathcal{H}(h)$ using three types of arguments. First we have an incompatibility condition: assume h is such that

$$\psi_U|_{U \cap {}^{h^{-1}}U_P} \neq 1. \tag{2.6}$$

In this case we can take a subgroup $Y < U$ such that ${}^hY < U_P$ and $\psi_U|_Y \neq 1$. Then $Y < P_h$ and both ${}^{h^{-1}}(|\det|^{s-1/2}\rho \otimes \eta)$ and $\pi_1^\vee \otimes \pi_2^\vee \otimes \Lambda_\nu$ are trivial on Y (because ${}^hY < U_P$ and $Y < U$), hence the action on the left hand side in $\mathcal{H}_\nu(h)$ is given by ψ_U, which is nontrivial by (2.6). However, the action on the right hand side is trivial, because it is given by a modulus character and $Y < U$. Thus $\mathcal{H}_\nu(h) = 0$ for all ν, and $\mathcal{H}(h) = 0$.

Note that while a priori (2.6) depends on h, we will actually prove it only depends on the double coset PhQ (this is only important for the archimedean parts).

Second, if any morphism in $\mathcal{H}(h)$ factors through $J_{V(\sigma),\psi'}(\rho)$, where $\sigma \gtrsim (k^c)$ and $\psi' \in \widehat{V}(\sigma)_{\mathrm{gen}}$, then $J_{V(\sigma),\psi'}(\rho) = 0$ because ρ is (k, c), and a fortiori $\mathcal{H}(h) = 0$.

Let us remark that these two methods for proving vanishing will be applied to all but finitely many representatives. In fact, consider the Bruhat decomposition $H = \coprod_{w'} Pw'Q$ where w' are representatives for Weyl elements of H, and let w_0 denote the representative of the longest reduced Weyl element. Then the orbit Pw_0Q is open. The above arguments prove vanishing on $H - Pw_0Q$. The remaining orbit Pw_0Q is the disjoint union of finitely many orbits PhD, namely $n+1$ orbits when $H \neq \mathrm{GL}_{2kc}$ and $(c+1)(c+2)/2$ orbits for $H = \mathrm{GL}_{2kc}$. In particular, as explained in §1.5, one can choose $\delta = \delta_0\delta_1$ with $\delta_0 = w_0$ and $\delta_1 \in N_{H_0}$, so $P\delta D \subset Pw_0Q$. The orbits in Pw_0Q must be handled using the third method, which we now describe.

Third, assume there is a composition β of kc and a character ψ of V_β, which may depend on h, such that any morphism in $\mathcal{H}(h)$ factors through $J_{V_\beta,\psi}(\rho)$. The vanishing argument in this case will be applicable to all but a discrete subset of s.

We first describe the nonarchimedean case. Assume there is a proper parabolic subgroup $R = M_R \ltimes U_R < G$, with M_R containing GL_l as a direct factor, $l \geq 1$, such that $J_{V_\beta,\psi}(\rho)$ is a trivial representation of ${}^h(1, U_R)$. Therefore any morphism in $\mathcal{H}(h)$ also factors through $J_{U_R}(\pi_2^\vee)$, which is an admissible finite length representation of M_R (if π_2 is supercuspidal, we immediately deduce $\mathcal{H}(h) = 0$). On each irreducible constituent of $J_{U_R}(\pi_2^\vee)$, as a representation of M_R, $C_{\mathrm{GL}_l} < C_{M_R}$ acts by a character, and there are only finitely many such characters possible, depending only on π_2 and U_R (thereby on h).

Also assume $J_{V_\beta,\psi}(\rho)$ admits a finite length filtration as a representation of ${}^h(1, \mathrm{GL}_l)$, and on each of the (not necessarily irreducible) constituents, ${}^h(1, C_{\mathrm{GL}_l})$ acts by a character. Again this character belongs to a finite set, now depending only on ρ and on the character ψ (which depends on h).

If $0 \neq \mathcal{T} \in \mathcal{H}(h)$, we can take constituents $\mathcal{V}$ of $J_{V_\beta,\psi}(\rho)$ and $\mathcal{V}'$ of $J_{U_R}(\pi_2^\vee)$ such that $\mathcal{T}$ is well defined and nonzero on $\mathcal{V} \otimes \pi_1^\vee \otimes \mathcal{V}'$. We then obtain a relation

$$\begin{aligned} \mu(a)|a|^{bs}\mathcal{T}(\xi) &= \mathcal{T}\Big(\big({}^{h^{-1}}(|\det|^{s-1/2}\rho \otimes \eta) \otimes (\psi_U \otimes \pi_1^\vee \otimes \pi_2^\vee)\big)(1,a)\xi\Big) \\ &= \theta_h((1,a))\mathcal{T}(\xi), \end{aligned} \tag{2.7}$$

where μ is a quasi-character of F^* which belongs to a finite set depending only on (π_2, η, ρ, h), and b is a constant which depends only on h, and we assume $b \neq 0$. We deduce $\mu(a)|a|^{bs} = \theta_h((1,a))$ for all $a \in F^*$. This excludes at most a discrete subset of s, and if we apply this argument to only finitely many representatives h, the set of these values of s can be taken to be our $\mathcal{B}$.

Now assume the field is archimedean. Let $\mathfrak{u}_R$ denote the Lie algebra of U_R. Assume ${}^h(1, \mathfrak{u}_R)$ acts locally nilpotently on $J_{V_\beta,\psi}(\rho)^*$. Then there is a countable increasing filtration of (closed subspaces) $\mathcal{W}_i$ of $J_{V_\beta,\psi}(\rho)^*$ by the order of nilpotency. The orthogonal complements $\mathcal{V}_i = (\mathcal{W}_i)_\perp \subset J_{V_\beta,\psi}(\rho)$ form a decreasing filtration of $J_{V_\beta,\psi}(\rho)$, exhausting in the sense that $\bigcap_i \mathcal{V}_i = 0$. For each i, $J_{V_\beta,\psi}(\rho)/\mathcal{V}_i$ is a quotient of a generalized Jacquet module of ρ with respect to ${}^h(1, \mathfrak{u}_R)$. Since any morphism in $\mathcal{H}_\nu(h)$ lies in some $\mathcal{W}_i$, it is annihilated by $\mathfrak{u}_R^i$. Thus it factors through a generalized Jacquet module $\pi_2^\vee/\overline{\mathfrak{u}_R^i\pi_2^\vee}$. The latter is an admissible finite length representation of M_R by Lemma 1.1, in particular it admits a finite filtration such that C_{GL_l} acts by a character on each constituent.

Assume in addition that there exists a parabolic subgroup of GL_{kc}, whose Levi part contains ${}^h(1, \mathrm{GL}_l)$ as a direct factor, such that the Lie algebra $\mathfrak{v}$ of its unipotent radical acts locally nilpotently on $J_{V_\beta,\psi}(\rho)^*$. Repeating the argument in the last paragraph, any morphism in $\mathcal{H}_\nu(h)$ factors through a generalized Jacquet module $\rho/\overline{\mathfrak{v}^j\rho}$, and the latter—by Lemma 1.1—has a finite filtration with ${}^h(1, C_{\mathrm{GL}_l})$ acting by a character on its constituents.

Now if $0 \neq \mathcal{T} \in \mathcal{H}_\nu(h)$, there are constituents $\mathcal{V}$ of $\rho/\overline{\mathfrak{v}^j\rho}$, $\mathcal{V}'$ of $\pi_2^\vee/\overline{\mathfrak{u}_R^i\pi_2^\vee}$ and $\mathcal{V}''$ of Λ_ν (considering Λ_ν as a representation of $(1, \mathrm{GL}_l)$) such that $\mathcal{T}$ is well defined and nonzero on $\mathcal{V} \otimes \pi_1^\vee \otimes \mathcal{V}' \otimes \mathcal{V}''$. Again we can apply (2.7) and obtain a relation $\mu(a)|a|^{bs} = \theta_h((1,a))$ (with $b \neq 0$) for all $a \in F^*$. Here μ is uniquely determined by $\mathcal{V}, \mathcal{V}', \mathcal{V}''$ and h. In the archimedean case this condition excludes one s.

As we vary $\mathcal{V}$ and $\mathcal{V}'$ over the finite filtrations of $\rho/\overline{\mathfrak{v}^j\rho}$ and $\pi_2^\vee/\overline{\mathfrak{u}_R^i\pi_2^\vee}$, and also vary j and i, the actions of ${}^h(1, C_{\mathrm{GL}_l})$ and C_{GL_l} are given by a discrete set of characters, by Lemma 1.2. The action of $(1, C_{\mathrm{GL}_l})$ on $\mathcal{V}''$ is also given by a discrete set of characters, because the central characters of the set of irreducible constituents of $\{\Lambda_\nu\}_\nu$ as representations of $(1, \mathrm{GL}_l)$ form a lattice. Thus the total subset of s we exclude is still discrete (for each h). Again, repeating this for finitely many h, we will obtain a discrete set $\mathcal{B}$.

2.1.3. The space (2.1) *is at most one-dimensional outside $\mathcal{B}$: archimedean case.* Let the Bruhat cells appearing in the decomposition $P\backslash H/Q$ be $Y_0, \ldots, Y_l$, numbered so that if $Y_i \subset \overline{Y_j}$ then $i \geq j$. In particular, Y_0 is the open Bruhat cell (i.e., $Y_0 = Pw_0Q$). We have

$$\begin{aligned}\mathrm{Hom}_D(V(s,\rho\otimes\eta),\psi_U^{-1}\otimes\pi_1\otimes\pi_2) &\cong ((V(s,\rho\otimes\eta)\otimes\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee)^*)^{\Delta D}\\ &\cong \mathcal{D}'\big(H,(|\det|^{s-1/2}\rho\otimes\eta)\otimes\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee\big)^{D\times P}.\end{aligned}$$

First we show that outside $\mathcal{B}$,

$$\mathcal{D}'_{\overline{Y_1}}\big(H,(|\det|^{s-1/2}\rho\otimes\eta)\otimes\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee\big)^{D\times P}=0. \tag{2.8}$$

For any $i>0$, let $X_i=\bigcup_{j=0}^{i}Y_j$; it is an open subset of H, and Y_i is a closed submanifold of X_i. It is enough to show that for any $i>0$, outside $\mathcal{B}$ we have

$$\mathcal{D}'_{Y_i}\big(X_i,(|\det|^{s-1/2}\rho\otimes\eta)\otimes\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee\big)^{D\times P}=0. \tag{2.9}$$

Indeed, we show by induction on i that for any distribution $\mathcal{T}$ belonging to the left hand side of (2.9), the restriction $\mathcal{T}|_{X_i}$ vanishes. The base case $i=0$ holds by definition, and the induction step is (2.9). Since $X_k=H$ we get $\mathcal{T}=0$.

To prove (2.9), we divide Y_i into two cases depending on the first two vanishing arguments from §2.1.2 (which apply to all s). Assume (2.6) holds and recall this condition only depends on the double coset (this is proved in Proposition 2.7 below). In this case we show, for all s,

$$\mathcal{D}'_{Y_i}\big(X_i,(|\det|^{s-1/2}\rho\otimes\eta)\otimes\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee\big)^{U\times U_P}=0.$$

Indeed, by [68, §2, p. 70], the left hand side can be identified with the subspace of $(U\times U_P)$-invariant maps from $C_c^\infty(X_i,\psi_U)$ supported on Y_i to $((|\det|^{s-1/2}\rho\otimes\eta)\otimes\pi_1^\vee\otimes\pi_2^\vee)^*$ (recall that over archimedean fields * denotes the continuous dual). Since $U\times U_P$ acts trivially on $(|\det|^{s-1/2}\rho\otimes\eta)\otimes\pi_1^\vee\otimes\pi_2^\vee$, such a nonzero map $\mathcal{L}$ would define a nonzero distribution in $\mathcal{D}'_{Y_i}(X_i,\psi_U)^{U\times U_P}$ (e.g., fix some functional which is nonzero on the image of $\mathcal{L}$). But $\mathcal{D}'_{Y_i}(X_i,\psi_U)^{U\times U_P}=0$ by [68, Theorem 3.15, case (iii)] (in their notation $M_y^{(r)}=\Lambda_v$ which can be taken to be trivial as explained above, and $\mathcal{O}=Y_i$).

Now assume (2.6) does not hold. We prove a more general result: for all s,

$$\mathcal{D}'_{Y_i}\big(X_i,(|\det|^{s-1/2}\rho\otimes\eta)\otimes\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee\big)^{U\times P}=0. \tag{2.10}$$

We deduce it from Theorem 1.4 as follows. Let $X=X_i$, $Y=Y_i$, $C=U\times P$; $E=(|\det|^{s-1/2}\rho\otimes\eta)\otimes\pi_1^\vee\otimes\pi_2^\vee$ with U acting trivially and P acting only on $|\det|^{s-1/2}\rho\otimes\eta$; and $\chi=\psi_U^{-1}\times 1$. Let $A=Q\times P$ and extend the action of C on E to an action of A by letting Q act trivially. Condition (1.1) follows from our proof of $\mathcal{H}(h)=0$ in this case (which uses the fact that ρ is (k,c)). Note that $\mathcal{H}(h)$ can indeed be identified with the space of distributions on the orbit PhD by, e.g., [111, Theorem 5.2.4.5]. The set $\{\chi^a|_{C_z}:a\in A\}$ is finite: first, $\psi_U|_{U\cap{}^{h^{-1}}U_P}=1$ and because this condition is independent of the representative h in the double coset PhQ, χ^a is trivial on $U\cap{}^{h^{-1}}U_P$; and second,

$Q \cap {}^{h^{-1}}M_P$ is a parabolic subgroup of ${}^{h^{-1}}M_P$ and $U \cap {}^{h^{-1}}M_P$ is its unipotent radical (see (2.27) below). From this and Lemma 1.7 we deduce that the set $\{\chi^a|_{C_z} : a \in A\}$ is a finite union of orbits. All these orbits are locally closed since they are orbits of an algebraic action of an algebraic group (note that the characters are unitary). Thus Theorem 1.4 implies (2.10).

Altogether we have shown (2.8). Therefore restriction of $D \times P$-equivariant distributions from H to Y_0 is injective. Now $D \times P$ acts on Y_0 with finitely many orbits $Z_0, \ldots, Z_r$, enumerated such that $Z_i \subset \overline{Z_j}$ implies $i \geq j$, in particular, $Z_0 = P\delta D$ is the open orbit. As above is suffices to prove that for any $i > 0$, but for $s \notin \mathcal{B}$,

$$\mathcal{D}'_{Z_i}\Big(\bigcup_{j=0}^{i} Z_j, (|\det|^{s-1/2}\rho \otimes \eta) \otimes \psi_U \otimes \pi_1^\vee \otimes \pi_2^\vee\Big)^{D\times P} = 0. \tag{2.11}$$

Let $A = C = D \times P$, $E = (|\det|^{s-1/2}\rho \otimes \eta) \otimes \pi_1^\vee \otimes \pi_2^\vee$ with D acting only on $\pi_1^\vee \otimes \pi_2^\vee$ (U acting trivially), P acting only on $|\det|^{s-1/2}\rho \otimes \eta$; and $\chi = \psi_D \times 1$, where ψ_D is the character of D defined by ψ_U^{-1} extended trivially to (G, G). Our proof of $\mathcal{H}(h) = 0$ in this case (using (2.7)) implies (1.1) for $s \notin \mathcal{B}$, and Theorem 1.4 implies (2.11). It then follows that restriction of $D \times P$-equivariant distributions from H to Z_0 is injective. Combining this with the fact that $\dim \mathcal{H}(\delta) \leq 1$, we are done.

2.1.4. The main result. We now formulate our main result. Define

$$d(s, \rho, \eta, \pi_1, \pi_2) = \dim \mathrm{Hom}_{(G,G)}\big(J_{U,\psi_U^{-1}}(V(s, \rho \otimes \eta)), \pi_1 \otimes \pi_2\big).$$

Theorem 2.1. *Let π_1, π_2 and ρ be as above.*

(1) *Outside a discrete subset of s, $d(s, \rho, \eta, \pi_1, \pi_2) \leq \dim \mathrm{Hom}_G(\chi_0\pi_1^\vee, \pi_2^\iota)$.*

(2) *If π_1 and π_2 are irreducible, outside a discrete subset of s, $d(s, \rho, \eta, \pi_1, \pi_2) = 0$ unless $\pi_1 = \chi_0(\pi_2^\iota)^\vee$, in which case $d(s, \rho, \eta, \pi_1, \pi_2) \leq 1$.*

Furthermore, assume π_2 is supercuspidal and ρ is not necessarily of finite length. Then the assertions of (1) and (2) hold for all s, granted one of the following:

(a) *$H \neq \mathrm{GL}_{2kc}$ and $c > 2$; or $H = \mathrm{Sp}_{4k}$ ($G = \mathrm{Sp}_2$); or $H \neq \mathrm{GL}_{2kc}$, $c = 2$ and $\rho = \rho_c(\tau)$ for an irreducible supercuspidal representation τ of GL_k and $k > 1$.*

(b) *$H = \mathrm{GL}_{2kc}$, $ck > 1$, π_1 is also supercuspidal, and $\rho_i = \rho_c(\tau_i)$ for irreducible supercuspidal representations τ_1 and τ_2 of GL_k.*

Remark 2.2. If $\eta \neq \chi_{\pi_1}^{-1}$, then $d(s, \rho, \eta, \pi_1, \pi_2) = 0$ outside a discrete subset of s.

The proof of the theorem occupies §2.2–§2.3. Note that the case of $\mathrm{GL}_1 \times \mathrm{GL}_k$ over nonarchimedean fields was proved in [18, Lemma 35] for $\pi_1 = \pi_2^\vee$ and $\chi_0 = 1$ ($P\backslash H/D$ is finite in this case).

Recall the representations π and ρ defined in §1.5: π is an irreducible admissible representation of G, and ρ is either an admissible finite length (k, c) representation of GL_{kc} which admits a central character, or the tensor product $\rho_1 \otimes \chi^{-1}\rho_2$ of two such representations ρ_i of GL_{kc}, with a quasi-character χ of GL_k, in which case $\chi_0 = \chi^{-k}$ (the central character of ρ_1 is the inverse of the central character of ρ_2). This is a minor generalization of [20], where ρ was taken to be $\rho_c(\tau)$ (or $\rho_i = \rho_c(\tau_i)$, $i = 1, 2$).

Combining Theorem 2.1 with the doubling integral, we obtain the following.

Corollary 2.3. *Let F be nonarchimedean and consider (1.4) for the representations π and ρ defined in §1.5.*

(1) *If f is a rational section in q^{-s}, then $Z(s, \omega, f)$ admits meromorphic continuation to a rational function in q^{-s}.*

(2) *$d(s, \rho, \chi_\pi, \chi_0\pi^\vee, \pi^\iota) \geq 1$ for all s.*

Proof. For part (1), by Theorem 2.1 with $\pi_1 = \chi_0\pi^\vee$ and $\pi_2 = \pi^\iota$ (then $\eta = \chi_{\pi_1}^{-1} = \chi_\pi$), the dimension of (1.5) is at most 1 outside a discrete subset of s. Now the meromorphic continuation follows from Theorem 1.9 and Bernstein's continuation principle [7].

For part (2), fix some s_0. Consider the family $\mathcal{I}$ of integrals $Z(s, \omega, f)$, where ω varies over the matrix coefficients of $\pi^\vee$, and f varies over the sections of $V(s, \rho \otimes \chi_\pi)$ that are polynomial in $q^{\mp s}$. The set of poles of $Z(s, \omega, f) \in \mathcal{I}$ belongs to a finite set of values of q^{-s} which depends only on the representations π and ρ, by [7] (we do not claim the multiplicity of a pole is bounded independently of ω and f). Therefore, there is $r > 0$ such that all integrals of $\mathcal{I}$ are holomorphic in the punctured disk of radius $2r$ around s_0. Let γ be the boundary of the disk of radius r around s_0. Moreover, by Theorem 1.9 (3), there is $Z(s, \omega, f) \in \mathcal{I}$ which is a nonzero constant at s_0. Thus Cauchy's integral formula gives a nonzero morphism $(\omega, f) \mapsto \frac{1}{2\pi i} \oint_\gamma \frac{Z(s,\omega,f)}{s-s_0}\, ds$ in (1.5). ∎

Corollary 2.4. *Consider (1.4) for the representations π and ρ defined in §1.5. Assume π is irreducible supercuspidal and the additional assumptions (a) or (b) of Theorem 2.1 hold.*

(1) *$d(s, \rho, \chi_\pi, \chi_0\pi^\vee, \pi^\iota) = 1$ for all s.*

(2) *If f is a polynomial section in $q^{\mp s}$, then $Z(s, \omega, f)$ admits analytic continuation to a polynomial function in $q^{\mp s}$.*

Proof. The first assertion follows from Theorem 2.1 combined with Corollary 2.3 (2). The second holds because when $d(s, \rho, \chi_\pi, \chi_0\pi^\vee, \pi^\iota) = 1$ for all s, by the corollary in [7] the continuation is to a polynomial. ∎

2.2. *The case $H \neq \mathrm{GL}_{2kc}$*

As explained in §2.1, we will consider each $\mathcal{H}(h)$ separately. We prove that all but finitely many spaces $\mathcal{H}(h)$ vanish using the first two methods, show the vanishing of the remaining $\mathcal{H}(h)$ with $h \nsim \delta$ outside a discrete subset $\mathcal{B}$, then prove $\dim \mathcal{H}(\delta) \leq 1$.

Recall $n = \lfloor c/2 \rfloor$, and since we prove the result for both odd and even c simultaneously, we also use $\lceil c/2 \rceil$, which is n when c is even and $n + 1$ otherwise.

We start by describing a choice of representatives. Since $P\backslash H/N_H$ can be identified with $W(M_P)\backslash W(H)$ and $Q = M_Q \ltimes U$, we can write $P\backslash H/D = \coprod_h PhD$ with

$h = wu$, where w is a representative from $W(M_P)\backslash W(H)$ and $u \in M_Q \cap N_H$. Since $W(M_P)\backslash W(H)$ is embedded in $\mathbb{Z}_2^{kc}$, we can identify w with a kc-tuple of 0's and 1's, where the i-th coordinate (from the left) corresponds to the permutation matrix

$$\begin{pmatrix} I_{kc-i} &&&& \\ & 0 && 1 & \\ && I_{2(i-1)} && \\ & \epsilon_0 && 0 & \\ &&&& I_{kc-i} \end{pmatrix}.$$

(E.g., (0^{kc}) is the identity.) If $H \neq \mathrm{Sp}_{2kc}$, only even products of such matrices can appear in w. In this case denote for an integer $a \geq 0$, $\jmath_a = (1, 0^{kc-1})$ if a is odd otherwise $\jmath_a = (0^{kc})$. Note that $\jmath_a$ normalizes N_H. If $H = \mathrm{Sp}_{2kc}$, we set $\jmath_a = (0^{kc})$ for uniformity. We use $*$ in an expression for w to signify an undetermined coordinate (either 0 or 1).

For the case $k = 1$, we can parametrize $P\backslash H/D = P\backslash H/(G, G)$ using the elements

$$\jmath_l(0^{c-l}, 1^l)({}^{\jmath_l}u_l), \quad 0 \leq l \leq n,$$

where

$$(0^{c-l}, 1^l) = \begin{pmatrix} && J_l \\ & I_{2(c-l)} & \\ \epsilon_0 J_l && \end{pmatrix} \tag{2.12}$$

and

$$u_l = \begin{pmatrix} I_l && I_l &&&& \\ & I_{n-l} &&&&& \\ && I_l &&&& \\ &&& I_{2c-2(n+l)} &&& \\ &&&& I_l && -I_l \\ &&&&& I_{n-l} & \\ &&&&&& I_l \end{pmatrix}. \tag{2.13}$$

Here if $l < \lceil c/2 \rceil$, ${}^{\jmath_l}u_l = u_l$ (always the case for an odd c). The double cosets for $k = 1$ were described in [90, §2]; see also [45, §4] for Sp_{2c}; SO_{2c} and GSpin_{2c} with even c are similar, and for odd c also refer to the description of the embedding of $\mathrm{SO}_c \times \mathrm{SO}_c$ in SO_{2c} given in [20, Example 15] (see Example 1.8).

We start by generalizing this description, to some extent, to all $k \geq 1$. For $x \in M_Q$, denote its projection into the direct product of $k-1$ copies of GL_c by $\ell(x)$; then $x = \ell(x)\ell_0(x)$, where $\ell_0(x) \in H_0$. For $k = 1$, $x = \ell_0(x)$ and $\ell(x)$ is trivial. If $y \in M_Q$, then

$$\ell({}^x y) = {}^{\ell(x)}\ell(y), \quad \ell_0({}^x y) = {}^{\ell_0(x)}\ell_0(y).$$

For any $(g_1, g_2) \in (G, G)$,

$${}^{(g_1,g_2)}x = \ell({}^{(g_1,g_2)}x)\ell_0({}^{(g_1,g_2)}x), \tag{2.14}$$

but because $(1, g_2) \in H_0$,

$${}^{(1,g_2)}x = \ell(x)({}^{(1,g_2)}\ell_0(x)). \tag{2.15}$$

Proposition 2.5. *Let $h = wu$, where w is a representative from $W(M_P)\backslash W(H)$ and $u \in M_Q \cap N_H$. Then $h \sim \hat{w}\hat{u}$ with the following properties. There is $0 \leq l \leq n$ such that*

$$\hat{w} = \jmath_a(0^{c-l}, 1^l, w_2, \ldots, w_k), \quad \forall i, w_i \in \{0, 1\}^c, \tag{2.16}$$

where a is the sum of coordinates 1 in $(0^{c-l}, 1^l, w_2, \dots, w_k)$. Additionally $\hat{u} \in M_Q$, there is $\sigma = (g, 1) \in (G, 1)$ such that g is a representative of an element in $W(G)$ and ${}^{\sigma}\hat{u} \in M_Q \cap N_H$, ${}^{j_a}\ell_0(\hat{u})$ takes the form

$$\begin{pmatrix} I_l & & A_l & & & & \\ & I_{n-l} & & & & & \\ & & I_l & & & & \\ & & & I_{2c-2(n+l)} & & & \\ & & & & I_l & & A'_l \\ & & & & & I_{n-l} & \\ & & & & & & I_l \end{pmatrix}, \tag{2.17}$$

and there are no zero rows in A_l.

Proof. Let $E = M_E \ltimes U_E$ denote the standard parabolic subgroup of H_0 such that $M_E = \mathrm{GL}_n \times \mathcal{G}_{2(c-n)}$, and identify N_{GL_n} with its natural image in M_E. According to the description of (G, G) in §1.5, $N_{\mathrm{GL}_n} \ltimes C_{U_E} < \ell_0((G, 1))$. Let $g \in N_G$ be with $\ell_0((g, 1)) \in N_{\mathrm{GL}_n} \ltimes C_{U_E}$ and such that the projection of $\ell_0(u(g, 1))$ into N_{H_0} is trivial on N_{GL_n}. Put $u_1 = u(g, 1) \in M_Q \cap N_H$, $wu \sim wu_1$. We also have some control over the projection of the unipotent part of the representative into C_{U_E} (see below).

If c is even, then also $N_{\mathcal{G}_{2(c-n)}} < (1, G)$, and for $g \in N_{\mathcal{G}_{2(c-n)}}$ such that the projection of $\ell_0(u_1(1, g))$ into $N_{\mathcal{G}_{2(c-n)}}$ is trivial, $wu_1 \sim wu_2$ with $u_2 = u_1(1, g)$. The projection of $\ell_0(u_2)$ into $N_{\mathrm{GL}_n} \ltimes C_{U_E}$ coincides with that of $\ell_0(u_1)$.

If c is odd, $N_{\mathcal{G}_{2(c-n)}}/(1, N_G) \cong \mathrm{Mat}_{n\times 1}$. Choosing $g \in N_G$ and taking $u_2 = u_1(1, g)$, we can assume the projection of $\ell_0(u_2)$ into $N_{\mathcal{G}_{2(c-n)}}$ takes the form

$$\begin{pmatrix} I_n & y_1 & y_2 & y'' \\ & 1 & & y'_1 \\ & & 1 & y'_2 \\ & & & I_n \end{pmatrix} \in N_{\mathcal{G}_{2(n+1)}}, \tag{2.18}$$

where y'_i, y'' uniquely depend on y_1, y_2 and H, and we can choose either $y_1 = 0$ or $y_2 = 0$ (see Example 1.8). Observe that w conjugates precisely one of the columns y_1 or y_2 in (2.18) into P, so that if we choose the other column to be zero (i.e., define $g \in N_G$ accordingly), then now we already have both zero. In other words, we can write $u_3 = z^{-1}u_2$ for z defined by y_1 or y_2 such that ${}^{w}z \in P$, so $wu_1 \sim wu_2 = wzu_3 \sim wu_3$. If c is even, put $u_3 = u_2$, so that the projection of $\ell_0(u_3)$ into $N_{\mathcal{G}_{2(c-n)}}$ is trivial now for both odd and even c.

One can take a representative g of an element in $W(G)$ such that

$$w_1 = w(1, g) = j_a(0^{\lceil c/2 \rceil}, *^{kc-\lceil c/2 \rceil}). \tag{2.19}$$

Then $wu_3 \sim wu_3(1, g) = w_1({}^{(1,g)^{-1}}u_3)$. Put $u_4 = {}^{(1,g)^{-1}}u_3$; it is of the same form as u_3: this conjugation merely permutes the columns in the projection of $\ell_0(u_3)$ into $C_{U_E}\backslash U_E$. Now we can write

$$\ell_0(u_4) = \begin{pmatrix} I_n & v & v'' \\ & I_{2(c-n)} & v' \\ & & I_n \end{pmatrix} \in N_{H_0},$$

where $v \in \mathrm{Mat}_{n\times 2(c-n)}$ is arbitrary and v', v'' are uniquely defined given v and H.

Put $v = (v_1, v_2)$, $v_1 = (z_1, z_2)$ and $v_2 = (z_3, z_4)$, where $z_1, z_4 \in \mathrm{Mat}_{n\times\lceil c/2\rceil-1}$ and $z_2, z_3 \in \mathrm{Mat}_{n\times 1}$. The element w_1 does not permute any column of z_1 or z_4, and conjugates the block z_4 into P. Hence one can write $u_5 = z^{-1}u_4$, where $z \in N_{H_0}$ is defined by z_4, and the corresponding block z_4 of $\ell_0(u_5)$ is 0, so $w_1u_4 = w_1zu_5 \sim w_1u_5$.

Next we see that w_1 also conjugates precisely one column z_j out of $\{z_2, z_3\}$ into P. If a is even, then $j = 3$ and we can assume $z_3 = 0$. Otherwise $j = 2$, and we can assume $z_2 = 0$. In both cases we multiply u_5 on the left by a suitable matrix z^{-1}, and $w_1u_5 \sim w_1u_6$ with $u_6 = z^{-1}u_5$. We deduce

$$ {}^{j_a}\ell_0(u_6) = \begin{pmatrix} I_n & v & 0 & v'' \\ & I_{c-n} & & 0 \\ & & I_{c-n} & v' \\ & & & I_n \end{pmatrix} \in N_{H_0}. $$

If c is odd, v contains $n+1$ columns. We show that the rightmost column of ${}^{j_a}\ell_0(u_6)$ can be made 0. Indeed, we can take $g \in N_G$ such that

$$ \ell_0((g,1)) = \begin{pmatrix} I_n & & \epsilon_2 x & -\epsilon_1 x & & y \\ & I_n & & & & \\ & & 1 & & & \epsilon_1 x' \\ & & & 1 & & -\epsilon_2 x' \\ & & & & I_n & \\ & & & & & I_n \end{pmatrix} \in N_{H_0} $$

(see Example 1.8). The element w_1 permutes precisely one of the middle two columns into P, either the column with $\epsilon_2 x$ or with $-\epsilon_1 x$. Then if $\ell_0((g,1))$ is chosen such that the other column is 0 in $u_6(g,1)$, and $z_j \in N_{H_0}$ is defined by the column of $\ell_0((g,1))$ which is permuted by w_1 into P (thus ${}^{w_1}z_j \in P$), then

$$ w_1u_6 \sim w_1u_6(g,1) = w_1z_jz_j^{-1}u_6(g,1) \sim w_1z_j^{-1}u_6(g,1). $$

Put $u_7 = z_j^{-1}u_6(g,1)$. Then

$$ {}^{j_a}\ell_0(u_7) = \begin{pmatrix} I_n & v & & & v'' \\ & I_n & & & \\ & & I_{2(c-2n)} & & \\ & & & I_n & v' \\ & & & & I_n \end{pmatrix} \in N_{H_0}. $$

For uniformity, denote $u_7 = u_6$ when c is even.

By the definition of H, the block v'' in ${}^{j_a}\ell_0(u_7)$ above can be taken independently of v. Hence we can multiply u_7 on the right by $(g,1)$ with $g \in N_G$, where $\ell_0((g,1)) \in C_{U_E}$ is defined using v'', and obtain $u_8 = u_7(g,1)$ such that ${}^{j_a}\ell_0(u_8)$ is of the form

$$ \begin{pmatrix} I_n & v & & & \\ & I_n & & & \\ & & I_{2(c-2n)} & & \\ & & & I_n & v' \\ & & & & I_n \end{pmatrix} \in H_0. \tag{2.20} $$

Then $w_1u_7 \sim w_1u_8$. If c is odd, $\ell_0(u_8)$ commutes with j_a (since then $2(c-2n) = 2$).

At this point we still have $u_8 \in M_Q \cap N_H$, since the only changes from u to u_8 involve multiplying by elements of $M_Q \cap N_H$ (on the right or left).

For any matrix u_0 of the form (2.20), denote the block of v by $v(u_0)$. For any representative $w' = (*^{kc})$, let $\mathcal{R}(w')$ denote the set of $1 \le i \le n$ such that w' permutes the

i-th row of the block v of a general matrix (2.20). Note that $\mathcal{R}(w')$ only depends on the coordinates $\lceil c/2\rceil+1,\dots,c$ of w' (enumerating the coordinates of w' from left to right).

For each row i of $v({}^{J_a}\ell_0(u_8))$, one can always write $u_9 = z_i^{-1}u_8$, where the i-th row of $v({}^{J_a}\ell_0(u_9))$ is zero, ${}^{J_a}z_i$ is of the form (2.20) and any row $j\neq i$ in $v({}^{J_a}z_i)$ is zero. Moreover, $z_i\in P$, and if $i\notin\mathcal{R}(w_1)$, then w_1 commutes with z_i. Hence $w_1u_8 = z_iw_1u_9\sim w_1u_9$. Since we can apply this separately to each row, we can assume that for each $1\le i\le n$, either the i-th row of $v({}^{J_a}\ell_0(u_9))$ is zero or $i\in\mathcal{R}(w_1)$. The difference between u_8 and u_9 is that the nonzero rows of $v({}^{J_a}\ell_0(u_9))$ occur only at rows i which w_1 permutes.

Consider i such that both the i-th row of $v({}^{J_a}\ell_0(u_9))$ is zero and $i\in\mathcal{R}(w_1)$. In this case take $\sigma_1=(g,1)$ where $g\in G$ is a representative of an element of $W(G)$ of minimal length such that $\mathcal{R}(\sigma_1)=\{i\}$. More specifically, take g with $\ell_0((g,1)) = \jmath_1(0^{c-i},1,0^{i-1})$ (if c is odd, the right hand side is multiplied by $\mathrm{diag}(I_{c-1},2\epsilon_1^2,2\epsilon_2^2,I_{c-1})$, see Example 1.8). Since $\sigma_1=\ell(\sigma_1)\ell_0(\sigma_1)$ and $\ell(\sigma_1)\in P$,

$$\begin{aligned}
w_1u_9 &\sim w_1u_9\sigma_1 = w_1\sigma_1({}^{\sigma_1^{-1}}u_9) = \ell(\sigma_1)({}^{\ell(\sigma_1)^{-1}}w_1)\ell_0(\sigma_1)({}^{\sigma_1^{-1}}u_9)\\
&\sim ({}^{\ell(\sigma_1)^{-1}}w_1)\ell_0(\sigma_1)({}^{\sigma_1^{-1}}u_9).
\end{aligned}$$

Put $w_2=({}^{\ell(\sigma_1)^{-1}}w_1)\ell_0(\sigma_1)$, it is again a representative from $W(M_P)\backslash W(H)$ and

$$\mathcal{R}(w_2)=\mathcal{R}(w_1\ell_0(\sigma_1))=\mathcal{R}(w_1)-\{i\}.$$

Let $u_{10}={}^{\sigma_1^{-1}}u_9$. We have $\ell_0(u_{10})=\ell_0(u_9)$ if $H=\mathrm{Sp}_{2kc}$ or c is odd, otherwise $\ell_0(u_{10})$ differs from $\ell_0(u_9)$ only in the middle two columns: these columns are exchanged because of $\jmath_1$. The element $\ell(u_{10})$ need not be in N_H anymore, only in M_Q, but ${}^{\sigma_1}u_{10}\in M_Q\cap N_H$ and by (2.14), also ${}^{\ell(\sigma_1)}\ell(u_{10})\in(H_0\backslash M_Q)\cap N_H$. Since we can apply this procedure separately to each row i, we can assume the i-th row of $v({}^{J_a}\ell_0(u_{10}))$ is nonzero if and only if $i\in\mathcal{R}(w_2)$. However, we can no longer assume $\ell(u_{10})\in N_H$.

Regard GL_n as the direct factor of the standard Levi subgroup $\mathrm{GL}_n\times\mathcal{G}_{c-2n}$ of G. For any representative g of an element of $W(\mathrm{GL}_n)$, set $\sigma_2=(g,1)$. Given arbitrary sets $\mathcal{R}(w')$ and $\mathcal{R}'\subset\{1,\dots,n\}$ of the same size, one can find σ_2 for which $\mathcal{R}({}^{\sigma_2^{-1}}w')=\mathcal{R}'$. Because $i\in\mathcal{R}(w_2)$ if and only if the i-th row of $v({}^{J_a}\ell_0(u_{10}))$ is nonzero, we can choose σ_2 such that $\mathcal{R}({}^{\sigma_2^{-1}}w_2)=\{1,\dots,l\}$, where $0\le l\le n$ is the size of $\mathcal{R}(w_2)$, and simultaneously

$$\ell_0({}^{J_a}({}^{\sigma_2^{-1}}u_{10})) = \begin{pmatrix}
I_l & & v & & & & \\
& I_{n-l} & & & & & \\
& & I_n & & & & \\
& & & I_{2(c-2n)} & & & \\
& & & & I_n & & v' \\
& & & & & I_{n-l} & \\
& & & & & & I_l
\end{pmatrix} \tag{2.21}$$

where none of the rows of v are zero. Set $w_3={}^{\sigma_2^{-1}}w_2$ and $u_{11}={}^{\sigma_2^{-1}}u_{10}$. Since $\sigma_2\in(G,1)\cap P$, we have

$$w_2u_{10}\sim w_2u_{10}\sigma_2=\sigma_2({}^{\sigma_2^{-1}}w_2)({}^{\sigma_2^{-1}}u_{10})\sim w_3u_{11}.$$

Now $\mathcal{R}(w_3) = \{1, \dots, l\}$ and note that w_3 is still of the form (2.19) (with possibly a different a, but of the same parity), because when we pass to w_2 and then to w_3, we do not change the coordinates $2, \dots, \lceil c/2 \rceil$ of w_1 ($c - i \geq \lceil c/2 \rceil$ for $i \leq n$). Moreover, ${}^{w_3}({}^{j_a}(1, g)) \in M_P$ for any $g \in \mathrm{GL}_n$ (where $\mathrm{GL}_n < \mathrm{GL}_n \times \mathcal{G}_{c-2n} < G$); if j_a is trivial, w_3 simply commutes with $(1, \mathrm{GL}_n)$. Also

$$ {}^{\sigma_1\sigma_2}u_{11} = {}^{\sigma_1}u_{10} \in M_Q \cap N_H. \tag{2.22} $$

The rank of v in (2.21) is at most l, whence we can further use ${}^{j_a}(1, g_0)$ with $g_0 \in \mathrm{GL}_n$ to reduce v to an $l \times l$ block (e.g., in a column reduced echelon form). Denote $\hat{w} = w_3$ and $\hat{u} = {}^{j_a(1,g_0)^{-1}}u_{11}$. Now $\mathcal{R}(\hat{w}) = \{1, \dots, l\}$ and $\hat{w}$ takes the form (2.16), namely $j_a(0^{c-l}, 1^l, *^{(k-1)c})$. Since ${}^{w_3 j_a}(1, g) \in M_P$ for any $g \in \mathrm{GL}_n$, we have $w_3 u_{11} \sim \hat{w}\hat{u}$.

Regarding $\hat{u}$, ${}^{j_a}\ell_0(\hat{u})$ takes the form (2.17) with $A_l = v$. Denote $\sigma = \sigma_1\sigma_2$ with the notation above. We claim ${}^{\sigma}\hat{u} \in N_H$ (clearly ${}^{\sigma}\hat{u} \in M_Q$). Since the conjugation by ${}^{j_a}(1, g_0)^{-1}$ only affects the columns of v and rows of v' in (2.21), the result follows from (2.22). ∎

While it is relatively straightforward to obtain condition (2.6) when $h = w$, the representatives wu are more difficult to describe, because of the form of $\ell(u)$. The following lemma implies that (with our current structure of u) it is sufficient to obtain (2.6) for w.

Lemma 2.6. *Let $h = wu$, where w and u are given by Proposition 2.5. Assume*

$$ \psi_U|_{U \cap {}^{w^{-1}}U_P} \neq 1. \tag{2.23} $$

Then (2.6) holds as well, that is, $\psi_U|_{U \cap {}^{h^{-1}}U_P} \neq 1$.

Proof. By (2.23) there exists a root in U, such that for the subgroup $Y < U$ generated by this root, ${}^{w}Y < U_P$ and $\psi_U|_Y \neq 1$. Since $u \in M_Q$, it normalizes U, whence ${}^{u^{-1}}Y < U$, and also ${}^{h}({}^{u^{-1}}Y) = {}^{w}Y < U_P$. It remains to show $\psi_U|_{{}^{u^{-1}}Y} \neq 1$, since then (2.6) holds.

We can identify the quotient of U by its commutator subgroup with the direct product of $k-2$ copies of Mat_c, and one copy of $\mathrm{Mat}_{c \times 2c}$. The root defining Y corresponds to a coordinate (i, j) in one of these copies. Looking at the definition of ψ_U, we can be more specific. Either (i, j) belongs to one of the $k-2$ blocks of size $c \times c$, on which ψ_U is given by $\psi \circ \mathrm{tr}$, or (i, j) belongs to one of two $n \times n$ blocks inside $\mathrm{Mat}_{c \times 2c}$ ($u^{1,1}$ or $u^{2,2}$, see (1.2)), and again ψ_U restricts to $\psi \circ \mathrm{tr}$ on these blocks. Denote the block by B. In both cases, since $\psi_U|_Y \neq 1$, the coordinate (i, j) appears as a diagonal coordinate of B. When c is odd there is a third possibility, that (i, j) appears in the block $B \in \mathrm{Mat}_{1 \times 2}$ on which ψ_U is given by $\psi(\epsilon_1 B_{1,1} - \epsilon_2 B_{1,2})$, and (i, j) is either the coordinate of $B_{1,1}$ or $B_{1,2}$. In this case also note that w of the prescribed structure cannot permute both $B_{1,1}$ and $B_{1,2}$ into U_P.

Write ${}^{\sigma}u = u_0^{-1} \in M_Q \cap N_H$, where $\sigma \in (G, 1)$ is given by Proposition 2.5. Then ${}^{\sigma}Y$ is again a root subgroup, and since conjugation by σ permutes the coordinates of U and stabilizes ψ_U, ${}^{\sigma}Y$ is still defined by a coordinate (i, j) which belongs to one of the blocks B' described above. In fact, if $B \in \mathrm{Mat}_c$, we must have $B' = B$; if $B \in \mathrm{Mat}_n$, there are two options for B', one of which is B; and for $B \in \mathrm{Mat}_{1\times 2}$, $B' = B$. Of course ψ_U is nontrivial on ${}^{\sigma}Y$.

Conjugation of ${}^{\sigma}Y$ by u_0 must be performed with more care, because u_0 normalizes U but may not stabilize ψ_U. First consider the case where $B = B' \in \mathrm{Mat}_c$. Then ${}^{\sigma}Y$ is the root subgroup defined by the (d,d)-th diagonal coordinate in B, for some $1 \le d \le c$. For a fixed element $y \in Y$, assume the (d,d)-th coordinate of ${}^{\sigma}y$ is $x \ne 0$. It is the only nonzero coordinate in the projection of y to B. Since $u_0 \in M_Q$, the nontrivial coordinates of ${}^{u_0}({}^{\sigma}y)$ are still contained in B. This means that the only nonzero coordinates of ${}^{u_0}({}^{\sigma}y)$ on which ψ_U can possibly be nontrivial are coordinates in the block B. Because $u_0 \in M_Q \cap N_H$, the (d,d)-th coordinate of ${}^{u_0}({}^{\sigma}y)$ is still x, and all other nontrivial coordinates belong to the set of coordinates in B of the form $\{(i',j') \ne (d,d) : i' \le d, j' \ge d\}$, i.e., are above or to the right of the (d,d)-th coordinate. Therefore $\psi_U({}^{u_0}({}^{\sigma}y)) = \psi(x)$, hence ψ_U is nontrivial on ${}^{u_0\sigma}Y$.

Next assume $B' \in \mathrm{Mat}_n$ and proceed with similar notation. Now ${}^{u_0}({}^{\sigma}y)$ can contain nontrivial coordinates outside B'. Assume B' is the top left $n \times n$ block in $\mathrm{Mat}_{c\times 2c}$ (i.e., $u^{1,1}$). Then ${}^{u_0}({}^{\sigma}y)$ contains x in the (d,d)-th coordinate, $1 \le d \le n$, and arbitrary elements in the coordinates $(i',j') \ne (d,d)$, where $i' \le d$ only varies over the rows of B', but j' varies over all columns $j' \ge d$ of B' and also the columns to the right of B', up to the rightmost column of $\mathrm{Mat}_{c\times 2c}$ (this is the $(k+1)c$-th column for a matrix in U). Otherwise B' is the bottom right block (which is $u^{2,2}$). Then ${}^{u_0}({}^{\sigma}y)$ contains x in the (d,d)-th coordinate and may contain nontrivial coordinates for $(i',j') \ne (d,d)$, where i' varies over the rows $i' \le d$ of B' and the rows above B', up to the first row of $\mathrm{Mat}_{c\times 2c}$ (row $(k-2)c+1$ for matrices in U), and $j' \ge d$ only varies over columns of B'. In both cases ψ_U is trivial on all of the possibly nonzero coordinates (i',j'), and the (d,d)-th coordinate is x, thus $\psi_U|_{{}^{u_0\sigma}Y} \ne 1$.

If c is odd we also consider $B = B' \in \mathrm{Mat}_{1\times 2}$. Observe that now ψ_U is trivial on all coordinates above or to the right of $B_{1,2}$, and also on all coordinates above or to the right of $B_{1,1}$ except $B_{1,2}$. Hence if the nonzero coordinate x of ${}^{\sigma}y$ is in $B_{1,2}$, $\psi_U|_{{}^{u_0\sigma}Y} \ne 1$, but also if x is in $B_{1,1}$ we have $\psi_U|_{{}^{u_0\sigma}Y} \ne 1$, because multiplying $u_0({}^{\sigma}y)$ on the right by u_0^{-1} leaves $B_{1,2}$ zero (when c is odd, the $(kc, kc+1)$-th coordinate of any element of N_H is zero).

Now because $\sigma \in (G,1)$, it immediately follows that ψ_U is nontrivial on ${}^{\sigma^{-1}u_0\sigma}Y = {}^{u^{-1}}Y$, completing the proof of the lemma. ∎

Let now $h = wu$ where w and u satisfy the properties of Proposition 2.5. In particular, w defines the integer $0 \le l \le n$.

Proposition 2.7. *We have $\mathcal{H}(h) = 0$ unless*

$$w_i = (1^{\lceil c/2\rceil}, *^{n-l}, 1^l), \quad \forall 1 < i \le k. \tag{2.24}$$

Proof. For $k = 1$ there is nothing to prove, so assume $k > 1$. Write $w_2 = (w_2', w_2'')$ with $w_2' \in \{0,1\}^{\lceil c/2\rceil}$, $w_2'' \in \{0,1\}^n$. If w_2' is not of the form $(1^l, *^{\lceil c/2\rceil - l})$, then $l > 0$ (for $l = 0$, w_2' is automatically of the form $(1^l, *^{\lceil c/2\rceil - l}) = (*^{\lceil c/2\rceil})$). Let $Y < U$ be the subgroup of elements with the middle $2(c+l)\times 2(c+l)$ block of the form

$$\begin{pmatrix}
I_l &&&&& y && yA'_l & \\
& I_l &&&&&&& -A_l y' \\
&& I_{n-l} &&&&&& \\
&&& I_l &&&&& y' \\
&&&& I_{2c-2(n+l)} &&&& \\
&&&&& I_l &&& \\
&&&&&& I_{n-l} && \\
&&&&&&& I_l & \\
&&&&&&&& I_l
\end{pmatrix}.$$

Recall ψ_U restricts to $\psi\circ\mathrm{tr}$ on the block $u^{2,2}$ (see (1.2)). The block yA'_l above occupies the bottom right $l\times l$ block of $u^{2,2}$. Since there are no zero rows in A_l, there are no zero columns in A'_l. Hence for each $1\le i\le l$, the form $y\mapsto (yA'_l)_{i,i}$ on Mat_l is not identically 0. Then if one of the first l coordinates of w'_2 is 0, we can take a subgroup $Y_i<Y$ with $\psi_U|_{Y_i}\ne 1$ and ${}^hY_i<U_P$, hence $\mathcal{H}(h)=0$ by (2.6). Thus we can write $w'_2=(1^l,*^{\lceil c/2\rceil-l})$ (whether $l>0$ or $l=0$).

If $w'_2\ne(1^n,*^{\lceil c/2\rceil-n})$, one of the rows from the top left $n-l\times n-l$ block of $u^{2,2}$ is conjugated by w into U_P. Hence we can take a subgroup $Y<U$ such that $\psi_U|_Y\ne 1$ and ${}^wY<U_P$, then $\mathcal{H}(h)=0$ by (2.23).

If c is odd, ψ_U restricts to a nontrivial character on the middle two coordinates (u^3,u^4) of row $n+1$ in (1.2), and the columns of u^3 and u^4 are either swapped or remain unchanged by w. Then if $w'_2\ne(1^{\lceil c/2\rceil})$, one of these coordinates is conjugated by w into U_P, and if $Y<U$ is defined by this coordinate, we have $\psi_U|_Y\ne 1$ and ${}^wY<U_P$. Thus $\mathcal{H}(h)=0$ by (2.23). (Because $2c-2(n+l)\ge 2$, ${}^uY=Y$ and we can also apply (2.6) directly.) Thus $w'_2=(1^{\lceil c/2\rceil})$ whether c is even or odd. We proceed for all c.

Recall ψ_U restricts to $\psi\circ\mathrm{tr}$ on the top left $l\times l$ block of $u^{1,1}$. Since the first c coordinates of w are $J_a(0^{c-l},1^l)$, w permutes the columns of this block into columns in U_P, hence if $w''_2\ne(*^{n-l},1^l)$, we can again find $Y<U_P$ such that $\psi_U|_Y\ne 1$ and ${}^wY<U_P$, so that $\mathcal{H}(h)=0$ by (2.23). Altogether, $w_2=(1^{\lceil c/2\rceil},*^{n-l},1^l)$.

If $k=2$ we are done, so assume $k>2$. We show $w_3=(1^{\lceil c/2\rceil},*^{n-l},1^l)$. Recall $V_{(c^{k-1})}<U$. Because $w_2=(1^{\lceil c/2\rceil},*^{n-l},1^l)$, w conjugates the last $\lceil c/2\rceil$ and first l columns of $v_{k-2,k-1}$ (see §1.2 for this notation) into U_P. Hence if $w_3\ne(1^{\lceil c/2\rceil},*^{n-l},1^l)$, a diagonal coordinate of one of the blocks inside $v_{k-2,k-1}$, namely the bottom right $\lceil c/2\rceil\times\lceil c/2\rceil$ block if $w_3\ne(1^{\lceil c/2\rceil},*^n)$, or the top left $l\times l$ block if $w_3\ne(*^{c-l},1^l)$, is conjugated by w into U_P, so that if $Y<U$ is generated by this coordinate, then $\mathcal{H}(h)=0$ by (2.23). Proceeding in this manner for $3<i\le k$, each time using $v_{k-i+1,k-i+2}$ and (2.23), we deduce $w_i=(1^{\lceil c/2\rceil},*^{n-l},1^l)$. ∎

For each $1<i\le k$, since w_i takes the form (2.24), we can uniquely identify a maximal integer $0\le d_{i-1}\le n-l$ such that $w_i=(1^{\lceil c/2\rceil},*^{n-l-d_{i-1}},1^{l+d_{i-1}})$. By maximality $(*^{n-l-d_{i-1}})=(*^{n-l-d_{i-1}-1},0)$ (if $d_{i-1}<n-l$), but the remaining coordinates are still undetermined. As we show next, if $\mathcal{H}(h)\ne 0$, we can replace h by a representative for

which $(*^{n-l-d_{i-1}}) = (0^{n-l-d_{i-1}})$ (this may mean the integers d_{i-1} are larger), and even fix an ascending order on d_{i-1}. Note that for $k = 1$, the integers d_{i-1} are undefined.

Proposition 2.8. *We have $\mathcal{H}(h) = 0$ unless $h \sim \hat{w}\hat{u}$ where*

$$\begin{aligned} &\hat{w} = j_a(0^{c-l}, 1^l, w_2, \dots, w_k), \\ &\forall 1 < i \leq k,\ w_i = (1^{\lceil c/2 \rceil}, 0^{n-l-d_{i-1}}, 1^{l+d_{i-1}}), \quad d_1 \leq \dots \leq d_{k-1}, \end{aligned} \tag{2.25}$$

and $\hat{u}$ satisfies the conditions of Proposition 2.5, in particular $\ell_0(\hat{u})$ takes the form (2.17) (and A_l does not have any zero row).

Proof. Write $w_i = (1^{\lceil c/2 \rceil}, w_i', 1^l)$ with $w_i' \in \{0, 1\}^{n-l}$. The rightmost d_{i-1} coordinates of w_i' are 1. We start with the following observation. Let $1 \leq j \leq n - l$ and assume $1 < i_0 \leq k$ is minimal such that $w_{i_0}'[j]$ (the j-th coordinate of w_{i_0}') equals 1. We claim $\mathcal{H}(h) = 0$ unless $w_i'[j] = 1$ for all $i \geq i_0$. Otherwise, assume $i > i_0$ is minimal with $w_i'[j] = 0$. Write the top left $n \times n$ block of the block $v_{k-i+1,k-i+2}$ of $V_{(c^{k-1})}$ in the form $\left(\begin{smallmatrix} v^1 & v^2 \\ v^3 & v^4 \end{smallmatrix}\right)$ with $v^1 \in \mathrm{Mat}_l$ and $v^4 \in \mathrm{Mat}_{n-l}$. Then a unipotent subgroup Y containing coordinates from v^4 will satisfy $\psi_U|_Y \neq 1$ and ${}^{w}Y < U_P$, whence $\mathcal{H}(h) = 0$ by (2.23).

We proceed to show that we can sort the coordinates of $w_2, \dots, w_k$ to obtain (2.25). Identify GL_{n-l} with its natural image in the middle factor of the standard Levi subgroup $\mathrm{GL}_l \times \mathrm{GL}_{n-l} \times \mathcal{G}_{c-2n}$ of G. Then $P \cap (\mathrm{GL}_{n-l}, 1)$ contains a full set of representatives for $W(\mathrm{GL}_{n-l})$. Given such a representative g, we have $h \sim h(g, 1)^{-1} \sim ({}^{(g,1)}w)({}^{(g,1)}u)$ where $\hat{u} = {}^{(g,1)}u$ still satisfies the conditions of Proposition 2.5 and $\ell_0(\hat{u}) = \ell_0(u)$ ($(g, 1)$ commutes with (2.17)). Hence one can use such conjugations to permute the entries in each w_i, while maintaining the prescribed structure of u. Using transpositions from $W(\mathrm{GL}_{n-l})$ we can permute each consecutive pair $(w_i'[j], w_i'[j + 1])$. If $w_i'[j] = w_i'[j + 1]$, the conjugation has no affect on this pair. Choose some j such that there is a minimal i_0 with $(w_{i_0}'[j], w_{i_0}'[j + 1]) = (1, 0)$. If j does not exist, then $w_i' = (0^{n-l-d_{i-1}}, 1^{d_{i-1}})$ for all $1 < i \leq k$, and by what we have proved, if $\mathcal{H}(h) \neq 0$, then $d_1 \leq \dots \leq d_{k-1}$ so that (2.25) holds. If j exists, then again by the above observation (assuming $\mathcal{H}(h) \neq 0$), for $i > i_0$, either $(w_i'[j], w_i'[j + 1]) = (1, 0)$, in which case the order is swapped, or $w_i'[j] = w_i'[j + 1] = 1$. Proceeding in this manner we obtain (2.25). ∎

Let now $h = wu$, with w and u given by Proposition 2.8, in particular w satisfies (2.25). Recall that in general if $Y < {}^{h}U \cap M_P$, ${}^{h^{-1}}Y < P_h$ and by definition any morphism in $\mathcal{H}(h)$ factors through $J_{Y,{}^{h}\psi_U^{-1}}(\rho)$ (see §2.1.1). We turn to computing ${}^{h}U \cap M_P$. To simplify the presentation we slightly alter w, using multiplication on the left by representatives of $W(M_P)$, which we identify with permutation matrices in GL_{kc}. First, we multiply w on the left by $\mathrm{diag}(I_{(k-1)c}, J_l, I_{c-l})$; this changes the innermost block J_l into I_l (see (2.12)). Then for w_i, $1 < i \leq k$, we multiply w on the left by

$$\mathrm{diag}(I_{(k-i)c}, J_{l+d_{i-1}}, I_{n-l-d_{i-1}}, J_{\lceil c/2 \rceil}, I_{c(i-1)}).$$

For example if $k = 2$,

$$w = j_a \begin{pmatrix} &&&&&&&& I_{l+d_1} \\ & I_{n-l-d_1} &&&&&&& \\ &&&&&& I_{\lceil c/2 \rceil} && \\ &&&&& I_l &&& \\ &&&& I_{2(c-l)} &&&& \\ &&& \epsilon_0 I_l &&&&& \\ && \epsilon_0 I_{\lceil c/2 \rceil} &&&&&& \\ &&&&&&& I_{n-l-d_1} & \\ \epsilon_0 I_{l+d_1} &&&&&&&& \end{pmatrix}. \tag{2.26}$$

For $1 \leq j \leq k-1$, define $\gamma_j \in \mathrm{GL}_{kc}$ by

$$\begin{aligned} \gamma_j = \mathrm{diag}\Big(& I_{n-l-d_{k-j}+\sum_{i=1}^{j-1}(n-l-d_{k-i})}, \begin{pmatrix} & I_{kc-\lceil c/2 \rceil-(j-1)c-n} \\ I_{\lceil c/2 \rceil} & \end{pmatrix}, \\ & I_{l+d_{k-j}+\sum_{i=1}^{j-1}(\lceil c/2 \rceil+l+d_{k-i})} \Big) \\ \times \mathrm{diag}\Big(& I_{\sum_{i=1}^{j-1}(n-l-d_{k-i})}, \begin{pmatrix} & I_{kc-(j-1)c-l-d_{k-j}} \\ I_{l+d_{k-j}} & \end{pmatrix}, I_{\sum_{i=1}^{j-1}(\lceil c/2 \rceil+l+d_{k-i})} \Big). \end{aligned}$$

For example,

$$\gamma_1 = \mathrm{diag}\Big(I_{n-l-d_{k-1}}, \begin{pmatrix} & I_{kc-\lceil c/2 \rceil-n} \\ I_{\lceil c/2 \rceil} & \end{pmatrix}, I_{l+d_{k-1}} \Big) \begin{pmatrix} & I_{kc-l-d_{k-1}} \\ I_{l+d_{k-1}} & \end{pmatrix}.$$

Further multiply w on the left by $\gamma_{k-1} \cdot \ldots \cdot \gamma_1$ (henceforth we only use this form for w). For the computation of ${}^hU \cap M_P$ also note that ${}^hU = {}^wU$. Now we see that ${}^hU \cap M_P = V_\beta$, where β is the composition of kc given by

$$\beta = (n-l-d_{k-1}, \ldots, n-l-d_1, c, \lceil c/2 \rceil + l + d_1, \ldots, \lceil c/2 \rceil + l + d_{k-1}). \tag{2.27}$$

(The purpose of the elements γ_i was to obtain an upper triangular ${}^hU \cap M_P$.) The character ${}^h\psi_U$ is a character of V_β by restriction; denote $\psi_{V_\beta} = {}^h\psi_U|_{V_\beta}$. We cannot fully describe ψ_{V_β} without determining $\ell(u)$, but the lemma below will provide the information we need. First we describe ${}^{w\ell_0(u)}\psi_U|_{V_\beta}$. For $v \in V_\beta$ write

$$v = \begin{pmatrix} I_{n-l-d_{k-1}} & b_1 & \cdots &&&& \\ & \ddots & \ddots &&&& \\ && I_{n-l-d_1} & b_{k-1} & \cdots && \\ &&& I_c & b_k & \cdots & \\ &&&& \ddots & \ddots & \\ &&&&& I_{\lceil c/2 \rceil+l+d_{k-2}} & b_{2k-2} \\ &&&&&& I_{\lceil c/2 \rceil+l+d_{k-1}} \end{pmatrix}. \tag{2.28}$$

Here

$$(b_i)_{1 \leq i \leq 2k-2} = (b_1, \ldots, b_{k-2}, b_{k-1}, b_k, b_{k+1}, \ldots, b_{2k-2})$$

is a general element of the product

$$\begin{aligned} \prod_{j=k-1}^{2} \mathrm{Mat}_{n-l-d_j \times n-l-d_{j-1}} \times \mathrm{Mat}_{n-l-d_1 \times c} \times \mathrm{Mat}_{c \times \lceil c/2 \rceil+l+d_1} & \\ \times \prod_{j=1}^{k-2} \mathrm{Mat}_{\lceil c/2 \rceil+l+d_j \times \lceil c/2 \rceil+l+d_{j+1}}&. \end{aligned}$$

Note that for $k = 2$, $(b_i)_{1 \leq i \leq 2k-2} \in \mathrm{Mat}_{n-l-d_1 \times c} \times \mathrm{Mat}_{c \times \lceil c/2 \rceil+l+d_1}$. Then

$$
\begin{aligned}
{}^{w\ell_0(u)}\psi_U(v) = \psi\Bigg(&\sum_{j=k-1}^{2}\mathrm{tr}\Big(b_{k-j}\begin{pmatrix}0_{d_j-d_{j-1}\times n-l-d_j}\\ I_{n-l-d_j}\end{pmatrix}\Big)+\mathrm{tr}\Big(b_{k-1}\begin{pmatrix}0_{l+d_1\times n-l-d_1}\\ I_{n-l-d_1}\\ 0_{\lceil c/2\rceil\times n-l-d_1}\end{pmatrix}\Big)\\
&-\mathrm{tr}\left(b_k\begin{pmatrix}0&0&-\epsilon_0 A_l&0&0\\ 0&\epsilon_0 I_{n-l}&0&0&0\\ 0&0&0&0&I_{c-2n}\\ 0&0&0&0_{d_1\times n-l}&0\\ I_l&0&0&0&0\end{pmatrix}\right)\\
&-\sum_{j=1}^{k-2}\mathrm{tr}\left(b_{k+j}\begin{pmatrix}I_{\lceil c/2\rceil}&0_{\lceil c/2\rceil\times d_j+l}\\ 0_{d_{j+1}-d_j\times\lceil c/2\rceil}&0_{d_{j+1}-d_j\times d_j+l}\\ 0_{d_j+l\times\lceil c/2\rceil}&I_{d_j+l}\end{pmatrix}\right)\Bigg). \qquad (2.29)
\end{aligned}
$$

Here the sum $\sum_{j=k-1}^{2}$ is omitted if $k=2$; and if $c-2n=1$ ($0\le c-2n\le 1$), the coordinate $I_{c-2n}=1$ initially depends on the constants ϵ_1,ϵ_2 (see §1.5; $2\epsilon_1\epsilon_2=1$), but we can use another conjugation of w by an element of M_P to fix this coordinate to be 1 (without otherwise changing (2.29)). Additionally, for $l=n$ and A_l of rank l, the character (2.29) belongs to the orbit of ψ_k^{-1}.

Example 2.9. For $k=2$ and an even a, after multiplying (2.26) on the left by γ_1 we have

$$
w=\begin{pmatrix}
0&I_{n-l-d_1}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&I_l&0&0&0\\
0&0&0&0&I_{c-l}&0&0&0&0&0\\
0&0&0&0&0&0&0&I_{\lceil c/2\rceil}&0&0\\
0&0&0&0&0&0&0&0&0&I_{l+d_1}\\
\epsilon_0 I_{l+d_1}&0&0&0&0&0&0&0&0&0\\
0&0&\epsilon_0 I_{\lceil c/2\rceil}&0&0&0&0&0&0&0\\
0&0&0&0&0&I_{c-l}&0&0&0&0\\
0&0&0&\epsilon_0 I_l&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&I_{n-l-d_1}&0
\end{pmatrix}, \qquad (2.30)
$$

$$
V_\beta=V_{(n-l-d_1,c,\lceil c/2\rceil+l+d_1)}=\left\{\begin{pmatrix}I_{n-l-d_1}&b_1&*\\ &I_c&b_2\\ &&I_{\lceil c/2\rceil+l+d_1}\end{pmatrix}\right\}
$$

and ψ_{V_β} depends only on b_1 and b_2. For example, if $u=\ell_0(u)$, its restriction to b_1 is given by ψ composed with the trace of the $n-l-d_1\times n-l-d_1$ block of b_1 starting at column $l+d_1+1$ of b_1. For $k=3$ and again an even a, after multiplying w on the left by $\gamma_2\gamma_1$ we obtain

$$
\begin{pmatrix}
0&I_{n-l-d_2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&I_{n-l-d_1}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&I_l&0&0&0&0&0&0\\
0&0&0&0&0&0&0&I_{c-l}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&I_{\lceil c/2\rceil}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&I_{l+d_1}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&I_{\lceil c/2\rceil}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&I_{l+d_2}\\
\epsilon_0 I_{l+d_2}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\epsilon_0 I_{\lceil c/2\rceil}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\epsilon_0 I_{l+d_1}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\epsilon_0 I_{\lceil c/2\rceil}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&I_{c-l}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&\epsilon_0 I_l&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&I_{n-l-d_1}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&I_{n-l-d_2}&0
\end{pmatrix},
$$

$\beta=(n-l-d_2,n-l-d_1,c,\lceil c/2\rceil+l+d_1,\lceil c/2\rceil+l+d_2)$.

Remark 2.10. It is convenient to compute V_β in two steps: first compute ${}^wU \cap M_P$ using w without the elements γ_i, e.g., (2.26), then conjugate by these elements in order to obtain V_β.

Proposition 2.11. *Assume $k > 1$ and $l < n$. If $\mathcal{H}(h) \neq 0$, then ψ_{V_β} belongs to the orbit of*

$$v \mapsto \psi\Bigg(\sum_{j=k-1}^{2} \operatorname{tr}(b_{k-j}(*_{n-l-d_{j-1}\times n-l-d_j})) + \operatorname{tr}\left(b_{k-1}\begin{pmatrix} *_{l+d_1\times n-l-d_1} \\ I_{n-l-d_1} \\ *_{\lceil c/2\rceil\times n-l-d_1}\end{pmatrix}\right)$$
$$- \operatorname{tr}\left(b_k\begin{pmatrix} 0 & 0 & -\epsilon_0 A_l & 0 & 0 \\ 0 & \epsilon_0 I_{n-l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c-2n} \\ 0 & 0 & 0 & 0_{d_1\times n-l} & 0 \\ I_l & 0 & 0 & 0 & 0 \end{pmatrix}\right)$$
$$- \sum_{j=1}^{k-2}\operatorname{tr}\left(b_{k+j}\begin{pmatrix} I_{\lceil c/2\rceil} & 0_{\lceil c/2\rceil\times d_j+l} \\ *_{d_{j+1}-d_j\times\lceil c/2\rceil} & *_{d_{j+1}-d_j\times d_j+l} \\ *_{d_j+l\times\lceil c/2\rceil} & *_{d_j+l}\end{pmatrix}\right)\Bigg). \tag{2.31}$$

Here $*$ means undetermined block entries. When $\ell(u)$ is the identity element, all coordinates were computed above and (2.31) *coincides with* (2.29).

Proof. We introduce notation for blocks of unipotent matrices in M_Q and U. Recall from Lemma 2.6 that ψ_U is defined by $k-2$ blocks $B_i \in \mathrm{Mat}_c$, $1 \le i \le k-2$, two blocks $B'_1, B'_2 \in \mathrm{Mat}_n$ and when c is odd also by $B'' \in \mathrm{Mat}_{1\times 2}$. Set $d_0 = 0$ and $d_k = d_{k-1}$. For each $1 \le i \le k-2$, B_i is further divided into subblocks by writing it as the upper right block of

$$\begin{pmatrix} I_{l+d_{k-i-1}} & & & & B_i^{1,1} & B_i^{1,2} & B_i^{1,3} & B_i^{1,4} \\ & I_{d_{k-i}-d_{k-i-1}} & & & B_i^{2,1} & B_i^{2,2} & B_i^{2,3} & B_i^{2,4} \\ & & I_{n-l-d_{k-i}} & & B_i^{3,1} & B_i^{3,2} & B_i^{3,3} & B_i^{3,4} \\ & & & I_{\lceil c/2\rceil} & B_i^{4,1} & B_i^{4,2} & B_i^{4,3} & B_i^{4,4} \\ & & & & I_{l+d_{k-i-1}} & & & \\ & & & & & I_{d_{k-i}-d_{k-i-1}} & & \\ & & & & & & I_{n-l-d_{k-i}} & \\ & & & & & & & I_{\lceil c/2\rceil} \end{pmatrix}.$$

The blocks B'_1, B'_2 are contained in the following blocks:

$$\begin{pmatrix} I_l & & & & B_1'^{1,1} & B_1'^{1,2} & B_1'^{1,3} \\ & I_{d_1} & & & B_1'^{2,1} & B_1'^{2,2} & B_1'^{2,3} \\ & & I_{n-l-d_1} & & B_1'^{3,1} & B_1'^{3,2} & B_1'^{3,3} \\ & & & I_{\lceil c/2\rceil} & B_1'^{4,1} & B_1'^{4,2} & B_1'^{4,3} \\ & & & & I_l & & \\ & & & & & I_{d_1} & \\ & & & & & & I_{n-l-d_1} \end{pmatrix},$$

$$\begin{pmatrix} I_{l+d_1} & & & & & B_2'^{1,1} & B_2'^{1,2} & B_2'^{1,3} \\ & I_{n-l-d_1} & & & & B_2'^{2,1} & B_2'^{2,2} & B_2'^{2,3} \\ & & I_{\lceil c/2\rceil-l} & & & B_2'^{3,1} & B_2'^{3,2} & B_2'^{3,3} \\ & & & I_l & & B_2'^{4,1} & B_2'^{4,2} & B_2'^{4,3} \\ & & & & I_{2c-n-l} & & & \\ & & & & & I_l & & \\ & & & & & & I_{n-l} & \\ & & & & & & & I_l \end{pmatrix}.$$

If c is odd, we also have the $c \times 2$ block containing B'' which we write in the form

$$\begin{pmatrix} B''^1 \\ B''^2 \\ B''^3 \\ B''^4 \end{pmatrix}, \quad B''^1 \in \mathrm{Mat}_{l+d_1\times 2},\ B''^2 \in \mathrm{Mat}_{n-l-d_1\times 2},\ B''^3 \in \mathrm{Mat}_{1\times 2},\ B''^4 \in \mathrm{Mat}_{n\times 2}.$$

For $1 \le i \le 4$, $B''^i = (B''^{i,1}, B''^{i,2})$. In terms of the blocks B_i, B'_i and B'', ψ_U is given by

$$\psi\Big(\sum_{i=1}^{k-2}\sum_{j=1}^{4}\mathrm{tr}(B_i^{j,j})+\sum_{j=1}^{3}\mathrm{tr}(B_1'^{j,j})+\mathrm{tr}(\begin{pmatrix} 0_{n-l\times c-2n} & I_{n-l}\end{pmatrix} B_2'^{3,2})+\mathrm{tr}(B_2'^{4,3})+B''^3\begin{pmatrix}\epsilon_1 \\ -\epsilon_2\end{pmatrix}\Big). \tag{2.32}$$

Let $\mathscr{M}_P$, $\mathscr{U}_P$ and $\mathscr{U}_P^-$ denote the lists of blocks $B_i^{t,t'}$, $B_i'^{t,t'}$, $B''^{t,t'}$ conjugated by w into M_P, U_P and U_P^-, respectively (these can still be computed using (2.25); w differs from (2.25) by an element of M_P). If c is odd, let $a_0 \in \{1,2\}$ be the column of B'' which w conjugates into column $kc+1$, it consists of the blocks $(B''^{1,a_0}, B''^{2,a_0}, B''^{3,a_0}, B''^{4,a_0})$. We see that

$$\begin{aligned}
\mathscr{M}_P &= \{B_i^{1,1}, B_i^{1,4}, B_i^{2,1}, B_i^{2,4}, B_i^{3,2}, B_i^{3,3}, B_i^{4,1}, B_i^{4,4} : 1 \le i \le k-2\}\\
&\quad \amalg \{B_1'^{1,1}, B_1'^{2,1}, B_1'^{3,2}, B_1'^{3,3}, B_1'^{4,1}, B_2'^{1,1}, B_2'^{1,2}, B_2'^{2,3}, B_2'^{3,1}, B_2'^{3,2}, B_2'^{4,1}, B_2'^{4,2}\}\\
&\quad \amalg \{B''^{1,a_0}, B''^{2,3-a_0}, B''^{3,a_0}, B''^{4,a_0}\},\\
\mathscr{U}_P &= \{B_i^{3,1}, B_i^{3,4} : 1 \le i \le k-2\} \amalg \{B_1'^{3,1}, B_2'^{2,1}, B_2'^{2,2}, B''^{2,a_0}\},\\
\mathscr{U}_P^- &= \{B_i^{1,2}, B_i^{1,3}, B_i^{2,2}, B_i^{2,3}, B_i^{4,2}, B_i^{4,3} : 1 \le i \le k-2\}\\
&\quad \amalg \{B_1'^{1,2}, B_1'^{1,3}, B_1'^{2,2}, B_1'^{2,3}, B_1'^{4,2}, B_1'^{4,3}, B_2'^{1,3}, B_2'^{3,3}, B_2'^{4,3}, B''^{1,3-a_0},\\
&\qquad\qquad B''^{3,3-a_0}, B''^{4,3-a_0}\}.
\end{aligned}$$

Since ${}^{\ell(\sigma)}\ell(u) \in M_Q \cap N_H$ with $\sigma \in (G,1)$, we can write $\ell(u) = \mathrm{diag}(z_1,\ldots,z_{k-1})$ for $z_i = {}^{w_\sigma}v_i$ where $w_\sigma \in W(\mathrm{GL}_c)$ corresponds to the projection of $(\sigma,1)^{-1}$ into the i-th copy of GL_c and $v_i \in N_{\mathrm{GL}_c}$. Note that if we write a general element of ${}^{w_\sigma}N_{\mathrm{GL}_c}$ in the form

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}$$

where X_1, X_5 and X_9 are square matrices (of arbitrary sizes), then X_1, X_5, X_9 are already invertible, and so are $\begin{pmatrix} I & X_2 \\ -X_4 & I\end{pmatrix}$ and $\begin{pmatrix} I & X_6 \\ -X_8 & I\end{pmatrix}$, whence $I + X_2X_4$, $I + X_4X_2$, $I + X_6X_8$ and $I + X_8X_6$ are also invertible (X_i, X_j need not be square matrices).

Since the left coset of w in $W(M_P)\backslash W(H)$ is still represented by (2.25), we can write $z_i = z'_i m_i$ where ${}^{w}\mathrm{diag}(z'_1,\ldots,z'_{k-1}, I_{2c}, z_1'^*,\ldots,z_{k-1}'^*) \in M_P$ and

$$m_i = \begin{pmatrix} I_{l+d_{k-i}}+M_i^1M_i^2 & M_i^1 & 0 \\ M_i^2 & I_{n-l-d_{k-i}}+M_i^3M_i^4 & M_i^3 \\ 0 & M_i^4 & I_{\lceil c/2\rceil}\end{pmatrix} \in \mathrm{GL}_c,$$

$$I_{l+d_{k-i}} + M_i^1M_i^2 \in \mathrm{GL}_{l+d_{k-i}}, \quad I_{n-l-d_{k-i}} + M_i^3M_i^4 \in \mathrm{GL}_{n-l-d_{k-i}}.$$

These matrices are invertible because $m_i = {}^{w_\sigma}v'_i$ where $v'_i \in N_{\mathrm{GL}_c}$. We have

$$m_i^{-1} = \begin{pmatrix} I_{l+d_{k-i}} & -M_i^1 & M_i^1 M_i^3 \\ -M_i^2 & I_{n-l-d_{k-i}}+M_i^2M_i^1 & -(I_{n-l-d_{k-i}}+M_i^2M_i^1)M_i^3 \\ M_i^4M_i^2 & -M_i^4(I_{n-l-d_{k-i}}+M_i^2M_i^1) & I_{\lceil c/2\rceil}+M_i^4(I_{n-l-d_{k-i}}+M_i^2M_i^1)M_i^3 \end{pmatrix}.$$

Since $h \sim ph$ for any $p \in P$, we can already assume $z_i = m_i$.

We show that ψ_{V_β} belongs to the orbit of a character whose restriction to the blocks $b_{k-1}, b_k, b_{k+1}, \ldots, b_{2k-2}$ agrees with (2.31), otherwise $\mathcal{H}(h) = 0$. This will complete the proof. To this end it suffices to compute ${}^u\psi_U$ on the blocks of U conjugated by w into $b_{k-1}, b_k, b_{k+1}, \ldots, b_{2k-2}$. The contribution of $\ell_0(u)$ is easy to compute and was essentially given in (2.29). To determine ${}^{\ell(u)}\psi_U$ (thereby ${}^u\psi_U$) we compute

$$m_{k-1}^{-1}B'_1, \quad m_{k-1}^{-1}B'_2, \quad m_{k-1}^{-1}B'', \quad m_i^{-1}B_i m_{i+1}, \quad \forall 1 \le i \le k-2.$$

Columns $l + d_1 + 1, \ldots, n$ of b_{k-1} (the only columns of b_{k-1} where (2.31) is determined) consist of the block $B_1'^{3,3}$, conjugated to b_{k-1} by w (other columns are conjugated from $B_1'^{3,2}$, $B_2'^{2,3}$ and columns between the columns of $u^{1,1}$ and $u^{2,2}$). The coordinates of b_k are uniquely defined by

$$B_1'^{1,1}, B_1'^{2,1}, B_1'^{4,1}, B_2'^{1,1}, B_2'^{1,2}, B_2'^{3,1}, B_2'^{3,2}, B_2'^{4,1}, B_2'^{4,2}, B''^{1,a_0}, \\ B''^{2,a_0}, B''^{3,3-a_0}, B''^{4,a_0}$$

and by additional $l + d_1 + \lceil c/2\rceil \times n - l$ coordinates appearing to the left of $B_2'^{1,1}$, $B_2'^{3,1}$, $B_2'^{4,1}$ (ψ_U and ${}^{\ell_0(u)}\psi_U$ are trivial on the corresponding columns, thereby also ${}^u\psi_U$ because multiplying on the left by m_{k-1}^{-1} cannot introduce a character on a column where ${}^{\ell_0(u)}\psi_U$ was trivial, so we do not provide notation for these), as well as the form defining H. Note that B'' is omitted if c is even.

When we multiply $m_{k-1}^{-1}B'_1$ we see that if the top l rows of M_{k-1}^1 are nonzero, ${}^u\psi_U$ is nontrivial on $B_1'^{3,1} \in \mathscr{U}_P$ and then $\mathcal{H}(h) = 0$ by (2.6). Hence we can assume the top l rows of M_{k-1}^1 are 0, which implies ${}^u\psi_U$ is trivial on the coordinates of b_k obtained from B'_1, namely $B_1'^{1,1}, B_1'^{2,1}, B_1'^{4,1}$ (ψ_U and ${}^{\ell_0(u)}\psi_U$ are also trivial there). Additionally ${}^u\psi_U$ restricts to $\psi(\mathrm{tr}((I_{n-l-d_1} + M_{k-1}^2M_{k-1}^1)B_1'^{3,3}))$ on $B_1'^{3,3}$, and since

$${}^w\mathrm{diag}(I_{(k-2)c+l+d_1}, I_{n-l-d_1} + M_{k-1}^2M_{k-1}^1, I_{2(\lceil c/2\rceil+c)}, (I_{n-l-d_1} + M_{k-1}^2M_{k-1}^1)^*, \\ I_{l+d_1+(k-2)c}) \in M_P,$$

ψ_{V_β} belongs to the orbit of a character which agrees with (2.31) on b_{k-1} and the coordinates of b_k conjugated from B'_1.

The character ${}^{\ell_0(u)}\psi_U$ is given on the blocks of B'_2, which w conjugates into b_k, by

$$\psi(\mathrm{tr}(\varphi_k B_2'^{\circ})), \quad \varphi_k = \begin{pmatrix} 0_{l\times l+d_1} & 0_{l\times n-l-d_1} & 0_{l\times\lceil c/2\rceil-l} & A(X) \\ 0_{n-l\times l+d_1} & 0_{n-l\times n-l-d_1} & (0_{n-l\times c-2n}\ I_{n-l}) & 0_{n-l\times l} \end{pmatrix}.$$

Here $B_2'^{\circ}$ is the $c \times n$ block consisting of $B_2'^{t,t'}$ with $1 \le t \le 4$ and $1 \le t' \le 2$ (all of these blocks except for $t = 2$ are conjugated into b_k). Multiplying $\varphi_k m_{k-1}^{-1}$ we deduce

$\mathcal{H}(h) = 0$, unless the product of φ_k and columns $l + d_1 + 1, \ldots, n$ of m_{k-1}^{-1} defines a trivial character on $(B_2^{\prime 2,1}, B_2^{\prime 2,2}) \in \mathcal{U}_P$, which amounts to

$$\begin{pmatrix} 0_{l\times\lceil c/2\rceil-l} & A(X) \\ (0_{n-l\times c-2n}\ I_{n-l}) & 0_{n-l\times l} \end{pmatrix}(-M_{k-1}^4(I_{n-l-d_1} + M_{k-1}^2 M_{k-1}^1)) = 0_{\lceil c/2\rceil}.$$

Hence the product of φ_k and the last $\lceil c/2\rceil$ columns of m_{k-1}^{-1} equals

$$\begin{pmatrix} 0_{l\times\lceil c/2\rceil-l} & A(X) \\ (0_{n-l\times c-2n}\ I_{n-l}) & 0_{n-l\times l} \end{pmatrix}(I_{\lceil c/2\rceil} + M_{k-1}^4(I_{n-l-d_1} + M_{k-1}^2 M_{k-1}^1)M_{k-1}^3) = \begin{pmatrix} 0_{l\times\lceil c/2\rceil-l} & A(X) \\ (0_{n-l\times c-2n}\ I_{n-l}) & 0_{n-l\times l} \end{pmatrix},$$

thus ${}^u\psi_U$ agrees with ψ_U on the blocks contained in B_2'.

If c is odd, the restriction of ${}^u\psi_U$ to B'' is given by

$$\psi\left(\mathrm{tr}\left(\begin{pmatrix} 0_{2\times n} & \begin{pmatrix}\epsilon_1 \\ -\epsilon_2\end{pmatrix} & 0_{2\times n}\end{pmatrix} m_{k-1}^{-1} B''\right)\right).$$

Since $B''^{2,a_0} \in \mathcal{U}_P$ and $\epsilon_1\epsilon_2 \neq 0$, we deduce the first row of $-M_{k-1}^4(I_{n-l-d_1} + M_{k-1}^2 M_{k-1}^1)$ is 0, and because $I_{n-l-d_1} + M_{k-1}^2 M_{k-1}^1$ is invertible we find that the first row of M_{k-1}^4 is 0. Then the first row of m_{k-1}^{-1} is $(0_n\ 1\ 0_n)$, whence ${}^u\psi$ and ψ_U agree on B''.

Altogether we have shown that ψ_{V_β} belongs to the orbit of a character which agrees with (2.31) on b_{k-1} and b_k.

Consider b_{k+i}, $1 \le i \le k-2$. The coordinates of b_{k+i} are uniquely defined by the blocks

$$B_{k-i-1}^{1,1}, B_{k-i-1}^{1,4}, B_{k-i-1}^{2,1}, B_{k-i-1}^{2,4}, B_{k-i-1}^{4,1}, B_{k-i-1}^{4,4}.$$

More precisely, if we denote for $X \in \mathrm{Mat}_{a\times b}$, $X' = -J_b\,{}^tXJ_a$, then

$$b_{k+i} = \begin{pmatrix} (B_{k-i-1}^{4,4})' & (B_{k-i-1}^{2,4})' & (B_{k-i-1}^{1,4})' \\ (B_{k-i-1}^{4,1})' & (B_{k-i-1}^{2,1})' & (B_{k-i-1}^{1,1})' \end{pmatrix}. \tag{2.33}$$

We multiply $m_{k-i-1}^{-1} B_{k-i-1} m_{k-i}$. Since ψ_U restricts to $\psi \circ \mathrm{tr}$ on B_{k-i-1}, the restriction of ${}^u\psi_U$ to B_{k-i-1} is given by $\psi(\mathrm{tr}(m_{k-i} m_{k-i-1}^{-1} B_{k-i-1}))$. If this restriction is nontrivial on $B_{k-i-1}^{3,1}, B_{k-i-1}^{3,4} \in \mathcal{U}_P$, we obtain $\mathcal{H}(h) = 0$.

On $B_{k-i-1}^{3,4}$, ${}^u\psi_U$ is given by the product of the last $\lceil c/2\rceil$ rows of m_{k-i} and columns $l + d_{i+1} + 1, \ldots, n$ of m_{k-i-1}^{-1}; $\mathcal{H}(h) = 0$ unless this product vanishes:

$$\begin{pmatrix} 0_{\lceil c/2\rceil\times l+d_i} & M_{k-i}^4 & I_{\lceil c/2\rceil} \end{pmatrix}\begin{pmatrix} -M_{k-i-1}^1 \\ I_{n-l-d_{i+1}} + M_{k-i-1}^2 M_{k-i-1}^1 \\ -M_{k-i-1}^4(I_{n-l-d_{i+1}} + M_{k-i-1}^2 M_{k-i-1}^1) \end{pmatrix} = 0.$$

Thus the product of the last $\lceil c/2\rceil$ rows of m_{k-i} and the last $\lceil c/2\rceil$ columns of m_{k-i-1}^{-1} is

$$\begin{pmatrix} 0_{\lceil c/2\rceil\times l+d_i} & M_{k-i}^4 & I_{\lceil c/2\rceil} \end{pmatrix}\begin{pmatrix} M_{k-i-1}^1 M_{k-i-1}^3 \\ -(I_{n-l-d_{i+1}} + M_{k-i-1}^2 M_{k-i-1}^1)M_{k-i-1}^3 \\ I_{\lceil c/2\rceil} + M_{k-i-1}^4(I_{n-l-d_{i+1}} + M_{k-i-1}^2 M_{k-i-1}^1)M_{k-i-1}^3 \end{pmatrix} = I_{\lceil c/2\rceil}.$$

This means that the restriction of ${}^u\psi_U$ to $B_{k-i-1}^{4,4}$, which corresponds to the bottom right $\lceil c/2\rceil \times \lceil c/2\rceil$ block of $m_{k-i} m_{k-i-1}^{-1}$, is $\psi \circ \mathrm{tr}$, so that it agrees with ψ_U on this block.

On $B^{3,1}_{k-i-1}$, ${}^{u}\psi_U$ is defined by the product of the first $l+d_i$ rows of m_{k-i} and columns $l+d_{i+1}+1,\ldots,n$ of m^{-1}_{k-i-1}. Then $\mathcal{H}(h)=0$ unless

$$\begin{pmatrix} I_{l+d_i}+M^1_{k-i}M^2_{k-i} & M^1_{k-i} & 0_{l+d_i\times\lceil c/2\rceil}\end{pmatrix}\begin{pmatrix} -M^1_{k-i-1} \\ I_{n-l-d_{i+1}}+M^2_{k-i-1}M^1_{k-i-1} \\ -M^4_{k-i-1}(I_{n-l-d_{i+1}}+M^2_{k-i-1}M^1_{k-i-1})\end{pmatrix}=0.$$

Hence

$$\begin{aligned}&\begin{pmatrix} I_{l+d_i}+M^1_{k-i}M^2_{k-i} & M^1_{k-i} & 0_{l+d_i\times\lceil c/2\rceil}\end{pmatrix}\\ &\qquad\times\begin{pmatrix} M^1_{k-i-1}M^3_{k-i-1} \\ -(I_{n-l-d_{i+1}}+M^2_{k-i-1}M^1_{k-i-1})M^3_{k-i-1} \\ I_{\lceil c/2\rceil}+M^4_{k-i-1}(I_{n-l-d_{i+1}}+M^2_{k-i-1}M^1_{k-i-1})M^3_{k-i-1}\end{pmatrix}=0.\end{aligned}$$

Therefore the restrictions of ${}^{u}\psi_U$ and ψ_U to $B^{4,1}_{k-i-1}$, which correspond to the top right $l+d_i\times\lceil c/2\rceil$ block of $m_{k-i}m^{-1}_{k-i-1}$, are both trivial.

Finally, using (2.33) and noting that the leftmost l columns of X are the bottom l rows of X' (and the entries are permuted), we find that ${}^{u}\psi_U$ is given on the blocks which w conjugates into b_{k+i} by

$$\psi\left(\mathrm{tr}\left(\begin{pmatrix} (B^{4,4}_{k-i-1})' & (B^{2,4}_{k-i-1})' & (B^{1,4}_{k-i-1})' \\ (B^{4,1}_{k-i-1})' & (B^{2,1}_{k-i-1})' & (B^{1,1}_{k-i-1})'\end{pmatrix}\begin{pmatrix} I_{\lceil c/2\rceil} & 0_{\lceil c/2\rceil\times l+d_i} \\ *_{l+d_{i+1}\times\lceil c/2\rceil} & *_{l+d_{i+1}\times l+d_i}\end{pmatrix}\right)\right).$$

We conclude that ψ_{V_β} belongs to the orbit of (2.31). ∎

Proposition 2.12. *Assume $d_1<n-l$ (in particular $k>1$ and $l<n$, because $d_1\geq 0$). Then $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)=0$, in particular $\mathcal{H}(h)=0$.*

Proof. Any morphism in $\mathcal{H}(h)$ factors through $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$. We show $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)=0$. Suppose otherwise. The subgroup V_β and character $\psi^{-1}_{V_\beta}$ define a degenerate Whittaker model in the sense of [46, 82]. The character $\psi^{-1}_{V_\beta}$ uniquely defines a nilpotent element ${}^{t}\varphi\in\mathrm{Mat}_{kc}$ such that $\psi^{-1}_{V_\beta}(v)=\psi(\mathrm{tr}(v({}^{t}\varphi)))$ for all $v\in V_\beta$. Then $\varphi\in\mathrm{Mat}_{kc}$ is an upper triangular nilpotent matrix. We prove φ is nilpotent of order at least $k+1$. By [46, Theorem E], the orbit of φ belongs to the closure of the wave-front set $\mathrm{WF}(\rho)$ of ρ, but this orbit is greater than or incomparable with (k^c), contradicting the fact that ρ is (k,c). When π_2 is supercuspidal (in particular, the field is nonarchimedean) and ρ is not necessarily of finite length, we derive the same contradiction from [46, Theorem A].

By Proposition 2.11, we can assume ψ_{V_β} is given by (2.31). Since $\varphi+I_{kc}\in V_\beta$, we let $b_1,\ldots,b_{2k-2}$ denote the blocks of φ above the principal diagonal (see (2.28)). These can be read off from (2.31), namely b_i is the transpose of the block appearing to the right of b_i in (2.31) (one should include the signs appearing in (2.31) before tr). For example, $b_{k-1}=-\begin{pmatrix} *_{n-l-d_1\times l+d_1} & I_{n-l-d_1} & *_{n-l-d_1\times\lceil c/2\rceil}\end{pmatrix}$.

We apply a sequence of conjugations to φ, conjugating k nonzero coordinates from φ, one coordinate from each block $b_{k-1},b_k,\ldots,b_{2k-2}$: since $d_1<n-l$, ψ_{V_β} is nontrivial on b_{k-1}, so that the block b_{k-1} of φ is nonzero.

The only nonzero blocks of φ are the blocks $b_1, \dots, b_{2k-2}$, and these blocks contain nonzero entries at the coordinates defined by (2.31). Define the following partial sums d^j of the integers appearing in the composition β, from right to left (see (2.27)):

$$d^j = \begin{cases} \sum_{i=1}^{j}(\lceil c/2\rceil + l + d_{k-i}), & 1 \le j \le k-1, \\ c + d^{k-1}, & j = k, \\ n - l - d_1 + d^k, & j = k+1. \end{cases}$$

First we conjugate φ by

$$\varepsilon_1 = \mathrm{diag}\Big(I_{kc-d^{k-1}-\lceil c/2\rceil-1}, \begin{pmatrix} & I_{\lceil c/2\rceil-1} & 1 \\ 1 & & \end{pmatrix}, I_{d^{k-1}}\Big).$$

Note that ε_1 normalizes V_β. The (n,n)-th coordinate of b_k, which is $\epsilon_0 = \pm 1$ because $l < n$, becomes the (c,n)-th coordinate, and the $(n-l-d_1, n)$-th coordinate of b_{k-1}, which is -1, becomes the $(n-l-d_1, c)$-th coordinate—the bottom right coordinate. Both of these coordinates are independent of the blocks where ψ_{V_β} is undetermined (denoted $*$ in (2.31)), and are the only nonzero entries on their columns. We can further conjugate ${}^{\varepsilon_1}\varphi$ by an element of the group

$$\Big\{\mathrm{diag}\Big(I_{(\sum_{i=1}^{k-1}(n-l-d_i))+c}, \begin{pmatrix} b & \\ & I_{l+d_1}\end{pmatrix}, \dots, \begin{pmatrix} b & \\ & I_{l+d_{k-1}}\end{pmatrix}\Big) : b \in \mathrm{GL}_{\lceil c/2\rceil}\Big\} < M_\beta,$$

to take the (c,n)-th coordinate of b_k into the $(c,1)$-th coordinate, without affecting any of the blocks $b_{k-1}, \dots, b_{2k-2}$ of ${}^{\varepsilon_1}\varphi$ except the block b_k (the diagonal embedding of $b \in \mathrm{GL}_n$ instead of $b \in \mathrm{GL}_{\lceil c/2\rceil}$ in each of the last $k-1$ blocks is sufficient). Now let

$$\varepsilon_2 = \mathrm{diag}\Big(I_{(\sum_{i=1}^{k-1}(n-l-d_i))+c}, \begin{pmatrix} & I_{\lceil c/2\rceil+l+d_1-2} & 1 \\ 1 & & \end{pmatrix}, \dots, \begin{pmatrix} & I_{\lceil c/2\rceil+l+d_{k-1}-2} & 1 \\ 1 & & \end{pmatrix}\Big) \in M_\beta,$$

where if $\lceil c/2\rceil + l + d_{k-i} = 1$, the corresponding block of size $1 + (\lceil c/2\rceil + l + d_{k-i} - 2) + 1$ is I_1. Again ε_2 normalizes V_β. Conjugating ${}^{\varepsilon_1}\varphi$ by ε_2, the $(c,1)$-th coordinate of b_k is taken into the $(c, \lceil c/2\rceil + l + d_1)$-th coordinate (the bottom right coordinate), and the top left coordinate of b_{k+i}, which is independent of the undetermined blocks and is the only nonzero coordinate on its column, is taken into the bottom right coordinate for each $1 \le i \le k-2$. We conclude that the bottom right coordinate of each of the blocks $b_{k-1}, \dots, b_{2k-2}$ of ${}^{\varepsilon_2\varepsilon_1}\varphi$ is nonzero (-1 for b_{k-1}, ϵ_0 for b_k, 1 for all other blocks), and in each of these blocks, the bottom right coordinate is the only nonzero entry in its column. Therefore φ is nilpotent of order at least $k+1$. ∎

Example 2.13. Consider $c = k = 2$, $G = \mathrm{Sp}_2$, $H = \mathrm{Sp}_8$ and $l = 0$. Then $0 \le d_1 \le n - l = 1$. For $d_1 = 0$, $w \in M_P(0^2, w_2)$ with $w_2 = (1, 0)$. One can take

$$w = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & I_4 & \\ & -1 & & \\ & & & & 1 \end{pmatrix}.$$

Then

$$u = \begin{pmatrix} I_2 & x & y \\ & I_4 & x' \\ & & I_2 \end{pmatrix}, \quad \psi_U(u) = \psi(x_{1,1} + x_{2,4}), \quad {}^{w}U \cap M_P = \left\{\begin{pmatrix} 1 & y_{1,1} & x_{1,1} & x_{1,2} \\ & 1 & & \\ & x_{2,4} & 1 & \\ & x_{2,3} & & 1 \end{pmatrix}\right\}.$$

Multiplying w on the left by $\gamma_1 = \mathrm{diag}\big(1, \left(\begin{smallmatrix} & I_2 \\ 1 & \end{smallmatrix}\right)\big)$, we see that

$$V_\beta = V_{(1,2,1)} = \left\{\begin{pmatrix} 1 & x_{1,1} & x_{1,2} & y_{1,1} \\ & 1 & & x_{2,4} \\ & & 1 & x_{2,3} \\ & & & 1 \end{pmatrix}\right\}, \quad \varphi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ & 0 & 0 & -1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}.$$

The nilpotency order of φ is 3, and since ρ is a $(k, c) = (2, 2)$ representation, we have $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho) = 0$.

Now consider the cases $d_1 = n - l$ or $k = 1$. There are only finitely many representatives (i.e., representatives h_i, h_j with $h_i \nsim h_j$) satisfying this condition. This is trivial when $k = 1$. For $k > 1$ recall $h = wu$ with $\ell_0(u)$ of the form (2.17). Since $d_1 = n - l$, and thereby $d_1 = \cdots = d_{k-1} = n - l$, we finally have for any $m \in M_Q$, ${}^w\ell(m) \in P$, in particular ${}^w\ell(u) \in P$ hence $h \sim w\ell_0(u)$. We simplify $\ell_0(u)$ and deduce that there are only finitely many representatives remaining. Regard GL_l as the direct factor of the standard Levi subgroup $\mathrm{GL}_l \times \mathcal{G}_{c-2l}$ of G. For $g_1, g_2 \in \mathrm{GL}_l$, because (finally) ${}^w(g_1, g_2) \in P$,

$$w\ell_0(u) \sim w\ell_0(u)(g_1, g_2) \sim w({}^{(g_1,g_2)^{-1}}\ell_0(u)).$$

Looking at (2.17), we can now assume $A_l = \mathrm{diag}(I_{l'}, 0_{l-l'})$, where $l' \le l$ is the rank of A_l. There are only finitely many such representatives. Furthermore if $l' < l$, take a representative g of an element of $W(G)$ such that $w\ell_0((g, 1))$ does not permute the rows $l' + 1, \ldots, l$ of (2.17). Since (as opposed to the proof of Proposition 2.5) ${}^w\ell((g, 1)) \in P$, we have $w(g, 1) \sim w\ell_0((g, 1))$. Then

$$w({}^{(g_1,g_2)^{-1}}\ell_0(u)) \sim w(g,1)({}^{(g,1)^{-1}(g_1,g_2)^{-1}}\ell_0(u)) \sim w\ell_0((g,1))({}^{(g,1)^{-1}(g_1,g_2)^{-1}}\ell_0(u)).$$

Since $l' \le l$, $w\ell_0((g, 1))$ now trivially satisfies (2.25) for l' with $d_1 = \cdots = d_{k-1} = n - l'$ (the multiplications on the left by elements of $W(M_P)$ do not matter for this), and if we reset $w := w\ell_0((g, 1))$ and $l := l'$, we have $h = w({}^{j_{(k-1)c+l}}u_l)$. If c is odd, then u_l commutes with any j_a, otherwise $j_{(k-1)c+l} = j_l$. Hence $h = w({}^{j_l}u_l)$ (as in the $k = 1$ case).

Thus there are only $n + 1$ representatives h to consider, and note that the representative $w({}^{j_n}u_n)$ satisfies $h \sim \delta$.

Proposition 2.14. *Assume $d_1 = n - l$ or $k = 1$, and $l < n$. Then $\mathcal{H}(h) = 0$ outside a discrete subset of s. Moreover, under each one of the conditions of (a), e.g., when π_2 is supercuspidal (and $c > 2$ or $G = \mathrm{Sp}_2$), $\mathcal{H}(h) = 0$ for all s.*

Proof. Now $V_\beta = V_{(c^k)}$, which is trivial when $k = 1$, in which case we set $\psi_{V_\beta} = 1$. If $k > 1$, since now $\ell(u)$ is trivial, one can read off ψ_{V_β} directly from (2.29), then

$$\psi_{V_\beta}(v) = \psi\left(-\mathrm{tr}\left(b_k\begin{pmatrix} 0 & 0 & -\epsilon_0 I_l & 0 & 0 \\ 0 & \epsilon_0 I_{n-l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c-2n} \\ 0 & 0 & 0 & 0_{n-l} & 0 \\ I_l & 0 & 0 & 0 & 0 \end{pmatrix}\right) - \sum_{j=1}^{k-2}\mathrm{tr}(b_{k+j})\right). \qquad (2.34)$$

(Note that A_l was replaced by I_l in the first matrix.)

Consider the parabolic subgroup $R = M_R \ltimes U_R < G$ where $M_R \cong \mathrm{GL}_{n-l} \times \mathscr{G}_{c-2(n-l)}$ and

$$ {}^{j_{(c+1)l}}U_R = \left\{ \begin{pmatrix} I_l & & & u_1 & 0 \\ u_2 & I_{n-l} & u_4 & u_3 & u_1' \\ & & I_{c-2n} & u_4' & \\ & & & I_{n-l} & \\ & & & u_2' & I_l \end{pmatrix} \right\}. $$

Here if c is even, then $j_{(c+1)l} = j_l$, otherwise $j_{(c+1)l}$ is trivial. Since $l < n$, this is a nontrivial parabolic subgroup unless $c = 2$ and $G \neq \mathrm{Sp}_2$. If c is odd, the image of U_R in H_0 is given by (see Example 1.8)

$$ \left\{ \begin{pmatrix} I_l & & \epsilon_2 u_4 & \epsilon_1 u_4 & u_1 & 0 \\ u_2 & I_{n-l} & & & u_3 & u_1' \\ & & 1 & & & \epsilon_1 u_4' \\ & & & 1 & & \epsilon_2 u_4' \\ & & & & I_{n-l} & \\ & & & & u_2' & I_l \end{pmatrix} \right\}. $$

Denote the Lie algebra of U_R by $\mathfrak{u}_R$.

Lemma 2.15. *The following holds for all s.*

(1) *If F is nonarchimedean, then $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is a trivial representation of ${}^h(1, {}^{j_{(c+1)l}}U_R)$.*

(2) *For an archimedean field, ${}^h(1, {}^{j_{(c+1)l}}\mathfrak{u}_R)$ acts locally nilpotently on $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)^*$.*

The proofs of this lemma and the following one appear after the proof of the proposition. If π_2 is supercuspidal (nontrivially), the proposition follows immediately from Lemma 2.15, in particular we do not need to exclude any s (see also the discussion preceding (2.7)).

Identify the group GL_{n-l} with its image in M_R,

$$ {}^h(1, \mathrm{GL}_{n-l}) = \{\mathrm{diag}(I_{n+l}, a, I_{c-2n+(k-1)c}) : a \in \mathrm{GL}_{n-l}\}, $$

where the right hand side is implicitly regarded as a subgroup of M_P. By (2.34), ${}^h(1, \mathrm{GL}_{n-l})$ stabilizes ψ_{V_β}, and because it also normalizes V_β, ${}^h(1, \mathrm{GL}_{n-l})$ acts on $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$.

Lemma 2.16. *The following holds for all s.*

(1) *If F is nonarchimedean, then $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ admits a finite length filtration as a representation of ${}^h(1, \mathrm{GL}_{n-l})$, where ${}^h(1, C_{\mathrm{GL}_{n-l}})$ acts by a character on each constituent.* (*The constituents need not be irreducible.*)

(2) *Over archimedean fields, there is a maximal parabolic subgroup of GL_{kc} whose Levi part contains ${}^h(1, \mathrm{GL}_{n-l})$ as a direct factor, such that the Lie algebra $\mathfrak{v}$ of its unipotent radical acts locally nilpotently on $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)^*$.*

(3) *If $c = 2$ and $\rho = \rho_c(\tau)$ for an irreducible supercuspidal representation τ of GL_k with $k > 1$, then $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho) = 0$.*

This implies $\mathcal{H}(h)=0$ outside a discrete subset of s, because ${}^{h^{-1}}(|\det|^{s-1/2})(1, aI_{n-l}) = |a|^{(n-l)(s-1/2)}$ for all $a \in F^*$, and $l < n$, then we can apply (2.7). Also Lemma 2.16 (3) implies $\mathcal{H}(h) = 0$ for all s, in the remaining case of (a). ■

Proof of Lemma 2.15. First consider a nonarchimedean field. We show $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is a trivial representation of ${}^h(1, {}^{J_{(c+1)l}}U_R)$. For $z \in U_R$, ${}^h(1, {}^{J_{(c+1)l}}z) = m_z z'$ where $z' \in U_P$ and

$$m_z = \operatorname{diag}\left(\begin{pmatrix} I_l &&&&& \\ & I_{n-l} &&&& \\ && I_l &&& \\ u_1' && u_2 & I_{n-l} & \epsilon u_4 & \\ &&&& I_{c-2n} \end{pmatrix}, I_{(k-1)c}\right).$$

Here $\epsilon = \epsilon_1$ or ϵ_2 depending on j_l; also for the computation note that since $l < n$, j_l commutes with u_l. When $l = 0$ and c is even, m_z is trivial, whence ${}^h(1, {}^{J_{(c+1)l}}U_R) < U_P$, so that $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is immediately a trivial representation of ${}^h(1, {}^{J_{(c+1)l}}U_R)$.

Let Z be the subgroup of M_P generated by the matrices m_z as z varies in U_R. It is an abelian group. The rank of Z is $(2l + c - 2n)(n - l)$. Since $\psi^{-1}_{V_\beta}$ restricts to a trivial character on rows $2n, \ldots, 2n + 1 - (n - l)$ of b_k (which are the last $n - l$ rows if c is even), Z stabilizes $\psi^{-1}_{V_\beta}$ (it clearly normalizes V_β). Thus $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is a representation of Z and for each character χ of Z,

$$J_{Z,\chi}(J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)) = J_{V_\beta \ltimes Z,\psi^{-1}_{V_\beta}\otimes\chi}(\rho).$$

A similar identity holds for any subgroup of Z.

For $b \in \mathrm{GL}_c$, denote $b^\triangle = \operatorname{diag}(b, b^{\triangle'}) \in M_{(c^k)}$ where $b^{\triangle'}$ is the diagonal embedding of GL_c in $M_{(c^{k-1})}$. The group $\operatorname{diag}(I_c, \mathrm{GL}_c^{\triangle'})$ stabilizes the restriction of $\psi^{-1}_{V_\beta}$ to the blocks $b_{k+1}, \ldots, b_{2k-2}$ (but not to b_k). The group $\mathrm{GL}_l \times \mathrm{GL}_l \times \mathrm{GL}_{n-l} \times \mathrm{GL}_{c-2n}$ embedded in $M_{(c^k)}$ by

$$[x_1, x_2, x_3, x_4] = \operatorname{diag}\left(\begin{pmatrix} x_1 &&&& \\ & I_{n-l} &&& \\ && x_2 && \\ &&& x_3 & \\ &&&& x_4 \end{pmatrix}, \begin{pmatrix} x_2 &&&& \\ & I_{n-l} &&& \\ && x_4 && \\ &&& I_{n-l} & \\ &&&& x_1 \end{pmatrix}^{\triangle'}\right), \tag{2.35}$$

where $x_1, x_2 \in \mathrm{GL}_l$, $x_3 \in \mathrm{GL}_{n-l}$ and $x_4 \in \mathrm{GL}_{c-2n}$, acts on the set of characters of Z with infinitely many orbits, but precisely two orbits separately on each block $Z' = u_1', u_2$ and ϵu_4, and stabilizes $\psi^{-1}_{V_\beta}$. It is enough to prove that each block Z' acts trivially on $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$. By [11, §§5.9–5.12], it suffices to show that for each nontrivial character χ' of Z',

$$J_{V_\beta \ltimes Z',\psi^{-1}_{V_\beta}\otimes\chi'}(\rho) = 0, \tag{2.36}$$

since then for each Z', $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho) = J_{V_\beta \ltimes Z',\psi^{-1}_{V_\beta}}(\rho)$, and thus $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho) = J_{V_\beta \ltimes Z,\psi^{-1}_{V_\beta}}(\rho)$. This implies that ${}^h(1, {}^{J_{(c+1)l}}U_R)$ acts trivially on $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$.

Let χ' be a nontrivial character of Z'. As in the proof of Proposition 2.12, we let φ denote the transpose of the nilpotent element defined by the character $\psi^{-1}_{V_\beta} \otimes \chi'$ of

$V_\beta \ltimes Z'$, and show that φ is nilpotent of order at least $k+1$, then (2.36) essentially follows from [46, Theorems A, E] because ρ is (k,c), but an additional argument is used because $V_\beta \ltimes Z'$ does not correspond to a unipotent orbit (see below).

Conjugating φ by a suitable element (2.35), we can assume the bottom left coordinate of Z' in φ is 1, and all other coordinates in the same column of Z' are 0. Assume (momentarily) c is even or $l>0$. One can permute ϵu_4 (for odd c) with the first column of u_2 using conjugation by

$$\mathrm{diag}\left(\begin{pmatrix} I_n && \\ && -1 \\ & I_{n-1} & \\ -1 && \end{pmatrix}, \begin{pmatrix} && 1 & \\ & I_{n-1} && \\ 1 &&& \\ &&& I_n \end{pmatrix}^{\Delta'}\right).$$

This element normalizes V_β and stabilizes the blocks $b_k,\ldots,b_{2k-2}$ of φ. Moreover, we can exchange the blocks u_1' and u_2 using conjugation by

$$\mathrm{diag}\left(\begin{pmatrix} && I_l & \\ & I_{n-l} && \\ I_l &&& \\ &&& I_{\lceil c/2\rceil-l} \end{pmatrix}, I_{(k-1)c}\right), \tag{2.37}$$

hence we can always assume, for any Z', that after a conjugation the bottom left coordinate of the block u_1' of φ is 1. The matrix (2.37) normalizes V_β, fixes the blocks $b_{k+1},\ldots,b_{2k-2}$ of φ, but permutes the coordinates of the block b_k of φ. In particular, the $(n+1,1)$-th coordinate in the block b_k of φ, which is $-\epsilon_0$, is conjugated into the $(1,1)$-th coordinate of this block. If $Z'=u_1'$ we can skip this conjugation. When c is odd and $l=0$, in which case $Z'=\epsilon u_4$ (evidently u_1' and u_2 are trivial when $l=0$), we use conjugation by

$$\mathrm{diag}\left(\begin{pmatrix} && 1 \\ & I_{c-2} & \\ 1 && \end{pmatrix}, \begin{pmatrix} && 1 \\ & I_{c-2} & \\ 1 && \end{pmatrix}^{\Delta'}\right),$$

so that the $(c,n+1)$-th coordinate of the block b_k of φ, which is 1, is permuted to the $(1,n+1)$-th coordinate, and the bottom left coordinate of Z' in φ is permuted to the $(2n,1)$-th coordinate of φ. At any rate, φ has a nonzero coordinate in the first row of b_k.

Conjugating φ by an element of $\mathrm{diag}(I_c, \mathrm{GL}_c^{\Delta'})$, we can always assume φ contains 1 in the top right coordinate of b_k, and additionally (still) contains 1 in the $(2n,1)$-th coordinate (the bottom left coordinate of u_1' if $l>0$). If c is odd, we can permute this coordinate to the $(c,1)$-th coordinate using conjugation by $\mathrm{diag}(I_{c-2}, \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right), I_{(k-1)c})$.

Now conjugating φ by

$$\mathrm{diag}\left(\begin{pmatrix} && 1 \\ & I_{c-2} & \\ 1 && \end{pmatrix}, I_{(k-1)c}\right),$$

the top right coordinate of b_k is permuted into its bottom right, so that now φ has 1 in the bottom right coordinate of each of the blocks $b_k,\ldots,b_{2k-2}$, and the $(c,1)$-th coordinate of φ is permuted into the $(1,c)$-th coordinate. Moreover, the only nonzero entry in column c is the $(1,c)$-th coordinate, and in each $b_k,\ldots,b_{2k-2}$ the only nonzero entry in the rightmost column is the bottom right one. This brings us to the situation in the proof of Proposition 2.12, with two exceptions. First, instead of -1 in the bottom right coordinate of the block b_{k-1}, we have 1 in the $(1,c)$-th coordinate (the first coordinate to the left of the top left coordinate of b_k). It again follows that φ is nilpotent of order at least $k+1$.

Second and more importantly, the unipotent subgroup $V_\beta \ltimes Z'$ does not correspond to a unipotent orbit (i.e., it is not of the form $V(\sigma)$, see §1.4). However, we reduced (2.36) to the vanishing of $J_{V_\beta \ltimes Z'', \psi_{V_\beta}^{-1} \otimes \chi''}(\rho)$, where $Z'' < N_{\mathrm{GL}_{kc}}$ corresponds to the $(1, c)$-th coordinate and χ'' is a nontrivial character of Z''. We see that $V_\beta \ltimes Z'' < V_{(1,c-1,c^{k-1})}$ and any character ψ' of $V_{(1,c-1,c^{k-1})}$ which extends $\psi_{V_\beta}^{-1} \otimes \chi''$ is still nilpotent of order $k+1$. Also, the torus $T_{\mathrm{GL}_{c-1}}$ acts on the added $c-2$ new coordinates of $V_{(1,c-1,c^{k-1})}$ with finitely many orbits (one can identify each diagonal coordinate from $T_{\mathrm{GL}_{c-1}}$ with an element of $T_{\mathrm{GL}_{kc}}$ which fixes $\psi_{V_\beta}^{-1} \otimes \chi''$). Thus (by [11, §§5.9–5.12]) $J_{V_\beta \ltimes Z'', \psi_{V_\beta}^{-1} \otimes \chi''}(\rho) = 0$ if for any ψ' as above, $J_{V_{(1,c-1,c^{k-1})}, \psi'}(\rho) = 0$. The latter holds by [46, Theorems A, E] because ρ is (k, c). We conclude (2.36) for any nontrivial χ' and any block Z'.

For the archimedean case, again the result is trivial if $l = 0$ and c is even. The (abelian) Lie algebra $\mathfrak{z}$ of Z decomposes into the direct sum of one-dimensional Lie algebras $\mathfrak{z}_{i,j}$, corresponding to the coordinates $Z_{i,j}$ of Z (which can be identified with roots of GL_{kc}).

For each (i, j), there is a subgroup of (2.35) which acts on the characters of $Z_{i,j}$ with two orbits; one can identify this subgroup with T_{GL_2}. Then we can proceed as in the nonarchimedean case and prove $J_{V_\beta \ltimes Z_{i,j}, \psi_{V_\beta}^{-1} \otimes \chi'}(\rho) = 0$ for any nontrivial character χ', where to deduce this from the vanishing of $J_{V_{(1,c-1,c^{k-1})}, \psi'}(\rho) = 0$ for all ψ' we apply [47, Corollary 3.0.2] (instead of [11]). By the transitivity of the Jacquet functor, this implies that there are no continuous distributions on $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ that transform on the left under $Z_{i,j}$ by χ', i.e., $(J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)^*)^{(Z_{i,j}, \chi')} = 0$. Hence by [47, Proposition 3.0.1], $\mathfrak{z}_{i,j}$ acts locally nilpotently on $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)^*$. Note that for the proof of (2.36) above we really used only one coordinate, the bottom left one, for each block, and using conjugation we can assume this coordinate is $Z_{i,j}$.

We deduce that each $\mathfrak{z}_{i,j}$ acts locally nilpotently, hence so does $\mathfrak{z}$. ∎

Proof of Lemma 2.16. Consider a nonarchimedean field. We prove that $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ admits a finite length filtration as a representation of ${}^h(1, \mathrm{GL}_{n-l})$, and on each constituent ${}^h(1, C_{\mathrm{GL}_{n-l}})$ acts by a character, by showing $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ factors through a Jacquet module along a unipotent radical of a certain parabolic subgroup, with respect to a trivial character.

After conjugating $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ by

$$\kappa = \mathrm{diag}\left(\begin{pmatrix} & & I_{n-l} & \\ & I_n & & \\ I_l & & & \\ & & & I_{c-2n} \end{pmatrix}, I_{(k-1)c}\right),$$

we regard $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ as a representation of ${}^\kappa({}^h(1, \mathrm{GL}_{n-l})) = \mathrm{diag}(\mathrm{GL}_{n-l}, I_{kc-(n-l)})$. In addition, this conjugation only changes the restriction of $\psi_{V_\beta}^{-1}$ to b_k, now given by

$$\psi\left(\mathrm{tr}\left(b_k \begin{pmatrix} 0 & 0 & -\epsilon_0 I_l & 0 & 0 \\ 0 & \epsilon_0 I_{n-l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{c-2n} \\ 0_{n-l} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_l & 0 \end{pmatrix}\right)\right)$$

(use (2.34)). Now $\psi_{V_\beta}^{-1}$ is trivial on the top $n-l$ rows of b_k, hence $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)$ is a representation of $V_{(n-l,c-(n-l))}$, which we identify with its image in the top left $c\times c$ block of GL_{kc}.

We claim

$$J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)=J_{V_{(n-l,c-(n-l),c^{k-1})},\psi_{V_\beta}^{-1}}(\rho). \tag{2.38}$$

Here $\psi_{V_\beta}^{-1}$ is extended trivially to $V_{(n-l,c-(n-l))}$. Before proving (2.38), we explain how it leads to the result. Because $V_{(n-l,kc-(n-l))}<V_{(n-l,c-(n-l),c^{k-1})}$, the right hand side of (2.38) becomes

$$J_{\mathrm{diag}(I_{n-l},V_{(c-(n-l),c^{k-1})}),\psi_{V_\beta}^{-1}}(J_{V_{(n-l,kc-(n-l))}}(\rho)).$$

Since ρ is an admissible finite length representation of GL_{kc}, $J_{V_{(n-l,kc-(n-l))}}(\rho)$ is an admissible finite length representation of $M_{(n-l,kc-(n-l))}$. As such, it admits a finite filtration with irreducible admissible constituents. On each constituent $\mathcal{V}$, $C_{M_{(n-l,kc-(n-l))}}$ acts by a character, and because ${}^{\kappa h}(1,C_{\mathrm{GL}_{n-l}})<C_{M_{(n-l,kc-(n-l))}}$, ${}^{\kappa h}(1,C_{\mathrm{GL}_{n-l}})$ also acts by a character. Note that $\mathcal{V}$ may certainly be reducible (or not admissible) as a representation of ${}^{\kappa h}(1,\mathrm{GL}_{n-l})$. By the exactness of the Jacquet functor, $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)$ admits a finite filtration where on each constituent ${}^{\kappa h}(1,C_{\mathrm{GL}_{n-l}})$ still acts by the same character. This completes the proof of the main assertion—part (1)—for the nonarchimedean case. Regarding (3), when $c=2$, $\rho=\rho_c(\tau)$ for an irreducible supercuspidal representation of GL_k and $k>1$, the Jacquet module $J_{V_{(n-l,kc-(n-l))}}(\rho)$ vanishes since $n-l=1$ [12, Theorem 2.13 (a)].

Write $v\in V_{(n-l,c-(n-l))}$ in the form $v=(v_1,v_2,v_3,v_4)$ with $v_1\in\mathrm{Mat}_{n-l}$, $v_2,v_3\in\mathrm{Mat}_{n-l\times l}$ and $v_4\in\mathrm{Mat}_{n-l\times c-2n}$. The group ${}^{\kappa}[x_1,x_2,x_3,x_4]$ (see (2.35)), together with the group GL_{n-l} embedded in $M_{(c^k)}$ by $x_5\mapsto\mathrm{diag}(I_{n-l},x_5,I_{2l+c-2n})^{\triangle}$, stabilizes $\psi_{V_\beta}^{-1}$ and acts on the set of characters of $V_{(n-l,c-(n-l))}$ with infinitely many orbits, but only two on each block v_i separately. Using the transitivity of the Jacquet functor and [11, §§5.9–5.12], (2.38) follows at once if we prove separately, that for each block $Z'=v_i$ and nontrivial character χ' of Z',

$$J_{V_\beta\ltimes Z',\psi_{V_\beta}^{-1}\otimes\chi'}(\rho)=0. \tag{2.39}$$

Let φ denote the transpose of the nilpotent element defined by the character $\psi_{V_\beta}^{-1}\otimes\chi'$ of $V_\beta\ltimes Z'$. We prove φ is nilpotent of order at least $k+1$; then since ρ is (k,c), the results of [46, Theorems A, E] imply (2.39).

First we show that, after possibly a suitable conjugation, φ is nontrivial on the $(1,c)$-th coordinate, and all other blocks remain unchanged except the block b_k, where there is a nonzero entry in the bottom right coordinate.

We can assume the top right coordinate of Z' in φ is nonzero, and it is the only nonzero entry in that column. If $Z'=v_4$, the $(1,c)$-th entry of φ is nonzero. Using conjugation by an element of $\mathrm{diag}(I_c,\mathrm{GL}_c^{\triangle'})$, the $(c,n+1)$-th coordinate of b_k becomes its (c,c)-th coordinate, and the other blocks $b_{k+1},\dots,b_{2k-2}$ are unchanged.

For $Z' = \mathfrak{v}_2$, we conjugate φ by

$$\mathrm{diag}\left(\begin{pmatrix} I_{n-l} &&&& \\ & I_{n-l} &&& \\ &&& I_l & \\ && I_l && \\ &&&& I_{c-2n}\end{pmatrix}, \begin{pmatrix} && I_l \\ & I_{c-2l} & \\ I_l && \end{pmatrix}^{\triangle'}\right),$$

and for $Z' = \mathfrak{v}_1$ we conjugate by

$$\mathrm{diag}\left(\begin{pmatrix} I_{n-l} &&&& \\ &&& I_l & \\ && I_l && \\ & I_{n-l} &&& \\ &&&& I_{c-2n}\end{pmatrix}, \begin{pmatrix} I_l &&& \\ &&& I_l \\ && I_{c-n-l} & \\ & I_{n-l} && \end{pmatrix}^{\triangle'}\right).$$

Both conjugations preserve $b_{k+1}, \ldots, b_{2k-2}$, the $(1, 2n)$-th coordinate of φ becomes nontrivial, and the rightmost column of b_k has (precisely) one nontrivial coordinate, which is the $(c-1, c)$-th coordinate if c is odd, otherwise it is the bottom coordinate (the (c, c)-th coordinate). For an even c, φ is of the prescribed form; if c is odd, using another conjugation by $\mathrm{diag}\left(I_{c-2}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, I_{(k-1)c}\right)$, the $(c-1, c)$ entry of block b_k is permuted into its bottom right coordinate, and the $(1, 2n)$-th coordinate of φ becomes its $(1, c)$-th coordinate.

We conclude that in all cases of Z', when χ' is nontrivial, the $(1, c)$-th coordinate of φ and the bottom right coordinate of each block $b_k, \ldots, b_{2k-2}$ of φ are nonzero (these coordinates are all 1 except for b_k, where the coordinate is ± 1), and the corresponding nonzero entry is the unique one in its column. Thus as in the proof of Lemma 2.15 (and again considering all extensions to characters of $V_{(1,c-1,c^{k-1})}$ in order to "adjust" $V_\beta \ltimes Z'$ to a unipotent radical), φ is nilpotent of order at least $k+1$ and (2.39) follows.

Over archimedean fields, as in the proof of Lemma 2.15, we deduce that the Lie algebra $\mathfrak{v}_{(n-l,c-(n-l))}$ of $V_{(n-l,c-(n-l))}$ acts locally nilpotently on $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)^*$ by carrying out the proof of (2.39) and applying [47, Proposition 3.0.1] separately for each coordinate of each $\mathfrak{v}_i$. Let $\mathfrak{v}'$ denote the Lie algebra of the unipotent subgroup $V_{(n-l,kc-(n-l))} \cap V_\beta$. Since $\mathfrak{v}'$ acts trivially on $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)$, $\mathfrak{v}_{(n-l,kc-(n-l))} = \mathfrak{v}_{(n-l,c-(n-l))} \oplus \mathfrak{v}'$ acts locally nilpotently on $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)^*$. ∎

Example 2.17. Consider $c = 4$, $k = 2$, $G = \mathrm{Sp}_4$, $H = \mathrm{Sp}_{16}$ and $l = 1$. Assume $d_1 = n - l = 1$. Then $w \in M_P(0^3, 1, w_2)$, $w_2 = (1^4)$ and $u = \ell_0(u) \in H_0$ is given by

$$u = \mathrm{diag}\left(I_4, \begin{pmatrix} 1 && 1 \\ & 1 & \\ && 1 \end{pmatrix}, I_2, \begin{pmatrix} 1 && -1 \\ & 1 & \\ && 1 \end{pmatrix}, I_4\right).$$

The element w is given by (2.30), note that $\gamma_1 = \mathrm{diag}\left(\begin{pmatrix} & I_4 \\ I_2 & \end{pmatrix}, I_2\right)\begin{pmatrix} & I_6 \\ I_2 & \end{pmatrix}$ (see Remark 2.10). We have $\beta = (4^2)$, and if we write an element of V_β in the form $v = \begin{pmatrix} I_4 & x \\ & I_4 \end{pmatrix}$, $\psi_{V_\beta}(v) = \psi(-x_{1,4} + x_{2,2} - x_{3,1})$. The Jacquet module $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)$ is a representation of ${}^h(1, U_R)$ with

$$U_R = \left\{\begin{pmatrix} 1 && u_1 & 0 \\ u_2 & 1 & u_3 & u_1 \\ && 1 & \\ && -u_2 & 1 \end{pmatrix}\right\}.$$

We see that for $z \in U_R$,

$$ {}^{u}(1,z) = \mathrm{diag}\left(I_4, \begin{pmatrix} 1 &&&& u_1 &&& \\ & 1 &&&&&& \\ && 1 && u_1 &&& \\ && u_2 & 1 & u_3 & u_1 && u_1 \\ &&&& 1 &&& \\ &&&& -u_2 & 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}, I_4 \right). $$

Then ${}^{h}(1,z) = m_z z'$ with $m_z \in M_P$ and $z' \in U_P$, and such that as an element of GL_8,

$$ m_z = \mathrm{diag}\left(\begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ u_1 && u_2 & 1 \end{pmatrix}, I_4 \right). $$

Denote the subgroup of elements of this form by Z, then $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is a representation of Z. We proceed over nonarchimedean fields. To show that $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is a trivial representation of ${}^{h}(1,U_R)$ amounts to proving $J_{V_\beta \ltimes Z,\psi^{-1}_{V_\beta}\otimes\chi}(\rho) = 0$ for any nontrivial character χ of Z. Combining Z and V_β together we are considering the following unipotent subgroup and character:

$$ \left\{ v = \begin{pmatrix} 1 &&& & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ & 1 && & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ && 1 & & x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ u_1 && u_2 & 1 & x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix} \right\}, $$

$$ (\psi^{-1}_{V_\beta} \otimes \chi)(v) = \psi(x_{1,4} - x_{2,2} + x_{3,1} + \alpha_1 u_1 + \alpha_2 u_2). $$

We have an action of T_{GL_2} on u_1 and u_2 separately, given by $\mathrm{diag}(x_1, I_2, x_3, I_3, x_1)$ (for u_1) and $\mathrm{diag}(I_2, x_2, x_3, x_2, I_3)$. When considering each coordinate separately, there are two orbits. The corresponding φ takes the form

$$ \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ &&&& 0 &&& \\ &&&&& 0 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{pmatrix}, $$

and a nontrivial χ means $(\alpha_1, \alpha_2) \neq (0,0)$. Using conjugations by $\mathrm{diag}(J_3, I_5)$ and by $\mathrm{diag}(I_4, g)$ for a permutation matrix $g \in \mathrm{GL}_4$ if necessary, we can assume the $(4,1)$-th and $(1,8)$-th coordinates of φ are nonzero; then conjugating by $\mathrm{diag}\left(\begin{pmatrix} && 1 \\ & I_2 & \\ 1 && \end{pmatrix}, I_4 \right)$, we see that φ is nilpotent of order at least 3. Thus $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is a trivial representation of ${}^{h}(1,U_R)$ (ρ is $(k,c) = (2,4)$).

The Jacquet module $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ is also a representation of

$$ {}^{h}(1, \mathrm{GL}_{n-l}) = \{\mathrm{diag}(I_3, a, I_4) : a \in F^*\}. $$

Conjugating by $\kappa = \mathrm{diag}\left(\begin{pmatrix} && 1 \\ & I_2 & \\ 1 && \end{pmatrix}, I_4 \right)$, we can regard $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho)$ as a representation of $\mathrm{diag}(V_{(1,3)}, I_4)$. Combining the coordinates of $V_{(1,3)}$ and V_β, we then have

$$\left\{ v = \begin{pmatrix} 1 & v_1 & v_2 & v_3 & x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} \\ & 1 & & & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ & & 1 & & x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \\ & & & 1 & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \right\},$$

and note that we permuted the coordinates of $x_{i,j}$ (so that $\psi_{V_\beta}^{-1}$ remains as above). The tensor product of an arbitrary character χ of $V_{(1,3)}$ with $\psi_{V_\beta}^{-1}$ takes the form

$$(\psi_{V_\beta}^{-1} \otimes \chi)(v) = \psi(x_{1,4} - x_{2,2} + x_{3,1} + \vartheta_1 v_1 + \vartheta_2 v_2 + \vartheta_3 v_3).$$

As above, we have the action of T_{GL_2} on each of the coordinates v_i: $\mathrm{diag}(x_3, x_5, I_3, x_5, I_2)$ for v_1, $\mathrm{diag}(x_3, 1, x_2, 1, x_2, I_3)$ for v_2, and $\mathrm{diag}(x_3, I_2, x_1, I_3, x_1)$ for v_3. The corresponding φ is then nilpotent of order at least 3 when $\vartheta_1\vartheta_2\vartheta_3 \neq 0$. This is immediate when $\vartheta_3 \neq 0$, using $x_{1,4}$. If $\vartheta_3 = 0$ and $\vartheta_2 \neq 0$, we conjugate by $\mathrm{diag}\left(I_2, J_2, \begin{pmatrix} & & 1 \\ & I_2 & \\ 1 & & \end{pmatrix}\right)$ and use $x_{3,1}$, otherwise $\vartheta_1 \neq 0$ and we conjugate by $\mathrm{diag}(1, J_3, 1, J_3)$ and use $x_{2,2}$.

Therefore $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ factors through $V_{(1,7)}$, so that the original module $J_{V_\beta, \psi_{V_\beta}^{-1}}(\rho)$ factors through ${}^{\kappa^{-1}}V_{(1,7)}$. Since $\det(\mathrm{diag}(I_3, a, I_4)) = a$, we can use (2.7) to deduce $\mathcal{H}(h) = 0$.

Propositions 2.5–2.14 imply $\mathcal{H}(h) = 0$ for all h such that $h \nsim \delta$. Finally, consider $h = w({}^{J_n}u_n) \sim \delta$. We prove that for all s, $\dim \mathcal{H}(\delta) = \dim \mathrm{Hom}_G(\pi_1^\vee, \pi_2^\iota)$. In this case $P_\delta = (G^\iota \times C_H^\circ) \ltimes V_{(c^k)}$, where $G^\iota = \{(g, {}^\iota g) : g \in G\}$, $\psi_{V_\beta}^{-1}$ belongs to the orbit of ψ_k (the choice of δ of [20] gives precisely ψ_k), and any morphism in $\mathcal{H}(\delta)$ factors through $J_{V_{(c^k)}, \psi_k}(\rho)$. Note that C_H° is trivial unless $H = \mathrm{GSpin}_{2kc}$, in which case $C_H^\circ < P_\delta$ because C_G° is mapped by the embedding $g \mapsto (g, 1)$ bijectively into C_H° (see also (1.3)). Therefore

$$\mathcal{H}(\delta) = \mathrm{Hom}_{G^\iota \times C_H^\circ}({}^{\delta^{-1}}J_{V_{(c^k)}, \psi_k}(\rho) \otimes \eta \otimes \pi_1^\vee \otimes \pi_2^\vee, 1).$$

Here $|\det|^{s-1/2}$ and θ_h are absent because they are trivial on $G^\iota \times C_H^\circ$.

For $H = \mathrm{GSpin}_{2kc}$ we assumed $\chi_{\pi_1}, \chi_{\pi_2}$ exist; then $\mathcal{H}(\delta) = 0$ unless $\eta = \chi_{\pi_1}^{-1} = \chi_{\pi_2}$, because for $z_1, z_2 \in C_G^\circ$, (z_1, z_2) is the element $z_1^{-1}z_2$ of C_H°. When this condition holds, we can finally ignore C_H° and η altogether.

Recall that $\mathrm{GL}_c^\triangle$ denotes the diagonal embedding of G in $M_{(c^k)}$, and $J_{V_{(c^k)}, \psi_k}(\rho)$ is a trivial representation of $\mathrm{SL}_c^\triangle$. Since ${}^\delta G^\iota < \mathrm{SL}_c^\triangle$ (for $H \neq \mathrm{GSpin}_{2kc}$, ${}^\delta G^\iota = G^\triangle$), the action of G^ι on ${}^{\delta^{-1}}J_{V_{(c^k)}, \psi_k}(\rho)$ is trivial, and because $\dim J_{V_{(c^k)}, \psi_k}(\rho) = 1$ (see §1.4),

$$\mathrm{Hom}_{G^\iota}({}^{\delta^{-1}}J_{V_{(c^k)}, \psi_k}(\rho) \otimes \pi_1^\vee \otimes \pi_2^\vee, 1) = \mathrm{Hom}_G(\pi_1^\vee \otimes (\pi_2^\vee)^\iota, 1) = \mathrm{Hom}_G(\pi_1^\vee, \pi_2^\iota).$$

This completes the proof of the first part of the theorem. For the second part, clearly when π_1 and π_2 are irreducible, $\dim \mathrm{Hom}_G(\pi_1^\vee, \pi_2^\iota) \leq 1$ and is zero unless $\pi_1 = (\pi_2^\iota)^\vee$. Under the assumptions of (a) (e.g., π_2 is supercuspidal and $c > 2$), we do not need to exclude any s, by Proposition 2.14 (which is the only case where the vanishing depends on s).

2.3. The case $H = \mathrm{GL}_{2kc}$

Write $P\backslash H/D = \coprod_h PhD$ with $h = wu$, where w is a representative from $W(M_P)\backslash W(H)$ and $u \in M_Q \cap N_H$. Throughout this section we fix the standard identification of $W(H)$ with permutation matrices in H. One can still describe w as a $2kc$-tuple of $\{0, 1\}$: if the i-th coordinate of w is 1, w permutes the i-th row into one of the first kc rows, and if it is 0, then w permutes this row into one of the last kc rows. Of course only vectors whose total sum of coordinates is kc are permissible (U_P contains kc nontrivial rows). Let $p_0(w)$ denote the middle $2c$ coordinates of w, and $p_1(w)$ (resp., $p_2(w)$) denote the first (resp., last) $(k-1)c$ coordinates. Also note that in general, w permutes row i into U_P if and only if w^{-1} permutes column i out of U_P.

For the case $k = 1$, we can parametrize $P\backslash H/(G \times G)$ using the elements

$$(0^l, 1^{c-l}, 0^{c-l}, 1^l)u_{l,j}, \quad 0 \le l \le c, \quad 0 \le j \le l,$$

where

$$(0^l, 1^{c-l}, 0^{c-l}, 1^l) = \begin{pmatrix} & & I_l \\ & I_{2(c-l)} & \\ I_l & & \end{pmatrix}, \quad u_{l,j} = \begin{pmatrix} I_l & & A_{l,j} \\ & I_{2(c-l)} & \\ & & I_l \end{pmatrix}, \quad A_{l,j} = \begin{pmatrix} & I_j \\ 0 & \end{pmatrix}.$$

The choice of matrix for $(0^l, 1^{c-l}, 0^{c-l}, 1^l)$ is not canonical, but can be used for convenience.

Assume $k \ge 1$. Recall

$$M_Q = \mathrm{GL}_c \times \cdots \times \mathrm{GL}_c \times H_0 \times \mathrm{GL}_c \times \cdots \times \mathrm{GL}_c, \quad H_0 = \mathrm{GL}_{2c}\,.$$

Given $x \in M_Q$, denote its projection into the left (resp., right) direct product of $k-1$ copies of GL_c by $\ell_1(x)$ (resp., $\ell_2(x)$), put $\ell(x) = \ell_1(x)\ell_2(x)$, and let $\ell_0(x)$ be the projection into H_0. We have the analogs of (2.14) and (2.15), in particular since $(1, G) < H_0$, conjugation by elements of $(1, G)$ does not affect $\ell(x)$.

Proposition 2.18. *Let $h = wu$, where w is a representative from $W(M_P)\backslash W(H)$ and $u \in M_Q \cap N_H$. Then $h \sim \hat{w}\hat{u}$, where $p_0(\hat{w}) = (0^l, 1^{c-l}, 0^{c-l'}, 1^{l'})$ for some $0 \le l, l' \le c$, $\hat{u} \in M_Q$, there is $\sigma \in (W(G), W(G))$ with ${}^{\sigma}\hat{u} \in M_Q \cap N_H$, and $\ell_0(\hat{u})$ takes the form*

$$\begin{pmatrix} I_l & & X \\ & I_{2c-l-l'} & \\ & & I_{l'} \end{pmatrix} \tag{2.40}$$

for some X.

Proof. Identify $N_{\mathrm{GL}_c} \times N_{\mathrm{GL}_c}$ with its image in $M_{(c,c)} < H_0$. Since $N_{\mathrm{GL}_c} \times N_{\mathrm{GL}_c} < \ell_0((G, G))$, one can assume $\ell_0(u) \in V_{(c,c)}$, possibly multiplying $\ell(u)$ by an element in $M_Q \cap N_H$.

One can find $\sigma \in (W(G), W(G))$ such that $w\sigma = w_1$ satisfies $p_0(w_1) = (0^l, 1^{c-l}, 0^{c-l'}, 1^{l'})$ for some $0 \le l, l' \le c$. Denote $u_1 = {}^{\sigma^{-1}}u$. Since $\ell_0((G, G)) < M_{(c,c)}$, we have $\ell_0(u_1) \in V_{(c,c)}$. Also $\ell(u_1)$ is in M_Q, but might not be in N_H. Then $wu \sim wu\sigma = w_1 u_1$.

Write the top right $c \times c$ block of $\ell_0(u_1)$ in the form $\begin{pmatrix} X^1 & X^2 \\ X^3 & X^4 \end{pmatrix}$, $X^2 \in \mathrm{Mat}_{l \times l'}$. Now w_1 conjugates the blocks X^i, $i \neq 2$, into P. Hence if $u_2 = z^{-1}u_1$ with $z \in V_{(c,c)}$ defined by these blocks, then $w_1u_1 = w_1zu_2 \sim w_1u_2$. Now $\ell_0(u_2)$ takes the form (2.40). Note that $\ell(u_2) = \ell(u_1)$, whence $\ell({}^{\sigma}u_2) = \ell({}^{\sigma}u_1) = \ell(u) \in M_Q \cap N_H$, and $\ell_0({}^{\sigma}u_2) \in N_{H_0}$ because $\ell_0((G, G)) < M_{(c,c)}$. Thus ${}^{\sigma}u_2 \in M_Q \cap N_H$. Then $\hat{w} = w_1$ and $\hat{u} = u_2$ satisfy the required properties. ∎

Lemma 2.19. *Let $h = wu$, where w and u are given by Proposition 2.18. Assume*

$$\psi_U|_{U \cap {}^{w^{-1}}U_P} \neq 1. \tag{2.41}$$

Then (2.6) also holds, i.e., $\psi_U|_{U \cap {}^{h^{-1}}U_P} \neq 1$.

Proof. The proof is a repetition of the proof of Lemma 2.6, with only one case to consider. In the notation of that proof, we only have to consider the case where the coordinate (i, j) defining Y belongs to a block $B \in \mathrm{Mat}_c$. Then ${}^{\sigma}Y$ is also defined by a coordinate in the same block B. One change here is that σ is in $(W(G), W(G))$ instead of $(W(G), 1)$, but this does not make any difference. In fact, $(1, G)$ commutes with all of the blocks of U where ψ_U is nontrivial. ∎

Let now $h = wu$ where w and u satisfy the properties of Proposition 2.18. In particular, w defines the integers $0 \leq l', l \leq c$. Write

$$w = (w_k^1, \ldots, w_1^1, w_1^2, \ldots, w_k^2), \quad \forall i, j, w_i^j \in \{0, 1\}^c.$$

With this notation

$$p_0(w) = (w_1^1, w_1^2), \quad w_1^1 = (0^l, 1^{c-l}), \quad w_1^2 = (0^{c-l'}, 1^{l'}).$$

Proposition 2.20. *We have $\mathcal{H}(h) = 0$ unless*

$$w_i^1 = (0^l, *^{c-l}), \quad w_i^2 = (*^l, 1^{c-l}), \quad \forall 1 < i \leq k. \tag{2.42}$$

Proof. For $k = 1$ there is nothing to prove, so assume $k > 1$. Since $w_1^1 = (0^l, 1^{c-l})$, conjugation by w leaves the last $c - l$ rows of $u^{2,2}$ (see (1.2)) in U_P; these are rows

$$(k-1)c + l + 1, \ldots, kc.$$

The character ψ_U is nontrivial on the bottom right $c - l \times c - l$ block of $u^{2,2}$, whence $\mathcal{H}(h) = 0$ by (2.41) unless w^{-1} permutes the last $c - l$ columns of $u^{2,2}$, columns

$$(k+1)c + l + 1, \ldots, (k+2)c$$

outside of U_P. This means w permutes rows

$$(k+1)c + l + 1, \ldots, (k+2)c$$

into U_P, i.e., $w_2^2 = (*^l, 1^{c-l})$. But these are also the last $c-l$ rows of the block $v_{1,2}$ of the bottom right copy of $V_{(c^{k-1})}$ in U. Since ψ_U restricts to $\psi \circ \mathrm{tr}$ on $v_{1,2}$, $\mathcal{H}(h) = 0$ by (2.41) unless w^{-1} permutes the last $c-l$ columns of $v_{1,2}$, columns

$$(k+2)c + l + 1, \ldots, (k+3)c$$

outside of U_P. Thus w permutes rows

$$(k+2)c + l + 1, \ldots, (k+3)c$$

into U_P, i.e. $w_3^2 = (*^l, 1^{c-l})$, and these are the bottom $c-l$ rows of $v_{2,3}$. Proceeding similarly (ψ_U is $\psi \circ \mathrm{tr}$ on $v_{j,j+1}$) we obtain $w_i^2 = (*^l, 1^{c-l})$ for all $1 < i \le k$.

In addition, $w_1^1 = (0^l, 1^{c-l})$ implies w^{-1} permutes the first l columns of $u^{1,1}$, columns

$$(k-1)c + 1, \ldots, (k-1)c + l,$$

into U_P. Since ψ_U restricts to $\psi \circ \mathrm{tr}$ on $u^{1,1}$, $\mathcal{H}(h) = 0$ by (2.41) unless w permutes the first l rows of $u^{1,1}$ outside of U_P; these are rows

$$(k-2)c + 1, \ldots, (k-2)c + l,$$

and we obtain $w_2^1 = (0^l, *^{c-l})$. Then w^{-1} permutes the first l columns of the block $v_{k-1,k}$ of the top left copy of $V_{(c^{k-1})}$ in U, columns

$$(k-2)c + 1, \ldots, (k-2)c + l$$

into U_P, so that $\mathcal{H}(h) = 0$ by (2.41) unless w permutes the first l rows of $v_{k-1,k}$, rows

$$(k-3)c + 1, \ldots, (k-3)c + l,$$

outside of U_P, i.e., $w_3^1 = (0^l, *^{c-l})$. Similarly, we deduce $w_i^1 = (0^l, *^{c-l})$ for all $1 < i \le k$. ∎

For each $1 < i \le k$, let $0 \le d_{i-1}^1 \le c - l$ and $0 \le d_{i-1}^2 \le l$ be maximal such that for all $1 < i \le k$,

$$w_i^1 = (0^{l+d_{i-1}^1}, *^{c-l-d_{i-1}^1}), \quad w_i^2 = (*^{l-d_{i-1}^2}, 1^{c-l+d_{i-1}^2}).$$

The integer d_{i-1}^j is defined since w_i^j takes the form (2.42).

Proposition 2.21. *We have $\mathcal{H}(h) = 0$ unless $h \sim \hat{w}\hat{u}$, $p_0(\hat{w}) = (0^l, 1^{c-l}, 0^{c-l'}, 1^{l'})$, for each $1 < i \le k$,*

$$\begin{aligned} &w_i^1 = (0^{l+d_{i-1}^1}, 1^{c-l-d_{i-1}^1}), \quad w_i^2 = (0^{l-d_{i-1}^2}, 1^{c-l+d_{i-1}^2}), \\ &d_1^1 \le \cdots \le d_{k-1}^1, \quad d_1^2 \le \cdots \le d_{k-1}^2, \end{aligned} \tag{2.43}$$

and $\hat{u}$ satisfies the conditions of Proposition 2.18, in particular $\ell_0(\hat{u})$ takes the form (2.40).

Proof. For each $1 < i \le k$, put $w_i^2 = ((w^2)'_i, 1^{c-l})$ with $(w^2)'_i \in \{0,1\}^l$. Let $1 \le j \le l$ and assume $1 < i_0 \le k$ is minimal such that $(w^2)'_{i_0}[j] = 1$. Assume $i > i_0$ is minimal with $(w^2)'_i[j] = 0$. Since $(w^2)'_{i-1}[j] = 1$, w permutes row $(k+i-2)c + j$ into U_P. This row contains coordinates of a row from $v_{i-2,i-1}$, and ψ_U is $\psi \circ \mathrm{tr}$ on $v_{i-2,i-1}$, so that on row $(k+i-2)c + j$ it is nontrivial on column $(k+i-1)c + j$. Thus (2.41) implies $\mathcal{H}(h) = 0$, unless w^{-1} permutes column $(k+i-1)c + j$ outside of U_P, which means that w permutes row $(k+i-1)c + j$ into U_P, contradicting the assumption $(w^2)'_i[j] = 0$. Therefore $(w^2)'_i[j] = 1$ for all $i \ge i_0$ (or $\mathcal{H}(h) = 0$).

Now we are in a situation similar to the proof of Proposition 2.8. If i_0 is minimal with $(w^2)'_{i_0}[j] = 1$ and $(w^2)'_{i_0}[j+1] = 0$, then for each $i > i_0$, either $(w^2)'_i[j] = 1$, $(w^2)'_i[j+1] = 0$ or $(w^2)'_i[j] = (w^2)'_i[j+1] = 1$. Using transpositions from $(\mathrm{diag}(W(\mathrm{GL}_l), I_{c-l}), 1)$, one can sort the coordinates of the blocks w_i^2 so that $d_1^2 \le \cdots \le d_{k-1}^2$. Any $b \in (\mathrm{diag}(W(\mathrm{GL}_l), I_{c-l}), 1)$ fixes the last $c - l$ rows of w_i^1 and keeps the first l rows of w_i^1 in w_i^1, for each $1 < i \le k$, and since w_i^1 starts with (0^l), we have $p_1({}^b w) = p_1(w)$ (for brevity, we identify w_i^1 with the rows it is affecting: $(k-i)c + 1, \dots, (k-i)c + c$). Additionally b fixes the last $2c - l$ rows of $p_0(w)$ while keeping the first l rows in $p_0(w)$, thus $p_0({}^b w) = p_0(w)$.

Similarly denote $w_i^1 = (0^l, (w^1)'_i)$ with $(w^1)'_i \in \{0,1\}^{c-l}$, and consider $1 \le j \le c - l$. Suppose $i_0 > 1$ is minimal with $(w^1)'_{i_0}[j] = 0$ and $i > i_0$ is minimal with $(w^1)'_i[j] = 1$. On the one hand, $(w^1)'_{i-1}[j] = 0$, hence w permutes row $(k-i+1)c + j$ outside of U_P, so that w^{-1} permutes column $(k-i+1)c + j$ into U_P. On the other hand, $(w^1)'_i[j] = 1$, whence w permutes row $(k-i)c + j$ into U_P. Again $\mathcal{H}(h) = 0$ because of (2.41), otherwise we deduce that if i_0 exists then $(w^1)'_i[j] = 0$ for all $i \ge i_0$.

This means that unless $\mathcal{H}(h) = 0$, if i_0 is minimal with $(w^1)'_{i_0}[j] = 1$ and $(w^1)'_{i_0}[j+1] = 0$, then for each $i > i_0$, either $(w^1)'_i[j] = 1$, $(w^1)'_i[j+1] = 0$ or $(w^1)'_i[j] = (w^1)'_i[j+1] = 0$. Again we use transpositions, now from $(\mathrm{diag}(I_l, W(\mathrm{GL}_{c-l})), 1)$, to rearrange the coordinates of the blocks w_i^1 and obtain $d_1^1 \le \cdots \le d_{k-1}^1$. For $b \in (\mathrm{diag}(I_l, W(\mathrm{GL}_{c-l})), 1)$, b fixes the first l rows of w_i^2 and leaves the last $c - l$ rows in w_i^2 for $1 < i \le k$, and because w_i^2 (still) ends with (1^{c-l}), we have $p_2({}^b w) = w$. Also $p_0({}^b w) = p_0(w)$.

Now condition (2.43) holds, and note that the conjugations affect u, but it still satisfies the conditions of Proposition 2.18. As opposed to the proof of Proposition 2.8, we do not claim $\ell_0(\hat{u}) = \ell_0(u)$, it might not hold because $(\mathrm{diag}(W(\mathrm{GL}_l), I_{c-l}), 1)$ does not commute with (2.40). ∎

Let now $h = wu$, with w and u given by Proposition 2.21. Since the total sum of coordinates of w must be kc, we have

$$c - l + l' + \sum_{i=1}^{k-1} (c - l - d_i^1) + \sum_{i=1}^{k-1} (c - l + d_i^2) = kc,$$

hence

$$\sum_{i=1}^{k-1} (d_i^2 - d_i^1) = (k-1)(2l - c) + l - l'. \tag{2.44}$$

We can multiply w on the left by an element of $W(M_P)$, so that the matrix corresponding to w takes the form

$$\begin{pmatrix}
0 & & & & & & & & I_{c-l+d^2_{k-1}} \\
& I_{c-l-d^1_{k-1}} & & & & & & 0 & \\
& & 0 & & & & \iddots & & \\
& & & \ddots & & 0 & & & \\
& & & & \begin{pmatrix} & & I_l \\ & I_{2c-l-l'} & \\ I_{l'} & & \end{pmatrix} & & & & \\
& & & 0 & & I_{l-d^2_1} & & & \\
& & \iddots & & & & 0 & & \\
& 0 & & & & & & \ddots & \\
I_{l+d^1_{k-1}} & & & & & & & & 0
\end{pmatrix}.$$

For example, if $k = 2$, then

$$w = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & I_{c-l+d^2_1} \\
0 & I_{c-l-d^1_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_l & 0 & 0 \\
0 & 0 & 0 & I_{2c-l-l'} & 0 & 0 & 0 \\
0 & 0 & I_{l'} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{l-d^2_1} & 0 \\
I_{l+d^1_1} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and for $k = 3$,

$$w = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{c-l+d^2_2} \\
0 & I_{c-l-d^1_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{c-l+d^2_1} & 0 & 0 \\
0 & 0 & 0 & I_{c-l-d^1_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_{l'} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{2c-l-l'} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_l & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{l-d^2_1} & 0 & 0 & 0 \\
0 & 0 & I_{l+d^1_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{l-d^2_2} & 0 \\
I_{l+d^1_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

For $1 \le j \le k-1$, denote

$$\gamma_j = \operatorname{diag}\left(I_{\sum_{i=1}^{j-1}(c-l-d^1_{k-i})}, \begin{pmatrix} & I_{kc-(2j-1)(c-l)-d^2_{k-j}+\sum_{i=1}^{j-1}(d^1_{k-i}-d^2_{k-i})} \\ I_{c-l+d^2_{k-j}} & \end{pmatrix}, \right.$$
$$\left. I_{\sum_{i=1}^{j-1}(c-l+d^2_{k-i})}\right) \in W(\mathrm{GL}_{kc}),$$

$$\gamma'_j = \operatorname{diag}\left(I_{\sum_{i=1}^{j-1}(l+d^1_{k-i})}, \begin{pmatrix} & I_{l+d^1_{k-j}} \\ I_{kc-(2j-1)l-d^1_{k-j}+\sum_{i=1}^{j-1}(d^2_{k-i}-d^1_{k-i})} & \end{pmatrix}, \right.$$
$$\left. I_{\sum_{i=1}^{j-1}(l-d^2_{k-i})}\right) \in W(\mathrm{GL}_{kc}),$$

and multiply w on the left by $\mathrm{diag}(\gamma_{k-1}\cdot\ldots\cdot\gamma_1, \gamma'_{k-1}\cdot\ldots\cdot\gamma'_1)$. Now it follows that ${}^{h}U \cap M_P = V_\beta \times V_{\beta'}$, where β and β' are the compositions of kc given by

$$\begin{aligned}
\beta &= (c-l-d^1_{k-1}, \ldots, c-l-d^1_1, l'+c-l, c-l+d^2_1, \ldots, c-l+d^2_{k-1}),\\
\beta' &= (l+d^1_{k-1}, \ldots, l+d^1_1, c-l'+l, l-d^2_1, \ldots, l-d^2_{k-1}).
\end{aligned}$$

Both β and β' are indeed compositions of kc, by (2.44). Put $\psi_{V_\beta\times V_{\beta'}} = {}^{h}\psi_U|_{V_\beta\times V_{\beta'}}$, $\psi_{V_\beta} = {}^{h}\psi_U|_{V_\beta\times I_{kc}}$ and $\psi_{V_{\beta'}} = {}^{h}\psi_U|_{I_{kc}\times V_{\beta'}}$, and note that

$$J_{V_\beta\times V_{\beta'},\psi_{V_\beta\times V_{\beta'}}}(\rho) = J_{V_\beta,\psi_{V_\beta}}(\rho_1)\otimes J_{V_{\beta'},\psi_{V_{\beta'}}}(\rho_2). \tag{2.45}$$

We start with describing ${}^{w\ell_0(u)}\psi_U|_{V_\beta\times V_{\beta'}}$. For $v \in V_\beta$ and $v' \in V_{\beta'}$, write

$$v = \begin{pmatrix}
I_{c-l-d^1_{k-1}} & b_1 & \cdots & & & & & & \\
& \ddots & \ddots & & & & & & \\
& & I_{c-l-d^1_2} & b_{k-2} & \cdots & & & & \\
& & & I_{c-l-d^1_1} & e & b_{k-1} & \cdots & & \\
& & & & I_{l'} & 0 & b_k & \cdots & \\
& & & & & I_{c-l} & b_{k+1} & \cdots & \\
& & & & & & \ddots & \ddots & \\
& & & & & & & I_{c-l+d^2_{k-2}} & b_{2k-1} \\
& & & & & & & & I_{c-l+d^2_{k-1}}
\end{pmatrix},$$

$$v' = \begin{pmatrix}
I_{l+d^1_{k-1}} & b'_1 & \cdots & & & & & & \\
& \ddots & \ddots & & & & & & \\
& & I_{l+d^1_2} & b'_{k-2} & \cdots & & & & \\
& & & I_{l+d^1_1} & e' & b'_{k-1} & \cdots & & \\
& & & & I_{c-l'} & 0 & f' & \cdots & \\
& & & & & I_{l} & b'_{k} & \cdots & \\
& & & & & & \ddots & \ddots & \\
& & & & & & & I_{l-d^2_{k-2}} & b'_{2k-2} \\
& & & & & & & & I_{l-d^2_{k-1}}
\end{pmatrix}.$$

The dimensions of the blocks b_i and b'_i are clear; note that b_k and b_{k+1} (resp., b'_k) have $c-l+d^2_1$ (resp., $l-d^2_1$) columns. Then

$$\begin{aligned}
&{}^{w\ell_0(u)}\psi_U(\mathrm{diag}(v, I_{kc}))\\
&= \psi\Bigg(-\sum_{j=k-1}^{2}\mathrm{tr}\Bigg(b_{k-j}\begin{pmatrix}0_{d^1_j-d^1_{j-1}\times c-l-d^1_j}\\ I_{c-l-d^1_j}\end{pmatrix}\Bigg) - \mathrm{tr}\Bigg(b_{k-1}\begin{pmatrix}0_{d^1_1\times c-l-d^1_1}\\ I_{c-l-d^1_1}\end{pmatrix}\Bigg)\\
&\quad -\mathrm{tr}\Bigg(b_k\begin{pmatrix}A(X)\\ 0_{c-l\times l'}\end{pmatrix}\Bigg) + \mathrm{tr}\Bigg(b_{k+1}\begin{pmatrix}0_{d^2_1\times c-l}\\ I_{c-l}\end{pmatrix}\Bigg) - \sum_{j=1}^{k-2}\mathrm{tr}\Bigg(b_{k+1+j}\begin{pmatrix}0_{d^2_{j+1}-d^2_j\times c-l+d^2_j}\\ I_{c-l+d^2_j}\end{pmatrix}\Bigg)\Bigg),
\end{aligned} \tag{2.46}$$

$$
\begin{aligned}
&{}^{w\ell_0(u)}\psi_U(\mathrm{diag}(I_{kc}, v'))\\
&= \psi\Big(-\sum_{j=k-1}^{2} \mathrm{tr}\Big(b'_{k-j}\begin{pmatrix} I_{l+d^1_{j-1}} & 0_{l+d^1_{j-1}\times d^1_j-d^1_{j-1}}\end{pmatrix}\Big) - \mathrm{tr}\big(b'_{k-1}\begin{pmatrix} I_l & 0_{l\times d^1_1}\end{pmatrix}\big)\\
&\quad + \mathrm{tr}\big(b'_k\begin{pmatrix} I_{l-d^2_1} & 0_{l-d^2_1\times d^2_1}\end{pmatrix}\big) - \sum_{j=1}^{k-2}\mathrm{tr}\Big(b'_{k+j}\begin{pmatrix} I_{l-d^2_{j+1}} & 0_{l-d^2_{j+1}\times d^2_{j+1}-d^2_j}\end{pmatrix}\Big)\Big). \qquad (2.47)
\end{aligned}
$$

In both formulas, the sums $\sum_{j=k-1}^{2}$ are omitted when $k=2$. The matrix $A(X)\in\mathrm{Mat}_{d^2_1\times l'}$ in (2.46) is uniquely defined given the block X of $\ell_0(u)$, and in particular $A(0)=0$ and when $d^2_1 = l = l'$, $A(I_l) = I_l$. For $l = c$, we have $d^1_i = 0$ for all i, and since $d^2_i \le c$ and $l' \le c$, (2.44) implies $d^2_i = c$ for all i and $l = l'$, then (2.47) becomes ψ_k^{-1}, and when $X = I_c$, (2.46) also becomes ψ_k^{-1}.

Proposition 2.22. *Assume $k > 1$ and $\mathcal{H}(h) \neq 0$. The character ψ_{V_β} belongs to the orbit of*

$$
\begin{aligned}
v \mapsto \psi\Big(&-\sum_{j=k-1}^{2}\mathrm{tr}(b_{k-j}\,*) - \mathrm{tr}\Big(b_{k-1}\begin{pmatrix} *_{d^1_1\times c-l-d^1_1}\\ I_{c-l-d^1_1}\end{pmatrix}\Big)\\
&-\mathrm{tr}\Big(b_k\begin{pmatrix} *_{d^2_1\times l'}\\ 0_{c-l\times l'}\end{pmatrix}\Big) + \mathrm{tr}\Big(b_{k+1}\begin{pmatrix} 0_{d^2_1\times c-l}\\ I_{c-l}\end{pmatrix}\Big) - \sum_{j=1}^{k-2}\mathrm{tr}\Big(b_{k+1+j}\begin{pmatrix} *_{d^2_{j+1}-d^2_j\times c-l+d^2_j}\\ I_{c-l+d^2_j}\end{pmatrix}\Big)\Big),
\end{aligned}
\qquad (2.48)
$$

and $\psi_{V_{\beta'}}$ belongs to the orbit of

$$
\begin{aligned}
v' \mapsto \psi\Big(&-\sum_{j=k-1}^{2}\mathrm{tr}\big(b'_{k-j}\begin{pmatrix} I_{l+d^1_{j-1}} & *_{l+d^1_{j-1}\times d^1_j-d^1_{j-1}}\end{pmatrix}\big) - \mathrm{tr}\big(b'_{k-1}\begin{pmatrix} I_l & 0_{l\times d^1_1}\end{pmatrix}\big)\\
&+\mathrm{tr}\big(b'_k\begin{pmatrix} I_{l-d^2_1} & *_{l-d^2_1\times d^2_1}\end{pmatrix}\big) - \sum_{j=1}^{k-2}\mathrm{tr}(b'_{k+j}\,*)\Big).
\end{aligned}
\qquad (2.49)
$$

Here $*$ means undetermined block entries. When $\ell(u)$ is the identity, (2.48)–(2.49) coincide with (2.46)–(2.47).

Proof. The proof is similar to the proof of Proposition 2.11. Now ψ_U is defined by $2(k-1)$ blocks in Mat_c. Let $B_{1,k-2}$ (resp., $B_{2,0}$) be the block corresponding to $u^{1,1}$ (resp., $u^{2,2}$), and $B_{1,0},\dots,B_{1,k-3}$ (resp., $B_{2,1},\dots,B_{2,k-2}$) be the blocks corresponding to the top left (resp., bottom right) embedding of $V_{(c^{k-1})} < M_P$ (see §1.5).

Set $d^1_0 = d^2_0 = 0$. For $0 \le i \le k-2$, write $B_{1,i}$ as the upper right block of

$$
\begin{pmatrix}
I_{l+d^1_{k-i-2}} & & & B^{1,1}_{1,i} & B^{1,2}_{1,i} & B^{1,3}_{1,i}\\
& I_{d^1_{k-i-1}-d^1_{k-i-2}} & & B^{2,1}_{1,i} & B^{2,2}_{1,i} & B^{2,3}_{1,i}\\
& & I_{c-l-d^1_{k-i-1}} & B^{3,1}_{1,i} & B^{3,2}_{1,i} & B^{3,3}_{1,i}\\
& & & I_{l+d^1_{k-i-2}} & & \\
& & & & I_{d^1_{k-i-1}-d^1_{k-i-2}} & \\
& & & & & I_{c-l-d^1_{k-i-1}}
\end{pmatrix}
$$

and $B_{2,i}$ as the upper right block of

$$\begin{pmatrix} I_{l-d^2_{i+1}} & & & B^{1,1}_{2,i} & B^{1,2}_{2,i} & B^{1,3}_{2,i} \\ & I_{d^2_{i+1}-d^2_i} & & B^{2,1}_{2,i} & B^{2,2}_{2,i} & B^{2,3}_{2,i} \\ & & I_{c-l+d^2_i} & B^{3,1}_{2,i} & B^{3,2}_{2,i} & B^{3,3}_{2,i} \\ & & & I_{l-d^2_{i+1}} & & \\ & & & & I_{d^2_{i+1}-d^2_i} & \\ & & & & & I_{c-l+d^2_i} \end{pmatrix}.$$

Recall ψ_U is given by

$$\psi\Big(-\sum_{i=0}^{k-2}\sum_{t=1}^{3}\mathrm{tr}(B^{t,t}_{1,i}) + \sum_{t=1}^{3}\mathrm{tr}(B^{t,t}_{2,0}) - \sum_{i=1}^{k-2}\sum_{t=1}^{3}\mathrm{tr}(B^{t,t}_{2,i})\Big). \tag{2.50}$$

Let $\mathscr{M}_P$ (resp., $\mathscr{U}_P$, $\mathscr{U}_P^-$) denote the list of blocks conjugated by w into M_P (resp., U_P, U_P^-):

$$\begin{aligned} \mathscr{M}_P &= \{B^{1,1}_{j,i}, B^{2,1}_{j,i}, B^{3,2}_{j,i}, B^{3,3}_{j,i} : 1 \le j \le 2,\ 0 \le i \le k-2\}, \\ \mathscr{U}_P &= \{B^{3,1}_{j,i} : 1 \le j \le 2,\ 0 \le i \le k-2\}, \\ \mathscr{U}_P^- &= \{B^{1,2}_{j,i}, B^{1,3}_{j,i}, B^{2,2}_{j,i}, B^{2,3}_{j,i} : 1 \le j \le 2,\ 0 \le i \le k-2\}. \end{aligned}$$

We can assume $\ell_j(u) = \mathrm{diag}(z_{j,0},\dots,z_{j,k-2})$, $j=1,2$, with $z_{j,i} \in {}^{w_\sigma}N_{\mathrm{GL}_c}$. Here w_σ is the projection of $(\sigma,1)^{-1}$ into the i-th copy of GL_c ($(1,\sigma)$ commutes with $\ell(u)$). We can then write $z_{j,i} = z'_{j,i}m_{j,i}$ where

$${}^{w}\mathrm{diag}(m_{1,0},\dots,m_{1,k-2}, I_{2c}, m_{2,0},\dots,m_{2,k-2}) \in M_P$$

and for all $0 \le i \le k-2$,

$$\begin{aligned} m_{1,i} &= \begin{pmatrix} I_{l+d^1_{k-i-1}} & M^1_{1,i} \\ M^2_{1,i} & I_{c-l-d^1_{k-i-1}} + M^2_{1,i}M^1_{1,i} \end{pmatrix} \in \mathrm{GL}_c, \\ m_{2,i} &= \begin{pmatrix} I_{l-d^2_{i+1}} & M^1_{2,i} \\ M^2_{2,i} & I_{c-l+d^2_{i+1}} + M^2_{2,i}M^1_{2,i} \end{pmatrix} \in \mathrm{GL}_c, \\ & I_{c-l-d^1_{k-i-1}} + M^2_{1,i}M^1_{1,i} \in \mathrm{GL}_{l+d^1_{k-i-1}}, \\ & I_{c-l+d^2_{i+1}} + M^2_{2,i}M^1_{2,i} \in \mathrm{GL}_{c-l+d^2_{i+1}}. \end{aligned}$$

Then

$$\begin{aligned} m_{1,i}^{-1} &= \begin{pmatrix} I_{l+d^1_{k-i-1}} + M^1_{1,i}M^2_{1,i} & -M^1_{1,i} \\ -M^2_{1,i} & I_{c-l-d^1_{k-i-1}} \end{pmatrix} \in \mathrm{GL}_c, \\ m_{2,i}^{-1} &= \begin{pmatrix} I_{l-d^2_{i+1}} + M^1_{2,i}M^2_{2,i} & -M^1_{2,i} \\ -M^2_{2,i} & I_{c-l+d^2_{i+1}} \end{pmatrix} \in \mathrm{GL}_c. \end{aligned}$$

Henceforth we assume $z_{j,i} = m_{j,i}$. To compute ${}^{\ell(u)}\psi_U$ we calculate

$$m_{1,k-2}^{-1}B_{1,k-2}, \quad m_{1,i}^{-1}B_{1,i}m_{1,i+1}, \quad B_{2,0}m_{2,0}, \quad m_{2,i}^{-1}B_{2,i+1}m_{2,i+1}, \quad \forall 0 \le i \le k-3.$$

We start with ψ_{V_β} and show that it belongs to the orbit of (2.48), otherwise $\mathcal{H}(h)=0$. This amounts to the description of its restriction to $b_{k-1},\dots,b_{2k-1}$ and e. The rightmost $c-l$ columns of b_{k-1} consist of $B^{3,2}_{1,k-2}$ and $B^{3,3}_{1,k-2}$ (the leftmost l' columns are conjugated from the $c\times c$ block to the right of $u^{1,1}$). Looking at $m^{-1}_{1,k-2}B_{1,k-2}$, if the top l rows of $M^1_{1,k-2}$ are nonzero, ${}^u\psi_U$ is nontrivial on $B^{3,1}_{1,k-2}\in\mathscr{U}_P$. Hence $\mathcal{H}(h)=0$ by (2.6) in this case. Also ${}^u\psi_U$ restricts to $\psi\circ\mathrm{tr}$ on $B^{3,3}_{1,k-2}$. Hence ψ_{V_β} agrees with (2.48) on b_{k-1}.

The block b_{k+1} consists of $(B^{3,2}_{2,0},B^{3,3}_{2,0})$ and we consider $B_{2,0}m_{2,0}$. If the last $c-l$ columns of $M^1_{2,0}$ are nonzero, ${}^u\psi_U$ is nontrivial on $B^{3,1}_{2,0}\in\mathscr{U}_P$. Unless $\mathcal{H}(h)=0$, we find that the last $c-l$ columns of $M^1_{2,0}$ are 0; then it follows that ${}^u\psi_U$ and ψ_U coincide on $(B^{3,2}_{2,0},B^{3,3}_{2,0})$.

Regarding b_k, it is conjugated by w from the $c\times c$ block below $u^{2,2}$. Denote this block by B_0; we further divide it by writing it as the upper right block of

$$\begin{pmatrix} I_{c-l'} & & & B_0^{1,1} & B_0^{1,2} & B_0^{1,3}\\ & I_{l'-d_1^2} & & B_0^{2,1} & B_0^{2,2} & B_0^{2,3}\\ & & I_{d_1^2} & B_0^{3,1} & B_0^{3,2} & B_0^{3,3}\\ & & & I_{l-d_1^2} & & \\ & & & & I_{d_1^2} & \\ & & & & & I_{c-l}\end{pmatrix}.$$

Here

$$B_0^{1,1},B_0^{2,2},B_0^{2,3},B_0^{3,2},B_0^{3,3}\in\mathscr{M}_P,\quad B_0^{2,1},B_0^{3,1}\in\mathscr{U}_P,\quad B_0^{1,2},B_0^{1,3}\in\mathscr{U}_P^-.$$

The blocks conjugated into b_k are $B_0^{2,2}$, $B_0^{2,3}$, $B_0^{3,2}$ and $B_0^{3,3}$. The conjugation of U by $\ell(u)$ multiplies B_0 on the right by $m^{-1}_{2,0}$. The restriction of ${}^{\ell_0(u)}\psi_U$ to $B_0^{2,2}$ and $B_0^{3,2}$ is defined by $A(X)$, but ${}^{\ell_0(u)}\psi_U$ can also be nontrivial on $B_0^{2,1}$ or $B_0^{2,2}$ (or we could have, e.g., $d_1^2=0,l$). We can assume ${}^{\ell_0(u)}\psi_U$ is given on B_0 by

$$\psi\left(\mathrm{tr}\left(\varphi_k\begin{pmatrix}B_0^{1,1}&B_0^{1,2}&B_0^{1,3}\\B_0^{2,1}&B_0^{2,2}&B_0^{2,3}\\B_0^{3,1}&B_0^{3,2}&B_0^{3,3}\end{pmatrix}\right)\right),\quad \varphi_k=\begin{pmatrix}0_{l-d_1^2\times c-l'}&A_1(X)\\0_{d_1^2\times c-l'}&A(X)\\0_{c-l\times c-l'}&0_{c-l\times l'}\end{pmatrix},\quad A_1(X)\in\mathrm{Mat}_{l-d_1^2\times l'}. \tag{2.51}$$

Here $A_1(X)$ defines the restriction of ${}^{\ell_0(u)}\psi_U$ to $B_0^{2,1}$ and $B_0^{3,1}$. When we consider $m_{2,0}\varphi_k$ we see that the restriction of ${}^u\psi_U$ to $B_0^{2,1},B_0^{3,1}\in\mathscr{U}_P$ is given by the first $l-d_1^2$ rows of $m_{2,0}$ multiplied by the last l' columns of φ_k; this restriction should vanish, and the restriction to the blocks conjugated into b_k corresponds to the last $c-l+d_1^2$ rows of $m_{2,0}$ multiplied by the last l' columns of φ_k. Since the last $c-l$ columns of $M^1_{2,0}$ are 0, we can denote $M^1_{2,0}=\begin{pmatrix}\alpha & 0_{l-d_1^2\times c-l}\end{pmatrix}$. Also put $M^2_{2,0}=\begin{pmatrix}\beta^1\\ \beta^2\end{pmatrix}$ with $\beta^1\in\mathrm{Mat}_{d_1^2\times l-d_1^2}$; then

$$m_{2,0}\begin{pmatrix}A_1(X)\\A(X)\\0_{c-l\times l'}\end{pmatrix}=\begin{pmatrix}I_{l-d_1^2}&\alpha&0\\ \beta^1& I_{d_1^2}+\beta^1\alpha&0\\ \beta^2&\beta^2\alpha&I_{c-l}\end{pmatrix}\begin{pmatrix}A_1(X)\\A(X)\\0_{c-l\times l'}\end{pmatrix}=\begin{pmatrix}A_1(X)+\alpha A(X)\\ \beta^1A_1(X)+I_{d_1^2}+\beta^1\alpha A(X)\\ \beta^2A_1(X)+\beta^2\alpha A(X)\end{pmatrix}.$$

Now if $\mathcal{H}(h) \neq 0$, we must have $A_1(X) + \alpha A(X) = 0$, so $\beta^2 A_1(X) + \beta^2\alpha A(X) = 0$, in which case ψ_{V_β} agrees with (2.48) on b_k. Thus both characters agree on b_{k-1}, b_k and b_{k+1}.

Consider b_{k+i}, $2 \le i \le k-1$. We multiply $m_{2,i-2}^{-1} B_{2,i-1} m_{2,i-1}$. The block b_{k+i} consists of $B_{2,i-1}^{3,2}$ and $B_{2,i-1}^{3,3}$. Recall $B_{2,i-1}^{3,1} \in \mathcal{U}_P$. Then $\mathcal{H}(h) = 0$ unless the top right $l - d_i^2 \times c - l + d_{i-1}^2$ block of $m_{2,i-1} m_{2,i-2}^{-1}$ is 0:

$$\begin{pmatrix} I_{l-d_i^2} & M_{2,i-1}^1 \end{pmatrix} \begin{pmatrix} -M_{2,i-2}^1 \\ I_{c-l+d_{i-1}^2} \end{pmatrix} = 0.$$

In this case the restriction of ${}^u\psi_U$ to $(B_{2,i-1}^{3,2}, B_{2,i-1}^{3,3})$, which corresponds to the bottom right $c - l + d_i^2 \times c - l + d_{i-1}^2$ block of $m_{2,i-1} m_{2,i-2}^{-1}$, is defined by

$$\begin{aligned} \begin{pmatrix} M_{2,i-1}^2 & I_{c-l+d_i^2} + M_{2,i-1}^2 M_{2,i-1}^1 \end{pmatrix} \begin{pmatrix} -M_{2,i-2}^1 \\ I_{c-l+d_{i-1}^2} \end{pmatrix} &= \begin{pmatrix} 0_{c-l+d_i^2 \times l-d_i^2} & I_{c-l+d_i^2} \end{pmatrix} \begin{pmatrix} -M_{2,i-2}^1 \\ I_{c-l+d_{i-1}^2} \end{pmatrix} \\ &= \begin{pmatrix} *_{d_i^2 - d_{i-1}^2 \times c-l+d_{i-1}^2} \\ I_{c-l+d_{i-1}^2} \end{pmatrix}. \end{aligned}$$

Therefore ψ_{V_β} agrees with (2.48) on b_{k+i}.

Also $\psi_{V_\beta}|_e = 1$, because e is conjugated from the $c \times c$ block to the right of $u^{1,1}$. This completes the proof for ψ_{V_β}.

We turn to the restriction of $\psi_{V_{\beta'}}$ to $b'_1, \ldots, b'_k$, e' and f', and prove that unless $\mathcal{H}(h) = 0$, $\psi_{V_{\beta'}}$ and (2.49) coincide. The block b'_k corresponds to $B_{2,0}^{1,1}$ and $B_{2,0}^{2,1}$. Considering $B_{2,0} m_{2,0}$, the restriction of ${}^u\psi_U$ to these blocks is given by the top left $l - d_1^2 \times l$ block of $m_{2,0}$, namely

$$\psi\left(\mathrm{tr}\left(\begin{pmatrix} I_{l-d_1^2} & *_{l-d_1^2 \times d_1^2} \end{pmatrix} \begin{pmatrix} B_{2,0}^{1,1} \\ B_{2,0}^{2,1} \end{pmatrix}\right)\right).$$

Hence $\psi_{V_{\beta'}}$ and (2.49) coincide on b'_k.

The block b'_{k-1} is conjugated from $B_{1,k-2}^{1,1}$ and $B_{1,k-2}^{2,1}$. This is similar to b_{k+1}. We multiply $m_{1,k-2}^{-1} B_{1,k-2}$ and if the first l rows of $M_{1,k-2}^1$ are nonzero, ${}^u\psi_U$ is nontrivial on $B_{1,k-2}^{3,1} \in \mathcal{U}_P$ whence $\mathcal{H}(h) = 0$. Henceforth we can assume the first l rows of $M_{1,k-2}^1$ are 0; then the top left $l \times l + d_1^1$ block of $m_{1,k-2}^{-1}$ equals $\begin{pmatrix} I_l & 0_{l \times l + d_1^1} \end{pmatrix}$, so that the restriction of ${}^u\psi_U$ to $B_{1,k-2}^{1,1}$ and $B_{1,k-2}^{2,1}$ coincides with the restriction of ψ_U ($\psi \circ \mathrm{tr}$ on the former, trivial on the latter).

Consider b'_i, $1 \le i \le k-2$. We multiply $m_{1,i-1}^{-1} B_{1,i-1} m_{1,i}$. The block b'_i is conjugated from $(B_{1,i-1}^{1,1}, B_{1,i-1}^{2,1})$. This is similar to the case of b_{k+i}. Since $B_{1,i-1}^{3,1} \in \mathcal{U}_P$, $\mathcal{H}(h) = 0$ unless the top right $l + d_{k-i-1}^1 \times c - l - d_{k-i}^1$ block of $m_{1,i} m_{1,i-1}^{-1}$ is 0, i.e.,

$$\begin{pmatrix} I_{l+d_{k-i-1}^1} & M_{1,i}^1 \end{pmatrix} \begin{pmatrix} -M_{1,i-1}^1 \\ I_{c-l-d_{k-i}^1} \end{pmatrix} = 0.$$

Then the restriction of ${}^{u}\psi_U$ to $(B^{1,1}_{1,i-1}, B^{2,1}_{1,i-1})$, which corresponds to the top left $l + d^1_{k-i-1} \times l + d^1_{k-i}$ block of $m_{1,i}m^{-1}_{1,i-1}$ becomes

$$\begin{pmatrix} I_{l+d^1_{k-i-1}} & M^1_{1,i} \end{pmatrix} \begin{pmatrix} I_{l+d^1_{k-i}} + M^1_{1,i-1}M^2_{1,i-1} \\ -M^2_{1,i-1} \end{pmatrix} = \begin{pmatrix} I_{l+d^1_{k-i-1}} & M^1_{1,i} \end{pmatrix} \begin{pmatrix} I_{l+d^1_{k-i}} \\ 0_{c-l-d^1_{k-i} \times l+d^1_{k-i}} \end{pmatrix}$$
$$= \begin{pmatrix} I_{l+d^1_{k-i-1}} & *_{l+d^1_{k-i-1} \times d^1_{k-i}-d^1_{k-i-1}} \end{pmatrix},$$

hence $\psi_{V_{\beta'}}$ agrees with (2.49) on b'_i.

The character $\psi_{V_{\beta'}}$ is trivial on f', because f' is conjugated from $B^{1,1}_0$ (see (2.51), the top left $l - d^2_1 \times c - l'$ block of φ_k). It is also trivial on e' since it is conjugated from the $c \times c$ block to the right of $u^{1,1}$ (this is similar to e). ∎

Proposition 2.23. *Consider $k > 1$. Assume $d^1_1 < c - l$ (in particular $l < c$) or $d^2_1 < l$ (in particular $0 < l$). Then $J_{V_\beta \times V_{\beta'}, \psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho) = 0$ and $\mathcal{H}(h) = 0$.*

Proof. We argue as in the proof of Proposition 2.12. By Proposition 2.22, we can assume ψ_{V_β} (resp., $\psi_{V_{\beta'}}$) is given by (2.48) (resp., (2.49)). Let φ be the transpose of the nilpotent element defined by $\psi^{-1}_{V_\beta}$ (resp., $\psi^{-1}_{V_{\beta'}}$). By (2.45) and [46, Theorems A, E], because ρ_1 (resp., ρ_2) is (k, c), it is enough to show that φ is nilpotent of order at least $k + 1$.

Consider $d^1_1 < c - l$. Looking at (2.48), we have k nontrivial blocks $b_{k-1}, b_{k+1}, \ldots, b_{2k-1}$, and for each block, the bottom right coordinate is nontrivial and the other coordinates in its column in φ are 0. This does not depend on the undetermined coordinates of the character. To see this use the assumption $c - l - d^1_1 > 0$ for b_{k-1} and $l < c$ for b_{k+1}, and the bottom right coordinate of b_{k+1} is the only nonzero coordinate in its column in φ because on the $l' \times c - l + d^2_1$ block b_k above b_{k+1}, φ is 0 in the last $c - l$ columns (if $l' = 0$, this is trivially true). It follows that φ is nilpotent of order at least $k + 1$.

For the case $d^2_1 < l$, the blocks $b'_1, \ldots, b'_k$ are k nontrivial blocks, the top left coordinate of each block is nontrivial (use $l > 0$ and $d^2_1 < l$) independently of undetermined coordinates, and is the only nonzero coordinate in its row (for b'_{k-1} use the fact that (2.49) is trivial on e'!). Again φ is nilpotent of order at least $k + 1$. ∎

Remark 2.24. If $l = l'$, the conditions $d^1_1 = c - l$ and (2.44) already imply $d^2_i = l$ for all i.

For the remaining cases $k = 1$ or both $d^1_1 = c - l$ and $d^2_1 = l$, in which case $d^1_i = c - l$ and $d^2_i = l$ for all i, whence by (2.44) we have, for all $k \geq 1$, $l' = l$. Up to left multiplication by an element of $W(M_P)$, w equals

$$\begin{pmatrix} &&&&&&& I_c \\ &&&&&& \iddots & \\ &&&&& I_c && \\ &&& \begin{pmatrix} && I_l \\ & I_{2(c-l)} & \\ I_l && \end{pmatrix} &&&& \\ && I_c &&&&& \\ & \iddots &&&&&& \\ I_c &&&&&&& \end{pmatrix},$$

so that ${}^{w}\ell(u) \in P$ and $h \sim w\ell_0(u)$ (we still do not change w, in order to use β, β' and the formulas for the characters given above). Therefore ψ_{V_β} and $\psi_{V_{\beta'}}$ are already given by (2.46) and (2.47). Considering the action of $(\mathrm{GL}_l, \mathrm{GL}_l)$, where GL_l is the natural subgroup of $M_{(l,c-l)}$, we can already write $X = A_{l,j} = \begin{pmatrix} & I_j \\ 0_{l-j} & \end{pmatrix}$ with $0 \le j \le l$. We deduce there are only finitely many more representatives to analyze, but as opposed to §2.2, we must handle each $0 \le j \le l$ separately (i.e., we cannot easily reduce to $j = l$). The form of representatives is finally similar to the case $k = 1$. For the representative h such that $j = l = c$ we have $h \sim \delta$.

Proposition 2.25. *Assume $d_1^1 = c - l$ or $k = 1$, and $0 \le l < c$. Then $\mathcal{H}(h) = 0$ outside a discrete subset of s. Furthermore, if $l > 0$ (forcing $c > 1$) and π_2 is supercuspidal, or $ck > 1$, $l = 0$, π_1 and π_2 are supercuspidal and $\rho_2 = \rho_c(\tau_2)$ for an irreducible supercuspidal representation τ_2 of GL_k, then $\mathcal{H}(h) = 0$ for all s.*

Proof. Now $V_\beta = V_{\beta'} = V_{(c^k)}$. Consider the parabolic subgroup $R < G$ with $M_R = M_{(c-l,l)}$ and $U_R = V^-_{(c-l,l)}$. Note that $V^-_{(c-l,l)}$ is trivial if $l = 0$. Identify the group GL_{c-l} (nontrivial for $0 \le l < c$) with its natural image in M_R.

For convenience, we multiply w on the left by $\mathrm{diag}\left(I_{2kc-c}, \begin{pmatrix} & I_l \\ I_{c-l} & \end{pmatrix}\right)$. This permutation normalizes $V_\beta \times V_{\beta'}$, fixes ψ_{V_β} and conjugates $\psi_{V_{\beta'}}$ into

$$\begin{pmatrix} I_c & b'_1 & & & \\ & \ddots & \ddots & & \\ & & I_c & b'_{k-1} & e' \\ & & & I_l & \\ & & & & I_{c-l} \end{pmatrix} \mapsto \psi\Big(-\sum_{j=k-1}^{2} \mathrm{tr}(b'_{k-j}) - \mathrm{tr}(b'_{k-1} \left(\begin{smallmatrix} I_l & 0_{l\times c-l} \end{smallmatrix}\right))\Big). \tag{2.52}$$

Now ${}^{h}(1, \mathrm{GL}_{c-l}) = \mathrm{diag}(I_{2kc-(c-l)}, \mathrm{GL}_{c-l})$, and since the character (2.52) is trivial on e', $J_{V_\beta \times V_{\beta'}, \psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)$ is a well defined representation of ${}^{h}(1, \mathrm{GL}_{c-l})$.

Over nonarchimedean fields, we simultaneously prove that $J_{V_\beta \times V_{\beta'}, \psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)$ is, for all s, a trivial representation of ${}^{h}(1, U_R)$, and admits a finite length filtration as a representation of ${}^{h}(1, \mathrm{GL}_{n-l})$, where ${}^{h}(1, C_{\mathrm{GL}_{n-l}})$ acts by a character on each constituent. For archimedean fields we prove that ${}^{h}(1, \mathfrak{u}_R)$ acts locally nilpotently on $J_{V_\beta \times V_{\beta'}, \psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)^*$, and the Lie algebra $\mathfrak{v}_{((k-1)c+l,c-l)}$ of $\mathrm{diag}(I_{kc}, V_{((k-1)c+l,c-l)})$ acts locally nilpotently on $J_{V_\beta \times V_{\beta'}, \psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)^*$. Note that ${}^{h}(1, \mathrm{GL}_{n-l})$ is a direct factor of $\mathrm{diag}(I_{kc}, M_{((k-1)c+l,c-l)})$. (Cf. Lemmas 2.15 and 2.16.)

Granted that, since ${}^{h^{-1}}(|\det|^{s-1/2})(1, aI_{c-l}) = |a|^{-(c-l)(s-1/2)}$, one can apply (2.7) to deduce $\mathcal{H}(h) = 0$ outside a discrete subset of s. For $l > 0$, if π_2 is supercuspidal, $\mathcal{H}(h) = 0$ for all s (because $J_{U_R}(\pi_2^\vee) = 0$).

Henceforth we identify GL_{kc} with the bottom right block of M_P. For $u \in U_R$, ${}^{h}(1, u) = m_u u'$ with $u' \in U_P$ and $m_u = \mathrm{diag}\left(I_{(k-1)c}, \begin{pmatrix} I_l & A_{l,j}u \\ & I_{c-l} \end{pmatrix}\right)$. Let $Z = \mathrm{diag}(I_{(k-1)c}, V_{(l,c-l)})$. This (abelian) group stabilizes (2.52). In addition, the subgroups $\mathrm{diag}(I_{kc-(c-l)}, \mathrm{GL}_{c-l})$ and $\mathrm{diag}(\mathrm{GL}_l, I_{c-l})^\triangle < \mathrm{GL}_{kc}$ stabilize (2.52), and act on the characters of Z with two orbits.

Over nonarchimedean fields, we show that for any nontrivial character χ of Z,

$$J_{V_{\beta'}\ltimes Z,\psi^{-1}_{V_{\beta'}}\otimes\chi}(\rho_2) = 0, \tag{2.53}$$

which implies (by [11, §§5.9–5.12])

$$J_{V_{\beta'},\psi^{-1}_{V_{\beta'}}}(\rho_2) = J_{V_{\beta'}\ltimes Z,\psi^{-1}_{V_{\beta'}}}(\rho_2). \tag{2.54}$$

Thus $J_{V_{\beta'},\psi^{-1}_{V_{\beta'}}}(\rho_2)$ is a trivial representation of ${}^{h}(1, U_R)$ and this Jacquet module factors through $J_{V_{((k-1)c+l,c-l)}}(\rho_2)$, which is an admissible finite length representation of $M_{((k-1)c+l,c-l)}$. By exactness $J_{V_{\beta'},\psi^{-1}_{V_{\beta'}}}(\rho_2)$ admits a finite length filtration such that on each constituent, ${}^{h}(1, C_{\mathrm{GL}_{c-l}})$ acts by a character. Now by (2.45), $J_{V_{\beta}\times V_{\beta'},\psi^{-1}_{V_{\beta}\times V_{\beta'}}}(\rho)$ is a trivial representation of ${}^{h}(1, U_R)$ and admits a finite filtration with ${}^{h}(1, C_{\mathrm{GL}_{c-l}})$ acting by a character on each constituent.

For the proof of (2.53) we can assume $l > 0$, otherwise (2.54) is trivial. Identifying Z with $\mathrm{Mat}_{l\times c-l}$, we can assume χ is nontrivial on the bottom right coordinate of Z, and trivial on the other coordinates of the rightmost column. Let φ be the transpose of the nilpotent element defined by the inverse of (2.52) and by χ. Note that φ is independent of $A_{l,j}$. After conjugating φ by $\mathrm{diag}\left(\begin{pmatrix} & I_{c-l} \\ I_l & \end{pmatrix}^{\triangle'}, I_c\right)$ ($\mathrm{GL}_c^{\triangle'}$ is the diagonal embedding of GL_c in $\mathrm{GL}_{(k-1)c}$), it has nontrivial entries on the bottom right coordinates of k blocks: $b'_1, \ldots, b'_{k-1}$ and Z, and in each block there is only one nontrivial entry in the rightmost column. Therefore φ is nilpotent of order at least $k+1$ and (2.53) holds, because ρ_2 is (k, c).

Over archimedean fields we repeat the proof of (2.53) and apply [47, Proposition 3.0.1] for each coordinate of Z separately, exactly as in the proofs of Lemmas 2.15 and 2.16.

It remains to prove the stronger assertion when $ck > 1$, $l = 0$, both π_1 and π_2 are supercuspidal and $\rho_2 = \rho_c(\tau_2)$ for an irreducible supercuspidal τ_2. Since τ_2 is supercuspidal and $J_{V_{\beta'},\psi^{-1}_{V_{\beta'}}}(\rho_2)$ factors through $J_{V_{((k-1)c,c)}}(\rho_2) = J_{V_{((k-1)c,c)}}(\rho_c(\tau_2))$, we obtain $\mathcal{H}(h) = 0$ for all s, unless $c = tk$ for some integer $t \geq 1$ (use [12, Theorem 2.13 (a)]).

If $t > 1$, in particular $c > k$, and we claim $J_{V_{((k-1)c,c)}}(\rho_2)$ is trivial on ${}^{h}(1, V_\delta)$ for some composition δ of c, then because π_2 is supercuspidal, $\mathcal{H}(h) = 0$ for all s. This follows by repeatedly applying the derivatives of Bernstein and Zelevinsky [11, 12] to $J_{V_{((k-1)c,c)}}(\rho_2)$. Indeed, for $1 \leq i \leq c$, let χ_i be the character of $V_{((k-1)c,c-i,1^i)}$ given by $\chi_i(z) = \psi(\sum_{i'=1}^{i} z_{kc-i',kc-i'+1})$. Then either

$$J_{V_{((k-1)c,c)}}(\rho_2) = J_{V_{((k-1)c,c-1,1)}}(\rho_2),$$

in which case our claim is proved with $\delta = (c-1, 1)$, or

$$J_{V_{((k-1)c,c)}}(\rho_2) = J_{V_{((k-1)c,c-1,1)},\chi_1}(\rho_2).$$

We proceed with $i = 2$. After i steps, our claim is either proved with $\delta = (c-i, 1^i)$, or

$$J_{V_{((k-1)c,c)}}(\rho_2) = J_{V_{((k-1)c,c-i,1^i)},\chi_i}(\rho_2).$$

However, since $c > k$, for $i = c$ we already obtain $J_{V_{((k-1)c,1^c)},\chi_c}(\rho_2) = 0$, because the highest derivative of ρ_2 is k (put differently, the suitable φ is nilpotent of order at least $k+1$).

Lastly, for $c = k > 1$, $J_{V_{((k-1)c,c)}}(\rho_2) = |\det|^{\alpha_1}\rho_{c-1}(\tau_2) \otimes |\det|^{\alpha_2}\tau_2$, where $\alpha_1, \alpha_2 \in \frac{1}{2}\mathbb{Z}$ (see [115, Theorem 3.4]) and $\rho_{c-1}(\tau_2)$ is $(k, c-1)$, then (because $l = 0$)

$$J_{V_{\beta'},\psi^{-1}_{V_{\beta'}}}(\rho_2) = |\det|^{\alpha_1} J_{V_{(c^{k-1})},\psi_{k-1}}(\rho_2) \otimes |\det|^{\alpha_2}\tau_2.$$

Hence $\mathrm{diag}(\mathrm{SL}_c^{\triangle'}, I_c)$ acts trivially on $J_{V_{\beta'},\psi^{-1}_{V_{\beta'}}}(\rho_2)$. Additionally because $\psi^{-1}_{V_\beta}$ belongs to the orbit of ψ_k ($l = 0$, see (2.46)), $\mathrm{SL}_c^{\triangle}$ acts trivially on $J_{V_\beta,\psi^{-1}_{V_\beta}}(\rho_1)$. Thus ${}^h(\mathrm{SL}_c, 1)$ acts trivially on $J_{V_\beta \times V_{\beta'},\psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)$, in particular $\mathcal{H}(h) = 0$ for all s, because π_1 is supercuspidal. ∎

For the remaining cases, $l = c$ and $0 \le j \le c$ (recall j is the rank of $A_{l,j}$). The cases $j < c$ are similar to $l < c$, but involve V_β and ψ_{V_β}.

Proposition 2.26. *Assume $0 \le j < l = c$ or $k = 1$. Then $\mathcal{H}(h) = 0$ outside a discrete subset of s. Furthermore, if $j > 0$ and π_2 is supercuspidal, or $ck > 1$, $j = 0$, π_1 and π_2 are supercuspidal and $\rho_1 = \rho_c(\tau_1)$ for an irreducible supercuspidal representation τ_1 of GL_k, then $\mathcal{H}(h) = 0$ for all s.*

Proof. In this case $X = A_{c,j} = \begin{pmatrix} & I_j \\ 0_{c-j} & \end{pmatrix}$, so that if we consider $R < G$ with $M_R = M_{(c-j,j)}$ and $U_R = V^-_{(c-j,j)}$, we can repeat most of the proof of Proposition 2.25 (with j instead of l), except we use V_β instead of $V_{\beta'}$ (hence, e.g., τ_1 instead of τ_2).

Identify GL_{c-j} with its natural image in M_R. We note that ${}^h(1, \mathrm{GL}_{c-j}) = \mathrm{diag}(\mathrm{GL}_{c-j}, I_{(2k-1)c})$. The character ψ_{V_β} is now given by

$$\begin{pmatrix} I_c & b_k & & & \\ & I_c & b_{k+2} & & \\ & & \ddots & \ddots & \\ & & & I_c & b_{2k-1} \\ & & & & I_c \end{pmatrix} \mapsto \psi\Big(-\mathrm{tr}(b_k A_{c,j}) - \sum_{j=2}^{k-1} \mathrm{tr}(b_{k+j})\Big),$$

and $\psi_{V'_\beta} = \psi_k^{-1}$. Then $J_{V_\beta \times V_{\beta'},\psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)$ is a well defined representation of ${}^h(1, \mathrm{GL}_{c-j})$.

We proceed over nonarchimedean fields, and prove that for all s, $J_{V_\beta \times V_{\beta'},\psi^{-1}_{V_\beta \times V_{\beta'}}}(\rho)$ is a trivial representation of ${}^h(1, U_R)$ and factors through $J_{V_{(c-j,j+(k-1)c)}}(\rho_1)$.

Since ${}^{h^{-1}}(|\det|^{s-1/2})(1, aI_{c-j}) = |a|^{(c-j)(s-1/2)}$, $\mathcal{H}(h) = 0$ outside a discrete subset of s by (2.7). For $j > 0$ (then $l = c > 1$), if π_2 is supercuspidal, then $\mathcal{H}(h) = 0$ for all s. For more details and the archimedean case see the proof of Proposition 2.25.

Identify GL_{kc} with the top left block of M_P. Let $Z = \mathrm{diag}(V_{(c-j,j)}, I_{(k-1)c}) \cong \mathrm{Mat}_{c-j\times j}$. For $u \in U_R$ we have ${}^h(1,u) = m_u u'$ with $m_u \in Z$ and $u' \in U_P$. The group Z stabilizes ψ_{V_β}, and the set of characters of Z is partitioned into two orbits with respect to the action of $\mathrm{diag}(\mathrm{GL}_{c-j}, I_{kc-(c-j)})$ and $\{\mathrm{diag}(I_{c-j}, g, \mathrm{diag}(g, I_{c-j})^{\Delta'}) : g \in \mathrm{GL}_j\}$. We show that for any character $\chi \neq 1$ of Z,

$$J_{V_\beta \ltimes Z, \psi^{-1}_{V_\beta}\otimes\chi}(\rho_1) = 0. \tag{2.55}$$

This implies that $J_{V_\beta\times V_{\beta'},\psi^{-1}_{V_\beta\times V_{\beta'}}}(\rho)$ is a trivial representation of ${}^h(1, U_R)$ and factors through $J_{V_{(c-j,j+(k-1)c)}}(\rho_1)$.

For the proof of (2.55) assume $j > 0$. We can assume χ is nontrivial on the bottom right coordinate of Z, and trivial on the other coordinates in the rightmost column. Let φ be the transpose of the nilpotent element defined by $\psi^{-1}_{V_\beta} \otimes \chi$, which now depends on $A_{l,j}$ (as opposed to the proof of Proposition 2.25). Using a conjugation by $\mathrm{diag}\left(I_c, \begin{pmatrix} & I_{c-j} \\ I_j & \end{pmatrix}^{\Delta'}\right)$, we obtain φ which has nontrivial entries on the bottom right coordinates of $b_k, b_{k+2}, \ldots, b_{2k-1}$ and Z (k blocks). This proves (2.55), because ρ_1 is (k, c).

The proof of the assertion for the case $ck > 1$, $j = 0$, supercuspidal representations π_1 and π_2, and $\rho_1 = \rho_c(\tau_1)$ for an irreducible supercuspidal τ_1, proceeds as in the proof of Proposition 2.25. Since now $J_{V_{(c-j,j+(k-1)c)}}(\rho_1) = J_{V_{(c,(k-1)c)}}(\rho_1)$, we have $\mathcal{H}(h) = 0$ for all s unless $c = tk$, $t \geq 1$. For $t > 1$ we use derivatives along $V_{(1^i,c-i,(k-1)c)}$, $1 \leq i \leq c$. For $c = k > 1$, $J_{V_{(c,(k-1)c)}}(\rho_1) = |\det|^{\alpha_3}\tau_1 \otimes |\det|^{\alpha_4}\rho_{c-1}(\tau_1)$ and $\rho_{c-1}(\tau_1)$ is $(k, c-1)$, hence ${}^h(\mathrm{SL}_c, 1)$ acts trivially on $J_{V_\beta\times V_{\beta'},\psi^{-1}_{V_\beta\times V_{\beta'}}}(\rho)$. ■

Propositions 2.18–2.26 imply $\mathcal{H}(h) = 0$ for all h unless $h \sim \delta$. We prove $\dim \mathcal{H}(\delta) = \dim \mathrm{Hom}_G(\chi_0\pi_1^\vee, \pi_2)$, for all s. Now $P_\delta = G^\iota \ltimes (V_{(c^k)} \times V_{(c^k)})$ with $G^\iota = \{(g, g) : g \in G\}$ (for $H = \mathrm{GL}_{2kc}$ one can take $\iota = I_c$; we keep the notation G^ι for uniformity) and any morphism in $\mathcal{H}(\delta)$ factors through $J_{V_{(c^k)}\times V_{(c^k)},\psi_k\otimes\psi_k}(\rho)$. Hence

$$\mathcal{H}(\delta) = \mathrm{Hom}_{G^\iota}({}^{\delta^{-1}}J_{V_{(c^k)}\times V_{(c^k)},\psi_k\otimes\psi_k}(\rho) \otimes \pi_1^\vee \otimes \pi_2^\vee, 1).$$

Note that $|\det|^{s-1/2} \otimes |\det|^{-s+1/2}$ and θ_h are trivial on G^ι.

We can assume δ commutes with G^ι (G^ι is simply the diagonal embedding of G in H). Then as a representation of G^ι,

$${}^{\delta^{-1}}J_{V_{(c^k)}\times V_{(c^k)},\psi_k\otimes\psi_k}(\rho) = J_{V_{(c^k)},\psi_k}(\rho_1) \otimes J_{V_{(c^k)},\psi_k}(\rho_2).$$

Recall that the action of G^ι on $J_{V_{(c^k)},\psi_k}(\rho_1) \otimes J_{V_{(c^k)},\psi_k}(\rho_2)$ is given by $g \mapsto \chi_0(\det g)$ ($g \in G$) for some quasi-character χ_0 of F^* (see §2.1), therefore

$$\mathrm{Hom}_{G^\iota}({}^{\delta^{-1}}J_{V_{(c^k)}\times V_{(c^k)},\psi_k\otimes\psi_k}(\rho) \otimes \pi_1^\vee \otimes \pi_2^\vee, 1) = \mathrm{Hom}_G(\chi_0\pi_1^\vee, \pi_2).$$

The remaining parts of the proof now follow as in §2.2, and note that when $ck > 1$, π_1 and π_2 are supercuspidal and $\rho_i = \rho_c(\tau_i)$ for irreducible supercuspidal representations τ_i of GL_k, $i = 1, 2$, we do not need to exclude any s.

3. Applications

3.1. Covering groups

In this section we describe the extension of Theorem 2.1 to certain covering groups. We proceed with the definitions and notation of §1.5. Let $m \geq 1$. Assume F^* contains the full group μ_m of m-th roots of unity. A topological central extension of $G(F)$ by μ_m is an exact sequence of topological groups

$$1 \to \mu_m \xrightarrow{i} G^{(m)} \xrightarrow{p} G(F) \to 1,$$

where i and p are continuous, $i(\mu_m)$ is closed and belongs to the center of $G^{(m)}$, and p induces an isomorphism $i(\mu_m)\backslash G^{(m)} \cong G(F)$ of topological groups. We call $G^{(m)}$ an *m-fold covering group* of $G(F)$; it is in general not unique, but for $G(F) = \mathrm{Sp}_c(F)$ it is uniquely defined given a Steinberg symbol (e.g., a Hilbert m-th order symbol). The covering groups under consideration here were constructed, in increasing level of generality, through a series of works including [16,67,70,71,80,85,101,112]. For further references see [8,81].

In this section we assume the field is nonarchimedean; then $G^{(m)}$ is an l-group in the sense of [11]. For $m > 2$, an archimedean field is already complex; then the cover is split over the group so that the results in this case are immediate from the linear case. As above, we identify F-groups with their F-points. Of course this only applies to G and its subgroups; $G^{(m)}$ is not an algebraic group.

In general for $X < G$, $\widetilde{X}$ denotes the covering of X (more precisely, of $X(F)$) defined by restriction from $G^{(m)}$. This covering depends on the embedding of X inside G. We say that $\widetilde{X}$ is *split* over X if there is a group embedding $X \to \widetilde{X}$. If X is perfect (as an F-group), such a splitting, if it exists, is unique. Note that since F is of characteristic 0, Sp_c and SL_c are perfect. The coverings under consideration are split canonically over unipotent subgroups, hence the notions of Jacquet functors and unipotent orbits extend to the covering in the obvious way. If Y is a unipotent subgroup of G, denote by $\varphi_Y : Y \to \widetilde{Y}$ the splitting of Y. Since φ_Y is canonical, we usually omit it from the notation, e.g., if $R < G$ is a parabolic subgroup and we consider a genuine representation σ of $\widetilde{M}_R$, for the induced representation $\mathrm{Ind}_{\widetilde{R}}^{G^{(m)}}(\sigma)$ we extend σ trivially on U_R, more precisely on $\varphi_Y(U_R)$. Since we are considering central coverings, G acts on $G^{(m)}$ by conjugation. In particular,

$$^{h}\varphi_Y(y) = \varphi_{^{h}Y}(^{h}y), \quad \forall y \in Y. \tag{3.1}$$

We describe a general system of assumptions for covering groups, under which the doubling construction is well defined, and then state the analog of Theorem 2.1. For the particular cases of the covering $\mathrm{Sp}_c^{(m)}$ of [80] and the covering $\widetilde{\mathrm{GL}}_c$ obtained by restriction from $\mathrm{Sp}_{2c}^{(m)}$, these assumptions were verified in [64]. More details are given below; see also Corollary 3.5.

Fix a covering group $G^{(m)}$. Assume there is a covering $\widetilde{H}$ of H (typically $\widetilde{H} = H^{(m)}$) with the following properties.

(1) For $H = \mathrm{GSpin}_{2kc}$, the preimage $\widetilde{C}_H^{\circ}$ of C_H° in $\widetilde{H}$ belongs to the center of $\widetilde{H}$, and $\widetilde{C}_H^{\circ}$ is split over C_H°. The same properties are satisfied by the preimage $\widetilde{C}_G^{\circ}$ of C_G° in $\widetilde{G}$.
(2) Let $\mathrm{e}_1(g) = (g, 1)$ and $\mathrm{e}_2(g) = (1, g)$. These are the embeddings of G into M_Q in the linear case. Assume they extend to embeddings $\widetilde{\mathrm{e}}_i : G^{(m)} \to \widetilde{e_i(G)}$.
(3) The restriction of $\widetilde{\mathrm{e}}_2$ to μ_m is the identity. (Here we regard μ_m as a subgroup of $G^{(m)}$.)
(4) The images of $\widetilde{\mathrm{e}}_1$ and $\widetilde{\mathrm{e}}_2$ commute in $\widetilde{H}$, and give rise to a homomorphism

$$\{(\epsilon_1, \epsilon_2) \in \mu_m \times \mu_m : \epsilon_1 = \epsilon_2\}\backslash G^{(m)} \times G^{(m)} \to \widetilde{M}_Q. \tag{3.2}$$

This is (automatically) an embedding unless $H = \mathrm{GSpin}_{2kc}$, in which case we further assume that $\widetilde{\mathrm{e}}_1(z)\widetilde{\mathrm{e}}_2(z)$ is the identity for $z \in C_G^{\circ}$; then the left hand side of (3.2) is further divided by the subgroup $\{(z, z) : z \in C_G^{\circ}\}$ (a subgroup by (1)). Cf. (1.3). Denote the left hand side of (3.2) by $(G, G)^{(m)}$.
(5) For $H = \mathrm{GL}_{2kc}$, the preimages of the direct factors GL_{kc} of M_P commute in $\widetilde{H}$, and the coverings $\widetilde{\mathrm{GL}}_{kc}$ of each copy of GL_{kc} are isomorphic.
(6) Identify $\widetilde{\mathrm{GL}}_{kc}$ with $\widetilde{M}_P$ if $H \neq \mathrm{GL}_{2kc}$, or with the covering of one of the copies of GL_{kc} in M_P for $H = \mathrm{GL}_{2kc}$. Assume $\widetilde{\mathrm{GL}}_{kc}$ is split over $\mathrm{SL}_c^{\triangle}$.
(7) For $H = \mathrm{GL}_{2kc}$, assume $\widetilde{M}_P$ is split over $\{\mathrm{diag}(g^{\triangle}, g^{\triangle}) : g \in \mathrm{GL}_c\}$.
(8) The involution ι extends to an involution of $G^{(m)}$ and for a genuine representation π of $G^{(m)}$, $(\pi^{\vee})^{\iota} = (\pi^{\iota})^{\vee}$.
(9) For any maximal parabolic subgroup $R < G$ whose Levi part contains GL_l, the covering $\widetilde{\mathrm{GL}}_l$ has the property that for a sufficiently large integer d, the preimage of $C_{\mathrm{GL}_l}^d = \{x^d : x \in C_{\mathrm{GL}_l}\}$ belongs to the center of $\widetilde{\mathrm{GL}}_l$.

First we use these properties to construct the basic data for the doubling method. Define $\widetilde{\mathrm{GL}}_{kc}$ by (6). Let ρ be a genuine representation of $\widetilde{\mathrm{GL}}_{kc}$. We say that ρ is a (k, c) *representation* if $\mathrm{Hom}_{V(\sigma)}(\rho, \psi') = 0$ for all $\sigma \gtrsim (k^c)$ and $\psi' \in \widehat{V}(\sigma)_{\mathrm{gen}}$, and $\dim \mathrm{Hom}_{V_{(c^k)}}(\rho, \psi_k) = 1$. By (6), the action of $\mathrm{SL}_c^{\triangle}$ on $J_{V_{(c^k)}, \psi_k}(\rho)$ is well defined, then it is trivial.

If $H \neq \mathrm{GL}_{2kc}$, let ρ be a genuine admissible finite length (k, c) representation of $\widetilde{\mathrm{GL}}_{kc}$. For $H = \mathrm{GSpin}_{2kc}$, by (1) the irreducible representations of $\widetilde{C}_H^{\circ}$ are the lifts of quasi-characters of F^* to genuine characters; therefore, if η is a quasi-character of F^* which we regard also as a character of $\widetilde{C}_H^{\circ}$, the representation $\rho \otimes \eta$ is well defined. For $H = \mathrm{GL}_{2kc}$, by (5) we have

$$\{(\epsilon_1, \epsilon_2) \in \mu_m \times \mu_m : \epsilon_1\epsilon_2 = 1\}\backslash \widetilde{\mathrm{GL}}_{kc} \times \widetilde{\mathrm{GL}}_{kc} \cong \widetilde{M}_P.$$

Hence $\rho = \rho_1 \otimes \rho_2$ is defined for genuine representations ρ_1 and ρ_2, which we take to be admissible finite length and (k, c). The space $V(s, \rho \otimes \eta)$ is now defined as in §1.5, with induction from $\widetilde{P}$ to $\widetilde{H}$.

If $H = \mathrm{GL}_{2kc}$, according to (7) there is a quasi-character χ_0 of F^* such that the action of $\{\mathrm{diag}(g^{\triangle}, g^{\triangle}) : g \in \mathrm{GL}_c\}$ on ${}^{\delta^{-1}}(J_{V_{(c^k)}, \psi_k}(\rho_1) \otimes J_{V_{(c^k)}, \psi_k}(\rho_2))$ is given by $g \mapsto \chi_0(\det g)$.

Let π_1 (resp., π_2) be an anti-genuine (resp., genuine) finite length admissible representation of $G^{(m)}$. If $H = \mathrm{GSpin}_{2kc}$, we assume π_1 and π_2 admit central characters. Then by (1) these characters restrict to genuine characters of $\widetilde{C}_H^\circ$, denoted χ_{π_1} and χ_{π_2}, which we can identify with quasi-characters of F^*. We assume $\chi_{\pi_1}^{-1} = \chi_{\pi_2}$ and put $\eta = \chi_{\pi_1}^{-1}$. Consider the space

$$\mathrm{Hom}_{(G,G)^{(m)}}\big(J_{U,\psi_U^{-1}}(V(s,\rho\otimes\eta)),\pi_1\otimes\pi_2\big). \tag{3.3}$$

The representation $V(s,\rho\otimes\eta)$ is a priori a representation of $(G,G)^{(m)}$ by (4). Since π_1 is anti-genuine and π_2 is genuine, $\pi_1\otimes\pi_2$ factors through $(G,G)^{(m)}$, and it follows that (3.3) is well defined.

Recall $D = U \rtimes (G,G)$ and denote $D^{(m)} = U \rtimes (G,G)^{(m)}$. Then (3.3) is isomorphic to

$$\mathrm{Hom}_{D^{(m)}}(V(s,\rho\otimes\eta),\psi_U^{-1}\otimes\pi_1\otimes\pi_2). \tag{3.4}$$

By (3), $V(s,\rho\otimes\eta)$ is a genuine representation of the right copy of $G^{(m)}$, and so is π_2. Combining (3) with (4), $\epsilon_1 \in \mu_m$ is mapped to ϵ_1^{-1} under $\widetilde{e}_1$, whence $V(s,\rho\otimes\eta)$ is an anti-genuine representation of the left copy of $G^{(m)}$, as is π_1. Therefore the representation

$$V(s,\rho\otimes\eta)\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee)$$

of $D^{(m)}$ factors through D. Hence (3.4) equals

$$\begin{aligned}&\mathrm{Hom}_D\big(V(s,\rho\otimes\eta)\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee),1\big)\\ &\qquad = \mathrm{Hom}_D\Big(\mathrm{Ind}_{\widetilde{P}\times D^{(m)}}^{\widetilde{H}\times D^{(m)}}\big((|\det|^{s-1/2}\rho\otimes\eta)\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee)\big),1\Big)\end{aligned}$$

(cf. (2.2)). Recall $P_h = {}^{h^{-1}}P \cap D$. The covering $\widetilde{P}_h$ obtained by restriction from $\widetilde{H}$ coincides with the covering restricted from $D^{(m)}$, by (4). The space of distributions on $\widetilde{P}hD^{(m)}$ corresponds to

$$\mathrm{Hom}_D\Big(\mathrm{ind}_{\widetilde{P}_h}^{D^{(m)}}\big({}^{h^{-1}}((|\det|^{s-1/2}\rho\otimes\eta)\delta_P^{1/2})\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee)\big),1\Big),$$

which by the Frobenius reciprocity is equal to

$$\mathcal{H}(h) = \mathrm{Hom}_{P_h}\big({}^{h^{-1}}(|\det|^{s-1/2}\rho\otimes\eta)\otimes(\psi_U\otimes\pi_1^\vee\otimes\pi_2^\vee),\theta_h\big) \tag{3.5}$$

(cf. (2.4)). We can now use the theory of distributions on l-sheafs of [11]. Recall that the right action of D on $P\backslash H$ is constructive, i.e., the graph of the action is a finite union of locally closed sets (see [11, §§6.1–6.6] for more details on these notions). This follows from [11, Theorem A], because $P\backslash H$ is an algebraic F-variety. Since

$$(\widetilde{P}\times D^{(m)})\backslash(\widetilde{H}\times D^{(m)}) \cong \widetilde{P}\backslash\widetilde{H} \cong P\backslash H$$

(as topological spaces), the right action of D on $(\widetilde{P}\times D^{(m)})\backslash(\widetilde{H}\times D^{(m)})$ is also constructive, justifying the application of [11, Theorem 6.9] (note that the action of $D^{(m)}$ on the quotient factors through D).

The arguments of §2.1 showing the vanishing of $\mathcal{H}(h)$ also remain valid. We explain this in more detail. First, if $Y < {}^{h}U \cap M_P$, then by (3.1), ${}^{h^{-1}}\varphi_Y(Y) = \varphi_{h^{-1}Y}({}^{h^{-1}}Y) = \varphi_U({}^{h^{-1}}Y)$. Hence (2.5) holds and any morphism in $\mathcal{H}(h)$ factors through $J_{Y,{}^{h}\psi_U^{-1}}(\rho)$.

Condition (2.6) is independent of the covering. Since $\rho \otimes \eta$ is trivial on $\varphi_{N_H}(U_P)$, the condition ${}^{h}Y < U_P$ and (3.1) imply ${}^{h^{-1}}(|\det|^{s-1/2}\rho \otimes \eta)$ is trivial on $\varphi_Y(Y)$, so we can deduce $\mathcal{H}(h) = 0$. The second method, where we show that any morphism in $\mathcal{H}(h)$ factors through $J_{V(\sigma),\psi'}(\rho)$ with $\sigma \gtrsim (k^c)$ and $\psi' \in \widehat{V}(\sigma)_{\mathrm{gen}}$, also implies $\mathcal{H}(h) = 0$, as in the linear case.

The only change concerns the third condition, where it is not necessarily true that the preimage of ${}^{h}(1, C_{\mathrm{GL}_l})$ in $\widetilde{H}$ acts by a character, because this preimage might not be abelian. However, we can instead use the preimage $\widetilde{C}^d_{\mathrm{GL}_l}$ of $C^d_{\mathrm{GL}_l}$ (for a large integer d), which is abelian and belongs to the center of $\widetilde{\mathrm{GL}}_l$, by assumption (9). Then $\widetilde{C}^d_{\mathrm{GL}_l}$ acts by a character on each irreducible constituent of $J_{U_R}(\pi_2^\vee)$, and the preimage of ${}^{h}(1, C^d_{\mathrm{GL}_l})$ in $\widetilde{H}$ acts by a character on each of finitely many constituents. The only change to (2.7) is that now we replace $a \in F^*$ with a^d, but this still implies the vanishing outside a discrete subset of s.

Define

$$d(s, \rho, \eta, \pi_1, \pi_2) = \dim \mathrm{Hom}_{(G,G)^{(m)}}(J_{U,\psi_U^{-1}}(V(s, \rho \otimes \eta)), \pi_1 \otimes \pi_2).$$

We are ready to prove Theorem 2.1 for covering groups.

Theorem 3.1. *Let π_1, π_2 and ρ be as above.*

(1) *Outside a discrete subset of s, $d(s, \rho, \eta, \pi_1, \pi_2) \leq \dim \mathrm{Hom}_{G^{(m)}}(\chi_0\pi_1^\vee, \pi_2^\iota)$.*

(2) *If π_1 and π_2 are irreducible, then outside a discrete subset of s, $d(s, \rho, \eta, \pi_1, \pi_2) = 0$ unless $\pi_1 = \chi_0(\pi_2^\iota)^\vee$ in which case $d(s, \rho, \eta, \pi_1, \pi_2) \leq 1$.*

Furthermore, assume π_2 is supercuspidal and ρ is not necessarily of finite length. Then the assertions of (1) and (2) hold for all s, granted either $H \neq \mathrm{GL}_{2kc}$ and $c > 2$, or $H = \mathrm{Sp}_{4k}$.

Remark 3.2. Evidently, there is no essential difference between the statements in the linear setting and the covering (for $m = 1$, $G^{(m)} = G$), except the supercuspidal cases, where we excluded the conditions depending on ρ. This is because we are not discussing $\rho_c(\tau)$ for covering groups here; the definition of this representation is thus far clear only when τ is a genuine unramified principal series (see [64]). Once the details are worked out, the arguments here are expected to extend to these cases as well.

Proof of Theorem 3.1. Since $\widetilde{P}\backslash\widetilde{H}/D^{(m)} = P\backslash H/D$, we can use the same description for the representatives h. The arguments of Propositions 2.5–2.11 and Propositions 2.18–2.22 extend to the covering.

For Propositions 2.12, 2.14, 2.23, 2.25–2.26 we used two types of arguments. First, we showed that the Jacquet module $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)$ vanishes, because the order of nilpotency of φ is at least $k+1$. The arguments involving the action of a normalizer on the set of characters of an abelian unipotent subgroup carry over to the covering. This is because in general, if a subgroup $A < H$ normalizes a unipotent subgroup $Y < H$, and thereby acts on its set of characters, then $\widetilde{A}$ also acts on the set of characters of Y with the same orbits (because $\widetilde{H}$ is split canonically over Y). The arguments of [11, §§5.9–5.12] still apply. Then we used [46, Theorems A, E], in which strictly speaking covering groups were not discussed. However, one can still use conjugations and [11, §§5.9–5.12] to show that $J_{V_\beta,\psi_{V_\beta}^{-1}}(\rho)$ factors through a Jacquet module with respect to a unipotent orbit which is greater than or incomparable with (k^c). See Example 3.3 below. Second, we used (2.7), which is still applicable with the minor change explained above.

It remains to consider $\mathcal{H}(h)$ where $h \sim \delta$. Consider $H \neq \mathrm{GL}_{2kc}$. Then

$$\mathcal{H}(\delta) = \mathrm{Hom}_{G^\iota \times C_H^\circ}({}^{\delta^{-1}}J_{V_{(c^k)},\psi_k}(\rho) \otimes \eta \otimes \pi_1^\vee \otimes \pi_2^\vee, 1).$$

For $H = \mathrm{GSpin}_{2kc}$ the assumption $\eta = \chi_{\pi_1}^{-1} = \chi_{\pi_2}$ implies that this space equals

$$\mathrm{Hom}_{G^\iota}({}^{\delta^{-1}}J_{V_{(c^k)},\psi_k}(\rho) \otimes \pi_1^\vee \otimes \pi_2^\vee, 1).$$

Then since $J_{V_{(c^k)},\psi_k}(\rho)$ is a trivial representation of $\mathrm{SL}_c^\triangle$ (see (6)) and by virtue of (8),

$$\mathcal{H}(\delta) = \mathrm{Hom}_{G^\iota}(\pi_1^\vee \otimes \pi_2^\vee, 1) = \mathrm{Hom}_G(\pi_1^\vee \otimes (\pi_2^\vee)^\iota, 1) = \mathrm{Hom}_{G^{(m)}}(\pi_1^\vee, \pi_2^\iota).$$

For $H = \mathrm{GL}_{2kc}$ we first have

$$\mathcal{H}(\delta) = \mathrm{Hom}_{G^\iota}({}^{\delta^{-1}}J_{V_{(c^k)}\times V_{(c^k)},\psi_k\otimes\psi_k}(\rho) \otimes \pi_1^\vee \otimes \pi_2^\vee, 1).$$

The action of G^ι on ${}^{\delta^{-1}}(J_{V_{(c^k)},\psi_k}(\rho_1) \otimes J_{V_{(c^k)},\psi_k}(\rho_2))$ is given by $g \mapsto \chi_0(\det g)$, and we obtain $\mathrm{Hom}_{G^{(m)}}(\chi_0\pi_1^\vee, \pi_2)$. The remaining parts of the proof now follow as in the linear case. ■

Example 3.3. Consider a Jacquet module of a $(2,2)$ representation ρ with respect to the unipotent subgroup $Y < \mathrm{GL}_4$ and character ψ given by

$$Y = \left\{ y = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & x_4 & x_5 \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}, \qquad \psi(y) = \psi(x_1 + x_5).$$

It suffices to show the vanishing with respect to the subgroup of Y with $x_4 = 0$. Using conjugation by $\mathrm{diag}\big(1, \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right), 1\big)$, we obtain

$$Y' = \left\{ y = \begin{pmatrix} 1 & x_2 & x_1 & x_3 \\ & 1 & & \\ & & 1 & x_5 \\ & & & 1 \end{pmatrix} \right\}.$$

The Jacquet module $J_{Y',\psi}(\rho)$ (ψ does not change) is a representation of

$$X = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & x_6 \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}.$$

The preimage of the subgroup $\{\mathrm{diag}(1, t, I_2) : t \in F^*\}$, which also acts on $J_{Y',\psi}(\rho)$, acts on the set of characters of X with two orbits (for an action of T_{GL_2} use $\mathrm{diag}(t', t, t', t')$). Both orbits can be conjugated into $J_{Y'\rtimes X,\psi}(\rho)$ with still the same ψ using $\mathrm{diag}\big(1, \left(\begin{smallmatrix} 1 & \\ z & 1 \end{smallmatrix}\right), 1\big)$. It remains to prove $J_{Y'\rtimes X,\psi}(\rho) = 0$. Passing to the subgroup with $x_2 = 0$ and conjugating by $\mathrm{diag}\left(\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right), I_2\right)$, it is enough to prove $J_{Y'',\psi}(\rho) = 0$ with

$$Y'' = \left\{ y = \begin{pmatrix} 1 & & & x_6 \\ & 1 & x_1 & x_3 \\ & & 1 & x_5 \\ & & & 1 \end{pmatrix} \right\}, \quad \psi(y) = \psi(x_1 + x_5).$$

As with x_6, one can fill in the missing coordinate above x_1 using

$$X' = \left\{ \begin{pmatrix} 1 & & x_0 & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}, \quad \{\mathrm{diag}(t, I_3) : t \in F^*\}, \quad \mathrm{diag}\left(\left(\begin{smallmatrix} 1 & \\ z & 1 \end{smallmatrix}\right), I_2\right).$$

We have shown that $J_{Y,\psi}(\rho)$ factors through $J_{V_{(2,1,1)},\psi}(\rho)$. This module is filtered by the third and fourth derivatives of ρ (in the sense of [11]), both of which vanish because ρ is $(2, 2)$.

We briefly describe the applicability of Theorem 3.1 to the construction of [64]. Henceforth assume -1 is an m-th root of unity in F^* (this is a technical assumption, used in [64] and several other works, to simplify some of the computations). For any integer l, let $\mathrm{Sp}_{2l}^{(m)}$ denote the covering of [80] defined using the m-th order Hilbert symbol $(\,,\,)_m$. For GL_l, let $\mathrm{GL}_l^{(m)}$ denote the covering obtained by restriction from $\mathrm{Sp}_{2l}^{(m)}$, when we identify GL_l with the standard Siegel Levi subgroup of Sp_{2l} by $g \mapsto \mathrm{diag}(g, g^*)$.

Let $r = m$ if m is odd, otherwise $r = m/2$. Let k_0 be a positive integer, and put $k = rk_0$. The above list of properties were verified in [64] when $G = \mathrm{Sp}_c$ or GL_c, for the covering $G^{(m)}$, with $\widetilde{H} = H^{(m)}$.

Remark 3.4. The group $\mathrm{GL}_l^{(m)}$ was denoted $\mathrm{GL}_l^{(m,r)}$ in [64], to underline the difference between this covering and the coverings of [67], and k of [64] is k_0 here.

Assume we have a $(k, c) = (rk_0, c)$ representation ρ (admissible of finite length). It is at present not clear how to construct such representations in general (e.g., from representations of a covering of GL_k), but in the unramified setting this was obtained in [64] (following [105]). Note that here the "unramified setting" includes the assumptions $|m| = 1$ and $q > 4$. Briefly, given a genuine unramified principal series representation τ of $\mathrm{GL}_{k_0}^{(m)}$, one can choose an unramified character of $T_{\mathrm{GL}_{k_0}}$ associated with the inducing data of τ (the correspondence is not unique). Using this character and an exceptional representation of $\mathrm{GL}_{rc}^{(m)}$ (exceptional in the sense of [67], see [37]), the prescribed ρ was constructed in [64, §2.2].

For $H = \mathrm{GL}_{2kc}$, χ_0 was taken to be trivial (see [64, (3.34)]).

Also let π be a genuine irreducible representation of $G^{(m)}$. The integral $Z(s, \omega, f)$, with a holomorphic or rational section f, was defined in [64] (using notation similar to §1.5). Formally, it belongs to (3.3) with $\pi_1 = \pi^\vee$ and $\pi_2 = \pi^\iota$. This was proved in [64, Propositions 68, 75] (in [64, (3.21) and (3.36)], $G^{(m)} \times G^{(m)}$ should be replaced with $(G, G)^{(m)}$).

Corollary 3.5. *$Z(s, \omega, f)$ admits meromorphic continuation to a function in q^{-s}.*

Proof. This follows from Theorem 3.1 and Bernstein's continuation principle [7]; see [64, Remark 72] and [64, §3.3] (cf. Corollary 2.3 here). ∎

Corollary 3.6. *One can define a local γ-factor $\gamma(s,\pi\times\tau,\psi)$ by virtue of* (0.2).

Note that the additional normalization of the intertwining operator appearing in (0.2) can be applied to the covering case as well; but we are not proving the multiplicativity properties of the γ-factor here, and at any rate, we are still limited to the unramified setting. The point here is that the proportionality factor exists.

3.2. Global unfolding

The global doubling construction in the linear case for arbitrary k was first described in [18] mainly for the symplectic group, with some details also for the special even orthogonal group, then briefly explained in [20] for the other cases appearing here. The covering version for the symplectic group was described in [64].

Let F_0 be a number field with a ring of adeles $\mathbb{A}$. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$, and let $\mathcal{E}_\tau$ denote the generalized Speh representation of $\mathrm{GL}_{kc}(\mathbb{A})$ corresponding to c copies of τ, constructed by Jacquet [55]. According to [18–20, 41, 60], the representation $\mathcal{E}_\tau$ is a global (k,c) representation: it does not support any Fourier coefficient along an orbit greater than or incomparable with (k^c), it supports a Fourier coefficient along (k^c), and all of its local components are (k,c). See [18, §2.2] and the references therein for more details on the global notions. Moreover, if $\tau=\otimes'_\nu\tau_\nu$ as a restricted tensor product, $(\mathcal{E}_\tau)_\nu=\rho_c(\tau_\nu)$ for any place ν of F_0.

One can readily globalize our arguments used for the proof of Theorem 2.1 to obtain a proof of the unfolding of the global doubling integral, for all of the groups under consideration here (and in [20]). At the same time, since local vanishing of Jacquet modules implies global vanishing of the corresponding Fourier coefficients (even one local (k,c) component suffices for this), the proof of Theorem 2.1 also provides a proof of the global unfolding. In addition we obtain the following corollary, which for brevity, is stated here in the symplectic or special orthogonal cases, but the other cases are evident as well.

We use the notation and definitions of §1.5, in the global context. Let K_H be a standard maximal compact subgroup, in a "good position" with respect to T_H. Let f be a K_H-finite section of $\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(|\det|^{s-1/2}\mathcal{E}_\tau)$, whose restriction to K_H is independent of s. We regard f as a complex-valued function.

Recall the definition (2.34) of a character ψ_{V_β} when $\beta=(c^k)$, defined with respect to $0\le l\le n$, which we rename here as $\psi_{(c^k),l}$ (in the context of (2.34), l was fixed). In particular, $\psi_{(c^k),n}$ is in the orbit of ψ_k^{-1} ($\psi_{(c^k),n}=\psi_k^{-1}$ when c is even). For $k=1$, $\psi_{(c^k),l}$ is trivial. Then we have the Fourier coefficients of f along $(V_{(c^k)},\psi_{(c^k),l})$, defined by

$$f^{V_{(c^k)},\psi_{(c^k),l}}(s,x)=\int_{V_{(c^k)}(F_0)\backslash V_{(c^k)}(\mathbb{A})} f(s,vx)\,\psi_{(c^k),l}(v)\,dv.$$

In particular, $f^{V_{(c^k)},\psi_{(c^k),n}}$ is the coefficient $f_{W(\mathcal{E}_\tau)}$ appearing in [18, Theorem 1], i.e., the composition of f with the global (k,c) functional on the space of $\mathcal{E}_\tau$ given by a Fourier coefficient (if c is odd, this is true up to a conjugation which identifies $\psi_{(c^k),n}$ with ψ_k^{-1}).

The Eisenstein series corresponding to f is defined by

$$E(x; s, f) = \sum_{\gamma \in P(F_0)\backslash H(F_0)} f(s, \gamma x), \quad x \in H(\mathbb{A}). \tag{3.6}$$

The series is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to $\mathbb{C}$. Consider the Fourier coefficient of $E(x; s, f)$ along (U, ψ_U), given by

$$E^{U,\psi_U}(x; s, f) = \int_{U(F_0)\backslash U(\mathbb{A})} E(ux; s, f)\psi_U(u)\, du. \tag{3.7}$$

The definitions imply that $E^{U,\psi}(\cdot; s, f)$ is an automorphic form on $G(\mathbb{A}) \times G(\mathbb{A})$.

For $0 \le l \le n$, let w_l be the representative w chosen after the proof of Proposition 2.8 (used for the computation of (2.27)), but with $d_1 = \cdots = d_{k-1} = n - l$. Using Example 2.9 we see that

$$w_l = \begin{pmatrix} 0 & 0 & 0 & 0 & I_l & 0 \\ 0 & 0 & I_{c-l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{(k-1)c} \\ \epsilon_0 I_{(k-1)c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{c-l} & 0 & 0 \\ 0 & \epsilon_0 I_l & 0 & 0 & 0 & 0 \end{pmatrix} J_{(k-1)c+l}.$$

A quick computation implies ${}^{w_l^{-1}}V_{(c^k)} = {}^{w''}V_{(c^k)}$, where

$$w'' = {}^{J_{(k-1)c+l}}\mathrm{diag}\Big(I_{(k-1)c+l}, \begin{pmatrix} & I_{c-l} \\ \epsilon_0 I_{c-l} & \end{pmatrix}, I_{(k-1)c+l}\Big).$$

Then $U = {}^{w_l^{-1}}V_{(c^k)} \ltimes U_{n-l}$ for the subgroup

$$U_{n-l} = {}^{w''}(U \cap U_P) = {}^{J_{(k-1)c+l}}\begin{pmatrix} I_{(k-1)c} & 0 & u_1 & 0 & u_2 & u_3 \\ & I_l & & & & u_2' \\ & & I_{c-l} & & & 0 \\ & & & I_{c-l} & & u_1' \\ & & & & I_l & 0 \\ & & & & & I_{(k-1)c} \end{pmatrix}$$

(if we replace w_l by δ_0, $U_0 = U \cap {}^{J_{kc}}U_P$), and $P_{w_l} \cap U = {}^{w_l^{-1}}V_{(c^k)}$. Denote $P'_{w_l} = P_{w_l} \cap (G, G)$.

Corollary 3.7. *In* $\mathrm{Re}(s) \gg 0$,

$$E^{U,\psi_U}(x; s, f)$$
$$= \sum_{l=0}^{n} \sum_{y \in P'_{w_l}(F_0)\backslash(G(F_0), G(F_0))} \int_{U_{n-l}(\mathbb{A})} f^{V_{(c^k)}, \psi_{(c^k),l}}(s, w_l({}^{J_l}u_l)uyx)\psi_U(u)\, du.$$

Proof. We can assume $x = I_{2kc}$. Write the sum (3.6) over $P(F_0)\backslash H(F_0)/D(F_0) \times P_h(F_0)\backslash D(F_0)$. In a right half-plane we can exchange summation and integration. Thus

$$E^{U,\psi_U}(I_{2kc}; s, f)$$
$$= \sum_{h \in P(F_0)\backslash H(F_0)/D(F_0)} \int_{U(F_0)\backslash U(\mathbb{A})} \sum_{y \in P_h(F_0)\backslash D(F_0)} f(s, hyu)\psi_U(u)\, du.$$

Next because $w^{-1}h \in M_Q$ and $P \cap {}^{w}Q = (P \cap {}^{w}M_Q) \ltimes (P \cap {}^{w}U)$, we have

$$^{h^{-1}}P \cap Q = {}^{h^{-1}}(P \cap {}^{w}Q) = ({}^{h^{-1}}P \cap M_Q) \ltimes ({}^{h^{-1}}P \cap U).$$

Since $P_h < Q$ and $(G, G) < M_Q$, we deduce

$$P_h = (P_h \cap (G, G)) \ltimes (P_h \cap U) = P'_h \ltimes P''_h,$$

whence we can collapse the du-integral, exchange $yu \mapsto uy$ and take the integral inside:

$$E^{U,\psi_U}(I_{2kc}; s, f)$$
$$= \sum_{h \in P(F_0)\backslash H(F_0)/D(F_0)} \sum_{y \in P'_h(F_0)\backslash (G(F_0),G(F_0))} \int_{P''_h(F_0)\backslash U(\mathbb{A})} f(s, huy)\psi_U(u)\, du.$$

Now the proof of Theorem 2.1, more specifically Propositions 2.5, 2.7–2.12, imply the inner du-integral vanishes unless $h \sim w_l({}^{J_l}u_l)$, $0 \le l \le n$. The corresponding summand is

$$\sum_{y \in P'_{w_l}(F_0)\backslash (G(F_0),G(F_0))} \int_{U_{n-l}(\mathbb{A})} f^{V_{(c^k)},\psi_{(c^k),l}}(s, w_l({}^{J_l}u_l)uy)\psi_U(u)\, du.$$

This completes the proof. ∎

Now let π_1 and π_2 be irreducible cuspidal automorphic representations of $G(\mathbb{A})$, and φ_1 and φ_2 be two cusp forms in the corresponding spaces. Assume G admits nontrivial unipotent subgroups (i.e., exclude some low rank cases). Denote ${}^{\iota}\varphi_2(g) = \varphi_2({}^{\iota}g)$ and

$$\langle \varphi_1, \varphi_2 \rangle = \int_{G(F_0)\backslash G(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\, dg.$$

Then by Corollary 3.7 and Lemma 2.15, (3.7) pairs with φ_1 and φ_2, in the sense that for $\mathrm{Re}(s) \gg 0$,

$$\int_{G(F_0)\backslash G(\mathbb{A}) \times G(F_0)\backslash G(\mathbb{A})} \varphi_1(g_1)\overline{{}^{\iota}\varphi_2(g_2)} E^{U,\psi}((g_1, g_2); s, f)\, dg_1\, dg_2$$
$$= \int_{G(\mathbb{A})} \int_{U_0(\mathbb{A})} \langle \varphi_1, \pi_2(g)\varphi_2 \rangle f^{V_{(c^k)},\psi_{(c^k),n}}(s, \delta u_0(1, {}^{\iota}g))\psi_U(u_0)\, du_0\, dg. \tag{3.8}$$

Indeed, consider one of the summands appearing in Corollary 3.7 with $l < n$. Set $U_R^l = {}^{w_l({}^{J_l}u_l)}(1, {}^{J_{(c+1)l}}U_R)$, with the notation of the proof of Proposition 2.14. The Fourier coefficient of f along $(V_{(c^k)}, \psi_{(c^k),l})$ is left invariant under $U_R^l(\mathbb{A})$. To see this consider a second Fourier expansion of this coefficient, along U_R^l. All terms but the constant one vanish, because by Lemma 2.15, at the nonarchimedean places v of F_0 the action of $U_R^l((F_0)_v)$ on $J_{V_{(c^k)},\psi^{-1}_{(c^k),l}}((\mathcal{E}_\tau)_v)$ is trivial. Since π_2 is cuspidal, the summand itself vanishes.

Of course (3.8) is plainly the main global identity of [18]: the left hand side is the global doubling integral $Z(s, \varphi_1, \varphi_2, f)$, and it is nontrivial when $\pi_1 = \pi_2$ according to the computations of the local integrals appearing in the Euler product on the right hand side.

One can include low rank cases, e.g., $c = 2$ and $G = \mathrm{SO}_2$, by globalizing the argument from Lemma 2.16 (the constant term of the Eisenstein series defining $\mathcal{E}_\tau$ along $V_{(1,kc-1)}$ vanishes when $k > 1$). The low rank arguments of Propositions 2.25 and 2.26 can be globalized using the constant term computation of $\mathcal{E}_\tau$ given by [61, Lemma 4.1].

The results of this section also hold in the covering case of [64], but to formulate them properly one must check the validity of certain properties of the global covering, which are the analogs of the list from §3.1 (this was carried out in [64]).

Avraham Aizenbud and Dmitry Gourevitch

Appendix A. Vanishing of vector-valued distributions on smooth manifolds

Let a Lie group C act on a smooth manifold X. Let $Z \subset X$ be a locally closed C-invariant subset. Let $\mathcal{F}$ be a possibly infinite-dimensional C-equivariant bundle on X (see §A.1.3 below for this notion). Assume that for any $z \in Z$ and $k \in \mathbb{Z}_{\geq 0}$ we have

$$((\mathcal{F}|_z \otimes \mathrm{Sym}^k(CN^X_{z,Cz}) \otimes ((\Delta_C)|_{C_z}/\Delta_{C_z}))^*)^{C_z} = 0\,. \tag{A.1}$$

In this appendix we show that

$$\mathcal{D}'_Z(X, \mathcal{F})^C = 0, \tag{A.2}$$

under certain additional finiteness conditions, generalizing Theorem 1.4.

In §A.1 we will explain the notation of (A.1)–(A.2), and give the definitions of the main notions used in this appendix, as well as some basic properties of these objects. In particular, we use the theory of infinite-dimensional bundles developed in [69], define generalized sections of such bundles, and construct pullbacks and pushforwards for such sections.

In the case when C has finitely many orbits on Z, and the bundle is trivial, the implication (A.1)⇒(A.2) is classical (see [15]). In [24] a cohomological version of the implication is proven, in a semialgebraic setting and assuming finitely many orbits. In [1] a similar implication is proven in the semialgebraic setting, with $\mathcal{F}$ finite-dimensional. In [68] several special cases of the implication are proven, in particular the case in which $\mathcal{F}$ has the form $\mathcal{E} \otimes V$, where $\mathcal{E}$ is finite-dimensional and V is a fixed representation in a Fréchet space, and the action of C on X, $\mathcal{E}$ and $\mathcal{F}$ can be extended to an action of a group G that includes C as a normal subgroup, preserves Z and acts on it with finitely many orbits, each orbit locally closed.

We prove that (A.1)⇒(A.2) in a similar generality, but with two essential differences. First, we allow $\mathcal{E}$ to be a general Fréchet bundle (which makes V obsolete). Second, we allow twisting the action by an additional C-equivariant line bundle L on which the C-action does not necessarily extend to G. However, we put an additional finiteness condition on the pullbacks of L under the action of G on X. The twist by a line bundle L is crucial for our application. We use both the result and the method of [68] in our proof.

We do not know whether the vanishing (A.1) implies the vanishing (A.2) in general. This is probably a very difficult analytic question.

Structure of our proof. Let us first describe the finiteness condition we require. For any $z \in Z$ denote by C_z and G_z the stabilizers of z in C and G respectively. Let L_z denote the character of C_z defined by L. For any $g \in G$ denote by L_z^g the character of C_z given by $L_z^g(c) := L_{gz}(gcg^{-1})$. We require that for any $z \in Z$, the set $\{L_z^g : g \in G\}$ is a finite union of locally closed orbits of G_z.

We first solve the case in which G acts trivially on X, and $\mathcal{E}$ is constant as a G-equivariant bundle. For this case we remove the assumption that Z lies in a finite union of G-orbits. Localizing the problem, we assume X to be $\mathbb{R}^n$, and let it act on itself by translations, and on $\mathcal{E}$ trivially. We translate the problem to a problem on $X \times C$-equivariant distributions on $X \times \hat{C}_{\mathbb{C}}$, where $\hat{C}_{\mathbb{C}}$ denotes the set of all (continuous) characters of C. In this new problem, the $X \times C$-equivariant structure on the line bundle extends to an action of $X \times G$, and thus the space of equivariant distributions vanishes by [68].

The next case we resolve is when G acts transitively on X. Then we have $X = G/H$. We construct a bundle $\mathcal{F}_1$ on $X_1 := C\backslash G$ such that the space of C-invariant $\mathcal{F}$-distributions on X is isomorphic to the space of H-invariant $\mathcal{F}_1$-distributions on X_1. We show that already the space of $H \cap C$-invariant distributions vanishes, using the previous case. The argument here is somewhat similar to the argument in [68]. However, it is complicated by the presence of the line bundle L, and by $\mathcal{F}$ not being constant.

The next case we treat is the case of Z being a single G-orbit. As in [68], it reduces to the previous one using the transverse symbol of distributions.

Finally, we prove the general case by induction on the number of G-orbits on Z.

A.1. Preliminaries

A.1.1. Topological vector spaces. All topological vector spaces considered in this appendix will be complete, Hausdorff, and locally convex. For such a space V, V^* will denote the strong dual, and for two spaces V and W, $V \widehat{\otimes} W$ will denote the completed projective tensor product (this is the same tensor product denoted $\otimes$ in the body of the paper, for convenience). The projective topology on $V \widehat{\otimes} W$ is generated by the family of seminorms which are the largest cross-norms corresponding to pairs of generating seminorms on V and W [107, §43]. In particular, if V and W are Fréchet spaces, then so is $V \widehat{\otimes} W$. If V (or W) is nuclear then the projective tensor product is naturally isomorphic to the injective one [107, Theorem 50.1]. This is the case in all our theorems. The tensor product of nuclear spaces is nuclear. A Fréchet space is nuclear if and only if its dual space is. For more information on nuclear spaces we refer the reader to [107, §50] or [23, Appendix A].

A.1.2. General topology. We will use the following elementary lemma.

Lemma A.1. *Let X be a topological space, and let $\{X_i\}_{i=1}^k$ be disjoint locally closed subsets such that $X = \bigcup_{i=1}^k X_i$. Then there exists i such that the interior of X_i is nonempty.*

Corollary A.2. *Let a topological group G act continuously on a topological space X with finitely many locally closed orbits. Then one of the orbits is open.*

A.1.3. Infinite-dimensional smooth bundles over smooth manifolds. We will use the results and terminology from [69], which considers infinite-dimensional smooth manifolds and bundles over them. In our case the base manifolds will be finite-dimensional, but we will consider infinite-dimensional bundles. All the vector spaces we consider are complete, thus sequentially complete, and thus c^∞-complete [69, Lemma 2.2 and Theorem 2.14]. Therefore all the results of [69] are applicable to them. We use the notion of infinite-dimensional vector bundles and their spaces of smooth sections and spaces of compactly supported sections [69, §§29, 30].

Let X be a smooth manifold and $\mathcal{E}$ be a vector bundle over X, possibly infinite-dimensional. We define the space $\mathcal{D}'(X,\mathcal{E})$ of $\mathcal{E}$-distributions to be the continuous dual space $C_c^\infty(X,\mathcal{E})^*$ equipped with the strong topology. For a closed subset $Z \subset X$, we denote by $\mathcal{D}'_Z(X,\mathcal{E}) \subset \mathcal{D}'(X,\mathcal{E})$ the subspace of distributions supported on Z. For a locally closed subset $Z \subset X$, we denote

$$\mathcal{D}'_Z(X,\mathcal{E}) := \mathcal{D}'_Z(U,\mathcal{E}), \quad \text{where} \quad U := X \setminus (\overline{Z} \setminus Z). \tag{A.3}$$

By [49, II.§3.3], if $\mathcal{E}$ is a trivial bundle with Fréchet fiber V, then $C_c^\infty(X,\mathcal{E}) \cong C_c^\infty(X) \widehat{\otimes} V$.

For any smooth map $\nu : Y \to X$ and a bundle $\mathcal{E}$ over X, the pullback bundle $\nu^*\mathcal{E}$ over Y is defined in [69, §29.6]. For a smooth section f of $\mathcal{E}$ we denote the corresponding section of $\nu^*\mathcal{E}$ by $\nu^* f$.

We define $\nu^!\mathcal{E} := \nu^*\mathcal{E} \otimes D_Y \otimes D_X^{-1}$, where D_X, D_Y denote the bundles of densities on X and Y respectively. In case ν is a submersion and $\mathcal{E}$ has Fréchet fibers, we can also define $\nu_* : C_c^\infty(Y,\nu^!\mathcal{E}) \to C_c^\infty(X,\mathcal{E})$ in the following way. For a trivial bundle $\mathcal{E}$ with Fréchet fiber V, we use the identification

$$C_c^\infty(X,\mathcal{E}) \cong C_c^\infty(X,V) \cong C_c^\infty(X) \widehat{\otimes} V,$$

and the classical pushforward $\nu_* : C_c^\infty(Y, D_Y \otimes \nu^* D_X^{-1}) \to C_c^\infty(X)$. For a locally trivial bundle we use the partition of unity to trivialize $\mathcal{E}$.

We denote the map dual to $\nu_* : C_c^\infty(Y,\nu^!\mathcal{E}) \to C_c^\infty(X,\mathcal{E})$ by

$$\nu^* : \mathcal{D}'(X,\mathcal{E}) \to \mathcal{D}'(Y,\nu^!\mathcal{E}).$$

If Z is a smooth submanifold regularly embedded in X, and V is a Fréchet space, then for any $\xi \in \mathcal{D}_Z(X,V)$ and $z \in X$, [68, §2] defines a *transversal degree* $d \in \mathbb{Z}_{\geq 0}$ and a transverse symbol $\sigma_d(\xi) \in V^* \otimes \operatorname{Sym}^d(N^X_{z,Z})$, where Sym^d denotes symmetric power, $N^X_{z,Z}$ denotes the normal space to Z in X at z, and $CN^X_{z,Z} := (N^X_{z,Z})^*$ is the conormal bundle. Denote by $\mathcal{D}'^{\leq d}_Z(X,V)$ the space of distributions that have transversal degree at most d for any $z \in Z$. By [68, Theorem 2.1], σ_d defines a natural embedding

$$\sigma_d : \mathcal{D}'^{\leq d}_Z(X,V)/\mathcal{D}'^{\leq d-1}_Z(X,V) \hookrightarrow \mathcal{D}'(X, V \otimes \operatorname{Sym}^d(CN^X_{z,Z})).$$

Using a partition of unity, this construction extends to any bundle $\mathcal{E}$ with Fréchet fibers.

Let a Lie group G act on X, let $a : G \times X \to X$ denote the action map and $p : G \times X \to X$ denote the projection. A G-equivariant bundle $\mathcal{E}$ on X is a bundle $\mathcal{E}$ on X together with an isomorphism $a^*\mathcal{E} \simeq p^*\mathcal{E}$ satisfying the usual cocycle condition. Note that this structure also defines an isomorphism $a^!\mathcal{E} \simeq p^!\mathcal{E}$. Note also that the dual of an equivariant bundle has a canonical equivariant structure.

We denote by Δ_G the modular function of G.

We define a smooth representation of G to be a G-equivariant bundle on a point. Note that a smooth Fréchet representation of moderate growth is also a smooth representation according to this definition.

Note that if G is a Lie group, X is a G-manifold, $p : X \to Y$ is a G-invariant map (i.e. $p(gx) = p(x)$), and $\mathcal{F}$ is a bundle on Y, then $p^!\mathcal{F}$ has a natural G-equivariant structure.

Lemma A.3. *Let G be a Lie group, Y be a smooth manifold, and $p : X \to Y$ be a G-principal space over Y. Let $\mathcal{E}$ be a G-equivariant bundle on X. Then there exists a natural bundle $\mathcal{F}$ over Y and an isomorphism of G-equivariant bundles $p^!\mathcal{F} \cong \mathcal{E}$ such that p^* defines an isomorphism $\mathcal{D}'(Y, \mathcal{F}) \cong \mathcal{D}'(X, \mathcal{E})^G$.*

The proof is standard, but we will include it here since our bundles are infinite-dimensional.

We will need the following notation and lemmas.

Notation A.4. For any continuous representation A of G, denote $A(G) := \mathrm{Span}\{v - gv : g \in G,\ v \in A\}$, and $A_G := A/\overline{A(G)}$.

Lemma A.5. *Let A be a continuous representation of G, and B be a nuclear space. Let G act on $A \widehat{\otimes} B$ by acting on A. Then the natural map $\alpha : (A_G \widehat{\otimes} B)^* \to ((A \widehat{\otimes} B)^*)^G$ is an isomorphism.*

Proof. We have $(A_G \widehat{\otimes} B)^* \cong \mathrm{Bil}(A_G, B)$ and $(A \widehat{\otimes} B)^* \cong \mathrm{Bil}(A, B)$, where $\mathrm{Bil}(A, B)$ denotes the space of continuous bilinear maps $A \times B \to \mathbb{C}$ (see e.g. [107, Ch. 41]). The map α is defined by the map $\alpha' : \mathrm{Bil}(A_G, B) \to \mathrm{Bil}(A, B)^G$ which in turn is given by the projection $\mathrm{pr} : A \times B \to A_G \times B$. Since pr is onto, α' is injective. To show that α' is onto, choose $\omega \in \mathrm{Bil}(A, B)^G$. Since the left kernel of ω includes $A(G)$, ω factors through a bilinear map $\omega' : A_G \times B \to \mathbb{C}$. Since $\mathrm{pr} : A \times B \to A_G \times B$ is open and surjective, ω' is continuous. ∎

Lemma A.6. *Let G be a Lie group, and Y be a smooth manifold. Let G act on $Y \times G$ by left shifts on G, and let $p : Y \times G \to Y$ denote the projection. Then p_* defines an isomorphism of topological vector spaces*

$$C_c^\infty(Y \times G, D_G)_G \cong C_c^\infty(Y).$$

Proof. Denote by $C_c^\infty(Y \times G, D_G)_0$ the kernel of p_*. Let us first show that

$$\overline{C_c^\infty(Y \times G, D_G)(G)} = C_c^\infty(Y \times G, D_G)_0. \tag{A.4}$$

Since p_* is a G-invariant morphism, the inclusion $\subset$ follows. For the other inclusion, let $f \in C_c^\infty(Y \times G, D_G)_0$, and approximate it by a sequence f^j of the form $f^j = \sum_{j=1}^{n_j} q_i^j \otimes h_i^j$, with $q_i^j \in C_c^\infty(Y)$ and $h_i^j \in C_c^\infty(G, D_G)$. Fix $\rho \in C_c^\infty(G, D_G)$ with $\int_G \rho = 1$ and let $F^j := f^j - p_*(f^j) \otimes \rho$. Then we have

$$\begin{aligned} F^j &= f^j - \left(\sum_{i=1}^{n_j} \left(\int_G h_i^j\right) q_i^j\right) \otimes \rho \\ &= \sum_{i=1}^{n_j} q_i^j \otimes \left(h_i^j - \left(\int_G h_i^j\right)\rho\right) \in C_c^\infty(Y) \otimes C_c^\infty(G, D_G)_0, \end{aligned}$$

where $C_c^\infty(G, D_G)_0$ denotes smooth compactly supported measures on G with zero integral.

By [13, Theorem 1] we have

$$C_c^\infty(G, D_G)_0 = C_c^\infty(G, D_G)(G),$$

and thus $F^j \in C_c^\infty(Y \times G, D_G)(G)$. Now, since $p_*(f) = 0$, we have $p_*(f^j) \to 0$, so $F^j - f^j \to 0$, and thus $F^j \to f$. Hence $f \in \overline{C_c^\infty(Y \times G, D_G)(G)}$ and (A.4) holds. This shows that p_* defines a continuous linear isomorphism between $C_c^\infty(Y \times G, D_G)_G$ and $C_c^\infty(Y)$. To see that its inverse is continuous, it is enough to construct a continuous section of p_*. One such section is given by $f \mapsto f \otimes \rho$. ■

Proof of Lemma A.3. Let us first consider the case of $X = Y \times G$. Let $i : Y \to X$ be defined by $i(y) := (y, 1)$ and $\mathcal{F} := i^!\mathcal{E}$. The isomorphism $p^!\mathcal{F} \cong \mathcal{E}$ is given by the G-equivariant structure of $\mathcal{E}$. By a partition of unity, we can assume that $\mathcal{F}$ is a constant bundle and denote its fiber by V. We have to show that p^* defines an isomorphism $\mathcal{D}'(Y, V) \cong \mathcal{D}'(Y \times G, V \otimes D_{G,1})^G$, where $D_{G,1}$ is the fiber of the bundle D_G at 1. By Lemma A.6 we have

$$C_c^\infty(Y \times G, D_{G,1})_G \cong C_c^\infty(Y).$$

Then the map p^* decomposes as

$$\begin{aligned} \mathcal{D}'(Y, V) &= (C_c^\infty(Y, V))^* \cong (C_c^\infty(Y) \mathbin{\widehat{\otimes}} V)^* \cong (C_c^\infty(Y \times G, D_{G,1})_G \mathbin{\widehat{\otimes}} V)^* \\ &\cong ((C_c^\infty(Y \times G, D_{G,1}) \mathbin{\widehat{\otimes}} V)^*)^G \cong (C_c^\infty(Y \times G, V \otimes D_{G,1})^*)^G \\ &= \mathcal{D}'(Y \times G, V \otimes D_{G,1})^G. \end{aligned}$$

We now turn to the general case.

Note that if there exists an $\mathcal{F}$ and an isomorphism $\nu : p^!\mathcal{F} \cong \mathcal{E}$ of G-equivariant bundles then such $\mathcal{F}$ and ν are unique, in the sense that for any other such pair $(\mathcal{F}', \nu')$ there exists an isomorphism $\mu : \mathcal{F} \cong \mathcal{F}'$ such that $\nu = \nu' \circ p^!(\mu)$, and such that μ is unique. Thus it is enough to construct $\mathcal{F}$ locally, which is done in the first case considered above. Now, $p^* : \mathcal{D}'(Y, \mathcal{F}) \cong \mathcal{D}'(X, \mathcal{E})^G$ is an isomorphism by a partition of unity and the first case. ■

A.2. Distribution vanishing theorems and their proofs

Theorem A.7. *Let X be a smooth manifold and $Z \subset X$ be a locally closed subset. Let C be a Lie group with finitely many connected components. Let C act trivially on X. Let L be a C-equivariant line bundle on X. Let H be a Lie group with a smooth action on C. For any $z \in Z$ let L_z denote the character by which C acts on the fiber $L|_z$. Let $\widehat{C}_{\mathbb{C}}$ denote the manifold of all characters of C. Assume that the set $\{L_z : z \in Z\}$ lies in a finite union of locally closed H-orbits in $\widehat{C}_{\mathbb{C}}$. Let V be a smooth representation of $H \ltimes C$ in a Fréchet space. Assume that for any $z \in Z$ we have $((V \otimes L|_z)^*)^C = 0$. Then*

$$\mathcal{D}'_Z(X, V \otimes L)^C = 0.$$

Proof. By a partition of unity, we may assume that L is trivial as a line bundle, that $X = \mathbb{R}^n$ and Z is compact. Let L_0 denote the natural C-equivariant line bundle on $\widehat{C}_{\mathbb{C}}$. There exists a unique smooth map $\psi : X \to \widehat{C}_{\mathbb{C}}$ such that $L \cong \psi^*(L_0)$. Define $\Gamma \subset Z \times \widehat{C}_{\mathbb{C}} \subset X \times \widehat{C}_{\mathbb{C}}$ to be the graph of the restriction $\psi|_Z$. Let $\tilde{L} := \mathbb{C}_X \boxtimes L_0$. It is enough to show that $\mathcal{D}'_\Gamma(X \times \widehat{C}_{\mathbb{C}}, V \otimes \tilde{L})^C = 0$.

Assume the contrary. Let $0 \neq \xi \in \mathcal{D}'_\Gamma(X \times \widehat{C}_{\mathbb{C}}, V \otimes \tilde{L})^C$. Let $G := X \times H \ltimes C$ act on $X \times \widehat{C}_{\mathbb{C}}$ by

$$(x, h, c)(y, \chi) := (x + y, \chi \circ a(h^{-1}))$$

where $a(h)$ denotes the action of H on C. Define a structure of a G-equivariant bundle on $\tilde{L}$ through the action on the total space by

$$(x, h, c)(y, \chi, \alpha) := (x + y, \chi \circ a(h^{-1}), \chi(c)\alpha).$$

Define a representation of G on V by letting X act trivially. By our assumptions, $\mathrm{Supp}(\xi)$ lies in the union of finitely many locally closed G-orbits on $X \times \widehat{C}_{\mathbb{C}}$. By Corollary A.2, for one of those orbits $\mathcal{O}$, the intersection $\mathcal{O} \cap \mathrm{Supp}(\xi)$ is open and nonempty in $\mathrm{Supp}(\xi)$. Let $x \in \mathcal{O} \cap \mathrm{Supp}(\xi)$. There exists a cutoff function ρ such that $x \in \mathrm{Supp}(\rho\xi) \subset \mathcal{O}$. This leads to a contradiction since by [68, Theorem 3.15 (i)], $\rho\xi = 0$. ■

By a partition of unity, we obtain the following corollary.

Corollary A.8. *Let X, Z, C, H, and L be as in Theorem A.7. Let $H \ltimes C$ act trivially on X, and let $\mathcal{E}$ be a locally constant $H \ltimes C$-equivariant bundle on X, i.e. an equivariant bundle that is locally given by a single representation of $H \ltimes C$. Assume that for any $z \in Z$ we have $(((\mathcal{E} \otimes L)|_z)^*)^C = 0$. Then*

$$\mathcal{D}'_Z(X, \mathcal{E} \otimes L)^C = 0.$$

We will need the following corollary of Lemma A.3.

Corollary A.9. *Let G be a Lie group and H_1, H_2 be closed Lie subgroups. Consider the two-sided action of $H_1 \times H_2$ on G, and let $\mathcal{E}$ be an $H_1 \times H_2$-equivariant bundle on G. Let $p : G \to G/H_2$ denote the natural projection. Then there exists a natural H_1-equivariant bundle $\mathcal{F}$ on G/H_2 and a natural isomorphism $p^!\mathcal{F} \cong \mathcal{E}$ such that p^* defines an isomorphism*

$$\mathcal{D}'(G/H_2, \mathcal{F})^{H_1} \cong \mathcal{D}'(G, \mathcal{E})^{H_1 \times H_2}.$$

Corollary A.10. *Let G be a Lie group and H_1, H_2 be closed Lie subgroups. Let $\mathcal{F}_1$ be an H_1-equivariant bundle on G/H_2. Let $p_1 : G \to H_1\backslash G$ and $p_2 : G \to G/H_2$ denote the natural projections.*

Then there exists a natural H_2-equivariant bundle $\mathcal{F}_2$ on $H_1\backslash G$ such that $p_1^!\mathcal{F}_2 \cong p_2^!\mathcal{F}_1$ as $H_1 \times H_2$-equivariant bundles, and

$$\mathcal{D}'(G/H_2, \mathcal{F}_1)^{H_1} \cong \mathcal{D}'(H_1\backslash G, \mathcal{F}_2)^{H_2}.$$

Notation A.11. Let a Lie group G act on a smooth manifold X. Let $C \subset G$ be a subgroup. Let L be a C-equivariant line bundle on X. Then for any $x \in X$ and $g \in G$ we define a character $L_x^g : C_x \to \mathbb{C}^\times$ by letting $L_x^g(c)$ be the scalar by which gcg^{-1} acts on the fiber L_{gx}.

Definition A.12. Let a Lie group C act on a smooth manifold X. Let $\mathcal{F}$ be a C-equivariant Fréchet bundle over X. Let $Z \subset X$ be a locally closed C-invariant subset. We call the quadruple $(C, X, Z, \mathcal{F})$ *convenient* if there exist

- a Lie group $G \supset C$ acting smoothly on X extending the action of C,
- a G-equivariant Fréchet bundle $\mathcal{E}$ on X, and
- a C-equivariant line bundle L on X

such that the following holds:

(i) $\mathcal{F} \cong \mathcal{E} \otimes L$ as a C-equivariant bundle.

(ii) C is a normal subgroup of G.

(iii) For any $z \in Z$, the collection $\{L_z^g \in \widehat{(C_z)}_{\mathbb{C}} : g \in G\}$ lies in a finite union of locally closed G_z-orbits.

(iv) Z is contained in a union of finitely many locally closed G-orbits.

Theorem A.13. *Let $(C, X, Z, \mathcal{F})$ be a convenient quadruple. Suppose that for any $z \in Z$ and any $k \geq 0$ we have*

$$(\mathcal{F}|_z^* \otimes \mathrm{Sym}^k(N_{Cz,z}^X) \otimes (\Delta_C^{-1})|_{C_z} \otimes \Delta_{C_z})^{C_z} = 0. \tag{A.5}$$

Then $\mathcal{D}'_Z(X, \mathcal{F})^C = 0$.

Proof. We divide the proof into several cases.

(1) *G acts transitively on X.* Fix $x_0 \in X$ and let $\nu : G \to X$ be the corresponding action map. Let $G' := X_1 := C\backslash G$ and $Z_1 := C\backslash \nu^{-1}(Z)$. By Corollary A.10, there exists a G_{x_0}-equivariant bundle $\mathcal{F}_1$ on X_1 such that $\mathcal{D}'(X, \mathcal{F})^C \cong \mathcal{D}'(X_1, \mathcal{F}_1)^{G_{x_0}}$ and $\nu^!\mathcal{F} \cong p^!\mathcal{F}_1$, where $p : G \to X_1$ is the projection. We construct bundles $\mathcal{E}_1$ and L_1 on X_1 in a similar way, and have $\mathcal{F}_1 = \mathcal{E}_1 \otimes L_1$.

Let $H_1 := G_{x_0}$ and $C_1 := H_1 \cap C$. Then it is enough to show that $\mathcal{D}'(X_1, \mathcal{F}_1)^{C_1} = 0$. We deduce this from Corollary A.8. For this we need to show that

(a) $\mathcal{E}_1$ is locally constant as a C_1-equivariant bundle.

(b) The action of C_1 on the fibers of $\mathcal{E}_1$ extends to H_1.

(c) The set $\{(L_1)_z : z \in X_1\}$ lies in a finite union of locally closed H_1-orbits in $\widehat{(C_1)}_{\mathbb{C}}$.

(d) For any $z \in Z_1$ we have $(((\mathcal{E}_1 \otimes L_1)|_z)^*)^{C_1} = 0$.

Proof of (a). It is enough to show that $\mathcal{E}_1' := \mathcal{E}_1 \otimes D_{X_1}^{-1}$ is locally constant as a C_1-equivariant bundle. Since p locally has a section, it is enough to show that $p^*(\mathcal{E}_1')$ is constant as an H_1-equivariant bundle with respect to the action of H_1 on G by right multiplications. We have $p^*(\mathcal{E}_1') = \nu^*(\mathcal{E}')$, where $\mathcal{E}' := \mathcal{E} \otimes D_X^{-1}$. This gives a structure of a $G \times H_1$-equivariant bundle on $p^*(\mathcal{E}_1')$ with respect to the two-sided action. This implies that $p^*(\mathcal{E}_1')$ is constant as an H_1-equivariant bundle.

Proof of (b). The fiber of $\mathcal{E}_1$ at $[1]$ is isomorphic to $\mathcal{E}|_{x_0} \otimes \Delta_{H_1}|_{C_1} \otimes \Delta_C^{-1}|_{C_1}$ as a representation of C_1. Since C is normal in G, we have $\Delta_C = \Delta_G|_C$, and thus the representation $\mathcal{E}_{x_0} \otimes \Delta_{H_1}|_{C_1} \otimes \Delta_C^{-1}|_{C_1}$ extends to H_1.

Proof of (c). L satisfies condition (iii) of Definition A.12. Thus so does $L' := L \otimes D_X^{-1}$, since the action of C on D_X^{-1} can be extended to G. It is enough to show (c) with L_1 replaced by $L_1' := L \otimes D_{X_1}$. Now, we have $p^*(L_1') = \nu^*(L')$. Thus for any $g \in G$ we have $(L')^g_{x_0} = (L_1')_{[g]}$ and thus

$$\{(L_1')_y \in \widehat{C}_{\mathbb{C}} : y \in X_1\} = \{(L'_{x_0})^g \in \widehat{C}_{\mathbb{C}} : g \in G\}.$$

Statement (d) follows from (A.5) by a straightforward computation.

(2) *Z lies in a single closed G-orbit $\mathcal{O}$.* Suppose by way of contradiction that there exists $0 \neq \xi \in \mathcal{D}'_Z(X, \mathcal{F})^C$ and let $z \in \operatorname{Supp}(\xi)$. Let $X_1 := \mathcal{O}$. Let d be the transversal degree of ξ to $\mathcal{O}$ at z. Let $Z_1 := \{p \in Z : \deg_{p,\mathcal{O}}(\xi) \le d\}$. Consider

$$\sigma_d(\xi|_{Z_1}) \in \mathcal{D}'_{Z_1}(X_1, \mathcal{F} \otimes \operatorname{Sym}^d(CN_{X_1}^X))^C.$$

By the previous case we obtain $\sigma_d(\xi|_{Z_1}) = 0$, a contradiction.

(3) *The general case.* We prove this step by induction on the number n of orbits of G in GZ. When $n = 0$, Z is empty and the statement is obvious. For $n \ge 1$, Corollary A.2 implies that there exists an open orbit $\mathcal{O} \subset Z$. Let $Z' := Z \setminus \mathcal{O}$, $X' := X \setminus Z'$. Then we have the exact sequence

$$0 \to \mathcal{D}'_{Z'}(X, \mathcal{F})^C \to \mathcal{D}'_Z(X, \mathcal{F})^C \to \mathcal{D}'_{\mathcal{O}}(X', \mathcal{F})^C.$$

We have $\mathcal{D}'_{\mathcal{O}}(X', \mathcal{F})^C = 0$ by the previous case, and $\mathcal{D}'_{Z'}(X, \mathcal{F})^C = 0$ by the induction hypothesis. ∎

Remark A.14. Substituting for $\mathcal{E}$ the constant bundle with fiber V and for L a constant line bundle, we obtain Theorem 1.4.

Acknowledgments. We are grateful to the referees for their interest in this work and helpful remarks, which helped improve the presentation. Aizenbud and Gourevitch thank Alexander Shamov for fruitful discussions.

Funding. This research was supported by the ERC, StG grant number 637912 (Gourevitch), and by the ISRAEL SCIENCE FOUNDATION, grant numbers 249/17 (Aizenbud and Gourevitch) and 421/17 (Kaplan).

References

[1] Aizenbud, A., Gourevitch, D.: Smooth transfer of Kloosterman integrals (the Archimedean case). Amer. J. Math. **135**, 143–182 (2013) Zbl 1329.11049 MR 3022961

[2] Aizenbud, A., Gourevitch, D., Rallis, S., Schiffmann, G.: Multiplicity one theorems. Ann. of Math. (2) **172**, 1407–1434 (2010) Zbl 1202.22012 MR 2680495

[3] Aizenbud, A., Gourevitch, D., Sahi, S.: Derivatives for smooth representations of $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. Israel J. Math. **206**, 1–38 (2015) Zbl 1317.22008 MR 3319629

[4] Aizenbud, A., Gourevitch, D., Sahi, S.: Twisted homology for the mirabolic nilradical. Israel J. Math. **206**, 39–88 (2015) Zbl 1311.22020 MR 3319630

[5] Arthur, J.: The Endoscopic Classification of Representations. Amer. Math. Soc. Colloq. Publ. 61, Amer. Math. Soc., Providence, RI (2013) Zbl 1310.22014 MR 3135650

[6] Asgari, M., Shahidi, F.: Generic transfer for general spin groups. Duke Math. J. **132**, 137–190 (2006) Zbl 1099.11028 MR 2219256

[7] Banks, W. D.: A corollary to Bernstein's theorem and Whittaker functionals on the metaplectic group. Math. Res. Lett. **5**, 781–790 (1998) Zbl 0944.22007 MR 1671189

[8] Banks, W. D., Levy, J., Sepanski, M. R.: Block-compatible metaplectic cocycles. J. Reine Angew. Math. **507**, 131–163 (1999) Zbl 0914.20039 MR 1670203

[9] Barbasch, D., Vogan, D. A., Jr.: The local structure of characters. J. Funct. Anal. **37**, 27–55 (1980) Zbl 0436.22011 MR 576644

[10] Ben-Artzi, A., Soudry, D.: L-functions for $\mathrm{U}_m \times R_{E/F}\mathrm{GL}_n$ $(n \le [\frac{m}{2}])$. In: Automorphic Forms and L-functions I. Global Aspects, Contemp. Math. 488, Amer. Math. Soc., Providence, RI, 13–59 (2009) Zbl 1187.11018 MR 2522026

[11] Bernšteĭn, I. N., Zelevinskiĭ, A. V.: Representations of the group $GL(n,F)$, where F is a local non-Archimedean field. Uspekhi Mat. Nauk **31**, no. 3, 5–70 (1976) (in Russian) Zbl 0348.43007 MR 0425030

[12] Bernstein, I. N., Zelevinsky, A. V.: Induced representations of reductive $\mathfrak{p}$-adic groups. I. Ann. Sci. École Norm. Sup. (4) **10**, 441–472 (1977) Zbl 0412.22015 MR 579172

[13] Blanc, P., Wigner, D.: Homology of Lie groups and Poincaré duality. Lett. Math. Phys. **7**, 259–270 (1983) Zbl 0548.57026 MR 706216

[14] Böcherer, S., Schmidt, C.-G.: p-adic measures attached to Siegel modular forms. Ann. Inst. Fourier (Grenoble) **50**, 1375–1443 (2000) Zbl 0962.11023 MR 1800123

[15] Bruhat, F.: Sur les représentations induites des groupes de Lie. Bull. Soc. Math. France **84**, 97–205 (1956) Zbl 0074.10303 MR 84713

[16] Brylinski, J.-L., Deligne, P.: Central extensions of reductive groups by $\mathbf{K}_2$. Publ. Math. Inst. Hautes Études Sci. **94**, 5–85 (2001) Zbl 1093.20027 MR 1896177

[17] Cai, Y.: Twisted doubling integrals for classical groups. Math. Z. **297**, 1075–1104 (2021) Zbl 1478.11077 MR 4229593

[18] Cai, Y., Friedberg, S., Ginzburg, D., Kaplan, E.: Doubling constructions and tensor product L-functions: the linear case. Invent. Math. **217**, 985–1068 (2019) Zbl 1480.11060 MR 3989257

[19] Cai, Y., Friedberg, S., Gourevitch, D., Kaplan, E.: The generalized doubling method: (k,c) models. arXiv:2109.11309 (2021)

[20] Cai, Y., Friedberg, S., Kaplan, E.: Doubling constructions: local and global theory, with an application to global functoriality for non-generic cuspidal representations. arXiv:1802.02637 (2020)

[21] Carter, R. W.: Finite Groups of Lie Type. Wiley Classics Library, Wiley, Chichester (1993) Zbl 0567.20023 MR 1266626

[22] Casselman, W.: Canonical extensions of Harish-Chandra modules to representations of G. Canad. J. Math. **41**, 385–438 (1989) Zbl 0702.22016 MR 1013462

[23] Casselman, W., Hecht, H., Miličić, D.: Bruhat filtrations and Whittaker vectors for real groups. In: The Mathematical Legacy of Harish-Chandra (Baltimore, MD, 1998), Proc. Sympos. Pure Math. 68, Amer. Math. Soc., Providence, RI, 151–190 (2000) Zbl 0959.22010 MR 1767896

[24] Chen, Y., Sun, B.: Schwartz homologies of representations of almost linear Nash groups. J. Funct. Anal. **280**, art. 108817, 50 pp. (2021) Zbl 1478.22008 MR 4211018

[25] Cogdell, J. W., Kim, H. H., Piatetski-Shapiro, I. I., Shahidi, F.: On lifting from classical groups to GL_N. Publ. Math. Inst. Hautes Études Sci. **93**, 5–30 (2001) Zbl 1028.11029 MR 1863734

[26] Cogdell, J. W., Kim, H. H., Piatetski-Shapiro, I. I., Shahidi, F.: Functoriality for the classical groups. Publ. Math. Inst. Hautes Études Sci. **99**, 163–233 (2004) Zbl 1090.22010 MR 2075885

[27] Cogdell, J. W., Piatetski-Shapiro, I. I.: Converse theorems for GL_n. Inst. Hautes Études Sci. Publ. Math. **79**, 157–214 (1994) Zbl 0814.11033 MR 1307299

[28] Cogdell, J. W., Piatetski-Shapiro, I. I.: Converse theorems for GL_n. II. J. Reine Angew. Math. **507**, 165–188 (1999) Zbl 0912.11022 MR 1670207

[29] Collingwood, D. H., McGovern, W. M.: Nilpotent Orbits in Semisimple Lie Algebras. Van Nostrand Reinhold, New York (1993) Zbl 0972.17008 MR 1251060

[30] du Cloux, F.: Sur les représentations différentiables des groupes de Lie algébriques. Ann. Sci. École Norm. Sup. (4) **24**, 257–318 (1991) Zbl 0748.22006 MR 1100992

[31] Eischen, E., Harris, M., Li, J., Skinner, C.: p-adic L-functions for unitary groups. Forum Math. Pi **8**, art. e9, 160 pp. (2020) Zbl 1473.11118 MR 4096618

[32] Feigon, B., Lapid, E., Offen, O.: On representations distinguished by unitary groups. Publ. Math. Inst. Hautes Études Sci. **115**, 185–323 (2012) Zbl 1329.11053 MR 2930996

[33] Friedberg, S., Jacquet, H.: Linear periods. J. Reine Angew. Math. **443**, 91–139 (1993) Zbl 0782.11033 MR 1241129

[34] Gan, W. T.: Doubling zeta integrals and local factors for metaplectic groups. Nagoya Math. J. **208**, 67–95 (2012) Zbl 1280.11028 MR 3006697

[35] Gan, W. T., Gross, B. H., Prasad, D.: Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. Astérisque **346**, 1–109 (2012) Zbl 1280.22019 MR 3202556

[36] Gan, W. T., Savin, G.: Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence. Compos. Math. **148**, 1655–1694 (2012) Zbl 1325.11046 MR 2999299

[37] Gao, F.: Distinguished theta representations for certain covering groups. Pacific J. Math. **290**, 333–379 (2017) Zbl 1395.11082 MR 3681111

[38] Gao, F., Shahidi, F., Szpruch, D.: Local coefficients and gamma factors for principal series of covering groups. Mem. Amer. Math. Soc., to appear; arXiv:1902.02686

[39] Gao, F., Shahidi, F., Szpruch, D.: On the local coefficients matrix for coverings of SL_2. In: Geometry, Algebra, Number Theory, and Their Information Technology Applications, Springer Proc. Math. Statist. 251, Springer, Cham, 207–244 (2018) Zbl 1475.11096 MR 3880389

[40] Gelbart, S., Piatetski-Shapiro, I., Rallis, S.: Explicit Constructions of Automorphic L-functions. Lecture Notes in Math. 1254, Springer, Berlin (1987) Zbl 0612.10022 MR 892097

[41] Ginzburg, D.: Certain conjectures relating unipotent orbits to automorphic representations. Israel J. Math. **151**, 323–355 (2006) Zbl 1128.11028 MR 2214128

[42] Ginzburg, D., Jiang, D., Rallis, S., Soudry, D.: L-functions for symplectic groups using Fourier–Jacobi models. In: Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. 19, Int. Press, Somerville, MA, 183–207 (2011) Zbl 1325.11042 MR 2906909

[43] Ginzburg, D., Piatetski-Shapiro, I., Rallis, S.: L functions for the orthogonal group. Mem. Amer. Math. Soc. **128**, no. 611, viii+218 pp. (1997) Zbl 0884.11022 MR 1357823

[44] Ginzburg, D., Rallis, S., Soudry, D.: On a correspondence between cuspidal representations of GL_{2n} and $\widetilde{\mathrm{Sp}}_{2n}$. J. Amer. Math. Soc. **12**, 849–907 (1999) Zbl 0928.11027 MR 1671452

[45] Ginzburg, D., Rallis, S., Soudry, D.: On explicit lifts of cusp forms from GL_m to classical groups. Ann. of Math. (2) **150**, 807–866 (1999) Zbl 0949.22019 MR 1740991

[46] Gomez, R., Gourevitch, D., Sahi, S.: Generalized and degenerate Whittaker models. Compos. Math. **153**, 223–256 (2017) Zbl 1384.20039 MR 3705224

[47] Gomez, R., Gourevitch, D., Sahi, S.: Whittaker supports for representations of reductive groups. Ann. Inst. Fourier (Grenoble) **71**, 239–286 (2021) Zbl 07362103 MR 4275869

[48] Gourevitch, D., Sahi, S.: Generalized and degenerate Whittaker quotients and Fourier coefficients. In: Representations of Reductive Groups, Proc. Sympos. Pure Math. 101, Amer. Math. Soc., Providence, RI, 133–154 (2019) Zbl 1469.20046 MR 3930016

[49] Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc. **16** (1955) Zbl 0064.35501 MR 75539

[50] Harish-Chandra: The characters of reductive p-adic groups. In: Contributions to Algebra (collection of papers dedicated to Ellis Kolchin), 175–182 (1977) Zbl 0365.22016 MR 0492086

[51] Harris, M., Kudla, S. S., Sweet, W. J.: Theta dichotomy for unitary groups. J. Amer. Math. Soc. **9**, 941–1004 (1996) Zbl 0870.11026 MR 1327161

[52] Harris, M., Li, J.-S., Skinner, C. M.: The Rallis inner product formula and p-adic L-functions. In: Automorphic Representations, L-functions and Applications: Progress and Prospects, Ohio State Univ. Math. Res. Inst. Publ. 11, de Gruyter, Berlin, 225–255 (2005) Zbl 1103.11017 MR 2192825

[53] Harris, M., Li, J.-S., Skinner, C. M.: p-adic L-functions for unitary Shimura varieties. I. Construction of the Eisenstein measure. Doc. Math. 393–464 (2006) Zbl 1143.11019 MR 2290594

[54] Howe, R.: The Fourier transform and germs of characters (case of Gl_n over a p-adic field). Math. Ann. **208**, 305–322 (1974) Zbl 0266.43007 MR 342645

[55] Jacquet, H.: On the residual spectrum of $\mathrm{GL}(n)$. In: Lie Group Representations, II (College Park, MD, 1982/1983), Lecture Notes in Math. 1041, Springer, Berlin, 185–208 (1984) MR 748508

[56] Jacquet, H.: Représentations distinguées pour le groupe orthogonal. C. R. Acad. Sci. Paris Sér. I Math. **312**, 957–961 (1991) Zbl 0745.11031 MR 1113084

[57] Jacquet, H.: Distinction by the quasi-split unitary group. Israel J. Math. **178**, 269–324 (2010) Zbl 1208.11068 MR 2733072

[58] Jacquet, H., Piatetskii-Shapiro, I. I., Shalika, J. A.: Rankin–Selberg convolutions. Amer. J. Math. **105**, 367–464 (1983) Zbl 0525.22018 MR 701565

[59] Jacquet, H., Rallis, S.: Symplectic periods. J. Reine Angew. Math. **423**, 175–197 (1992) Zbl 0734.11035 MR 1142486

[60] Jiang, D., Liu, B.: On Fourier coefficients of automorphic forms of $\mathrm{GL}(n)$. Int. Math. Res. Notices **2013**, 4029–4071 Zbl 1359.11039 MR 3096918

[61] Jiang, D., Liu, B.: On Fourier coefficients of certain residual representations of symplectic groups. Pacific J. Math. **281**, 421–466 (2016) Zbl 1343.11054 MR 3463042

[62] Jiang, D., Sun, B., Zhu, C.-B.: Uniqueness of Bessel models: the Archimedean case. Geom. Funct. Anal. **20**, 690–709 (2010) Zbl 1200.22008 MR 2720228

[63] Jiang, D., Zhang, L.: A product of tensor product L-functions of quasi-split classical groups of Hermitian type. Geom. Funct. Anal. **24**, 552–609 (2014) Zbl 1298.11046 MR 3192035

[64] Kaplan, E.: Doubling constructions and tensor product L-functions: coverings of the symplectic group. arXiv:1902.00880v2 (2020)

[65] Kaplan, E.: On the gcd of local Rankin–Selberg integrals for even orthogonal groups. Compos. Math. **149**, 587–636 (2013) Zbl 1330.11036 MR 3049697

[66] Kaplan, E.: On the local theory of Rankin–Selberg convolutions for $\mathrm{SO}_{2l} \times \mathrm{GL}_n$. Thesis, Tel Aviv Univ. (2013)

[67] Kazhdan, D. A., Patterson, S. J.: Metaplectic forms. Inst. Hautes Études Sci. Publ. Math. **59**, 35–142 (1984) Zbl 0559.10026 MR 743816

[68] Kolk, J. A. C., Varadarajan, V. S.: On the transverse symbol of vectorial distributions and some applications to harmonic analysis. Indag. Math. (N.S.) **7**, 67–96 (1996) Zbl 0892.22010 MR 1621372

[69] Kriegl, A., Michor, P. W.: The Convenient Setting of Global Analysis. Math. Surveys Monogr. 53, Amer. Math. Soc., Providence, RI (1997) Zbl 0889.58001 MR 1471480

[70] Kubota, T.: Topological covering of SL(2) over a local field. J. Math. Soc. Japan **19**, 114–121 (1967) Zbl 0166.29603 MR 204422

[71] Kubota, T.: On Automorphic Functions and the Reciprocity Law in a Number Field. Kinokuniya, Tokyo (1969) Zbl 0231.10017 MR 0255490

[72] Kudla, S. S., Rallis, S.: Poles of Eisenstein series and L-functions. In: Festschrift in Honor of I. I. Piatetski-Shapiro on the Occasion of his Sixtieth Birthday, Part II (Ramat Aviv, 1989), Israel Math. Conf. Proc. 3, Weizmann, Jerusalem, 81–110 (1990) Zbl 0712.11029 MR 1159110

[73] Lapid, E. M., Rallis, S.: On the local factors of representations of classical groups. In: Automorphic Representations, L-functions and Applications: Progress and Prospects, Ohio State Univ. Math. Res. Inst. Publ. 11, de Gruyter, Berlin, 309–359 (2005) Zbl 1188.11023 MR 2192828

[74] Liu, Y., Sun, B.: Uniqueness of Fourier–Jacobi models: the Archimedean case. J. Funct. Anal. **265**, 3325–3344 (2013) Zbl 1286.22012 MR 3110504

[75] Matringe, N.: Conjectures about distinction and local Asai L-functions. Int. Math. Res. Notices **2009**, 1699–1741 Zbl 1225.22014 MR 2500974

[76] Matringe, N.: Distinction of some induced representations. Math. Res. Lett. **17**, 77–97 (2010) Zbl 1222.22014 MR 2592729

[77] Matringe, N.: Distinguished representations and exceptional poles of the Asai-L-function. Manuscripta Math. **131**, 415–426 (2010) Zbl 1218.11051 MR 2592088

[78] Matringe, N.: Distinguished generic representations of $\mathrm{GL}(n)$ over p-adic fields. Int. Math. Res. Notices **2011**, 74–95 Zbl 1223.22015 MR 2755483

[79] Matringe, N.: On the local Bump-Friedberg L-function. J. Reine Angew. Math. **709**, 119–170 (2015) Zbl 1398.11080 MR 3430877

[80] Matsumoto, H.: Sur les sous-groupes arithmétiques des groupes semi-simples déployés. Ann. Sci. École Norm. Sup. (4) **2**, 1–62 (1969) Zbl 0261.20025 MR 240214

[81] McNamara, P. J.: Principal series representations of metaplectic groups over local fields. In: Multiple Dirichlet Series, L-functions and Automorphic Forms, Progr. Math. 300, Birkhäuser/Springer, New York, 299–327 (2012) Zbl 1283.11081 MR 2963537

[82] Mœglin, C., Waldspurger, J.-L.: Modèles de Whittaker dégénérés pour des groupes p-adiques. Math. Z. **196**, 427–452 (1987) Zbl 0612.22008 MR 913667

[83] Mœglin, C., Waldspurger, J.-L.: Le spectre résiduel de $\mathrm{GL}(n)$. Ann. Sci. École Norm. Sup. (4) **22**, 605–674 (1989) Zbl 0696.10023 MR 1026752

[84] Mok, C. P.: Endoscopic classification of representations of quasi-split unitary groups. Mem. Amer. Math. Soc. **235**, no. 1108, vi+248 pp. (2015) Zbl 1316.22018 MR 3338302

[85] Moore, C. C.: Group extensions of p-adic and adelic linear groups. Inst. Hautes Études Sci. Publ. Math. **35**, 157–222 (1968) Zbl 0159.03203 MR 244258

[86] Offen, O.: Residual spectrum of GL_{2n} distinguished by the symplectic group. Duke Math. J. **134**, 313–357 (2006) Zbl 1220.11072 MR 2248833

[87] Offen, O.: Unitary periods and Jacquet's relative trace formula. In: Automorphic Forms and L-functions I. Global Aspects, Contemp. Math. 488, Amer. Math. Soc., Providence, RI, 183–236 (2009) Zbl 1234.11067 MR 2522031

[88] Offen, O., Sayag, E.: On unitary representations of GL_{2n} distinguished by the symplectic group. J. Number Theory **125**, 344–355 (2007) Zbl 1147.11028 MR 2332593

[89] Offen, O., Sayag, E.: Uniqueness and disjointness of Klyachko models. J. Funct. Anal. **254**, 2846–2865 (2008) Zbl 1143.22013 MR 2414223

[90] Piatetski-Shapiro, I., Rallis, S.: L-functions for the classical groups. In: Explicit Constructions of Automorphic L-functions, Lecture Notes in Math. 1254, Springer, New York, 1–52 (1987) Zbl 0612.10022 MR 0892097

[91] Richardson, R. W.: Finiteness theorems for orbits of algebraic groups. Indag. Math. **47**, 337–344 (1985) Zbl 0595.20039 MR 814886

[92] Shahidi, F.: Functional equation satisfied by certain L-functions. Compos. Math. **37**, 171–207 (1978) Zbl 0393.12017 MR 498494

[93] Shahidi, F.: On certain L-functions. Amer. J. Math. **103**, 297–355 (1981) Zbl 0467.12013 MR 610479

[94] Shahidi, F.: Local coefficients and normalization of intertwining operators for $\mathrm{GL}(n)$. Compos. Math. **48**, 271–295 (1983) Zbl 0506.22020 MR 700741

[95] Shahidi, F.: Local coefficients as Artin factors for real groups. Duke Math. J. **52**, 973–1007 (1985) Zbl 0674.10027 MR 816396

[96] Shahidi, F.: A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. Ann. of Math. (2) **132**, 273–330 (1990) Zbl 0780.22005 MR 1070599

[97] Soudry, D.: Rankin–Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$: local theory. Mem. Amer. Math. Soc. **105**, no. 500, vi+100 pp. (1993) Zbl 0805.22007 MR 1169228

[98] Soudry, D.: On the archimedean theory of Rankin–Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. Ann. Sci. École Norm. Sup. (4) **28**, 161–224 (1995) Zbl 0824.11034 MR 1318068

[99] Soudry, D.: The unramified computation of Rankin–Selberg integrals expressed in terms of Bessel models for split orthogonal groups: Part I. Israel J. Math. **222**, 711–786 (2017) Zbl 1433.11065 MR 3722265

[100] Soudry, D.: The unramified computation of Rankin–Selberg integrals expressed in terms of Bessel models for split orthogonal groups: Part II. J. Number Theory **186**, 62–102 (2018) Zbl 1433.11066 MR 3758206

[101] Steinberg, R.: Lectures on Chevalley Groups. Yale Univ., New Haven, CN (1968) Zbl 1196.22001 MR 0466335

[102] Sun, B.: Multiplicity one theorems for Fourier–Jacobi models. Amer. J. Math. **134**, 1655–1678 (2012) Zbl 1280.22022 MR 2999291

[103] Sun, B.: Almost linear Nash groups. Chinese Ann. Math. Ser. B **36**, 355–400 (2015) Zbl 1322.22009 MR 3341164

[104] Sun, B., Zhu, C.-B.: Multiplicity one theorems: the Archimedean case. Ann. of Math. (2) **175**, 23–44 (2012) Zbl 1239.22014 MR 2874638

[105] Suzuki, T.: Distinguished representations of metaplectic groups. Amer. J. Math. **120**, 723–755 (1998) Zbl 0921.11028 MR 1637947

[106] Szpruch, D.: On Shahidi local coefficients matrix. Manuscripta Math. **159**, 117–159 (2019) Zbl 1417.22011 MR 3936136

[107] Trèves, F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, New York (1967) Zbl 0171.10402 MR 0225131

[108] Waldspurger, J.-L.: La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux. Astérisque **347**, 103–165 (2012) Zbl 1276.22010 MR 3155345

[109] Waldspurger, J.-L.: Une variante d'un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann. Astérisque **346**, 313–318 (2012) Zbl 1308.22008 MR 3202559

[110] Wallach, N. R.: Real Reductive Groups. I. Pure Appl. Math. 132, Academic Press, Boston, MA (1988) Zbl 0666.22002 MR 929683

[111] Warner, G.: Harmonic Analysis on Semi-simple Lie Groups. I. Grundlehren Math. Wiss. 188, Springer, New York (1972) Zbl 0265.22021 MR 0498999

[112] Weil, A.: Sur la formule de Siegel dans la théorie des groupes classiques. Acta Math. **113**, 1–87 (1965) Zbl 0161.02304 MR 223373

[113] Yamana, S.: L-functions and theta correspondence for classical groups. Invent. Math. **196**, 651–732 (2014) Zbl 1303.11054 MR 3211043

[114] Yamana, S.: Local symmetric square L-factors of representations of general linear groups. Pacific J. Math. **286**, 215–256 (2017) Zbl 1421.11044 MR 3582406

[115] Zelevinsky, A. V.: Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. Ann. Sci. École Norm. Sup. (4) **13**, 165–210 (1980) Zbl 0441.22014 MR 584084