J. Eur. Math. Soc. 25, 3177–3228 (2023) DOI 10.4171/JEMS/1212

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The Zakharov system in dimension $d \geq 4$

Received November 12, 2020

Abstract. The sharp range of Sobolev spaces is determined in which the Cauchy problem for the classical Zakharov system is well-posed, which includes existence of solutions, uniqueness, persistence of initial regularity, and real-analytic dependence on the initial data. In addition, under a condition on the data for the Schrödinger equation at the lowest admissible regularity, global well-posedness and scattering are proved. The results cover energy-critical and energy-supercritical dimensions $d \geq 4$.

Keywords. Zakharov system, local well-posedness, global well-posedness, ill-posedness, scattering

1. Introduction

Consider an at most weakly magnetized plasma with ion density fluctuation $v : \mathbb{R}^{1+d} \to \mathbb{R}$ and complex envelope $u : \mathbb{R}^{1+d} \to \mathbb{C}$ of the electric field. In [37] Zakharov derived the equations for the dynamics of Langmuir waves, which are rapid oscillations of the electric field in a conducting plasma. A scalar version of his model, called the Zakharov system, is given by

$$\begin{aligned} i\partial_t u + \Delta u &= vu, \\ \Box v &= \Delta |u|^2 \end{aligned} \tag{1.1}$$

with the d'Alembertian $\Box = \partial_t^2 - \Delta$. We refer to [8, 36, 37] and the books [17, 35] for more details of the model and its derivation.

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Mathematics Subject Classification (2020): Primary 35Q55; Secondary 42B37, 35L70, 35P25

The Zakharov system is Lagrangian, and formally the L^2-norm of u and the energy

$$E_Z(u(t), v(t), \partial_t v(t)) := \int_{\mathbb{R}^d} \big(\tfrac{1}{2}|\nabla u(t)|^2 + \tfrac{1}{4}\big||\nabla|^{-1}\partial_t v(t)\big|^2 + \tfrac{1}{4}|v(t)|^2 + \tfrac{1}{2}v(t)|u(t)|^2\big)\,dx$$

are constant in time.

The Zakharov system (1.1) is typically studied as a Cauchy problem by prescribing initial data in Sobolev spaces, i.e.

$$u(0) = f \in H^s(\mathbb{R}^d) \quad \text{and} \quad (v, |\nabla|^{-1}\partial_t v)(0) = (g_0, g_1) \in H^\ell(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d). \tag{1.2}$$

In recent years, this initial value problem has attracted considerable attention, partly driven by the close connection with the focusing cubic nonlinear Schrödinger equation (NLS) which arises as a subsonic limit of the Zakharov system (1.1) [1, 26, 29, 32, 34]. In addition, bound states for the focusing cubic NLS are closely intertwined with the global dynamics of (1.1). More precisely, if $Q_\omega : \mathbb{R}^d \to \mathbb{R}$ is a bound state for the focusing cubic NLS, in other words if Q_ω solves

$$-\Delta Q_\omega + \omega Q_\omega = Q_\omega^3,$$

then $(u, v) = (e^{it\omega}Q_\omega, -Q_\omega^2)$ is a global (non-dispersive) solution of (1.1). This connection has been used to analyze the blow-up behaviour [15, 16, 30] in dimension $d = 2$, and also in the periodic case [28]. Furthermore, we can write the Zakharov energy as

$$E_Z(u(t), v(t), \partial_t v(t)) = E_S(u(t)) + \frac{1}{4}\int_{\mathbb{R}^d} \big|(1 - i|\nabla|^{-1}\partial_t)v(t) + |u|^2\big|^2\,dx$$

where

$$E_S(u(t)) := \int_{\mathbb{R}^d} \big(\tfrac{1}{2}|\nabla u(t)|^2 - \tfrac{1}{4}|u(t)|^4\big)\,dx$$

is the energy for the focusing cubic NLS. As the cubic NLS is energy-critical in $d = 4$, the Zakharov system is also frequently referred to as energy-critical in dimension $d = 4$, although, in contrast to the cubic NLS, the Zakharov system lacks scale-invariance; see [20] for further discussion.

In the Zakharov system, the interplay between the different dispersive effects of solutions to Schrödinger and wave equations leads to a rich local and global well-posedness theory [1–5, 9, 12, 13, 26, 27, 31]. In particular, it turned out that the required regularity of the Schrödinger component can go below the scaling critical one ($s = d/2 - 1$) for the cubic nonlinear Schrödinger equation. Concerning the asymptotic behaviour of global solutions, scattering results have been proven in certain cases [2, 14, 18–22, 24, 33].

The aim of this paper is twofold. First, we give a complete answer to the question of local well-posedness in dimension $d \geq 4$, i.e. the energy-critical and supercritical dimensions. Second, we prove that these local solutions are global in time and scatter, provided that the Schrödinger part is small enough. To be more precise, consider the case $d \geq 4$, and (s, ℓ) satisfying

$$\ell \geq \frac{d}{2} - 2, \quad \max\left\{\ell - 1, \frac{\ell}{2} + \frac{d-2}{4}\right\} \leq s \leq \ell + 2, \quad (s, \ell) \neq \left(\frac{d}{2}, \frac{d}{2} - 2\right), \left(\frac{d}{2}, \frac{d}{2} + 1\right). \tag{1.3}$$

Our first main result is

Theorem 1.1. *The Zakharov system* (1.1) *with initial condition* (1.2) *is locally well-posed with a real-analytic flow map if and only if* $(s, \ell) \in \mathbb{R}^2$ *satisfies* (1.3).

To be more precise, we consider mild solutions to an equivalent first order system (2.1), as usual. For this we show a local well-posedness result, Theorem 7.6, which applies to the non-endpoint case, and Theorem 7.7, for the endpoint case. Finally, we provide two examples in Section 9.1, which show that if the flow map exists for (s, ℓ) in the exterior of the region defined by (1.3), it does not have bounded directional derivatives of second order at the origin. Partial ill-posedness results have been obtained earlier in [4,11,13,23]. At the specific point $(s, \ell) = (2, 3)$ in $d = 4$ a stronger form of ill-posedness was proved in [2, Section 7], namely that there is no distributional solution of this regularity.

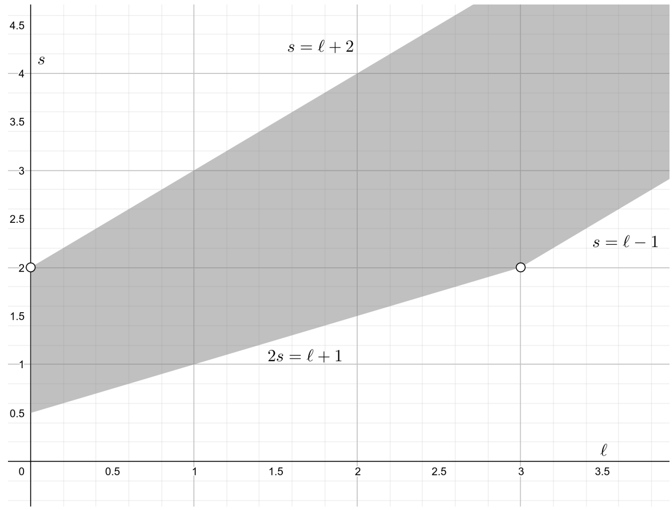

Fig. 1. In dimension $d = 4$: Local well-posedness and small data global well-posedness within grey region, ill-posedness elsewhere.

Our second main result is

Theorem 1.2. *Let* $d \geq 4$ *and* (s, ℓ) *satisfy* (1.3). *For any data* $(g_0, g_1) \in H^\ell(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d)$, *there exists* $\varepsilon > 0$ *such that for any* $f \in H^s(\mathbb{R}^d)$ *satisfying* $\|f\|_{H^{(d-3)/2}} \leq \varepsilon$, *we have a global solution* $u \in C(\mathbb{R}, H^s(\mathbb{R}^d))$, $(v, |\nabla|^{-1}\partial_t v) \in C(\mathbb{R}, H^\ell(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d))$ *to* (1.1) *and* (1.2), *which is unique under the condition*

$$u \in L^2_{t,\mathrm{loc}}(\mathbb{R}, W_x^{\frac{d-3}{2}, \frac{2d}{d-2}}(\mathbb{R}^d)),$$

and depends real-analytically on the initial data. This solution scatters as $t \to \pm\infty$.

Theorem 1.2 is a consequence of Theorems 7.6, 7.7 and 8.1, which again apply to the first order system (2.1) in the mild formulation (see Section 9.2). In fact, we prove something stronger: the smallness condition in Theorem 1.2 can be replaced with the weaker condition

$$\|f\|_{H^{\frac{d-3}{2}}} \|e^{it\Delta} f\|^7_{L^2_t W_x^{\frac{d-3}{2}, \frac{2d}{d-2}}} \leq \varepsilon.$$

We remark that Theorems 7.6 and 7.7 (with $g_* = 0$) also imply that the smallness condition on f does not depend on (g_0, g_1) provided it is also sufficiently small, i.e. $\|(g_0, g_1)\|_{H^{(d-4)/2}} \ll 1$. For readers interested in this important and much easier case only, we provide a simplified approach and results in Section 5.

In general, $\varepsilon > 0$ in Theorem 1.2 must depend on the wave initial data (g_0, g_1), and it is not even uniform with respect to its norm, at least when (s, ℓ) is on a segment of the lowest regularity ($\ell = d/2 - 2$ and $(d-3)/2 \leq s < d/2 - 1$): Take any non-negative $f_0 \in C_0^\infty(\mathbb{R}^d) \setminus \{0\}$. Multiplying it by a large number $a \gg 1$, we can make the NLS energy negative, $E_S(af_0) < 0$. Imposing $g_0 = -|af_0|^2$ and $g_1 = 0$ makes the Zakharov energy the same: $E_Z(af_0, g_0, g_1) = E_S(af_0)$. When the energy is negative, scattering is impossible, because the global dispersion would send the negative nonlinear part to zero as $t \to \infty$. Finally, to make the Schrödinger data small, we can use the scaling-invariance of the NLS: Let $f(x) = \lambda a f_0(\lambda x)$ with $\lambda \to \infty$. Since this is the $\dot{H}^{d/2-1}$-invariant scaling, all $\dot{H}^s$ norms with $s < d/2 - 1$ tend to zero as the data concentrate, including the L^2 norm ($s = 0$). For the wave component, the scaling leaves $\dot{H}^{d/2-2}$ invariant, which is the lowest (critical) regularity. In other words, we can make the Schrödinger data as small in H^s as we like for $s < d/2 - 1$, while keeping the wave norm in $\dot{H}^{d/2-2}$.

Further, in the energy-critical case ($d = 4$), we observe that there exist non-scattering solutions as long as $\|g_0\|_{L^2} > \|W^2\|_{L^2}$, where $W(x) = (|x|^2/(d(d-2)) + 1)^{-1}$ is the ground state of the NLS. To see this, start with $f(x) = aW\chi(x/R)$ with a smooth cut-off function χ (which is needed since W barely fails to be in $L^2(\mathbb{R}^4)$). Choosing $a > 1$, and then $R > 1$ large enough depending on a, we obtain $E_S(f) < E_S(W)$ and $\||f|^2\|_{L^2} > \|W^2\|_{L^2}$, so that we can apply the grow-up result (with $g_0 = -|f|^2$ and $g_1 = 0$ as above) in the radial case obtained in [20]. The large data case in the energy-critical dimension $d = 4$ is addressed in a follow-up paper [7].

The key contributions of Theorems 1.1 and 1.2 are firstly that we give a complete characterisation of the region of well-posedness in arbitrary space dimension $d \geq 4$, and secondly that we obtain global well-posedness and scattering for wave data of arbitrary size, only requiring the Schrödinger data to be small enough. In particular, in the energy-critical dimension $d = 4$ this extends [2] to the subregion where $(s, \ell) = (1, 0)$ or $s \geq 4\ell + 1$ or $s > 2\ell + \frac{11}{8}$ and the scattering to wave data of arbitrary size. Note that [2] covers the energy space $(s, \ell) = (1, 0)$ but by a compactness argument, from which it is not immediately clear whether the solution map is analytic. Further, if $d = 4$, the large data threshold result in [20] is restricted to radial data. In higher dimensions, this is an extension of the local well-posedness results in [13], which apply in the subregion where $\ell \leq s \leq \ell + 1$ and $2s > \ell + \frac{d-2}{2}$, and the global well-posedness and scattering result in [24], which applies if $(s,\ell) = (\frac{d-3}{2}, \frac{d-4}{2})$ and both the wave and the Schrödinger data are small.

The recent well-posedness results cited above rely on a partial normal form transformation. This strategy introduces certain boundary terms which are non-dispersive and difficult to deal with in the low regularity setup. In this paper, we introduce a new perturbative approach which is based on Strichartz and maximal $L^2_{t,x}$ norms with additional temporal derivatives allowing us to exploit the different dispersive properties of the wave and the Schrödinger equation. Further, the global well-posedness result allows for wave data of arbitrary size, which is achieved by treating the free wave evolution as a potential term in the Schrödinger equation.

One of the main challenges in proving the global well-posedness results in Theorems 1.1 and 1.2 in the range $s > \ell + 1$ lies in the fact that it seems impossible to control the endpoint Strichartz norm, i.e. to prove that $\langle\nabla\rangle^s u \in L^2_t L^{\frac{2d}{d-2}}_x$. To some extent, this is explained by considering

$$(i\partial_t + \Delta)u = \phi_\lambda \psi_\mu$$

as a toy model for (1.1), where $\phi_\lambda = e^{it|\nabla|} f_\lambda$ is a free wave, $\psi_\mu = e^{it\Delta} g_\mu$ is a free solution to the Schrödinger equation, the wave data f_λ has spatial frequencies $|\xi| \approx \lambda$, and the Schrödinger data g_μ has spatial frequencies $|\xi| \approx \mu$ with $\mu \ll \lambda$. Note that this is essentially the first Picard iterate for (1.1). A computation shows that the product $\phi_\lambda\psi_\mu$ has space-time Fourier support in the set $\{|\tau| \ll \lambda^2, |\xi| \approx \lambda\}$ and hence (modulo a free Schrödinger wave) we can write

$$u \sim (i\partial_t + \Delta)^{-1}(\phi_\lambda\psi_\mu) \sim \lambda^{-2}(\phi_\lambda\psi_\mu).$$

In particular, we expect that (in the case $d = 4$ for ease of notation)

$$\|\langle\nabla\rangle^s u\|_{L^2_t L^4_x(\mathbb{R}^{1+4})} \approx \lambda^{s-2}\|\phi_\lambda\psi_\mu\|_{L^2_t L^4_x(\mathbb{R}^{1+4})}.$$

If we assume the wave endpoint regularity, in $d = 4$ we can only place $\phi_\lambda \in L^\infty_t L^2_x$. Thus applying Hölder's inequality together with the sharp Sobolev embedding and the endpoint Strichartz estimate for the free Schrödinger equation we see that

$$\|\phi_\lambda\psi_\mu\|_{L^2_t L^4_x(\mathbb{R}^{1+4})} \lesssim \|\phi_\lambda\|_{L^\infty_t L^4_x(\mathbb{R}^{1+4})}\|\psi_\mu\|_{L^2_t L^\infty_x(\mathbb{R}^{1+4})} \lesssim \lambda\|f_\lambda\|_{L^2(\mathbb{R}^4)}\mu\|g_\mu\|_{L^2(\mathbb{R}^4)}.$$

Note that the above chain of inequalities is essentially forced if we may only assume the regularity $\phi_\lambda \in L^\infty_t L^2_x$. Consequently, we obtain

$$\|\langle\nabla\rangle^s u\|_{L^2_t L^4_x} \lesssim \left(\frac{\lambda}{\mu}\right)^{s-1}\|f_\lambda\|_{L^2}\|g_\mu\|_{H^s}.$$

Again, as we can only place $f_\lambda \in L^2_x$, this imposes the restriction $s \leq 1$. It is very difficult to see a way to improve the above computation, and in fact this high-low interaction is essentially what led to the restriction $s < 1$ in [2, 24]. Note however that this obstruction only leads to $\langle\nabla\rangle^s u \notin L^2_t L^4_x(\mathbb{R}^{1+4})$, and is *not* an obstruction to well-posedness. In other words, provided only that $s \leq 2$ we still have $u \in L_t^\infty H_x^s$ since similar to the above computation,

$$\|u\|_{L_t^\infty H_x^s(\mathbb{R}^{1+4})} \approx \lambda^{s-2} \|\phi_\lambda u_\mu\|_{L_t^\infty L_x^2} \lesssim \left(\frac{\lambda}{\mu}\right)^{s-2} \|f_\lambda\|_{L_x^2} \|g_\mu\|_{H^s}.$$

In summary, the above example strongly suggests that it is not possible to construct solutions to the Zakharov system by iterating in the endpoint Strichartz norms $L_t^2 W^{s,4}(\mathbb{R}^{1+4})$, or even any space which contains the endpoint Strichartz space. Thus an alternative space is required, and this is what we construct in this paper.

A partial solution to the above problem of obtaining well-posedness in the regularity region $s \geq \ell + 1$ was given in [2]. The approach taken there was to replace the endpoint Strichartz space $L_t^2 W_x^{s,4}$ with the intermediate Strichartz spaces $L_t^q W_x^{s,r}$ for appropriate (non-endpoint, i.e. $q > 2$) Schrödinger admissible (q, r). However, the argument given in [2] requires additional regularity for the wave component v as it exploits Strichartz estimates for the wave equation to compensate for the loss in decay in the intermediate Schrödinger Strichartz spaces, and thus misses a neighbourhood of the corner $(s, l) = (d/2, d/2 - 2)$.

The key observation that gives well-posedness in the full region (1.3) is that the output of the above high-low interaction has small temporal frequencies. Consequently, the endpoint Strichartz space only loses regularity at small temporal frequencies. This observation can be exploited by using norms of the form

$$\|(\langle \nabla \rangle + |\partial_t|)^a u\|_{L_t^2 W_x^{s-2a,4}(\mathbb{R}^{1+4})}. \tag{1.4}$$

Note that if $u = e^{it\Delta} f$ is a free solution to the Schrödinger evolution, then u has temporal Fourier support in $\{|\tau| \approx |\xi|^2\}$ and hence

$$\|(\langle \nabla \rangle + |\partial_t|)^a u\|_{L_t^2 W_x^{s-2a,4}(\mathbb{R}^{1+4})} \approx \|u\|_{L_t^2 W_x^{s,4}}.$$

Thus the norm (1.4) is equivalent to the standard endpoint Strichartz space for free Schrödinger waves. On the other hand, if u has Fourier support in $\{|\tau| \lesssim |\xi|\}$, i.e. u has only small temporal frequencies, then

$$\|(\langle \nabla \rangle + |\partial_t|)^a u\|_{L_t^2 W_x^{s-2a,4}(\mathbb{R}^{1+4})} \approx \|u\|_{L_t^2 W_x^{s-a,4}}.$$

In other words, we only have $\langle \nabla \rangle^{s-a} u \in L_t^2 L_x^4(\mathbb{R}^{1+4})$ and thus we allow for a loss of regularity in the small temporal frequency region of the Strichartz norm. Moreover, again considering the above high-low interaction, we can control the output $(i\partial_t + \Delta)^{-1}(\phi_\lambda \psi_\mu)$ in the temporal derivative Strichartz space (1.4) provided that $a \geq s - 1$. In particular choosing $a \sim 1$ gives the full range $s < 2$. Thus roughly speaking, the norm (1.4) matches the standard endpoint Strichartz space for the Schrödinger-like portion of the evolution of u (i.e. when $|\tau| \approx |\xi|^2$), but allows for a loss of regularity in the small temporal frequency regions $|\tau| \ll |\xi|^2$ of u which are strongly influenced by nonlinear wave-Schrödinger interactions. We refer to estimate (2.5) and Remark 7.3 below for further related comments.

1.1. Outline of the paper

In Section 2, notation is introduced, the crucial function spaces are defined, and their key properties are discussed. Further, a product estimate for fractional time-derivatives is proved. Bilinear estimates for the Schrödinger and the wave nonlinearities are proved in Sections 3 and 4, respectively. In Section 5 we provide a shortcut to simplified local and small data global well-posedness and scattering results which do not use the refined results of the following sections. Local versions of the bilinear estimates in the endpoint case are proved in Section 6. In Section 7 the technical well-posedness results are established, most notably Theorems 7.6 and 7.7. Persistence of regularity is established in Section 8. Finally, the proofs of Theorems 1.1 and 1.2 are completed in Section 9.

2. Notation and preliminaries

The Zakharov system has an equivalent first order formulation which is slightly more convenient to work with. Suppose that (u, v) is a solution to (1.1) and let $V = v - i|\nabla|^{-1}\partial_t v$. Then (u, V) solves the first order problem

$$\begin{aligned} i\partial_t u + \Delta u &= \Re(V)u, \\ i\partial_t V + |\nabla| V &= -|\nabla|\,|u|^2. \end{aligned} \tag{2.1}$$

Conversely, given a solution (u, V) to (2.1), the pair $(u, \Re(V))$ solves the original Zakharov equation (1.1).

2.1. Fourier multipliers

Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi \geq 0$, $\operatorname{supp}\varphi \subset \{1/2 < r < 2\}$ and

$$1 = \sum_{\lambda \in 2^{\mathbb{Z}}} \varphi\Big(\frac{r}{\lambda}\Big) \quad \text{for } r > 0.$$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $\lambda \in 2^{\mathbb{N}}$, define the spatial Fourier multipliers

$$P_\lambda = \varphi\Big(\frac{|\nabla|}{\lambda}\Big) \quad \text{if } \lambda > 1, \quad P_1 = \sum_{\lambda \in 2^{\mathbb{Z}}, \lambda \leq 1} \varphi\Big(\frac{|\nabla|}{\lambda}\Big).$$

Thus P_λ is (inhomogeneous) Fourier multiplier localising the spatial Fourier support to the set $\{\lambda/2 < |\xi| < 2\lambda\}$ if $\lambda > 1$ and $\{|\xi| < 2\}$ if $\lambda = 1$. Further, for $\lambda \in 2^{\mathbb{Z}}$, we define

$$P^{(t)}_\lambda = \varphi\Big(\frac{|\partial_t|}{\lambda}\Big), \quad C_\lambda = \varphi\Big(\frac{|i\partial_t + \Delta|}{\lambda}\Big).$$

$P^{(t)}_\lambda$ localises the temporal Fourier support to the set $\{\lambda/2 < |\tau| < 2\lambda\}$, and C_λ localises the space-time Fourier support to distances $\approx \lambda$ from the paraboloid.

To restrict the Fourier support to larger sets, we use the notation

$$P_{\le\lambda} = \sum_{\mu\in 2^{\mathbb{Z}},\, \mu\le\lambda} \varphi\Big(\frac{|\nabla|}{\mu}\Big), \quad P^{(t)}_{\le\lambda} = \sum_{\mu\in 2^{\mathbb{Z}},\, \mu\le\lambda} \varphi\Big(\frac{|\partial_t|}{\mu}\Big),$$
$$C_{\le\lambda} = \sum_{\mu\in 2^{\mathbb{Z}},\, \mu\le\lambda} \varphi\Big(\frac{|i\partial_t + \Delta|}{\mu}\Big),$$

and define $C_{>\mu} = I - C_{\le\mu}$. For ease of notation, for $\lambda \in 2^{\mathbb{N}}$ we often use the shorthand $P_\lambda f = f_\lambda$. In particular, note that $u_1 = P_1 u$ has Fourier support in $\{|\xi| < 2\}$, and we have the identity

$$f = \sum_{\lambda\in 2^{\mathbb{N}}} f_\lambda \quad \text{for any } f \in L^2(\mathbb{R}^d).$$

For brevity, let us denote the frequently used decomposition into high and low modulation by

$$P^N_\lambda u := C_{\le(\lambda/2^8)^2} P_\lambda u, \quad P^F_\lambda u := C_{>(\lambda/2^8)^2} P_\lambda u, \tag{2.2}$$

so that $u_\lambda = P^N_\lambda u + P^F_\lambda u$. Similarly, we take

$$P^N := \sum_{\lambda\in 2^{\mathbb{N}}} P^N_\lambda, \quad P^F = \sum_{\lambda\in 2^{\mathbb{N}}} P^F_\lambda, \quad P^F_{\le\lambda} = \sum_{\mu\in 2^{\mathbb{N}},\, \mu\le\lambda} P^F_\lambda, \quad \text{etc.}$$

Note that $u = P^N u + P^F u$, and these multipliers all obey the Schrödinger scaling, for instance

$$(P^N_\lambda u)(t/\lambda^2, x/\lambda) = P^N_2(u(4t/\lambda^2, 2x/\lambda)), \tag{2.3}$$

where P^N_2 is a space-time convolution with a Schwartz function, so that we can easily deduce that P^N_λ and P^F_λ are bounded on any $L^p_t L^q_x$ uniformly in $\lambda \in 2^{\mathbb{N}}$, and that P^N and P^F are bounded on any $L^2_t B^s_{q,2}$.

2.2. *Function spaces*

In what follows, by default we consider tempered distributions. We define the inhomogeneous Besov spaces $B^s_{q,r}$ and Sobolev spaces $W^{s,p}$ via the norms

$$\|f\|_{B^s_{q,r}} = \Big(\sum_{\lambda\in 2^{\mathbb{N}}} \lambda^{sr} \|f_\lambda\|^r_{L^q}\Big)^{1/r}, \quad \|f\|_{W^{s,p}} = \|\langle\nabla\rangle^s f\|_{L^p}.$$

We use the notation $2^* = \frac{2d}{d-2}$ and $2_* = (2^*)' = \frac{2d}{d+2}$ for the endpoint Strichartz exponents for the Schrödinger equation. Thus for $d \ge 3$ we have

$$\|e^{it\Delta} f\|_{L^\infty_t L^2_x \cap L^2_t L^{2^*}_x} + \Big\|\int_0^t e^{i(t-s)\Delta} F(s)\, ds\Big\|_{L^\infty_t L^2_x \cap L^2_t L^{2^*}_x} \lesssim \|f\|_{L^2_x} + \|F\|_{L^2_t L^{2*}_x}$$

by the (double) endpoint Strichartz estimate [25]. To control the frequency localised Schrödinger component of the Zakharov evolution, we take parameters $s, a, b \in \mathbb{R}$, $\lambda \in 2^{\mathbb{N}}$ and define

$$\|u\|_{S^{s,a,b}_\lambda} = \lambda^s \|u\|_{L^\infty_t L^2_x} + \lambda^{s-2a} \|(\lambda + |\partial_t|)^a u\|_{L^2_t L^{2^*}_x} + \lambda^{s-1+b} \Big\| \Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|} \Big)^a (i\partial_t + \Delta) u \Big\|_{L^2_{t,x}}.$$

The parameters $a, b \in \mathbb{R}$ are required to prove the bilinear estimates in the full admissible region (1.3). Roughly speaking, a measures a loss of regularity in the small temporal frequency regime $|\tau| \ll \langle \xi \rangle$, for instance (when $b = 0$) if $\operatorname{supp} \widetilde{u} \subset \{\tau \lesssim \langle \xi \rangle \approx \lambda\}$ we have

$$\lambda^{s-2a} \|(\lambda + |\partial_t|)^a u\|_{L^2_t L^{2^*}_x} + \lambda^{s-1+b} \Big\| \Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|} \Big)^a (i\partial_t + \Delta) u \Big\|_{L^2_{t,x}} \approx \lambda^{-a} \big(\lambda^s \|u\|_{L^2_t L^{2^*}_x} + \lambda^{s-1} \|(i\partial_t + \Delta) u\|_{L^2_{t,x}} \big).$$

Thus, when the temporal frequencies are small, the non-$L^\infty_t H^s_x$ component of the norm $S^{s,a,b}_\lambda$ loses λ^{-a} derivatives when compared to the standard scaling for the Schrödinger equation. On the other hand, the b parameter simply gives a gain in regularity in the high modulation regime, for instance we have $\|P^F u\|_{L^\infty_t H^{s+b}_x} \lesssim \|u\|_{S^{s,0,b}}$.

The choice of a and b will depend on (s, ℓ), there is some flexibility here, but one option is to choose

$$a = a^* := \begin{cases} \frac{3}{4}(s - \ell) - \frac{1}{2} & \text{if } s - \ell \geq 1, \\ 0 & \text{if } s - \ell < 1, \end{cases} \qquad b = b^* := \begin{cases} 0 & \text{if } s - \ell > 0, \\ \frac{1}{2}(\ell - s) + \frac{1}{2} & \text{if } s - \ell \leq 0. \end{cases} \tag{2.4}$$

Thus in the region $\ell + 1 \leq s \leq \ell + 2$, when the Schrödinger component of the evolution is more regular, we require $a > 0$ (depending on the size of $s - \ell$) and can take $b = 0$. On the other hand, in the "balanced region" $\ell < s < \ell + 1$ we can simply take $a = b = 0$. In the final region $\ell - 1 \leq s \leq \ell$, when the wave is more regular, we can take $a = 0$ and require $b > 0$.

Remark 2.1. It is worth noting that due to the factor $(\lambda^2 + |\partial_t|)^{-a}(\lambda + |\partial_t|)^a$, the norm $\| \cdot \|_{S^{s,a,b}_\lambda}$ only controls the endpoint Strichartz estimate without loss when $a = 0$. In particular, if $0 \leq a \leq 1$, we only have

$$\lambda^{s-a} \|u_\lambda\|_{L^2_t L^{2^*}_x} \lesssim \lambda^{s-2a} \|(\lambda + |\partial_t|)^a u_\lambda\|_{L^2_t L^{2^*}_x} \lesssim \|u_\lambda\|_{S^{s,a,0}_\lambda}. \tag{2.5}$$

In view of the choice (2.4), this means that in the region $s - \ell \geq 1$ we no longer have control over the endpoint Strichartz space $L^2_t W^{s,2^*}_x$. On the other hand, in the small modulation regime, we retain control of the endpoint Strichartz space. More precisely, provided that $0 \leq a \leq 1$, an application of Bernstein's inequality gives the characterisation

$$\|u_\lambda\|_{S^{s,a,b}_\lambda} \approx \lambda^s \big(\|u_\lambda\|_{L^\infty_t L^2_x} + \|P^N_\lambda u\|_{L^2_t L^{2^*}_x} \big) + \lambda^{s-1+b} \Big\| \Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|} \Big)^a (i\partial_t + \Delta) u_\lambda \Big\|_{L^2_{t,x}}. \tag{2.6}$$

To control the Schrödinger nonlinearity we take

$$\|F\|_{N_\lambda^{s,a,b}} = \lambda^{s-2}\|P^{(t)}_{\leq(\lambda/2^8)^2}F\|_{L^\infty_t L^2_x} + \lambda^s\|C_{\leq(\lambda/2^8)^2}F\|_{L^2_t L^{2*}_x} + \lambda^{s-1+b}\left\|\left(\frac{\lambda+|\partial_t|}{\lambda^2+|\partial_t|}\right)^a F\right\|_{L^2_{t,x}}.$$

Remark 2.2. In the special case $0 \leq a < 1/2$ we have

$$\|F_\lambda\|_{N_\lambda^{s,a,b}} \approx \lambda^s\|C_{\leq(\lambda/2^8)^2}F_\lambda\|_{L^2_t L^{2*}_x} + \lambda^{s-1+b}\left\|\left(\frac{\lambda+|\partial_t|}{\lambda^2+|\partial_t|}\right)^a F_\lambda\right\|_{L^2_{t,x}}. \tag{2.7}$$

To see this, let $1/r = 1/2 - a$ and apply Bernstein's inequality together with the Sobolev embedding to obtain

$$\begin{aligned}\lambda^{s-2}\|P^{(t)}_{\leq(\lambda/2^8)^2}F_\lambda\|_{L^\infty_t L^2_x} &\lesssim \lambda^{s-2+2/r}\|P^{(t)}_{\leq(\lambda/2^8)^2}F_\lambda\|_{L^2_x L^r_t}\\ &\lesssim \lambda^{s-1-2a}\|(\lambda+|\partial_t|)^a P^{(t)}_{\leq(\lambda/2^8)^2}F_\lambda\|_{L^2_{t,x}},\end{aligned}$$

which implies the claim, since $b \geq 0$.

We also require a suitable space in which to control the evolution of the wave component. To this end, for $\ell, \alpha, \beta \in \mathbb{R}$, we let

$$\|V\|_{W_\lambda^{\ell,\alpha,\beta}} = \lambda^\ell\|V\|_{L^\infty_t L^2_x} + \lambda^{\ell-\alpha}\|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\leq(\lambda/2^8)^2}V\|_{L^\infty_t L^2_x} + \lambda^{\beta-1}\|(i\partial_t+|\nabla|)V\|_{L^2_{t,x}}.$$

Thus for small temporal frequencies we essentially take $(\frac{\lambda+|\partial_t|}{\lambda})^\alpha V \in L^\infty_t H^\ell_x$, while for large temporal frequencies (in the Schrödinger-like regime) the wave component V has roughly β derivatives. Eventually we will take $\alpha = a$ and $\beta = s - 1/2$. Consequently, in the high temporal frequency regime, the wave component V essentially inherits the regularity of the Schrödinger evolution u. To bound the right hand side of the half-wave equation at frequency λ, we define

$$\|G\|_{R_\lambda^{\ell,\alpha,\beta}} = \lambda^{\ell-2}\|G\|_{L^\infty_t L^2_x} + \lambda^{\ell-\alpha}\|(|\partial_t|+\lambda)^\alpha P^{(t)}_{\leq(\lambda/2^8)^2}G\|_{L^1_t L^2_x} + \lambda^{\beta-1}\|G\|_{L^2_{t,x}}.$$

Lemma 2.3 (Nested embeddings). *Let $s, a, a', b, b' \in \mathbb{R}$ with $a' \leq a$ and $b' \leq b$. Then*

$$\|u_\lambda\|_{S_\lambda^{s,a,b}} \lesssim \|u_\lambda\|_{S_\lambda^{s,a',b}}, \quad \|u_\lambda\|_{S_\lambda^{s,a,b'}} \leq \|u_\lambda\|_{S_\lambda^{s,a,b}}.$$

Similarly, if $\ell, \alpha, \alpha', \beta, \beta' \in \mathbb{R}$ with $\alpha' \leq \alpha$ and $\beta' \leq \beta$, we have

$$\|V_\lambda\|_{W_\lambda^{\ell,\alpha',\beta}} \lesssim \|V_\lambda\|_{W_\lambda^{\ell,\alpha,\beta}}, \quad \|V_\lambda\|_{W_\lambda^{\ell,\alpha,\beta'}} \leq \|V_\lambda\|_{W_\lambda^{\ell,\alpha,\beta}}.$$

Proof. The first claim follows from the characterisation (2.6). The remaining inequalities are clear from the definitions. ∎

To control the evolution of the full solution, we sum the dyadic terms in ℓ^2, and define the norms

$$\|u\|_{S^{s,a,b}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}}} \|u_\lambda\|^2_{S^{s,a,b}_\lambda}\Big)^{1/2}, \quad \|F\|_{N^{s,a,b}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}}} \|F_\lambda\|^2_{N^{s,a,b}_\lambda}\Big)^{1/2},$$

$$\|V\|_{W^{\ell,\alpha,\beta}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}}} \|V_\lambda\|^2_{W^{\ell,\alpha,\beta}_\lambda}\Big)^{1/2}, \quad \|G\|_{R^{\ell,\alpha,\beta}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}}} \|G_\lambda\|^2_{R^{\ell,\alpha,\beta}_\lambda}\Big)^{1/2} + \|G_{\leq 2^{16}}\|_{L^1_t L^2_x}.$$

Then, we define the corresponding spaces as the collection of all tempered distributions with finite norm.

Let $I \subset \mathbb{R}$ be an open interval, i.e. a connected open subset of the real line $\mathbb{R}$. We localise the norms and spaces to time intervals $I \subset \mathbb{R}$ via restriction norms. For instance, we define the restriction norm

$$\|u\|_{S^{s,a,b}(I)} = \inf_{u' \in S^{s,a,b} \text{ and } u'|_I = u} \|u'\|_{S^{s,a,b}},$$

provided that such an extension $u' \in S^{s,a,b}$ exists. The norms $\|\cdot\|_{N^{s,a,b}(I)}$, $\|\cdot\|_{W^{\ell,\alpha,\beta}(I)}$, and $\|\cdot\|_{R^{\ell,\alpha,\beta}(I)}$ and the corresponding spaces are defined similarly.

2.3. *Duhamel formulae and energy inequalities*

The solution operator for the inhomogeneous Schrödinger equation is denoted by

$$\mathcal{I}_0[F](t) = -i \int_0^t e^{i(t-s)\Delta} F(s)\, ds.$$

For a general potential $V \in L^\infty_t L^2_x$, we let

$$\mathcal{I}_V[F](t) = -i \int_0^t \mathcal{U}_V(t,s) F(s)\, ds$$

where $\mathcal{U}_V(t,s) f$ denotes the homogeneous solution operator for the Cauchy problem

$$(i\partial_t + \Delta - \Re V)u = 0, \quad u(s) = f.$$

We show later that the operators $\mathcal{U}_V$ and $\mathcal{I}_V$ are well-defined on suitable function spaces, provided only that $V \approx e^{it\Delta} f \in L^\infty_t H^{(d-4)/2}_x$, i.e. V is close to a $L^\infty_t H^{(d-4)/2}_x$ solution to the wave equation.

Similarly, we define the solution operator for the inhomogeneous half-wave equation by

$$\mathcal{J}_0[F](t) = -i \int_0^t e^{i(t-s)|\nabla|} F(s)\, ds.$$

We record here two straightforward energy inequalities which we exploit in what follows.

Lemma 2.4. *Let $s \in \mathbb{R}$, $0 \le a, b \le 1$. For any $\lambda \in 2^{\mathbb{N}}$ we have*

$$\|e^{it\Delta} f_\lambda\|_{S_\lambda^{s,a,b}} \lesssim \lambda^s \|f_\lambda\|_{L_x^2}, \quad \|\mathcal{I}_0[F_\lambda]\|_{S_\lambda^{s,a,b}} \lesssim \|F_\lambda\|_{N_\lambda^{s,a,b}}.$$

Moreover, if $0 \in I \subset \mathbb{R}$ is an open interval and $F \in N^{s,a,b}(I)$, then $\mathcal{I}_0[F] \in C(I, H^s)$.

Proof. The estimate for the free solutions follows from the fact that the temporal frequency is of size λ^2 and the endpoint Strichartz estimate.

In order to prove the estimate for the Duhamel term, in view of the characterisation (2.6) it suffices to bound the high modulation contribution $\lambda^s \|\mathcal{I}_0[P_\lambda^F F]\|_{L_t^\infty L_x^2}$ due to the (double) endpoint Strichartz estimate. To this end, we first claim that for any $\mu > 0$ and $G \in L_t^\infty L_x^2$ we have

$$\|\mathcal{I}_0[C_{>\mu} G]\|_{L_t^\infty L_x^2} \lesssim \mu^{-1} \|C_{>\mu} G\|_{L_t^\infty L_x^2}. \tag{2.8}$$

Assuming (2.8) for the moment, we conclude that

$$\|\mathcal{I}_0[C_{>(\lambda/2^8)^2} F_\lambda]\|_{L_t^\infty L_x^2} \lesssim \lambda^{-2} \|C_{>(\lambda/2^8)^2} F_\lambda\|_{L_t^\infty L_x^2}. \tag{2.9}$$

To improve this, we again use (2.8) and observe that

$$\begin{aligned}
\|\mathcal{I}_0[P^{(t)}_{>(\lambda/2^8)^2} C_{>(\lambda/2^8)^2} F_\lambda]\|_{L_t^\infty L_x^2} &\le \sum_{\nu \gtrsim \lambda^2} \|\mathcal{I}_0[P_\nu^{(t)} C_{\approx \nu} C_{>(\lambda/2^8)^2} F_\lambda]\|_{L_t^\infty L_x^2} \\
&\lesssim \sum_{\nu \gtrsim \lambda^2} \nu^{-1} \|P_\nu^{(t)} C_{>(\lambda/2^8)^2} F_\lambda\|_{L_t^\infty L_x^2} \lesssim \sum_{\nu \gtrsim \lambda^2} \nu^{-1/2} \|P_\nu^{(t)} C_{>(\lambda/2^8)^2} F_\lambda\|_{L_{t,x}^2} \\
&\lesssim \lambda^{-s-b} \|F_\lambda\|_{N_\lambda^{s,a,b}}.
\end{aligned}$$

Hence the claimed inequality follows.

To complete the proof of the norm bounds, it only remains to verify the claimed bound (2.8). Define $H(t) = (\partial_t^{-1} P^{(t)}_{>\mu}[e^{-it\Delta} G])(t)$. A computation gives the $L_t^\infty L_x^2$ bound

$$\|H\|_{L_t^\infty L_x^2} \lesssim \mu^{-1} \|e^{-it\Delta} G\|_{L_t^\infty L_x^2} = \mu^{-1} \|G\|_{L_t^\infty L_x^2}$$

and, since $C_{>\mu} G = e^{it\Delta} P^{(t)}_{>\mu}[e^{-it\Delta} G]$, the identity

$$\partial_t H(t) = P^{(t)}_{>\mu}[e^{-it\Delta} G](t) = e^{-it\Delta} C_{>\mu} G.$$

Then (2.8) follows by writing $\mathcal{I}_0[C_{>\mu} G](t) = \mathcal{I}_0[e^{it\Delta} \partial_t H](t) = -i e^{it\Delta}(H(t) - H(0))$.

We now turn to the proof of continuity. In view of the definition of the time restricted space $N^{s,a,b}(I)$, it suffices to consider the case $I = \mathbb{R}$. Moreover, the norm bound proved implies that it is enough to prove that if $\lambda \in 2^{\mathbb{N}}$ and $F_\lambda \in N^{0,a,b}$ then $\mathcal{I}_0[F_\lambda] \in C(I, L^2)$. If $\|F_\lambda\|_{N_\lambda^{s,a,b}} < \infty$ for $a, b \ge 0$, then $F_\lambda \in L^1_{t,\mathrm{loc}} L_x^2$ and the continuity follows from the dominated convergence theorem. ∎

The energy inequality has the following useful consequence.

Lemma 2.5. *Let $s, a, b \in \mathbb{R}$ with $b \geq 0$. If $F \in N^{s,a,b}$ then*

$$\lim_{t,t'\to\infty} \left\| \int_t^{t'} e^{-is\Delta} F(s)\, ds \right\|_{H^s} = 0.$$

Proof. After writing $\int_t^{t'} e^{-is\Delta} F(s)\, ds = e^{-it'\Delta} \mathcal{I}_0[F](t') - e^{-it\Delta} \mathcal{I}_0[F](t)$, the energy inequality in Lemma 2.4 implies that it suffices to prove that for every $\lambda \in 2^{\mathbb{N}}$ we have

$$\lim_{t,t'\to\infty} \left\| \int_t^{t'} e^{-is\Delta} F_\lambda(s)\, ds \right\|_{L^2_x} = 0.$$

We decompose into low and high modulation contributions, $F_\lambda = P^N_\lambda F + P^F_\lambda F$. For the former term, we observe that the endpoint Strichartz estimate gives

$$\left\| \int_t^{t'} e^{-is\Delta} P^N_\lambda F(s)\, ds \right\|_{L^2_x} \lesssim \| P^N_\lambda F \|_{L^2_t L^{2*}_x((t,t')\times\mathbb{R}^d)},$$

which vanishes as $t, t' \to \infty$ since $P^N_\lambda F \in L^2_t L^{2*}_x$. For the remaining high modulation contribution $P^F_\lambda F$, we let $G(t) = \partial_t^{-1} P^{(t)}_{\gtrsim \lambda^2}(e^{-it\Delta} P^F_\lambda F)$. Then $e^{-it\Delta} P^F_\lambda F = \partial_t G$ and therefore an application of Sobolev embedding gives, uniformly for $M \geq 1$,

$$\begin{aligned}
\left\| \int_t^{t'} e^{-is\Delta} P^F_\lambda F(s)\, ds \right\|_{L^2_x} &= \|G(t') - G(t)\|_{L^2_x} \\
&\leq \|(P^{(t)}_{\leq M} G)(t')\|_{L^2_x} + \|(P^{(t)}_{\leq M} G)(t)\|_{L^2_x} + \|(P^{(t)}_{> M} G)(t')\|_{L^2_x} + \|(P^{(t)}_{> M} G)(t)\|_{L^2_x} \\
&\lesssim_\lambda \|(P^{(t)}_{\leq M} G)(t')\|_{L^2_x} + \|(P^{(t)}_{\leq M} G)(t)\|_{L^2_x} + \|C_{>M} P^F_\lambda F\|_{L^2_{t,x}}.
\end{aligned}$$

Since $\widetilde{P^{(t)}_{\leq M} G} \in L^1_\tau L^2_\xi$ and $P^F_\lambda F \in L^2_{t,x}$, for any $\varepsilon > 0$, by choosing M sufficiently large and letting $t, t' \to \infty$ the Riemann–Lebesgue lemma implies that

$$\limsup_{t,t'\to\infty} \left\| \int_t^{t'} e^{-is\Delta} P^F_\lambda F(s)\, ds \right\|_{L^2_x} \leq \varepsilon.$$

As this holds for every $\varepsilon > 0$, the result follows. ■

We also require an energy type inequality for the wave equation.

Lemma 2.6. *Let $0 \leq \alpha \leq 1$ and $\beta, \ell \in \mathbb{R}$. Then, for all $\lambda \in 2^{\mathbb{N}}$,*

$$\|e^{it|\nabla|} g_\lambda\|_{W^{\ell,\alpha,\beta}_\lambda} \lesssim \lambda^\ell \|g_\lambda\|_{L^2},$$

and for $\lambda > 2^{16}$,

$$\|\mathcal{J}_0[G_\lambda]\|_{W^{\ell,\alpha,\beta}_\lambda} \lesssim \|G_\lambda\|_{R^{\ell,\alpha,\beta}_\lambda}.$$

Moreover, if $0 \in I \subset \mathbb{R}$ is an open interval and $G \in R^{\ell,\alpha,\beta}(I)$, then $\mathcal{J}_0[G] \in C(I, H^\ell)$.

Proof. The estimate for free solutions follows from the fact that their temporal frequencies are of size λ.

For the Duhamel integral we have

$$\lambda^{\ell}\|\mathcal{J}_0[P^{(t)}_{\ll\lambda^2}G_\lambda]\|_{L^\infty_t L^2_x} \lesssim \lambda^{\ell}\|P^{(t)}_{\ll\lambda^2}G_\lambda\|_{L^1_t L^2_x} \lesssim \lambda^{\ell-\alpha}\|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}G_\lambda\|_{L^1_t L^2_x}.$$

Similarly to (2.8) above we also obtain, for $\lambda > 2^{16}$,

$$\lambda^{\ell}\|\mathcal{J}_0[P^{(t)}_{\gtrsim\lambda^2}G_\lambda]\|_{L^\infty_t L^2_x} \lesssim \lambda^{\ell-2}\|P^{(t)}_{\gtrsim\lambda^2}G_\lambda\|_{L^\infty_t L^2_x},$$

and deduce

$$\lambda^{\ell}\|\mathcal{J}_0[G_\lambda]\|_{L^\infty_t L^2_x} \lesssim \lambda^{\ell-\alpha}\|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}G_\lambda\|_{L^1_t L^2_x} + \lambda^{\ell-2}\|P^{(t)}_{\gtrsim\lambda^2}G_\lambda\|_{L^\infty_t L^2_x}. \tag{2.10}$$

Since the bound for the $L^2_{t,x}$ component of the norm $\|\cdot\|_{W^{\ell,\alpha,\beta}}$ follows directly from the definition, it only remains to bound

$$\begin{aligned}\lambda^{\ell-\alpha}&\|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}\mathcal{J}_0[G_\lambda]\|_{L^\infty_t L^2_x}\\ &\lesssim \lambda^{\ell}\|P^{(t)}_{\lesssim\lambda}\mathcal{J}_0[G_\lambda]\|_{L^\infty_t L^2_x} + \lambda^{\ell-\alpha}\|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}P^{(t)}_{\gg\lambda}\mathcal{J}_0[G_\lambda]\|_{L^\infty_t L^2_x}.\end{aligned} \tag{2.11}$$

The first term on the right hand side of (2.11) can be bounded directly from (2.10). We turn to the second contribution in (2.11) and write

$$P^{(t)}_{\ll\lambda^2}P^{(t)}_{\gg\lambda}\mathcal{J}_0[G_\lambda] = P^{(t)}_{\ll\lambda^2}P^{(t)}_{\gg\lambda}\mathcal{J}_0[P^{(t)}_{\ll\lambda^2}G_\lambda],$$

where the identity is due to the fact that $d_\lambda = P^{(t)}_{\ll\lambda^2}P^{(t)}_{\gg\lambda}\mathcal{J}_0[P^{(t)}_{\gtrsim\lambda^2}G_\lambda]$ solves the equation $(i\partial_t + |\nabla|)d_\lambda = 0$, therefore $d_\lambda = e^{it|\nabla|}d_\lambda(0)$ and since d_λ has temporal frequencies $\gg \lambda$ it must vanish identically.

Let $e_\lambda := \mathcal{J}_0[P^{(t)}_{\ll\lambda^2}G_\lambda]$ and $f_\lambda := \mathcal{J}_0[(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}G_\lambda]$. Then it follows that $(i\partial_t + |\nabla|)((\lambda+|\partial_t|)^\alpha e_\lambda - f_\lambda) = 0$, and therefore

$$(\lambda+|\partial_t|)^\alpha e_\lambda - f_\lambda = e^{it|\nabla|}z_\lambda, \quad z_\lambda = \big((\lambda+|\partial_t|)^\alpha e_\lambda - f_\lambda\big)\big|_{t=0}.$$

Again, since the temporal frequencies of $e^{it|\nabla|}z$ are $\approx \lambda$, we conclude that

$$(\lambda+|\partial_t|)^\alpha P^{(t)}_{\gg\lambda}\mathcal{J}_0[P^{(t)}_{\ll\lambda^2}G_\lambda] = P^{(t)}_{\gg\lambda}\mathcal{J}_0[(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}G_\lambda],$$

hence

$$\begin{aligned}\|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\gg\lambda}P^{(t)}_{\ll\lambda^2}\mathcal{J}_0[G_\lambda]\|_{L^\infty_t L^2_x} &\lesssim \|\mathcal{J}_0[(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}G_\lambda]\|_{L^\infty_t L^2_x}\\ &\lesssim \|(\lambda+|\partial_t|)^\alpha P^{(t)}_{\ll\lambda^2}G_\lambda\|_{L^1_t L^2_x}.\end{aligned}$$

Concerning the continuity, we observe that if $\|G_\lambda\|_{R^{0,a,b}_\lambda} < \infty$ for $a, b \geq 0$, then $G_\lambda \in L^1_{t,\mathrm{loc}}L^2_x$ and the continuity follows from the dominated convergence theorem as in Lemma 2.4. ∎

2.4. *A product estimate for fractional derivatives*

The definition of the norms $\|\cdot\|_{S^{s,a,b}_\lambda}$ involves three distinct regions of temporal frequencies: the low modulation case $|\tau + |\xi|^2| \ll \lambda^2$, the medium modulation case $|\tau| \ll \lambda^2$, and the high modulation case $|\tau| \gg \lambda^2$. When estimating bilinear quantities, this leads to a large number of possible frequency interactions. To help alleviate the number of possible cases we have to consider, we prove the following bilinear estimate which we later exploit as a black box.

Lemma 2.7. *Let $a \in \mathbb{R}$, $\mu > 0$, and $1 \leq \tilde{p}, \tilde{q}, \tilde{r}, p, q, r \leq \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $\frac{1}{\tilde{p}} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}}$. Then*

$$\|(\mu + |\partial_t|)^a (vu)\|_{L_t^{\tilde{p}} L_x^p} \lesssim \mu^{-|a|} \|(\mu + |\partial_t|)^{|a|} v\|_{L_t^{\tilde{r}} L_x^r} \|(\mu + |\partial_t|)^a u\|_{L_t^{\tilde{q}} L_x^q}.$$

Proof. The proof is essentially known, and thus we shall be somewhat brief. The main obstruction is that we allow the endpoint case $\tilde{r} = \infty$, and, as we are working with fractional derivatives in time, this causes the usual difficulties due to the failure of the Littlewood–Paley theory. In particular, to avoid summation issues, we closely follow the proof of the endpoint Kato–Ponce type inequality in [6].

To simplify notation, and in contrast to the rest of the paper, we temporarily adopt the convention that the temporal frequency multipliers $P_\nu^{(t)}$ give an inhomogeneous decomposition over $\nu \in 2^{\mathbb{N}}$, thus

$$P_1^{(t)} = \sum_{\lambda \in 2^{\mathbb{Z}}, \lambda \leq 1} \varphi\Big(\frac{|\partial_t|}{\lambda}\Big), \quad f = \sum_{\nu \in 2^{\mathbb{N}}} P_\nu^{(t)} f$$

where φ is as in Section 2.1.

We first consider the case $a > 0$ and prove the stronger estimate

$$\begin{aligned} &\|(1 + |\partial_t|)^a (vu)\|_{L_t^{\tilde{p}} L_x^p} \\ &\quad \lesssim \|v\|_{L_t^{\tilde{r}} L_x^r} \|(1 + |\partial_t|)^a u\|_{L_t^{\tilde{q}} L_x^q} + \|(1 + |\partial_t|)^a v\|_{L_t^{\tilde{r}} L_x^r} \|u\|_{L_t^{\tilde{q}} L_x^q}. \end{aligned} \tag{2.12}$$

Clearly, after rescaling, this implies the required estimate for $a > 0$. The proof of the estimate (2.12) is a straightforward adaption of the argument in [6]. In more detail, we decompose

$$vu = \sum_{\nu \in 2^{\mathbb{N}}} P_\nu^{(t)} v P_{\leq \nu}^{(t)} u + \sum_{\nu \in 2^{\mathbb{N}}} P_{<\nu}^{(t)} v P_\nu^{(t)} u.$$

By symmetry, it is enough to consider the first term. To deal with the problem of summation over frequencies, we introduce a commutator term and write

$$\begin{aligned} &\sum_{\nu \in 2^{\mathbb{N}}} (1 + |\partial_t|)^a (P_\nu^{(t)} v P_{\leq \nu}^{(t)} u) \\ &\qquad = \sum_{\nu \in 2^{\mathbb{N}}} \big[(1 + |\partial_t|)^a (P_\nu^{(t)} v P_{\leq \nu}^{(t)} u) - ((1 + |\partial_t|)^a P_\nu^{(t)} v) P_{\leq \nu}^{(t)} u\big] \\ &\qquad\quad + \sum_{\nu \in 2^{\mathbb{N}}} ((1 + |\partial_t|)^a P_\nu^{(t)} v) P_{\leq \nu}^{(t)} u \\ &\qquad = \sum_{\nu \in 2^{\mathbb{N}}} \big[(1 + |\partial_t|)^a (P_\nu^{(t)} v P_{\leq \nu}^{(t)} u) - ((1 + |\partial_t|)^a P_\nu^{(t)} v) P_{\leq \nu}^{(t)} u\big] \\ &\qquad\quad + ((1 + |\partial_t|)^a v) u + \sum_{\nu \in 2^{\mathbb{N}}} ((1 + |\partial_t|)^a P_\nu^{(t)} v) P_{> \nu}^{(t)} u. \end{aligned} \tag{2.13}$$

The bound for the second term in (2.13) follows directly from Hölder's inequality. To bound the third term in (2.13), we note that for any $M \in 2^{\mathbb{N}}$ we have

$$\begin{aligned}
&\sum_{\nu\in 2^{\mathbb{N}}} \|(1+|\partial_t|)^a P^{(t)}_\nu v P^{(t)}_{>\nu} u\|_{L^{\tilde p}_t L^p_x} \\
&\qquad \lesssim \sum_{\nu\le M} \nu^a \|v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x} + \sum_{\nu>M} \nu^{-a} \|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_r} \\
&\qquad \lesssim M^a \|v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x} + M^{-a} \|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_r}.
\end{aligned}$$

Optimising in M then gives

$$\begin{aligned}
&\Big\| \sum_{\nu\in 2^{\mathbb{N}}} (1+|\partial_t|)^a P^{(t)}_\nu v P^{(t)}_{>\nu} u \Big\|_{L^{\tilde p}_t L^p_x} \\
&\qquad \lesssim \big(\|v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x} \|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_r} \big)^{1/2},
\end{aligned}$$

and hence (2.12) follows for the third term in (2.13). Finally, to bound the first term in (2.13), we first claim that for any $0 < \theta < 1/a$ we have the commutator estimates

$$\begin{aligned}
&\|(1+|\partial_t|)^a (P^{(t)}_\nu v P^{(t)}_{\le\nu} u) - ((1+|\partial_t|)^a P^{(t)}_\nu v) P^{(t)}_{\le\nu} u\|_{L^{\tilde p}_t L^p_x} \\
&\qquad \lesssim \nu^{-\theta a} \|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|^{\theta}_{L^{\tilde q}_t L^q_x} \|u\|^{1-\theta}_{L^{\tilde q}_t L^q_x}
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\|(1+|\partial_t|)^a (P^{(t)}_\nu v P^{(t)}_{\le\nu} u) - ((1+|\partial_t|)^a P^{(t)}_\nu v) P^{(t)}_{\le\nu} u\|_{L^{\tilde p}_t L^p_x} \\
&\qquad \lesssim \nu^{a} \|v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x}.
\end{aligned} \tag{2.15}$$

Assuming these bounds for the moment, we have, for any $M \in 2^{\mathbb{N}}$,

$$\begin{aligned}
&\sum_{\nu\in 2^{\mathbb{N}}} \|(1+|\partial_t|)^a (P^{(t)}_\nu v P^{(t)}_{\le\nu} u) - ((1+|\partial_t|)^a P^{(t)}_\nu v) P^{(t)}_{\le\nu} u\|_{L^{\tilde p}_t L^p_x} \\
&\quad \lesssim \sum_{\nu\le M} \nu^a \|v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x} \\
&\qquad + \sum_{\nu>M} \nu^{-\theta a} \|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|^{\theta}_{L^{\tilde q}_t L^q_x} \|u\|^{1-\theta}_{L^{\tilde q}_t L^q_x} \\
&\quad \lesssim M^a \|v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x} + M^{-\theta a} \|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|^{\theta}_{L^{\tilde q}_t L^q_x} \|u\|^{1-\theta}_{L^{\tilde q}_t L^q_x}.
\end{aligned}$$

Optimising in M, we conclude that

$$\begin{aligned}
&\Big\| \sum_{\nu\in 2^{\mathbb{N}}} P^{(t)}_\nu v P^{(t)}_{\le\nu} u \Big\|_{L^{\tilde p}_t L^p_x} \\
&\qquad \lesssim \big(\|(1+|\partial_t|)^a v\|_{L^{\tilde r}_t L^r_x} \|u\|_{L^{\tilde q}_t L^q_x} \big)^{\frac{1}{1+\theta}} \big(\|v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_x} \big)^{1-\frac{1}{1+\theta}}
\end{aligned}$$

and hence (2.12) follows.

It only remains to prove the standard commutator bounds (2.14) and (2.15). We begin by noting that for any $a \in \mathbb{R}$, we have the related estimate

$$\begin{aligned}\big\|(1+|\partial_t|)^a (P^{(t)}_\nu v P^{(t)}_{\ll \nu} u) - ((1+|\partial_t|)^a P^{(t)}_\nu v) P^{(t)}_{\ll\nu} u\big\|_{L^{\tilde p}_t L^p_x} \\ \lesssim \nu^{a-1} \|P^{(t)}_\nu v\|_{L^{\tilde r}_t L^r_x} \|\partial_t P^{(t)}_{\ll\nu} u\|_{L^{\tilde q}_t L^q_x}, \qquad (2.16)\end{aligned}$$

which follows by writing

$$\begin{aligned}(1+|\partial_t|)^a (P^{(t)}_\nu v P^{(t)}_{\ll\nu} u) &- ((1+|\partial_t|)^a P^{(t)}_\nu v) P^{(t)}_{\ll\nu} u \\ &= \nu^a \int_{\mathbb{R}} \nu \psi_1(s\nu)(P^{(t)}_\nu v)(t-s)\big((P^{(t)}_{\ll\nu}u)(t-s) - (P^{(t)}_{\ll\nu}u)(t)\big)\, ds \\ &= -\nu^{a-1} \int_{\mathbb{R}} \int_0^1 \nu\, \psi_2(s\nu)(P^{(t)}_\nu v)(t-s)(\partial_t P^{(t)}_{\ll\nu} u)(t-ss')\, ds'\, ds\end{aligned}$$

for some $\psi_1 \in \mathcal{S}(\mathbb{R})$ (i.e. some smooth rapidly decreasing kernel independent of ν, u, and v), $\psi_2(s) = s\psi_1(s)$, and so applying Hölder's inequality and using translation invariance, we obtain (2.16). To conclude the proof of (2.14), we note that if $a > 0$, then (2.16) also holds with $P^{(t)}_{\ll\nu} u$ replaced with $P^{(t)}_{\leq\nu} u$ (this is simply another application of Hölder and Bernstein), and hence (2.14) follows from the interpolation type bound

$$\begin{aligned}\|\partial_t P^{(t)}_{\leq\nu} u\|_{L^{\tilde q}_t L^q_x} &\leq \sum_{\nu' \in 2^{\mathbb{N}},\, \nu' \leq \nu} \nu' \|P^{(t)}_{\nu'} u\|_{L^{\tilde q}_t L^q_x} \\ &\lesssim \sum_{\nu' \in 2^{\mathbb{N}},\, \nu' \leq \nu} (\nu')^{1-\theta a} \|(1+|\partial_t|)^a u\|^{\theta}_{L^{\tilde q}_t L^q_x} \|u\|^{1-\theta}_{L^{\tilde q}_t L^q_x} \\ &\lesssim \nu^{1-\theta a} \|(1+|\partial_t|)^a u\|^{\theta}_{L^{\tilde q}_t L^q_x} \|u\|^{1-\theta}_{L^{\tilde q}_t L^q_x},\end{aligned}$$

which holds for any $0 \leq \theta < 1/a$. Finally, the second commutator bound (2.15) follows by simply discarding the commutator structure and applying Hölder's and Bernstein's inequalities. This completes the proof of (2.12) and hence the required estimate holds in the case $a > 0$.

It only remains to consider the case $a < 0$, but this follows by arguing via duality. Namely, the estimate (2.12) gives

$$\begin{aligned}&\|(1+|\partial_t|)^a (vu)\|_{L^{\tilde p}_t L^p_x} \\ &\quad = \sup_{\|w\|_{L^{\tilde p'}_t L^{p'}_x} \leq 1} \left| \int_{\mathbb{R}^{1+d}} ((1+|\partial_t|)^a w) v u \, dx\, dt \right| \\ &\quad \leq \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_x} \sup_{\|w\|_{L^{\tilde p'}_t L^{p'}_x} \leq 1} \|(1+|\partial_t|)^{|a|} (v (1+|\partial_t|)^a w)\|_{L^{\tilde q'}_t L^{q'}_x} \\ &\quad \lesssim \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_x} \\ &\qquad \times \sup_{\|w\|_{L^{\tilde p'}_t L^{p'}_x} \leq 1} \big(\|v\|_{L^{\tilde r}_t L^r_x} \|w\|_{L^{\tilde p'}_t L^{p'}_x} + \|(1+|\partial_t|)^{|a|} v\|_{L^{\tilde r}_t L^r_x} \|(1+|\partial_t|)^a w\|_{L^{\tilde p'}_t L^{p'}_x} \big) \\ &\quad \lesssim \|(1+|\partial_t|)^a u\|_{L^{\tilde q}_t L^q_x} \|(1+|\partial_t|)^{|a|} v\|_{L^{\tilde r}_t L^r_x}\end{aligned}$$

as required. ∎

2.5. Decomposability of norms

Given open intervals $I_1, I_2 \subset \mathbb{R}$ we would like to bound the norm $\|u\|_{S^{s,a,b}(I_1\cup I_2)}$ in terms of the norms $\|u\|_{S^{s,a,b}(I_1)}$ and $\|u\|_{S^{s,a,b}(I_2)}$ on the small intervals I_1 and I_2.

Lemma 2.8 (Decomposability). *There exists a constant $C > 0$ such that for any $s \in \mathbb{R}$, $0 \le a, b \le 1$, any open intervals $I_1, I_2 \subset \mathbb{R}$ with $I_1 \cap I_2 \neq \emptyset$, and any $u \in S^{s,a,b}(I_1) \cap S^{s,a,b}(I_2)$ we have*

$$\|u\|_{S^{s,a,b}(I_1\cup I_2)} \le C(1 + |I_1 \cap I_2|^{-a_\sharp})(\|u\|_{S^{s,a,b}(I_1)} + \|u\|_{S^{s,a,b}(I_2)}),$$

for $a_\sharp := \max\{a, 1/2\}$.

Proof. Let $\rho \in C^\infty(\mathbb{R})$ with $\rho(t) = 1$ for $t \le -1$, $\rho(t) = 0$ for $t \ge 1$, and for every $t \in \mathbb{R}$,

$$\rho(t) + \rho(-t) = 1.$$

After a shift, we may assume that $(-\varepsilon, \varepsilon) \subset I_1 \cap I_2$ for some $\varepsilon > 0$, and that I_1 lies to the left of I_2 (i.e. $\inf I_1 \le \inf I_2$). Define $\rho_1(t) = \rho(\varepsilon^{-1}t)$ and $\rho_2(t) = \rho(-\varepsilon^{-1}t)$ and let u^j be an extension of $u|_{I_j}$ to $\mathbb{R}$ such that $\|u\|_{S^{s,a,b}(I_j)} \sim \|u^j\|_{S^{s,a,b}}$. By construction we have $u = \rho_1 u^1 + \rho_2 u^2$ on $I_1 \cup I_2$, and hence by definition of the restriction norm

$$\begin{aligned}\|u\|_{S^{s,a,b}(I_1\cup I_2)} &\le \|\rho_1 u^1\|_{S^{s,a,b}} + \|\rho_2 u^2\|_{S^{s,a,b}} \lesssim (1+\varepsilon^{-a_\sharp})(\|u^1\|_{S^{s,a,b}} + \|u^2\|_{S^{s,a,b}}) \\ &\lesssim (1+\varepsilon^{-a_\sharp})(\|u\|_{S^{s,a,b}(I_1)} + \|u\|_{S^{s,a,b}(I_2)}),\end{aligned}$$

provided that $S^{s,a,b}$ enjoys a localisability estimate of the form

$$\|\rho_j u\|_{S^{s,a,b}} \lesssim (1 + \varepsilon^{-a_\sharp})\|u\|_{S^{s,a,b}}.$$

Taking $\varepsilon > 0$ as large as possible (namely $\varepsilon \approx |I_1 \cap I_2|$) leads to the desired estimate.

It remains to prove the above localisability, which follows from the product estimate Lemma 2.7. Indeed, for every frequency $\lambda \in 2^{\mathbb{N}}$, we have

$$\|(\lambda + |\partial_t|)^a(\rho_1 u_\lambda)\|_{L^2_t L^{2*}_x} \lesssim \|\lambda^{-a}(\lambda + |\partial_t|)^a \rho_1\|_{L^\infty_t} \|(\lambda + |\partial_t|)^a u_\lambda\|_{L^2_t L^{2*}_x},$$

where the norm of ρ_1 is bounded uniformly in λ by $\|(1 + |\partial_t|)^a[\rho(\varepsilon^{-1}t)]\|_{L^\infty_t} \lesssim 1 + \varepsilon^{-a}$. The $L^\infty_t L^2_x$ component is trivially localisable. For the remaining $L^2_{t,x}$ component of $S^{s,a,b}$, we have

$$\begin{aligned}&\Big\|\Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|}\Big)^a (i\partial_t + \Delta)(\rho_1 u_\lambda)\Big\|_{L^2_{t,x}} \\ &\quad\le \Big\|\Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|}\Big)^a [\rho_1(i\partial_t + \Delta)u_\lambda]\Big\|_{L^2_{t,x}} + \Big\|\Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|}\Big)^a [\dot\rho_1 u_\lambda]\Big\|_{L^2_{t,x}}.\end{aligned}$$

To bound the first term, we decompose u into high and low temporal frequencies and observe that another application of Lemma 2.7 gives

$$
\begin{aligned}
&\Big\| \Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|} \Big)^a [\rho_1 (i\partial_t + \Delta) u_\lambda] \Big\|_{L^2_{t,x}} \\
&\qquad \lesssim \lambda^{-2a} \| (\lambda + |\partial_t|)^a [\rho_1 (i\partial_t + \Delta) P^{(t)}_{\ll \lambda^2} u_\lambda] \|_{L^2_{t,x}} + \| \rho_1 (i\partial_t + \Delta) P^{(t)}_{\gtrsim \lambda^2} u_\lambda \|_{L^2_{t,x}} \\
&\qquad \lesssim \| \lambda^{-a} (\lambda + |\partial_t|)^a \rho_1 \|_{L^\infty_t} \lambda^{-2a} \| (\lambda + |\partial_t|)^a (i\partial_t + \Delta) P^{(t)}_{\ll \lambda^2} u_\lambda \|_{L^2_{t,x}} \\
&\qquad \quad + \| \rho_1 \|_{L^\infty_t} \| (i\partial_t + \Delta) P^{(t)}_{\gtrsim \lambda^2} u_\lambda \|_{L^2_{t,x}} \\
&\qquad \lesssim (1 + \varepsilon^{-a}) \lambda^{1-s-b} \| u_\lambda \|_{S^{s,a,b}_\lambda}.
\end{aligned}
$$

On the other hand, for the second term, we have

$$
\Big\| \Big(\frac{\lambda + |\partial_t|}{\lambda^2 + |\partial_t|} \Big)^a [\dot{\rho}_1 u_\lambda] \Big\|_{L^2_{t,x}} \lesssim \| \dot{\rho}_1 u_\lambda \|_{L^2_{t,x}} \leq \| \dot{\rho}_1 \|_{L^2_x} \| u_\lambda \|_{L^\infty_t L^2_x} \lesssim \varepsilon^{-1/2} \lambda^{-s} \| u_\lambda \|_{S^{s,a,b}_\lambda},
$$

which implies the required bound, since $b \leq 1$. ∎

3. Bilinear estimates for Schrödinger nonlinearity

In this section we prove that we can bound the Schrödinger nonlinearity in the space $N^{s,a,b}$.

Theorem 3.1 (Bilinear estimate for Schrödinger nonlinearity). *Let $d \geq 4$, $0 \leq s \leq \ell + 2$, $\beta \geq 0$, and $0 \leq a, b \leq 1$ such that*

$$
\ell \geq b + \tfrac{d-4}{2}, \quad s - \ell \leq a + 1 - b, \quad s + \ell \geq 2a, \quad \beta \geq \max\big\{ s - 1, \tfrac{d-4}{2} + b \big\}.
$$

and

$$
(s, \ell) \neq \big(\tfrac{d-2}{2} + a, \tfrac{d-4}{2} + b \big), \quad (\beta, b) \neq \big(\tfrac{d-2}{2}, 1 \big).
$$

Then

$$
\| \Re(V) u \|_{N^{s,a,b}} \lesssim \| V \|_{W^{\ell,a,\beta}} \| u \|_{S^{s,a,0}}.
$$

Proof. In view of the definition of $N^{s,a,b}$ and $W^{\ell,a,\beta}$, a short computation shows that it suffices to prove the bounds

$$
\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2(s-1-2a+b)} \| (\lambda_0 + |\partial_t|)^a P_{\lambda_0}(vu) \|^2_{L^2_{t,x}} \Big)^{1/2} \lesssim \Big(\sum_\mu \| (\mu + |\partial_t|)^a v_\mu \|^2_{L^\infty_t H^{\ell-a}_x} \Big)^{1/2} \| u \|_{S^{s,a,0}}, \tag{3.1}
$$

$$
\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \big(\lambda_0^{2s} \| P_{\lambda_0}(vu) \|^2_{L^2_t L^{2*}_x} + \lambda_0^{2(s-1+b)} \| P_{\lambda_0}(vu) \|^2_{L^2_{t,x}} \big) \Big)^{1/2} \lesssim \| v \|_{L^2_t H^{\beta+1}_x} \| u \|_{S^{s,a,0}}, \tag{3.2}
$$

$$
\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2(s-2)} \| P_{\lambda_0}(vu) \|^2_{L^\infty_t L^2_x} \Big)^{1/2} \lesssim \Big(\sum_\mu \| v_\mu \|^2_{L^\infty_t H^\ell} \Big)^{1/2} \| u \|_{L^\infty_t H^s_x}, \tag{3.3}
$$

and, under the additional assumption that $\operatorname{supp} \widetilde{v} \subset \{|\tau| \ll \langle\xi\rangle^2\}$, to prove that

$$\Big(\sum_{\lambda_0} \lambda_0^{2s} \|P^N_{\lambda_0}(vu)\|^2_{L^2_t L^{2*}_x}\Big)^{1/2} \lesssim \Big(\sum_{\mu} \|(\mu + |\partial_t|)^a v_\mu\|^2_{L^\infty_t H^{\ell-a}_x}\Big)^{1/2} \|u\|_{S^{s,a,0}}. \tag{3.4}$$

More precisely, assuming that the bounds (3.1)–(3.4) hold, we decompose

$$\begin{aligned} V = \sum_{\mu\in 2^{\mathbb{N}}} V_\mu &= \sum_{\mu\in 2^{\mathbb{N}}} P^{(t)}_{\ll \mu^2} V_\mu + \sum_{\mu\in 2^{\mathbb{N}}} P^{(t)}_{\gg\mu} P^{(t)}_{\gtrsim \mu^2} V_\mu + \sum_{\mu\in 2^{\mathbb{N}}} P^{(t)}_{\lesssim\mu} P^{(t)}_{\gtrsim \mu^2} V_\mu \\ &= V_1 + V_2 + V_3. \end{aligned}$$

An application of (3.1), (3.3), and (3.4) (together with the invariance of the right hand side with respect to complex conjugation) gives

$$\begin{aligned} \|\Re(V_1)u\|_{N^{s,a,b}} &\lesssim \Big(\sum_{\mu} \|(\mu + |\partial_t|)^a P^{(t)}_{\ll\mu^2} V_\mu\|^2_{L^\infty_t H^{\ell-a}_x}\Big)^{1/2} \|u\|_{S^{s,a,0}} \\ &\lesssim \|V\|_{W^{\ell,a,\beta}} \|u\|_{S^{s,a,0}}. \end{aligned}$$

On the other hand, for the V_2 contribution, we note that since

$$\begin{aligned} \|V_2\|_{L^2_t H^{\beta+1}_x} &\approx \Big(\sum_{\mu} \mu^{2(\beta+1)} \|P^{(t)}_{\gg\mu} V_\mu\|^2_{L^2_{t,x}}\Big)^{1/2} \\ &\lesssim \Big(\sum_{\mu} \mu^{2(\beta-1)} \|(i\partial_t + |\nabla|) P^{(t)}_{\gg\mu} V_\mu\|^2_{L^2_{t,x}}\Big)^{1/2} \lesssim \|V\|_{W^{\ell,a,\beta}} \end{aligned}$$

an application of (3.2) and (3.3) implies that

$$\|\Re(V_2)u\|_{N^{s,a,b}} \lesssim (\|V_2\|_{L^\infty_t H^\ell_x} + \|V_2\|_{L^2_t H^{\beta+1}_x}) \|u\|_{S^{s,a,0}} \lesssim \|V\|_{W^{\ell,a,\beta}} \|u\|_{S^{s,a,0}}$$

as required. Finally, the bound for the V_3 contribution follows from the fact that $\operatorname{supp} \widetilde{V}_3 \subset \{|\tau| + |\xi| \lesssim 1\}$ together with (3.1), (3.3), and the estimate (3.6) below.

We now turn to the proof of the bounds (3.1)–(3.4). For the first estimate (3.1), we begin by decomposing the product vu into

$$\begin{aligned} P_{\lambda_0}(vu) &= \sum_{\lambda_1\in 2^{\mathbb{N}}} P_{\lambda_0}(vu_{\lambda_1}) \\ &= \sum_{\lambda_1\ll\lambda_0} P_{\lambda_0}(vu_{\lambda_1}) + \sum_{\lambda_1\gg\lambda_0} P_{\lambda_0}(vu_{\lambda_1}) + \sum_{\lambda_1\approx\lambda_0} P_{\lambda_0}(vu_{\lambda_1}) \end{aligned} \tag{3.5}$$

and consider the high-low interactions $\lambda_0 \gg \lambda_1$, low-high interactions $\lambda_0 \ll \lambda_1$, and the balanced interactions case $\lambda_0 \approx \lambda_1$.

Case 1: $\lambda_0 \gg \lambda_1$**.** Applying the product estimate of Lemma 2.7 together with Sobolev embedding gives

$$\begin{aligned} &\lambda_0^{s-1-2a+b} \|(\lambda_0 + |\partial_t|)^a P_{\lambda_0}(vu_{\lambda_1})\|_{L^2_{t,x}} \\ &\quad\lesssim \lambda_0^{s-1-2a+b} \lambda_0^{-a} \|(\lambda_0 + |\partial_t|)^a v_{\approx\lambda_0}\|_{L^\infty_t L^2_x} \lambda_1^{\frac{d-2}{2}} \|(\lambda_0 + |\partial_t|)^a u_{\lambda_1}\|_{L^2_t L^{2*}_x} \\ &\quad\lesssim \lambda_0^{s-\ell-1-a+b} \lambda_1^{\frac{d-2}{2}+a-s} \|(\lambda_0 + |\partial_t|)^a v_{\approx\lambda_0}\|_{L^\infty_t H^{\ell-a}_x} \lambda_1^{s-2a} \|(\lambda_1 + |\partial_t|)^a u_{\lambda_1}\|_{L^2_t L^{2*}_x}. \end{aligned}$$

Therefore, provided that

$$s - \ell \leq a + 1 - b, \quad \ell \geq \tfrac{d-4}{2} + b, \quad (s, \ell) \neq \big(\tfrac{d-2}{2} + a, \tfrac{d-4}{2} + b\big),$$

we obtain

$$\begin{aligned}
&\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2(s-1-2a+b)} \Big\| \sum_{\lambda_1 \ll \lambda_0} (\lambda_0 + |\partial_t|)^a P_{\lambda_0}(v u_{\lambda_1}) \Big\|_{L^2_{t,x}}^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \Big(\sum_{\lambda_1 \ll \lambda_0} \lambda_0^{s-\ell-1-a+b} \lambda_1^{\frac{d-2}{2}+a-s} \|(\lambda_0 + |\partial_t|)^a v_{\approx \lambda_0}\|_{L^\infty_t H^{\ell-a}_x} \|u_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}} \Big)^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{\mu \in 2^{\mathbb{N}}} \|(\mu + |\partial_t|)^a v_\mu\|^2_{L^\infty_t H^{\ell-a}} \Big)^{1/2} \sup_{\lambda_1} \|u_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}}
\end{aligned}$$

as required.

Case 2: $\lambda_0 \ll \lambda_1$. We begin by observing that an application of the Sobolev embedding $W^{\frac{d-2}{2}, \frac{d}{d-1}}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ implies that

$$\begin{aligned}
\Big(\sum_{\lambda_0 \lesssim \lambda_1} \lambda_0^{2(s-1-2a+b)} \|F_{\lambda_0}\|^2_{L^2_{t,x}} \Big)^{1/2} &\lesssim \|F_{\lesssim \lambda_1}\|_{L^2_t H^{s-1-2a+b}_x} \\
&\lesssim \|F_{\lesssim \lambda_1}\|_{L^2_t W_x^{\frac{d-2}{2}+s-1-2a+b, \frac{d}{d-1}}} \lesssim \lambda_1^{(s+\frac{d-4}{2}-2a+b)_+} \|F\|_{L^2_t L_x^{\frac{d}{d-1}}}.
\end{aligned}$$

On the other hand, again applying Lemma 2.7 gives

$$\begin{aligned}
&\|(\lambda_1 + |\partial_t|)^a (v_{\approx \lambda_1} u_{\lambda_1})\|_{L^2_t L_x^{\frac{d}{d-1}}} \\
&\qquad \lesssim \lambda_1^{-a} \|(\lambda_1 + |\partial_t|)^a v_{\approx \lambda_1}\|_{L^\infty_t L^2_x} \|(\lambda_1 + |\partial_t|)^a u_{\lambda_1}\|_{L^2_t L^{2^*}_x} \\
&\qquad \lesssim \lambda_1^{2a-s-\ell} \|(\lambda_1 + |\partial_t|)^a v_{\approx \lambda_1}\|_{L^\infty_t H^{\ell-a}_x} \|u_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}}.
\end{aligned}$$

Hence, provided that

$$s + \ell \geq 2a, \quad \ell \geq \tfrac{d-4}{2} + b,$$

we see that

$$\begin{aligned}
&\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2(s-1-2a+b)} \Big\| (\lambda_0 + |\partial_t|)^a \sum_{\lambda_1 \gg \lambda_0} P_{\lambda_0}(v u_{\lambda_1}) \Big\|^2_{L^2_{t,x}} \Big)^{1/2} \\
&\qquad \lesssim \sum_{\lambda_1 \in 2^{\mathbb{N}}} \Big(\sum_{\lambda_0 \lesssim \lambda_1} \lambda_0^{2(s-1-2a+b)} \|(\lambda_1 + |\partial_t|)^a P_{\lambda_0}(v_{\approx \lambda_1} u_{\lambda_1})\|^2_{L^2_{t,x}} \Big)^{1/2} \\
&\qquad \lesssim \sum_{\lambda_1 \in 2^{\mathbb{N}}} \lambda_1^{(s+\frac{d-4}{2}-2a+b)_+} \|(\lambda_1 + |\partial_t|)^a (v_{\approx \lambda_1} u_{\lambda_1})\|_{L^2_t L_x^{\frac{d}{d-1}}} \\
&\qquad \lesssim \sum_{\lambda_1 \in 2^{\mathbb{N}}} \lambda_1^{(s+\frac{d-4}{2}-2a+b)_+ + 2a-s-\ell} \|(\lambda_1 + |\partial_t|)^a v_{\approx \lambda_1}\|_{L^\infty_t H^{\ell-a}_x} \|u_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}} \\
&\qquad \lesssim \Big(\sum_{\mu \in 2^{\mathbb{N}}} \|(\mu + |\partial_t|)^a v_\mu\|^2_{L^\infty_t H^{\ell-a}_x} \Big)^{1/2} \|u\|_{S^{s,a,0}}.
\end{aligned}$$

Case 3: $\lambda_0 \approx \lambda_1$**.** Similar to the above, we have

$$\begin{aligned}
&\lambda_0^{s-1-2a+b} \|(\lambda_0 + |\partial_t|)^a P_{\lambda_0}(v u_{\lambda_1})\|_{L^2_{t,x}} \\
&\qquad \lesssim \lambda_0^{s-1-2a+b} \lambda_0^{-a} \|(\lambda_0 + |\partial_t|)^a v_{\lesssim \lambda_0}\|_{L^\infty_t L^d_x} \|(\lambda_1 + |\partial_t|)^a u_{\lambda_1}\|_{L^2_t L^{2^*}_x} \\
&\qquad \lesssim \lambda_0^{b-1} \|(\langle \nabla \rangle + |\partial_t|)^a v_{\lesssim \lambda_0}\|_{L^\infty_t H_x^{\frac{d-2}{2}-a}} \lambda_1^{s-2a} \|(\lambda_1 + |\partial_t|)^a u_{\lambda_1}\|_{L^2_t L^{2^*}_x},
\end{aligned}$$

which is summable provided that

$$b \le 1, \quad \ell \ge \tfrac{d-4}{2} + b.$$

This completes the proof of (3.1).

We now turn to the proof of the second estimate (3.2). As previously, we apply the frequency decomposition (3.5) and consider each frequency interaction separately.

Case 1: $\lambda_0 \gg \lambda_1$**.** We start by noting that an application of Sobolev embedding gives

$$\sup_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{s-\beta-1} (\|u_{\ll \lambda_0}\|_{L^\infty_t L^d_x} + \lambda_0^{b-1} \|u_{\ll \lambda_0}\|_{L^\infty_{t,x}}) \lesssim \|u\|_{L^\infty_t H^s_x} \lesssim \|u\|_{S^{s,a,0}}$$

provided that

$$s \le \beta + 1, \quad \beta \ge \tfrac{d-4}{2} + b, \quad (\beta, b) \ne (\tfrac{d-2}{2}, 1).$$

Hence via Hölder's inequality we obtain

$$\begin{aligned}
&\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} (\lambda_0^{2s} \|P_{\lambda_0}(v u_{\ll \lambda_0})\|^2_{L^2_t L^{2*}_x} + \lambda_0^{2(s-1+b)} \|P_{\lambda_0}(v u_{\ll \lambda_0})\|^2_{L^2_{t,x}}) \Big)^{1/2} \\
&\quad \lesssim \Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2(\beta+1)} \|v_{\approx \lambda_0}\|^2_{L^2_{t,x}} \Big)^{1/2} \sup_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{s-\beta-1} (\|u_{\ll \lambda_0}\|_{L^\infty_t L^d_x} + \lambda_0^{b-1} \|u_{\ll \lambda_0}\|_{L^\infty_{t,x}}) \\
&\quad \lesssim \|v\|_{L^2_t H^{\beta+1}_x} \|u\|_{S^{s,a,0}}.
\end{aligned}$$

Case 2: $\lambda_0 \ll \lambda_1$**.** An application of Bernstein's inequality together with the square function characterisation of L^p_x gives

$$\begin{aligned}
\Big(\sum_{\lambda_0 \ll \lambda_1} (\lambda_0^{2s} \|F_{\lambda_0}\|^2_{L^2_t L^{2*}_x} + \lambda_0^{2(s-1+b)} \|F_{\lambda_0}\|^2_{L^2_{t,x}}) \Big)^{1/2} &\lesssim \lambda_1^{s+b} \Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \|F_{\lambda_0}\|^2_{L^2_t L^{2*}_x} \Big)^{1/2} \\
&\lesssim \lambda_1^{s+b} \Big\| \Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} |F_{\lambda_0}|^2 \Big)^{1/2} \Big\|_{L^2_t L^{2*}_x} \lesssim \lambda_1^{s+b} \|F\|_{L^2_t L^{2*}_x}.
\end{aligned}$$

Therefore applying Bernstein's inequality and Hölder's inequality we conclude that

$$\begin{aligned}
&\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2s} \Big\| \sum_{\lambda_1 \gg \lambda_0} P_{\lambda_0}(v u_{\lambda_1}) \Big\|^2_{L^2_t L^{2*}_x} + \lambda_0^{2(s-1+b)} \Big\| \sum_{\lambda_1 \gg \lambda_0} P_{\lambda_0}(v u_{\lambda_1}) \Big\|^2_{L^2_{t,x}} \Big)^{1/2} \\
&\quad \lesssim \sum_{\lambda_1 \in 2^{\mathbb{N}}} \Big(\sum_{\lambda_0 \ll \lambda_1} (\lambda_0^{2s} \|P_{\lambda_0}(v_{\approx \lambda_1} u_{\lambda_1})\|^2_{L^2_t L^{2*}_x} + \lambda_0^{2(s-1+b)} \|P_{\lambda_0}(v_{\approx \lambda_1} u_{\lambda_1})\|^2_{L^2_{t,x}}) \Big)^{1/2}
\end{aligned}$$

$$\lesssim \sum_{\lambda_1 \in 2^{\mathbb{N}}} \lambda_1^{s+b} \|v_{\approx \lambda_1} u_{\lambda_1}\|_{L_t^2 L_x^{2*}}$$
$$\lesssim \sum_{\lambda_1 \in 2^{\mathbb{N}}} \lambda_1^{s+b+\frac{d-2}{2}} \|v_{\approx \lambda_1}\|_{L_{t,x}^2} \|u_{\lambda_1}\|_{L_t^\infty L_x^2} \lesssim \|v\|_{L_t^2 H_x^{\beta+1}} \|u\|_{S^{s,a,0}}$$

provided that

$$\beta \geq \tfrac{d-4}{2} + b.$$

Case 3: $\lambda_0 \approx \lambda_1$**.** Let $\frac{1}{r} = \frac{1-b}{d}$. Similar to the above, an application of Sobolev embedding gives

$$\|v\|_{L_t^2 L_x^d} + \|v\|_{L_t^2 L_x^r} \lesssim \|v\|_{L_t^2 H_x^{\beta+1}}$$

provided that

$$\beta \geq \tfrac{d-4}{2} + b, \quad (\beta, b) \neq \big(\tfrac{d-2}{2}, 1\big).$$

Consequently, via Bernstein's inequality we have

$$\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2s} \|v u_{\approx \lambda_0}\|_{L_t^2 L_x^{2*}}^2 + \lambda_0^{2(s-1+b)} \|v u_{\approx \lambda_0}\|_{L_{t,x}^2}^2 \Big)^{1/2}$$
$$\lesssim (\|v\|_{L_t^2 L_x^d} + \|v\|_{L_t^2 L_x^r}) \Big(\sum_{\lambda_0 \in 2^{\mathbb{N}}} \lambda_0^{2s} \|u_{\approx \lambda_0}\|_{L_t^\infty L_x^2}^2 \Big)^{1/2} \lesssim \|v\|_{L_t^2 H_x^{\beta+1}} \|u\|_{S^{s,a,0}}.$$

This completes the proof of (3.2).

The $L_t^\infty L_x^2$ bound (3.3) holds provided that $s \leq \ell + 2$, $\ell \geq \frac{d-4}{2}$, and $(s, \ell) \neq (\frac{d}{2}, \frac{d-4}{2})$. The proof is standard, and follows by adapting the proof of the product estimate $\|fg\|_{H^{s-2}} \lesssim \|f\|_{H^\ell} \|g\|_{H^s}$.

We now turn to the proof of the final estimate (3.4). As before, we decompose the inner sum into high-low interactions $\lambda_0 \gg \lambda_1$, low-high interactions $\lambda_0 \ll \lambda_1$, and the balanced interactions case $\lambda_0 \approx \lambda_1$, and consider each case separately.

Case 1: $\lambda_0 \gg \lambda_1$**.** The assumption on the Fourier support of v implies the non-resonant identity

$$C_{\leq (\lambda_0/2^8)^2} P_{\lambda_0}(v u_{\lambda_1}) = C_{\leq (\lambda_0/2^8)^2} P_{\lambda_0}(v_{\approx \lambda_0} P_{\approx \lambda_0^2}^{(t)} C_{>(\lambda_1/2^8)^2} u_{\lambda_1}).$$

Hence the disposability of the multiplier $P_{\lambda_0}^N$ and Bernstein's inequality give

$$\|P_{\lambda_0}^N(v u_{\lambda_1})\|_{L_t^2 L_x^{2*}} \lesssim \lambda_1^{d/2-1} \|v_{\approx \lambda_0}\|_{L_t^\infty L_x^2} \|P_{\approx \lambda_0^2}^{(t)} C_{>(\lambda_1/2^8)^2} u_{\lambda_1}\|_{L_{t,x}^2}$$
$$\lesssim \lambda_1^{d/2-1} \lambda_0^{-2} \|v_{\approx \lambda_0}\|_{L_t^\infty L_x^2} \Big\| \Big(\frac{\lambda_1 + |\partial_t|}{\lambda_1^2 + |\partial_t|} \Big)^a (i\partial_t + \Delta) u_{\lambda_1} \Big\|_{L_{t,x}^2}.$$

Consequently, we conclude that

$$\lambda_0^s \|P_{\lambda_0}^N(v u_{\lambda_1})\|_{L_t^2 L_x^{2*}} \lesssim \lambda_0^{s-\ell-2} \lambda_1^{d/2-s} \|v_{\approx \lambda_0}\|_{L_t^\infty H_x^\ell} \|u\|_{S^{s,a,0}},$$

which is summable provided that

$$s \leq \ell + 2, \quad \ell \geq \tfrac{d-4}{2}, \quad (s, \ell) \neq \left(\tfrac{d}{2}, \tfrac{d-4}{2}\right).$$

Case 2: $\lambda_0 \ll \lambda_1$. We first observe that the Fourier support assumption on v implies that

$$C_{\leq(\lambda_0/2^8)^2} P_{\lambda_0}(v u_{\lambda_1}) = C_{\leq(\lambda_0/2^8)^2} P_{\lambda_0}(v_{\approx \lambda_1} C_{\approx \lambda_1^2} u_{\lambda_1}).$$

Bernstein's inequality and the temporal product estimate in Lemma 2.7 implies

$$\begin{aligned}
\lambda_0^s \| P_{\lambda_0}^N (v u_{\lambda_1}) \|_{L_t^2 L_x^{2*}} &\lesssim \lambda_0^{s + \frac{d-2}{2} - 2a} \| (\lambda_1 + |\partial_t|)^a (v_{\approx \lambda_1} C_{\approx \lambda_1^2} u_{\lambda_1}) \|_{L_t^2 L_x^1} \\
&\lesssim \lambda_0^{s + \frac{d-2}{2} - 2a} \lambda_1^{-a} \| (\lambda_1 + |\partial_t|)^a v_{\approx \lambda_1} \|_{L_t^\infty L_x^2} \| (\lambda_1 + |\partial_t|)^a C_{\approx \lambda_1^2} u_{\lambda_1} \|_{L_{t,x}^2} \\
&\lesssim \lambda_0^{s + \frac{d-2}{2} - 2a} \lambda_1^{2a - s - \ell - 1} \| (\lambda_1 + |\partial_t|)^a v_{\approx \lambda_1} \|_{L_t^\infty H_x^{\ell - a}} \| u_{\lambda_1} \|_{S_{\lambda_1}^{s,a,0}},
\end{aligned}$$

which is certainly summable under the assumption that

$$s + \ell \geq 2a, \quad \ell \geq \tfrac{d-4}{2}.$$

Case 3: $\lambda_0 \approx \lambda_1$. We now consider the remaining high-high interactions. Via the product estimate in Lemma 2.7 we obtain

$$\begin{aligned}
\lambda_0^s \| P_{\lambda_0}^N (v u_{\lambda_1}) \|_{L_t^2 L_x^{2*}} &\lesssim \lambda_1^{s - 2a} \| (\lambda_1 + |\partial_t|)^a (v_{\lesssim \lambda_1} u_{\lambda_1}) \|_{L_t^2 L_x^{2*}} \\
&\lesssim \lambda_1^{s - 2a} \lambda_1^{-a} \| (\lambda_1 + |\partial_t|)^a v_{\lesssim \lambda_1} \|_{L_t^\infty L_x^{d/2}} \| (\lambda_1 + |\partial_t|)^a u_{\lambda_1} \|_{L_t^2 L_x^{2*}} \\
&\lesssim \| (\langle \nabla \rangle + |\partial_t|)^a v \|_{L_t^\infty H_x^{\ell - a}} \| u_{\lambda_1} \|_{S_{\lambda_1}^{s,a,0}},
\end{aligned} \tag{3.6}$$

where we have used $\ell \geq \frac{d-4}{2}$ for the Sobolev embedding, and the summation is trivial in this case. ∎

We require a local version of the bilinear estimate, with the advantage that we can place v in dispersive norms of the form $L_t^\infty L_x^2 + L_t^2 L_x^d$.

Corollary 3.2. *Let $d \geq 4$. Assume that $\beta \geq \max\{\frac{d-4}{2}, s-1\}$ and*

$$0 \leq a \leq 1, \quad 0 \leq s \leq \ell + 2, \quad \ell \geq \tfrac{d-4}{2}, \quad s - \ell \leq a + 1, \quad s + \ell \geq 2a, \tag{3.7}$$

with $(s, \ell) \neq (\frac{d-2}{2} + a, \frac{d-4}{2})$. There exists $C > 0$ such that for any interval $0 \in I \subset \mathbb{R}$,

$$\| \mathcal{I}_0 (\Re(V) u) \|_{S^{s,a,0}(I)} \leq C \| V \|_{W^{\ell,a,\beta}(I) + L_t^2 W_x^{s,d}(I \times \mathbb{R}^d)} \| u \|_{S^{s,a,0}(I)}.$$

Proof. In view of Lemma 2.4 and Theorem 3.1, it suffices to prove that for any $s \geq 0$ and $0 \leq a \leq 1$ we have

$$\| \mathcal{I}_0 (\Re(V) u) \|_{S^{s,a,0}(I)} \lesssim \| V \|_{L_t^2 W_x^{s,d}(I \times \mathbb{R}^d)} \| u \|_{S^{s,a,0}(I)}. \tag{3.8}$$

An application of Bernstein's inequality together with (2.6) gives

$$\|u_\lambda\|_{S^{s,a,0}_\lambda} \approx \lambda^s(\|u_\lambda\|_{L^\infty_t L^2_x} + \|P^N_\lambda u\|_{L^2_t L^{2*}_x}) + \lambda^{s-1}\left\| \left(\frac{\lambda+|\partial_t|}{\lambda^2+|\partial_t|}\right)^a (i\partial_t+\Delta)u_\lambda \right\|_{L^2_{t,x}}$$
$$\lesssim \lambda^s\big(\|u_\lambda\|_{L^\infty_t L^2_x} + \|u_\lambda\|_{L^2_t L^{2*}_x} + \|(i\partial_t+\Delta)u_\lambda\|_{L^2_t L^{2*}_x}\big)$$

and hence the endpoint Strichartz estimate implies that after extending F from I to $\mathbb{R}$ by zero,

$$\|\mathcal{I}_0[F]\|_{S^{s,a,0}(I)} \leq \|\mathcal{I}_0[\mathbb{1}_I F]\|_{S^{s,a,0}} \lesssim \Big(\sum_{\lambda\in 2^{\mathbb{N}}} \lambda^{2s}\|F_\lambda\|^2_{L^2_t L^{2*}_x(I\times\mathbb{R}^d)}\Big)^{1/2}$$
$$\lesssim \|F\|_{L^2_t W^{s,2*}_x(I\times\mathbb{R}^d)}.$$

Inequality (3.8) then follows from the elementary product estimate

$$\|fg\|_{W^{s,2*}(\mathbb{R}^d)} \lesssim \|f\|_{W^{s,d}(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)},$$

which holds for any $s \geq 0$. ■

4. Bilinear estimates for the wave nonlinearity

Here we give the bilinear estimates required to control solutions to

$$(i\partial_t + |\nabla|)v = |\nabla|(\overline{\varphi}\psi), \quad v(0) = 0$$

with $\varphi, \psi \in S^{s,a,b}$. The main estimate we prove is the following.

Theorem 4.1 (Bilinear estimate for wave nonlinearity). *Let $d \geq 4$, $s, \ell, \beta \geq 0$, and $0 \leq a, b \leq 1$ satisfy*

$$\beta \leq \min\big\{s, 2s - \tfrac{d-2}{2} - a\big\}, \quad 2a \leq 2s - \ell - \tfrac{d-2}{2}, \quad a - b \leq s - \ell$$

and

$$(s,\ell) \neq \big(\tfrac{d}{2}, \tfrac{d+2}{2}\big), \big(\tfrac{d-2}{2} + a, \tfrac{d-2}{2} + b\big), \quad (s,\beta) \neq \big(\tfrac{d-2}{2} + a, \tfrac{d-2}{2} + a\big).$$

If $\varphi, \psi \in S^{s,a,b}$, then

$$\|\mathcal{J}_0(|\nabla|(\overline{\varphi}\psi))\|_{W^{\ell,a,\beta}} \lesssim \|\varphi\|_{S^{s,a,b}}\|\psi\|_{S^{s,a,b}}.$$

Proof. An application of the energy inequality of Lemma 2.6 implies that it suffices to prove the bounds

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}} \mu^{2(\ell-a+1)}\|(\mu+|\partial_t|)^a P_\mu P^{(t)}_{\ll\mu^2}(\overline{\varphi}\psi)\|^2_{L^1_t L^2_x}\Big)^{1/2} \lesssim \|\varphi\|_{S^{s,a,b}}\|\psi\|_{S^{s,a,b}}, \quad (4.1)$$

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}} \mu^{2(\ell-1)} \|P_\mu(\overline{\varphi}\psi)\|^2_{L^\infty_t L^2_x}\Big)^{1/2} \lesssim \|\varphi\|_{S^{s,a,b}} \|\psi\|_{S^{s,a,b}}, \tag{4.2}$$

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}} \mu^{2\beta} \|P_\mu(\overline{\varphi}\psi)\|^2_{L^2_{t,x}}\Big)^{1/2} \lesssim \|\varphi\|_{S^{s,a,b}} \|\psi\|_{S^{s,a,b}}, \tag{4.3}$$

$$\|P_{\leq 2^{16}}(\overline{\varphi}\psi)\|_{L^1_t L^2_x} \lesssim \|\varphi\|_{S^{s,a,b}} \|\psi\|_{S^{s,a,b}}. \tag{4.4}$$

We start with the proof of (4.1) and decompose the product $\overline{\varphi}\psi$ into the standard frequency trichotomy

$$P_\mu(\overline{\varphi}\psi) = P_\mu(\overline{\varphi}\psi_{\ll\mu}) + \sum_{\lambda_1\approx\lambda_2\gtrsim\mu} P_\mu(\overline{\varphi}_{\lambda_1}\psi_{\lambda_2}) + P_\mu(\overline{\varphi}_{\ll\mu}\psi). \tag{4.5}$$

In view of the fact that the left hand side of (4.1) is invariant with respect to complex conjugation, it suffices to consider the first two terms in (4.5), i.e. the high-low and high-high frequency interactions.

Proof of (4.1)*, case 1: high-low interactions.* Note that in this case we must have $\mu \gg 1$. A computation then gives the non-resonant identity

$$\begin{aligned} P^{(t)}_{\ll\mu^2} P_\mu(\overline{\varphi}\psi_{\ll\mu}) &= P^{(t)}_{\ll\mu^2} P_\mu(\overline{C_{\ll\mu^2}\varphi}_{\approx\mu} P^{(t)}_{\approx\mu^2}\psi_{\ll\mu}) + P^{(t)}_{\ll\mu^2} P_\mu(\overline{C_{\gtrsim\mu^2}\varphi}_{\approx\mu}\psi_{\ll\mu}) \\ &= A_1 + A_2. \end{aligned}$$

To bound the A_1 term, we observe that

$$\begin{aligned} &\mu^{\ell+1-a}\|(|\partial_t| + \mu)^a P^{(t)}_{\ll\mu^2} P_\mu(\overline{C_{\ll\mu^2}\varphi}_{\approx\mu} P^{(t)}_{\approx\mu^2}\psi_{\ll\mu})\|_{L^1_t L^2_x} \\ &\qquad \lesssim \mu^{\ell+1+a}\|C_{\ll\mu^2}\varphi_{\approx\mu}\|_{L^2_t L^{2^*}_x} \|P^{(t)}_{\approx\mu^2}\psi_{\ll\mu}\|_{L^2_t L^d_x} \\ &\qquad \lesssim \mu^{\ell-s-1+a}\mu^{s-2a}\|(\mu + |\partial_t|)^a\varphi_{\approx\mu}\|_{L^2_t L^{2^*}_x} \|(i\partial_t + \Delta) P^{(t)}_{\approx\mu^2}\psi_{\ll\mu}\|_{L^2_t H^{d-2/2}_x} \\ &\qquad \lesssim \mu^{\ell-s-1+a+(d/2-s)_+}\mu^{s-2a}\|(\mu + |\partial_t|)^a\varphi_{\approx\mu}\|_{L^2_t L^{2^*}_x} \|\psi\|_{S^{s,a,0}}. \end{aligned}$$

Provided that

$$s-\ell \geq a-b, \quad 2s-\ell-\tfrac{d-2}{2} \geq a$$

we can sum up over $\mu \gg 1$ to obtain (4.1) for the A_1 contribution. To bound A_2, we apply the temporal product estimate in Lemma 2.7 to obtain

$$\begin{aligned} &\mu^{\ell+1-a}\|(\mu + |\partial_t|)^a P^{(t)}_{\ll\mu^2} P_\mu(\overline{C_{\gtrsim\mu^2}\varphi}_{\approx\mu}\psi_{\ll\mu})\|_{L^1_t L^2_x} \\ &\quad \lesssim \mu^{\ell+1-2a}\|(\mu + |\partial_t|)^a C_{\gtrsim\mu^2}\varphi_{\approx\mu}\|_{L^2_{t,x}} \|(\mu + |\partial_t|)^a\psi_{\ll\mu}\|_{L^2_t L^\infty_x} \\ &\quad \lesssim \mu^{\ell-1}\Big\|\Big(\frac{\mu + |\partial_t|}{\mu^2 + |\partial_t|}\Big)^a (i\partial_t + \Delta)\varphi_{\approx\mu}\Big\|_{L^2_{t,x}} \sum_{\lambda\ll\mu}\Big(\frac{\mu}{\lambda}\Big)^a \lambda^{\frac{d-2}{2}}\|(\lambda + |\partial_t|)^a\psi_\lambda\|_{L^2_t L^{2^*}_x} \\ &\quad \lesssim \mu^{\ell-s-b+a}\sum_{\lambda\ll\mu}\lambda^{\frac{d-2}{2}+a-s}\Big\|\Big(\frac{\mu + |\partial_t|}{\mu^2 + |\partial_t|}\Big)^a (i\partial_t + \Delta)\varphi_{\approx\mu}\Big\|_{L^2_t H^{s+b-1}_x} \|\psi\|_{S^{s,a,0}}. \end{aligned}$$

This can be summed up over $\mu \gg 1$ to give (4.1) for the A_2 contribution provided that

$$s-\ell \geq a-b, \quad 2s-\ell-\tfrac{d-2}{2} \geq 2a-b, \quad (s,\ell) \neq \big(\tfrac{d-2}{2}+a, \tfrac{d-2}{2}+b\big).$$

Proof of (4.1)*, case 2: high-high interactions.* An application of the product estimate in Lemma 2.7 together with Bernstein's inequality gives

$$\begin{aligned}\lambda_1^{\ell+1-a}\|(\lambda_1+|\partial_t|)^a(\overline{\varphi}_{\lambda_1}\psi_{\lambda_2})\|_{L^1_tL^2_x} &\lesssim \lambda_1^{\ell+1-2a}\lambda_1^{\frac{d-4}{2}}\|(\lambda_1+|\partial_t|)^a\varphi_{\lambda_1}\|_{L^2_tL^{2^*}_x}\|(\lambda_2+|\partial_t|)^a\psi_{\lambda_2}\|_{L^2_tL^{2^*}_x}\\ &\lesssim \lambda_1^{\ell+\frac{d-2}{2}+2a-2s}\|\varphi_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,a,0}_{\lambda_2}}.\end{aligned}$$

On the other hand, since $\ell+1-a \geq 0$, we have

$$\begin{aligned}&\Big(\sum_{\mu\lesssim\lambda_1}\mu^{2(\ell+1-a)}\|(\mu+|\partial_t|)^aP_\mu P^{(t)}_{\ll\mu^2}F\|^2_{L^1_tL^2_x}\Big)^{1/2}\\ &\qquad\lesssim \lambda_1^{\ell+1-a}\Big(\sum_{\mu\lesssim\lambda_1}\|(\lambda_1+|\partial_t|)^aP_\mu F\|^2_{L^1_tL^2_x}\Big)^{1/2} \lesssim \lambda_1^{\ell+1-a}\|(\lambda_1+|\partial_t|)^aF\|_{L^1_tL^2_x}.\end{aligned}$$

Therefore summing up gives

$$\begin{aligned}&\Big(\sum_{\mu\gg1}\mu^{2(\ell+1-a)}\Big\|\sum_{\lambda_1\approx\lambda_2\gtrsim\mu}(\mu+|\partial_t|)^aP_\mu P^{(t)}_{\ll\mu^2}(\overline{\varphi}_{\lambda_1}\psi_{\lambda_2})\Big\|^2_{L^1_tL^2_x}\Big)^{1/2}\\ &\qquad\lesssim \sum_{\lambda_1\approx\lambda_2}\lambda_1^{\ell+1-a}\|(\lambda_1+|\partial_t|)^a(\overline{\varphi}_{\lambda_1}\psi_{\lambda_2})\|_{L^1_tL^2_x}\\ &\qquad\lesssim \sum_{\lambda_1\approx\lambda_2}\lambda_1^{\ell+\frac{d-2}{2}+2a-2s}\|\varphi_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,a,0}_{\lambda_2}} \lesssim \|\varphi\|_{S^{s,a,b}}\|\psi\|_{S^{s,a,b}}\end{aligned}$$

where we have used the assumption

$$2s-\ell-\tfrac{d-2}{2} \geq 2a.$$

This completes the proof of (4.1).

Proof of (4.2). This is slightly easier than the previous estimate (4.1) as we no longer have to deal with the temporal weight $(\mu+|\partial_t|)^a$. To bound the high-low interactions, we observe that

$$\begin{aligned}\mu^{\ell-1}\|P_\mu(\overline{\varphi}\psi_{\ll\mu})\|_{L^\infty_tL^2_x} &\lesssim \mu^{\ell-1}\sum_{\lambda\ll\mu}\lambda^{d/2}\|\varphi_{\approx\mu}\|_{L^\infty_tL^2_x}\|\psi_\lambda\|_{L^\infty_tL^2_x}\\ &\lesssim \mu^{\ell-s-1}\sum_{\lambda\ll\mu}\lambda^{d/2-s}\|\varphi_{\approx\mu}\|_{S^{s,a,0}}\|\psi\|_{S^{s,a,0}}\end{aligned}$$

and hence provided that

$$s+1 \geq \ell, \quad 2s \geq \ell+\tfrac{d-2}{2}, \quad (s,\ell) \neq \big(\tfrac{d}{2}, \tfrac{d}{2}+1\big),$$

we obtain

$$\Big(\sum_{\mu\gg 1}\mu^{2(\ell-1)}\|P_\mu(\overline{\varphi}\psi_{\ll\mu})\|^2_{L^\infty_t L^2_x}\Big)^{1/2}\lesssim\Big(\sum_{\mu\in 2^{\mathbb{N}}}\mu^{2s}\|\varphi_{\approx\mu}\|^2_{L^\infty_t L^2_x}\Big)^{1/2}\|\psi\|_{S^{s,a,b}}$$
$$\lesssim\|\varphi\|_{S^{s,a,b}}\|\psi\|_{S^{s,a,b}}.$$

Similarly, to deal with the high-high interactions, we note that for any $\lambda_1\approx\lambda_2$ since $\ell+d/2-1\geq 0$ an application of Bernstein's inequality gives

$$\Big(\sum_{\mu\lesssim\lambda_1}\mu^{2(\ell-1)}\|P_\mu(\overline{\varphi}_{\lambda_1}\psi_{\lambda_2})\|^2_{L^\infty_t L^2_x}\Big)^{1/2}$$
$$\lesssim\Big(\sum_{\mu\lesssim\lambda_1}\mu^{2(\ell+d/2-1)}\|\varphi_{\lambda_1}\|^2_{L^\infty_t L^2_x}\|\psi_{\lambda_2}\|^2_{L^\infty_t L^2_x}\Big)^{1/2}$$
$$\lesssim\lambda_1^{\ell+d-2/2-2s}\|\varphi_{\lambda_1}\|_{S^{s,a,b}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,a,b}_{\lambda_2}}$$

and therefore

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}}\mu^{2(\ell-1)}\Big\|\sum_{\lambda_1\approx\lambda_2\gtrsim\mu}P_\mu(\overline{\varphi}_{\lambda_1}\psi_{\lambda_2})\Big\|^2_{L^\infty_t L^2_x}\Big)^{1/2}$$
$$\lesssim\sum_{\lambda_1\approx\lambda_2}\lambda_1^{\ell+\frac{d-2}{2}-2s}\|\varphi_{\lambda_1}\|_{S^{s,a,b}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,a,b}_{\lambda_2}}\lesssim\|\varphi\|_{S^{s,a,b}}\|\psi\|_{S^{s,a,b}}$$

where we have used the assumption

$$2s-\ell-\tfrac{d-2}{2}\geq 0.$$

In view of the frequency decomposition (4.5), together with the invariance of the left hand side of (4.2) under complex conjugation, this completes the proof of the $L^\infty_t L^2_x$ bound (4.2).

Proof of (4.3). We now turn to the proof of the $L^2_{t,x}$ bound (4.3), and again decompose the product into the standard frequency trichotomy as in (4.5). For the high-low interaction terms, we note that

$$\mu^\beta\|P_\mu(\overline{\varphi}\psi_{\ll\mu})\|_{L^2_{t,x}}\lesssim\mu^\beta\|\varphi_{\approx\mu}\|_{L^\infty_t L^2_x}\|\psi_{\ll\mu}\|_{L^2_t L^\infty_x}$$
$$\lesssim\mu^\beta\sum_{1\leq\lambda\ll\mu}\lambda^{\frac{d-2}{2}+a-s}\|\varphi_{\approx\mu}\|_{L^\infty_t L^2_x}\|(\lambda+|\partial_t|)^a\psi_\lambda\|_{L^2_t L^{2*}_x}$$

and hence, provided that

$$\beta\leq s,\quad 2s-\beta-\tfrac{d-2}{2}\geq a,\quad (s,\beta)\neq\big(\tfrac{d-2}{2}+a,\tfrac{d-2}{2}+a\big),$$

summing gives

$$\Big(\sum_{\mu\gg 1}\mu^{2\beta}\|P_\mu(\overline{\varphi}\psi_{\ll\mu})\|^2_{L^2_{t,x}}\Big)^{1/2}\lesssim\Big(\sum_{\mu\gg 1}\mu^{2s}\|\varphi_{\approx\mu}\|^2_{L^\infty_t L^2_x}\Big)^{1/2}\|\psi\|_{S^{s,a,b}}$$
$$\lesssim\|\varphi\|_{S^{s,a,b}}\|\psi\|_{S^{s,a,b}}.$$

Similarly, to bound the high-high interaction terms, for any $\lambda_1 \approx \lambda_2$ we have

$$\lambda_1^{\beta} \|\overline{\varphi}_{\lambda_1} \psi_{\lambda_2}\|_{L^2_{t,x}} \lesssim \lambda_1^{\beta} \|\varphi_{\lambda_1}\|_{L^2_t L^{2^*}_x} \|\psi_{\lambda_2}\|_{L^\infty_t L^d_x} \lesssim \lambda_1^{\beta+a+\frac{d-2}{2}-2s} \|\varphi_{\lambda_1}\|_{S^{s,a,0}_{\lambda_1}} \|\psi_{\lambda_2}\|_{S^{s,a,0}_{\lambda_2}}.$$

Therefore, noting that since $\beta \geq 0$ we have

$$\Big(\sum_{\mu \lesssim \lambda_1} \mu^{2\beta} \|P_\mu F\|^2_{L^2_{t,x}} \Big)^{1/2} \lesssim \lambda_1^{\beta} \Big(\sum_{\mu \in 2^{\mathbb{N}}} \|P_\mu F\|^2_{L^2_{t,x}} \Big)^{1/2} \lesssim \lambda_1^{\beta} \|F\|_{L^2_{t,x}},$$

we conclude that

$$\Big(\sum_{\mu \in 2^{\mathbb{N}}} \mu^{2\beta} \Big\| \sum_{\lambda_1 \approx \lambda_2 \gtrsim \mu} P_\mu(\overline{\varphi}_{\lambda_1} \psi_{\lambda_2}) \Big\|^2_{L^2_{t,x}} \Big)^{1/2} \lesssim \sum_{\lambda_1 \approx \lambda_2} \Big(\sum_{\mu \lesssim \lambda_1} \mu^{2\beta} \|P_\mu(\overline{\varphi}_{\lambda_1} \psi_{\lambda_1})\|^2_{L^2_{t,x}} \Big)^{1/2}$$
$$\lesssim \sum_{\lambda_1 \approx \lambda_2} \lambda_1^{\beta+a+\frac{d-2}{2}-2s} \|\varphi_{\lambda_1}\|_{S^{s,a,b}_{\lambda_1}} \|\psi_{\lambda_2}\|_{S^{s,a,b}_{\lambda_2}} \lesssim \|\varphi\|_{S^{s,a,b}} \|\psi\|_{S^{s,a,b}}$$

provided that

$$2s - \beta - \tfrac{d-2}{2} \geq a.$$

This completes the proof of (4.3).

Proof of (4.4). To prove the remaining estimate (4.4), we can simply use Bernstein's and Hölder's inequalities and the endpoint Strichartz estimate with loss (2.5):

$$\|P_{\leq 2^{16}}(|\nabla|(\overline{\varphi}\psi))\|_{L^1_t L^2_x} \lesssim \|\overline{\varphi}\psi\|_{L^1_t L_x^{\frac{d}{d-2}}} \lesssim \|\varphi\|_{L^2_t L^{2^*}_x} \|\psi\|_{L^2_t L^{2^*}_x} \lesssim \|\varphi\|_{S^{a,a,b}} \|\psi\|_{S^{a,a,b}},$$

since $s \geq a$. ∎

As in the Schrödinger case, we additionally provide a local version of the bilinear estimate which contains a dispersive norm.

Corollary 4.2. *Let $d \geq 4$, $s, \ell, \beta \geq 0$, and $0 \leq a \leq 1$ satisfy*

$$\beta < \min\{s, 2s - \tfrac{d-2}{2} - a\}, \quad 2a < 2s - \ell - \tfrac{d-2}{2}, \quad a < s - \ell.$$

There exist $0 < \theta < 1$ and $C > 0$ such that for any interval $0 \in I \subset \mathbb{R}$, if $\varphi, \psi \in S^{s,a,0}(I)$, then

$$\|\mathcal{J}_0[\nabla(\overline{\varphi}\psi)]\|_{W^{\ell,a,\beta}(I)} \leq C(\|\varphi\|_{S^{s,a,0}(I)} \|\psi\|_{S^{s,a,0}(I)})^{1-\theta} (\|\varphi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)})^{\theta}.$$

Proof. Let $\lambda_1, \lambda_2 \in 2^{\mathbb{N}}$. It suffices to show that there exist $\delta, N > 0$ such that

$$\|\mathcal{J}_0[|\nabla|(\overline{\psi_{\lambda_1}}\varphi_{\lambda_2})]\|_{W^{\ell,a,\beta}} \lesssim (\max\{\lambda_1, \lambda_2\})^{-\delta} \|\psi\|_{S^{s,a,0}} \|\varphi\|_{S^{s,a,0}} \tag{4.6}$$

together with an estimate with a derivative loss, but the Strichartz norm on the right hand side:

$$\|\mathcal{J}_0[|\nabla|(\overline{\psi_{\lambda_1}}\varphi_{\lambda_2})]\|_{W^{\ell,a,\beta}(I)} \lesssim (\max\{\lambda_1,\lambda_2\})^N \big(\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\|\varphi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\|\psi\|_{S^{s,a,0}(I)}\|\varphi\|_{S^{s,a,0}(I)}\big)^{1/2}. \tag{4.7}$$

We start with the proof of (4.6). Choose $s' < s$ such that

$$\beta < \min\{s', 2s' - \tfrac{d-2}{2} - a\}, \quad 2a < 2s' - \ell - \tfrac{d-2}{2}, \quad a < s' - \ell.$$

An application of Theorem 4.1 implies that

$$\|\mathcal{J}_0[|\nabla|(\overline{\psi_{\lambda_1}}\varphi_{\lambda_2})\|_{W^{\ell,a,\beta}} \lesssim \|\psi_{\lambda_1}\|_{S^{s',a,0}}\|\varphi_{\lambda_2}\|_{S^{s',a,0}} \lesssim (\lambda_1\lambda_2)^{s'-s}\|\psi\|_{S^{s,a,0}}\|\varphi\|_{S^{s,a,0}}$$

and hence (4.6) follows.

We now turn to the proof of (4.7). An application of the standard energy inequality for the wave equation together with the convexity of L^p_t and Bernstein's inequality implies that

$$\begin{aligned}\|\mathcal{J}_0[G_\lambda]\|_{W^{\ell,a,\beta}_\lambda} &\lesssim \lambda^{\ell+a}\|\mathcal{J}_0[G_\lambda]\|_{L^\infty_t L^2_x} + \lambda^{\beta-1}\|G_\lambda\|_{L^2_{t,x}} \\ &\lesssim \lambda^{\ell+a+\beta+d/4}\big(\|G_\lambda\|_{L^1_t L^2_x} + \|G_\lambda\|^{1/2}_{L^1_t L^2_x}\|G_\lambda\|^{1/2}_{L^\infty_t L^1_x}\big).\end{aligned}$$

Therefore there exists $N > 0$ such that

$$\begin{aligned}&\|\mathcal{J}_0[|\nabla|(\overline{\psi_{\lambda_1}}\varphi_{\lambda_2})]\|_{W^{\ell,a,\beta}(I)} \le \|\mathcal{J}_0[\mathbb{1}_I|\nabla|(\overline{\psi_{\lambda_1}}\varphi_{\lambda_2})]\|_{W^{\ell,a,\beta}} \\ &\lesssim (\max\{\lambda_1,\lambda_2\})^N\big(\|\psi_{\lambda_1}\varphi_{\lambda_2}\|_{L^1_t L^2_x(I\times\mathbb{R}^d)} \\ &\qquad + \|\psi_{\lambda_1}\varphi_{\lambda_2}\|^{1/2}_{L^1_t L^2_x(I\times\mathbb{R}^d)}\|\psi_{\lambda_1}\varphi_{\lambda_2}\|^{1/2}_{L^\infty_t L^1_x(I\times\mathbb{R}^d)}\big) \\ &\lesssim (\max\{\lambda_1,\lambda_2\})^N\big(\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\|\varphi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\|\psi\|_{S^{s,a,0}(I)}\|\varphi\|_{S^{s,a,0}(I)}\big)^{1/2},\end{aligned}$$

and the proof is complete. ∎

5. Simplified small data global theory and large data local theory

As a warm up to the proof of the main results contained in Theorems 1.1 and 1.2, we show how the bilinear estimates in the previous two sections can be used to prove a simplified small data global well-posedness and scattering result and a large data local well-posedness result in the non-endpoint case.

Recall that $\mathcal{J}_0[F]$ denotes the solution to the inhomogeneous Schrödinger equation

$$(i\partial_t + \Delta)\psi = F, \quad \psi(0) = 0,$$

and similarly $\mathcal{J}_0[G]$ denotes the solution to the inhomogeneous wave equation

$$(i\partial_t + |\nabla|)\phi = G, \quad \phi(0) = 0.$$

Given data $(f, g) \in H^s \times H^\ell$, define the functional

$$\Phi(f, g; \psi) := e^{it\Delta} f + \mathcal{I}_0[\Re(e^{it|\nabla|} g)\psi] - \mathcal{I}_0[\Re(\mathcal{J}_0(|\nabla|\, |\psi|^2))\psi].$$

Suppose that (s, ℓ) lies in the region (1.3), and define the parameters (a, b) as in (2.4). A computation shows that the energy inequality in Lemma 2.4 together with the bilinear estimates in Theorems 3.1 and 4.1 implies that we have the bound

$$\begin{aligned}\|\Phi(f, g; \psi)\|_{S^{s,a,b}} &\lesssim \|f\|_{H^s} + \|\Re(e^{it|\nabla|} g)\psi\|_{N^{s,a,b}} + \|\Re(\mathcal{J}_0(|\nabla|\, |\psi|^2))\psi\|_{N^{s,a,b}} \\ &\lesssim \|f\|_{H^s} + \|g\|_{W^{\ell,a,s-1/2}} \|\psi\|_{S^{s,a,b}} + \|\mathcal{J}_0(|\nabla|\, |\psi|^2)\|_{W^{\ell,a,s-1/2}} \|\psi\|_{S^{s,a,b}} \\ &\lesssim \|f\|_{H^s} + \|g\|_{H^\ell} \|\psi\|_{S^{s,a,b}} + \|\psi\|^3_{S^{s,a,b}}.\end{aligned}$$

Therefore $\Phi : S^{s,a,b} \to S^{s,a,b}$. Repeating this argument with differences shows that provided $\|f\|_{H^s} + \|g\|_{H^\ell}$ is sufficiently small, there exists a fixed point $u \in \{\psi \in S^{s,a,b} \mid \|\psi\|_{S^{s,a,b}} \lesssim \|f\|_{H^s}\}$ to Φ. Defining

$$V = e^{it|\nabla|} g - \mathcal{J}_0(|\nabla|\, |u|^2)$$

and again applying Theorem 4.1, we then obtain a solution $(u, V) \in C(\mathbb{R}, H^s \times H^\ell)$ to the Zakharov system (2.1). The scattering property follows from Lemma 2.5 and an analogue for the wave part.

Note that the above argument requires the smallness condition $\|f\|_{H^s} + \|g\|_{H^\ell} \ll 1$. Our later arguments will significantly improve this to just requiring $g \in H^{(d-4)/2}$ and $\|f\|_{H^{(d-3)/2}} \ll_g 1$. In other words, we only require smallness of f in the *endpoint* Sobolev space. In addition, we also obtain a stronger uniqueness claim, as well as persistence of regularity.

Let us now sketch a simplified, large data local well-posedness result in the non-endpoint case $s > \frac{d-3}{2}$. Suppose that (s, ℓ) satisfies (1.3) with $s > \frac{d-3}{2}$, and take (a, b) as in (2.4). Define $\tilde{\ell} = \min\{s - 1/2, \ell\}$ and take the map Φ as above. The non-endpoint condition $s > \frac{d-3}{2}$ is due to the use of Corollary 4.2, while the choice of $\tilde{\ell}$ is made to ensure that we can construct a fixed point for Φ in $S^{s,a,0}(I)$ via Corollary 3.2. Once we have a fixed point $u \in S^{s,a,0}(I)$, we use an additional argument to upgrade this to $u \in S^{s,a,b}(I)$, which is needed to get the correct regularity for the wave component V.

Fix $(f, g) \in H^s \times H^\ell$. Choose an interval $0 \in I \subset \mathbb{R}$ such that

$$\|e^{it\Delta} f\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} + \|e^{it|\nabla|} g\|_{W^{\tilde{\ell},a,s-1/2}(I) + L^2_t W^{s,d}_x(I\times\mathbb{R}^d)} < \varepsilon,$$

where $\varepsilon > 0$ is fixed later (depending on f, g, and the absolute constants in the above bilinear estimates). Define the subset $\Omega \subset S^{s,a,0}(I)$ as

$$\Omega = \{\psi \in S^{s,a,0}(I) : \|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \lesssim (1 + \|g\|_{L^2_x})\varepsilon,\ \|\psi\|_{S^{s,a,0}(I)} \lesssim \|f\|_{H^s}\}.$$

An application of Corollaries 3.2 and 4.2 gives $\theta > 0$ such that for every $\psi \in \Omega$ we have

$$\begin{aligned}\|\Phi(f,g;\psi)\|_{S^{s,a,0}(I)} &\lesssim \|f\|_{H^s} + (\|e^{it|\nabla|}g\|_{W^{\tilde{\ell},a,s-1/2}(I)+L^2_t W^{s,d}_x(I)} \\ &\qquad + \|\mathcal{J}_0(|\nabla|\,|\psi|^2)\|_{W^{\tilde{\ell},a,s-1/2}(I)})\|\psi\|_{S^{s,a,0}(I)} \\ &\lesssim \|f\|_{H^s} + \varepsilon\|\psi\|_{S^{s,a,0}(I)} + \varepsilon^{2\theta}(1+\|g\|_{L^2_x})^{2\theta}\|\psi\|^{3-2\theta}_{S^{s,a,0}(I)}.\end{aligned}$$

On the other hand, in view of the endpoint Strichartz estimate we have

$$\begin{aligned}&\|\Phi(f,g;\psi)\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \\ &\quad\lesssim \|e^{it\Delta}f\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} + \|\Re(e^{it|\nabla|}g)\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \\ &\qquad + \|\Re(\mathcal{J}_0[|\nabla|\,|\psi|^2])\psi\|_{L^2_t L^{2^*}_x} \\ &\quad\lesssim \varepsilon + \|g\|_{L^2_x}\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} + \|\mathcal{J}_0[|\nabla|\,|\psi|^2]\|_{L^\infty_t L^2_x(I\times\mathbb{R}^d)}\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \\ &\quad\lesssim \varepsilon + \varepsilon\|g\|_{L^2_x} + \varepsilon^{1+2\theta}(1+\|g\|_{L^2})^{1+2\theta}\|\psi\|^{2-2\theta}_{S^{s,a,0}(I)}.\end{aligned}$$

Consequently, choosing $\varepsilon > 0$ sufficiently small, we see that $\Phi : \Omega \to \Omega$. A similar argument shows that Φ is a contraction on Ω (with respect to the norm $\|\cdot\|_{S^{s,a,0}(I)}$), and hence there exists a fixed point $u \in \Omega \subset S^{s,a,0}(I)$ for Φ.

We now upgrade the (far paraboloid) regularity to $u \in S^{s,a,b}(I)$. Note that this is immediate if $s > \ell$ since $b = 0$ in this case. If $s \leq \ell$, then an application of Theorem 3.1 together with Lemma 2.4 gives

$$\begin{aligned}\|u\|_{S^{s,a,b}(I)} &\lesssim \|f\|_{H^s} + \|\Re(e^{it|\nabla|}g)u\|_{N^{s,a,b}(I)} + \|\Re(\mathcal{J}_0[|\nabla|\,|u|^2])u\|_{N^{s,a,b}(I)} \\ &\lesssim \|f\|_{H^s} + \|g\|_{H^\ell}\|u\|_{S^{s,a,0}(I)} \\ &\quad + \|\mathcal{J}_0[|\nabla|\,|u|^2]\|_{W^{s-(1-b)/2,a,s-1/2}(I)}\|u\|_{S^{s,a,0}(I)}.\end{aligned}$$

To check conditions of Theorem 3.1, it is helpful to note that $a = 0$ when $s \leq \ell$. Theorem 4.1 implies

$$\|\mathcal{J}_0[|\nabla|\,|u|^2]\|_{W^{s-(1-b)/2,a,s-1/2}(I)} \lesssim \|u\|^2_{S^{s,a,0}(I)}$$

and hence we conclude that

$$\|u\|_{S^{s,a,b}(I)} \lesssim \|f\|_{H^s} + \|g\|_{H^\ell}\|u\|_{S^{s,a,0}(I)} + \|u\|^3_{S^{s,a,0}(I)}.$$

Consequently, if $u \in S^{s,a,0}(I)$ is a solution to $\Phi(f,g;u) = u$, then we have the improved regularity $u \in S^{s,a,b}(I)$. As above, we now define

$$V = e^{it|\nabla|}g - \mathcal{J}_0(|\nabla|\,|u|^2).$$

Since $u \in S^{s,a,b}(I)$, an application of Theorem 4.1 then gives $V \in W^{\ell,a,s-1/2}(I)$. In particular, we have a local solution $(u,V) \in C(I, H^s \times H^\ell)$ to the Zakharov system (2.1).

6. Local bilinear estimates in the endpoint case

In this section, we deal with the endpoint case $(s, \ell) = (\frac{d-3}{2}, \frac{d-4}{2})$ and establish bilinear estimates which include dispersive norms on the right hand side. We start with an improvement of Theorem 3.1 in the endpoint case.

Proposition 6.1. *Let $d \geq 4$ and $(s, \ell) = (\frac{d-3}{2}, \frac{d-4}{2})$. Let $I \subset \mathbb{R}$ be an interval. Then*

$$\|vu\|_{N^{s,0,0}(I)} \lesssim \|v\|_{W^{\ell,0,s-1/2}(I)} \|u\|^{1/2}_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|u\|^{1/2}_{S^{s,0,0}(I)}. \tag{6.1}$$

Proof. Suppose for the moment that we can prove that for any $\alpha \gg 1$ we have

$$\Big\| \sum_{\lambda \geq \alpha} P_\lambda(vu_{\lambda/\alpha}) \Big\|_{N^{s,0,0}(I)} \lesssim \alpha^{1/2} \|v\|_{L^\infty_t H^\ell(I\times\mathbb{R}^d)} \|u\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)}, \tag{6.2}$$

$$\Big\| \sum_{\lambda \geq \alpha} P_\lambda(vu_{\lambda/\alpha}) \Big\|_{N^{s,0,0}} \lesssim \alpha^{-1/2} \|v\|_{W^{\ell,0,s-1/2}} \|u\|_{S^{s,0,0}}. \tag{6.3}$$

Then since

$$\begin{aligned}
\Big\| \sum_{\lambda} P_\lambda(vu_{\gtrsim\lambda}) \Big\|_{N^{s,0,0}(I)} &\lesssim \Big(\sum_{\lambda} \lambda^{2s} \|\mathbb{1}_I v u_{\gtrsim\lambda}\|^2_{L^2_t L^{2*}_x} \Big)^{1/2} \\
&\lesssim \Big\| v \Big(\sum_{\lambda} \lambda^{2s} |u_{\gtrsim\lambda}|^2 \Big)^{1/2} \Big\|_{L^2_t L^{2*}_x(I\times\mathbb{R}^d)} \\
&\lesssim \|v\|_{L^\infty_t L^{d/2}_x(I\times\mathbb{R}^d)} \Big\| \Big(\sum_{\lambda} \lambda^{2s} |u_{\gtrsim\lambda}|^2 \Big)^{1/2} \Big\|_{L^2_t L^{2*}_x(I\times\mathbb{R}^d)} \\
&\lesssim \|v\|_{L^\infty_t H^\ell_x(I\times\mathbb{R}^d)} \|u\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)}
\end{aligned}$$

an application of (6.2) and (6.3), together with the definition of the restricted spaces $N^{s,0,0}(I)$, $S^{s,0,0}(I)$, and $W^{\ell,0,s-1/2}(I)$ implies that for any $M \gg 1$ we have

$$\begin{aligned}
&\|vu\|_{N^{s,0,0}(I)} \\
&\quad \leq \Big\| \sum_{\lambda} P_\lambda(vu_{\gtrsim\lambda}) \Big\|_{N^{s,0,0}(I)} + \sum_{1\ll\alpha\leq M} \Big\| \sum_{\lambda\geq\alpha} P_\lambda(vu_{\lambda/\alpha}) \Big\|_{N^{s,0,0}(I)} \\
&\qquad + \sum_{\alpha>M} \Big\| \sum_{\lambda\geq\alpha} P_\lambda(vu_{\lambda/\alpha}) \Big\|_{N^{s,0,0}(I)} \\
&\quad \lesssim M^{1/2} \|v\|_{W^{\ell,0,s-1/2}(I)} \|u\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} + M^{-1/2} \|v\|_{W^{\ell,0,s-1/2}(I)} \|u\|_{S^{s,0,0}(I)}
\end{aligned}$$

Optimising in M then gives (6.1). Thus it remains to prove the bounds (6.2) and (6.3). For the former estimate, we observe that since $s = \ell + 1/2$,

$$\begin{aligned}
\Big\| \sum_{\lambda\geq\alpha} P_\lambda(vu_{\lambda/\alpha}) \Big\|_{N^{s,0,0}(I)} &\lesssim \Big(\sum_{\lambda\geq\alpha} \lambda^{2s} \|vu_{\lambda/\alpha}\|^2_{L^2_t L^{2*}_x(I\times\mathbb{R}^d)} \Big)^{1/2} \\
&\lesssim \Big\| \Big(\sum_{\lambda\geq\alpha} \lambda^{2s} |v_{\approx\lambda}|^2 |u_{\lambda/\alpha}|^2 \Big)^{1/2} \Big\|_{L^2_t L^{2*}_x(I\times\mathbb{R}^d)}
\end{aligned}$$

$$\lesssim \alpha^{1/2} \Big\| \sup_{\mu} \mu^{\ell} |v_\mu| \Big(\sum_{\lambda} \lambda |u_\lambda|^2 \Big)^{1/2} \Big\|_{L^2_t L^{2*}_x (I \times \mathbb{R}^d)}$$
$$\lesssim \alpha^{1/2} \|v\|_{L^\infty_t H^\ell} \|u\|_{L^2_t W^{s,2*}_x (I \times \mathbb{R}^d)}$$

where the last line followed via Hölder's inequality and Sobolev embedding. The proof of (6.3) is more involved, and exploits the fact that the high-low interactions are non-resonant. In particular, since $\lambda \geq \alpha \gg 1$, the non-resonant identity

$$P^N_\lambda (P^{(t)}_{\ll \lambda^2} v u_{\lambda/\alpha}) = P^N_\lambda (P^{(t)}_{\ll \lambda^2} v_{\approx \lambda} P^{(t)}_{\approx \lambda^2} u_{\lambda/\alpha})$$

implies that

$$\begin{aligned} \Big\| \sum_{\lambda \geq \alpha} P_\lambda (v u_{\lambda/\alpha}) \Big\|^2_{N^{s,0,0}} &\lesssim \sum_{\lambda \geq \alpha} \lambda^{2s} \| P^N_\lambda (v u_{\lambda/\alpha}) \|^2_{L^2_t L^{2*}_x} + \sum_{\lambda \geq \alpha} \lambda^{2(s-1)} \| P^F_\lambda (v u_{\lambda/\alpha}) \|^2_{L^2_{t,x}} \\ &\lesssim \sum_{\lambda \geq \alpha} \lambda^{2s} \| P^{(t)}_{\ll \lambda^2} v_{\approx \lambda} P^{(t)}_{\approx \lambda^2} u_{\lambda/\alpha} \|^2_{L^2_t L^{2*}_x} \\ &\quad + \sum_{\lambda \geq \alpha} \lambda^{2s} \| P^{(t)}_{\gtrsim \lambda^2} v_{\approx \lambda} u_{\lambda/\alpha} \|^2_{L^2_t L^{2*}_x} + \sum_{\lambda \geq \alpha} \lambda^{2(s-1)} \| v_{\approx \lambda} u_{\lambda/\alpha} \|^2_{L^2_{t,x}}. \end{aligned} \tag{6.4}$$

To estimate the first term in (6.4), we observe that since $s = \ell + 1/2$, we have

$$\begin{aligned} &\Big(\sum_{\lambda \geq \alpha} \lambda^{2s} \| P^{(t)}_{\ll \lambda^2} v_{\approx \lambda} P^{(t)}_{\approx \lambda^2} u_{\lambda/\alpha} \|^2_{L^2_t L^{2*}_x} \Big)^{1/2} \\ &\qquad = \Big\| \Big(\sum_{\lambda \geq \alpha} \lambda^{2s} \| P^{(t)}_{\ll \lambda^2} v_{\approx \lambda} P^{(t)}_{\approx \lambda^2} u_{\lambda/\alpha} \|^2_{L^{2*}_x} \Big)^{1/2} \Big\|_{L^2_t} \\ &\qquad \lesssim \|v\|_{L^\infty_t H^\ell} \Big(\sum_{\lambda \geq \alpha} \lambda^{2(s-\ell)} \| P^{(t)}_{\approx \lambda} u_{\lambda/\alpha} \|^2_{L^2_t L^d_x} \Big)^{1/2} \\ &\qquad \lesssim \|v\|_{L^\infty_t H^\ell} \Big(\sum_{\lambda \geq \alpha} \lambda^{2(s-\ell-2)} \| (i\partial_t + \Delta) u_{\lambda/\alpha} \|^2_{L^2_t L^d_x} \Big)^{1/2} \\ &\qquad \lesssim \alpha^{-3/2} \|v\|_{L^\infty_t H^\ell} \| (i\partial_t + \Delta) u \|_{L^2_t H^{s-1}_x}. \end{aligned}$$

To bound the second term in (6.4), again using the fact that $\ell + 1/2 = s = \frac{d-3}{2}$, we have

$$\begin{aligned} &\Big(\sum_{\lambda \geq \alpha} \lambda^{2s} \| P^{(t)}_{\gtrsim \lambda^2} v_{\approx \lambda} u_{\lambda/\alpha} \|^2_{L^2_t L^{2*}_x} \Big)^{1/2} \\ &\qquad \lesssim \Big(\sum_{\lambda \geq \alpha} \lambda^{2(s+1/2)} \| P^{(t)}_{\gtrsim \lambda^2} v_{\approx \lambda} \|^2_{L^2_{t,x}} \Big)^{1/2} \sup_{\lambda \geq \alpha} \lambda^{-1/2} \| u_{\lambda/\alpha} \|_{L^\infty_t L^d_x} \\ &\qquad \lesssim \alpha^{-1/2} \Big(\sum_{\lambda} \lambda^{2(s-3/2)} \| (i\partial_t + |\nabla|) P^{(t)}_{\gtrsim \lambda^2} v_{\approx \lambda} \|^2_{L^2_{t,x}} \Big)^{1/2} \|u\|_{L^\infty_t H^s_x} \\ &\qquad \lesssim \alpha^{-1/2} \|v\|_{W^{\ell,0,s-1/2}} \|u\|_{L^\infty_t H^s_x}. \end{aligned}$$

Finally, for the last term in (6.4), since $s - \ell - 1 = -1/2$, an application of Bernstein's inequality gives

$$\Big(\sum_{\lambda \geq \alpha} \lambda^{2(s-1)} \|v_{\approx\lambda} u_{\lambda/\alpha}\|_{L^2_{t,x}}^2\Big)^{1/2} \lesssim \bigg\| \Big(\sum_{\lambda \geq \alpha} \lambda^{2(s-1)} \Big(\frac{\lambda}{\alpha}\Big)^{d-2} \|v_{\approx\lambda}\|_{L^2_x}^2 \|u_{\lambda/\alpha}\|_{L^{2^*}_x}^2\Big)^{1/2}\bigg\|_{L^2_t}$$
$$\lesssim \alpha^{-1/2} \|v\|_{L^\infty_t H^\ell} \|\sup_\lambda \lambda^s \|u_\lambda\|_{L^{2^*}_x}\|_{L^2_t}$$
$$\lesssim \alpha^{-1/2} \|v\|_{L^\infty_t H^\ell} \|u\|_{L^2_t W^{s,2^*}_x}. \qquad \blacksquare$$

We have a related estimate to deal with the wave nonlinearity.

Proposition 6.2. *Let $d \geq 4$, $(s, \ell) = (\frac{d-3}{2}, \frac{d-4}{2})$. Let $0 \in I \subset \mathbb{R}$ be an interval. Then*

$$\|\mathcal{J}_0(|\nabla|(\overline{\varphi}\psi))\|_{W^{\ell,0,\ell}(I)} \lesssim \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)}\big)^{1/2} \big(\|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}\big)^{1/2}. \tag{6.5}$$

Proof. Suppose for the moment that for any $\alpha \gg 1$ we have the bounds

$$\Big\|\sum_{\lambda\geq\alpha} |\nabla| \mathcal{J}_0(\overline{\varphi}_\lambda \psi_{\lambda/\alpha})\Big\|_{W^{\ell,0,\ell}(I)} \lesssim \alpha^{1/2} \|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)}$$
$$+ \alpha^{-1/2} \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}\big)^{1/2}, \tag{6.6}$$

$$\Big\|\sum_{\lambda\geq\alpha} |\nabla| \mathcal{J}_0(\overline{\varphi}_\lambda \psi_{\lambda/\alpha})\Big\|_{W^{\ell,0,\ell}(I)} \lesssim \alpha^{-1/2} \|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}, \tag{6.7}$$

$$\Big\|\sum_{\lambda} |\nabla| \mathcal{J}_0(\overline{\varphi}_\lambda \psi_{\approx\lambda})\Big\|_{W^{\ell,0,\ell}(I)} \lesssim \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}\big)^{1/2}. \tag{6.8}$$

Let $M \gg 1$. As in Proposition 6.1, by decomposing

$$\overline{\varphi}\psi = \sum_{\alpha\gg1}\sum_{\lambda\geq\alpha} \overline{\varphi}_\lambda \psi_{\lambda/\alpha} + \sum_\lambda \overline{\varphi}_\lambda \psi_{\approx\lambda} + \sum_{\alpha\gg1}\sum_{\lambda\geq\alpha} \overline{\varphi}_{\lambda/\alpha} \psi_\lambda$$

and using symmetry, an application of (6.6) for $\alpha < M$, (6.7) for $\alpha \geq M$, and (6.8) for the remaining high-high interactions gives

$$\|V\|_{W^{\ell,0,\ell}(I)} \lesssim \big\||\nabla|\mathcal{J}_0(\overline{\varphi}\psi)\big\|_{W^{\ell,0,\ell}(I)}$$
$$\lesssim \sum_{\alpha\gg1}\Big(\Big\|\sum_{\lambda\geq\alpha} |\nabla|\mathcal{J}_0(\overline{\varphi}_\lambda\psi_{\lambda/\alpha})\Big\|_{W^{\ell,0,\ell}(I)} + \Big\|\sum_{\lambda\geq\alpha} |\nabla|\mathcal{J}_0(\overline{\varphi}_{\lambda/\alpha}\psi_\lambda)\Big\|_{W^{\ell,0,\ell}(I)}\Big)$$
$$+ \Big\|\sum_\lambda |\nabla|\mathcal{J}_0(\overline{\varphi}_\lambda\psi_{\approx\lambda})\Big\|_{W^{\ell,0,\ell}(I)}$$
$$\lesssim M^{1/2} \|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)}$$
$$+ \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}\big)^{1/2}$$
$$+ M^{-1/2} \|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}$$
$$+ \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\varphi\|_{S^{s,0,0}(I)} \|\psi\|_{S^{s,0,0}(I)}\big)^{1/2}.$$

Optimising in M then gives (6.5).

It remains to prove the bounds (6.6)–(6.8). We begin by noting that an application of Hölder's inequality and the Sobolev embedding together with the assumptions on (s, ℓ) imply that

$$\Big(\sum_{\lambda\geq\alpha} \lambda^{2(\ell+1)} \|\overline{\varphi}_\lambda \psi_{\lambda/\alpha}\|^2_{L^1_t L^2_x(I\times\mathbb{R}^d)}\Big)^{1/2}$$
$$\lesssim \alpha^{\ell+1-s} \Big\| \Big(\sum_{\lambda} \lambda^{2s}|\varphi_\lambda|^2\Big)^{1/2} \Big(\sum_{\lambda} \lambda^{2(\ell+1-s)}|\psi_\lambda|^2\Big)^{1/2} \Big\|_{L^1_t L^2_x(I\times\mathbb{R}^d)}$$
$$\lesssim \alpha^{\ell+1-s} \|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{\ell+1-s,d}_x(I\times\mathbb{R}^d)}$$
$$\lesssim \alpha^{1/2} \|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)}.$$

On the other hand, again applying a combination of Hölder's inequality and Sobolev embedding, we have

$$\Big(\sum_{\lambda\geq\alpha} \lambda^{2\ell} \|\mathbb{1}_I \overline{\varphi}_\lambda \psi_{\lambda/\alpha}\|^2_{L^2_{t,x}}\Big)^{1/2} \lesssim \alpha^{\ell-s} \Big\| \Big(\sum_{\lambda} \lambda^{2s}|\varphi_\lambda|^2\Big)^{1/2} \sup_\lambda \lambda^{\ell-s}|\psi_\lambda| \Big\|_{L^2_{t,x}(I\times\mathbb{R}^d)}$$
$$\lesssim \alpha^{\ell-s} \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^\infty_t W^{\ell-s,d}_x(I\times\mathbb{R}^d)}\big)^{1/2}$$
$$\times \Big(\|\varphi\|_{L^\infty_t H^s_x} \Big\| \sup_\lambda \lambda^{\ell-s} \|\psi_\lambda\|_{L^\infty_x} \Big\|_{L^2_t(I)}\Big)^{1/2}$$
$$\lesssim \alpha^{-1/2} \big(\|\varphi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\psi\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} \|\varphi\|_{L^\infty_t H^s_x(I\times\mathbb{R}^d)} \|\psi\|_{L^\infty_t H^s_x(I\times\mathbb{R}^d)}\big)^{1/2}. \tag{6.9}$$

The bound (6.6) now follows from the standard energy inequality as

$$\|\mathcal{J}_0(G)\|_{W^{\ell,0,\ell}(I)} \leq \|\mathcal{J}_0(\mathbb{1}_I G)\|_{W^{\ell,0,\ell}}$$
$$\lesssim \Big(\sum_{\mu\in 2^{\mathbb{N}}} \mu^{2\ell} \|\mathcal{J}_0(\mathbb{1}_I G_\mu)\|^2_{L^\infty_t L^2_x} + \mu^{2(\ell-1)} \|\mathbb{1}_I G_\mu\|^2_{L^2_{t,x}}\Big)^{1/2}$$
$$\lesssim \Big(\sum_{\mu\in 2^{\mathbb{N}}} \mu^{2\ell} \|G_\mu\|^2_{L^1_t L^2_x(I\times\mathbb{R}^d)} + \mu^{2(\ell-1)} \|\mathbb{1}_I G_\mu\|^2_{L^2_{t,x}}\Big)^{1/2}. \tag{6.10}$$

We now turn to the proof of (6.7); this requires exploiting the fact that the high-low interactions are non-resonant. More precisely, since $\lambda \geq \alpha \gg 1$, the non-resonant identity

$$P^{(t)}_{\ll\lambda^2}(\overline{\varphi}_\lambda \psi_{\lambda/\alpha}) = P^{(t)}_{\ll\lambda^2}(C_{\gtrsim\lambda^2}\overline{\varphi}_\lambda \psi_{\lambda/\alpha}) + P^{(t)}_{\ll\lambda^2}(C_{\ll\lambda^2}\overline{\varphi}_\lambda P^{(t)}_{\approx\lambda^2}\psi_{\lambda/\alpha})$$

together with

$$\Big(\sum_{\lambda\geq\alpha} \lambda^{2(\ell+1)} \|P^{(t)}_{\ll\lambda^2}(C_{\gtrsim\lambda^2}\overline{\varphi}_\lambda \psi_{\lambda/\alpha})\|^2_{L^1_t L^2_x}\Big)^{1/2}$$
$$\lesssim \alpha^{\ell-s} \Big(\sum_{\lambda\geq\alpha} \lambda^{2(s+1)} \|C_{\gtrsim\lambda^2}\overline{\varphi}_\lambda\|^2_{L^2_{t,x}}\Big)^{1/2} \sup_\lambda \lambda^{\ell-s} \|\psi_\lambda\|_{L^2_t L^\infty_x} \lesssim \alpha^{-1/2} \|\varphi\|_{S^{s,0,0}} \|\psi\|_{S^{s,0,0}}$$

and

$$\Big(\sum_{\lambda\geq\alpha}\lambda^{2(\ell+1)}\|P^{(t)}_{\ll\lambda^2}(C_{\ll\lambda^2}\overline{\varphi}_\lambda P^{(t)}_{\approx\lambda^2}\psi_{\lambda/\alpha})\|^2_{L^1_tL^2_x}\Big)^{1/2}$$
$$\lesssim \sup_\lambda \lambda^s\|\varphi_\lambda\|_{L^2_tL^{2^*}_x}\Big(\sum_{\lambda\geq\alpha}\lambda^{2(\ell+1-s)}\|P^{(t)}_{\approx\lambda^2}\psi_{\lambda/\alpha}\|^2_{L^2_tL^d_x}\Big)^{1/2}$$
$$\lesssim \alpha^{-3/2}\|\varphi\|_{L^2_tW^{s,2^*}_x}\|P^{(t)}_{\approx\lambda^2}(i\partial_t+\Delta)\psi\|_{L^2_tH^{s-1}_x}$$

and

$$\Big(\sum_{\lambda\geq\alpha}\lambda^{2(\ell-2)}\|\overline{\varphi}_\lambda\psi_{\lambda/\alpha}\|^2_{L^\infty_tL^2_x}\Big)^{1/2}\lesssim\Big(\sum_{\lambda\geq\alpha}\lambda^{2(\ell-2)}\Big(\frac{\lambda}{\alpha}\Big)^d\|\overline{\varphi}_\lambda\|^2_{L^\infty_tL^2_x}\|\psi_{\lambda/\alpha}\|^2_{L^\infty_tL^2_x}\Big)^{1/2}$$
$$\lesssim \alpha^{-3/2}\|\varphi\|_{S^{s,0,0}}\|\psi\|_{S^{s,0,0}}$$

and (6.9) in the special case $I=\mathbb{R}$, implies that (6.7) follows from an application of the high frequency energy estimate in Lemma 2.6.

The final estimate is the high-high case (6.8). An application of Sobolev embedding gives

$$\Big(\sum_{\lambda}\lambda^{2(\ell+1)}\|\mathbb{1}_I\overline{\varphi}_\lambda\psi_{\approx\lambda}\|^2_{L^1_tL^2_x}\Big)^{1/2}$$
$$\lesssim\Big\|\Big(\sum_\lambda\lambda^{2s}|\varphi_\lambda|^2\Big)^{1/2}\Big(\sum_\lambda\lambda^{2(\ell+1-s)}|\psi_\lambda|^2\Big)^{1/2}\Big\|_{L^1_tL^2_x}$$
$$\lesssim\|\varphi\|_{L^2_tW^{s,2^*}_x(I\times\mathbb{R}^d)}\|\psi\|_{L^2_tW^{s,2^*}_x(I\times\mathbb{R}^d)}$$

and hence (6.8) follows from the energy estimate (6.10) together with the $L^2_{t,x}$ bound (6.9) in the special case $\alpha\approx 1$. ∎

7. Well-posedness results

7.1. Global well-posedness for the model problem

The first step in the proof of Theorem 1.2 is to prove the following global result for the model problem

$$(i\partial_t+\Delta-\Re(V))u=F,\quad u(0)=f,$$

where we assume that $f\in H^s$ and $\|F\|_{N^{s,a,0}}<\infty$. In particular, this shows that the Duhamel operators $\mathcal{I}_V$ are well-defined maps from $N^{s,a,0}$ to $S^{s,a,0}$, even for large wave potentials V.

Theorem 7.1. *Let* $0\leq s\leq\ell+2$ *and* $\ell\geq\frac{d-4}{2}$ *with* $(s,\ell)\neq(\frac{d}{2},\frac{d}{2}-2)$. *Let* $\beta=\max\{\frac{d-4}{2},s-1\}$ *and* $a=a^*(s,\ell)$ *where* $a^*(s,\ell)$ *is as in* (2.4). *There exists* $\varepsilon>0$ *such that if* $0\in I\subset\mathbb{R}$ *is an open interval, and*

$$f\in H^s(\mathbb{R}^d),\quad V_L=e^{it|\nabla|}g\in L^\infty_tH^\ell_x,\quad \|V-V_L\|_{W^{\ell,a,\beta}(I)}<\varepsilon,\quad F\in N^{s,a,0}(I)$$

then there exists a unique solution $u \in C(I, H^s(\mathbb{R}^d)) \cap L^2_t L^{2^*}_x(I \times \mathbb{R}^d)$ *to the Cauchy problem*

$$(i\partial_t + \Delta - \Re(V))u = F, \quad u(0) = f.$$

Moreover, there exists a constant $C = C(V_L) > 0$ (*independent of* I, f, V, *and* F) *such that*

$$\|u\|_{S^{s,a,0}(I)} \leq C(V_L)(\|f\|_{H^s} + \|F\|_{N^{s,a,0}(I)})$$

and, writing $I = (T_-, T_+)$ *with* $-\infty \leq T_- < T_+ \leq \infty$, *there exists* $f_\pm \in H^s$ *such that*

$$\lim_{t \to T_\pm} \|e^{-it\Delta}u(t) - f_\pm\|_{H^s_x} = 0.$$

Remark 7.2 (Free wave potentials). The potential V in Theorem 7.1 should be thought of as a small perturbation of the free wave $V_L = e^{it|\nabla|}g$. In particular, in the special case where the potential is simply a free wave, i.e. $V = V_L$, the smallness condition is trivially satisfied. Consequently, for any $f \in H^s$, $g \in H^\ell$, $F \in N^{s,a,0}$, Theorem 7.1 gives a global solution $u \in S^{s,a,0}$ to the Schrödinger equation

$$(i\partial_t + \Delta - \Re(V_L))u = F, \quad u(0) = f. \tag{7.1}$$

Thus no smallness condition is required on the potential V_L or the data f. Moreover, for any open interval $I \subset \mathbb{R}$ and $g \in H^\ell$, the Duhamel integral is a continuous map $\mathcal{I}_{V_L} : N^{s,a,0}(I) \to S^{s,a,0}(I)$, and we have the bound

$$\|\mathcal{I}_{V_L}[F]\|_{S^{s,a,0}(I)} \lesssim \|F\|_{N^{s,a,0}(I)}.$$

Remark 7.3 (Strichartz control). When $a > 0$, the solution space $S^{s,a,0}$ does not control the Strichartz space $L^2_t W^{s,2^*}_x$. On the other hand, when $0 \leq s < \ell + 1$, we have $a^*(s,\ell) = 0$. Therefore, an application of (2.5) and Theorem 7.1 implies that solutions to the Schrödinger equation (7.1) satisfy the (global) Strichartz estimate

$$\Big(\sum_{\lambda \in 2^{\mathbb{N}}} \lambda^{2s} \|u\|^2_{L^2_t L^{2^*}_x(\mathbb{R}^{1+d})}\Big)^{1/2} \lesssim \|u\|_{S^{s,0,0}} \lesssim \|f\|_{H^s} + \|F\|_{N^{s,0,0}}.$$

In particular, for any $0 \leq s < \ell + 1$, $\ell \geq \frac{d-4}{2}$, and $(f, g) \in H^s \times H^\ell$ we have

$$\|u\|_{L^2_t W^{s,2^*}_x} \lesssim \|f\|_{H^s} + \|(i\partial_t + \Delta - \Re(V_L))u\|_{L^2_t W^{s,2*}_x}.$$

The first step in the proof of Theorem 7.1 is to prove a local version with the additional assumption that the potential V is small in some dispersive type norm.

Proposition 7.4. *Let* $0 \leq s \leq \ell + 2$ *and* $\ell \geq d/2 - 2$ *with* $(s, \ell) \neq (d/2, d/2 - 2)$. *Let* $\beta = \max\{\frac{d-4}{2}, s - 1\}$ *and define* $a = a^*(s, \ell)$ *as in* (2.4). *There exists* $\varepsilon > 0$ *and* $C > 0$ *such that if* $0 \in I \subset \mathbb{R}$ *is an open interval, and*

$$f \in H^s(\mathbb{R}^d), \quad F \in N^{s,a,0}(I), \quad V \in W^{\ell,a,\beta}(I), \quad \|V\|_{W^{\ell,a,\beta}(I) + L^2_t W^{s,d}_x(I \times \mathbb{R}^d)} < \varepsilon,$$

then the Cauchy problem

$$(i\partial_t + \Delta)u = \Re(V)u + F, \quad u(0) = f,$$

has a unique solution $u \in C(I, H^s(\mathbb{R}^d)) \cap L^2_t L^{2^*}_x(I \times \mathbb{R}^d)$ *and we have the bound*

$$\|u\|_{S^{s,a,0}(I)} \leq C(\|f\|_{H^s} + \|F\|_{N^{s,a,0}(I)}).$$

Moreover, writing $I = (T_-, T_+)$ *with* $-\infty \leq T_- < T_+ \leq \infty$, *there exists* $f_\pm \in H^s_x$ *such that*

$$\lim_{t \to T_\pm} \|e^{-it\Delta}u(t) - f_\pm\|_{H^s_x} = 0.$$

Proof. This is a direct application of Lemma 2.4, Lemma 2.5, and Corollary 3.2. Define the sequence $u_j \in S^{s,a,0}(I)$ for $j \geq 1$ by solving

$$(i\partial_t + \Delta)u_j = \Re(V)u_{j-1} + F, \quad u_j(0) = f,$$

and let $u_0 = 0$. An application of Corollary 3.2 together with the smallness assumption on V implies that

$$\|u_j\|_{S^{s,a,0}(I)} \lesssim \|f\|_{H^s} + \varepsilon\|u_{j-1}\|_{S^{s,a,0}(I)} + \|F\|_{N^{s,a,0}(I)}$$

and

$$\|u_j - u_{j-1}\|_{S^{s,a,0}(I)} \lesssim \varepsilon\|u_{j-1} - u_{j-2}\|_{S^{s,a,0}(I)}.$$

Thus provided $\varepsilon > 0$ is sufficient small (depending only on the constant in Corollary 3.2), the sequence u_j is a Cauchy sequence and hence converges to a (unique) solution $u \in S^{s,a,0}(I)$. Uniqueness in the larger space $L^\infty_t L^2_x \cap L^2_t L^{2^*}_x$ follows by standard arguments from the Strichartz estimate

$$\begin{aligned}
&\|\mathcal{I}_0[\Re(V)u]\|_{L^\infty_t L^2_x \cap L^2_t L^{2^*}_x(I \times \mathbb{R}^d)} \\
&\qquad \lesssim \|\Re(V)u\|_{L^2_t L^{2_*}_x(I \times \mathbb{R}^d)} \lesssim \|V\|_{(L^\infty_t L^2_x + L^2_t L^d_x)(I \times \mathbb{R}^d)} \|u\|_{L^\infty_t L^2_x \cap L^2_t L^{2^*}_x(I \times \mathbb{R}^d)} \\
&\qquad \lesssim \|V\|_{W^{\ell,a,\beta}(I) + L^2_t W^{s,d}_x(I \times \mathbb{R}^d)} \|u\|_{L^\infty_t L^2_x \cap L^2_t L^{2^*}_x(I \times \mathbb{R}^d)}.
\end{aligned}$$

Finally, to prove the existence of the limits $\lim_{t \to T_\pm} e^{-it\Delta}u(t)$, it suffices to show that $e^{-it\Delta}u$ is a Cauchy sequence as $t \to T_+$. To this end, we first observe that by Corollary 3.2 we have $G = \Re(V)u + F \in N^{s,a,0}(I)$. Let $G' \in N^{s,a,0}$ be any extension of G from I to $\mathbb{R}$. Then for any $t, t' \in I$,

$$\begin{aligned}
\|e^{-it\Delta}u(t) - e^{-it'\Delta}u(t')\|_{H^s} &= \|e^{-it\Delta}\mathcal{I}_0[G](t) - e^{-it'\Delta}\mathcal{I}_0[G](t')\|_{H^s} \\
&= \|e^{-it\Delta}\mathcal{I}_0[G'](t) - e^{-it'\Delta}\mathcal{I}_0[G'](t')\|_{H^s} \\
&= \left\| \int_{t'}^{t} e^{-is\Delta}G'(s)\, ds \right\|_{H^s},
\end{aligned}$$

and therefore an application of Lemmas 2.4 and 2.5 implies that $e^{-it\Delta}u(t)$ is a Cauchy sequence as required. ∎

To apply the previous proposition, we need to decompose $\mathbb{R}$ into intervals on which V_L is small. This exploits the dispersive properties of the free wave $V_L = e^{it|\nabla|}g$. More precisely, we have the following minor variation of [2, Lemma 4.1].

Lemma 7.5 ([2, Lemma 4.1]). *Let $\ell, s, a \geq 0$, $\varepsilon > 0$, and $V_L = e^{it|\nabla|}g \in L^\infty_t H^\ell_x$. Then there exists a finite collection $(I_j)_{j=1,\dots,N}$ of open intervals such that $\mathbb{R} = \bigcup_{j=1}^N I_j$, $\min |I_j \cap I_{j+1}| > 0$, and*

$$\sup_{j=1,\dots,N} \|V_L\|_{W^{\ell,a,\beta}(I_j) + L^2_t W^{s,d}_x(I_j\times\mathbb{R}^d)} < \varepsilon.$$

Proof. Decompose $g = g_1 + g_2$ where $g_2 \in C^\infty_0(\mathbb{R}^d)$ and $\|g_1\|_{H^\ell} < \varepsilon$. Since g_2 is smooth and compactly supported, the dispersive estimate for the free wave equation gives $e^{it|\nabla|}g_2 \in L^2_t W^{s,d}_x(\mathbb{R}^{1+d})$ and hence we can find a collection $(I_j)_{j=1,\dots,N}$ of open intervals such that $\mathbb{R} = \bigcup_{j=1}^N I_j$, $\min |I_j \cap I_{j+1}| > 0$, and

$$\sup_{j=1,\dots,N} \|e^{it|\nabla|}g_2\|_{L^2_t W^{s,d}_x(I_j\times\mathbb{R}^d)} < \varepsilon.$$

On the other hand, the definition of the norm $W^{\ell,a,\beta}$ implies that

$$\|e^{it|\nabla|}g_1\|_{W^{\ell,a,\beta}(I_j)} \leq \|e^{it|\nabla|}g_1\|_{W^{\ell,a,\beta}} \lesssim \|g_1\|_{H^\ell} \lesssim \varepsilon.$$

Therefore, for every $j = 1,\dots,N$, we have

$$\|V_L\|_{W^{\ell,a,\beta}(I_j) + L^2_t W^{s,d}_x(I_j\times\mathbb{R}^d)} \leq \|e^{it|\nabla|}g_1\|_{W^{\ell,a,\beta}(I_j)} + \|e^{it|\nabla|}g_2\|_{L^2_t W^{s,d}_x(I_j\times\mathbb{R}^d)} \lesssim \varepsilon.$$

∎

The proof of Theorem 7.1 now follows by repeatedly applying Proposition 7.4 together with the decomposability property in Lemma 2.8.

Proof of Theorem 7.1. Let $\varepsilon > 0$ and suppose that

$$\|V - V_L\|_{W^{\ell,a,\beta}} < \varepsilon.$$

An application of Lemma 7.5 gives a finite collection $(I_j)_{j=1,\dots,N}$ of open intervals and points $t_j \in I_{j-1} \cap I_j$ such that $I = \bigcup_{j=1}^N I_j$, $\min |I_j \cap I_{j+1}| > 0$, and

$$\begin{aligned}\sup_{j=1,\dots,N} &\|V\|_{W^{\ell,a,\beta}(I_j) + L^2_t W^{s,d}_x(I_j\times\mathbb{R}^d)} \\ &\leq \|V - V_L\|_{W^{\ell,a,\beta}(I)} + \sup_{j=1,\dots,N} \|V_L\|_{W^{\ell,a,\beta}(I_j) + L^2_t W^{s,d}_x(I_j\times\mathbb{R}^d)} < 2\varepsilon.\end{aligned}$$

Assuming $\varepsilon > 0$ is sufficiently small, Proposition 7.4 gives a (unique) solution $u \in C(I_j, H^s) \cap L^2_t L^{2^*}_x(I_j\times\mathbb{R}^d)$ on the interval $I_j \ni 0$ to the Cauchy problem

$$(i\partial_t + \Delta)u = \Re(V)u + F, \quad u(0) = f \tag{7.2}$$

such that

$$\|u\|_{S^{s,a,0}(I_j)} \lesssim \|f\|_{H^s} + \|F\|_{N^{s,a,0}(I_j)} \lesssim \|f\|_{H^s} + \|F\|_{N^{s,a,0}(I)}.$$

Taking new data $u(t_j)$ and $u(t_{j-1})$, and again applying Proposition 7.4, we get a unique solution

$$u \in C(I_{j-1} \cup I_j \cup I_{j+1}, H^s) \cap L_t^2 L_x^{2^*}((I_{j-1} \cup I_j \cup I_{j+1}) \times \mathbb{R}^d)$$

with

$$\sup_{k=j-1,j,j+1} \|u\|_{S^{s,a,0}(I_k)} \lesssim \|f\|_{H^s} + \|F\|_{N^{s,a,0}(I)}.$$

Continuing in this manner, after at most N steps, we obtain a unique solution $u \in C(I, H^s) \cap L_t^2 L_x^{2^*}(I \times \mathbb{R}^d)$ such that

$$\|u\|_{S^{s,a,0}(I)} \lesssim_N \sup_{j=1,\dots,N} \|u\|_{S^{s,a,0}(I_j)} \lesssim_N \|f\|_{H^s} + \|F\|_{N^{s,a,0}(I)}$$

where the first inequality is a consequence of Lemma 2.8. Finally, to show that the claimed limits as $t \to \sup I$ and $t \to \inf I$ exist, we simply repeat the argument at the end of the proof of Proposition 7.4. ∎

7.2. *Local and small data global results for the Zakharov system*

We first consider the non-endpoint case $s > \frac{d-3}{2}$.

Theorem 7.6 (LWP and small data GWP: non-endpoint case). *Let $d \geq 4$ and suppose that (s, ℓ) satisfies the conditions (1.3) and $s > \frac{d-3}{2}$. Let $a = a^*(s, \ell)$ and $b = b^*(s, \ell)$ as in (2.4). For some $0 < \theta < 1$ and any $g_* \in H^\ell(\mathbb{R}^d)$ there exists $\varepsilon > 0$ such that if $f_* \in H^s(\mathbb{R}^d)$ satisfies*

$$\|f_*\|_{H^s}^{2-2\theta} \|e^{it\Delta} f_*\|_{L_t^2 L_x^{2^*}(I \times \mathbb{R}^d)}^{2\theta} < \varepsilon \quad \textit{for an interval } 0 \in I \subset \mathbb{R}, \tag{7.3}$$

then for all (f, g) in

$$D_\varepsilon(f_*, g_*) := \{H^s \times H^\ell : \|f - f_*\|_{H^s} < \varepsilon, \|g - g_*\|_{H^\ell} < \varepsilon\},$$

there exists a unique solution $(u, V) \in S^{s,a,b}(I) \times W^{\ell,a,s-1/2}(I)$ to (2.1). The flow map

$$H^s(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d) \supset D \ni (f, g) \mapsto (u, V) \in S^{s,a,b}(I) \times W^{\ell,a,s-1/2}(I)$$

is real-analytic, where $D = D_\varepsilon(f_*, g_*)$ is the open bi-disc defined above. Moreover, if $I = \mathbb{R}$, then there exists $(f_\pm, g_\pm) \in H^s \times H^\ell$ such that

$$\lim_{t \to \pm\infty} \left(\|u(t) - e^{it\Delta} f_\pm\|_{H^s} + \|V(t) - e^{it|\nabla|} g_\pm\|_{H^\ell}\right) = 0.$$

Proof. Fix (s, ℓ) satisfying the conditions (1.3) and $s > \frac{d-3}{2}$, and define $a = a^*(s, \ell)$ and $b = b^*(s, \ell)$ as in (2.4). Let $\tilde{\ell} = \min\{\ell, s - 1/2\}$ and define $V_L = e^{it|\nabla|} g_*$ to be the free wave evolution of $g_* \in H^\ell$, and similarly $u_L = e^{it\Delta} f_*$ for $f_* \in H^s$ in the case of the free Schrödinger evolution.

Let us recall that $\mathcal{I}_{V_L}[F]$ denotes the solution to the inhomogeneous Schrödinger equation

$$(i\partial_t + \Delta - \Re(V_L))\psi = F, \quad \psi(0) = 0,$$

and similarly, $\mathcal{J}_0[G]$ denotes the solution to the inhomogeneous wave equation

$$(i\partial_t + |\nabla|)\phi = G, \quad \phi(0) = 0.$$

We claim there exists $C_{g_*} > 0$ and $0 < \theta < 1$ such that

$$\|\mathcal{I}_{V_L}[\Re(\phi)\psi]\|_{S^{s,a,0}(I)} \le C_{g_*}\|\phi\|_{W^{\tilde{\ell},a,s-1/2}(I)}\|\psi\|_{S^{s,a,0}(I)}, \tag{7.4}$$

$$\|\mathcal{I}_{V_L}[\Re(\phi)\psi]\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \le C_{g_*}\|\phi\|_{W^{\tilde{\ell},0,0}(I)}\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}, \tag{7.5}$$

$$\begin{aligned}\|\mathcal{J}_0[|\nabla|(\overline{\psi}\varphi)]\|_{W^{\tilde{\ell},a,s-1/2}(I)} &\le C\big(\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\|\varphi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\big)^{\theta} \\ &\quad \times \big(\|\psi\|_{S^{s,a,0}(I)}\|\varphi\|_{S^{s,a,0}(I)}\big)^{1-\theta}.\end{aligned} \tag{7.6}$$

The estimate (7.4) follows from Theorems 7.1 and 3.1. To prove (7.5), we again apply Theorem 7.1 and observe that via the Littlewood–Paley square function estimate and Bernstein's inequality,

$$\begin{aligned}\|\mathcal{I}_{V_L}[F]\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} &\lesssim \Big(\sum_{\lambda\in 2^{\mathbb{N}}}\|P_\lambda \mathcal{I}_{V_L}[F]\|^2_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\Big)^{1/2} \\ &\lesssim_{g_*} \|F\|_{N^{0,0,0}(I)} \lesssim_{g_*} \|F\|_{L^2_t L^{2_*}_x(I\times\mathbb{R}^d)}\end{aligned}$$

(see also (2.7)). Therefore

$$\begin{aligned}\|\mathcal{I}_{V_L}[\Re(\phi)\psi]\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} &\lesssim_{g_*} \|\phi\psi\|_{L^2_t L^{2_*}_x} \lesssim_{g_*} \|\phi\|_{L^\infty_t L^{d/2}_x(I\times\mathbb{R}^d)}\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \\ &\lesssim_{g_*} \|\phi\|_{W^{\tilde{\ell},0,0}(I)}\|\psi\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}\end{aligned}$$

and so (7.5) follows. The final estimate (7.6) is a direct application of Corollary 4.2.

Set $\rho = V - V_L$ and $g^* = g - g_*$. Then we want to solve

$$\begin{aligned}(i\partial_t + \Delta - \Re(V_L))u &= \Re(\rho)u, \qquad u(0) = f, \\ (i\partial_t + |\nabla|)\rho &= -|\nabla|\,|u|^2, \quad \rho(0) = g^*.\end{aligned}$$

Since $\rho = e^{it|\nabla|}g^* - \mathcal{J}_0(|\nabla|\,|u|^2)$, we want to solve $u = \Phi(f, g; u)$ for u, where

$$\Phi(f, g; u) := e^{it\Delta}f + \mathcal{I}_{V_L}(\Re(e^{it|\nabla|}g^*)u) - \mathcal{I}_{V_L}(\mathcal{J}_0(|\nabla|\,|u|^2)u).$$

Also, let $f^* = f - f_*$. Then, by the endpoint Strichartz estimate

$$\|e^{it\Delta}f^*\|_{L^2_t L^{2^*}_x} \le C_{\mathrm{Str}}\|f^*\|_{L^2}$$

and estimates (7.4)–(7.6) above we obtain

$$\begin{aligned}\|\Phi(f, g; u)\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} &\le C_{\mathrm{Str}}\|f^*\|_{H^s} + \|e^{it\Delta}f_*\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \\ &\quad + C_{g_*}\|g^*\|_{H^{\tilde{\ell}}}\|u\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} + C_{g_*}C\|u\|^{2-2\theta}_{S^{s,a,0}}\|u\|^{1+2\theta}_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}.\end{aligned}$$

Similarly, we have

$$\|\Phi(f,g;u)\|_{S^{s,a,0}(I)} \leq (1+2C_{\mathrm{Str}})\|f_*\|_{H^s} + (1+2C_{\mathrm{Str}})\|f^*\|_{H^s} + C_{g_*}\|g^*\|_{H^{\tilde{\ell}}}\|u\|_{S^{s,a,0}(I)} + C_{g_*}C\|u\|^{3-2\theta}_{S^{s,a,0}}\|u\|^{2\theta}_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}.$$

In addition,

$$\|\Phi(f,g;u_1)-\Phi(f,g;u_2)\|_{S^{s,a,0}(I)} \leq C_{g_*}\|g^*\|_{H^{\tilde{\ell}}}\|u_1-u_2\|_{S^{s,a,0}(I)} + 3C_{g_*}C(\|u_1\|_{S^{s,a,0}}+\|u_2\|_{S^{s,a,0}})^{2-\theta} \times (\|u_1\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}+\|u_2\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)})^{\theta}\|u_1-u_2\|_{S^{s,a,0}(I)}.$$

Let $R \geq 1$ be chosen such that $1+2C_{\mathrm{Str}}, C_{g_*}, CC_{g_*} \leq R$. Consider the complete space K defined by all $S^{s,a,0}(I)$ satisfying

$$\|u\|_{S^{s,a,0}(I)} \leq 2R\|f_*\|_{H^s}, \quad \|u\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)} \leq 2R\|e^{it\Delta}f_*\|_{L^2_t L^{2^*}_x(I\times\mathbb{R}^d)}$$

with the distance defined by the norm $\|u\|_{S^{s,a,0}(I)}$ (which dominates $\|u\|_{L^2_t L^{2^*}_x}$).

Recall that $\|f^*\|_{H^s}, \|g^*\|_{H^\ell} < \varepsilon$. Therefore, for small enough $\varepsilon > 0$, we conclude that $\Phi(f,g;\cdot): K \to K$ is a contraction. Hence, there is a unique fixed point $u \in K \subset S^{s,a,0}(I)$ of $\Phi(f,g;\cdot)$.

In addition, as a consequence of the above estimates, for $(f,g) \in D$ and $u \in K$, we see that for any $v \in S^{s,a,0}(I)$, the linear map $Tv = v - D_v\Phi(f,g;u)$ is a small perturbation of the identity, and hence T is a linear homeomorphism onto $S^{s,a,0}(I)$. Furthermore, the map Φ is real-analytic (as a composition of linear, bi- and trilinear maps over $\mathbb{R}$). If $u[f,g]$ denotes the solution with initial data (f,g), the implicit function theorem [10, Theorem 15.3] implies that the flow map $D \ni (f,g) \mapsto u[f,g] \in S^{s,a,0}(I)$ is real-analytic. Define $V = e^{it|\nabla|}g - \mathcal{J}_0(|\nabla|\,|u|^2)$. Estimate (7.6) implies that $V \in W^{\tilde{\ell},a,s-1/2}(I)$ and (u,V) is a solution of (2.1). Also, $D \ni (f,g) \mapsto V[f,g] = e^{it|\nabla|}g - \mathcal{J}_0(|\nabla|\,|u[f,g]|^2) \in W^{\tilde{\ell},a,s-1/2}(I)$ is a composition of real-analytic maps and therefore real-analytic. In the case $s \geq \ell + 1/2$ we have $\ell = \tilde{\ell}$ and $b = 0$, so this is the claim.

In the remaining case $s < \ell + 1/2$, we have $a = 0$. Define $\kappa = \ell$ if $s > \ell$ and $\kappa = s - \frac{1}{2}(1-b)$ if $s \leq \ell$. An application of Theorem 3.1 gives

$$\|\mathcal{I}_0[\Re(\phi)\psi]\|_{S^{s,0,b}(I)} \lesssim \|\phi\|_{W^{\kappa,0,s-1/2}(I)}\|\psi\|_{S^{s,0,0}(I)},$$

while Theorem 4.1 implies that

$$\|\mathcal{J}_0[|\nabla|(\overline{\psi}\varphi)]\|_{W^{\kappa,0,s-1/2}(I)} \lesssim \|\psi\|_{S^{s,0,0}(I)}\|\varphi\|_{S^{s,0,0}(I)}.$$

For $(f,g) \in D$ and the solution $u \in K$ we have

$$u = e^{it\Delta}f + \mathcal{I}_0(\Re(e^{it|\nabla|}g)u) - \mathcal{I}_0(\mathcal{J}_0(|\nabla|\,|u|^2)u). \tag{7.7}$$

Thus, we conclude that

$$\|u\|_{S^{s,0,b}(I)} \lesssim \|f\|_{H^s} + \|g\|_{H^\ell}\|u\|_{S^{s,0,0}(I)} + \|u\|^3_{S^{s,0,0}(I)} \lesssim (1+\|g\|_{H^\ell}+\|f\|^2_{H^s})\|f\|_{H^s}.$$

Equation (7.7) also shows that $D \ni (f, g) \mapsto u[f, g] \in S^{s,0,b}(I)$ is a composition of real-analytic maps, hence real-analytic. Theorem 4.1 again implies that

$$\|\mathcal{J}_0[|\nabla|(\overline{\psi}\varphi)]\|_{W^{\ell,0,s-1/2}(I)} \lesssim \|\psi\|_{S^{s,0,b}(I)}\|\varphi\|_{S^{s,0,b}(I)}.$$

We conclude

$$\|V\|_{W^{\ell,0,s-1/2}(I)} \lesssim \|g\|_{H^\ell} + (1 + \|g\|_{H^\ell} + \|f\|_{H^s}^2)^2\|f\|_{H^s}^2$$

and, as above, $D \ni (f, g) \mapsto V[f, g] \in W^{\ell,0,s-1/2}(I)$ is real-analytic.

Finally, we remark that if $I = \mathbb{R}$, then the solution scatters. This follows from an analogous argument to that used in the proof of Theorem 7.1 (i.e. one shows that $u(t)$ forms a Cauchy sequence as $t \to \infty$). It only remains to prove uniqueness in $S^{s,a,b}(I) \times W^{\ell,a,s-1/2}(I)$, but this is again a consequence of the estimates proved above. ■

We now consider the endpoint case $(s, \ell) = (\frac{d-3}{2}, \frac{d-4}{2})$.

Theorem 7.7 (LWP and small data GWP: endpoint case). *Let $d \geq 4$ and fix $(s, \ell) = (\frac{d-3}{2}, \frac{d-4}{2})$. For any $g_* \in H^\ell$ there exists $\varepsilon > 0$ such that if $f_* \in H^s$ and $0 \in I \subset \mathbb{R}$ is an interval with*

$$\|e^{it\Delta} f_*\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}\|f_*\|_{H^s}^7 < \varepsilon, \tag{7.8}$$

then for all (f, g) in

$$D_\varepsilon(f_*, g_*) = \{(f, g) \in H^s \times H^\ell : \|f - f_*\|_{H^s} < \varepsilon,\ \|g - g_*\|_{H^\ell} < \varepsilon\},$$

there exists a unique solution $u \in C(I, H^s) \cap L_t^2 W_x^{s,2^*}(I \times \mathbb{R}^d)$, $V \in C(I, H^\ell)$ to (2.1). Moreover, $(u, V) \in S^{s,0,0}(I) \times W^{\ell,0,s-1/2}(I)$ and the flow map

$$H^s(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d) \supset D \ni (f, g) \mapsto (u, V) \in S^{s,0,0}(I) \times W^{\ell,0,s-1/2}(I)$$

is real-analytic, where $D = D_\varepsilon(f_*, g_*)$ is the open bi-disc defined above. If $I = \mathbb{R}$, then there exists $(f_\pm, g_\pm) \in H^s \times H^\ell$ such that

$$\lim_{t\to\pm\infty}\big(\|u(t) - e^{it\Delta} f_\pm\|_{H^s} + \|V(t) - e^{it|\nabla|}g_\pm\|_{H^\ell}\big) = 0.$$

Proof. Let $g_* \in H^\ell$ and $\varepsilon > 0$ to be fixed later depending only on g_* and the implicit constants in Theorem 7.1 and Propositions 6.1 and 6.2. Let $f_* \in H^s$ and $0 \in I \subset \mathbb{R}$ satisfy the smallness condition (7.8). As in the proof of Theorem 7.6, we let $V_L = e^{it|\nabla|}g_*$, $\rho = V - V_L$, $g^* = g - g_*$ and $f^* = f - f_*$. We want to solve $u = \Phi(f, g; u)$ for u, where

$$\Phi(f, g; u) := e^{it\Delta} f + \mathcal{I}_{V_L}(\Re(e^{it|\nabla|}g^*)u) - \mathcal{I}_{V_L}(\mathcal{J}_0(|\nabla|\,|u|^2)u).$$

and obtain $\rho = e^{it|\nabla|}g^* - \mathcal{J}_0(|\nabla|\,|u|^2)$. An application of Theorem 7.1 implies that

$$\|\mathcal{I}_{V_L}[F]\|_{S^{s,0,0}(I)} \lesssim_{g_*} \|F\|_{N^{s,0,0}(I)}$$

and therefore Propositions 6.1 and 6.2 give

$$\|\Phi(f,g;u)\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)} \lesssim_{g_*} \|f_*\|_{H^s}^{1/2} \|e^{it\Delta} f_*\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}^{1/2} + \|f^*\|_{H^s} + \|g^*\|_{H^\ell} \|u\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}^{1/2} \|u\|_{S^{s,0,0}(I)}^{1/2} + \|u\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}^{3/2} \|u\|_{S^{s,0,0}(I)}^{3/2}$$

and similarly

$$\|\Phi(f,g;u)\|_{S^{s,0,0}(I)} \lesssim_{g_*} \|f_*\|_{H^s} + \|f^*\|_{H^s} + \|g^*\|_{H^\ell} \|u\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}^{1/2} \|u\|_{S^{s,0,0}(I)}^{1/2} + \|u\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}^{3/2} \|u\|_{S^{s,0,0}(I)}^{3/2}$$

and

$$\begin{aligned}\|\Phi(f,g;u_1) - \Phi(f,g;u_2)\|_{S^{s,0,0}(I)} &\lesssim_{g_*} \|g^*\|_{H^\ell} \|u_1 - u_2\|_{S^{s,0,0}(I)} \\ &+ (\|u_1\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)} + \|u_2\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)})(\|u_1\|_{S^{s,0,0}(I)} + \|u_2\|_{S^{s,0,0}(I)}) \\ &\times \|u_1 - u_2\|_{S^{s,0,0}(I)}.\end{aligned}$$

As in the proof of Theorem 7.6, a routine contraction argument then implies that, provided $\varepsilon > 0$ is sufficiently small, there is a unique fixed point $u \in S^{s,0,0}(I)$. Setting $V = V_L - \mathcal{J}_0[|\nabla|\,|u|^2]$, we get a solution $(u, V) \in S^{s,0,0}(I) \times W^{\ell,0,s-1/2}(I)$ due to Proposition 6.2. Also, the flow map is real-analytic; we omit the details.

To prove that the solution scatters, we note that writing $I = (T_0, T_1)$, then as in the proof of Theorem 7.1, a computation shows that for any sequence of times $t_j \nearrow T_1$, the sequence $(e^{-it_j\Delta} u(t_j), e^{-it_j|\nabla|} V(t_j))$ forms a Cauchy sequence in $H^s \times H^\ell$. In particular, the limits

$$\lim_{t\nearrow T_1} (e^{-it\Delta} u(t), e^{-it|\nabla|} V(t)) \quad \text{and} \quad \lim_{t\searrow T_0} (e^{-it\Delta} u(t), e^{-it|\nabla|} V(t))$$

exist in $H^s \times H^\ell$. Therefore, if $I = \mathbb{R}$, the solution scatters to free solutions as $t \to \pm\infty$.

To check the uniqueness claim, we note that the above bounds together with a continuity argument give uniqueness in $S^{s,0,0}(I) \times W^{\ell,0,s-1/2}(I)$. If (u, V) is a solution with $u \in S^{s,0,0}(I)$, then $V \in W^{\ell,0,s-1/2}(I)$. In particular, to prove uniqueness, it suffices to show that if (u, V) with $u \in L_t^\infty H^s \cap L_t^2 W_x^{s,2^*}(I \times \mathbb{R}^d)$ and $V \in L_t^\infty H_x^\ell$ is a solution, then $u \in S^{s,0,0}(I)$. To this end, we note that a standard product estimate gives

$$\begin{aligned}\|(i\partial_t + \Delta)u\|_{L_t^2 H_x^{s-1}(I\times\mathbb{R}^d)} &= \|\Re(V)u\|_{L_t^2 H_x^{s-1}(I\times\mathbb{R}^d)} \\ &\lesssim \|V\|_{L_t^\infty H^\ell(I\times\mathbb{R}^d)} \|u\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)}\end{aligned}$$

and therefore

$$\|u\|_{L_t^\infty H^s(I\times\mathbb{R}^d)} + \|u\|_{L_t^2 W_x^{s,2^*}(I\times\mathbb{R}^d)} + \|(i\partial_t + \Delta)u\|_{L_t^2 H_x^{s-1}(I\times\mathbb{R}^d)} < \infty.$$

Consequently, extending u from the interval $I = (T_0, T_1)$ to $\mathbb{R}$ using free Schrödinger waves

$$u' = \mathbb{1}_{(-\infty,T_0)}(t) e^{i(t-T_0)\Delta} u(T_0) + \mathbb{1}_I(t) u(t) + \mathbb{1}_{(T_1,\infty)}(t) e^{i(t-T_1)\Delta} u(T_1)$$

($u(T_0)$ and $u(T_1)$ are well-defined by the above) we see that by definition of the norm $S^{s,0,0}(I)$, together with the endpoint Strichartz estimate, we have

$$\|u\|_{S^{s,0,0}(I)} \le \|u'\|_{L^\infty_t H^s} + \|u'\|_{L^2_t W^{s,2^*}_x} + \|(i\partial_t + \Delta)u'\|_{L^2_t H^{s-1}_x}$$
$$\lesssim \|u\|_{L^\infty_t H^s(I\times\mathbb{R}^d)} + \|u\|_{L^2_t W^{s,2^*}_x(I\times\mathbb{R}^d)} + \|(i\partial_t + \Delta)u\|_{L^2_t H^{s-1}_x(I\times\mathbb{R}^d)} < \infty.$$

Therefore $u \in S^{s,0,0}(I)$ as required. ∎

8. Persistence of regularity

In this section our goal is show that under suitable assumptions on a solution (u, V) to (2.1), any additional regularity of the data $(u, V)(0)$ persists in time.

Theorem 8.1. *Let (s,ℓ) satisfy (1.3) and fix $a = a^*(s,\ell)$, $b = b^*(s,\ell)$ as in (2.4). Suppose that (u, V) is a solution to the Zakharov system (2.1) on some interval $I \ni 0$ with*

$$\|u\|_{L^\infty_t H_x^{\frac{d-3}{2}}(I\times\mathbb{R}^d)} + \|u\|_{L^2_t W_x^{\frac{d-3}{2},2^*}(I\times\mathbb{R}^d)} + \|V\|_{L^\infty_t H_x^{\frac{d-4}{2}}(I\times\mathbb{R}^d)} < \infty.$$

If $(u, V)(0) \in H^s \times H^\ell$, then $(u, V) \in S^{s,a,b}(I) \times W^{\ell,a,s-1/2}(I)$, and the flow map is real-analytic with respect to the $H^s \times H^\ell$ and $S^{s,a,b}(I) \times W^{\ell,a,s-1/2}(I)$ topologies.

We break the proof of Theorem 8.1 into three main steps.

(i) (Improving Schrödinger regularity when $s \ge \ell + 1/2$) If (s,ℓ) and $(\tilde{s},\ell)$ satisfy (1.3) and $\ell + 1/2 \le s < \tilde{s}$, then

$$(u, V) \in S^{s,a,0}(I) \times W^{\ell,a,s-1/2}(I) \text{ and } u(0) \in H^{\tilde{s}}$$
$$\implies (u, V) \in S^{\tilde{s},\tilde{a},0}(I) \times W^{\ell,\tilde{a},s-1/2}(I)$$

where $a = a^*(s,\ell)$ and $\tilde{a} = a^*(\tilde{s},\ell)$.

(ii) (Improving wave regularity when $s \ge \ell + 1/2$) If (s,ℓ) and $(s,\tilde{\ell})$ satisfy (1.3) and $\ell < \tilde{\ell} \le s - 1/2$, then

$$(u, V) \in S^{s,a,0}(I) \times W^{\ell,a,s-1/2}(I) \text{ and } V(0) \in H^{\tilde{\ell}}$$
$$\implies (u, V) \in S^{s,\tilde{a},0}(I) \times W^{\tilde{\ell},\tilde{a},s-1/2}(I)$$

where now $a = a^*(s,\ell)$ and $\tilde{a} = a^*(s,\tilde{\ell})$.

(iii) (Improving wave regularity when $\ell > s - 1$) If (s,ℓ) and $(s,\tilde{\ell})$ satisfy (1.3) and $s - 1 < \ell < \tilde{\ell}$, then

$$(u, V) \in S^{s,0,b}(I) \times W^{\ell,0,s-1/2}(I) \text{ and } V(0) \in H^{\tilde{\ell}}$$
$$\implies (u, V) \in S^{s,0,\tilde{b}}(I) \times W^{\tilde{\ell},0,s-1/2}(I)$$

where $b = b^*(s,\ell)$ and $\tilde{b} = b^*(s,\tilde{\ell})$.

Theorem 8.1 then follows by repeatedly applying the implications (i)–(iii) and using the fact that the assumptions on (u, V) in Theorem 8.1 imply that $(u, V) \in S^{\frac{d-3}{2},0,0}(I) \times W^{\frac{d-4}{2},0,\frac{d-4}{2}}(I)$.

We give the proof of the implications (i)–(iii) in Sections 8.1–8.3 respectively. The proof of Theorem 8.1 is then given in Section 8.4.

8.1. Improving Schrödinger regularity

Our goal here is to prove the implication (i). Let (s,ℓ) and $(\tilde{s},\ell)$ satisfy (1.3) and $\ell + 1/2 \leq s < \tilde{s}$. Let $\tilde{a} = a^*(\tilde{s},\ell)$ and $a = a^*(s,\ell)$. Clearly we may also assume that $\tilde{s} < s + 1/8$, since the general case follows by repeatedly applying this special case. The key point is to prove that there exists $\theta > 0$ such that for any interval $\tilde{I} \subset \mathbb{R}$,

$$\|\mathcal{J}_0(|\nabla|\,|u|^2)\|_{W^{\ell,\tilde{a},\beta}(\tilde{I})} \lesssim \|u\|^{\theta}_{L^2_t W_x^{\frac{d-3}{2},2^*}(\tilde{I}\times\mathbb{R}^d)} \|u\|^{2-\theta}_{S^{s,a,0}(\tilde{I})} \tag{8.1}$$

where $\beta = \max\{\frac{d-4}{2}, \tilde{s} - 1\}$. Supposing (8.1) holds, decomposing $I = \bigcup_{j=1}^N I_j$ with $\min|I_j \cap I_{j+1}| > 0$, we may assume that on each interval I_j we have

$$\|u\|^{\theta}_{L^2_t W_x^{\frac{d-3}{2},2^*}(I_j\times\mathbb{R}^d)} \|u\|^{2-\theta}_{S^{s,a,0}(I_j)} \ll \varepsilon$$

where $\varepsilon > 0$ is as in Theorem 7.6. Choose $t_j \in I_j \cap I_{j+1}$. Applying (8.1) and a time translated version of Theorem 7.6 then implies that $u \in S^{\tilde{s},\tilde{a},0}(I_j)$ with real-analytic dependence on $(u(t_j), V(t_j))$ for every $j = 1,\ldots,N$. Taking the union of the finite number of intervals I_j via Lemma 2.8 then gives $u \in S^{\tilde{s},\tilde{a},0}(I)$ and real-analytic dependence on $(u(0), V(0))$. In particular, we have the implication (i) under the additional assumption that $s < \tilde{s} < s + 1/8$. But this implies (i) after repeatedly applying the above argument.

We now turn to the proof of (8.1). In view of the bound $\|V\|_{W^{\ell,\tilde{a},\beta}} \lesssim \|V\|_{W^{\ell+\tilde{a}-a,a,\beta}}$, it suffices to show that

$$\|\mathcal{J}_0(|\nabla|\,|u|^2)\|_{W^{\ell+\tilde{a}-a,a,\beta}(\tilde{I})} \lesssim \|u\|^{\theta}_{L^2_t W_x^{\frac{d-3}{2},2^*}(\tilde{I}\times\mathbb{R}^d)} \|u\|^{2-\theta}_{S^{s,a,0}(\tilde{I})}. \tag{8.2}$$

If $s > \frac{d-3}{2}$, then a computation shows that

$$\beta < \min\left\{s, 2s - \tfrac{d-2}{2} - a\right\}, \quad 2a < 2s - (\ell + \tilde{a} - a) - \tfrac{d-2}{2}, \quad a < s - (\ell + \tilde{a} - a)$$

and hence (8.2) follows from Corollary 4.2. On the other hand, in the endpoint case $s = \frac{d-3}{2}$, we have $a = \tilde{a} = 0$ and $\ell = \frac{d-4}{2}$, and hence (8.2) follows from Proposition 6.2.

8.2. Improving wave regularity I

Our goal here is to prove the implication (ii). Let (s,ℓ) and $(s,\tilde{\ell})$ satisfy (1.3) and $\ell < \tilde{\ell} \leq s - 1/2$. Without loss of generality, we may additionally assume that $\tilde{\ell} < \ell + 1/2$, as the general case again follows by repeating this special case. Let $a = a^*(s,\ell)$ and $\tilde{a} = a^*(s,\tilde{\ell})$. A computation shows that

$$2a < 2s - \tilde{\ell} - \tfrac{d-2}{2}, \quad a < s - \tilde{\ell}.$$

In particular, since $\tilde{a} \leq a$, an application of Corollary 4.2 implies that there exists $\theta > 0$ such that

$$\begin{aligned}\|\mathcal{J}_0(|\nabla|\,|u|^2)\|_{W^{\tilde{\ell},\tilde{a},s-1/2}(I)} &\leq \|\mathcal{J}_0(|\nabla|\,|u|^2)\|_{W^{\tilde{\ell},a,s-1/2}(I)} \\ &\lesssim \|u\|^{\theta}_{L^2_t W_x^{\frac{d-3}{2},2^*}(I\times\mathbb{R}^d)} \|u\|^{2-\theta}_{S^{s,a,0}(I)}\end{aligned} \tag{8.3}$$

and hence $V = e^{it|\nabla|}V(0) + \mathcal{J}_0(|\nabla|\,|u|^2) \in W^{\tilde{\ell},\tilde{a},s-1/2}(I)$. It only remains to improve the Schrödinger regularity to $u \in S^{s,\tilde{a},0}(I)$; but this follows by arguing as in (i). Namely, we can decompose the interval $I = \bigcup_{j=1}^{N} I_j$ into a finite number of intervals I_j satisfying $\min |I_j \cap I_{j+1}| > 0$ and

$$\|u\|^{\theta}_{L^2_t W_x^{\frac{d-3}{2},2*}(I_j\times\mathbb{R}^d)} \|u\|^{2-\theta}_{S^{s,a,0}(I)} \ll \varepsilon$$

where $\varepsilon > 0$ is as in Theorem 7.6. Choose $t_j \in I_j \cap I_{j+1}$. Applying the estimate (8.3) together with Theorem 7.6, we conclude that $u \in S^{s,\tilde{a},0}(I_j)$ with real-analytic dependence on $(u(t_j), V(t_j))$ for $j = 1,\dots,N$ and hence $u \in S^{s,\tilde{a},0}(I)$ by Lemma 2.8 and real-analytic dependence on $(u(0), V(0))$. Therefore the implication (ii) follows.

8.3. *Improving wave regularity II*

Our goal here is to prove the implication (iii). Let (s,ℓ) and $(s,\tilde{\ell})$ satisfy (1.3) and $s-1 < \ell < \tilde{\ell}$. Let $b = b^*(s,\ell)$ and $\tilde{b} = b^*(s,\tilde{\ell})$. Suppose that $(u,V) \in S^{s,0,b}(I) \times W^{\ell,0,s-1/2}(I)$, we would like to improve this to $(u,v) \in S^{s,0,\tilde{b}}(I) \times W^{\tilde{\ell},0,s-1/2}(I)$, again with real-analytic dependence. In view of Theorem 4.1, it suffices to show that $u \in S^{s,0,\tilde{b}}(I)$. Choose $\ell \le \ell' \le \tilde{\ell}$ such that

$$\max\left\{\tfrac{d-4}{2} + \tilde{b}, s-1+\tilde{b}\right\} \le \ell' \le \min\left\{2s - \tfrac{d-2}{2}, s+b\right\}, \quad (s,\ell') \neq \left(\tfrac{d-2}{2}, \tfrac{d-2}{2} + b\right).$$

An application of Theorem 4.1 gives

$$\|V\|_{W^{\ell',0,s-1/2}(I)} \lesssim \|V(0)\|_{H^{\tilde{\ell}}} + \|u\|^2_{S^{s,0,b}(I)}$$

and thus, via Theorem 3.1, we conclude that

$$\begin{aligned}\|u\|_{S^{s,0,\tilde{b}}(I)} &\lesssim \|u(0)\|_{H^s} + \|V\|_{W^{\ell',0,s-1/2}(I)}\|u\|_{S^{s,0,0}(I)} \\ &\lesssim \|u(0)\|_{H^s} + (\|V(0)\|_{H^{\tilde{\ell}}} + \|u\|^2_{S^{s,0,b}(I)})\|u\|_{S^{s,0,0}(I)}.\end{aligned}$$

Therefore $u \in S^{s,0,\tilde{b}}(I)$ as required.

8.4. *Proof of Theorem 8.1*

In view of the implications (i)–(iii), it suffices to show that if $(u,V) \in C(I, H_x^{\frac{d-3}{2}} \times H^{\frac{d-4}{2}})$ is a solution to the Zakharov equation (2.1) with $u \in L^2_t W_x^{\frac{d-3}{2},2*}(I\times\mathbb{R}^d)$, then $(u,V) \in S^{\frac{d-3}{2},0,0}(I) \times W^{\frac{d-4}{2},0,\frac{d-4}{2}}(I)$. But this implication is contained in the argument used to prove uniqueness in Theorem 7.7.

9. Proofs of the main results

9.1. *Proof of Theorem 1.1*

Suppose that (s,ℓ) satisfies (1.3). Then Theorem 7.6 or Theorem 7.7 implies well-posedness on a (small enough) interval $I \ni 0$.

Now, we prove the converse implication. More precisely, we prove that the flow map of (2.1) is not of class C^2 for (s, ℓ) which do not satisfy (1.3). Fix (s, ℓ) and assume the contrary. Fix $t > 0$ and consider

$$I_\lambda(t) = -i \int_0^t e^{i(t-t')\Delta}\big(\Re(e^{it'|\nabla|}g_\lambda)e^{it'\Delta} f_\lambda\big)\, dt',$$
$$J_\lambda(t) = i \int_0^t e^{i(t-t')|\nabla|}|\nabla|\big(e^{it'\Delta}h_\lambda \overline{e^{it'\Delta}h_\lambda}\big)\, dt'$$

for certain $\|g_\lambda\|_{H^\ell(\mathbb{R}^d)} \approx \|f_\lambda\|_{H^s(\mathbb{R}^d)} \approx \|h_\lambda\|_{H^s(\mathbb{R}^d)} \approx 1$, to be chosen later. (I_λ, J_λ) corresponds to a second order directional derivative (Gâteaux derivative) at the origin, which must be uniformly (in λ) bounded by our hypothesis.

We first prove lower bounds on ℓ. Choose

$$\widehat{g_\lambda}(\xi) = \lambda^{-\ell-d/2}\mathbb{1}_{G_\lambda}(\xi), \quad G_\lambda = \{\xi \in \mathbb{R}^d : \lambda \leq |\xi| \leq 2\lambda\}$$

and

$$\widehat{f_\lambda}(\xi) = \frac{\mathbb{1}_{F_\lambda}(\xi)}{|\xi|^{d/2+s}\log(|\xi|)}, \quad F_\lambda = \{\xi \in \mathbb{R}^d : 2 \leq |\xi| \leq \lambda/4\}.$$

We compute

$$\int_{F_\lambda} \widehat{f_\lambda}(\xi)\, d\xi \approx \begin{cases} \frac{\lambda^{d/2-s}}{\log \lambda} & \text{if } s < d/2, \\ \log\log\lambda & \text{if } s = d/2, \\ 1 & \text{if } s > d/2. \end{cases}$$

If $\frac{5}{4}\lambda \leq |\xi| \leq \frac{3}{2}\lambda$ and $\eta \in F_\lambda$, then $\xi - \eta \in G_\lambda$. Therefore,

$$\|I_\lambda(t)\|_{H^s(\mathbb{R}^d)} \lesssim \|g_\lambda\|_{H^\ell(\mathbb{R}^d)}\|f_\lambda\|_{H^s(\mathbb{R}^d)} \quad \text{for all } \lambda \gg 1$$

implies

$$\lambda^{-2+s-\ell}\int_{F_\lambda} \widehat{f_\lambda}(\xi)\, d\xi \lesssim \|I_\lambda(t)\|_{H^s(\mathbb{R}^d)} \lesssim 1,$$

which is true if and only if

$$\begin{cases} \ell \geq d/2-2 & \text{if } s < d/2, \\ \ell > d/2-2 & \text{if } s = d/2, \\ \ell \geq s-2 & \text{if } s > d/2. \end{cases}$$

Second, we prove lower bounds on s. Choose $h_\lambda = a_\lambda + b_\lambda$, where

$$\widehat{a_\lambda}(\xi) = \lambda^{-s-d/2}(\mathbb{1}_{A_\lambda}(\xi) + \mathbb{1}_{-A_\lambda}(\xi)), \quad A_\lambda = \{\xi \in \mathbb{R}^d : |\xi - e_1\lambda| \leq \lambda/4\},$$
$$\widehat{b_\lambda}(\xi) = \frac{\mathbb{1}_{B_\lambda}(\xi)}{|\xi|^{d/2+s}\log(|\xi|)}, \quad B_\lambda = \{\xi \in \mathbb{R}^d : 2 \leq |\xi| \leq \lambda/8\}.$$

We compute

$$\int_{B_\lambda} \widehat{b_\lambda}(\xi)\, d\xi = \begin{cases} \frac{\lambda^{d/2-s}}{\log \lambda} & \text{if } s < d/2, \\ \log\log\lambda & \text{if } s = d/2, \\ 1 & \text{if } s > d/2, \end{cases}$$

as above. The spatial Fourier transform of $e^{it'\Delta}a_\lambda\overline{e^{it'\Delta}a_\lambda}+e^{it'\Delta}b_\lambda\overline{e^{it'\Delta}b_\lambda}$ is zero within the set $C_\lambda=\{\xi\in\mathbb{R}^d:|\xi-\lambda e_1|\leq\frac{1}{8}\lambda\}$. Further, if $\xi\in C_\lambda$ and $\eta\in B_\lambda$, then $\xi-\eta\in A_\lambda$. Therefore, the bound

$$\|J_\lambda(t)\|_{H^\ell(\mathbb{R}^d)}\lesssim\|h_\lambda\|_{H^s(\mathbb{R}^d)}\|h_\lambda\|_{H^s(\mathbb{R}^d)}\quad\text{for all }\lambda\gg 1$$

implies

$$\Big(\int_{C_\lambda}\langle\xi\rangle^{2\ell}|\mathcal{F}J_\lambda^{ab}(t)(\xi)|^2\,d\xi\Big)^{1/2}\lesssim 1\quad\text{for all }\lambda\gg 1,$$

where

$$J_\lambda^{ab}(t)=\int_0^t e^{i(t-t')|\nabla|}|\nabla|\Re\big(e^{it'\Delta}a_\lambda\overline{e^{it'\Delta}b_\lambda}\big)\,dt'.$$

Since

$$\Big(\int_{C_\lambda}\langle\xi\rangle^{2\ell}|\mathcal{F}J_\lambda^{ab}(t)(\xi)|^2\,d\xi\Big)^{1/2}\approx\lambda^{\ell-s-1}\int_{B_\lambda}\widehat{b_\lambda}(\xi)\,d\xi$$

we must have

$$\begin{cases}2s\geq\ell+\frac{d-2}{2} & \text{if } s<d/2,\\ s>\ell-1 & \text{if } s=d/2,\\ s\geq\ell-1 & \text{if } s>d/2.\end{cases}$$

9.2. *Proof of Theorem 1.2*

Suppose that (s,ℓ) satisfies (1.3) and $f\in H^s(\mathbb{R}^d)$ and $(g_0,g_1)\in H^\ell(\mathbb{R}^d)\times H^\ell(\mathbb{R}^d)$. Define $g=g_0-ig_1\in H^\ell(\mathbb{R}^d)$. Suppose that $\|f\|_{H^s}\leq\varepsilon$. If $\varepsilon>0$ is small enough (depending on g), due to the endpoint Strichartz estimate, we see that (7.8) is satisfied for $I=\mathbb{R}$, and Theorem 7.7 yields a unique global solution $(u,V)\in C(\mathbb{R},H^s)\cap L_t^2W_x^{(d-3)/2,2^*}(\mathbb{R}\times\mathbb{R}^d)\times C(\mathbb{R},H^\ell)$ to the Zakharov equation (2.1). Note that $v=\Re V$, $|\nabla|^{-1}\partial_t v=\Im V$ have the same regularity. Also, by Theorem 8.1, the additional regularity persists, i.e. $(u,V)\in S^{s,a,b}\times W^{\ell,a,s-1/2}$ and we have real-analytic dependence. Further, this implies the scattering claim, as shown in the proof of Theorem 7.1.

Acknowledgments. The authors would like to thank the anonymous referee for his valuable suggestions and Akansha Sanwal for spotting a number of typos.

Funding. Financial support by the German Research Foundation (DFG) through the CRC 1283 “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications” is acknowledged.

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