JEMS

Oren Becker · Michael Chapman

Stability of approximate group actions: uniform and probabilistic

Abstract. We prove that every uniform approximate homomorphism from a discrete amenable group into a symmetric group is uniformly close to a homomorphism into a slightly larger symmetric group. That is, amenable groups are uniformly flexibly stable in permutations. This answers affirmatively a question of Kun and Thom and a slight variation of a question of Lubotzky. We also give a negative answer to Lubotzky's original question by showing that the group $\mathbb{Z}$ is not uniformly strictly stable. Furthermore, we show that $\mathrm{SL}_r(\mathbb{Z})$, $r \geq 3$, is uniformly flexibly stable, but the free group F_r, $r \geq 2$, is not. We define and investigate a probabilistic variant of uniform stability that has an application to property testing.

Keywords. Stability of homomorphisms, approximate actions, property testing

1. Introduction

In 1940, Ulam asked the following general question, usually referred to as *Ulam's stability problem* [21,35]: given two groups Γ and G and an approximate homomorphism $f : \Gamma \to G$, is f close to a homomorphism? The answer depends on the groups Γ and G as well as the chosen notions of an approximate homomorphism and proximity between functions.

The following theorem of Kazhdan tackles a particular instance of Ulam's problem: is every approximate unitary representation of a group close in operator norm to a unitary representation?

Theorem (Kazhdan 1982, [24]). *Let $f : \Gamma \to \mathrm{U}(\mathcal{H})$ be a function from an amenable group Γ into the group $\mathrm{U}(\mathcal{H})$ of unitary operators on the Hilbert space $\mathcal{H}$. Take $\delta < 1/200$ such that $\|f(\gamma_1\gamma_2) - f(\gamma_1)f(\gamma_2)\|_{\mathrm{op}} \leq \delta$ for all $\gamma_1, \gamma_2 \in \Gamma$. Then there is a group homomorphism $h : \Gamma \to \mathrm{U}(\mathcal{H})$ such that $\|h(\gamma) - f(\gamma)\|_{\mathrm{op}} \leq 2\delta$ for every $\gamma \in \Gamma$.*

Oren Becker: Department of Pure Mathematics and Mathematical Statistics,
Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom; oren.becker@gmail.com

Michael Chapman: Einstein Institute of Mathematics, The Hebrew University,
Jerusalem 91904, Israel; michael.chapman@mail.huji.ac.il

The present paper tackles a similar problem: is every approximate action on a finite set close to an action? That is, we replace the unitary groups in Kazhdan's theorem by finite symmetric groups. We refer to this version of Ulam stability as *stability in permutations*, and make use of the *normalized Hamming metric* on $\mathrm{Sym}(n)$:

$$d^H(\sigma, \tau) = \frac{1}{n}|\{x \in [n] \mid \sigma(x) \neq \tau(x)\}| \qquad \forall \sigma, \tau \in \mathrm{Sym}(n),$$

where $[n] = \{1, \ldots, n\}$. For a group Γ and a function $f : \Gamma \to \mathrm{Sym}(n)$, we define the *uniform local defect* of f to be

$$\mathrm{def}_\infty(f) = \sup\{d^H(f(\gamma_1 \gamma_2), f(\gamma_1)f(\gamma_2)) \mid \gamma_1, \gamma_2 \in \Gamma\}.$$

A basic result of Glebsky and Rivera studies stability in permutations when the domain group is finite.

Theorem (Glebsky and Rivera 2009, [17]). *Let Γ be a finite group and $f : \Gamma \to \mathrm{Sym}(n)$ a function, $n \in \mathbb{N}$. Then there is a group homomorphism $h : \Gamma \to \mathrm{Sym}(n)$ such that $d^H(h(\gamma), f(\gamma)) \leq C \mathrm{def}_\infty(f)$ for every $\gamma \in \Gamma$, where C depends only on the group Γ (but not on n).*

In Section 1.1, we recall the terminology of group-theoretic stability and two types of a uniformly stable group: strict and flexible. The above theorem says that each finite group is uniformly strictly stable. In the spirit of Kazhdan's Theorem, it is natural to ask whether the same is true for infinite amenable groups. The first test case, as raised by Alex Lubotzky, is the following.

Problem (Lubotzky 2018). Does the theorem of Glebsky and Rivera hold when the finite group Γ is replaced by $\mathbb{Z}$?

In Section 4 we give a negative answer to Lubotzky's question: $\mathbb{Z}$ is not uniformly strictly stable. In fact, we prove the following stronger result.

Theorem 1.1. *Let Γ be a group that has a transitive action on $[n]$. Then there is a function $f : \Gamma \to \mathrm{Sym}(n-1)$ such that $\mathrm{def}_\infty(f) \leq \frac{2}{n-1}$, but for every homomorphism $h : \Gamma \to \mathrm{Sym}(n-1)$ there is $\gamma \in \Gamma$ such that $d^H(f(\gamma), h(\gamma)) \geq \frac{1}{2} - \frac{1}{n-1}$.*

The theorem implies a negative answer to Lubotzky's question, not only for $\Gamma = \mathbb{Z}$, but for every group that has finite quotients of unbounded cardinality. Furthermore, by considering the action of a finite group on itself by left multiplication, Theorem 1.1 implies that in the theorem of Glebsky and Rivera the dependence of the constant C on the group Γ is essential.

These negative results might be discouraging at first. However, another stability theorem, proved by Gowers and Hatami [18] and generalized by De Chiffre, Ozawa and Thom [12], leads us in the right direction. Recall that the normalized Hilbert–Schmidt norm on $\mathrm{U}(n)$ is given by $\|A\|_{\mathrm{hs}} = (\frac{1}{n}\mathrm{tr}(A^*A))^{1/2}$.

Theorem (Gowers and Hatami 2017, [18]; De Chiffre, Ozawa and Thom 2019, [12]). *Let Γ be a discrete amenable group and $f:\Gamma \to U(n)$ a function. Take $\delta > 0$ such that $\|f(\gamma_1\gamma_2) - f(\gamma_1)f(\gamma_2)\|_{hs} \leq \delta$ for all $\gamma_1, \gamma_2 \in \Gamma$. Then there is a representation $h:\Gamma \to U(N)$ and an isometry $T:\mathbb{C}^n \to \mathbb{C}^N$ such that $\|h(\gamma) - T^*f(\gamma)T\|_{hs} \leq 211\delta$ for every $\gamma \in \Gamma$, where $n \leq N \leq (1 + 2500\delta^2)n$.*

It is also shown in [18] and [12] that allowing the increase in dimension from n to N is essential. Similarly, in the context of functions into $\mathrm{Sym}(n)$, we shall allow a controlled increase in the number of points n. We refer to this approach as *flexibility in the number of points.* It was proven to be effective and necessary in many instances [2, 8, 26]. We extend the definition of the normalized Hamming metric to measure distances between elements of symmetric groups of different cardinalities. For $n \leq N$, $\sigma \in \mathrm{Sym}(n)$ and $\tau \in \mathrm{Sym}(N)$, define

$$d^H(\sigma, \tau) = d^H(\tau, \sigma) = \frac{1}{N}(|\{x \in [n] \mid \sigma(x) \neq \tau(x)\}| + (N - n)).$$

The *uniform distance* between two functions $f:\Gamma \to \mathrm{Sym}(n)$ and $h:\Gamma \to \mathrm{Sym}(N)$ is

$$d_\infty(f, h) = d_\infty(h, f) = \sup\{d^H(f(\gamma), h(\gamma)) \mid \gamma \in \Gamma\}.$$

By Lemma A.1, d^H satisfies the triangle inequality and thus it is a metric on the disjoint union $\bigsqcup_{n=1}^{\infty} \mathrm{Sym}(n)$.

The following question was asked by Kun and Thom.

Problem (Kun and Thom 2019, [25, Remark 4.3]). Take a function $f:\Gamma \to \mathrm{Sym}(n)$, where Γ is a finite group and $n \in \mathbb{N}$. Is there a homomorphism $h:\Gamma \to \mathrm{Sym}(N)$ such that $d_\infty(h, f) \leq \varepsilon$ and $n \leq N \leq (1 + \varepsilon)n$, where ε depends only on $\mathrm{def}_\infty(f)$ and tends to zero as $\mathrm{def}_\infty(f)$ tends to zero?

Remark. For $N \geq n$, $\sigma \in \mathrm{Sym}(n)$ and $\tau \in \mathrm{Sym}(N)$, our definition of d^H forces $d^H(\sigma, \tau) \geq 1 - \frac{n}{N}$. Hence, the condition $d_\infty(h, f) \leq \varepsilon$ in the above problem implies that $N \leq \frac{n}{1-\varepsilon} \leq (1 + 2\varepsilon)n$ whenever $\varepsilon \leq 1/2$. Thus, the condition $N \leq (1 + \varepsilon)n$ in the problem statement is redundant. We state it solely for the sake of emphasis.

The problem of Kun and Thom is not solved by the theorem of Glebsky and Rivera because the latter provides ε that depends on the domain group Γ and not only on $\mathrm{def}_\infty(f)$. As already mentioned, the dependency on Γ is essential when flexibility in the number of points is not allowed. Our main result in the present paper is the next theorem. It gives an affirmative answer to Kun and Thom's problem and to a flexible variant of Lubotzky's problem. In fact, it only assumes that Γ is amenable (rather than finite or infinite cyclic), and provides an explicit ε which is linear in $\mathrm{def}_\infty(f)$.

Theorem 1.2 (Amenable groups are uniformly flexibly stable). *Let Γ be a discrete amenable group and $f:\Gamma \to \mathrm{Sym}(n)$ a function, $n \in \mathbb{N}$. Then there is a homomorphism $h:\Gamma \to \mathrm{Sym}(N)$ such that $d_\infty(h, f) \leq 2039\,\mathrm{def}_\infty(f)$ and $n \leq N \leq (1 + 1218\,\mathrm{def}_\infty(f))n$.*

Theorem 1.2 has the following useful corollary (see Theorem 2.20): for a finite-index normal subgroup Δ of Γ and a function $f \colon \Gamma \to \mathrm{Sym}(n)$, there is a homomorphism $h \colon \Gamma \to \mathrm{Sym}(N)$, $N \geq n$, such that

$$d_\infty(h, f) \leq C \cdot (\mathrm{def}_\infty(f) + \sup\{d^H(f(\gamma), \mathrm{id}) \mid \gamma \in \Delta\}),$$

where C is a universal constant. Notably, the bound on $d_\infty(h, f)$ does not depend on the index $[\Gamma : \Delta]$. In fact, it suffices to assume that Γ/Δ is a discrete amenable (rather than finite) group, and thus Theorem 2.20 is a strong form of Theorem 1.2 (up to the constants).

Using Theorems 1.2 and 2.20, and following the method of [10, Section 5], we deduce the following theorem.

Theorem 1.3 ($\mathrm{SL}_r(\mathcal{O}_K)$ is uniformly flexibly stable). *Let* $f \colon \mathrm{SL}_r(\mathcal{O}_K) \to \mathrm{Sym}(n)$ *be a function, where* $n \in \mathbb{N}$, $r \geq 3$ *and* $\mathcal{O}_K$ *is the ring of integers of a number field* K. *Then there is a homomorphism* $h \colon \mathrm{SL}_r(\mathcal{O}_K) \to \mathrm{Sym}(N)$ *such that* $d_\infty(h, f) \leq C \, \mathrm{def}_\infty(f)$ *and* $n \leq N \leq (1 + C \, \mathrm{def}_\infty(f))n$, *where* C *depends only on* r.

The next theorem shows that for some groups not every approximate homomorphism is close to a homomorphism, even when flexibility in the number of points is allowed.

Theorem 1.4 (Nonabelian free groups are not uniformly flexibly stable). *Let* Γ *be a group that surjects onto the free group* F_2 *of rank 2. Then there is a sequence* $(f_k)_{k=1}^\infty$ *of functions* $f_k \colon \Gamma \to \mathrm{Sym}(n_k)$, $n_k \xrightarrow{k \to \infty} \infty$, *such that* $\mathrm{def}_\infty(f_k) \leq 2/k$, *but* $d_\infty(h_k, f_k) \geq 1 - 5/k$ *for every homomorphism* $h_k \colon \Gamma \to \mathrm{Sym}(N_k)$ *for all* $N_k \geq n_k$.

Remark. Theorem 1.4 is analogous to a result of Burger, Ozawa and Thom [10, Proposition 3.3] that says that for $r \geq 2$, an approximate homomorphism $F_r \to U(n)$ need not be close to a homomorphism. More generally, the same is true whenever F_r is replaced by a group Γ such that the comparison map $H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is noninjective [10, Corollary 3.5]. This condition holds when Γ is a nonelementary word-hyperbolic group [16] (and was known earlier in the special case $\Gamma = F_r$ [9]).

The discrete nature of $\mathrm{Sym}(n)$ makes it difficult to prove Theorem 1.4 by following the lines of the proofs of [10, Proposition 3.3] or [10, Corollary 3.5]. Our proof does not use cohomological methods. We do draw inspiration from Rolli's construction of nontrivial quasimorphisms from F_r to $\mathbb{R}$ [33].

In light of the success of [16] in generalizing the cohomological result of [9] from nonabelian free groups to nonelementary word-hyperbolic groups, we pose the following open problem that asks whether Theorem 1.4 can be generalized as well.

Problem 1.5 (Are all nonelementary hyperbolic groups uniformly flexibly instable?). Let Γ be a nonelementary word-hyperbolic group. Is there a sequence $(f_k)_{k=1}^\infty$ of functions $f_k \colon \Gamma \to \mathrm{Sym}(n_k)$, $n_k \in \mathbb{N}$, such that $\mathrm{def}_\infty(f_k) \xrightarrow{k \to \infty} 0$, but $d_\infty(f_k, h_k) \geq C$ for every homomorphism $h_k \colon \Gamma \to \mathrm{Sym}(N_k)$, $N_k \geq n_k$, where $C > 0$ does not depend on k?

1.1. A framework for stability

The above results can be formulated in a single framework. We consider the following objects:

- Two classes of groups: $\mathcal{C}$ and $\mathcal{G}$.
- The full class of functions $\mathcal{F} = \{f : \Gamma \to G \mid \Gamma \in \mathcal{C}, G \in \mathcal{G}\}$.
- A function $\mathrm{def} : \mathcal{F} \to \mathbb{R}_{\geq 0}$, called the *local defect*.
- A distance function $d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$.

The *global defect* of $f \in \mathcal{F}$ with respect to d is

$$\mathrm{Def}_d(f) = \inf\{d(f, h) \mid h \in \mathcal{F} \text{ is a group homomorphism}\}.$$

We say that the class $\mathcal{C}$ of groups is $(\mathcal{G}, \mathrm{def}, d)$-*stable* if $\mathrm{Def}_d(f) \leq R(\mathrm{def}(f))$ for every $f \in \mathcal{F}$, where $R(\delta) \xrightarrow{\delta \to 0} 0$. We say that $\mathcal{C}$ is stable *with linear rate* if $R(\delta) \leq C\delta$ for a universal constant $C > 0$. A group Γ is $(\mathcal{G}, \mathrm{def}, d)$-*stable* if the same is true for the class $\{\Gamma\}$.

All of the results presented so far can be formulated using this framework. In the context of stability in permutations, the role of $\mathcal{G}$ is taken by $\mathcal{S} = \{\mathrm{Sym}(n)\}_{n=1}^{\infty}$. The role of def is taken by def_{∞}. Let $f, h \in \mathcal{F}$. We extend d_{∞} by setting $d_{\infty}(f, h) = \infty$ if the domains of f and h are different. We define $d_{\infty}^{\mathrm{strict}}(f, h) = d_{\infty}(f, h)$ if f and h have the same domain and range, and $d_{\infty}^{\mathrm{strict}}(f, h) = \infty$ otherwise. A class $\mathcal{C}$ of groups is *uniformly flexibly stable in permutations* if it is $(\mathcal{S}, \mathrm{def}_{\infty}, d_{\infty})$-stable. The class $\mathcal{C}$ is *uniformly strictly stable in permutations* if it is $(\mathcal{S}, \mathrm{def}_{\infty}, d_{\infty}^{\mathrm{strict}})$-stable. From now on, we omit the phrase "in permutations".

The results presented thus far about functions into symmetric groups are summarized in the following table, where the numbers on the right are theorem numbers or references (in brackets).

Uniformly strictly stable:	each finite group	[17]
Not uniformly strictly stable:	the class of finite groups	1.1
	each group with unbounded finite quotients	1.1
Uniformly flexibly stable:	the class of discrete amenable groups	1.2
	each $\mathrm{SL}_r(\mathbb{Z})$, $r \geq 3$	1.3
Not uniformly flexibly stable:	each group that surjects onto F_2	1.4

In fact, [17] and Theorems 1.2 and 1.3 prove stability with linear rate.

Remark (Pointwise stability). The related notion of *pointwise stability* has recently been under heavy investigation [1–4, 8, 11, 15, 17, 20, 22, 26–29, 34, 37]. A *stability challenge* for Γ is a sequence $(f_k)_{k=1}^{\infty}$ of functions $f_k : \Gamma \to \mathrm{Sym}(n_k)$, $n_k \in \mathbb{N}$, such that $d^H(f_k(\gamma_1 \gamma_2), f_k(\gamma_1) f_k(\gamma_2))$ tends to zero as $k \to \infty$ for all $\gamma_1, \gamma_2 \in \Gamma$. A *solution* for $(f_k)_{k=1}^{\infty}$ is a sequence $(h_k)_{k=1}^{\infty}$ of homomorphisms $h_k : \Gamma \to \mathrm{Sym}(N_k)$, $N_k \geq n_k$, such

that $d^H(f_k(\gamma), h_k(\gamma))$ tends to zero as $k \to \infty$ for each $\gamma \in \Gamma$. The group Γ is *pointwise stable in permutations* if every stability challenge for Γ has a solution.

For finitely presented groups, pointwise stability can be formalized using the above framework if we allow each object of $\mathcal{C}$ to be a group with a fixed presentation (rather than just a group). Proximity between functions is measured with respect to the images of the generators, and the local defect is defined using the relators.

In the context of pointwise stability, the strict and flexible versions are equivalent when the group Γ is amenable [22, Lemma 3.2 (1)]. The main result of [3] provides a useful equivalent condition for pointwise stability among amenable groups. In particular, and as has been known previously [1, 17], some amenable groups are pointwise stable and some are not (in contrast with the flexible uniform case, as shown by Theorem 1.2). For $r \geq 3$, the group $\mathrm{SL}_r(\mathbb{Z})$ is not strictly stable both in the pointwise sense [2] and in the uniform sense (by Theorem 1.1). By Theorem 1.3, $\mathrm{SL}_r(\mathbb{Z})$ is uniformly flexibly stable, but it is not known whether it is pointwise flexibly stable. By [8], if one can find $r \geq 5$ such that $\mathrm{PSL}_r(\mathbb{Z})$ is pointwise flexibly stable, it would solve a long-standing open problem by showing that not all groups are sofic.

1.2. Probabilistic stability and homomorphism testing

Let Γ be a discrete amenable group. Fix a finitely additive measure m on Γ that is either left or right invariant. The *mean local defect* of a function $f : \Gamma \to \mathrm{Sym}(n)$ is given by

$$\mathrm{def}_1(f) = \iint d^H(f(\gamma_1 \gamma_2), f(\gamma_1) f(\gamma_2)) \, dm(\gamma_1) \, dm(\gamma_2).$$

For $N \geq n$, $f : \Gamma \to \mathrm{Sym}(n)$ and $h : \Gamma \to \mathrm{Sym}(N)$, the *mean distance* between f and h is

$$d_1(f, h) = d_1(h, f) = \int d^H(f(\gamma), h(\gamma)) \, dm(\gamma).$$

It is well known that every discrete amenable group Γ admits a finitely additive measure that is simultaneously left, right and inverse invariant (see Section 2.1). Henceforth, we fix such a measure for each discrete amenable group. We prove that the class of discrete amenable groups is *probabilistically flexibly stable*. That is, it is $(\mathcal{S}, \mathrm{def}_1, d_1)$-stable. More precisely, we prove the following analogue of Theorem 1.2.

Theorem 1.6 (Amenable groups are probabilistically flexibly stable). *Let Γ be a discrete amenable group and $f : \Gamma \to \mathrm{Sym}(n)$ a function, $n \in \mathbb{N}$. Then there is a homomorphism $h : \Gamma \to \mathrm{Sym}(N)$ such that $d_1(h, f) \leq 2913 \, \mathrm{def}_1(f)$ and $n \leq N \leq (1 + 1740 \, \mathrm{def}_1(f))n$.*

Theorem 1.6 has an application to property testing in the case where the group Γ is finite. We begin by recalling a generalized version of the Blum–Luby–Rubinfeld Theorem [7] on homomorphism testing.

Theorem ([6], [19, Theorem 2.3]). *Let $f : \Gamma \to G$ be a function between finite groups. Let*

$$\delta = \frac{1}{|\Gamma \times \Gamma|} |\{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma \mid f(\gamma_1 \gamma_2) \neq f(\gamma_1) f(\gamma_2)\}|.$$

If $\delta < 1/6$, then there is a homomorphism $h\colon \Gamma \to G$ such that

$$\frac{1}{|\Gamma|}|\{\gamma \in \Gamma \mid h(\gamma) \neq f(\gamma)\}| \leq 2\delta.$$

Take f, Γ and G as in the theorem. The finite group Γ should be thought of as very large. Assume that f is either a homomorphism or disagrees with every homomorphism $\Gamma \to G$ on many elements of Γ. The goal in homomorphism testing is to distinguish between the two cases after reading $f(\gamma)$ for only a small number of elements $\gamma \in \Gamma$. This is achieved by the following algorithm.

Algorithm 1 Homomorphism testing

Input: A function $f\colon \Gamma \to G$ between finite groups
Output: Accept or Reject

```
1: Sample (γ₁, γ₂) ∈ Γ × Γ uniformly at random.
2: If f(γ₁γ₂) = f(γ₁)f(γ₂), return Accept.
3: Otherwise, return Reject.
```

Clearly, if f is a homomorphism then the algorithm accepts. Let $0 < \varepsilon \leq 1/3$ and assume that f disagrees with every group homomorphism $h\colon \Gamma \to G$ on at least $\varepsilon|\Gamma|$ elements of Γ. The generalized Blum–Luby–Rubinfeld Theorem implies that the algorithm rejects with probability at least $\varepsilon/2$. For $\alpha > 0$, we can amplify the rejection probability to be at least $1 - \alpha$ by running the algorithm for $k = \lceil \log_{1-\varepsilon/2} \alpha \rceil = O\left(\frac{2\log(1/\alpha)}{\varepsilon}\right)$ independent iterations and accepting if and only if all iterations accept. Note that k is independent of Γ and G.

Now consider the scenario where the group Γ is still very large and $G = \mathrm{Sym}(n)$, where n is also very large. In this case, reading the permutation $f(\gamma)$, even just for a single element $\gamma \in \Gamma$, may be too time consuming. The following algorithm is tailored for this situation.

Algorithm 2 Testing of homomorphisms into $\mathrm{Sym}(n)$

Input: A function $f\colon \Gamma \to \mathrm{Sym}(n)$, where Γ is a finite group and $n \in \mathbb{N}$
Output: Accept or Reject

```
1: Sample (γ₁, γ₂, x) ∈ Γ × Γ × [n] uniformly at random.
2: If f(γ₁γ₂)(x) = f(γ₁)f(γ₂)(x), return Accept.
3: Otherwise, return Reject.
```

Again, if f is a homomorphism then the algorithm always accepts. On the other hand, for $0 < \varepsilon \leq 1$, if $d_1(f, h) \geq \varepsilon$ for every homomorphism $h\colon \Gamma \to \mathrm{Sym}(N)$, $N \geq n$, then the probability that the algorithm rejects is at least $\frac{\varepsilon}{2913}$. As before, the rejection probability may be amplified by running the algorithm repeatedly.

Finally, we show that strict stability has the same caveats in the probabilistic setting as in the uniform setting. That is, our proof of Theorem 1.1 in Section 4 handles probabilistic stability in addition to uniform stability. Hence, Algorithm 2 is not a good tester in the strict model.

Remark (A recent breakthrough in quantum information theory). The recent solution [23] to Connes' embedding problem relies on a probabilistic variant of the Gowers–Hatami Theorem (see also [31, 32, 36]). The latter plays a fundamental role in the proof by forcing the shared state of nearly optimal provers to be close to a specific desired state. We would like to know whether there are similar applications where Theorem 1.6 can be used instead of the Gowers–Hatami Theorem.

1.3. The structure of the paper and some comments on the proofs

In Section 2 we prove Theorems 1.2 and 1.6. The proof takes a function $f: \Gamma \to \text{Sym}(n)$ and restricts each permutation $f(\gamma)$ to an injective function $f(\gamma)|_{D(\gamma)}: D(\gamma) \to [n]$, $D(\gamma) \subset [n]$, such that $f(\gamma_1)|_{D(\gamma_1)} \circ f(\gamma_2)|_{D(\gamma_2) \cap f(\gamma_2)^{-1}D(\gamma_1)}$ and $f(\gamma_1\gamma_2)|_{D(\gamma_1\gamma_2)}$ coincide on the intersection of their domains. We then extend each $f(\gamma)|_{D(\gamma)}$ to a permutation $h(\gamma): [N] \to [N]$, $N \geq n$, such that $h: \Gamma \to \text{Sym}(N)$ is a homomorphism. Amenability is used in the restriction step to ensure quantitative properties, such as the ratios $|D(\gamma)|/n$ being close to 1 (uniformly in γ or on average). After the restriction step we are left with a purely algebraic structure (a groupoid). The extension step does not make direct use of amenability. The proof is related to the proof of homomorphism testing for finite groups [19, Theorem 2.3] in the sense that both employ majority voting.

In Section 3 we prove a more general version of Theorem 1.3. To that end we use Theorem 1.2 on the subgroups of upper and lower triangular unipotent matrices of SL_r, apply bounded generation [30], and conclude by using Theorem 2.20. In Section 4 we prove Theorem 1.1 and its probabilistic version. To exclude strict stability for a group Γ, we take a homomorphism $f: \Gamma \to \text{Sym}(n)$ that defines a transitive action and deform it into a function $\hat{f}: \Gamma \to \text{Sym}(n-1)$ by bypassing n. We invoke [2, Proposition 2.4 (ii)] and some analysis to show that $\hat{f}$ is far from every homomorphism in the strict model. In Section 5 we prove Theorem 1.4. We show that the free group F_2 is not flexibly uniformly stable by constructing functions $f_k: F_2 \to \text{Sym}(n_k)$, with small $\text{def}_\infty(f_k)$, that grossly violate a group identity that holds in $\text{Sym}(N)$ for all $N \geq n$. In Appendix A we prove that d^H satisfies the triangle inequality. In Appendix B we prove an auxiliary result that says that a function $f: \Gamma \to \text{Sym}(n)$ with small local defect is close to a function that sends $1_\Gamma \mapsto \text{id}$ and respects inverses. This result is used in Section 2. Appendix B handles a more general case, where $\text{Sym}(n)$ is replaced by a metric group satisfying a mild condition.

2. Flexible stability of amenable groups

In this section we prove Theorems 1.2 and 1.6, and deduce Theorem 2.20.

2.1. Preliminaries on amenable groups

Let Γ be a discrete group. Write $\mathcal{P}(\Gamma)$ for the power set of Γ. A *finitely additive probability measure* m on Γ is a function $m \colon \mathcal{P}(\Gamma) \to [0, 1]$ such that $m(\Gamma) = 1$ and $m(A \cup B) = m(A) + m(B)$ whenever A and B are disjoint subsets of Γ. We call such an m a *measure* on Γ for short. We say that m is *left invariant* (resp. *right invariant*) if $m(\gamma A) = m(A)$ (resp. $m(A\gamma) = m(A)$) for every $\gamma \in \Gamma$ and $A \subset \Gamma$. The group Γ is *amenable* if it admits a left-invariant measure. Finite groups are amenable since we may take m to be the normalized counting measure. In this particular case, the left-invariant measure is unique. Other basic examples of amenable groups are abelian groups and, more generally, solvable groups. Examples of nonamenable groups include free groups on more than one generator and infinite groups with Property (T), such as $\mathrm{SL}_r(\mathbb{Z})$, $r \geq 3$.

Let us give an example of a left-invariant measure m on $\Gamma = \mathbb{Z}$. To each $A \subset \mathbb{Z}$ we attach a bounded sequence $(a_t)_{t=1}^{\infty}$, given by $a_t = \frac{|A \cap [-t,t]|}{2t+1}$. If $(a_t)_{t=1}^{\infty}$ converges, we set $m(A)$ to be its limit. More generally, we set $m(A)$ to be a carefully chosen accumulation point of $(a_t)_{t=1}^{\infty}$. This is made possible by the axiom of choice, and if done carefully, results in a finitely additive measure m. The left invariance of m follows from the fact that for a fixed $r \in \mathbb{Z}$, the ratio

$$\frac{|[-t,t] \triangle (r + [-t,t])|}{|[-t,t]|}$$

tends to zero as $t \to \infty$, where $\triangle$ denotes symmetric difference. In other words, $([-t,t])_{n=1}^{\infty}$ is a *Følner sequence* for $\mathbb{Z}$. It is possible to change our proofs of Theorems 1.2 and 1.6 to use $A \mapsto |A \cap [-t,t]|$ instead of m, for a carefully chosen large t. However, the use of a limit (in fact, an ultralimit) in the definition of m saves us the effort of tracking error terms in the course of the proof.

Let m be a measure on Γ. There is a notion of integration with respect to m of bounded functions $\Gamma \to \mathbb{C}$ (see [13, Section 1.2.2]). The integration functional $f \mapsto \int f \, dm \colon L^{\infty}(\Gamma) \to \mathbb{C}$ is a positive linear functional such that $\int \mathbf{1}_A \, dm = m(A)$. Positivity means that $\int f \, dm \geq 0$ whenever the image of f is contained in $\mathbb{R}_{\geq 0}$, and linearity means that integration commutes with finite sums and multiplication by scalars. We may write $dm(\gamma)$ instead of dm to indicate that γ is the variable of integration. If m is left invariant, then $\int f(\gamma_0 \gamma) \, dm(\gamma) = \int f(\gamma) \, dm(\gamma)$ for every $f \in L^{\infty}(\Gamma)$ and $\gamma_0 \in \Gamma$. For $\Omega \subset \Gamma$, write $\int_{\Omega} f \, dm$ for $\int f \cdot \mathbf{1}_{\Omega} \, dm$.

A left-invariant measure μ on Γ gives rise to $m \colon \mathcal{P}(\Gamma) \to [0, 1]$, given by

$$m(A) = \frac{1}{2} \int \left(\mu(A\gamma) + \mu(A^{-1}\gamma) \right) d\mu(\gamma).$$

Then m is a measure on Γ and it is *bi-invariant*, that is, simultaneously left and right invariant. Furthermore, it is *inverse invariant*, that is, $m(A) = m(A^{-1})$ for each $A \subset \Gamma$. Integration with respect to a right-invariant or inverse-invariant measure has the corresponding invariance property.

Finally, a trivial but useful consequence of left or right invariance of m is that for a subgroup H of Γ we have $[\Gamma : H] = \frac{1}{m(H)}$, interpreted as ∞ if $m(H) = 0$.

2.2. *The setup for the proof of Theorems* 1.2 *and* 1.6

Fix an amenable group Γ with a measure m that is left, right and inverse invariant. We consider the metrics d_∞ and d_1, and the local defects def_∞ and def_1, as defined in the introduction, where d_1 and def_1 are defined with respect to m.

2.2.1. *Symmetrization*

Definition 2.1. Let Γ and G be groups. A function $f: \Gamma \to G$ is *symmetric* if $f(1_\Gamma) = 1_G$ and $f(\gamma^{-1}) = (f(\gamma))^{-1}$ for all $\gamma \in \Gamma$.

Appendix B deals with deforming a function $f: \Gamma \to G$ into a symmetric function, and equips us with the following proposition.

Proposition 2.2. *Let* Γ *be a group and* $f: \Gamma \to \mathrm{Sym}(n)$ *a function. Then there exists a symmetric function* $f': \Gamma \to \mathrm{Sym}(n)$ *such that*

$$d_\infty(f, f') \le 2\,\mathrm{def}_\infty(f), \quad \mathrm{def}_\infty(f') \le 7\,\mathrm{def}_\infty(f),$$
$$d_1(f, f') \le 3\,\mathrm{def}_1(f), \quad \mathrm{def}_1(f') \le 10\,\mathrm{def}_1(f).$$

Proof. The claim is a special case of Proposition B.3 (see Lemma B.2). ∎

By virtue of Proposition 2.2, Theorems 1.2 and 1.6 follow from the following theorem.

Theorem 2.3. *Let* $n \in \mathbb{N}$ *and let* $f: \Gamma \to \mathrm{Sym}(n)$ *be a symmetric function. Write* $\delta_\infty = \mathrm{def}_\infty(f)$ *and* $\delta_1 = \mathrm{def}_1(f)$. *Then there is* $N \ge n$ *and a homomorphism* $h: \Gamma \to \mathrm{Sym}(N)$ *such that*

$$N \le (1 + 174\delta_\infty)n \quad and \quad d_\infty(h, f) \le 291\delta_\infty$$

and

$$N \le (1 + 174\delta_1)n \quad and \quad d_1(h, f) \le 291\delta_1 .$$

Note that $\delta_1 \le \delta_\infty$ and thus the claim $N \le (1 + 174\delta_\infty)n$ follows at once from $N \le (1 + 174\delta_1)n$.

2.2.2. Γ-*graphs.* The proof of Theorem 2.3 is based on a graph-theoretic approach.

Definition 2.4. A Γ-*graph* is a pair $X = (V, E)$, where V is a set and E is a subset of $V \times \Gamma \times V$. The elements of V are the *vertices* of X and the elements of E are the oriented Γ-labelled *edges* of X. We use the notation $x \xrightarrow{\gamma} y$ to denote the edge (x, γ, y) with *origin* x, *destination* y and *label* γ. We require that for every $x \in V$ and $\gamma \in \Gamma$ there exists at most one $y \in V$ such that $x \xrightarrow{\gamma} y \in E$.

With X as above, denote $V(X) := V$ and $E(X) := E$. We say that X is *finite* if $|V(X)| < \infty$. All Γ-graphs that appear in our argument are finite in this sense.

Definition 2.5. Let V be a set and take a function $f: \Gamma \to \mathrm{Sym}(V)$. The *function graph* X_f of f is the Γ-graph with vertex set V and edge set

$$E(X_f) = \{x \xrightarrow{\gamma} f(\gamma)(x) \mid x \in V, \gamma \in \Gamma\}.$$

The function f is symmetric if and only if all edges of X_f with label 1_Γ are loops and for every edge $x \xrightarrow{\gamma} y \in E(X_f)$, we have $y \xrightarrow{\gamma^{-1}} x \in E(X_f)$. In this case, f is a homomorphism if and only if every path in X_f of the form $\xrightarrow{\gamma_1} \xrightarrow{\gamma_2} \xrightarrow{\gamma_1^{-1}\gamma_2^{-1}}$ is closed (for every starting vertex). Informally, if $\mathrm{def}_1(f)$ is small, then almost all of these paths are closed, and the same is true if $\mathrm{def}_\infty(f)$ is small since $\mathrm{def}_1(f) \le \mathrm{def}_\infty(f)$.

Let $X = (V, E)$ be a Γ-graph. We write $x \xrightarrow{\gamma} \in E$ to indicate that there is a vertex $y \in V$ such that $x \xrightarrow{\gamma} y \in E$. For $\gamma \in \Gamma$, the *domain* of γ is $D_X(\gamma) := \{x \in V \mid x \xrightarrow{\gamma} \in E\}$. The *set of outgoing labels* from a vertex $x \in V$ is $\mathrm{OL}_X(x) := \{\gamma \in \Gamma \mid x \xrightarrow{\gamma} \in E\}$ and the *out degree* of a vertex $x \in V$ is $\deg_X(x) := m(\mathrm{OL}(x))$. In this way, the measure m enables us to define a useful notion of a degree in the graph X, where a vertex may have infinitely many incident edges. When the graph X is clear from the context, we may omit it from the notation in $D_X(\gamma)$ and $\mathrm{OL}_X(x)$.

Fact 2.6. *For a Γ-graph $X = (V, E)$, the sum of out degrees is equal to the integral of the cardinalities of domains. That is,*

$$\sum_{x \in V} \deg_X(x) = \int |D_X(\gamma)| \, dm(\gamma).$$

Fact 2.6 follows immediately from the definitions and the basic properties of integration discussed in Section 2.1.

Let $X = (V, E)$ and $X' = (V', E')$ be Γ-graphs. We say that X is a *subgraph* of X' if $V \subseteq V'$ and $E \subseteq E'$. A function $\varphi : V \to V'$ is a *morphism of Γ-graphs* from X to X' if $\varphi(x_1) \xrightarrow{\gamma} \varphi(x_2)$ is in E' whenever $x_1 \xrightarrow{\gamma} x_2$ is in E. Such a function φ is an *embedding* of X in X' if it is injective.

The heart of the proof of Theorem 2.3 lies in the proof of the following proposition.

Proposition 2.7. *Let $f : \Gamma \to \mathrm{Sym}(n)$ be a symmetric function, $n \in \mathbb{N}$. Write $\delta_\infty = \mathrm{def}_\infty(f)$ and $\delta_1 = \mathrm{def}_1(f)$, and assume that $\delta_1 \le 1/78$. Then there is a subgraph Z of the function graph X_f, a finite set V_1 and a homomorphism $g : \Gamma \to \mathrm{Sym}(V_1)$ such that:*

(i) *Z embeds in the function graph X_g.*

(ii) *$|V(Z)| \ge (1 - 96\delta_1)n$.*

(iii) *For every $\gamma_0 \in \Gamma$,*

$$|D_Z(\gamma_0)| \ge (1 - 117\delta_\infty)n \tag{2.1}$$

and

$$\int |D_Z(\gamma)| \, dm(\gamma) \ge (1 - 117\delta_1)n. \tag{2.2}$$

(iv) *$n \le |V_1| \le (1 + 78\delta_1)n$.*

Theorem 2.3 follows from Proposition 2.7 as follows.

Proof of Theorem 2.3. We are given a function $f : \Gamma \to \mathrm{Sym}(n)$ and need to define a homomorphism $h : \Gamma \to \mathrm{Sym}(N)$, $N \ge n$, such that h and f are close together. If $\delta_1 > 1/78$,

set $N = n$ and let $h \colon \Gamma \to \operatorname{Sym}(N)$ be the trivial homomorphism. Assume henceforth that $\delta_1 \leq 1/78$. Apply Proposition 2.7 to f to obtain a subgraph Z of the function graph X_f, a set V_1 and a homomorphism $g \colon \Gamma \to \operatorname{Sym}(V_1)$ such that (i)–(iv) of Proposition 2.7 are satisfied. In particular, there is an embedding $\varphi \colon V(Z) \to V_1$ of the Γ-graph Z in the function graph X_g. Let $N = |V_1| + (n - |V(Z)|)$. Then

$$n \leq N \leq (1 + (78 + 96)\delta_1)n = (1 + 174\delta_1)n.$$

Assume without loss of generality that $V_1 = V(Z) \amalg ([N] \setminus [n])$ and that φ is the inclusion map. For every $\gamma \in \Gamma$, define $h(\gamma) \in \operatorname{Sym}(N)$ by

$$h(\gamma)(x) = \begin{cases} g(\gamma)(x), & x \in V_1, \\ x, & x \in [n] \setminus V(Z), \end{cases} \qquad \forall x \in [N].$$

Let $\gamma \in \Gamma$ and $x \in D_Z(\gamma)$. Then $g(\gamma)(x) = f(\gamma)(x)$ since the inclusion map $\varphi \colon V(Z) \to V_1$ is an embedding of the Γ-graph Z in X_g. On the other hand, $h(\gamma)(x) = g(\gamma)(x)$ since $x \in V(Z) \subset V_1$. Hence, for every $\gamma_0 \in \Gamma$,

$$\begin{aligned} d^H(h(\gamma_0), f(\gamma_0)) &\leq \frac{1}{N}\big((n - |D_Z(\gamma_0)|) + (N - n)\big) \\ &= 1 - \frac{n}{N} \frac{|D_Z(\gamma_0)|}{n} \\ &\leq 1 - \frac{1}{1 + 174\delta_1} \cdot \frac{|D_Z(\gamma_0)|}{n}. \end{aligned} \qquad (2.3)$$

Hence,

$$\begin{aligned} d_\infty(h, f) &\leq 1 - \frac{1 - 117\delta_\infty}{1 + 174\delta_\infty} && \text{by (2.3) and (2.1)} \\ &\leq 1 - (1 - 174\delta_\infty)(1 - 117\delta_\infty) \\ &\leq 291\delta_\infty \end{aligned}$$

and

$$\begin{aligned} d_1(h, f) &\leq \int \left(1 - \frac{1}{1 + 174\delta_1} \cdot \frac{|D_Z(\gamma)|}{n}\right) dm(\gamma) && \text{by (2.3)} \\ &\leq 1 - (1 - 174\delta_1)(1 - 117\delta_1) && \text{by (2.2)} \\ &\leq 291\delta_1. \quad \blacksquare \end{aligned}$$

2.3. Proof of Proposition 2.7

Fix $n \in \mathbb{N}$ and a symmetric function $f \colon \Gamma \to \operatorname{Sym}(n)$. Write $\delta_\infty = \operatorname{def}_\infty(f)$ and $\delta_1 = \operatorname{def}_1(f)$. We first construct the subgraph Z of the function graph X_f. Then, we proceed to define the set V_1 and the homomorphism $g \colon \Gamma \to \operatorname{Sym}(V_1)$. Finally, we show that (i)–(iv) of Proposition 2.7 are satisfied.

2.3.1. The construction of the subgraph Z. We begin by assigning a weight in the range $[0, 1]$ to each edge $x \xrightarrow{\gamma} y \in [n] \times \Gamma \times [n]$ (whether or not the edge belongs to X_f).

Definition 2.8. The *set of supporters* of an edge $x \xrightarrow{\gamma} y \in [n] \times \Gamma \times [n]$ is

$$T(x \xrightarrow{\gamma} y) = \{t \in \Gamma \mid f(t) f(t^{-1}\gamma)(x) = y\}$$

and the *weight* of this edge is

$$w(x \xrightarrow{\gamma} y) = m(T(x \xrightarrow{\gamma} y)).$$

Note that $T(x \xrightarrow{\gamma} y)$ consists of all elements $t \in \Gamma$ such that the path $x \xrightarrow{t^{-1}\gamma} \xrightarrow{t}$ in the function graph X_f ends at y. Recall that our eventual goal is to find a homomorphism g near f. One may think of the weight $w(x \xrightarrow{\gamma} y)$ as the (normalized) result of a vote, taken among the elements of Γ, on whether the permutation $g(\gamma)$ should send x to y. Intuitively, if δ_∞ (or δ_1) is small then almost all edges of X_f have high weight. Below, we consider subgraphs of X_f that include only the high-weight edges. In Section 2.3.2, we prove that two of these subgraphs admit an algebraic structure. In Section 2.3.3 the algebraic structure gives rise to the sought-after homomorphism g.

For a Γ-graph $X = (V, E)$ and a subset $V' \subset V$, the *induced subgraph* of X on V' is $X' = (V', E')$, where $E' = E \cap (V' \times \Gamma \times V')$. We say that a subgraph X'' of X is *induced* if there is a subset $V'' \subset V$ such that X'' is the induced subgraph of X on V''.

Definition 2.9. (i) For $\varepsilon > 0$, let X_ε be the subgraph of X_f with vertex set $[n]$ and edge set

$$E(X_\varepsilon) = \{x \xrightarrow{\gamma} y \in E(X_f) \mid w(x \xrightarrow{\gamma} y) > 1 - \varepsilon\}.$$

(ii) For $0 < \varepsilon \le 1/6$, let Y_ε be the induced subgraph of $X_{2\varepsilon}$ on the following vertex set:

$$V(Y_\varepsilon) = \{x \in [n] \mid \deg_{X_\varepsilon}(x) > 2/3\}.$$

Explicitly, the edge set $E(Y_\varepsilon)$ consists of all edges $x \xrightarrow{\gamma} f(\gamma)(x)$ for $x \in [n]$ and $\gamma \in \Gamma$ such that $\deg_{X_\varepsilon}(x) > 2/3$, $\deg_{X_\varepsilon}(f(\gamma)(x)) > 2/3$ and $w(x \xrightarrow{\gamma} y) > 1 - 2\varepsilon$.

(iii) For $0 < \varepsilon \le 1/6$, let Z_ε be the induced subgraph of Y_ε on the following vertex set:

$$V(Z_\varepsilon) = \{x \in V(Y_\varepsilon) \mid \deg_{Y_\varepsilon}(x) \ge 1/2\}.$$

(iv) Finally, set $Z = Z_{1/6}$.

Note that in the definition of Y_ε we use $\deg_{X_\varepsilon}$ to filter out the low-degree vertices of $X_{2\varepsilon}$. The interplay between ε and 2ε is crucial in our proof in the next section that Y_ε and Z_ε are well structured. In regard to the definition of Z_ε, in Section 2.3.3 we shall see that the degrees in Y_ε are constant within each connected component, and thus Z_ε is a union of components of Y_ε.

2.3.2. The Γ-graphs Y_ε and Z_ε are Γ-groupoids

Definition 2.10. A Γ-graph $X = (V, E)$ is a Γ-*groupoid* if the following conditions hold:

(i) **Symmetry:** for every edge $x\xrightarrow{\gamma}y \in E$, we have $y\xrightarrow{\gamma^{-1}}x \in E$.

(ii) **Triangles:** for all $x, y, z \in V$ and $\gamma_1, \gamma_2 \in \Gamma$, if $x\xrightarrow{\gamma_1}y \in E$ and $y\xrightarrow{\gamma_2}z \in E$ then $x\xrightarrow{\gamma_2\gamma_1}z \in E$.

If $h\colon \Gamma \to \mathrm{Sym}(n)$ is a homomorphism then the function graph X_h is a Γ-groupoid, usually referred to as an *action groupoid*. In Section 2.3.3, we investigate general properties of Γ-groupoids. Here, we prove that Y_ε and Z_ε are Γ-groupoids whenever $\varepsilon \le 1/6$.

Our assumption that f is symmetric comes into play in the next lemma, which will be used in what follows without reference.

Lemma 2.11. *Consider an edge* $x\xrightarrow{\gamma}y \in [n] \times \Gamma \times [n]$. *Then* $w(x\xrightarrow{\gamma}y) = w(y\xrightarrow{\gamma^{-1}}x)$.

Proof. Let $t \in \Gamma$. Then $t \in T(x\xrightarrow{\gamma}y)$ if and only if $f(t)f(t^{-1}\gamma)(x) = y$ if and only if $x = f(\gamma^{-1}t)f((\gamma^{-1}t)^{-1}\gamma^{-1})(y)$ if and only if $\gamma^{-1}t \in T(y\xrightarrow{\gamma^{-1}}x)$. Thus $\gamma^{-1}T(x\xrightarrow{\gamma}y) = T(y\xrightarrow{\gamma^{-1}}x)$, and since m is left invariant we conclude that $w(x\xrightarrow{\gamma}y) = w(y\xrightarrow{\gamma^{-1}}x)$. ∎

Lemma 2.12. *Consider a triplet of edges* $x\xrightarrow{\gamma_1}y$, $y\xrightarrow{\gamma_2}z$ *and* $x\xrightarrow{\gamma_2\gamma_1}u$ *in* $[n] \times \Gamma \times [n]$, *each of weight larger than* $2/3$. *Then* $z = u$.

Proof. Let $Q = T(x\xrightarrow{\gamma_1}y) \cap \gamma_2^{-1}T(y\xrightarrow{\gamma_2}z) \cap \gamma_2^{-1}T(x\xrightarrow{\gamma_2\gamma_1}u)$. Each set in the intersection has measure larger than $2/3$, so there exists an element t in Q. Hence,

$$
\begin{aligned}
z &= f(\gamma_2 t)f((\gamma_2 t)^{-1}\gamma_2)(y) & \gamma_2 t \in T(y\xrightarrow{\gamma_2}z)\\
&= f(\gamma_2 t)f(t^{-1})(y)\\
&= f(\gamma_2 t)f(t^{-1})f(t)f(t^{-1}\gamma_1)(x) & t \in T(x\xrightarrow{\gamma_1}y)\\
&= f(\gamma_2 t)f(t^{-1}\gamma_1)(x) & f \text{ is symmetric}\\
&= f(\gamma_2 t)f((\gamma_2 t)^{-1}\gamma_2\gamma_1)(x)\\
&= u. & \gamma_2 t \in T(x\xrightarrow{\gamma_2\gamma_1}u)
\end{aligned}
$$
∎

Lemma 2.13. *Consider* $\varepsilon_1, \varepsilon_2 > 0$ *and a pair of edges* $x\xrightarrow{\gamma_1}y$ *and* $y\xrightarrow{\gamma_2}z$ *in* $[n] \times \Gamma \times [n]$, *such that* $w(x\xrightarrow{\gamma_1}y) > 1 - \varepsilon_1$ *and* $w(y\xrightarrow{\gamma_2}z) > 1 - \varepsilon_2$. *Then* $w(x\xrightarrow{\gamma_2\gamma_1}z) > 1 - \varepsilon_1 - \varepsilon_2$.

Proof. Let $Q = \gamma_2 T(x\xrightarrow{\gamma_1}y) \cap T(y\xrightarrow{\gamma_2}z)$. Then $m(Q) > 1 - \varepsilon_1 - \varepsilon_2$, and thus it suffices to show that $Q \subset T(x\xrightarrow{\gamma_2\gamma_1}z)$. Indeed, if $t \in Q$ then $t \in T(x\xrightarrow{\gamma_2\gamma_1}z)$ because

$$
\begin{aligned}
f(t)f(t^{-1}\gamma_2\gamma_1)(x) &= f(t)f(t^{-1}\gamma_2)f(\gamma_2^{-1}t)f(t^{-1}\gamma_2\gamma_1)(x) & f \text{ is symmetric}\\
&= f(t)f(t^{-1}\gamma_2)(y) & \gamma_2^{-1}t \in T(x\xrightarrow{\gamma_1}y)\\
&= z. & t \in T(y\xrightarrow{\gamma_2}z)
\end{aligned}
$$
∎
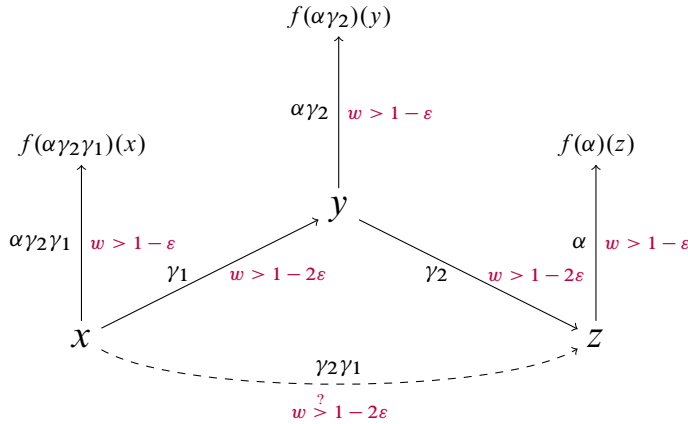
Fig. 2.1. The high degrees of x, y and z guarantee the existence of $\alpha \in \Gamma$ such that the three vertical edges have weight larger than $1 - \varepsilon$. Then, Lemma 2.12 ensures that the three vertices on top are the same vertex u. Finally, Lemma 2.13, applied to the path $x \xrightarrow{\alpha\gamma_2\gamma_1} u \xrightarrow{\alpha^{-1}} z$, implies that the bottom edge has weight larger than $1 - 2\varepsilon$.

The above lemma shows that the composition of two high-weight edges results in a high-weight edge, but with some decrease in weight. This deterioration makes it difficult to grow a large Γ-groupoid edge by edge inside $X_{2\varepsilon}$. This difficulty is addressed by the following lemma, which motivates the distinction between ε and 2ε in the definition of Y_ε.

Lemma 2.14. *Let $0 < \varepsilon \leq 1/6$ and consider a pair of edges $x \xrightarrow{\gamma_1} y$ and $y \xrightarrow{\gamma_2} z$ in $[n] \times \Gamma \times [n]$ such that*

$$\deg_{X_\varepsilon}(x), \deg_{X_\varepsilon}(y), \deg_{X_\varepsilon}(z) > 2/3,$$

$$w(x \xrightarrow{\gamma_1} y) > 1 - 2\varepsilon, \tag{2.4}$$

$$w(y \xrightarrow{\gamma_2} z) > 1 - 2\varepsilon. \tag{2.5}$$

Then $w(x \xrightarrow{\gamma_2\gamma_1} z) > 1 - 2\varepsilon$.

Proof. The proof is illustrated in Figure 2.1. Let

$$Q = (\mathrm{OL}_{X_\varepsilon}(x)\gamma_1^{-1}\gamma_2^{-1}) \cap (\mathrm{OL}_{X_\varepsilon}(y)\gamma_2^{-1}) \cap \mathrm{OL}_{X_\varepsilon}(z).$$

Each set in the intersection has measure larger than $2/3$, so there exists an element α in Q. Then

$$w(x \xrightarrow{\alpha\gamma_2\gamma_1} f(\alpha\gamma_2\gamma_1)(x)) > 1 - \varepsilon, \tag{2.6}$$

$$w(y \xrightarrow{\alpha\gamma_2} f(\alpha\gamma_2)(y)) > 1 - \varepsilon, \tag{2.7}$$

$$w(z \xrightarrow{\alpha} f(\alpha)(z)) > 1 - \varepsilon. \tag{2.8}$$

Lemma 2.12 applies to the triplet of edges $x \xrightarrow{\gamma_1} y$, $y \xrightarrow{\alpha\gamma_2} f(\alpha\gamma_2)(y)$ and $x \xrightarrow{\alpha\gamma_2\gamma_1} f(\alpha\gamma_2\gamma_1)(x)$ since $\varepsilon \leq 1/6$ and by virtue of (2.4), (2.7) and (2.6), and thus $f(\alpha\gamma_2)(y) = f(\alpha\gamma_2\gamma_1)(x)$. Similarly, the lemma applies to the triplet of edges $y \xrightarrow{\gamma_2} z$, $z \xrightarrow{\alpha} f(\alpha)(z)$ and $y \xrightarrow{\alpha\gamma_2} f(\alpha\gamma_2)(y)$ due to (2.5), (2.8) and (2.7), and thus $f(\alpha)(z) = f(\alpha\gamma_2)(y)$. Hence

$$f(\alpha\gamma_2\gamma_1)(x) = f(\alpha)(z). \tag{2.9}$$

By (2.9), Lemma 2.13 applies to the pair of edges $x \xrightarrow{\alpha\gamma_2\gamma_1} f(\alpha\gamma_2\gamma_1)(x)$ and $f(\alpha)(z) \xrightarrow{\alpha^{-1}} z$, and thus

$$w(x \xrightarrow{\gamma_2\gamma_1} z) > 1 - 2\varepsilon$$

by (2.6) and (2.8). ∎

Proposition 2.15. *Let $0 < \varepsilon \leq 1/6$. Then Y_ε and Z_ε are Γ-groupoids.*

Proof. The function graph X_f is a Γ-graph, so the same is true for its subgraph $X_{2\varepsilon}$. By Lemma 2.11, $X_{2\varepsilon}$ satisfies the symmetry condition of Definition 2.10, and thus so does its induced subgraph Y_ε. Furthermore, since $\varepsilon \leq 1/6$, Lemma 2.14 implies that Y_ε satisfies the triangles condition, and is thus a Γ-groupoid. Therefore, the induced subgraph Z_ε of Y_ε is a Γ-groupoid as well. ∎

2.3.3. Construction of the homomorphism g. Recall that we want to construct a finite set V_1, of cardinality not much larger than n, and a homomorphism $g \colon \Gamma \to \mathrm{Sym}(V_1)$ such that the subgraph $Z = Z_{1/6}$ of X_f embeds into the action groupoid X_g. First, we investigate further generalities on Γ-groupoids.

Let $X = (V, E)$ be a Γ-groupoid. Two vertices $x, y \in V$ are *connected* if there is $\gamma \in \Gamma$ such that $x \xrightarrow{\gamma} y \in E$. Connectedness induces an equivalence relation on V. The induced subgraphs of X on the equivalence classes are the *connected components*, or *components* for short, of the Γ-groupoid X. If there is just one component, we say that X is *connected*. Each component of X is a connected Γ-groupoid. For $x \in V$, write $C_{X,x}$, or C_x, for the component of x in X.

The *stabilizer* of $x \in V$ is $\Gamma_x = \{\gamma \in \Gamma \mid x \xrightarrow{\gamma} x \in E\}$. Note that Γ_x is a subgroup of Γ. Also, if $x \xrightarrow{\gamma} y \in E$, then $\Gamma_y = \gamma\Gamma_x\gamma^{-1}$ and $\mathrm{OL}(y)\gamma = \mathrm{OL}(x)$. Hence, if the Γ-groupoid X is connected, then the index $[\Gamma : \Gamma_x]$ is the same for all $x \in V$, and the same is true for the degree $\deg(x)$. These numbers are, respectively, the *index* $\mathrm{ind}(X)$ and *degree* $\deg(X)$ of the connected groupoid X.

By Proposition 2.15, if $0 < \varepsilon \leq 1/6$ then Y_ε is a Γ-groupoid. Recalling Definition 2.9 (iii), we see that in this case Z_ε is the union of the components of Y_ε that have degree at least $1/2$.

The proof of the following proposition is straightforward and is left to the reader.

Proposition 2.16. *Let $X = (V, E)$ be a connected Γ-groupoid and let $x \in V$. Let $h \colon \Gamma \to \mathrm{Sym}(\Gamma/\Gamma_x)$ be the action of Γ on Γ/Γ_x by left multiplication. Then the function*

$\varphi: V \to \Gamma / \Gamma_x$ defined by

$$\varphi(y) = \gamma \Gamma_x \quad \forall x \xrightarrow{\gamma} y \in E$$

is well defined and injective. Furthermore, it is an embedding of the Γ-graph X into the action groupoid X_h.

Let $X = (V, E)$ be a finite (i.e. $|V| < \infty$) connected groupoid. For $x \in V$, Proposition 2.16 embeds X, which has $|V|$ vertices, in an action groupoid that has $[\Gamma : \Gamma_x]$ vertices. We investigate the ratio $|V|/[\Gamma : \Gamma_x]$. For $x \in V$ and $\gamma \in \mathrm{OL}(x)$, write $\gamma \cdot x$ for the unique element y of V such that $x \xrightarrow{\gamma} y \in E$. For $x \in V$, $\mathrm{OL}(x)$ is a union of left cosets of Γ_x, and for $\gamma_1, \gamma_2 \in \mathrm{OL}(x)$ we have $\gamma_1 \cdot x = \gamma_2 \cdot x$ if and only if $\gamma_1 \Gamma_x = \gamma_2 \Gamma_x$. Since X is connected, this means that $\mathrm{OL}(x)$ is a union of $|V|$ distinct left cosets of Γ_x. Hence $|V| m(\Gamma_x) = m(\mathrm{OL}(x)) = \deg(X)$. But $[\Gamma : \Gamma_x] = 1/m(\Gamma_x)$, and thus if $\deg(X) > 0$ then $[\Gamma : \Gamma_x] < \infty$ and

$$\frac{|V|}{[\Gamma : \Gamma_x]} = \deg(X). \tag{2.10}$$

It is possible to apply Proposition 2.16 to each component of $Y_{1/6}$ in order to embed $Y_{1/6}$ into an action groupoid. However, in order to obtain sufficiently good bounds on the number of vertices in the action groupoid, we do the same to $Z = Z_{1/6}$ rather than $Y_{1/6}$.

The definition of the set V_1, the homomorphism $g: \Gamma \to \mathrm{Sym}(V_1)$ and the embedding of Z into the action groupoid X_g proceeds as follows. Let $\{C_i\}_{i=1}^m$ be the components of Z. Fix a vertex $x_i \in V(C_i)$ for each $1 \le i \le m$, and write $g_i: \Gamma \to \mathrm{Sym}(\Gamma / \Gamma_{x_i})$ for the action of Γ on Γ / Γ_{x_i} by left multiplication. By Proposition 2.16, each C_i embeds into X_{g_i}, and thus Z embeds into $\bigsqcup_{i=1}^m X_{g_i}$. More precisely, write $V_1 = \bigsqcup_{i=1}^m \Gamma / \Gamma_{x_i}$ and let $g: \Gamma \to \mathrm{Sym}(V_1)$ be the action by left multiplication. Each C_i embeds into X_{g_i} by an embedding $\varphi_i : V(C_i) \to \Gamma / \Gamma_{x_i}$. These embeddings give rise to an embedding $\varphi : V(Z) \to V_1$ of Z into X_g.

We evaluate the cardinality of V_1 as follows:

$$
\begin{aligned}
|V_1| &= \sum_{i=1}^c [\Gamma : \Gamma_x] \\
&= \sum_{i=1}^c |V(C_i)| \cdot (\deg(C_i))^{-1} \quad \text{by (2.10)} \\
&= \sum_{x \in V(Z)} (\deg_Z x)^{-1}. \tag{2.11}
\end{aligned}
$$

We shall see in Lemma 2.19 (iv) that (2.11) is bounded from above by $(1 + O(\delta_1))|V(Z)|$.

2.3.4. Proof of Proposition 2.7 (ii)–(iv).

We constructed the subgraph Z of X_f, the homomorphism $g: \Gamma \to \mathrm{Sym}(V_1)$ and an embedding $\varphi: V(Z) \to V_1$ of Z into X_g. To complete the proof of Proposition 2.7, it remains to prove the lower bounds on $|V(Z)|$ and $|D_Z(\gamma)|$ and the upper bound on $|V_1|$. To do so, we study quantitative properties of X_ε, Y_ε and Z_ε.

In the following three lemmas, we justify some of the steps by invoking Markov's inequality. By this we are referring to the fact that if $a_1, \ldots, a_m$ are real numbers in the

interval $[0, 1]$ and $\theta > 0$, then

$$\left|\{i \mid a_i > 1 - \theta\}\right| \geq \left(1 - \frac{1}{\theta}\left(1 - \frac{1}{m}\sum_{i=1}^{m}a_i\right)\right)m. \tag{2.12}$$

Lemma 2.17. *Let $\varepsilon > 0$. Then*

(i) $|D_{X_\varepsilon}(\gamma)| \geq (1 - \delta_\infty/\varepsilon)n$ *for every $\gamma \in \Gamma$.*

(ii) $\int |D_{X_\varepsilon}(\gamma)|\, dm(\gamma) \geq (1 - \delta_1/\varepsilon)n$.

Proof. For $\gamma, t \in \Gamma$ and $x \in [n]$, write

$$\mathbf{1}_{\gamma,t,x} = \begin{cases} 1, & f(\gamma)(x) = f(t)f(t^{-1}\gamma)(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{n}\sum_{x=1}^{n}\mathbf{1}_{\gamma,t,x} = 1 - d^H\left(f(t^{-1})f(\gamma), f(t^{-1}\gamma)\right) \geq 1 - \delta_\infty \quad \forall \gamma \in \Gamma\ \forall t \in \Gamma \tag{2.13}$$

and

$$\int \mathbf{1}_{\gamma,t,x}\, dm(t) = w(x\xrightarrow{\gamma}f(\gamma)(x)) \quad \forall \gamma \in \Gamma\ \forall x \in [n]. \tag{2.14}$$

Hence, for $\gamma \in \Gamma$,

$$\frac{1}{n}\sum_{x\in[n]}w(x\xrightarrow{\gamma}f(\gamma)(x)) = \int\left(\frac{1}{n}\sum_{x\in[n]}\mathbf{1}_{\gamma,t,x}\right)dm(t) \geq 1 - \delta_\infty,$$

and thus (i) follows from Markov's inequality (2.12):

$$|D_{X_\varepsilon}(\gamma)| = \left|\left\{x \in [n] \mid w(x\xrightarrow{\gamma}f(\gamma)(x)) > 1 - \epsilon\right\}\right| \geq (1 - \delta_\infty/\epsilon)n.$$

Now, using (2.14), (2.13) and the inverse invariance of m,

$$\frac{1}{n}\sum_{x\in[n]}\int w(x\xrightarrow{\gamma}f(\gamma)(x))\, dm(\gamma) = \int\int\left(\frac{1}{n}\sum_{x\in[n]}\mathbf{1}_{\gamma,t,x}\right)dm(t)\, dm(\gamma)$$

$$= \int\int\left(1 - d^H\left(f(t^{-1})f(\gamma), f(t^{-1}\gamma)\right)\right)dm(t)\, dm(\gamma)$$

$$= 1 - \delta_1. \tag{2.15}$$

Thus (ii) follows:

$$\int|D_{X_\varepsilon}(\gamma)|\, dm(\gamma)$$

$$= \int\left|\left\{x \in [n] \mid w(x\xrightarrow{\gamma}f(\gamma)(x)) > 1 - \varepsilon\right\}\right|dm(\gamma)$$

$$\geq \int\left(1 - \frac{1}{\varepsilon}\left(1 - \frac{1}{n}\sum_{x\in[n]}w(x\xrightarrow{\gamma}f(\gamma)(x))\right)\right)n\, dm(\gamma) \quad \text{by Markov (2.12)}$$

$$= \left(1 - \frac{\delta_1}{\varepsilon}\right)n. \qquad\qquad\qquad\qquad \text{by (2.15)} \qquad\blacksquare$$

Lemma 2.18. *Let* $0 < \varepsilon \leq 1/6$. *Then*

(i) $|V(Y_\varepsilon)| \geq (1 - 3\delta_1/\varepsilon)n$.

(ii) $|D_{Y_\varepsilon}(\gamma)| \geq (1 - 6.5\delta_\infty/\varepsilon)n$ *for every* $\gamma \in \Gamma$.

(iii) $\int |D_{Y_\varepsilon}(\gamma)| \, dm(\gamma) \geq (1 - 6.5\delta_1/\varepsilon)n$.

Proof. To prove (i), we compute

$$
\begin{aligned}
|V(Y_\varepsilon)| &= |\{x \in [n] \mid \deg_{X_\varepsilon}(x) > 1 - 1/3\}| \\
&\geq \left(1 - 3\left(1 - \frac{1}{n}\sum_{x \in [n]} \deg_{X_\varepsilon}(x)\right)\right)n && \text{by Markov (2.12)} \\
&= \left(1 - 3\left(1 - \frac{1}{n}\int |D_{X_\varepsilon}(\gamma)| \, dm(\gamma)\right)\right)n && \text{by Fact 2.6} \\
&\geq \left(1 - \frac{3\delta_1}{\varepsilon}\right)n. && \text{by Lemma 2.17 (ii)}
\end{aligned}
$$

Now, for $\gamma \in \Gamma$,

$$
\begin{aligned}
|D_{Y_\varepsilon}(\gamma)| &= |D_{X_{2\varepsilon}}(\gamma) \cap V(Y_\varepsilon) \cap f(\gamma)^{-1}V(Y_\varepsilon)| \\
&= \left|D_{X_{2\varepsilon}}(\gamma) \setminus \left(([n] \setminus V(Y_\varepsilon)) \cup ([n] \setminus f(\gamma^{-1})V(Y_\varepsilon))\right)\right| \\
&\geq |D_{X_{2\varepsilon}}(\gamma)| - 2(n - |V(Y_\varepsilon)|) \\
&\geq |D_{X_{2\varepsilon}}(\gamma)| - \frac{6\delta_1}{\varepsilon}n. && \text{by (i)} \quad (2.16)
\end{aligned}
$$

Then (ii) follows from (2.16) and Lemma 2.17 (i), and (iii) follows by integrating (2.16) and using Lemma 2.17 (ii). ∎

Proposition 2.7 (ii)–(iii) follows from Lemma 2.19 (i)–(iii) by plugging in $\varepsilon = 1/6$. In light of (2.11), Proposition 2.7 (iv) follows from Lemma 2.19 (iv).

Lemma 2.19. *Let* $0 < \varepsilon \leq 1/6$ *and assume that* $\delta_1 \leq \varepsilon/13$. *Then*

(i) $|Z_\varepsilon| \geq (1 - 16\delta_1/\varepsilon)n$.

(ii) $|D_{Z_\varepsilon}(\gamma)| \geq (1 - 19.5\delta_\infty/\varepsilon)n$ *for every* $\gamma \in \Gamma$.

(iii) $\int |D_{Z_\varepsilon}(\gamma)| \, dm(\gamma) \geq (1 - 19.5\delta_1/\varepsilon)n$.

(iv) $\sum_{x \in V(Z_\varepsilon)} (\deg_{Z_\varepsilon}(x))^{-1} \leq (1 + 13\delta_1/\varepsilon)n$.

Proof. We first bound the average degree in Y_ε:

$$
\begin{aligned}
\frac{1}{|V(Y_\varepsilon)|} \sum_{x \in V(Y_\varepsilon)} \deg_{Y_\varepsilon}(x) &\geq \frac{1}{n} \sum_{x \in V(Y_\varepsilon)} \deg_{Y_\varepsilon}(x) \\
&= \frac{1}{n} \int |D_{Y_\varepsilon}(\gamma)| \, dm(\gamma) && \text{by Fact 2.6} \\
&\geq 1 - 6.5\delta_1/\varepsilon && \text{by Lemma 2.18 (iii)} \quad (2.17) \\
&\geq 1/2. && \delta_1 \leq \varepsilon/13 \quad (2.18)
\end{aligned}
$$

Recall that Z_ε is attained from Y_ε by removing the components of degree smaller than $1/2$, and hence, by (2.18), the average degree in Z_ε is larger than or equal to that of Y_ε. Thus, by (2.17),

$$\frac{1}{|V(Z_\varepsilon)|} \sum_{x \in V(Z_\varepsilon)} \deg_{Z_\varepsilon}(x) \geq 1 - \frac{6.5\delta_1}{\varepsilon}. \tag{2.19}$$

To prove (i), we compute

$$
\begin{aligned}
|V(Z_\varepsilon)| &= \left| \{ x \in V(Y_\varepsilon) \mid \deg_{Y_\varepsilon}(x) \geq 1/2 \} \right| \\
&\geq \left(1 - 2 \left(1 - \frac{1}{|V(Y_\varepsilon)|} \sum_{x \in V(Y_\varepsilon)} \deg_{Y_\varepsilon}(x) \right) \right) |V(Y_\varepsilon)| \quad \text{by Markov (2.12)} \\
&\geq \left(1 - \frac{13\delta_1}{\varepsilon} \right) |V(Y_\varepsilon)| \qquad\qquad\qquad\qquad \text{by (2.17)} \tag{2.20} \\
&\geq \left(1 - \frac{16\delta_1}{\varepsilon} \right) n. \qquad\qquad\qquad\qquad\quad \text{by Lemma 2.18 (i)}
\end{aligned}
$$

Now, for $\gamma \in \Gamma$,

$$
\begin{aligned}
|D_{Z_\varepsilon}(\gamma)| &\geq |D_{Y_\varepsilon}(\gamma)| - (|V(Y_\varepsilon)| - |V(Z_\varepsilon)|) \\
&\geq |D_{Y_\varepsilon}(\gamma)| - \frac{13\delta_1}{\varepsilon} n. \qquad\qquad \text{by (2.20)} \tag{2.21}
\end{aligned}
$$

Then (ii) follows from (2.21) and Lemma 2.18 (ii), and (iii) follows from integrating (2.21) and using Lemma 2.18 (iii).

Finally, we prove (iv). For $x \in V(Z_\varepsilon)$ we have $\deg_{Z_\varepsilon}(x) \geq 1/2$ and thus $(\deg_{Z_\varepsilon}(x))^{-1} \leq 1 + 2(1 - \deg_{Z_\varepsilon}(x))$. Hence,

$$
\begin{aligned}
\sum_{x \in V(Z_\varepsilon)} (\deg_{Z_\varepsilon}(x))^{-1} &\leq \sum_{x \in V(Z_\varepsilon)} \left(1 + 2(1 - \deg_{Z_\varepsilon}(x)) \right) \\
&= |V(Z_\varepsilon)| \left(1 + \frac{2}{|V(Z_\varepsilon)|} \sum_{x \in V(Z_\varepsilon)} (1 - \deg_{Z_\varepsilon}(x)) \right) \\
&\leq n(1 + 13\delta_1/\varepsilon). \qquad\qquad\qquad\qquad \text{by (2.19)} \quad \blacksquare
\end{aligned}
$$

2.3.5. From almost vanishing on a coamenable subgroup to a nearby homomorphism. The following theorem strengthens Theorem 1.2 (up to the constants). We will use it in Section 3 in the special case where $[\Gamma : \Delta] < \infty$. For $\delta > 0$ and permutations σ and τ, the proof uses the following notation:

$$\sigma \approx_\delta \tau \quad \text{meaning} \quad d^H(\sigma, \tau) \leq \delta. \tag{2.22}$$

Theorem 2.20. *Let Γ be a group, $f : \Gamma \to \mathrm{Sym}(n)$ a function and $\Delta \lhd \Gamma$ a normal subgroup such that Γ/Δ is a discrete amenable group. Write $\delta_\infty = \mathrm{def}_\infty(f)$ and $\delta_\Delta = \sup\{d^H(f(\gamma), \mathrm{id}) \mid \gamma \in \Delta\}$. Then there is $n \leq N \leq (1 + 2436\delta_\infty + 1218\delta_\Delta)n$ and a homomorphism $h : \Gamma \to \mathrm{Sym}(N)$ such that $d_\infty(h, f) \leq 4079\delta_\infty + 2040\delta_\Delta$.*

Proof. Let T be a transversal for the set of left cosets of Δ in Γ. Define a function $\bar{f} \colon \Gamma/\Delta \to \mathrm{Sym}(n)$ by letting $\bar{f}(\gamma\Delta) = f(\gamma)$ for $\gamma \in T$. Let $\gamma_1, \gamma_2 \in T$ and take $\gamma \in T$ and $\alpha \in \Delta$ such that $\gamma = \gamma_1\gamma_2\alpha$. Then

$$\bar{f}((\gamma_1\Delta)(\gamma_2\Delta)) = f(\gamma) = f(\gamma_1\gamma_2\alpha) \approx_{2\delta_\infty} f(\gamma_1)f(\gamma_2)f(\alpha)$$
$$\approx_{\delta_\Delta} f(\gamma_1)f(\gamma_2) = \bar{f}(\gamma_1\Delta)\bar{f}(\gamma_2\Delta).$$

Hence, $\mathrm{def}_\infty(\bar{f}) \leq 2\delta_\infty + \delta_\Delta$. Therefore, by Theorem 1.2 applied to $\bar{f}$, there is $n \leq N \leq (1 + 1218(2\delta_\infty + \delta_\Delta))n$ and a homomorphism $\bar{h} \colon \Gamma/\Delta \to \mathrm{Sym}(N)$ such that $d_\infty(\bar{h}, \bar{f}) \leq 2039(2\delta_\infty + \delta_\Delta)$. Define $h \colon \Gamma \to \mathrm{Sym}(N)$ by letting $h(\gamma) = \bar{h}(\gamma\Delta)$ for $\gamma \in \Gamma$. Consider $\gamma \in \Gamma$ and take $\alpha \in \Delta$ such that $\gamma\alpha \in T$. Then

$$h(\gamma) = \bar{h}(\gamma\Delta) \approx_{4078\delta_\infty + 2039\delta_\Delta} \bar{f}(\gamma\Delta) = f(\gamma\alpha) \approx_{\delta_\infty} f(\gamma)f(\alpha) \approx_{\delta_\Delta} f(\gamma). \qquad \blacksquare$$

3. Flexible stability of special linear groups

Here we prove a more general version of Theorem 1.3 about the uniform flexible stability of $\mathrm{SL}_r(A)$, $r \geq 3$, where A is either $\mathbb{Z}$ or one of many other commutative rings, as discussed below. For many of those rings, our result applies to $\mathrm{SL}_2(A)$ as well. We follow the method of [10, Section 5] and use Theorems 1.2 and 2.20 together with a well-known theorem about bounded generation [30].

Fron now on, let $A = S^{-1}B$, where B is an order in the ring of integers $\mathcal{O}_K$ of an algebraic number field $K/\mathbb{Q}$ and S is a multiplicative subset of B. For example, we can take $A = \mathcal{O}_K$. Fix $r \geq 2$ and assume that at least one of the following holds:

- $r \geq 3$;
- A has infinitely many units.

The following result is a special case of [30, Theorem 6.1].

Theorem 3.1. *Let S be a conjugation-invariant subset of $\mathrm{SL}_r(A)$ that contains at least one nonscalar matrix. Then $\langle S \rangle$ is a finite-index normal subgroup of $\mathrm{SL}_r(A)$ and there is an integer C_r, depending only on r, such that every element of the group $\langle S \rangle$ is a product of at most C_r elements of S and their inverses.*

Let C_r be the constant provided by Theorem 3.1 (we fixed r, but we keep it in the notation for emphasis). The notation $O(F(x))$ is used in Theorem 3.2 to refer to an unspecified real-valued function $g(x)$ such that $|g(x)| \leq M \cdot F(x)$ for all $x \geq 0$, where M is an unspecified absolute constant. We also use the notation $\approx$ as in (2.22).

Theorem 3.2. *The group $\mathrm{SL}_r(A)$ is uniformly flexibly stable with linear rate. More explicitly, let $f \colon \mathrm{SL}_r(A) \to \mathrm{Sym}(n)$ be a function and write $\delta = \mathrm{def}_\infty(f)$. Then there is $n \leq N \leq (1 + O(\delta))n$ and a homomorphism $h \colon \mathrm{SL}_r(A) \to \mathrm{Sym}(N)$ such that $d_\infty(h, f) \leq O(C_r\delta)$.*

Proof. Let U^+ and U^- be the subgroups of $\mathrm{SL}_r(A)$ of upper and lower triangular unipotent matrices, respectively. Both U^+ and U^- are nilpotent, and thus they are amenable. Apply Theorem 1.2 to the restrictions $f|_{U^+}$ and $f|_{U^-}$. The theorem provides $N_1, N_2 \in \mathbb{N}$, $n \leq N_i \leq (1 + O(\delta))n$, and homomorphisms $g^+ : U^+ \to \mathrm{Sym}(N_1)$ and $g^- : U^- \to \mathrm{Sym}(N_2)$ such that $d_\infty(g^+, f|_{U^+}) \leq O(\delta)$ and $d_\infty(g^-, f|_{U^-}) \leq O(\delta)$. Let $N = \max \{N_1, N_2\}$.

For distinct $i, j \in [r]$, let E_{ij} be the $r \times r$ matrix with 1 in the (i, j) entry and 0 elsewhere, and let $U_{ij} = I_r + E_{ij}$. We consider the elements $\{U_{ij}\}_{i \neq j}$ and their powers. Let $i \neq j$. If $i < j$, then

$$f(U_{ij}^{\pm N!}) \approx_{O(\delta)} g^+(U_{ij}^{\pm N!}) = \mathrm{id}_{\mathrm{Sym}(N_1)} \approx_{O(\delta)} \mathrm{id}_{\mathrm{Sym}(n)}.$$

Similarly, $f(U_{ij}^{\pm N!}) \approx_{O(\delta)} \mathrm{id}_{\mathrm{Sym}(n)}$ for $i > j$. Hence, for $\gamma \in \mathrm{SL}_r(A)$ and $i \neq j$, we have

$$f(\gamma U_{ij}^{\pm N!} \gamma^{-1}) \approx_{O(\delta)} f(\gamma) f(U_{ij}^{\pm N!}) f(\gamma^{-1}) \approx_{O(\delta)} f(\gamma) f(\gamma^{-1}) \approx_{2\delta} \mathrm{id}_{\mathrm{Sym}(n)}$$

(see (B.2) for the last step). Let $S = \{\gamma U_{ij}^{\pm N!} \gamma^{-1} \mid i \neq j, \gamma \in \mathrm{SL}_r(A)\}$ and $\Delta = \langle S \rangle$. By Theorem 3.1, Δ is a finite-index normal subgroup of $\mathrm{SL}_r(A)$ and every element of Δ is a product of at most C_r elements of S. Hence,

$$f(\gamma) \approx_{O(C_r \delta)} \mathrm{id}_{\mathrm{Sym}(n)} \quad \forall \gamma \in \Delta.$$

The claim now follows from Theorem 2.20. ∎

Remark 3.3. Fix $r \geq 2$ and let $\mathcal{C}$ be the set consisting of all groups $\mathrm{SL}_r(A')$ such that the pair (A', r) satisfies the conditions from the beginning of the section. Then Theorem 3.2 says that the class $\mathcal{C}$ is uniformly flexibly stable with linear rate.

4. Counterexamples for strict stability: the integers and the class of finite groups

This section is devoted to the proof of Theorem 1.1 and its probabilistic version. Both versions are included in the statement of Theorem 4.3 below.

For the sake of the proof of the probabilistic version, we collect preliminary facts regarding integration on a space equipped with a finitely additive probability measure (or *measure* for short, see Section 2.1). In our case, the space is an amenable group Γ equipped with a left- or right-invariant measure m, but the preliminary facts hold regardless of the invariance property.

Recall that integration is a positive linear functional $L^\infty(\Gamma) \to \mathbb{C}$. That is, if the image of $f \in L^\infty(\Gamma)$ is contained in $\mathbb{R}_{\geq 0}$, then $\int f \, dm \geq 0$. By the proof of [14, IV.4.1], this is enough for the Cauchy–Schwarz inequality to hold: $|\int f \overline{g} \, dm|^2 \leq (\int |f|^2 \, dm)(\int |g|^2 \, dm)$ for bounded functions $f, g : \Gamma \to \mathbb{C}$. By taking $g = \mathbf{1}_\Gamma$ we deduce that $|\int f \, dm|^2 \leq \int |f|^2 \, dm$. For a bounded vector-valued function $f : \Gamma \to \mathbb{C}^d$, define $\int f \, dm$ by integrating coordinatewise. The aforementioned corollary of the Cauchy–Schwarz inequality extends to this setting: $\|\int f \, dm\|^2 \leq \int \|f\|^2 \, dm$, where $\|\cdot\|$ is the L^2 norm on $\mathbb{C}^d$. Finally, for a linear operator $A : \mathbb{C}^d \to \mathbb{C}^d$, we have $A \int f \, dm = \int A f \, dm$.

Write $\Psi_n \colon \mathrm{Sym}(n) \to \mathrm{Sym}(n-1)$ for the map given by

$$\Psi_n(\sigma)(x) = \begin{cases} \sigma(x), & \sigma(x) \neq n, \\ \sigma(\sigma(x)), & \sigma(x) = n, \end{cases} \quad \forall \sigma \in \mathrm{Sym}(n) \ \forall x \in [n-1].$$

For a group Γ and a homomorphism $f \colon \Gamma \to \mathrm{Sym}(n)$, define $\hat{f} = \Psi_n \circ f$. First, note that

$$\mathrm{def}_\infty(\hat{f}) \leq \frac{2}{n-1}, \tag{4.1}$$

a fortiori, $\mathrm{def}_1(\hat{f}) \leq \frac{2}{n-1}$ if Γ is equipped with a measure. Indeed, for $\gamma_1, \gamma_2 \in \Gamma$ and $x \in [n-1]$, if $x \notin \{f(\gamma_2)^{-1}(n), f(\gamma_1\gamma_2)^{-1}(n)\}$, then $\hat{f}(\gamma_1)\hat{f}(\gamma_2)(x) = \hat{f}(\gamma_1\gamma_2)(x)$. Thus $d^H(\hat{f}(\gamma_1)\hat{f}(\gamma_2), \hat{f}(\gamma_1\gamma_2)) \leq \frac{2}{n-1}$, and (4.1) follows. In Theorem 4.3 we show that if f defines a transitive action $\Gamma \curvearrowright [n]$ then $\hat{f}$ is far from every homomorphism $\Gamma \to \mathrm{Sym}(n-1)$. The proof of the theorem relies on the following observation.

Lemma 4.1. *Let $\rho \colon \Gamma \to \mathrm{U}(\mathcal{H})$ be a unitary representation of a (discrete) group Γ on a finite-dimensional complex Hilbert space $\mathcal{H}$, $\varepsilon > 0$ and $v \in \mathcal{H}$.*

(i) *If $\|\rho(\gamma)v - v\| \leq \varepsilon\|v\|$ for every $\gamma \in \Gamma$, then there is a Γ-invariant vector $u \in \mathcal{H}$ such that $\|u - v\| \leq \frac{\varepsilon}{\sqrt{2}}\|v\|$.*

(ii) *If Γ is an amenable group equipped with a right-invariant measure m such that $\int \|\rho(\gamma^{-1})v - v\|^2 \, dm(\gamma) \leq \epsilon^2\|v\|^2$, then the vector $w = \int \rho(\gamma^{-1})v \, dm(\gamma)$ is Γ-invariant and satisfies $\|w - v\| \leq \varepsilon\|v\|$.*

Proof. (i) The claim follows from the argument presented in [5, Propositions 1.1.5 and 1.1.9]. Here we recall a part of the argument, which yields a weaker bound. Let C be the closed convex hull of the orbit $\rho(\Gamma)v$ of v. Then C is contained in the closed ball of radius $\varepsilon\|v\|$ centred at v, and there is a unique point u in C of minimal norm. Clearly $\|u - v\| \leq \varepsilon\|v\|$. Furthermore, for $\gamma \in \Gamma$ we have $\rho(\gamma)(\rho(\Gamma)v) = \rho(\Gamma)v$ and thus $\rho(\gamma)C = C$. Then $\rho(\gamma)u = u$ since $\rho(\gamma)$ is norm preserving.

(ii) The right invariance of m implies that w is Γ-invariant. Indeed, for $\gamma_0 \in \Gamma$,

$$\rho(\gamma_0)w = \rho(\gamma_0)\int \rho(\gamma^{-1})v \, dm(\gamma) = \int \rho((\gamma\gamma_0^{-1})^{-1})v \, dm(\gamma)$$
$$= \int \rho(\gamma^{-1})v \, dm(\gamma) = w.$$

Furthermore,

$$\|w - v\|^2 = \left\|\int (\rho(\gamma^{-1})v - v) \, dm(\gamma)\right\|^2 \leq \int \|\rho(\gamma^{-1})v - v\|^2 \, dm(\gamma) \leq \varepsilon^2\|v\|^2. \quad \blacksquare$$

For a finite set X, write $e_x \colon X \to \mathbb{C}$ for the function given by

$$e_x(x') = \begin{cases} 1, & x' = x, \\ 0, & x' \neq x, \end{cases} \quad \text{for } x' \in X,$$

Write $L^2(X)$ for the finite-dimensional complex Hilbert space of functions $X \to \mathbb{C}$, endowed with the unique Hermitian product such that $\{e_x\}_{x \in X}$ is an orthonormal basis. Note that for $x, y \in X$,

$$\|e_x - e_y\|^2 = 2 \cdot \mathbf{1}_{x \neq y} = \begin{cases} 0, & x = y, \\ 2, & x \neq y. \end{cases}$$

For a group action $f \colon \Gamma \to \mathrm{Sym}(X)$, write $\rho_f \colon \Gamma \to \mathrm{U}(L^2(X))$ for the unitary representation given by $\rho_f(\gamma) e_x = e_{f(\gamma)x}$ for $x \in X$.

For finite sets X and Y, $x \in X$ and $y \in Y$, write $E_{x,y} \colon L^2(X) \to L^2(Y)$ for the linear map such that

$$E_{x,y}(e_{x'}) = \begin{cases} e_y, & x = x', \\ 0, & x \neq x', \end{cases} \quad \text{for all } x' \in X.$$

We make the space $\mathrm{Lin}_{X,Y} := \mathrm{Hom}_{\mathbb{C}}(L^2(X), L^2(Y))$ of linear maps into a complex Hilbert space by endowing it with the unique Hermitian product such that $\{E_{x,y} \mid x \in X, y \in Y\}$ is an orthonormal basis. A pair of group actions $h \colon \Gamma \to \mathrm{Sym}(X)$ and $f \colon \Gamma \to \mathrm{Sym}(Y)$ gives rise to a unitary representation $\rho_{h,f} \colon \Gamma \to \mathrm{U}(\mathrm{Lin}_{X,Y})$ defined by

$$\rho_{h,f}(\gamma) T = \rho_f(\gamma) \circ T \circ \rho_h(\gamma^{-1}) \quad \forall \gamma \in \Gamma \; \forall T \in \mathrm{Lin}_{X,Y}.$$

For $n \in \mathbb{N}$, write $T_{n-1} \colon L^2([n-1]) \to L^2([n])$ for the linear extension of the inclusion map $[n-1] \hookrightarrow [n]$. Then $\|T_{n-1}\|^2 = n - 1$. For $x \in [n-1]$, the notation e_x can be used both for a function $[n-1] \to \mathbb{C}$ and for a function $[n] \to \mathbb{C}$. The domain should be understood from the context. In particular, we write $T_{n-1}(e_x) = e_x$.

The proof of Theorem 4.3 makes use of the following lemma.

Lemma 4.2 ([2, Proposition 2.4 (ii)]). *Let Γ be a group. Take $n \geq 2$ and group homomorphisms $h \colon \Gamma \to \mathrm{Sym}(n-1)$ and $f \colon \Gamma \to \mathrm{Sym}(n)$ such that f defines a transitive action of Γ on $[n]$. Endow $L^2([n-1])$ and $L^2([n])$, respectively, with the representations ρ_h and ρ_f. Then $\|T_{n-1} - T'\| \geq \frac{1}{\sqrt{2}} \|T_{n-1}\|$ for every morphism of representations $T' \colon L^2([n-1]) \to L^2([n])$.*

Theorem 4.3. *Let Γ be a group and $f \colon \Gamma \to \mathrm{Sym}(n)$, $n \geq 2$, a homomorphism that defines a transitive action $\Gamma \curvearrowright [n]$. Let $h \colon \Gamma \to \mathrm{Sym}(n-1)$ be an arbitrary homomorphism. Then*

$$\mathrm{def}_\infty(\hat{f}) \leq \frac{2}{n-1} \quad \text{and} \quad d_\infty(h, \hat{f}) \geq \frac{1}{2} - \frac{1}{n-1}.$$

Furthermore, if Γ is a discrete amenable group equipped with a right-invariant measure m, then

$$\mathrm{def}_1(\hat{f}) \leq \frac{2}{n-1} \quad \text{and} \quad d_1(h, \hat{f}) \geq \frac{1}{4} - \frac{1}{n-1}.$$

Proof. By (4.1), $\mathrm{def}_\infty(\hat{f}) \leq \frac{2}{n-1}$. Now, we have an action $h \colon \Gamma \to \mathrm{Sym}([n-1])$ and a transitive action $f \colon \Gamma \to \mathrm{Sym}([n])$. Consider the representation $\rho := \rho_{h,f} \colon \Gamma \to \mathrm{U}(\mathrm{Lin}_{[n-1],[n]})$.

For every $\gamma \in \Gamma$,

$$
\begin{aligned}
\|\rho(\gamma^{-1})T_{n-1} - T_{n-1}\|^2 &= \sum_{x \in [n-1]} \|(\rho_f(\gamma^{-1}) \circ T_{n-1} \circ \rho_h(\gamma) - T_{n-1})e_x\|^2 \\
&= \sum_{x \in [n-1]} \|(T_{n-1} \circ \rho_h(\gamma) - \rho_f(\gamma) \circ T_{n-1})e_x\|^2 \quad \rho_f(\gamma) \text{ preserves norm} \\
&= \sum_{x \in [n-1]} \|e_{h(\gamma)(x)} - e_{f(\gamma)(x)}\|^2 \\
&= \sum_{x \in [n-1]} 2 \cdot \mathbf{1}_{h(\gamma)(x) \neq f(\gamma)(x)} \\
&\leq 2 \sum_{x \in [n-1]} (\mathbf{1}_{h(\gamma)(x) \neq \hat{f}(\gamma)(x)} + \mathbf{1}_{\hat{f}(\gamma)(x) \neq f(\gamma)(x)}) \quad \text{triangle inequality} \\
&\leq 2\Big(\sum_{x \in [n-1]} \mathbf{1}_{h(\gamma)(x) \neq \hat{f}(\gamma)(x)} + 1 \Big) \\
&= 2\Big(d^H(h(\gamma), \hat{f}(\gamma)) + \frac{1}{n-1} \Big) \|T_{n-1}\|^2. \tag{4.2}
\end{aligned}
$$

Hence, for $\varepsilon = (2(d_\infty(h, \hat{f}) + \frac{1}{n-1}))^{1/2}$ we have

$$
\|\rho(\gamma)T_{n-1} - T_{n-1}\| \leq \varepsilon \|T_{n-1}\| \quad \forall \gamma \in \Gamma.
$$

By Lemma 4.1 (i), there is $T' \in \mathrm{Lin}_{[n-1],[n]}$ such that $\|T' - T_{n-1}\| \leq \frac{\varepsilon}{\sqrt{2}}\|T_{n-1}\|$ and $\rho(\gamma)T' = T'$ for every $\gamma \in \Gamma$. The latter condition means that $T': L^2([n-1]) \to L^2([n])$ is a morphism of representations. Therefore, $\varepsilon \geq 1$ by Lemma 4.2. Hence $d_\infty(h, \hat{f}) \geq \frac{1}{2} - \frac{1}{n-1}$.

Now, assume that m is a right-invariant measure on Γ. Then (4.1) implies that $\mathrm{def}_1(\hat{f}) \leq \frac{2}{n-1}$. By integrating (4.2), we see that for $\alpha = \sqrt{2}(d_1(h, \hat{f}) + \frac{1}{n-1})^{1/2}$,

$$
\int \|\rho(\gamma^{-1})T_{n-1} - T_{n-1}\|^2 \, dm(\gamma) \leq 2\|T_{n-1}\|^2 \int \Big(d^H(h(\gamma), \hat{f}(\gamma)) + \frac{1}{n-1} \Big) \, dm(\gamma)
$$
$$
= \alpha^2 \|T_{n-1}\|^2.
$$

By Lemma 4.1 (ii), there is $T'' \in W$ such that $\|T'' - T_{n-1}\| \leq \alpha\|T_{n-1}\|$ and T'' is a morphism of representations. Using Lemma 4.2 as before, we see that $\alpha \geq \frac{1}{\sqrt{2}}$ and thus $d_1(h, \hat{f}) \geq \frac{1}{4} - \frac{1}{n-1}$. $\blacksquare$

Remark 4.4. Let Γ be a group with finite quotients of unbounded cardinality. Theorem 4.3 implies that Γ is not uniformly strictly stable. By [2, Theorem 1.4], if we assume further that Γ has property (τ), then Γ is not pointwise strictly stable. Both the uniform and the pointwise versions are proved by considering $\hat{f}: \Gamma \to \mathrm{Sym}(n-1)$, where $f: \Gamma \to \mathrm{Sym}(n)$ is a transitive action.

5. Counterexamples for flexible stability: free groups

This section is devoted to the proof of Theorem 1.4. The proof involves a construction that combines exponent reduction on words in a free group, inspired by [33], with a pinched grid construction, inspired by [4, Section 5].

Let F_2 be the free group on $\{x_1, x_2\}$. Let k be a positive integer. For an integer $t \geq 0$, let $\bar{t}$ be the unique element of $C_k = \{0, \ldots, k-1\}$ that is congruent to t modulo k. For $t < 0$ we let $\bar{t} = -\overline{(-t)}$. For example, if $k = 5$ then $\bar{7} = 2$ and $\overline{-12} = -2$. Define $\alpha'_1, \alpha_2 \in \mathrm{Sym}(C_k \times C_k)$ as follows:

$$\alpha'_1((i, j)) = (\overline{i+1}, j) \quad \text{and} \quad \alpha_2((i, j)) = (i, \overline{j+1}) \quad \forall (i, j) \in C_k \times C_k$$

and let $\alpha_1 = \alpha'_1 \circ \tau$, where $\tau \in \mathrm{Sym}(C_k \times C_k)$ is the transposition that swaps $(0, 0)$ and $(0, 1)$. The actions of α_1 and α_2 on $C_k \times C_k$ are depicted in Figure 5.1.
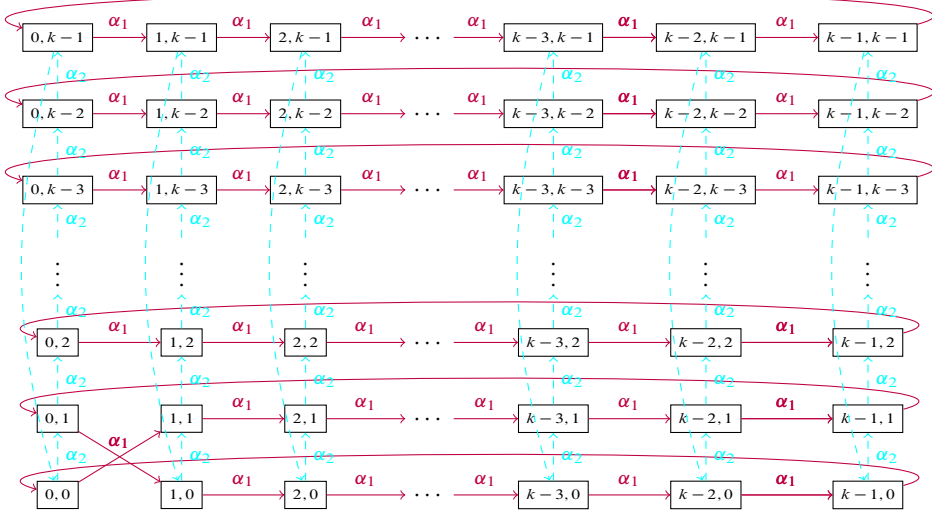

Fig. 5.1. The actions of α_1 (solid) and α_2 (dashed) on $C_k \times C_k$.

Define $g_k : F_2 \to \mathrm{Sym}(C_k \times C_k)$ as follows. Let $w \in F_2$ be a reduced word. Write $w = x_1^{d_1} x_2^{e_1} \cdots x_1^{d_r} x_2^{e_r} \in F_2, r \geq 0, d_i, e_i \in \mathbb{Z}$, where $d_i \neq 0$ for $i > 1$ and $e_i \neq 0$ for $i < r$. Define $g_k(w) = \alpha_1^{\overline{d_1}} \alpha_2^{\overline{e_1}} \cdots \alpha_1^{\overline{d_r}} \alpha_2^{\overline{e_r}}$. For example, if $k = 5$ then $g_k(x_1^{13} x_2^{-9} x_1^3 x_2 x_1^{-77}) = \alpha_1^3 \alpha_2^{-4} \alpha_1^3 \alpha_2 \alpha_1^{-2}$.

The following lemma shows that g_k has small local defect, but grossly violates an identity that holds in $\mathrm{Sym}(N)$ for every $N \geq k^2$. The lemma readily implies Theorem 1.4 (see below). We shall write $\ell(w)$ for the length of a reduced word w.

Lemma 5.1. *Let $k \geq 1$. Then* $\mathrm{def}_\infty(g_k) \leq 2/k$, *but*

$$d^H\big(g_k\big((x_1^{N!-k+1} x_2)^k (x_1^{-k+1} x_2)^{-k}\big), \mathrm{id}\big) \geq 1 - 5/k$$
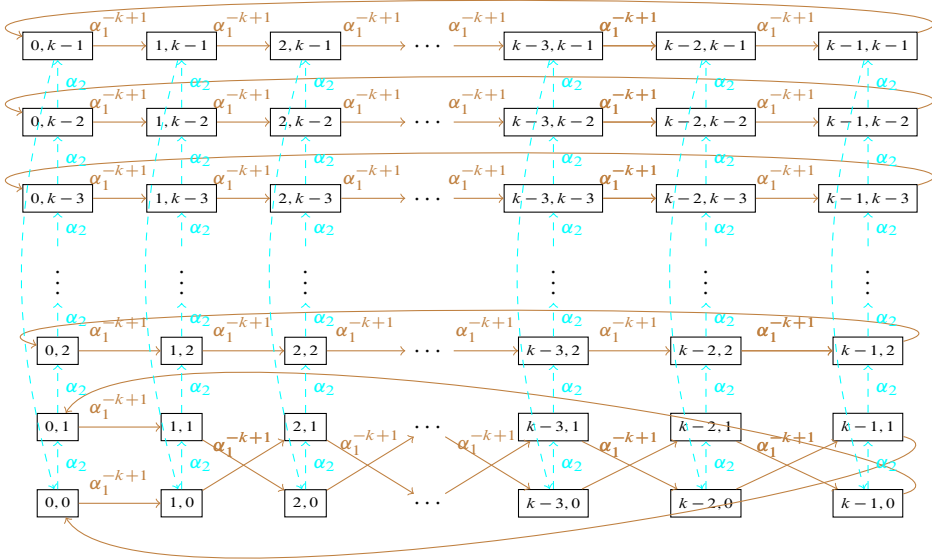
for every $N \geq k^2$.

Fig. 5.2. The actions of α_1^{-k+1} (solid) and α_2 (dashed) on $C_k \times C_k$.

Proof. Write $g = g_k$. Let $w_1, w_2 \in F_2$ be reduced words and write

$$\beta_{w_1,w_2} = g(w_1)^{-1} g(w_1 w_2) g(w_2)^{-1}.$$

We prove that $d^H(\beta_{w_1,w_2}, \mathrm{id}) \leq 2/k$ by induction on $\ell(w_1) + \ell(w_2)$. If either $w_1 = \mathrm{id}$ or $w_2 = \mathrm{id}$ then $\beta_{w_1,w_2} = \mathrm{id}$ and we are done. Assume that $w_1 \neq \mathrm{id}$ and $w_2 \neq \mathrm{id}$. If the last letter of w_1 and the first letter of w_2 are neither the same nor mutual inverses, then $g(w_1 w_2) = g(w_1)g(w_2)$, and thus $\beta_{w_1,w_2} = \mathrm{id}$ and we are done. Otherwise, we have cancellation-free concatenations $w_1 = \tilde{w}_1 \cdot x^{e_1}$ and $w_2 = x^{e_2} \cdot \tilde{w}_2$, where $\tilde{w}_1$ and $\tilde{w}_2$ are reduced words, $x \in \{x_1, x_2\}$ and $e_1, e_2 \in \mathbb{Z} \setminus \{0\}$. Write $\alpha = g(x) \in \{\alpha_1, \alpha_2\}$. If $e_2 \neq -e_1$ then $g(w_1 w_2) = g(\tilde{w}_1)\alpha^{e_1+e_2} g(\tilde{w}_2)$, and thus $\beta_{w_1,w_2} = \alpha^t$ for $t = \overline{e_1 + e_2} - (\overline{e_1} + \overline{e_2})$. Then $|t| \in \{0, k\}$. Since α_2^k is id and α_1^k fixes all elements outside $C_k \times \{0, 1\}$, we see that $d^H(\beta_{w_1,w_2}, \mathrm{id}) \leq 2/k$ as required. Finally, if $e_2 = -e_1$, then

$$\beta_{w_1,w_2} = \left(g(\tilde{w}_1)\alpha^{\overline{e_1}}\right)^{-1} g(\tilde{w}_1 \alpha^{\overline{e_1}} \cdot \alpha^{\overline{-e_1}} \tilde{w}_2)(\alpha^{\overline{-e_1}} g(\tilde{w}_2))^{-1} = \alpha^{-\overline{e_1}} \beta_{\tilde{w}_1, \tilde{w}_2} \alpha^{\overline{e_1}},$$

and thus $d^H(\beta_{w_1,w_2}, \mathrm{id}) = d^H(\beta_{\tilde{w}_1, \tilde{w}_2}, \mathrm{id}) \leq 2/k$ by the induction hypothesis. The upshot is that $\mathrm{def}_\infty(g) \leq 2/k$ as claimed.

Let $N \geq k^2$. Then $\overline{N! - k + 1} = 1$ and $\overline{-k + 1} = -k + 1$, and thus $g((x_1^{N!-k+1} x_2)^k) = (\alpha_1 \alpha_2)^k$ and $g((x_1^{-k+1} x_2)^k) = (\alpha_1^{-k+1} \alpha_2)^k$. For $i \in \mathbb{N}$, write

$$D_i = \{(x, y) \in C_k \times C_k \mid x - y \equiv i \pmod{k}\}.$$

By examining Figure 5.1, we see that $(\alpha_1 \alpha_2)^k$ fixes every element of $C_k \times C_k$ outside the diagonals D_0 and D_1, and thus $d^H((\alpha_1 \alpha_2)^k, \mathrm{id}) \leq 2/k$. By examining Figure 5.2, we see that $(\alpha_1^{-k+1} \alpha_2)^k$ does not fix any element outside the diagonal D_0, and hence

$d^H((\alpha_1^{-k+1}\alpha_2)^k, \mathrm{id}) \geq 1 - 1/k$. Therefore, by the triangle inequality,

$$d^H\big(g((x_1^{N!-k+1}x_2)^k), g((x_1^{-k+1}x_2)^k)\big) \geq 1 - 3/k.$$

Thus, since d^H is bi-invariant and $\mathrm{def}_\infty(g) \leq 2/k$, we have

$$d^H\big(g((x_1^{N!-k+1}x_2)^k(x_1^{-k+1}x_2)^{-k}), \mathrm{id}\big) \geq 1 - 5/k. \qquad \blacksquare$$

For $k \geq 1$, fix an arbitrary bijection between $[k^2]$ and $C_k \times C_k$. Henceforth, we use this bijection to view g_k as a function from Γ to $\mathrm{Sym}(k^2)$.

Proof of Theorem 1.4. We are given a group Γ and a surjective homomorphism $\pi: \Gamma \to F_2$. Define $f_k = g_k \circ \pi$. Then $\mathrm{def}_\infty(f_k) = \mathrm{def}_\infty(g_k) \leq 2/k$. Take $\gamma_1, \gamma_2 \in \Gamma$ such that $\pi(\gamma_1) = x_1$ and $\pi(\gamma_2) = x_2$. Let $N \geq k^2$, take a homomorphism $h_k: \Gamma \to \mathrm{Sym}(N)$, and write $\gamma_0 = (\gamma_1^{N!-k+1}\gamma_2)^k(\gamma_1^{-k+1}\gamma_2)^{-k} \in \Gamma$. Then $h_k(\gamma_0) = \mathrm{id}_N \in \mathrm{Sym}(N)$, while Lemma 5.1 says that $d^H(f_k(\gamma_0), \mathrm{id}_{k^2}) \geq 1 - 5/k$. Hence, $d_\infty(f_k, h_k) \geq d^H(f_k(\gamma_0), h_k(\gamma_0)) \geq 1 - 5/k.$ $\blacksquare$

Appendix A. The triangle inequality for d^H

It is clear that d^H is symmetric and that $d^H(\sigma_1, \sigma_2) = 0$ if and only if $\sigma_1 = \sigma_2$. In fact, d^H is a metric by the following lemma.

Lemma A.1. *For $i \in [3]$, let $n_i \geq 1$ and $\sigma_i \in \mathrm{Sym}(n_i)$. Then*

$$d^H(\sigma_1, \sigma_2) + d^H(\sigma_2, \sigma_3) \geq d^H(\sigma_1, \sigma_3).$$

Proof. For $i, j \in [3]$ write $n_{ij} = \max\{n_i, n_j\}$ and $n = \max\{n_1, n_2, n_3\}$. For a permutation $\sigma \in \mathrm{Sym}(N)$ and $x > N$ write $\sigma(x) = \star$, where $\star$ is a dummy object that is not a natural number. For $k \geq 1$ and permutations τ_1 and τ_2 (of possibly different sizes), write $d_k(\tau_1, \tau_2) = |\{x \in [k] \mid \tau_1(x) \neq \tau_2(x)\}|$. Then, for $k \geq n_{ij}$ we have $d^H(\sigma_i, \sigma_j) = \frac{1}{n_{ij}} d_k(\sigma_i, \sigma_j)$. Clearly, d_k satisfies the triangle inequality for each fixed k.

Using these notations, we have

$$d^H(\sigma_1, \sigma_2) + d^H(\sigma_2, \sigma_3) \geq \frac{1}{n}(d_n(\sigma_1, \sigma_2) + d_n(\sigma_2, \sigma_3)) \geq \frac{1}{n} d_{n_{13}}(\sigma_1, \sigma_3).$$

If $n_2 \leq n_{13}$ then $n = n_{13}$, and thus we are done by the above. On the other hand, if $n_2 > n_{13}$ then

$$
\begin{aligned}
d^H(\sigma_1, \sigma_2) + d^H(\sigma_2, \sigma_3) &= \frac{1}{n_2}(d_{n_2}(\sigma_1, \sigma_2) + d_{n_2}(\sigma_2, \sigma_3)) \\
&= \frac{1}{n_2}\big(d_{n_{13}}(\sigma_1, \sigma_2) + d_{n_{13}}(\sigma_2, \sigma_3) + 2(n_2 - n_{13})\big) \\
&\geq \frac{n_{13}}{n_2 n_{13}}\big(d_{n_{13}}(\sigma_1, \sigma_3) + (n_2 - n_{13})\big) \\
&\geq \frac{1}{n_2 n_{13}}\big(n_{13}d_{n_{13}}(\sigma_1, \sigma_3) + (n_2 - n_{13})d_{n_{13}}(\sigma_1, \sigma_3)\big) \\
&= \frac{1}{n_{13}} d_{n_{13}}(\sigma_1, \sigma_3) = d^H(\sigma_1, \sigma_3). \qquad \blacksquare
\end{aligned}
$$

Appendix B. Symmetrization

Let G be a group equipped with a bi-invariant metric $d : G \times G \to \mathbb{R}_{\geq 0}$. Bi-invariance means that $d(agb, ahb) = d(g, h)$ for all $g, h, a, b \in G$. For example, one can take G to be $\mathrm{Sym}(n)$ and d to be the (normalized) Hamming metric d^H on $\mathrm{Sym}(n)$. This section deals with deforming a function $f : \Gamma \to G$ into a symmetric function $f' : \Gamma \to G$ (see Definition 2.1). This is achieved by Proposition B.3 and is used in Section 2 via Proposition 2.2.

Throughout the section, the notation $(\triangle)$ indicates the use of the triangle inequality. We use the bi-invariance of the metric $d(\cdot, \cdot)$ freely, without further explanation.

Let Γ be a group and take functions $f, h : \Gamma \to G$. The *uniform local defect* of f is

$$\mathrm{def}_\infty(f) = \sup \{ d(f(\gamma_1 \gamma_2), f(\gamma_1) f(\gamma_2)) \mid \gamma_1, \gamma_2 \in \Gamma \}$$

and the *uniform distance* between f and h is

$$d_\infty(f, h) = \sup \{ d(f(\gamma), h(\gamma)) \mid \gamma \in \Gamma \}.$$

If Γ is a discrete amenable group equipped with a left- or right-invariant measure m, then the *mean local defect* of f is

$$\mathrm{def}_1(f) = \int\int d(f(\gamma_1 \gamma_2), f(\gamma_1) f(\gamma_2)) \, dm(\gamma_1) \, dm(\gamma_2)$$

and the *mean distance* between f and h is

$$d_1(f, h) = \int d(f(\gamma), h(\gamma)) \, dm(\gamma).$$

We begin with the following lemma, which says that if two functions are close together then their local defects are nearly the same.

Lemma B.1. *Let Γ be a group and let $f, f' : \Gamma \to G$ be functions. Then*

$$\mathrm{def}_\infty(f') \leq 3 d_\infty(f, f') + \mathrm{def}_\infty(f).$$

Furthermore, if Γ is a discrete amenable group equipped with a right-invariant measure m, then $\mathrm{def}_1(f') \leq 3 d_1(f, f') + \mathrm{def}_1(f)$.

Proof. For $\gamma_1, \gamma_2 \in \Gamma$, we have

$$
\begin{aligned}
d(f'(&\gamma_1 \gamma_2), f'(\gamma_1) f'(\gamma_2)) \\
&\leq d(f'(\gamma_1 \gamma_2), f(\gamma_1 \gamma_2)) + d(f(\gamma_1 \gamma_2), f(\gamma_1) f(\gamma_2)) && \text{by } (\triangle) \\
&\quad + \underbrace{d(f(\gamma_1) f(\gamma_2), f(\gamma_1) f'(\gamma_2))}_{= d(f(\gamma_2), f'(\gamma_2))} + \underbrace{d(f(\gamma_1) f'(\gamma_2), f'(\gamma_1) f'(\gamma_2))}_{= d(f(\gamma_1), f'(\gamma_1))}.
\end{aligned}
$$

Hence,

$$\sup \{ d(f'(\gamma_1 \gamma_2), f'(\gamma_1) f'(\gamma_2)) \mid \gamma_1, \gamma_2 \in \Gamma \} \leq 3 d_\infty(f, f') + \mathrm{def}_\infty(f)$$

and if m is a right-invariant measure on Γ, then

$$\iint d(f'(\gamma_1\gamma_2), f'(\gamma_1)f'(\gamma_2))\, dm(\gamma_1)\, dm(\gamma_2) \le 3d_1(f, f') + \mathrm{def}_1(f). \qquad \blacksquare$$

We turn to the task of symmetrizing a given function $f\colon \Gamma \to G$. Note that

$$d(f(1_\Gamma), 1_G) = d\big(f(1_\Gamma)f(1_\Gamma), f(1_\Gamma)\big) \le \mathrm{def}_\infty(f) \tag{B.1}$$

and thus for $\gamma \in \Gamma$,

$$\begin{aligned}
d\big(f(\gamma), (f(\gamma^{-1}))^{-1}\big) &= d(f(\gamma)f(\gamma^{-1}), 1_G) \\
&\le d(f(\gamma)f(\gamma^{-1}), f(1_\Gamma)) + d(f(1_\Gamma), 1_G) \quad \text{by } (\triangle) \\
&\le 2\,\mathrm{def}_\infty(f). \tag{B.2}
\end{aligned}$$

The following is a natural attempt to produce a symmetric function $f'\colon \Gamma \to G$ close to f.

- Let $\Gamma_2 = \{\gamma \in \Gamma \mid \gamma^2 = 1\}$ and fix a set B containing exactly one of γ and γ^{-1} for each $\gamma \in \Gamma \setminus \Gamma_2$.
- Set $f'(1_\Gamma) = 1_G$.
- For $\gamma \in B$, set $f'(\gamma) = f(\gamma)$ and $f'(\gamma^{-1}) = (f(\gamma))^{-1}$.
- For $\gamma \in \Gamma_2 \setminus \{1\}$, let $f'(\gamma)$ be an order-2 element of G that is close to $f(\gamma)$.

The function $f'\colon \Gamma \to G$ is symmetric by construction, and we would like to bound $d_\infty(f, f')$ and $d_1(f, f')$. We shall see that we can obtain good bounds if we can perform the last step efficiently, that is, if approximate square roots in G are close to square roots. First, we investigate approximate square roots in the case $G = \mathrm{Sym}(n)$.

Lemma B.2. *Let $\sigma \in \mathrm{Sym}(n)$. Then there is $\tau \in \mathrm{Sym}(n)$ such that $\tau^2 = \mathrm{id}$ and*

$$d^H(\sigma, \tau) = d^H(\sigma^2, \mathrm{id}).$$

Proof. Let $A = \{x \in [n] \mid \sigma^2(x) = x\}$. Note that $\sigma|_A$ is an involution $A \to A$. Define

$$\tau(x) = \begin{cases} \sigma(x), & x \in A, \\ x, & x \notin A. \end{cases}$$

Then $\tau^2 = \mathrm{id}$ and $d^H(\sigma, \tau) = 1 - |A|/n = d^H(\sigma^2, \mathrm{id})$. $\qquad \blacksquare$

In other words, the lemma says that the cyclic group C_2 of order 2 is stable with respect to $\mathrm{Sym}(n)$, and bounds the stability rate.

Assume that there is a real number M_2 such that for every $g \in G$ there is $h \in G$ satisfying $h^2 = 1_G$ and $d(g, h) \le M_2 d(g^2, 1_G)$. In the context of symmetrization of a function $\Gamma \to G$, we would like M_2 to be small. Lemma B.2 implies that in the case $G = \mathrm{Sym}(n)$, we may take $M_2 = 1$.

Proposition B.3. *With M_2 as above, write $C = 2 \max \{1, M_2\}$. Let Γ be a group and $f \colon \Gamma \to G$ a function. Then there is a symmetric function $f' \colon \Gamma \to G$ such that $d_\infty(f, f') \leq C\delta_\infty$ and $\mathrm{def}_\infty(f') \leq (3C + 1)\delta_\infty$, where $\delta_\infty = \mathrm{def}_\infty(f)$.*

Furthermore, if Γ is a discrete amenable group equipped with a right-invariant inverse-invariant measure m, then for the same function f' we have $d_1(f, f') \leq (C + 1)\delta_1$ and $\mathrm{def}_1(f, f') \leq (3C + 4)\delta_1$, where $\delta_1 = \mathrm{def}_1(f)$.

Proof. By Lemma B.1, the bounds on $\mathrm{def}_\infty(f')$ and $\mathrm{def}_1(f')$ follow from the bounds on $d_\infty(f, f')$ and $d_1(f, f')$, respectively. We turn to the proof of the latter.

Let Γ_2 and B be as in the discussion preceding Lemma B.2. For each $\gamma \in \Gamma_2 \setminus \{1_\Gamma\}$, take $\tau(\gamma) \in G$ such that

$$d(f(\gamma), \tau(\gamma)) \leq M_2 d((f(\gamma))^2, 1_G). \tag{B.3}$$

Define a function $f' \colon \Gamma \to G$ as follows:

$$f'(\gamma) = \begin{cases} 1_G, & \gamma = 1_\Gamma, \\ f(\gamma), & \gamma \in B, \\ (f(\gamma^{-1}))^{-1}, & \gamma \in \Gamma \setminus (\Gamma_2 \cup B), \\ \tau(\gamma), & \gamma \in \Gamma_2 \setminus \{1_\Gamma\}. \end{cases}$$

Then f' is symmetric by construction. By (B.1), we have

$$d(f(1_\Gamma), f'(1_\Gamma)) = d(f(1_\Gamma), 1_G) \leq \delta_\infty. \tag{B.4}$$

By (B.2), for $\gamma \in \Gamma \setminus (\Gamma_2 \cup B)$ we have

$$d(f(\gamma), f'(\gamma)) = d\left(f(\gamma), (f(\gamma^{-1}))^{-1}\right) \leq 2\delta_\infty.$$

Finally, for $\gamma \in \Gamma_2 \setminus \{1_\Gamma\}$ we have

$$\begin{aligned} d(f(\gamma), f'(\gamma)) &= d(f(\gamma), \tau(\gamma)) \\ &\leq M_2 d((f(\gamma))^2, 1_G) && \text{by (B.3)} \\ &\leq M_2 \cdot \left(d\left((f(\gamma))^2, f(\gamma^2)\right) + d(f(1_\Gamma), 1_G)\right) && \text{by ($\triangle$) and } \gamma^2 = 1_\Gamma \\ &\leq 2M_2\delta_\infty. && \text{by (B.4)} \end{aligned}$$

This finishes the proof that $d_\infty(f', f) \leq C\delta_\infty$.

Now, assume that m is a right-invariant inverse-invariant measure on Γ. First, we show that f approximately respects inverses on average:

$$\begin{aligned} \int d(f(\gamma)f(\gamma^{-1}), 1_G) \, dm(\gamma) &= \iint d(f(\gamma)f(\gamma^{-1}), 1_G) \, dm(\gamma') \, dm(\gamma) \\ &= \iint d\left(f(\gamma')f(\gamma)f(\gamma^{-1}), f(\gamma')\right) dm(\gamma') \, dm(\gamma) \\ &\leq \iint d\left(f(\gamma')f(\gamma)f(\gamma^{-1}), f(\gamma'\gamma)f(\gamma^{-1})\right) dm(\gamma') \, dm(\gamma) \quad \text{by ($\triangle$)} \\ &\quad + \iint d\left(f(\gamma'\gamma)f(\gamma^{-1}), f(\gamma')\right) dm(\gamma') \, dm(\gamma) \\ &= 2\delta_1, \tag{B.5} \end{aligned}$$

where for the last equality, we manipulate the first term by cancelling out $f(\gamma^{-1})$, and in the second term we apply $\gamma \mapsto \gamma^{-1}$ and then $\gamma' \mapsto \gamma'\gamma$. Furthermore, f approximately respects the identity elements in the following sense (which is significant only when Γ is finite):

$$
\begin{aligned}
m(\{1_\Gamma\})d(f(1_\Gamma),1_G) &= \int_{\{1_\Gamma\}}\int d(f(1_\Gamma),1_G)\,dm(\gamma')\,dm(\gamma) \\
&= \int_{\{1_\Gamma\}}\int d(f(\gamma')f(1_\Gamma),f(\gamma'\cdot 1_\Gamma))\,dm(\gamma')\,dm(\gamma) \\
&\leq \iint d(f(\gamma')f(\gamma),f(\gamma'\gamma))\,dm(\gamma')\,dm(\gamma) = \delta_1. \quad \text{(B.6)}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\int_{\Gamma_2\setminus\{1_\Gamma\}} &d(f(\gamma),\tau(\gamma))\,dm(\gamma) \\
&\leq M_2\int_{\Gamma_2\setminus\{1_\Gamma\}} d((f(\gamma))^2,\mathrm{id})\,dm(\gamma) \qquad \text{by (B.3)} \\
&= M_2\int_{\Gamma_2\setminus\{1_\Gamma\}} d(f(\gamma)f(\gamma^{-1}),1_G)\,dm(\gamma). \quad \gamma=\gamma^{-1} \text{ for } \gamma\in\Gamma_2 \quad \text{(B.7)}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
d_1(f,f') &= \int d(f(\gamma),f'(\gamma))\,dm(\gamma) \\
&= m(\{1_\Gamma\})d(f(1_\Gamma),1_G) + \int_{\Gamma\setminus(\Gamma_2\cup B)} d(f(\gamma)f(\gamma^{-1}),1_G)\,dm(\gamma) \\
&\quad + \int_{\Gamma_2\setminus\{1_\Gamma\}} d(f(\gamma),\tau(\gamma))\,dm(\gamma) \\
&\leq \delta_1 + \max\{1,M_2\}\int d(f(\gamma)f(\gamma^{-1}),1_G)\,dm(\gamma) \quad \text{by (B.6) and (B.7)} \\
&\leq (C+1)\delta_1. \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (B.5)}
\end{aligned}
$$

■

References

[1] Arzhantseva, G., Păunescu, L.: Almost commuting permutations are near commuting permutations. J. Funct. Anal. **269**, 745–757 (2015) Zbl 1368.20025 MR 3350728

[2] Becker, O., Lubotzky, A.: Group stability and Property (T). J. Funct. Anal. **278**, art. 108298, 20 pp. (2020) Zbl 07127791 MR 4027744

[3] Becker, O., Lubotzky, A., Thom, A.: Stability and invariant random subgroups. Duke Math. J. **168**, 2207–2234 (2019) Zbl 07145001 MR 3999445

[4] Becker, O., Mosheiff, J.: Abelian groups are polynomially stable. Int. Math. Res. Notices **2021**, 15574–15632 Zbl 07456010 MR 4329877

[5] Bekka, B., de la Harpe, P., Valette, A.: Kazhdan's Property (T). New Math. Monogr. 11, Cambridge Univ. Press, Cambridge (2008) Zbl 1146.22009 MR 2415834

[6] Ben-or, M., Coppersmith, D., Luby, M., Rubinfeld, R.: Non-abelian homomorphism testing, and distributions close to their self-convolutions. Random Structures Algorithms **32**, 49–70 (2008) Zbl 1135.68065 MR 2371051

[7] Blum, M., Luby, M., Rubinfeld, R.: Self-testing/correcting with applications to numerical problems. J. Computer Systems Sci. **47**, 549–595 (1993) Zbl 0795.68131 MR 1248868

[8] Bowen, L., Burton, P.: Flexible stability and nonsoficity. Trans. Amer. Math. Soc. **373**, 4469–4481 (2020) Zbl 1476.20046 MR 4105530

[9] Brooks, R.: Some remarks on bounded cohomology. In: Riemann Surfaces and Related Topics: (Stony Brook, NY, 1978), Ann. of Math. Stud. 97, Princeton Univ. Press, Princeton, NJ, 53–63 (1981) Zbl 0457.5502 MR 624804

[10] Burger, M., Ozawa, N., Thom, A.: On Ulam stability. Israel J. Math. **193**, 109–129 (2013) Zbl 1271.22003 MR 3038548

[11] De Chiffre, M., Glebsky, L., Lubotzky, A., Thom, A.: Stability, cohomology vanishing, and nonapproximable groups. Forum Math. Sigma **8**, art. e18, 37 pp. (2020) Zbl 1456.22002 MR 4080477

[12] De Chiffre, M., Ozawa, N., Thom, A.: Operator algebraic approach to inverse and stability theorems for amenable groups. Mathematika **65**, 98–118 (2019) Zbl 1416.22008 MR 3867328

[13] Druțu, C., Kapovich, M.: Geometric Group Theory. Amer. Math. Soc. Colloq. Publ. 63, Amer. Math. Soc., Providence, RI (2018) Zbl 1447.20001 MR 3753580

[14] Dunford, N., Schwartz, J. T.: Linear Operators. Part II. Wiley Classics Library, Wiley, New York (1988) Zbl 0635.47002 MR 1009163

[15] Eilers, S., Shulman, T., Sørensen, A. P. W.: C*-stability of discrete groups. Adv. Math. **373**, art. 107324, 41 pp. (2020) Zbl 1466.46044 MR 4131398

[16] Epstein, D. B. A., Fujiwara, K.: The second bounded cohomology of word-hyperbolic groups. Topology **36**, 1275–1289 (1997) Zbl 0884.55005 MR 1452851

[17] Glebsky, L., Rivera, L. M.: Almost solutions of equations in permutations. Taiwanese J. Math. **13**, 493–500 (2009) Zbl 1203.20003 MR 2500002

[18] Gowers, W. T., Hatami, O.: Inverse and stability theorems for approximate representations of finite groups. Mat. Sb. **208**, 70–106 (2017) (in Russian) Zbl 1427.20022 MR 3733361

[19] Goldreich, O.: Introduction to Property Testing. Cambridge Univ. Press, Cambridge (2017) Zbl 06797790 MR 3837126

[20] Hadwin, D., Shulman, T.: Stability of group relations under small Hilbert–Schmidt perturbations. J. Funct. Anal. **275**, 761–792 (2018) Zbl 1411.46046 MR 3807776

[21] Hyers, D. H.: On the stability of the linear functional equation. Proc. Nat. Acad. Sci. U.S.A. **27**, 222–224 (1941) Zbl 67.0424.01 MR 4076

[22] Ioana, A.: Stability for product groups and property (τ). J. Funct. Anal. **279**, art. 108729, 32 pp. (2020) Zbl 1485.20004 MR 4134896

[23] Ji, Z., Natarajan, A., Vidick, T., Wright, J., Yuen, H.: MIP* = RE. arXiv:2001.04383 (2020)

[24] Kazhdan, D.: On ε-representations. Israel J. Math. **43**, 315–323 (1982) Zbl 0518.22008 MR 693352

[25] Kun, G., Thom, A.: Inapproximability of actions and Kazhdan's property (T). arXiv:1901.03963 (2019)

[26] Lazarovich, N., Levit, A., Minsky, Y.: Surface groups are flexibly stable. arXiv:1901.07182 (2019)

[27] Levit, A., Lubotzky, A.: Infinitely presented stable groups and invariant random subgroups of metabelian groups, Ergodic Theory Dynam. Systems **42**, 2028–2063 (2022) Zbl 07543353 MR 4417344

[28] Levit, A., Lubotzky, A.: Uncountably many permutation stable groups. arXiv:1910.11722 (2019)

[29] Lubotzky, A., Oppenheim, I.: Non p-norm approximated groups. J. Anal. Math. **141**, 305–321 (2020) Zbl 07317680 MR 4174045

[30] Morris, D. W.: Bounded generation of $SL(n, A)$ (after D. Carter, G. Keller, and E. Paige). New York J. Math. **13**, 383–421 (2007) Zbl 1137.20045 MR 2357719

[31] Natarajan, A., Vidick, T.: A quantum linearity test for robustly verifying entanglement. In: STOC'17—Proc 49th Annual ACM SIGACT Symposium on Theory of Computing, ACM, New York, 1003–1015 (2017) Zbl 1370.81045 MR 3678246

[32] Natarajan, A., Vidick, T.: Low-degree testing for quantum states, and a quantum entangled games PCP for QMA. arXiv:1801.03821 (2018)

[33] Rolli, P.: Quasi-morphisms on free groups. arXiv:0911.4234 (2009)

[34] Thom, A.: Finitary approximations of groups and their applications. In: Proc. International Congress of Mathematicians—Rio de Janeiro 2018, Vol. III, Invited lectures, World Sci., Hackensack, NJ, 1779–1799 (2018) Zbl 1445.20039 MR 3966829

[35] Ulam, S. M.: A Collection of Mathematical Problems. Interscience Tracts in Pure Appl. Math. 8, Interscience Publ., New York (1960) Zbl 0086.24101 MR 0120127

[36] Vidick, T.: Pauli braiding (2017). https://mycqstate.wordpress.com/2017/06/28/pauli-braiding/

[37] Zheng, T.: On rigid stabilizers and invariant random subgroups of groups of homeomorphisms. arXiv:1901.04428 (2019)