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Smooth rigidity for very non-algebraic expanding maps

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Abstract. We show that the space of expanding maps contains an open and dense subset where smooth conjugacy classes of expanding maps are determined by the values of the Jacobians of return maps at periodic points.

Keywords. Expanding maps, rigidity, foliations, transfer operator, smooth conjugacy

1. Introduction

Let M be a smooth closed manifold. Recall that a C^r, $r \geq 1$, map $f: M \to M$ is called *expanding* if

$$\|Dfv\| > \|v\|$$

for all non-zero $v \in TM$ and some choice of Riemannian metric on M. It is easy to see that an expanding map is necessarily a covering map.

Recall that expanding maps have been classified up to topological conjugacy. Shub [33] proved that M is covered by the Euclidean space and also that an expanding endomorphism of M is topologically conjugate to an affine expanding endomorphism of an infranilmanifold if and only if the fundamental group $\pi_1(M)$ contains a nilpotent subgroup of finite index. Franks [15] showed that if M admits an expanding endomorphism then $\pi_1(M)$ has polynomial growth. Finally, in 1981, Gromov [19] completed the classification by showing that any finitely generated group of polynomial growth contains a nilpotent subgroup of finite index. Hence any expanding endomorphism is topologically conjugate to an affine expanding endomorphism of an infranilmanifold.

Let $f_i: M_i \to M_i$ be C^r-smooth, $r \geq 1$, expanding maps $i = 1, 2$. Also we will assume that f_1 and f_2 are conjugate via a homeomorphism $h: M_1 \to M_2$, i.e., $h \circ f_1 = f_2 \circ h$. For example, homotopic expanding maps on the same manifold are always conjugate.

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It is well known that h and its inverse are Hölder continuous. However, a priori h is not C^1-smooth with obvious obstructions carried by the eigendata of periodic points. That is, when h is C^1, the differential of the return map $Df_1^n(x)$ is conjugate to $Df_2^n(h(x))$ whenever $x = f_1^n(x)$. A weaker necessary assumption is coincidence of *Jacobian data*, i.e.,

$$\mathrm{Jac}(f_1^n)(x) = \mathrm{Jac}(f_2^n)(h(x))$$

for all periodic points $x = f_1^n(x)$.

In this paper we offer the following progress for higher-dimensional expanding maps: *For any $r \geq 2$ there exists a C^r-dense and C^1-open subset $\mathcal{U}$ in the space of C^r expanding maps such that if $f_1 \in \mathcal{U}$ and f_2 is an expanding map which is conjugate to f_1 and has the same Jacobian data then the conjugacy is C^{r-1}.* In the proof we use the fact that f_1 lives on an infranilmanifold. In the next section we will give precise statements which, in particular, explicitly describe the set $\mathcal{U}$ in the next section. Our proof of this result was partially inspired by the Embedding Theorem (or Reconstruction Theorem) of Takens [36].

In dimension 1 smooth classification was already known. Indeed, Shub and Sullivan showed that for C^r, $r \geq 2$, expanding maps of the circle S^1 the above condition on coincidence of Jacobians implies that the conjugacy h is C^r-smooth [34]. In fact, they proved a stronger result that an absolutely continuous conjugacy (which is not, a priori, even continuous) must coincide a.e. with a smooth conjugacy provided that the Jacobian of one of the expanding maps is not cohomologous to a constant.

The analogous "smooth conjugacy problem" in the setting of Anosov diffeomorphisms was completely resolved by de la Llave, Marco and Moriyón in dimension 2 [6,7,9]. In higher dimensions there was much partial progress: see e.g. [8,18,23] and references therein. However, progress was made only for certain special classes of Anosov diffeomorphisms such as conformal ones or with a fine dominated splitting. When compared to this body of work, the current paper is very different. It relies on a fundamentally different approach – to examine matching functions rather than matching measures. And it yields a smooth classification on a large open set as opposed to characterization of smooth conjugacy classes of certain special maps.

The next section contains the statement of our main technical result, Theorem 2.1. Then we state a number of corollaries for the smooth conjugacy problem and discuss necessity of various assumptions. Section 3 is devoted to preliminaries on properties of the transfer operator associated to an expanding map. Section 4 contains the proof of the main theorem under an additional simplifying assumption that the underlying manifold is a torus; that assumption makes the proof much shorter and more transparent. Then in Section 5 we prove Theorem 2.1 in full generality. In Section 6 we derive all the corollaries on the smooth classification problem. Then, in Section 7, we give a number of examples of expanding maps illustrating various features of our results and proofs. Finally, in Section 8 we state a generalized factor version of Theorem 2.1 and also give an application.

2. The results

We adopt the standard convention and call a map $f\colon M \to M$ *C^r-smooth*, $r \geq 0$, or just C^r, if it is $\lfloor r \rfloor$ times continuously differentiable and its $\lfloor r \rfloor$-th differential is Hölder continuous with exponent $r - \lfloor r \rfloor$. We also allow $r = \infty$ and $r = \omega$ (real analytic maps). One defines C^r-smooth functions on M in a similar way.

Recall that we denote by $f_i\colon M_i \to M_i$, $i = 1, 2$, C^r-smooth expanding maps and we assume that f_1 and f_2 are conjugate by h, $h \circ f_1 = f_2 \circ h$. Given functions $\varphi_i\colon M_i \to \mathbb{R}$, $i = 1, 2$, we say that (f_1, φ_1) is *equivalent* to (f_2, φ_2) and write

$$(f_1, \varphi_1) \sim (f_2, \varphi_2)$$

if there exists a function $u\colon M_1 \to \mathbb{R}$ such that

$$\varphi_1 - \varphi_2 \circ h = u - u \circ f_1.$$

Then, by the Livshits theorem [26], $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ if and only if for every periodic point $x \in \mathrm{Fix}(f_1^n)$,

$$\sum_{k=0}^{n-1} \varphi_1(f_1^k(x)) = \sum_{k=0}^{n-1} \varphi_2(f_2^k(h(x))).$$

Further, if the functions φ_i are C^r-smooth then the transfer function u is also C^r-smooth.[1] The following is our main technical result.

Theorem 2.1. *Assume that M_i, $i = 1, 2$, are closed manifolds homeomorphic to a nilmanifold. Let $f_i\colon M_i \to M_i$, $i = 1, 2$, be C^r-smooth, $r \geq 1$, expanding maps and assume they are conjugate via a homeomorphism $h\colon M_1 \to M_2$. Then there exist manifolds $\bar{M}_i$ (which are homeomorphic to nilmanifolds) and C^r fibrations $p_i\colon M_i \to \bar{M}_i$, $i = 1, 2$, (whose fibers are homeomorphic to a nilmanifold) and C^r expanding maps $\bar{f}_i\colon \bar{M}_i \to \bar{M}_i$, such that f_i fibers over $\bar{f}_i$, i.e.,*

$$p_i \circ f_i = \bar{f}_i \circ p_i, \quad i = 1, 2.$$

The conjugacy h maps fibers to fibers, i.e.,

$$p_2 \circ h = \bar{h} \circ p_1,$$

where the induced conjugacy $\bar{h}\colon \bar{M}_1 \to \bar{M}_2$, $\bar{h} \circ \bar{f}_1 = \bar{f}_2 \circ \bar{h}$, is a C^r diffeomorphism.

Further, the fibrations p_i, $i = 1, 2$, have the following property. If $\varphi_i\colon M_i \to \mathbb{R}$, $i = 1, 2$, are C^r-smooth functions such that $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ then there exist C^r functions $\bar{\varphi}_i\colon \bar{M}_i \to \mathbb{R}$, $i = 1, 2$, such that φ_i is cohomologous to $\bar{\varphi}_i \circ p_i$ over f_i, $i = 1, 2$, and

$$\bar{\varphi}_2 \circ \bar{h} = \bar{\varphi}_1.$$

[1] The Livshits theorem for expanding maps can be proved using the standard transitive point argument [26]. There is no loss of regularity in the bootstrap argument for the transfer function; see e.g. [22].

All manifolds in the above theorem, including the fibers of the fibrations, are connected. All manifolds are homeomorphic to nilmanifolds but could carry exotic smooth structures.

At this point we recommend that the reader looks at Example 7.1 to better understand the statement of the above theorem.

Remark 2.2. The manifold $\bar{M}_1$ may be equal to M_1 or may be a point or may have some dimension inbetween. In the first case we find that f_1 and f_2 are C^r-smoothly conjugate and in the second case we find that the functions φ_1 and φ_2 are both cohomologous to a constant.

Also notice that the regularity of the f_i's and φ_i's may be different to start with. Then naturally one takes r to be the minimal value. Moreover, for a given pair of f_i, $i = 1, 2$, but different choices of $r \geq 1$, the resulting fibrations p_i, $i = 1, 2$, may in fact depend on the value of r.

Remark 2.3. It will become clear from the proof of Theorem 2.1 that the fibrations p_i are uniquely determined by f_i, h, and r. However, if one does not require the last property in the statement, i.e., that the "matching" functions φ_i are cohomologous to $\bar{\varphi}_i \circ p_i$, then the choice of fibrations, in general, is not unique. For example, there is always the trivial fibration whose fibers are points. In general there are finitely or infinitely many distinct smooth fibrations for a given expanding map and the maximal number of possible fibrations occurs when h is smooth. This maximal number of fibrations is determined by the linearization of f_i (see also Remark 7.2). There is also a naturally defined partial order on the set of fibrations with the trivial one being subordinate to any other fibration and the one given by Theorem 2.1 being the maximal one.

Remark 2.4. If one does not assume that the manifolds M_i are homeomorphic to nilmanolds then, instead of fibrations, the construction in the proof of Theorem 2.1 yields compact foliations $\mathcal{F}_i$, i.e., foliations with all leaves compact. Further, by improving the argument used to show that the leaves of $\mathcal{F}_i$ are compact, one can check that these foliations are generalized Seifert fibrations. The argument for compactness and the Seifert property of the foliation is independent of the classification of expanding maps. Example 7.8 (Klein bottle) shows that such foliations can indeed have exceptional leaves on infranilmanifolds, that is, they are not necessarily locally trivial fibrations. Hence the assumption that the manifolds M_i are homeomorphic to nilmanifolds is necessary. However, in practice, this assumption is not a big restriction. Indeed, by the classification, any manifold which supports an expanding map is homeomorphic to an infranilmanifold. Hence, one can always lift given expanding maps to finite nilmanifold covers and study the problem on the cover.

Remark 2.5. Recall that there exist expanding maps on exotic nilmanifolds, i.e., manifolds homeomorphic but not diffeomorphic (or even not PL-homeomorphic) to nilmanifolds [13, 14]. Our theorem applies to such examples. Moreover, by using Gromoll's filtration and following the strategy of [12], one can construct expanding map $f_1\colon M_1 \to M_1$

on a nilmanifold M_1 and an expanding map $f_2\colon M_2 \to M_2$ on an exotic nilmanifold M_2 in such a way that the fibrations $p_i\colon M_i \to \bar{M}_i$ are non-trivial, i.e., $\dim M_i > \dim \bar{M}_i > 0$. Also note that our theorem applies to the case when both M_1 and M_2 are exotic. We elaborate on this remark in Example 7.9.

An expanding linear endomorphism L of a d-dimensional torus M is called *irreducible* if the characteristic polynomial of the integer matrix defining L is irreducible over $\mathbb{Z}$; equivalently, L does not have non-trivial invariant rational subspaces. Recall that any expanding map $f\colon M \to M$ is conjugate to an expanding linear endomorphism L. We will say f is *irreducible* if L is irreducible.

Corollary 2.6. *Let M_i be manifolds homeomorphic to the d-dimensional torus. Assume that $f_i\colon M_i \to M_i$ are C^{r+1}-smooth, $r \geq 1$, expanding maps. Assume that they are conjugate via h. Also assume that f_1 is irreducible and that the entropy maximizing measure for f_1 is not absolutely continuous with respect to the Lebesgue measure. If* $\mathrm{Jac}(f_1^n)(x) = \mathrm{Jac}(f_2^n)(h(x))$ *for all* $x \in \mathrm{Fix}(f_1^n)$ *and n, then h is a C^r diffeomorphism.*

We make four remarks pertaining to this corollary.

Remark 2.7. The condition on the measure of maximal entropy can be detected from a pair of periodic points. Hence the space of expanding maps which satisfy this assumption is C^{r+1}-dense and C^1-open in the space of expanding maps.

Remark 2.8. The analogue of Corollary 2.6 for non-abelian nilmanifolds is vacuous. This is because every expanding linear map on a nilmanifold leaves invariant the fibration given by the center of the corresponding nilpotent Lie group. Indeed, the proof of Corollary 2.6 relies on absence of such fibrations (which is guaranteed by irreducibility in the toral case).

Remark 2.9. Recall that an *infratorus* M is a closed manifold covered by the torus $\mathbb{T}^d$. The deck transformations of the covering $\mathbb{T}^d \to M$ have the form $x \mapsto Qx + v$ and the linear parts Q form the so called *holonomy group* of M. We can define an expanding map $f\colon M \to M$ to be irreducible if its lift to $\mathbb{T}^d$ is irreducible. Then Corollary 2.6 holds for such irreducible expanding maps of infratori by first passing to the torus cover and then arguing in the same way.

However, the supply of irreducible examples of expanding endomorphisms of infratori which are not tori is rather limited. Notice that any $Q \neq \mathrm{id}$ from the holonomy group has 1 for an eigenvalue. Indeed, otherwise the corresponding affine map $x \mapsto Qx + v$ of the torus would have a fixed point by the Lefschetz formula. Further L acts on the holonomy group by conjugation. Hence, because the holonomy group is finite, for a sufficiently large k, L^k and Q commute, and hence L^k leaves invariant a non-trivial rational subspace, the eigenspace of eigenvalue 1 for Q. Hence all irreducible examples must become reducible after passing to a finite power. Still, examples like that exist and we present one in Example 7.7.

Remark 2.10. Define the *critical regularity* r_0 of f_1 by

$$r_0(f_1) = \min_{n\geq 1} \max_{x\in M_1} \frac{\log \|D_x f_1^n\|}{\log m(D_x f_1^n)}$$

where m is the conorm. Then by the argument of de la Llave [7, Section 6] one can rectify the loss of one derivative and bootstrap the regularity of the conjugacy. That is, if $r > r_0$ then the C^r conjugacy given by Corollary 2.6 is, in fact, C^{r+1}. The same observation applies to other statements in this section.

In fact, the number $r_0(f_1)$ admits an alternative expression

$$r_0(f_1) = \max_{p\in \mathrm{Per}(f_1)} \frac{\lambda^+(p)}{\lambda^-(p)}$$

where $\lambda^{\pm}(p)$ are, respectively, the largest and the smallest Lyapunov exponents of f_1 at p. Therefore $r_0(f_1)$ can be computed directly from the Lyapunov exponents along periodic orbits. To see that the two formulae give the same value $r_0(f_1)$ one can pass to the invertible solenoid diffeomorphism and apply the approximation result of [38].

Notice also that a priori it does not follow from the hypothesis of Corollary 2.6 that $r_0(f_1) = r_0(f_2)$, but a posteriori one obtains this equality from smoothness of the conjugacy.

We say that an expanding map $f: M \to M$ is *very non-algebraic* if for every $\lambda \in \mathbb{Z}$ and every m, $1 \leq m \leq \dim M$, there exists a periodic point x of period n such that λ^n is not an eigenvalue of the m-fold exterior power $\bigwedge^m D_x f^n$. Notice that this condition is open and dense.

Corollary 2.11. *Assume that $f_i: M_i \to M_i$ are C^{r+1}-smooth, $r \geq 1$, expanding maps. Assume that they are topologically conjugate, $f_1: M_1 \to M_1$ is very non-algebraic, and for every n-periodic point x of f_1,*

$$\mathrm{Jac}(f_1^n)(x) = \mathrm{Jac}(f_2^n)(h(x)).$$

Then h is a C^r diffeomorphism.

Remark 2.12. It will be clear from the proof that the very non-algebraic assumption can be weakened to asking that for $m = 1, \dots, \dim M$ if $\lambda \in \mathbb{Z}$ appears in the spectrum of

$$\bigwedge^m Df_{1*}$$

then λ^n does not appear in the spectrum of

$$\bigwedge^m D_x f_1^n$$

for some periodic point x, $x = f_1^n(x)$. Here f_{1*} stands for the expanding linear automorphism induced by f_1 on the nilpotent Lie group and Df_{1*} is the corresponding Lie algebra automorphism.

Note that the very non-algebraic condition prevents f_1 from being linear.

Given two linear maps $D_i: \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2$, we say that D_1 and D_2 have *disjoint spectrum* if for every $m = 1, \dots, d$, the m-th exterior powers $\bigwedge^m D_1$ and $\bigwedge^m D_2$ do not share any real eigenvalues. Given two periodic points $x = f^k(x)$ and $y = f^l(y)$ we say that they have *disjoint spectrum* if the differentials $D_x f^{kl}$ and $D_y f^{kl}$ have disjoint spectrum.

Corollary 2.13. *Assume that $f_i: M_i \to M_i$ are C^{r+1}-smooth, $r \geq 1$, expanding maps. Assume that they are conjugate and there exist f_1-periodic points x and y which have disjoint spectrum. If for every n-periodic point x of f_1 the Jacobians* $\mathrm{Jac}(f_1^n)(x)$ *and* $\mathrm{Jac}(f_2^n)(h(x))$ *coincide then f_1 is C^r-conjugate to f_2.*

Corollary 2.13 follows directly from Corollary 2.11 since the property of having two periodic points with disjoint spectrum directly implies the very non-algebraic property.

Recall that a homeomorphism is called *absolutely continuous* if it sends the Lebesgue measure to a measure which is absolutely continuous with respect to the Lebesgue measure.

Corollary 2.14. *Let $r \geq 1$. If two C^{r+1} very non-algebraic expanding maps are conjugate via an absolutely continuous homeomorphism h then h is in fact C^r-smooth.*

Corollary 2.14 follows directly from Corollary 2.11. Indeed, by ergodicity h must map the smooth absolutely continuous measure of f_1 to the smooth absolutely continuous measure for f_2. It follows that the Jacobians at corresponding periodic points must be equal.

3. Krzyżewski–Sacksteder theorem for expanding maps

Given a C^r, $r \geq 1$, expanding map $f: M \to M$ and a C^r potential $\varphi: M \to \mathbb{R}$ the *transfer operator* $\mathcal{L}_{\varphi,f}: C^k(M) \to C^k(M)$ given by

$$\mathcal{L}_{\varphi,f}\, w(x) = \sum_{y \in f^{-1}x} e^{\varphi(y)} w(y)$$

is defined for C^k functions w, where $k \leq r$. When no confusion is possible we abbreviate $\mathcal{L}_{\varphi,f}$ to $\mathcal{L}_{\varphi}$.

Theorem 3.1 (Ruelle–Perron–Frobenius/Krzyżewski–Sacksteder). *Let $f: M \to M$ be a C^r, $r \geq 1$, expanding map and let $\varphi: M \to \mathbb{R}$ be a C^r potential; let $0 \leq k \leq r$. Then the transfer operator $\mathcal{L}_{\varphi}: C^k(M) \to C^k(M)$ has a unique maximal positive eigenvalue e^c with*

$$\mathcal{L}_{\varphi} e^u = e^{c+u}.$$

The eigenfunction e^u is positive and is unique up to scaling. The eigenvalue e^c and the eigenfunction e^u are independent of the choice of $k \in [0, r]$. Furthermore, e^u is C^r-smooth.

Remark 3.2. Originally this theorem was established by Ruelle for a more general class of expanding maps and in Hölder regularity [30, 31] (see also [3, 1.7]). Sacksteder [32] and Krzyżewski [24] independently established regularity of the eigenfunction. Krzyżewski [25] handled the analytic case as well. We note that both Sacksteder and Krzyżewski only considered the case when $\varphi = -\log \operatorname{Jac}(f)$ because they were interested in regularity of the smooth invariant measure for f. However, the proofs work equally well for arbitrary smooth potentials. Note that the uniqueness of the eigenspace occurs already among continuous functions provided that the potential is at least Hölder.

Another comment is that when r is an integer, the proof of Sacksteder only yields $(r-1)+\text{Lip}$ regularity of the eigenfunction e^u. The C^r regularity of e^u was established by Szewc [35]; see also [4, Theorem 8.6.3] for an exposition in the one-dimensional case.

Corollary 3.3. *Let f and φ be as in Theorem 3.1. Then there exists a unique C^r-smooth function $\hat{\varphi}\colon M \to \mathbb{R}$ and a unique constant c given by Theorem 3.1 such that*

(1) *$\hat{\varphi} + c$ is cohomologous to φ;*

(2) *1 is the maximal eigenvalue of the transfer operator $\mathcal{L}_{\hat{\varphi}}$;*

(3) *$\mathcal{L}_{\hat{\varphi}} 1 = 1$.*

Proof. Let e^c be the maximal eigenvalue with eigenfunction e^u for $\mathcal{L}_\varphi$ given by Theorem 3.1,

$$\mathcal{L}_\varphi e^u = e^{c+u}.$$

Let $\hat{\varphi} = \varphi - c + u - u \circ f$. Then

$$\mathcal{L}_{\hat{\varphi}} 1 = 1.$$

It is also clear that 1 is the maximal eigenvalue of $\mathcal{L}_{\hat{\varphi}}$ since otherwise e^c would not be the maximal positive eigenvalue for $\mathcal{L}_\varphi$.

Further, assume that $c' \in \mathbb{R}$ and $\hat{\varphi}'$ continuous also satisfy the conclusion of the corollary with

$$\hat{\varphi}' = \varphi - c' + u' - u' \circ f.$$

Then by the same calculation we have

$$\mathcal{L}_\varphi e^{u'} = e^{c'+u'}$$

with $e^{c'}$ being the maximal positive eigenvalue. Hence, by the uniqueness part of Theorem 3.1 we find that $c = c'$ and $u = u'$. ∎

Such normalized potentials $\hat{\varphi}$ have recently been studied in the context of thermodynamic formalism [17].

Remark 3.4. The constant c equals the topological pressure $P(\varphi)$. It follows that if $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ then the maximal eigenvalue is the same for the corresponding operators and hence $(f_1, \hat{\varphi}_1) \sim (f_2, \hat{\varphi}_2)$. (But we will not use this fact.)

Remark 3.5. Let e^c be the maximal positive eigenvalue for $\mathcal{L}_\varphi$ with eigenfunction e^u and assume that e^w is another positive continuous eigenfunction for $\mathcal{L}_\varphi$, i.e., $\mathcal{L}_\varphi e^w = \sigma e^w$ for some $\sigma \in \mathbb{R}$. Then $w = u + k$ for some $k \in \mathbb{R}$ and σ is the maximal eigenvalue. Notice that it follows that condition (2) of Corollary 3.3 is automatic from condition (3) because a positive eigenfunction necessarily corresponds to the maximal eigenvalue. (But we will not use this fact.)

4. Proof of the Main Theorem: the torus case

The proof of Theorem 2.1 consists of two steps. The first step is to build the fibrations and the second step is to verify the posited property of the fibrations. In this section will prove Theorem 2.1 under the *additional assumption that the manifolds M_i are homeomorphic to a torus*. This assumption simplifies the construction of fibrations quite a bit. The second step is general and does not rely on the homotopy type of M_i. Building fibrations in the case when M_i are general nilmanifolds requires a more complicated argument that involves an inductive procedure on the degree of nilpotency of the fundamental group. This more general argument appears in Section 5.

4.1. Fibrations

We begin by explaining the construction of fibrations p_i, $i = 1, 2$, which appear in Theorem 2.1.

Recall that $h \circ f_1 = f_2 \circ h$ and consider the following space of pairs of smooth functions:

$$V = \{(\psi_1, \psi_2) \in C^r(M_1) \times C^r(M_2) : \psi_1 = \psi_2 \circ h\}.$$

This is a closed subspace of $C^r(M_1) \times C^r(M_2)$. Note that if $(\psi_1, \psi_2) \in V$ then $(\psi_1 \circ f_1, \psi_2 \circ f_2) \in V$. Also note that V always contains constants (c, c) and is an algebra. We denote by V_i the projection of V on $C^r(M_i)$, $i = 1, 2$.

Define the subspace fields $E_i(x) \subset T_x M_i$, $i = 1, 2$, by

$$E_i(x) = \bigcap_{\psi_i \in V_i} \ker d_x \psi_i .$$

Notice that if $x_n \to x$ as $n \to \infty$, then $\limsup E_i(x_n) \subset E_i(x)$. This property implies that $\dim E_i(x)$ is upper semicontinuous. Let $m_i = \min_{x \in M_i} \dim E_i(x)$; then upper semicontinuity implies that the set

$$U_i = \{x \in M_i : k_i(x) = m_i\}$$

is open.

Lemma 4.1. *$U_i = M_i$, $i = 1, 2$, and hence E_i is in fact a distribution.*

Proof. Let Γ_i^n be the group of deck transformations of the covering map $f_i^n\colon M_i \to M_i$, $i = 1, 2$. Deck transformations are C^r diffeomorphisms. By definition

$$\Gamma_i^n = \{T \in \mathrm{Diff}^r(M_i) : f_i^n \circ T = f_i^n\},$$

and hence $h_\#\colon \Gamma_1^n \to \Gamma_2^n$ given by $h_\#(T) = h \circ T \circ h^{-1}$ is an isomorphism. Indeed, if $T \in \Gamma_1^n$ then $f_2^n \circ (h \circ T \circ h^{-1}) = h \circ f_1^n \circ T \circ h^{-1} = h \circ f_1^n \circ h^{-1} = f_2^n$ and vice versa.

Now it is easy to see that V_i is Γ_i^n-invariant, that is, if $(\psi_1, \psi_2) \in V$ then $(\psi_1 \circ T, \psi_2 \circ h_\#(T)) \in V$ for all $T \in \Gamma_1^n$. Indeed,

$$\psi_2 \circ h_\#(T) \circ h = \psi_2 \circ h \circ T = \psi_1 \circ T.$$

Hence, for all $T \in \Gamma_i^n$ we have

$$\begin{aligned} E_i(T(x)) &= \bigcap_{\psi \in V_i} \ker d_{T(x)}\psi = \bigcap_{\psi \in V_i} \ker d_{T(x)}(\psi \circ T) = \bigcap_{\psi \in V_i} DT(\ker d_x\psi) \\ &= DT(E_i(x)). \end{aligned}$$

Hence E_i is Γ_i^n-invariant, and in particular U_i is Γ_i^n-invariant.

Because $\pi_1 M_i = \mathbb{Z}^d$ is abelian, the covering f_i^n is normal and $\Gamma_i^n(x) = f_i^{-n}(x)$. Hence the orbits $\Gamma_i^n(x)$ become arbitrarily dense as $n \to \infty$ and because U_i is open, for a sufficiently large n we have $U_i = \Gamma_i^n(U_i) = M_i$. ∎

It is easy to see now that the distributions E_i integrate to C^r foliations $\mathcal{F}_i$. Indeed, for every $x \in M_i$ there exist finitely many functions $\psi_i^1, \dots, \psi_i^{d-m_i} \in V_i$ such that

$$E_i(x) = \bigcap_{j=1}^{d-m_i} \ker d_x\psi_i^j.$$

Indeed, just take ψ_i^j such that $\{d_x\psi_i^j\}_j$ is a maximal linearly independent subset of $\{d_x\psi\}_{\psi \in V_i}$.

By continuity of $d\psi_i^j$ and since E_i has constant dimension, the same formula holds on a small neighborhood of x. That is, there exists a neighborhood $U_{i,x}$ of x such that

$$E_i(y) = \bigcap_{j=1}^{d-m_i} \ker d_y\psi_i^j \quad \text{for all } y \in U_{i,x}.$$

Therefore, by the implicit function theorem, the maps $\Psi_{i,x}\colon U_{i,x} \to \mathbb{R}^{d-m_i}$,

$$\Psi_{i,x}(y) = (\psi_i^1(y), \dots \psi_i^{d-m_i}(y)),$$

define a foliation atlas of a C^r foliation which is tangent to E_i. We denote these foliations by $\mathcal{F}_i$, $i = 1, 2$.

Lemma 4.2. *The leaves of $\mathcal{F}_i$ are compact. In fact, the leaf $\mathcal{F}_i(x)$ for $x \in M_i$ is the connected component of x of the intersection*

$$\bigcap_{\psi \in V_i} \psi^{-1}(\psi(x)).$$

Proof. Let ψ be a function in V_i and let $x \in M_i$. Then by the definition

$$T\mathcal{F}_i(y) = E_i(y) \subset \ker d_y \psi$$

for every $y \in \mathcal{F}_i(x)$. Hence ψ is constant on $\mathcal{F}_i(x)$ and $\mathcal{F}_i(x) \subset \psi^{-1}(\psi(x))$. Hence

$$\mathcal{F}_i(x) \subset \bigcap_{\psi \in V_i^r} \psi^{-1}(\psi(x)).$$

On the other hand, recall that, locally, for a sufficiently small neighborhood $U_{i,x} \ni x$ we have the foliation chart and hence

$$\mathcal{F}_i(x) \cap U_{i,x} = \Psi_{i,x}^{-1}(\Psi_{i,x}(x)) = \bigcap_{j=1}^{d-m_i} (\psi_i^j)^{-1}(\psi_i^j(x)) \cap U_{i,x} \supset \bigcap_{\psi \in V_i^r} \psi^{-1}(\psi(x)) \cap U_{i,x},$$

and the main claim of the lemma follows. ∎

Recall that for every function $\psi_1 \in V_1$ there is $\psi_2 \in V_2$ such that $\psi_2 \circ h = \psi_1$ and vice versa. This implies that $h(\mathcal{F}_1(x)) = \mathcal{F}_2(h(x))$ for every $x \in M_1$. Hence by invariance of domain we obtain $m_1 = m_2$, i.e., the dimensions of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are the same. Also note that Lemma 4.2 and $V_i \circ f_i \subset V_i$ immediately imply that $\mathcal{F}_i$ is invariant under f_i.

To conclude that compact C^r foliations $\mathcal{F}_i$ are, in fact, fibrations we need to rely on global structural stability of expanding maps and complete the argument on the "linear side". Namely, $h_i \circ f \circ h_i^{-1} = A\colon \mathbb{T}^d \to \mathbb{T}^d$ is an expanding endomorphism. Then $F = h_i(\mathcal{F}_i)$ is an A-invariant compact continuous foliation on $\mathbb{T}^d$. The action of Γ_i^n is conjugate via h_i to the translation action by the set $A^{-n}(\{x_0\})$, where x_0 is a fixed point of A which we identify with $0 \in \mathbb{T}^d$. Because the set $\bigcup_{n\geq 0} A^{-n}(\{x_0\})$ is dense in $\mathbb{T}^d$ we conclude that F is invariant under the $\mathbb{T}^d$-action on itself by translations. Hence, for all $y \in F(x_0)$ we have $y + F(x_0) = F(y + x_0) = F(y) = F(x_0)$ and $F(-y) = -y + F(x_0) = -y + F(y) = F(-y + y) = F(x_0)$; that is, $F(x_0)$ is a subgroup of $\mathbb{T}^d$. Also recall that $F(x_0)$ is compact and connected. Hence, one can easily check (or use Cartan's closed subgroup theorem) that $F(x_0)$ is a linearly embedded subtorus $\mathbb{T}^m \subset \mathbb{T}^d$. And because F is invariant under translations, we conclude that F is a linear fibration $\mathbb{T}^m \to \mathbb{T}^d \to \mathbb{T}^{d-m}$. It remains to recall that $\mathcal{F}_i = h_i^{-1}(F)$, and therefore $\mathcal{F}_i$ is a fibration whose fiber is homeomorphic to $\mathbb{T}^m$ and whose base $\bar{M}_i$ is a C^r manifold homeomorphic to $\mathbb{T}^{d-m}$.

Since h sends $\mathcal{F}_1$ to $\mathcal{F}_2$, it induces a homeomorphism $\bar{h}\colon \bar{M}_1 \to \bar{M}_2$. To see that $\bar{h}$ is smooth, consider foliation charts around x and $h(x)$, $x \in M_1$, given by

$$\Psi_{1,x}(y) = (\psi_{1,x}^1(y), \ldots, \psi_{1,x}^{d-m_1}(y)) \quad \text{and} \quad \Psi_{2,h(x)}(y) = (\psi_{2,h(x)}^1(y), \ldots, \psi_{2,h(x)}^{d-m_2}(y))$$

respectively. In these local coordinates $\bar{h}$ is given by $\bar{h}(\Psi_{1,x}(y)) = \Psi_{2,h(x)}(h(y))$. However, by definition, there exist C^r functions $\psi^j_{1,h(x)}$ which satisfy $\psi^j_{1,h(x)} = \psi^j_{2,h(x)} \circ h$, $j = 1, \dots, d - m_2$. Hence, $\bar{h}$ is given by

$$\bar{h}(\psi^1_{1,x}(y), \dots, \psi^{d-m_1}_{1,x}(y)) = (\psi^1_{1,h(x)}(y), \dots, \psi^{d-m_2}_{1,h(x)}(y))$$

and since $\Psi_{1,x}$ is a C^r submersion we conclude that $\bar{h}$ is C^r on a neighborhood of $p_1(x)$. A symmetric argument proves that $\bar{h}^{-1}$ is C^r.

4.2. *Second step of the proof of Theorem 2.1: verifying the matching property*

Finally, we need to show that if $(f_1, \varphi_1) \sim (f_2, \varphi_2)$ then the functions φ_i are cohomologous to functions in V_i.

By Corollary 3.3 there are C^r functions $\hat{\varphi}_i$ and constants $c_i \in \mathbb{R}$ such that φ_i is f_i-cohomologous to $\hat{\varphi}_i + c_i$, and also $\mathcal{L}_{\hat{\varphi}_1, f_1} 1 = 1$, $\mathcal{L}_{\hat{\varphi}_2, f_2} 1 = 1$. Moreover such $\hat{\varphi}_i$ and c_i are unique. We know that $\hat{\varphi}_2 \circ h$ is cohomologous to $\hat{\varphi}_1 + c_2 - c_1$. In fact, we will show that

$$\hat{\varphi}_2 \circ h = \hat{\varphi}_1.$$

By direct calculation, we have

$$(\mathcal{L}_{\hat{\varphi}_2, f_2} v) \circ h = \mathcal{L}_{\hat{\varphi}_2 \circ h, f_1}(v \circ h)$$

for every function v. In particular, for the constant function $v = 1$ we have

$$1 = 1 \circ h = (\mathcal{L}_{\hat{\varphi}_2, f_2} 1) \circ h = \mathcal{L}_{\hat{\varphi}_2 \circ h, f_1}(1 \circ h) = \mathcal{L}_{\hat{\varphi}_2 \circ h, f_1}(1).$$

Since $\hat{\varphi}_2 \circ h$ is cohomologous to φ_1 up to a constant we find that $\hat{\varphi}_2 \circ h = \hat{\varphi}_1$. Hence $(\hat{\varphi}_1, \hat{\varphi}_2) \in V$ and, by the definition of the foliations $\mathcal{F}_i$, we conclude that $\hat{\varphi}_i$ is constant on $\mathcal{F}_i$, $i = 1, 2$. It remains to set $\bar{\varphi}_i(p_i(x)) = \hat{\varphi}_i(x) + c_i$.

5. Proof of the Main Theorem: building fibrations on nilmanifolds

In this section we build the fibrations in the general case when the manifolds M_i are both homeomorphic to a nilmanifold N/Γ. Recall that, by classification, there is an expanding automorphism $A: N \to N$, $A(\Gamma) \subset \Gamma$, which induces an algebraic expanding map $N/\Gamma \to N/\Gamma$ topologically conjugate to $f_i: M_i \to M_i$, $i = 1, 2$. The rest of the proof of Theorem 2.1, that is, verification of the matching property of fibrations, was already done in the second half of Section 4.

Define the subspace fields $E_i(x) \subset T_x M_i$, $i = 1, 2$, and level sets as follows:

$$E_i(x) = \bigcap_{\psi \in V_i} \ker d_x \psi, \quad \mathcal{P}_i(x) = \bigcap_{\psi \in V_i} \psi^{-1}(\psi(x)),$$

and let $\mathcal{F}_i(x) = \mathrm{cc}_x(\mathcal{P}_i(x))$, where cc_x stands for the "connected component of x". Our goal is to show that the maps $\mathcal{F}_i$ are, in fact, C^r fibrations with fiber and base both homeomorphic to nilmanifolds.

Remark 5.1. If $\dim E_1(x) = 0$ at one point $x \in M_1$ then it is easy to conclude using the inverse function theorem that the conjugacy h is C^r on a neighborhood of x and then, using dynamics, that h is C^r globally. Thus the main interest of the proof to follow is in the case when $\dim E_i \geq 1$.

5.1. Algebraic lemmas

Recall that Γ is a lattice in a simply connected nilpotent Lie group N, and hence Γ is torsion free and nilpotent. Let $\Gamma^0 = \Gamma$ let $\Gamma^j = [\Gamma, \Gamma^{j-1}]$ be the lower central series. Denote by k the smallest number such that $\Gamma^{k+1} = \{0\}$. Recall that $A(\Gamma) \subset \Gamma$, and hence also $A(\Gamma^j) \subset \Gamma^j$. Now define the following lattice:

$$A^*\Gamma^j = A^{-1}(\Gamma^j) \cdot \Gamma.$$

Note that $A^*\Gamma^0 = A^{-1}(\Gamma)$ and $A^*\Gamma^{k+1} = \Gamma$. The following lemma implies that $A^*\Gamma^j$ is indeed a well-defined group.

Lemma 5.2. $A^{-1}(\Gamma^j) \cdot \Gamma = \Gamma \cdot A^{-1}(\Gamma^j)$*, $j = 0, \dots, k+1$.*

Proof. Let $\alpha \in \Gamma^j$ and $\gamma \in \Gamma$. Then

$$A^{-1}(\alpha)\gamma = A^{-1}(\alpha A(\gamma)) = A^{-1}(A(\gamma)\alpha c)$$

where c is a commutator, $c \in [\Gamma, \Gamma^j] = \Gamma^{j+1} \subset \Gamma^j$. Hence $A^{-1}(\alpha)\gamma = \gamma A^{-1}(\alpha c) \in \Gamma \cdot A^{-1}(\Gamma^j)$. This proves the inclusion $A^{-1}(\Gamma^j) \cdot \Gamma \subset \Gamma \cdot A^{-1}(\Gamma^j)$, and the reverse inclusion follows from a similar calculation. ∎

Lemma 5.3. *The group $A^*\Gamma^{j+1}$ is a normal subgroup of $A^*\Gamma^j$, $j = 0, \dots, k$.*

Proof. We will use a group element $\alpha\gamma \in A^{-1}\Gamma^j \cdot \Gamma$ to conjugate an element $\beta\delta \in A^{-1}\Gamma^{j+1} \cdot \Gamma$ and we will see that the result is in $A^*\Gamma^{j+1}$:

$$\alpha\gamma\beta\delta\gamma^{-1}\alpha^{-1} = (\alpha\gamma\alpha^{-1}\gamma^{-1})\gamma(\alpha\beta\alpha^{-1}\beta^{-1})\beta(\alpha\delta\gamma^{-1}\alpha^{-1}\gamma\delta^{-1})\gamma^{-1}\delta.$$

Indeed, we have written it as a product of elements in $A^*\Gamma^{j+1}$ and commutators from $[A^{-1}(\Gamma^j), \Gamma] = A^{-1}[\Gamma^j, A(\Gamma)] \subset A^{-1}[\Gamma^j, \Gamma] = A^{-1}(\Gamma^{j+1}) \subset A^*\Gamma^{j+1}$. ∎

We also recall that $\Gamma^j \subset A^{-1}\Gamma^j$ and $\Gamma \subset A^*\Gamma^j$ are finite index subgroups.

5.2. The setup on universal covers

The expanding maps f_i are conjugate to the algebraic expanding map via conjugacies $h_i \colon M_i \to N/\Gamma$. Let $x_i = h_i^{-1}(\mathrm{id}_N\Gamma)$ and let $\pi_i \colon (\tilde{M}_i, \tilde{x}_i) \to (M_i, x_i)$ be the universal covers, $i = 1, 2$. We denote by $\Gamma_i = \{T : \pi_i \circ T = \pi_i\} \simeq \Gamma$ the group of deck transformations of π_i, which we can also identify with the fundamental group $\pi_1(M_i, x_i)$. Next we lift h_i and f_i to the universal covers in such a way that $\tilde{h}_i(\tilde{x}_i) = \mathrm{id}_N$ and $\tilde{f}_i(\tilde{x}_i) = \tilde{x}_i$. Then we have

$$\tilde{h}_i \circ \tilde{f}_i = A \circ \tilde{h}_i, \quad \tilde{h} \circ \tilde{f}_1 = \tilde{f}_2 \circ \tilde{h},$$

where $\tilde{h} = \tilde{h}_2^{-1} \circ \tilde{h}_1 \colon \tilde{M}_1 \to \tilde{M}_2$. We also have $\tilde{f}_i \circ T = A(T) \circ \tilde{f}_i$ for $T \in \Gamma_i$.

The group $A^{-1}(\Gamma)$ acts on N by left translations $x \mapsto A^{-1}(T) \cdot x = (A^{-1} \circ T \circ A)(x)$, $T \in \Gamma$. Following the same idea as in Section 4 we can conjugate this action using $\tilde{h}_i$ and obtain actions on $\tilde{M}_i$:

$$A^{-1}\Gamma_i = \{\tilde{f}_i^{-1} \circ T \circ \tilde{f}_i : T \in \Gamma_i\}.$$

Furthermore, we can similarly consider the following groups for any $j = 0, \dots, k+1$:

$$A^{-1}\Gamma_i^j = \{\tilde{f}_i^{-1} \circ T \circ \tilde{f}_i : T \in \Gamma_i^j\},$$
$$A^*\Gamma_i^j = A^{-1}\Gamma_i^j \circ \Gamma_i = \{f_i^{-1} \circ T \circ f_i \circ S : T \in \Gamma_i^j,\ S \in \Gamma_i\}.$$

Clearly the actions of $A^*\Gamma_1^j$ and $A^*\Gamma_2^j$ are conjugate via $\tilde{h}$. Consider the orbits

$$\tilde{\mathcal{O}}_i^j(x) = A^*\Gamma_i^j(x), \quad x \in \tilde{M}_i.$$

It is immediate that the orbits are Γ_i-invariant: $\tilde{\mathcal{O}}_i^j(T(x)) = \tilde{\mathcal{O}}_i^j(x)$ for $T \in \Gamma_i$. Hence the projection of the orbit $\mathcal{O}_i^j(\pi_i(x)) = \pi_i(\tilde{\mathcal{O}}_i^j(x))$ is a finite set of cardinality $|A^*\Gamma^j/\Gamma|$. Further these partitions into orbits are invariant under the expanding maps: $\tilde{f}_i(\tilde{\mathcal{O}}_i^j(x)) \subset \tilde{\mathcal{O}}_i^j(\tilde{f}_i(x))$ and $f_i(\mathcal{O}_i^j(x)) \subset \mathcal{O}_i^j(f_i(x))$. Indeed, let $T \in \Gamma_i^j$ and $S \in \Gamma_i$; then

$$\tilde{f}_i((\tilde{f}_i^{-1} \circ T \circ \tilde{f}_i \circ S)(x)) = (T \circ \tilde{f}_i \circ S)(x) = (T \circ A(S))(\tilde{f}_i(x)) \in \tilde{\mathcal{O}}_i^j(\tilde{f}_i(x))$$

because $T \in \Gamma_i^j \subset A^{-1}\Gamma_i^j$.

Note that $\tilde{\mathcal{O}}_i^{k+1}(x)$ is just the Γ_i-orbit of x, and hence $\mathcal{O}_i^{k+1}(x) = \{x\}$; and $\tilde{\mathcal{O}}_i^0(x)$ is the $A^{-1}\Gamma_i$-orbit of x, and hence $\mathcal{O}_i^0(x) = f_i^{-1}(f_i(x))$, while $\mathcal{O}_i^j(x)$, $j = 1, \dots, k$, interpolate in between.

Remark 5.4. The fact that $\mathcal{O}_i^0(\pi_i(x))$ are not orbits of a finite deck group action on M_i is forcing us to work on the universal cover; cf. Section 4.

We now make an observation which will be very important in the sequel. Namely, we take any $n \geq 1$ and apply all prior constructions to the expanding maps f_1^n, f_2^n and the endomorphism A^n. In this way we obtain actions of $A^{-n}\Gamma_i^j$ and of $A^{*n}\Gamma_i^j = A^{-n}\Gamma_i^j \cdot \Gamma_i$ on the universal covers $\tilde{M}_i$, $i = 1, 2$, which are conjugate via $\tilde{h}$. Also, the action of $A^{*n}\Gamma_i^0 = A^{-n}\Gamma_i$ is conjugate via $\tilde{h}_i$ to the action by left translations by elements of $A^{-n}\Gamma$ on N. Hence we can consider the group

$$A^{-\infty}\Gamma = \bigcup_{n \geq 1} A^{-n}\Gamma,$$

which is a dense subgroup of N, and its versions $A^{-\infty}\Gamma_i$ acting on $\tilde{M}_i$, $i = 1, 2$.

5.3. Invariance of the level set partition

To set up an induction argument we introduce "interpolating" subspace fields and level sets as follows. Recall that

$$V = \{(\psi_1, \psi_2) \in C^r(M_1) \times C^r(M_2) : \psi_1 = \psi_2 \circ h\}.$$

Denote by $\tilde{V}$ the corresponding space of lifted pairs,

$$\tilde{V} = \{(\psi_1, \psi_2) \in C^r(\tilde{M}_1) \times C^r(\tilde{M}_2) : \psi_i \circ T = \psi_i \ \forall T \in \Gamma_i,\, i = 1, 2;\ \psi_1 = \psi_2 \circ \tilde{h}\},$$

and denote by $\tilde{V}_i$ the projection of V on the i-th coordinate.

Now consider the filtration $\tilde{V}^0 \subset \tilde{V}^1 \subset \cdots \subset \tilde{V}^{k+1} = \tilde{V}$ given by

$$\tilde{V}^j = \{(\psi_1, \psi_2) \in C^r(\tilde{M}_1) \times C^r(\tilde{M}_2) : \psi_i \circ T = \psi_i \ \forall T \in A^*\Gamma_i^j,\, i = 1, 2;\ \psi_1 = \psi_2 \circ \tilde{h}\}$$

As before, we will use $\tilde{V}_i^j$ to denote the projection of $\tilde{V}^j$ on the i-th coordinate. Let $V^j = (\pi_1^{-1}, \pi_2^{-1}) \circ \tilde{V}^j$, which is well defined due to equivariance. Hence we have the corresponding filtration on M_i, $V_i^0 \subset V_i^1 \subset \cdots \subset V_i^{k+1} = V_i$. Note that functions on V_i^j are precisely those functions from V_i which are constant on $\mathcal{O}_i^j(x)$, $x \in M_i$.

Define

$$\tilde{E}_i^j(x) = \bigcap_{\psi \in \tilde{V}_i^j} \ker d_x\psi, \quad \tilde{\mathcal{P}}_i^j(x) = \bigcap_{\psi \in \tilde{V}_i^j} \psi^{-1}(\psi(x)), \quad \tilde{\mathcal{F}}_i^j(x) = \mathrm{cc}_x(\tilde{\mathcal{P}}_i^j(x)).$$

In the same way define

$$E_i^j(x) = \bigcap_{\psi \in V_i^j} \ker d_x\psi, \quad \mathcal{P}_i^j(x) = \bigcap_{\psi \in V_i^j} \psi^{-1}(\psi(x)), \quad \mathcal{F}_i^j(x) = \mathrm{cc}_x(\mathcal{P}_i^j(x)).$$

Also given a set O define

$$\tilde{\mathcal{P}}_i^j(O) = \bigcup_{x \in O} \tilde{\mathcal{P}}_i^j(x),$$

and similarly define $\mathcal{P}_i^j(O)$.

Immediately from the definitions we have the following properties:

- $E_i^{k+1} = E_i$, $\mathcal{P}_i^{k+1} = \mathcal{P}_i$ and $\mathcal{F}_i^{k+1} = \mathcal{F}_i$;
- $\mathcal{P}_i^j$ and $\mathcal{F}_i^j$ are well-defined partitions of M_i, and $\tilde{\mathcal{P}}_i^j$ and $\tilde{\mathcal{F}}_i^j$ are well-defined partitions of $\tilde{M}_i$;
- $\tilde{E}_i^j$, $\tilde{\mathcal{P}}_i^j$ and $\tilde{\mathcal{F}}_i^j$ are $A^*\Gamma_i^j$-invariant;
- $Df_i(E_i^j(x)) \subset E_i^j(f_i(x))$, $f_i(\mathcal{P}_i^j(x)) \subset \mathcal{P}_i^j(f_i(x))$ and $f_i(\mathcal{F}_i^j(x)) \subset \mathcal{F}_i^j(f_i(x))$, $x \in M_i$, and similarly for $\tilde{E}_i^j$, $\tilde{\mathcal{P}}_i^j$ and $\tilde{\mathcal{F}}_i^j$;
- $h(\mathcal{P}_1^j(x)) = \mathcal{P}_2^j(h(x))$ and $h(\mathcal{F}_1^j(x)) = \mathcal{F}_2^j(h(x))$, and similarly for $\tilde{\mathcal{P}}_i^j$ and $\tilde{\mathcal{F}}_i^j$;
- $E_i(x) = E_i^{k+1} \subset E_i^k(x) \subset \cdots \subset E_i^0(x)$, $x \in M_i$, and similarly for $\tilde{E}_i^j$;
- $\mathcal{P}_i(x) = \mathcal{P}_i^{k+1}(x) \subset \mathcal{P}_i^k(x) \subset \cdots \subset \mathcal{P}_i^0(x)$, $x \in M_i$, and similarly for $\tilde{\mathcal{P}}_i^j$, $\mathcal{F}_i^j$ and $\tilde{\mathcal{F}}_i^j$;

- $\mathcal{O}_i^j(x) \subset \mathcal{P}_i^j(x)$, $x \in M_i$, and $\tilde{\mathcal{O}}_i^j(x) \subset \tilde{\mathcal{P}}_i^j(x)$, $x \in \tilde{M}_i$;
- $D\pi_i(\tilde{E}_i^j) = E_i^j$;
- $\pi_i^{-1}(\mathcal{P}_i^j(\pi_i(x))) = \tilde{\mathcal{P}}_i^j(x)$ and $\pi_i(\tilde{\mathcal{F}}_i^j(x)) \subset \mathcal{F}_i^j(\pi_i(x))$, $x \in M_i$.

Lemma 5.5. *For all $j = 0, 1, \dots, k$ and all $T \in A^*\Gamma_i^j$ we have $T(\tilde{\mathcal{P}}_i^{j+1}(x)) = \tilde{\mathcal{P}}_i^{j+1}(T(x))$ and $DT(\tilde{E}_i^j(x)) = \tilde{E}_i^j(T(x))$ for $x \in \tilde{M}_i$.*

Proof. It is sufficient to show that if $\psi \in \tilde{V}_i^{j+1}$ then $\psi \circ T \in \tilde{V}_i^{j+1}$ for all $T \in A^*\Gamma_i^j$, which implies that $\tilde{V}_i^{j+1} = \tilde{V}_i^{j+1} \circ T$. Indeed,

$$\tilde{\mathcal{P}}_i^{j+1}(T(x)) = \bigcap_{\psi \in \tilde{V}_i^{j+1}} \psi^{-1}(\psi(T(x))) = T\Big(\bigcap_{\psi \in \tilde{V}_i^{j+1}} (\psi \circ T)^{-1}\big((\psi \circ T)(x)\big)\Big)$$
$$= T\Big(\bigcap_{\psi \in \tilde{V}_i^{j+1}} \psi^{-1}(\psi(x))\Big) = T(\tilde{\mathcal{P}}_i^{j+1}(x)).$$

Similarly, $DT(\tilde{E}_i^j(x)) = \tilde{E}_i^j(T(x))$ (cf. the proof of Lemma 4.1).

Let $h_\#: A^*\Gamma_1^j \to A^*\Gamma_2^j$ be the isomorphism given by conjugation, $h_\#(T) = \tilde{h} \circ T \circ \tilde{h}^{-1}$. To complete the proof we have to show that if $(\psi_1, \psi_2) \in \tilde{V}_i^{j+1}$ then $(\psi_1 \circ T, \psi_2 \circ h_\#(T)) \in \tilde{V}_i^{j+1}$. We have

$$\psi_2 \circ h_\#(T) \circ \tilde{h} = \psi_2 \circ \tilde{h} \circ T = \psi_1 \circ T,$$

and it remains to check that the function $\psi_1 \circ T$, $T \in A^*\Gamma_i^j$, is $A^*\Gamma_i^{j+1}$-equivariant. Indeed, for $S \in A^*\Gamma_i^{j+1}$ we have

$$\psi_i \circ T \circ S = \psi_i \circ (T \circ S \circ T^{-1}) \circ T = \psi_i \circ T,$$

where the last equality holds because $T \circ S \circ T^{-1} \in A^*\Gamma_i^{j+1}$ by Lemma 5.3. ■

Lemma 5.6. *Let X and Y be finite subsets of M_i. If $X \cap \mathcal{P}_i^{j+1}(Y) = \emptyset$ then there exists a function $\psi \in V_i^{j+1}$ such that $\psi|_X = 0$ and $\psi|_Y = 1$.*

Proof. The proof is based on the observation that if $\psi \in V_i^{j+1}$ then $\varphi \circ \psi \in V_i^{j+1}$ for any C^r function $\varphi: \mathbb{R} \to \mathbb{R}$.

First consider the case when $X = \{x\}$ and $Y = \{y_1, \dots, y_p\}$. Then because $x \notin \mathcal{P}_i^{j+1}(Y)$ it can be separated from every point in Y by a function from V_i^{j+1}; that is, for all $t = 1, \dots, p$ there exists $\psi_t \in V_i^{j+1}$ such that $\psi_t(x) \neq \psi_t(y_t)$. By replacing ψ_t with an appropriate linear combination $A\psi_t + B$, we can assume that $\psi_t(x) = 0$ and $\psi_t(y_t) = 1$. Now let

$$\psi_x = \sum_{t=1}^{p} \psi_t^2.$$

Then $\psi_x(x) = 0$ and $\psi_x(y_t) \geq 1$ for all $t = 1, \dots, p$. Finally, we replace ψ_x with $\varphi \circ \psi_x$, where φ is a C^r function such that $\varphi(0) = 0$ and $\varphi(\xi) = 1$ for all $\xi \geq 1$. This completes the proof when $X = \{x\}$.

In the general case we have $X = \{x_1, \dots, x_q\}$ and $Y = \{y_1, \dots, y_p\}$ and we can apply the above construction to each x_s, $s = 1, \dots, q$, to obtain a function $\chi_s \in V_i^{j+1}$ such that $\chi_s(x_s) = 0$ and $\chi_s(y_t) = 1$ for all $t = 1, \dots, p$. Consider

$$\chi = \sum_{s=1}^{q} (1 - \chi_s)^2.$$

Obviously, $\chi(y_t) = 0$ for all t and $\chi(x_s) \geq 1$ for all s. We can use the C^r function φ to define the posited separating function as $\psi = 1 - \varphi \circ \chi$. ■

Lemma 5.7. *$\tilde{\mathcal{P}}_i^{j+1}(\tilde{\mathcal{O}}_i^j(x)) = \tilde{\mathcal{P}}_i^j(x)$ for $x \in \tilde{M}_i$ and $\mathcal{P}_i^{j+1}(\mathcal{O}_i^j(x)) = \mathcal{P}_i^j(x)$ for $x \in M_i$, for all $j = 0, 1, \dots, k$.*

Proof. The inclusion $\tilde{\mathcal{P}}_i^{j+1}(\tilde{\mathcal{O}}_i^j(x)) \subset \tilde{\mathcal{P}}_i^j(x)$ is straightforward. Indeed, $\tilde{\mathcal{O}}_i^j(x) \subset \tilde{\mathcal{P}}_i^j(x)$ and $\tilde{\mathcal{P}}_i^{j+1}(y) \subset \tilde{\mathcal{P}}_i^j(y) = \tilde{\mathcal{P}}_i^j(x)$ for all $y \in \tilde{\mathcal{O}}_i^j(x)$.

Assume the reverse inclusion does not hold. Then there exists a point x and $y \in \tilde{\mathcal{P}}_i^j(x)$ such that $y \notin \tilde{\mathcal{P}}_i^{j+1}(\tilde{\mathcal{O}}_i^j(x))$. By Lemma 5.5 the set $\tilde{\mathcal{P}}_i^{j+1}(\tilde{\mathcal{O}}_i^j(x))$ is $A^*\Gamma_i^j$-invariant and hence $\tilde{\mathcal{O}}_i^j(y) \cap \tilde{\mathcal{P}}_i^{j+1}(\tilde{\mathcal{O}}_i^j(x)) = \emptyset$. Then also $\mathcal{O}_i^j(\pi_i(y)) \cap \mathcal{P}_i^{j+1}(\mathcal{O}_i^j(\pi_i(x))) = \emptyset$ and we can apply Lemma 5.6 to $\mathcal{O}_i^j(\pi_i(y))$ and $\mathcal{O}_i^j(\pi_i(x))$, and obtain a function $\psi \in V_i^{j+1}$ such that $\psi|_{\mathcal{O}_i^j(\pi_i(x))} = 0$ and $\psi|_{\mathcal{O}_i^j(\pi_i(y))} = 1$. We now lift ψ to $\tilde{M}_i$ and consider the finite sum

$$\bar{\psi} = \sum_{[T] \in A^*\Gamma_i^j / A^*\Gamma_i^{j+1}} \psi \circ \pi_i \circ T.$$

Note that the summands are well-defined because $\psi \circ \pi_i \in \tilde{V}_i^{j+1}$, and hence are $A^*\Gamma_i^{j+1}$-equivariant. Notice that $\bar{\psi}|_{\tilde{\mathcal{O}}_i^j(x)} = 0$ and $\bar{\psi}|_{\tilde{\mathcal{O}}_i^j(y)} = |A^*\Gamma_i^j / A^*\Gamma_i^{j+1}| > 0$. Finally, notice that for any $[S] \in A^*\Gamma_i^j / A^*\Gamma_i^{j+1}$,

$$\bar{\psi} \circ S = \sum_{[T] \in A^*\Gamma_i^j / A^*\Gamma_i^{j+1}} \psi \circ \pi_i \circ T \circ S = \sum_{[T] \in A^*\Gamma_i^j / A^*\Gamma_i^{j+1}} \psi \circ \pi_i \circ T = \bar{\psi}.$$

Hence $\bar{\psi}$ belongs to $\tilde{V}_i^j$ and separates x and y, which yields a contradiction. ■

We finally arrive at the main lemma of this subsection. Recall that $\tilde{\mathcal{F}}_i = \tilde{\mathcal{F}}_i^{k+1}$.

Lemma 5.8. *For all $j = 0, 1, \dots, k$, $\tilde{\mathcal{F}}_i^j = \tilde{\mathcal{F}}_i$ and $\tilde{\mathcal{F}}_i$ is an $A^{-1}\Gamma_i$-invariant partition of M_i, $i = 1, 2$.*

Proof. Recall that cc_x stands for "connected component of x". Applying the previous lemma we have

$$\begin{aligned}\tilde{\mathcal{F}}_i^j(x) &= \mathrm{cc}_x(\tilde{\mathcal{P}}_i^j(x)) = \mathrm{cc}_x\Big(\bigcup_{y \in \tilde{\mathcal{O}}_i^j(x)} \tilde{\mathcal{P}}_i^{j+1}(y)\Big) \\ &= \mathrm{cc}_x\Big(\bigcup_{[T] \in A^*\Gamma_i^j / A^*\Gamma_i^{j+1}} T(\tilde{\mathcal{P}}_i^{j+1}(x))\Big) = \mathrm{cc}_x(\tilde{\mathcal{P}}_i^{j+1}(x)) = \tilde{\mathcal{F}}_i^{j+1}(x),\end{aligned}$$

where the first equality in the second line is due to invariance of $\tilde{\mathcal{P}}_i^{j+1}(y)$ under the action of $A^*\Gamma_i^{j+1}$. By induction on j we conclude that $\tilde{\mathcal{F}}_i^j(x) = \tilde{\mathcal{F}}_i^{k+1}(x)$. In particular, $\tilde{\mathcal{F}}_i^0 = \tilde{\mathcal{F}}_i$. It remains to recall that $\tilde{\mathcal{F}}_i^0$ is $A^*\Gamma_i^0 = A^{-1}\Gamma_i$-invariant. ∎

Using higher iterates of expanding maps we can prove, using exactly the same arguments, that the partitions $\tilde{\mathcal{F}}_i$ are invariant under the action of $A^{-n}\Gamma_i$ for all $n \geq 1$. Hence we have the following corollary. (Recall that $A^{-\infty}\Gamma_i = \bigcup_{n\geq 1} A^{-n}\Gamma_i$.)

Corollary 5.9. *The partition $\tilde{\mathcal{F}}_i$ of $\tilde{M}_i$ is invariant under the action of $A^{-\infty}\Gamma_i$.*

5.4. *Upgrading to a foliation*

Now we will prove that $\tilde{\mathcal{F}}_i$ is in fact a C^r-smooth foliation.

Consider the dimension function $\dim \tilde{E}_i$ and let $m_i = \min_{x\in\tilde{M}_i} \dim \tilde{E}_i(x)$. Pick a point $x \in \tilde{M}_i$ such that $\dim \tilde{E}_i(x) = m_i$. Then, by definition of $\tilde{E}_i(x)$, we can find functions $\psi_1, \dots, \psi_{d-m_i} \in \tilde{V}_i$ such that

$$\tilde{E}_i(x) = \bigcap_{j=1}^{d-m_i} \ker d_x\psi_j.$$

From the continuity of $d_x\psi_j$ and the fact that m_i is the minimal dimension, we also have the same formula

$$\tilde{E}_i(y) = \bigcap_{\psi\in\tilde{V}_i} \ker d_y\psi = \bigcap_{j=1}^{d-m_i} \ker d_y\psi_j$$

for all y in a sufficiently small open neighborhood B of x.

Consider the map $\Psi: B \to \mathbb{R}^{d-m_i}$ given by $\Psi(y) = (\psi_1(y), \dots, \psi_{d-m_i}(y))$. It is clear that the plaque $\Psi^{-1}(\Psi(y))$ is tangent to $\tilde{E}_i$ at every point of the plaque. By choosing B appropriately we may assume that the plaques $\Psi^{-1}(\Psi(y))$ are path-connected for all $y \in B$.

Lemma 5.10. *For every $y \in B$ we have $\Psi^{-1}(\Psi(y)) = \tilde{\mathcal{F}}_i(y) \cap B$.*

Proof. If a point $z \in B$ does not belong to a plaque $\Psi^{-1}(\Psi(y))$ then one of the functions ψ_j separates z and y. Hence $z \notin \tilde{\mathcal{P}}_i(y) \supset \tilde{\mathcal{F}}_i(y)$.

Now take $z \in \Psi^{-1}(\Psi(y))$ and consider any function $\psi \in \tilde{V}_i$. Connect z to y by a path. If $\psi(z) \neq \psi(y)$ then for some point q on the path the restriction of ψ to this path has non-zero derivative, and hence $\tilde{E}_i(q) \not\subset \ker d_q\psi$, giving a contradiction. Hence $\psi(z) = \psi(y)$ for all $\psi \in \tilde{V}_i$, which implies that $\Psi^{-1}(\Psi(y)) \subset \tilde{P}_i(y)$. Thus we also have $\Psi^{-1}(\Psi(y)) \subset \mathcal{F}_i(y)$ because $\Psi^{-1}(\Psi(y))$ is connected. ∎

Now the restriction $\tilde{\mathcal{F}}_i|_B$ is a foliation and we would like to spread the foliation structure to the whole $\tilde{M}_i$. For that we have to see that $A^{-n}\Gamma_i(B) = \tilde{M}_i$. If we have it, then using invariance under $A^{-n}\Gamma_i$ provided by Corollary 5.9, we can conclude that $\mathcal{F}_i$ has a C^r foliation structure in the neighborhood of every point. Hence, $\mathcal{F}_i$ is indeed a C^r foliation.

Recall that the action of $A^{-n}\Gamma_i$ on $\tilde{M}_i$ is conjugate via $\tilde{h}_i$ to the action by left translations by $A^{-n}(\Gamma) \subset N$ on N. To guarantee that $A^{-n}\Gamma_i(B) = \tilde{M}_i$ it suffices to choose a sufficiently large n so that the set $\tilde{h}_i(B)$ covers a fundamental domain of the lattice $A^{-n}(\Gamma)$.

5.5. Upgrading to a fibration and completing the proof

Now we know that both $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ are C^r foliations. Also, $\tilde{h}(\tilde{\mathcal{F}}_1) = \tilde{\mathcal{F}}_2$, and hence by invariance of domain, these foliations have the same dimension. It remains to show that $\pi_i(\tilde{\mathcal{F}}_i)$ (which is clearly also a C^r foliation) is, in fact, a C^r fibration. We also need to show that the fibers $\pi_i(\tilde{\mathcal{F}}_i)(x)$ and the bases of the fibrations are homeomorphic to nilmanifolds and that the induced conjugacy on the base is a C^r diffeomorphism. To do that we go to linearized dynamics on N/Γ, similarly to the argument in Section 4.

Let $\tilde{F} = \tilde{h}_1(\tilde{\mathcal{F}}_1) = \tilde{h}_2(\tilde{\mathcal{F}}_2)$. Then $\tilde{F}$ is a topological foliation with closed leaves which is invariant under the expanding automorphism $A\colon N \to N$. By Corollary 5.9 the foliation $\tilde{F}$ is also invariant under left translations by $A^{-\infty}(\Gamma)$. Because $\tilde{F}$ is continuous and $A^{-\infty}(\Gamma)$ is dense we conclude that $\tilde{F}$ is invariant by all left translations on N. This allows us to argue that $\tilde{F}(\mathrm{id}_N)$ is a group. Indeed, for all $x, y \in \tilde{F}(\mathrm{id}_N)$ we have $\tilde{F}(xy) = x\tilde{F}(y) = x\tilde{F}(\mathrm{id}_N) = \tilde{F}(x\mathrm{id}_N) = \tilde{F}(\mathrm{id}_N)$, and similarly for all $x \in \tilde{F}(\mathrm{id}_N)$ we have $\tilde{F}(x^{-1}) = x^{-1}\tilde{F}(\mathrm{id}_N) = x^{-1}\tilde{F}(x) = \tilde{F}(x^{-1}x) = \tilde{F}(\mathrm{id}_N)$. We can now apply Cartan's closed subgroup theorem (see e.g. [20]) to conclude that $\tilde{F}(\mathrm{id}_N)$ is a Lie subgroup of N.

So we denote the leaf through the identity by $G = \tilde{F}(\mathrm{id}_N)$. Hence, using translation invariance again, we conclude that $\tilde{F}$ is a smooth foliation by cosets of G.

Lemma 5.11. *Let F be the projection of $\tilde{F}$ on N/Γ. Then each leaf of F is either compact or it "accumulates on itself", that is, there exists $x \in N/\Gamma$ such that for an arbitrarily small neighborhood B of x the intersection $F(x) \cap B$ has infinitely many connected components.*

Proof. The leaves of F are orbits of the action of G, which is a nilpotent Lie group on N/Γ. So one can refer to Ratner theory, specifically to [29], which implies in particular that the closures of orbits of such a unipotent action are affinely embedded nilmanifolds. Hence each orbit is either compact or dense in its higher-dimensional closure, which implies the needed recurrence. It is also not hard to derive this lemma from earlier work of Parry on homogeneous flows on nilmanifolds: one needs to choose orbits which escape to infinity in non-compact leaves and use [28, Theorem 5].

However, the lemma can also be derived from more basic topological dynamics using work of Ellis and Furstenberg on distal actions [10, 16], which we proceed to explain. It is a well-known and simple fact that the nilpotent action of G on N/Γ is distal (this follows from the fact that a nil-translation is an iterated isometric extension). Based on work of Ellis, Furstenberg proved that a distal action can be decomposed into a disjoint union of minimal sets [16, Theorem 3.2]. Hence each leaf of F is either compact or has a non-trivial closure and is dense in the closure and hence recurrent. ■

Lemma 5.12. *There exists a non-empty open set $U \subset N/\Gamma$ such that each leaf of F that meets U is compact.*

Proof. We begin by noticing that the properties of being compact and of "accumulating on itself" are topological. Hence we have the property given by Lemma 5.11 on the non-linear side as well by applying h_i^{-1}: for all $x \in M_i$ either $\pi_i(\tilde{F}_i)(x)$ is compact or for all small neighborhoods B of x the intersection $(\pi_i(\tilde{F}_i)(x)) \cap B$ has infinitely many connected components.

Now we will argue in the same way as in Section 5.4, but on M_i instead of $\tilde{M}_i$. Let $U_i \subset M_i$ be the set where $\dim E_i$ achieves its minimum m_i. Recall that the dimension function $x \mapsto \dim E_i(x)$ is upper semicontinuous, which implies that U_i is open.

Take a point $x \in U_i$. Then we can construct a foliation chart for $\mathcal{F}_i$ about x which we denote by $\Psi\colon B \to \mathbb{R}^{d-m_i}$, $B \subset U_i$, such that $\Psi^{-1}(\Psi(x))$ is a connected subset of $\mathcal{F}_i(x)$ and for each $z \in B$ which does not belong to $\Psi^{-1}(\Psi(x))$ we have $z \notin \mathcal{P}_i(x)$ (see the proof of Lemma 5.10). Then we have

$$(\pi_i(\tilde{\mathcal{F}}_i)(x)) \cap B \subset \mathcal{F}_i(x) \cap B \subset \mathcal{P}_i(x) \cap B = \Psi^{-1}(\Psi(x)).$$

On the other hand, recalling that $D\pi_i(\tilde{E}_i) = E_i$ and by the discussion at the beginning of Section 5.4, $\pi_i(\tilde{\mathcal{F}}_i)$ is a $(d-m_i)$-dimensional foliation; that is, it has the same dimension as the plaque $\Psi^{-1}(\Psi(x))$, so $\pi_i(\tilde{\mathcal{F}}_i)(x) \cap B = \Psi^{-1}(\Psi(x))$. Hence $\pi_i(\tilde{\mathcal{F}}_i)(x) \cap B$ has only one connected component and we conclude, by the dichotomy of Lemma 5.11, that $\pi_i(\tilde{\mathcal{F}}_i)(x)$ is compact. Going back to the foliation F via h_i we obtain the same conclusion: all leaves of F which meet the non-empty open set $U = h_i(U_i)$ are compact. ∎

Lemma 5.13. *The group G is a normal subgroup of N.*

Proof. By Lemma 5.12 there exists a small open ball $B \subset U \subset N/\Gamma$ such that every leaf $F(x\Gamma) = Gx\Gamma$, $x \in B$, is compact. We consider the stabilizer group Γ_x of the leaf $\tilde{F}(x)$:

$$\begin{aligned}\Gamma_x &= \{\gamma \in \Gamma : Gx = Gx\gamma\} = \{\gamma \in \Gamma : x\gamma x^{-1} \in G\}\\ &= \Gamma \cap x^{-1}Gx \subset \Gamma.\end{aligned}$$

Thus $F(x\Gamma) = \tilde{F}(x)/\Gamma_x$ is homeomorphic to $x^{-1}Gx/\Gamma_x$, and hence Γ_x is a cocompact lattice in $x^{-1}Gx$, $x \in B$.

Now assume that for some $x_0, x_1 \in B$ we have $x_0^{-1}Gx_0 \neq x_1^{-1}Gx_1$. Then we can find a path $x_t \in B$, $t \in I$, such that $x_t^{-1}Gx_t$, $t \in I$, are all mutually distinct subgroups of N. (Indeed, just connect x_0 to x_1 by a path in B and then choose a small subpath in a neighborhood of a point where $x_t^{-1}Gx_t$ varies infinitesimally linearly with t.)

We can also see that all $\Gamma_{x_t} = \Gamma \cap x_t^{-1}Gx_t$ are mutually distinct. To see this, notice that because the exponential map provides a one-to-one correspondence between subalgebras of the Lie algebra of the simply connected nilpotent Lie group N and its connected Lie subgroups, the intersection subgroup $(x_t^{-1}Gx_t) \cap (x_s^{-1}Gx_s)$, $s \neq t$, is of codimension at least 1 in both $x_t^{-1}Gx_t$ and $x_s^{-1}Gx_s$. Because Γ_{x_t} is cocompact in $x_t^{-1}Gx_t$ it must have a non-trivial image in the non-compact quotient space $x_t^{-1}Gx_t/((x_t^{-1}Gx_t)\cap(x_s^{-1}Gx_s))$. Hence, indeed Γ_{x_t} contains elements which are not in Γ_{x_s}.

Thus we have obtained an uncountable family Γ_{x_t}, $t\in I$, of mutually distinct subgroups of Γ, which gives a contradiction. Indeed, Γ is finitely generated and nilpotent, and hence any subgroup of Γ is also finitely generated. So Γ only has countably many distinct subgroups. We conclude that $x_0^{-1}Gx_0 = x_1^{-1}Gx_1$ for all $x_0, x_1 \in B$. Hence $(x_1x_0^{-1})^{-1}Gx_1x_0^{-1} = G$ for all $x_1x_0^{-1}$ is a small neighborhood of id_N. And because such a neighborhood generates N, we conclude that G is normal. ■

Let $\Gamma_G = \Gamma\cap G$. Normality of G implies that $\Gamma_x = \Gamma_G$ for all x, and Γ_G is a cocompact lattice in G by Lemma 5.12. Again using normality of G (and of Γ_G in Γ) it is easy to check that the quotient homomorphism $N\to G\backslash N$ induces a well-defined fibration map $p\colon N/\Gamma\to \bar M$ with compact nilmanifold base $\bar M = (G\backslash N)/(\Gamma_G\backslash\Gamma)$ and nilmanifold fiber G/Γ_G. Conjugating back to M_i we obtain the posited fibrations $p_i\colon M_i\to \bar M_i$ whose fibers are the leaves of $\pi_i(\tilde{\mathcal{F}}_i)$ which are homeomorphic to G/Γ_G and whose base $\bar M_i$ is homeomorphic to $\bar M$. Note that p_i is C^r-smooth because we already know that $\pi_i(\tilde{\mathcal{F}}_i)$ is C^r.

It remains to check that $\bar h\colon \bar M_1\to\bar M_2$ induced by h is a C^r diffeomorphism. The argument follows closely the corresponding argument in Section 4. The only difficulty comes from the fact that we still do not know that $T(\pi_i(\tilde{\mathcal{F}}_i)) = E_i$ (and that $\mathcal{F}_i$ is a foliation). However, we do know, from the proof of Lemma 5.12, that $E_i = T\mathcal{F}_i = T(\pi_i(\tilde{\mathcal{F}}_i))$ on an open and dense set – the set where E_i achieves the minimal dimension m_i. Indeed, recall that m_i denotes the minimal dimension of E_i. We have shown that $m_1 = \dim\mathcal{F}_1 = \dim\mathcal{F}_2 = m_2$. Let $U_i = \{x\in M_i : \dim E_i(x) = m_i\}$. Recall that U_i is open (cf. Section 4.1). Further if $x\in U_i$ and $f_i(y) = x$ then, using $V_i\circ f_i\subset V_i$, we have

$$\begin{aligned} Df_i(E_i(y)) &= Df_i\Big(\bigcap_{\psi\in V_i}\ker d_y\psi\Big)\subset Df_i\Big(\bigcap_{\psi\in V_i}\ker d_y(\psi\circ f_i)\Big)\\ &= \bigcap_{\psi\in V_i} Df_i(\ker d_y(\psi\circ f_i)) = \bigcap_{\psi\in V_i}\ker d_x\psi = E_i(x). \end{aligned}$$

Hence, as $\dim E_i(x)$ is minimal, the above inclusion is in fact an equality and $y\in U_i$. We obtain $f_i^{-1}(U_i)\subset U_i$, which implies that U_i is dense in M_i. Therefore we can pick a point $x\in U_1$ such that $h(x)\in U_2$. Consequently, both points admit nice foliation charts and we can show, repeating verbatim the arguments of the last paragraph of Section 4.1, that $\bar h$ is a C^r diffeomorphism on a neighborhood B of $p_1(x)$.

Now recall that $\bar h$ conjugates the induced C^r expanding maps, $\bar h\circ\bar f_1 = \bar f_2\circ\bar h$. We can lift all the maps to the universal covers and express the lift of $\bar h$ as follows:

$$\tilde{\bar h} = \tilde{\bar f}_1^{\,n}\circ\tilde{\bar h}\circ\tilde{\bar f}_2^{\,-n}.$$

If $\tilde B$ is the lift of B, then the above equation implies that $\tilde{\bar h}$ is a C^r diffeomorphism on $\tilde{\bar f}_2^{\,n}(\tilde B)$. For a sufficiently large n the set $\tilde{\bar f}_2^{\,n}(\tilde B)$ contains a fundamental domain of the cover and we conclude that $\bar h$ is indeed a C^r diffeomorphism.

Remark 5.14. Once the proof is finished we can actually conclude that $\mathcal{F}_i = \pi_1(\tilde{\mathcal{F}}_i)$ and $E_i = T\mathcal{F}_i$. This fact would have been helpful in the proof, but it was out of reach and we can only obtain it a posteriori. To see that $\mathcal{F}_i = \pi_1(\tilde{\mathcal{F}}_i)$ and $E_i = T\mathcal{F}_i$ one can characterize V_i as the space of C^r functions which are constant on the fibers of p_i. Such a characterization easily follows from the fact that $\bar{h}$ is a C^r diffeomorphism. Note, however, that this fact is not needed in the statement of Theorem 2.1.

6. Proofs of corollaries

Proof of Corollary 2.6. We denote by L the linear endomorphism to which both f_1 and f_2 are conjugate.

By passing to the second iterate we may assume $\mathrm{Jac}(f_i) > 0$, $i = 1, 2$. Let $\varphi_i = -\log \mathrm{Jac}(f_i)$. By Theorem 2.1 we have C^r fibrations (with connected fiber) $p_i : M_i \to \bar{M}_i$ and functions $\bar{\varphi}_i : \bar{M}_i \to \mathbb{R}$ such that $\bar{\varphi}_i \circ p_i$ is cohomologous to φ_i and the induced conjugacy $\bar{h}: \bar{M}_1 \to \bar{M}_2$, $\bar{h} \circ p_1 = p_2 \circ h$, is a C^r diffeomorphism.

If $\dim \bar{M}_1 = 0$, then $\bar{\varphi}_1$ is constant, and hence φ_1 is cohomologous to a constant. Then the equilibrium state for φ_1, which is an absolutely continuous measure, equals the equilibrium state for the constant function, which is the entropy maximizing measure [3], contradicting the assumption of the corollary.

If $\dim \bar{M}_1 = d$, then p_1 and p_2 are diffeomorphisms (in fact, identity diffeomorphisms), and hence h is a C^r diffeomorphism since $h = p_2^{-1} \circ \bar{h} \circ p_1$.

It remains to consider the case when $0 < \dim \bar{M}_1 < d$. However, this is impossible due to irreducibility. Indeed, from the proof of Theorem 2.1 in Section 4.1 it is clear that L leaves invariant a torus of a positive dimension $m < d$, which contradicts irreducibility. We also provide an alternative self-contained short argument below.

Abbreviate $M = M_1$ and $\bar{M} = \bar{M}_1$. Let x be a fixed point of f_1 and let F be the fiber of p_1 which contains x. Recall that, by Theorem 2.1, $\bar{M}$ supports an expanding map $\bar{f}_1$, and hence is aspherical. Therefore the fundamental groups fit into the short exact sequence

$$0 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\bar{M}) \to 0.$$

Note that taking tensor product with $\mathbb{R}$ leaves the sequence exact.

Because $f_1(F) = F$ we have $(f_1)_*(\pi_1(F)) = L_*(\pi_1(F)) < \pi_1(F) < \pi_1(M) \simeq \mathbb{Z}^d$. Since $\dim \bar{M} < d$ we have $\dim F > 0$ and F is compact and also aspherical (because it supports the expanding map $f_1|_F$). It follows that $\pi_1(F) \otimes \mathbb{R}$ gives a non-zero rational invariant subspace for L. Because L is irreducible we conclude that $\pi_1(F) \otimes \mathbb{R} = \mathbb{R}^d$. Hence $\pi_1(\bar{M}) \otimes \mathbb{R} = 0$, i.e., $\pi_1(\bar{M})$ is a torsion finitely generated abelian group, hence finite. But any closed aspherical manifold of dimension > 0 has an infinite fundamental group, a contradiction. ∎

Proof of Corollary 2.11. By the classification of expanding maps, the manifolds M_i are homeomorphic to infranilmanifolds. Therefore we can pass to the nilmanifold covers and, accordingly, pass to the lifts of expanding maps. It is easy to see that the very non-algebraic assumption still holds for the lifted maps. From now on we assume that the manifolds M_i are homeomorphic to nilmanifolds, and hence Theorem 2.1 applies.

Recall that f_i are C^{r+1}, $r \geq 1$, very non-algebraic expanding maps and $h: M_1 \to M_2$ is a conjugacy. We apply Theorem 2.1 to f_i and r (not $r+1$!). Let $p_i: M_i \to \bar{M}_i$, $\bar{f}_i: \bar{M}_i \to \bar{M}_i$, $\bar{h}: \bar{M}_1 \to \bar{M}_2$ be the C^r maps given by Theorem 2.1. We shall show that $\dim \bar{M}_i = \dim M_i$. Then $h = \bar{h}$ and, by Theorem 2.1, the conjugacy is C^r.

Assume that $\dim M_i - \dim \bar{M}_i = m > 0$. Recall that by Theorem 2.1 the fibers $F_{i,x} = p_i^{-1}(p_i(x))$ are nilmanifolds, and hence are orientable. Moreover, the fibers can be simultaneously coherently oriented because the base space $\bar{M}_i$ is also an orientable nilmanifold. We fix a choice of orientation on the fibers and on the base. The expanding map f_i does not necessarily preserve any of the orientations. (And we cannot pass to finite iterates because such operation would not preserve the "very non-algebraic" condition.) Let $\boldsymbol{d}$ be the absolute value of the degree of the map between the fibers,

$$\boldsymbol{d} = \left|\deg(f_i|_{F_{i,x}}: F_{i,x} \to F_{i,f_i(x)})\right|.$$

Note that $\boldsymbol{d}$ is indeed independent of x by continuity and is independent of i because the maps f_i are conjugate. Further, if $\dim F_{i,x} > 0$ then $\boldsymbol{d} > 1$ because the expanding map on the fiber through a fixed point is a self-cover of degree > 1.

In the rest of the proof write $Jf := |\mathrm{Jac}(f)|$. Let $\psi_i = \log(Jf_i|_{\ker Dp_i})$. We note that these functions are only C^{r-1} because the distributions $\ker Dp_i$ are merely C^{r-1}.

First we pick Riemannian metrics on $\bar{M}_i$, $i = 1, 2$, so that $\bar{h}$ is volume preserving (e.g., an isometry), and hence

$$\log J\bar{f}_1 = \log J\bar{f}_2 \circ \bar{h}.$$

Then pick smooth connections $\mathcal{E}_i$ for p_i (subbundles transverse to $\ker Dp_i$) and then lift the Riemannian metrics from $\bar{M}_i$ to $\mathcal{E}_i$. Then consider Riemannian metrics on M_i which are direct sums of metrics on $\ker Dp_i$ and the lifted metrics on $\mathcal{E}_i$. By construction, the differentials Df_i have upper-triangular form and we have

$$\psi_i = \log Jf_i - \log J\bar{f}_i \circ p_i.$$

The Livshits theorem for expanding maps together with the assumption on Jacobians at periodic points implies that $\log Jf_1$ is cohomologous to $\log Jf_2 \circ h$. Note that $\log Jf_i$ is a C^r function. Hence, by the main property of $(p_i, \bar{f}_i, \bar{h})$ given by Theorem 2.1, $\log Jf_i$ is f_i-cohomologous to a C^r function which is constant on the fibers. Because $\log J\bar{f}_i \circ p_i$ is also constant on the fibers, it follows that ψ_i is cohomologous to $\bar{\psi}_i \circ p_i$.

In other words, there exists a C^r function u_i such that

$$\log(Jf_i|_{\ker Dp_i}) - u_i + u_i \circ f_i = \bar{\psi}_i \circ p_i.$$

Therefore by replacing the volume form ω on the fibers $F_{1,x}$ with the volume form

$$\bar{\omega} = e^{u_1}\omega$$

we can assume that the absolute value of the Jacobian of $f_1|_{\ker Dp_1}$ equals $e^{\bar{\psi}_1 \circ p_1}$. Denote by $\mathrm{vol}(F_{1,x})$ the total $\bar{\omega}$-volume of $F_{1,x}$. For any $x \in M_1$ we have

$$e^{\bar{\psi}_1 \circ p_1(x)} = \frac{1}{\mathrm{vol}(F_{1,x})} \int_{F_{1,x}} e^{\bar{\psi}_1 \circ p_1} \bar{\omega} = \frac{1}{\mathrm{vol}(F_{1,x})} \int_{F_{1,x}} J f_1|_{\ker Dp_1} \bar{\omega} = \boldsymbol{d}.$$

Hence for every periodic point x, with $f_1^k(x) = x$ we have $J f_1^k|_{\ker Dp_1} = \boldsymbol{d}^k$. This means that either $\boldsymbol{d}^k$ or $(-\boldsymbol{d})^k$ belongs to the spectrum of $\bigwedge^m Df_1^k(x)$, which contradicts f_1 being very non-algebraic. We conclude that $m = 0$, i.e., $\dim \bar{M}_1 = \dim M_1$ and we are done. ■

Note that even though the regularity of f_i is $r + 1$, we use Theorem 2.1 with regularity r because we work with the Jacobian of f_i.

7. Examples

Example 7.1 (Basic example). Here we give an explicit example where non-trivial fibrations $p_i \colon M_i \to \bar{M}_i$, $i = 1, 2$, with $\dim \bar{M}_i \neq 0, \dim M_i$ appear. Consider the expanding maps $L, f \colon \mathbb{T}^2 \to \mathbb{T}^2$ given by $L(x, y) = (2x, 2y)$ and $f(x, y) = (g(x), 2y)$, where g is conjugate to the $\times 2$ map via a nowhere differentiable conjugacy h_0, $h_0 \circ g = 2h_0$. For simplicity we may assume that $g(0) = 0$ and $g'(0) < 2$. Then $h = (h_0, \mathrm{id}_{S^1})$ is the conjugacy between f and L. Recall that the fibrations p_i arise from the space of pairs of C^r functions (ψ_1, ψ_2) which satisfy $\varphi_1 = \psi_2 \circ h$, i.e.,

$$\psi_1(x, y) = \psi_2(h_0(x), y).$$

Clearly any C^r function $\psi_1(x, y) = \psi(y)$ belongs to this space. We will show that these are the only functions which could appear. Then it immediately follows that $p_1(x, y) = p_2(x, y) = y$, that is, p_i are circle fibrations over S^1.

Denote by $\partial_{\inf} h_0$ the lower derivative of h_0 defined via lim inf. All periodic points which spend a sufficiently large proportion of time near 0 have Lyapunov exponent $< \log 2$. Such periodic points p are dense in S^1 and it is easy to see that $\partial_{\inf} h_0(p) = 0$ for any such p. Hence differentiating the relation between ψ_1 and ψ_2 with respect to x yields

$$\frac{\partial}{\partial x} \psi_1(p, y) = \frac{\partial}{\partial x} \psi_2(h_0(p), y) \partial_{\inf} h_0(p) = 0$$

for a dense set of p. Hence, indeed, ψ_1 and ψ_2 are functions of y only.

Remark 7.2. Any primitive vector $(m, n) \in \mathbb{Z}^2$ yields a fibration $S^1 \to \mathbb{T}^2 \to S^1$ whose fibers in the universal cover $\mathbb{R}^2$ are lines parallel to the vector (m, n). This gives infinitely different fibrations each of which is preserved by the conformal map L from Example 7.1. Further, similarly to the construction of Example 7.1, one can construct perturbations $f_{(m,n)}$ such that the fibration given by Theorem 2.1 is precisely the fibration coming from (m, n).

Example 7.3 (de la Llave example). A non-trivial fibration may appear in a more subtle way when full periodic Jacobian data match. Of course, this can only happen for expanding maps which are not very non-algebraic. The example presented here is due to de la Llave [7].

Consider the maps

$$L(x, y) = (dx, ay), \quad d \geq 2, a \geq 2,$$
$$f(x, y) = (dx + \alpha(y), ay).$$

Then the conjugacy between L and f has the form

$$h(x, y) = (x + \beta(y), y)$$

where β can be expressed explicitly as the series [7]

$$\beta(y) = \frac{1}{d} \sum_{i \geq 0} \frac{1}{d^i} \alpha(a^i y).$$

Notice that β is a Weierstrass function. Let

$$r_0 = \frac{\log d}{\log a}$$

and let $r_0 = n + \theta$ where $n \in \mathbb{N}_0$ and $\theta \in (0, 1]$.

To analyze the regularity of β there are several cases to consider which give different answers.

Lemma 7.4. *Assume that $\alpha \in C^r$, $r = k + \delta$ with $k \in \mathbb{N}_0$ and $\delta \in [0, 1)$ and let $r_0 = n + \theta$ with $n \in \mathbb{N}_0$ and $\theta \in (0, 1]$. Consider the following cases:*

Case I: *$r < r_0$ and $\beta \in C^r$,*

Case II: *$r > r_0$, $r_0 \notin \mathbb{N}$ and $\beta \in C^{r_0}$,*

Case III: *$r > r_0$, $r_0 = n + 1 \in \mathbb{N}$ and $\beta \in C^{n+x|\log x|}$,*

Case IV: *$r = r_0$ and $\beta \in C^{n+x^\theta|\log x|}$.*

In all cases, there is a generic set of $\alpha \in C^r$ where the regularity is optimal, in particular for such α, we have $\beta \notin C^{r_0+\varepsilon}$ for any $\varepsilon > 0$.

Proof. We give the proof for Case IV, all other cases being analogous.

By termwise differentiation we have

$$\beta^{(n)}(y) = \frac{1}{d} \sum_{i \geq 0} \left(\frac{a^n}{d} \right)^i \alpha^{(n)}(a^i y),$$

which is convergent because $r_0 > n$. Comparing the series for β and $\beta^{(n)}$, we can clearly assume $n = 0$ because the argument for $n > 0$ would be the same with $\beta^{(n)}$ in place of β.

Let $A = \max|\alpha|$ and C the θ-Hölder constant for α. Take $x \neq y$ and let N be such that

$$\frac{1}{a^{N-1}} \leq |x-y| \leq \frac{1}{a^N}.$$

Then

$$|\beta(x)-\beta(y)| \leq \sum_{k=0}^{N-1} \frac{1}{a^{\theta k}} |\alpha(a^k x) - \alpha(a^k y)| + \sum_{k \geq N} \frac{1}{a^{\theta k}} |\alpha(a^k x) - \alpha(a^k y)|.$$

The first summand is smaller than

$$C \sum_{k=0}^{N-1} \frac{1}{a^{\theta k}} a^{\theta k} |x-y|^\theta \leq CN|x-y|^\theta \leq C|x-y|^\theta \left|\log|x-y|\right|.$$

The second summand is smaller than

$$\frac{2A}{1-a^\theta} \frac{1}{a^{N\theta}} \leq C|x-y|^\theta.$$

Hence we obtain the posited $x^\theta|\log x|$ modulus of continuity for β.

On the other hand, assume, to simplify notation, that $\alpha(0) = 0$, and say $\alpha(x) > 0$ for $x > 0$, and $\liminf_{x \to 0} \frac{|\alpha(x)|}{|x|^\theta} > 0$. Pick $\varepsilon_0 > 0$ sufficiently small so that $K = \inf_{|x|<\varepsilon_0} |\alpha(x)|/|x|^\theta > 0$. Then, taking $x > 0$ very close to 0 and $N > 0$ first such that $a^N x \geq \varepsilon_0$, we obtain

$$\begin{aligned} |\beta(x)-\beta(0)| &\geq \sum_{k=0}^{N-1} \frac{1}{a^{\theta k}} \alpha(a^k x) - \sum_{k \geq N} \frac{1}{a^{\theta k}} \alpha(a^k x) \\ &\geq \left(KN - \frac{2A}{1-a^\theta}\right)|x|^\theta \geq \left(KC_{\varepsilon_0}\left|\log|x|\right| - \frac{2A}{(1-a^\theta)\varepsilon_0^\theta}\right)|x|^\theta. \end{aligned}$$

So, by taking x close enough to 0 we see that β is not C^θ at 0.

Now notice that

$$\beta(x) = \frac{1}{d}\beta(ax) + \frac{1}{d}\alpha(x)$$

and $\alpha \in C^{r_0}$. Let S be the set of x such that β is not C^{r_0} at x. Then, from the above equation, if $ax \in S$ then $x \in S$, i.e., S is backward invariant (and non-empty) and hence dense.

To show the genericity property we will use the following functional analysis lemma (a consequence of the proof of the open mapping theorem).

Lemma 7.5. *Let X and Y be Banach spaces and $L\colon X \to Y$ be a bounded linear map. Then either L is onto or the image of L is a first category set.*

Now consider $X = Y = C^\theta$ and $L(\beta) = d\beta - \beta \circ a$. The image of L is the set of $\alpha \in C^\theta$ such that the corresponding β belongs to C^θ. We have just showed that L is not surjective, hence the set of α such that the corresponding β belongs to C^θ is a first category set and so its complement is a second category set. ∎

Remark 7.6. Notice that if $r_0 \in \mathbb{N}$, then $d = a^{r_0}$. Further, if $\beta \in C^{r_0}$ then we can differentiate the above equation to find that $\beta^{(r_0)}$ solves the equation

$$d\beta^{(r_0)}(x) - a^{r_0}\beta^{(r_0)}(ax) = \alpha^{(r_0)}(x),$$

meaning that $\alpha^{(r_0)}/d$ is cohomologous to 0, which does not happen for generic α.

With Lemma 7.4 at hand we return to the discussion of Example 7.3. If we apply Theorem 2.1 to L, f and $r < r_0$ then the fibrations p_i are trivial with point fibers.

If $r \geq r_0$ we will have $p_1(x, y) = p_2(x, y) = y$, i.e., we get fibrations with circle fibers. Let us show this fact.

Differentiating $\psi_1(x, y) = \psi_2(x + \beta(y), y)$ with respect to y yields

$$\partial_y\psi_1(x, y) = \partial_y\psi_2(x + \beta(y), y) + \partial_x\psi_2(x + \beta(y), y)\beta'(y).$$

Notice that $\partial_x\psi_2(x + \beta(y), y)$ is C^{r_0-1}, and in particular continuous. Let

$$U = \{y : \partial_x\psi_2(x + \beta(y), y) \neq 0 \text{ for some } x\}.$$

Then U is open and for $y \in U$ and appropriate x,

$$\beta'(y) = \frac{\partial_y\psi_1(x, y) - \partial_y\psi_2(x + \beta(y), y)}{\partial_x\psi_2(x + \beta(y), y)}.$$

So, for $y \in U$ the right hand side is locally C^{r_0-1} and hence β' is locally C^{r_0-1} in U. Hence by Lemma 7.4, U is empty and hence $\frac{\partial}{\partial x}\psi_2(x + \beta(y), y) = 0$ for all x and a dense set of $y \in S^1$. We conclude that ψ_2 (and similarly ψ_1) depends solely on the y-coordinate.

Example 7.7 (Irreducible automorphism of an infratorus). We have explained in Remark 2.9 that (non-trivial) infratori do not support totally irreducible affine automorphisms. Here we show that one can still construct irreducible examples (which become reducible after passing to a finite iterate).

Define an expanding endomorphism of $\mathbb{T}^3$ by

$$L = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that L^3 is diagonal. Define the holonomy group $\{\mathrm{id}, \gamma_1, \gamma_2, \gamma_3\}$ as follows:

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, let

$$v_1 = (1/2, 0, 1/2)^t, \quad v_2 = (1/2, 1/2, 0)^t, \quad v_3 = (0, 1/2, 1/2)^t,$$

and $T_i(x) = \gamma_i(x) + v_i$, $i = 1, 2, 3$. We let Γ be the group of affine diffeomorphisms of $\mathbb{T}^3$ generated by the T_i's. It is easy to see that, in fact, $\Gamma = \{\mathrm{id}_{\mathbb{T}^3}, T_1, T_2, T_3\}$, and hence Γ acts freely on $\mathbb{T}^3$.

Finally, $L\Gamma L^{-1} \subset \Gamma$, and hence induces an expanding endomorphism of the infratorus $\mathbb{T}^3/\Gamma$. Indeed, $L \circ T_1 \circ L^{-1} = T_2 + (1,0,0)^t$, $L \circ T_2 \circ L^{-1} = T_3$ and $L \circ T_3 \circ L^{-1} = T_1 + (1,0,0)^t$.

Example 7.8 (Seifert fibration). Recall that in Theorem 2.1 we assume that the manifolds M_i are homeomorphic to nilmanifolds. If they are not, then the construction of compact foliations in the proof of Theorem 2.1 still works, but these foliations might fail to be fibrations. The example below illustrates this point.

Consider the Klein bottle $\mathbb{K}$ given as a quotient of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by the involution $T(x, y) = (x + 1/2, -y)$. We can also model $\mathbb{K}$ as the rectangle $[0, 1/2] \times [-1/2, 1/2]$ where the sides are identified by $(x, y) \mapsto (x - 1/2, -y)$ and $(x, y) \mapsto (x, y + 1)$. One can easily check that the expanding linear map

$$L = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

induces an expanding map $L: \mathbb{K} \to \mathbb{K}$. We foliate $\mathbb{K}$ by horizontal curves $\{y = \text{const}\}$. More precisely, for every $y \in [-1/2, 1/2]$ define the circles

$$\mathcal{C}_y = \{(t, y) : t \in [0, 1/2]\} \cup \{(t, -y) : t \in [0, 1/2]\}.$$

Notice that if $y \neq 0, 1/2$ then $\mathcal{C}_y$ consists of two segments on the rectangle. For $y = 0$, $\mathcal{C}_0$ is a singular curve that consists of only one segment $[0, 1/2] \times \{0\}$ and hence has half the length of the other leaves. The same happens for $y = 1/2$: $\mathcal{C}_{1/2}$ is a singular curve that consists of only one segment $[0, 1/2] \times \{1/2\} \sim [0, 1/2] \times \{-1/2\}$. Moreover, $\mathcal{C}_y = \mathcal{C}_{-y}$.

We have defined a foliation on $\mathbb{K}$ which is obviously not a fibration. Indeed, the quotient map $\pi: \mathbb{K} \to S^1/[y \sim -y]$ yields an orbifold structure on $S^1/[y \sim -y]$.

Notice that $L(\mathcal{C}_y) = \mathcal{C}_{(2y \bmod 1)}$, hence the foliation is L-invariant.

We now define expanding maps $f_i: \mathbb{K} \to \mathbb{K}$, $i = 1, 2$. We let $f_i(x, y) = (g_i(x), 2y)$, where $g_i(x) = 3x + \alpha_i(x)$ with $\alpha_i(0) = 0$ and $\alpha_i(x + 1/2) = \alpha_i(x)$ for every $x \in S^1$. Such formulae define maps on the Klein bottle which are homotopic to L. Moreover, these maps are expanding provided that the C^1 norms of α_i are sufficiently small. Also notice that the maps f_i preserve the foliation $\mathcal{C}$.

The conjugacy h between f_1 and f_2, $h \circ f_1 = f_2 \circ h$, has the form $h(x, y) = (h_0(x), y)$, where $h_0 \circ g_1 = g_2 \circ h_0$. Notice that by the symmetries of f_i, $h_0(x + 1/2) = h_0(x) + 1/2$, and hence h is indeed a conjugacy on the Klein bottle. We can assume that the functions α_i are chosen so that h_0, and hence h, is not C^1.

Take any $\varphi_0: \mathbb{R} \to \mathbb{R}$ such that $\varphi_0(y + 1) = \varphi_0(y)$ and $\varphi_0(-y) = \varphi_0(y)$, e.g., $\varphi_0(y) = \cos 2\pi y$. Then $\varphi(x, y) = \varphi_0(y)$ defines a function on $\mathbb{K}$ and $\varphi \circ h = \varphi$. On the other hand, if $\varphi_1 = \varphi_2 \circ h$ for some smooth functions φ_1 and φ_2 then both φ_1 and φ_2 must be constant on the leaves of $\mathcal{C}$ because h_0 is non-differentiable on a dense set of $x \in S^1$. So defining

$\varphi_i = \varphi$ for $i = 1, 2$ we are under the hypotheses of Theorem 2.1. We conclude that $\mathcal{C}$ is precisely the compact foliation given by the construction in the proof of Theorem 2.1.

Example 7.9 (Exotic examples). Here we explain that the fiber bundle structure given by Theorem 2.1 could be non-trivial even when the ambient manifold is an exotic torus. Examples of expanding maps on exotic tori were first constructed by Farrell and Jones [14] in dimensions $d \geq 7$. We explain how, with some extra care, the beautiful construction of Farrell–Jones can be adapted to our setting.

Let Σ^d be a d-dimensional, $d \geq 7$, homotopy sphere and let $\mathbb{T}^d$ be the standard torus. A simple way of constructing an exotic torus is by taking the connected sum $\mathbb{T}^d \# \Sigma$. If Σ^d is not homeomorphic to the standard sphere then $\mathbb{T}^d \# \Sigma^d$ is not homeomorphic to $\mathbb{T}^d$ [37, §15A]. Further, it is well-known that for $d \geq 7$, one can realize $\mathbb{T}^d \# \Sigma^d$ as $\mathbb{T}^d$ with a disk $\mathbb{D}^d$ removed and then glued back in using an orientation preserving "twist diffeomorphism" $\varphi \in \mathrm{Diff}(\mathbb{S}^{d-1})$,

$$\mathbb{T}^d \# \Sigma^d = (\mathbb{T}^d \setminus \mathbb{D}^d) \cup_\varphi \mathbb{D}^d.$$

It is easy to check that if φ' is isotopic to φ then the corresponding exotic tori are diffeomorphic.

We view the sphere $\mathbb{S}^{d-1} = \partial\mathbb{D}^d$ as the standard sphere in $\mathbb{R}^d$,

$$\mathbb{S}^{d-1} = \Big\{(x_1, \ldots, x_d) : \sum_i x_i^2 = 1\Big\}.$$

Cerf [5] showed that for every homotopy sphere Σ^d one can realize $\mathbb{T}^d \# \Sigma^d$ using a diffeomorphism $\varphi: \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ which preserves the first coordinate, i.e., has the form

$$\varphi(x_1, x_2, \ldots, x_d) = (x_1, x_2', x_3', \ldots, x_d').$$

Then φ can viewed as a path of diffeomorphisms and gives a representative of an element of $\pi_1(\mathrm{Diff}(\mathbb{S}^{d-2}))$. More generally, one can consider the space $\mathrm{Diff}_k(\mathbb{S}^{d-1})$ of orientation preserving diffeomorphisms which preserve the first k coordinates $x_1, \ldots, x_k$, and hence give an element of $\pi_k(\mathrm{Diff}(\mathbb{S}^{d-1-k}))$. Isotopy classes of such diffeomorphisms form a subgroup Γ^d_{k+1} of the group Θ_d of isotopy classes of all orientation preserving diffeomorphisms (which is identified with the group of homotopy spheres equipped with the connected sum operation). It is known that Γ^d_{k+1} is non-trivial in a certain range of pairs (k, d) [1].

Now we formulate the extra property of $\varphi \in \mathrm{Diff}_k(\mathbb{S}^{d-1})$ which we will need (and which is not needed in the original Farrell–Jones construction). Consider the obvious homomorphism

$$\gamma: \pi_k(\mathrm{Diff}(\mathbb{S}^{d-1-k})) \to \pi_0(\mathrm{Diff}(\mathbb{S}^{d-1})) \simeq \Theta_d.$$

Lemma 7.10 ([2, Proposition 1.2.3; §1.3]). *There exist pairs* (k, d)[2] *and a torsion element* $[\varphi] \in \pi_k(\mathrm{Diff}(\mathbb{S}^{d-1-k}))$, $[\varphi^p] = 0$, *whose image in* $\pi_0(\mathrm{Diff}(\mathbb{S}^{d-1}))$ *is non-trivial, i.e.,* $\gamma[\varphi] \neq 0$.

[2] For specific arithmetic conditions see [2, Corollary 1.3.6].

We proceed to briefly recall the Farrell–Jones construction [11, 14] and then explain how the above lemma allows us to produce an exotic example which admits invariant fibrations with $(d-k)$-dimensional fibers. The construction yields the $\times s$ map on $\pi_1(\mathbb{T}^d \# \Sigma^d)$ for a sufficiently large s which must also satisfy a certain congruence arithmetic condition.

We pick a $\varphi \in \mathrm{Diff}_k(\mathbb{S}^{d-1})$ given by Lemma 7.10 and realize $\mathbb{T}^d \# \Sigma^d$ by removing a disk $\mathbb{D}^d$ from $\mathbb{T}^d$ and then attaching it back with a twist $\cup_\varphi \mathbb{D}^d$. Given an integer $s \geq 2$ consider the manifold M_s which is diffeomorphic to $\mathbb{T}^d \# \Sigma^d$ and which is obtained by removing the conformally scaled disk $\frac{1}{s}\mathbb{D}^d$ and then attaching it back with a twist $\cup_\varphi \frac{1}{s}\mathbb{D}^d$. Because of our choice of φ both manifolds are naturally total spaces of smooth torus bundles

$$\mathbb{T}^{d-k} \to \mathbb{T}^d \# \Sigma^d \xrightarrow{p_1} \mathbb{T}^k, \quad \mathbb{T}^{d-k} \to M_s \to \mathbb{T}^k,$$

where the base space $\mathbb{T}^k$ corresponds to the first k coordinates fixed by φ.

Let $N \to \mathbb{T}^d \# \Sigma^d$ be the locally isometric cover which induces the $\times s$ map on the fundamental group. And let N_s be a copy of N with the Riemannian metric conformally scaled by $1/s$. Clearly both N_s and N smoothly fiber over $\mathbb{T}^k$. Then the posited expanding map is the composition

$$\mathbb{T}^d \# \Sigma^d \xrightarrow{F_s} M_s \xrightarrow{G_s} N_s \xrightarrow{\times s} N \to \mathbb{T}^d \# \Sigma^d.$$

The diffeomorphisms F_s and G_s are constructed with a uniform (in s) lower bound on minimal expansion. It immediately follows that for sufficiently large s the composite map $f\colon \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d \# \Sigma^d$ is uniformly expanding.

The diffeomorphism F_s which "shrinks" the exotic sphere is constructed using the "commutator trick" and it is easy to check that F_s is fiber preserving and fibers over the identity map $\mathrm{id}_{\mathbb{T}^k}$. We claim that the same is true for the diffeomorphism G_s. The purpose of G_s is to introduce a certain number of scaled exotic spheres $\cup_\varphi \frac{1}{s}\mathbb{D}^d$, and thus create N_s. These exotic spheres are introduced in groups of size b which is divisible by the order of φ in Θ_d [14, Lemma 3]. Alternatively one can think of G_s^{-1} as a diffeomorphism which removes exotic spheres in groups of size b. To remove one such group one uses a diffeomorphism given by the isotopy between φ^b and $\mathrm{id}_{\mathbb{S}^{d-1}}$. A priori such an isotopy does not preserve the fibers. However, we can require b to be divisible by the p which is given by Lemma 7.10. Then φ^b is isotopic to $\mathrm{id}_{\mathbb{S}^{d-1}}$ in the space $\mathrm{Diff}_k(\mathbb{S}^{d-1})$ and hence the resulting diffeomorphism $G_s\colon M_s \to N_s$ is fiber preserving and fibers over $\mathrm{id}_{\mathbb{T}^k}$. Finally, we notice that the covering map $N \to \mathbb{T}^d \# \Sigma^d$ and the expanding map $\times s\colon N_s \to N$ are fiber preserving as well. We conclude that the expanding map f fibers over the $\times s$ map on $\mathbb{T}^k$:

$$\begin{array}{ccc} \mathbb{T}^d \# \Sigma^d & \xrightarrow{f} & \mathbb{T}^d \# \Sigma^d \\ \downarrow p_1 & & \downarrow p_1 \\ \mathbb{T}^k & \xrightarrow{\times s} & \mathbb{T}^k \end{array}$$

The same diagram holds for the standard $\times s$ expanding map $E_s\colon \mathbb{T}^d \to \mathbb{T}^d$:

$$\begin{array}{ccc} \mathbb{T}^d & \xrightarrow{E_s} & \mathbb{T}^d \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{T}^k & \xrightarrow{\times s} & \mathbb{T}^k \end{array}$$

and it is easy to see that the conjugacy $h\colon \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d$, $h \circ f = E_s \circ h$, maps fibers to fibers and the induced conjugacy on $\mathbb{T}^k$ is the identity, i.e., $\bar{h} = \mathrm{id}_{\mathbb{T}^k}$.

We claim that one can perturb f along the fibers so that the fibrations p_1 and p_2 are precisely the ones appearing in Theorem 2.1. Indeed, consider the restrictions of $f_p\colon \mathbb{T}_p^{d-k} \to \mathbb{T}_p^{d-k}$ and $E_s\colon \mathbb{T}_{h(p)}^{d-k} \to \mathbb{T}_{h(p)}^{d-k}$ to the fibers through the corresponding fixed points. Denote by p_i', $i = 1, 2$, the fibrations produced by Theorem 2.1 applied to f and E_s. Then the fibers of p_i' refine the fibers of p_i, and hence we can restrict p_1' and p_2' to $\mathbb{T}_p^{d-k}$ and $\mathbb{T}_{h(p)}^{d-k}$, respectively. Denote by ℓ the dimension of the base space for these restricted fibrations. Recall that the induced conjugacy on the base space is smooth. It follows that $\bigwedge^\ell Df_p$ has s^ℓ as an eigenvalue. Hence we perturb f in the neighborhood of p so that $\bigwedge^\ell Df_p$ does not have s^ℓ for an eigenvalue for $\ell = 1, \dots, d-k$. Then $\ell = 0$, which means that $p_i' = p_i$.

Remark 7.11. Similarly, one can perturb f along the fibers to an expanding map $f_2\colon \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d \# \Sigma^d$ such that both p_i' given by Theorem 2.1 when applied to f and f_2 are equal to p_1.

Remark 7.12. An easier way of constructing an exotic expanding map with non-trivial fibration would be to take the product $f \times L$ of an exotic expanding map $f\colon \mathbb{T}^d \# \Sigma^d \to \mathbb{T}^d \# \Sigma^d$ and a linear expanding map $L\colon \mathbb{T}^m \to \mathbb{T}^m$. Smoothing theory implies that $(\mathbb{T}^d \# \Sigma^d) \times \mathbb{T}^m$ is not diffeomorphic to $\mathbb{T}^{d+m}$. Then $(\mathbb{T}^d \# \Sigma^d) \times \mathbb{T}^m$ fibers over $\mathbb{T}^m$ and one can arrange this fibration to be the fibration given by Theorem 2.1 in a similar way. The example which we described above is more interesting because the smooth structure on $\mathbb{T}^d \# \Sigma^d$ is *irreducible*, that is, $\mathbb{T}^d \# \Sigma^d$ is not diffeomorphic to a smooth product of two lower-dimensional smooth closed manifolds [12, Proposition 1.3].

8. Factor version

We formulate the following generalization of Theorem 2.1, where we replace the topological conjugacy by a continuous factor map. The proof follows the same lines with routine modifications and we omit it.

Theorem 8.1. *Assume that M_i, $i = 1, 2$, are closed manifolds homeomorphic to nilmanifolds. Let $f_i\colon M_i \to M_i$, $i = 1, 2$, be C^r-smooth, $r \ge 1$, expanding maps and assume that f_2 is a topological factor of f_1, that is, there exists a continuous map $h\colon M_1 \to M_2$ such that $h \circ f_1 = f_2 \circ h$.*

Then there exists a C^r expanding map $\bar{f}: \bar{M} \to \bar{M}$ where $\bar{M}$ is homeomorphic to a nilmanfold, and C^r fibrations (with connected fiber homeomorphic to a nilmanifold) $p_i: M_i \to \bar{M}$, $i = 1, 2$, such that

$$p_i \circ f_i = \bar{f} \circ p_i, \quad i = 1, 2,$$

Further the map h sends fibers to fibers:

$$p_2 \circ h = p_1,$$

and the fibrations p_i, $i = 1, 2$, have the following property. If $\varphi_i: M_i \to \mathbb{R}$, $i = 1, 2$, are C^r-smooth functions such that for every periodic point $x \in \mathrm{Fix}(f_1^n)$,

$$\sum_{k=0}^{n-1} \varphi_1(f_1^k(x)) = \sum_{k=0}^{n-1} \varphi_2(f_2^k(h(x))),$$

then there exist a C^r function $\bar{\varphi}: \bar{M} \to \mathbb{R}$ such that φ_i is cohomologous to $\bar{\varphi} \circ p_i$ over f_i.

Using Theorem 7.1 one can naturally study regularity properties of factor maps. We proceed to describe an application.

Let $M_1 = N \times M_2$, where N and M_2 are nilmanifolds, and let $L = (A, B): M_1 \to M_1$ be a product expanding map. Then, clearly, L factors over B. Hence if f_1 is an expanding map homotopic to L and f_2 is an expanding map homotopic to B then f_1 factors over f_2, $h \circ f_1 = f_2 \circ h$.

To define nice invariants of smooth conjugacy we need to introduce a restriction on L and f_1. Namely, we assume that the maximal expansion of A is greater than the minimal expansion of B. Then the "vertical foliation" $N \times \{x\}$, $x \in M_2$, is a weakly expanding foliation. It is easy to see that for any sufficiently C^1 small perturbation f_1 of L the weakly expanding foliation survives as an f_1-invariant foliation W^{wu}.

Corollary 8.2. *Consider C^{r+1} expanding maps L, f_1, f_2 and a factor map h, $h \circ f_1 = f_2 \circ h$. Assume that f_1 belongs to a sufficiently small C^1 neighborhood of L and f_2 is very non-algebraic. If for any periodic point $x = f_1^k(x)$,*

$$\frac{\mathrm{Jac}(f_1^k)(x)}{\mathrm{Jac}(f_1^k|_{W^{wu}})(x)} = \mathrm{Jac}(f_2^k)(h(x))$$

then the factor map h is C^r-smooth.

The proof is very similar to the proof of Corollary 2.11 and we merely provide a sketch. Also one can replace the very non-algebraic assumption on f_2 by requiring f_2 to be an irreducible toral diffeomorphism and assuming that the entropy maximizing measure for f_2 is not absolutely continuous.

Sketch of the proof. Let $p_i: M_i \to \bar{M}_i$ be fibrations given by Theorem 8.1 when applied to f_i and r. If $\dim \bar{M} = \dim M_2$ then p_2 is a diffeomorphism and so $h = p_2^{-1} \circ p_1$ is C^r.

Hence we need to rule out the possibility that $\dim \bar{M} < \dim M_2$, i.e., the case when the fiber of p_2 has dimension ≥ 1. In this case, following the proof of Corollary 2.11, we can apply Theorem 8.1 to $\log \mathrm{Jac}(f_2)$ to conclude that $\log \mathrm{Jac}(f_2|_{\ker p_2})$ is cohomologous to a function which is constant along the fibers of p_2, which yields a contradiction again, similarly to the proof of Corollary 2.11.

One subtle detail, however, is that in order to apply Theorem 8.1 one needs to have a pair of C^r functions (φ_1, φ_2). We let $\varphi_2 = \log \mathrm{Jac}(f_2)$ and $\varphi_1 = \log \mathrm{Jac}(f_1) - \log \mathrm{Jac}(f_1|_{W^{wu}})$. (Assume for simplicity that f_i's are orientation preserving.) It is clear from the assumption of the corollary that the sums of φ_i agree along the periodic orbits and it is clear that φ_2 is C^r. However, one also has to argue that φ_1 is C^r which is equivalent to $\log \mathrm{Jac}(f_1|_{W^{wu}})$ being C^r.

Smoothness of $\log \mathrm{Jac}(f_1|_{W^{wu}})$ can be established as follows. Pick a lift $\tilde{f}_1 : \tilde{M}_1 \to \tilde{M}_1$ to the universal cover $\tilde{M}_1$. The foliation W^{wu} lifts to $\tilde{W}^{wu}$. Because $\tilde{f}_1$ is invertible, the fast foliation $\tilde{W}^{uu}$ is also well defined by the standard cone argument. Notice that $\tilde{W}^{uu}$ is not equivariant under the group of deck transformations but this is not going to be important for what follows. Then, by the usual application of the C^r section theorem [21], we find that $\tilde{W}^{uu}$ is C^{r+1} and hence $\log \mathrm{Jac}(\tilde{f}_1|_{\tilde{W}^{uu}})$ is C^r. Finally, extending $\tilde{W}^{uu}$ to a smooth coordinate system, we see that Df_1 has an upper-triangular form and hence

$$\log \mathrm{Jac}(\tilde{f}_1) = \log \mathrm{Jac}(\tilde{f}_1|_{\tilde{W}^{uu}}) + \log \mathrm{Jac}(\tilde{f}_1|_{\tilde{W}^{wu}}),$$

which implies that $\log \mathrm{Jac}(\tilde{f}_1|_{\tilde{W}^{wu}})$, and hence $\log \mathrm{Jac}(f_1|_{W^{wu}})$, is C^r. ∎

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