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Extension theorem and representation formula in non-axially-symmetric domains for slice regular functions

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Abstract. Slice analysis is a generalization of the theory of holomorphic functions of one complex variable to quaternions. Among the new phenomena which appear in this context, there is the fact that the convergence domain of $f(q) = \sum_{n\in\mathbb{N}}(q-p)^{*n}a_n$, given by a σ-ball $\Sigma(p,r)$, is not open in $\mathbb{H}$ unless $p \in \mathbb{R}$. This motivates us to investigate, in this article, what is a natural topology for slice regular functions. It turns out that the natural topology is the so-called slice topology, which is different from the Euclidean topology and nicely adapts to the slice structure of quaternions. We extend the function theory of slice regular functions to any domains in the slice topology. Many fundamental results in the classical slice analysis for axially symmetric domains fail in our general setting. We can even construct a counterexample to show that a slice regular function in a domain cannot be extended to an axially symmetric domain. In order to provide positive results we need to consider so-called path-slice functions instead of slice functions. Along these lines, we can establish an extension theorem and a representation formula in a slice domain.

Keywords. Domains of holomorphy, quaternions, slice regular functions, representation formula, slice topology

1. Introduction

The richness of complex analysis makes it natural to look for generalizations to quaternions. Around the early thirties various people, among which Moisil and Fueter, considered possible definitions of analyticity over the quaternions. Since then, Fueter and his school started a systematic study, so the notion of 'regular' quaternionic function is the one associated with the so-called Cauchy–Riemann–Fueter equation [10],

$$\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2} + j\frac{\partial f}{\partial x_3} + k\frac{\partial f}{\partial x_4} = 0.$$

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This theory has been widely studied; see e.g. [9, 19, 23] but also [6, 17] and the references therein. Unfortunately, the class of Fueter regular functions does not contain the identity function $f(q) = q$ or any other polynomial in q. However, Fueter [10] found a powerful approach to construct functions in higher dimensions based on holomorphic function of one complex variable.

This approach was further developed by Sce [21], Rinehart [20] and resulted in the theory of intrinsic or stem functions. Later on, Cullen [8] defined another class of regular functions by intrinsic functions. Cullen regular functions contain quaternionic power series of the form $\sum_{n\in\mathbb{N}} q^n a_n$.

Following Cullen's approach, another theory, called slice quaternionic analysis, was started by Gentili and Struppa [13, 14] based on a more geometric formulation. This local theory has been well established first on balls centered at the origin [13, 14] and then over axially symmetric slice domains [4, 5]. Most of the local theory of holomorphic functions of one complex variable can be lifted to quaternions. It gives rise to the new notion of S-spectrum and has powerful applications in the quaternionic spectral theory (see e.g. [1,5]), and in quaternionic Hilbert spaces [2, 3, 5, 15]. See [7, 12] and the references therein for other information.

In contrast to its full development in local theory, the global one remains to be developed. The challenging task of establishing the global theory over quaternions can lead to some new theories such as slice Riemann surfaces, domains of slice regularity, and slice Dolbeault complexes. Therefore, the first natural question to be answered is:

What is the natural topology in slice analysis?

In [4], it has been claimed that any slice regular function on a domain of $\mathbb{H}$ can be extended to an axially symmetric domain. But this is not true and we provide a counterexample in Example 8.10. This means that axially symmetric slice domains are not the maximal domains of definition of a slice regular function. In other words, axially symmetric domains do not play the role of the natural maximal domains in slice analysis. On the other hand, the convergence domain of the Taylor expansion of a slice regular function

$$\sum_{n\in\mathbb{N}} (q-p)^{*n} \frac{f^n(p)}{n!},$$

completely described in terms of the σ-ball $\Sigma(p, r)$ (see [11]), may not be a Euclidean domain. Hence the Euclidean topology is not a natural topology in slice analysis.

To answer the above question, we observe that the slice book structure of quaternions plays a key role which makes it feasible to lift the theory of holomorphic functions in one complex variable to quaternions. The slice book structure comes from the following decomposition of quaternions into complex planes:

$$\mathbb{H} = \bigcup_{I\in\mathbb{S}} \mathbb{C}_I, \tag{1.1}$$

where $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ is the complex plane generated by the imaginary unit I and $\mathbb{S}$ consists of all imaginary units I of quaternions. As a result, the slice book structure of quaternions is a natural structure in slice analysis.

Motivated by the slice book structure, we can answer the main question of this article. It turns out that the natural topology in slice analysis is the so-called slice topology, which adapts nicely to the book structure of quaternions. We prove that the slice topology is finer than the Euclidean topology and all of the σ-balls $\Sigma(p, r)$ are domains in the slice topology.

With this slice topology, some natural questions arise. One can ask if the slice theory can be extended from axially symmetric domains to any domains in slice topology, but the answer is negative in general. As an example, one can consider the representation formula. This formula is the most important feature of the classical local theory of slice analysis. It states that any slice regular function over an axially symmetric slice domain is completely determined by its values on two pages, i.e. complex planes, of the book structure of $\mathbb{H}$. This result cannot be immediately extended to the case of open sets in the slice topology. Instead, we have to extend the theory of stem functions to a new one involving paths which produce path-slice functions.

Using the slice topology, one can also ask if any domain in the slice topology is a domain of holomorphy in some sense. The answer to this question is also negative, in general, in contrast to the case of holomorphic functions in one variable. This leads to the study of the characterization of domains of holomorphy just as in the case of holomorphic functions of several variables. We provide conditions for a domain to be such a domain of holomorphy.

The structure of the paper is the following. In Section 2, we introduce the slice topology on quaternions for slice regular functions and we describe our main results and ideas. In Section 3, we give some basic properties and examples for the slice topology. In Section 4, we prove an identity principle for slice regular functions on domains in the slice topology. In Section 5, we generalize the notion of slice function to any subset of $\mathbb{H}$ and give several equivalent definitions of slice functions. In Section 6, we prove a generalized extension formula. In Section 7, we define a class of functions, called path-slice functions. These functions play a similar role on slice domains to that of slice functions on axially symmetric slice domains. We also give several equivalent definitions of path-slice functions and prove our main theorem, i.e. the Representation Formula 2.11. In Section 8, we give an example to show that the classical general representation formula [4, Theorem 3.2] does not work on a non-axially-symmetric s-domain, using the new Representation Formula 2.11. Section 9 is devoted to domains of holomorphy for slice regular functions defined on slice-open sets among which there are axially symmetric st-domains and σ-balls.

We will continue our study of global slice analysis in forthcoming articles.

2. Main results

In this section, we state our main results. To this end, some notation and definitions from [13] are needed. Let

$$\mathbb{S} := \{q \in \mathbb{H} : q^2 = -1\}$$

be the sphere of imaginary units of $\mathbb{H}$. For any subset Ω of $\mathbb{H}$ and $I \in \mathbb{S}$, we call

$$\Omega_I := \Omega \cap \mathbb{C}_I$$

the I-*slice* (a slice) of Ω.

Definition 2.1. Assume that Ω is an open set in $\mathbb{C}_I$ for some $I \in \mathbb{S}$. A function $f : \Omega \to \mathbb{H}$ is said to be *left* $\mathbb{C}_I$-*holomorphic* (or simply holomorphic) if f has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(x+yI) = 0 \tag{2.1}$$

for any $x, y \in \mathbb{R}$ with $x + yI \in \Omega$.

The definition originally given in [13] is as follows:

Definition 2.2. Let Ω be a domain in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is said to be (left) *slice regular* if $f_I := f|_{\Omega_I}$ is left $\mathbb{C}_I$-holomorphic for any $I \in \mathbb{S}$.

[11, Theorem 8] shows that the convergence domain of the series

$$\sum_{n\in\mathbb{N}} (q-p)^{*n} a_n$$

is the σ-ball

$$\Sigma(p, r) := \{q \in \mathbb{H} : \sigma(p, q) < r\}.$$

with the σ-distance defined by

$$\sigma(q, p) := \begin{cases} |q-p|, & \exists I \in \mathbb{S} : p, q \in \mathbb{C}_I, \\ \sqrt{(\mathrm{Re}(q-p))^2 + (|\mathrm{Im}(q)| + |\mathrm{Im}(p)|)^2}, & \text{otherwise}, \end{cases}$$

for any $p, q \in \mathbb{H}$. A σ-ball is not a Euclidean domain when $p \in \mathbb{H} \setminus \mathbb{R}$. This illustrates the need to define 'slice regular' functions on more sets, such as the above σ-balls. Note that the 'holomorphic' condition of f in Definition 2.2 is limited to each slice $\mathbb{C}_I$, $I \in \mathbb{S}$. Thus in order to define 'slice regularity', we just need to guarantee that Ω_I is open in $\mathbb{C}_I$ for each $I \in \mathbb{S}$.

Definition 2.3. A subset Ω of $\mathbb{H}$ is called *slice-open* if Ω_I is open in $\mathbb{C}_I$ for any $I \in \mathbb{S}$.

It is clear that the σ-ball $\Sigma(p, r)$ is slice-open. Now we extend Definition 2.2 to slice-open sets.

Definition 2.4. Let Ω be a slice-open set in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is called (left) *slice regular* if f_I is left holomorphic for any $I \in \mathbb{S}$.

We note that, so far, in the literature, numerous results in slice quaternionic analysis (according to the definition in [13]) have been developed systematically over axially symmetric slice domains and this is basically enough for various purposes. Our goal is to generalize it to any slice-open set. Some properties can be proved as in the classical case, e.g. the following Splitting Lemma. Thus we state it without proof.

Lemma 2.5 (Splitting Lemma). *Let f be a function on a slice-open set Ω. Then f is slice regular if and only if for all $I, J \in \mathbb{S}$ with $I \perp J$, there are complex-valued holomorphic functions $F, G : \Omega_I \to \mathbb{C}_I$ such that $f_I = F + GJ$.*

The set of slice-open sets gives a topology on $\mathbb{H}$:

Lemma 2.6.

$$\tau_s(\mathbb{H}) := \{\Omega \subset \mathbb{H} : \Omega \textit{ is slice-open}\}$$

is a topology on $\mathbb{H}$.

Proof. This can be immediately verified directly or by observing that the slice topology is the final topology with respect to the inclusions $\{i_I : \mathbb{C}_I \to \mathbb{H}\}_{I\in\mathbb{S}}$. ∎

Definition 2.7. We call $\tau_s(\mathbb{H})$ the *slice topology*. Open sets, connected sets and paths in the slice topology are called slice-open sets, slice-connected sets and slice-paths.

Remark 2.8. In particular, a similar terminology will be used for all the other notions in the slice topology, with one remarkable exception. We will not use the term ‘slice-domain’ to denote a domain in the slice topology, since this notion is already used in the literature to denote something else (see Definition 2.9 below). We will use instead the term *slice topology-domain*, for short, *st-domain*.

Definition 2.9. A set Ω in $\mathbb{H}$ is called a *classical slice domain*, for short s-domain, if Ω is a domain in the Euclidean topology,

$$\Omega_{\mathbb{R}} := \Omega \cap \mathbb{R} \neq \emptyset,$$

and Ω_I is a domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$.

It is evident that an s-domain must be a domain in the slice topology, i.e. an st-domain, but the converse is not true (see Example 3.13).

The classical slice quaternionic analysis is established on axially symmetric s-domains. The slice quaternionic analysis on st-domains shows differences with respect to the classical one, since it relies on slice-connectedness. For example, the proof of the following generalized Identity Principle in Section 4 involves some properties of st-domains induced by slice-connectedness.

Theorem 2.10 (Identity Principle). *Let f and g be slice regular functions on an st-domain Ω in $\mathbb{H}$. If f and g coincide on a subset of Ω_I with an accumulation point in Ω_I for some $I \in \mathbb{S}$, then $f = g$ on Ω.*

Another fundamental result in the classical slice analysis is the general representation formula [4, Theorem 3.2]. Unfortunately, this formula fails, in general, on non-axially-symmetric domains (see Section 8).

To get the validity of the formula, we have to introduce the notion of path-slice functions (see Definition 7.1).

We consider the transform

$$\mathcal{P}_I : \mathbb{C} \to \mathbb{C}_I, \quad x + yi \mapsto x + yI,$$

for any $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$. For any path γ in $\mathbb{C}$, we define the corresponding path in $\mathbb{C}_I$ as

$$\gamma^I := \mathcal{P}_I \circ \gamma \quad \text{for any } I \in \mathbb{S}.$$

Theorem 2.11 (Representation Formula). *Assume that Ω is a slice-open set in $\mathbb{H}$ and suppose γ is a path in $\mathbb{C}$ satisfying the conditions*

$$\gamma(0) \in \mathbb{R}, \quad \gamma^I, \gamma^J, \gamma^K \subset \Omega,$$

for some $I, J, K \in \mathbb{S}$ with $J \neq K$. If f is a slice regular function on Ω, then

$$f \circ \gamma^I = (I - K)(J - K)^{-1} f \circ \gamma^J + (I - J)(K - J)^{-1} f \circ \gamma^K. \tag{2.2}$$

Remark 2.12. Although we only assume the domain Ω under consideration is slice-open, some restrictions related to slice-connectedness are implicitly involved as shown by the conditions

$$\gamma^I, \gamma^J, \gamma^K \subset \Omega.$$

The path γ^I in a slice can distinguish points of Ω more finely than $x + yI$ (by the Euclidean coordinate in $\mathbb{C}_I$; see Section 8). This ensures that the representation formula holds on non-axially-symmetric domains.

A function satisfying (2.2) is called path-slice in Section 7 based on an equivalent definition. It turns out that any slice regular function is a path-slice function. The proof of (2.2) will depend on a new approach; see Proposition 7.2(i, vi).

3. Slice topology

In this section, we study some properties of the slice topology $\tau_s(\mathbb{H})$. The slice structure induces the intricacy of the notion of slice-connectedness near the real axis. We tackle this issue in terms of slice-paths.

We denote by $\tau_s(\mathbb{H})$ and $\tau(\mathbb{H})$ the slice topology and the Euclidean topology of $\mathbb{H}$, respectively. Sometimes, we simply write τ_s and τ, for short.

Proposition 3.1. *$(\mathbb{H}, \tau_s)$ is a Hausdorff space and $\tau \subsetneq \tau_s$.*

Proof. Since every Euclidean open set in $\mathbb{H}$ is slice-open, we have $\tau \subset \tau_s$ and τ_s is Hausdorff. Note that $\Sigma(p, r)$ is slice-open and not open for any $p \in \mathbb{H} \setminus \mathbb{R}$ and $r \in \mathbb{R}_+$. It follows that the slice topology is strictly finer than the Euclidean topology. ∎

We remark that the slice topology locally coincides with the Euclidean topology on a slice complex plane for any point away from the real axis $\mathbb{R}$, because for any $I \in \mathbb{S}$ the subspace topologies on $\mathbb{C}_I$ of $\tau_s(\mathbb{H})$ and $\tau(\mathbb{H})$ coincide, i.e.

$$\tau_s(\mathbb{C}_I) = \tau(\mathbb{C}_I).$$

However, $\tau_s(\mathbb{H})$ is quite different from the Euclidean topology $\tau(\mathbb{H})$ near $\mathbb{R}$, as demonstrated by the following example.

Example 3.2. Fix $I \in \mathbb{S}$. We construct a slice-open set Ω in $\mathbb{H}$ as

$$\Omega := \bigcup_{J \in \mathbb{S}} \Omega_J, \tag{3.1}$$

where

$$\Omega_J := \begin{cases} \left\{ x + yJ \in \mathbb{C}_J : x^2 + \dfrac{y^2}{\operatorname{dist}(J, \mathbb{C}_I)} < 1 \right\}, & J \neq \pm I, \\ \{x + yJ \in \mathbb{C}_J : x^2 + y^2 < 1\}, & J = \pm I. \end{cases}$$

Here $\operatorname{dist}(J, \mathbb{C}_I)$ is the Euclidean distance from J to $\mathbb{C}_I$.

By construction, Ω is slice-open. But Ω is not open in $\mathbb{H}$ since $0 \in \Omega$ and 0 is not in the Euclidean interior of Ω. This is because Ω_J is an ellipse whose minor semi-axis $\sqrt{\operatorname{dist}(J, \mathbb{C}_I)}$ tends to 0 when J approaches I with $J \neq \pm I$.

The slice topology is finer than the topology induced by the σ-distance as proved in the following result:

Proposition 3.3. *$\tau_\sigma \subsetneq \tau_s$, where τ_σ is the topology on $\mathbb{H}$ induced by the σ-distance.*

Proof. Let $U \in \tau_\sigma$ and $I \in \mathbb{S}$. Then for any $q \in U_I$,

$$r := \sigma(q, \mathbb{H} \setminus U) > 0.$$

Note that for each $z, w \in \mathbb{C}_I$, $\sigma(z, w) = \operatorname{dist}_{\mathbb{C}_I}(z, w)$, where $\operatorname{dist}_{\mathbb{C}_I}(z, w)$ is the Euclidean distance in $\mathbb{C}_I$. Let $B_I(q, r)$ be the ball with center q and radius r in $\mathbb{C}_I$. It is clear that $B_I(q, r)$ is a subset of U_I, and q is a point in the interior of U_I. Hence U_I is open in $\mathbb{C}_I$, so that U is slice-open and $\tau_\sigma \subset \tau_s$.

To show that the slice topology is strictly finer, we consider the set Ω defined in (3.1), Example 3.2, which is slice-open. Let $J \in \mathbb{S}$. Since $\mathbb{H} \setminus \Omega \supset \mathbb{C}_J \setminus \Omega_J$,

$$\sigma(0, \mathbb{H} \setminus \Omega) \leq \sigma(0, \mathbb{C}_J \setminus \Omega_J) = \operatorname{dist}_{\mathbb{C}_J}(0, \mathbb{C}_J \setminus \Omega_J). \tag{3.2}$$

Note that

$$\lim_{J \to I,\, J \neq I} \operatorname{dist}_{\mathbb{C}_J}(0, \mathbb{C}_J \setminus \Omega_J) = 0. \tag{3.3}$$

From (3.2) and (3.3) we deduce that $\sigma(0, \mathbb{H} \setminus \Omega) = 0$. Hence, 0 is not an interior point in Ω under the topology τ_σ and so Ω is not open in τ_σ. However, Ω is a slice-open set and we conclude that $\tau_\sigma \neq \tau_s$. ∎

To deal with the difficulties of the topology near $\mathbb{R}$, a new notion, called real-connectedness, comes up. This provides an effective tool since the slice topology has a real-connected subbase.

Definition 3.4. A subset Ω of $\mathbb{H}$ is called *real-connected* if the set $\Omega_{\mathbb{R}} := \Omega \cap \mathbb{R}$ is connected in $\mathbb{R}$. In particular, when $\Omega \cap \mathbb{R} = \emptyset$, Ω is real-connected.

Proposition 3.5. *For any slice-open set Ω in $\mathbb{H}$ and $q \in \Omega$, there is a real-connected st-domain $U \subset \Omega$ containing q.*

Proof. We take U to be the slice-connected component of the set $(\Omega \setminus \Omega_{\mathbb{R}}) \cup A$ containing q. Here, when $q \in \mathbb{R}$, we take A to be the connected component of $\Omega_{\mathbb{R}}$ containing q in $\mathbb{R}$; otherwise, we set $A := \emptyset$.

It is easy to check that $q \in U$ and U is a real-connected st-domain. ∎

Now we describe slice-connectedness by means of slice-paths.

Definition 3.6. A path γ in $(\mathbb{H}, \tau)$ is said to be *on a slice* if $\gamma \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$.

Proposition 3.7. *Every path on a slice is a slice-path.*

Proof. This follows directly from the fact that $\tau_s(\mathbb{C}_I) = \tau(\mathbb{C}_I)$ for any $I \in \mathbb{S}$. ∎

Proposition 3.8. *Assume that an st-domain U is real-connected.*

(i) *If $U_{\mathbb{R}} = \emptyset$, then $U \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$.*

(ii) *If $U_{\mathbb{R}} \neq \emptyset$, then for any $q \in U$ and $x \in U_{\mathbb{R}}$, there exists a path on a slice from q to x.*

Proof. (i) If $U_{\mathbb{R}} = \emptyset$, then

$$U \subset \bigsqcup_{J \in \mathbb{S}} \mathbb{C}_J^+, \quad \text{where} \quad \mathbb{C}_J^+ := \{x + yJ \in \mathbb{H} : y > 0\}$$

is a slice-open set in $\mathbb{H}$ for any $J \in \mathbb{S}$. This means that $U \subset \mathbb{C}_I^+$ for some $I \in \mathbb{S}$ since U is slice-connected.

(ii) We fix $q \in U$ and $x \in U_{\mathbb{R}}$. Take $I \in \mathbb{S}$ such that $q \in \mathbb{C}_I$. Since U is an st-domain in $\mathbb{H}$, by definition U_I is an open set in the plane $\mathbb{C}_I$. Let V be the connected component of U_I containing q.

By definition the sets $\mathbb{C}_I \setminus \mathbb{R}$ and $\bigcup_{J \in \mathbb{S} \setminus \{\pm I\}} (\mathbb{C}_J \setminus \mathbb{R})$ are slice-open. If $V_{\mathbb{R}} = \emptyset$, then

$$V = U \cap (\mathbb{C}_I \setminus \mathbb{R}) \quad \text{and} \quad U \setminus V = U \cap \Big[\bigcup_{J \in \mathbb{S} \setminus \{\pm I\}} (\mathbb{C}_J \setminus \mathbb{R}) \Big]$$

are slice-open. Since U is slice-connected and nonempty, it follows from

$$U = V \sqcup (U \setminus V)$$

that $V = U$. This implies $U_{\mathbb{R}} = V_{\mathbb{R}} = \emptyset$, which is a contradiction. We thus conclude $V_{\mathbb{R}} \neq \emptyset$.

We take a point $x_0 \in V_{\mathbb{R}}$. Since V is the connected component of U_I containing q, there exists a path α in V from q to x_0. Because U is real-connected, we have a path β in $U_{\mathbb{R}}$ from x_0 to x. It is clear that $\alpha\beta$ is a path on a slice from q to x. ∎

Corollary 3.9. *Assume that an st-domain U is real-connected.*

(i) *U_I is a domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$.*

(ii) *For any $p, q \in U$, there exist paths γ_1, γ_2 such that each of them is a path on a slice in U, $\gamma_1(1) = \gamma_2(0)$, and $\gamma_1\gamma_2$ is a slice-path from p to q.*

Proof. This follows directly from Proposition 3.8. ∎

Proposition 3.10. *The topological space $(\mathbb{H}, \tau_s)$ is connected, locally path-connected and path-connected.*

Proof. It follows from Proposition 3.5 and Corollary 3.9 (ii) that $(\mathbb{H}, \tau_s)$ is locally path-connected. Since $\mathbb{H} \cap \mathbb{C}_I = \mathbb{C}_I \supset \mathbb{R}$ for any $I \in \mathbb{S}$, $(\mathbb{H}, \tau_s)$ is path-connected so that it is also connected. ∎

Corollary 3.11. *A set $\Omega \subset \mathbb{H}$ is an st-domain if $\Omega_{\mathbb{R}} \neq \emptyset$ and Ω_I is a domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$.*

Proof. If Ω_I is open for any $I \in \mathbb{S}$, then by definition Ω is slice-open. Since $\Omega_{\mathbb{R}} \neq \emptyset$, we can take a fixed point $x \in \Omega_{\mathbb{R}}$. By hypothesis, Ω_I is a domain in $\mathbb{C}_I$ for any $I \in \mathbb{S}$, there exists a path on a slice from x to each point of Ω. This implies that Ω is slice-path-connected, so that it is also slice-connected. Thus Ω is an st-domain. ∎

Note that there are sets Ω which are st-domains with $\Omega_{\mathbb{R}} = \emptyset$. For example, let us consider a fixed $J \in \mathbb{S}$. We set

$$\Omega := B_J(2J, 1) = \{q \in \mathbb{C}_J : |q - 2J| < 1\}.$$

It is evident that Ω is an st-domain and $\Omega_{\mathbb{R}} = \emptyset$.

Remark 3.12. By Corollary 3.11, any s-domain is an st-domain. Therefore the notion of st-domain is a generalization of the notion of s-domain.

However, not every st-domain Ω is an s-domain, even when Ω is a domain in $\mathbb{H}$, as we show in the following example.

Example 3.13. We fix $I \in \mathbb{S}$ and consider the domain in $\mathbb{H}$ defined by

$$\Omega := B(0,2) \cup B(6,2) \cup U, \quad \text{where} \quad U := \{q \in \mathbb{H} : \operatorname{dist}(q - I, [0,6]) < 1/2\}.$$

It is easy to check that

$$\Omega_J = B_J(0,2) \cup B_J(6,2)$$

for any $J \in \mathbb{S}$ with $J \perp I$. Hence Ω_J is not connected in $\mathbb{C}_J$ so that Ω is not an s-domain. However, Ω is slice-connected, because any point in Ω can be connected to 0 or 6 by a path in a slice, and 0 can be connected to 6 by a path in $\mathbb{C}_I$. And since Ω_J is open in $\mathbb{C}_J$ for any $J \in \mathbb{S}$, Ω is an st-domain.

4. Identity principle

In this section we provide an identity principle for slice regular functions defined on st-domains.

Since st-domains satisfy conditions weaker than those required of s-domains, the proof of the Identity Principle 2.10 is more difficult than the one for s-domains. We need to reduce the problem to the special case where the domain is real-connected.

Lemma 4.1. *Assume that an st-domain Ω is real-connected. Let f and g be slice regular functions on Ω. If f and g coincide on a subset of Ω_I with an accumulation point in Ω_I for some $I \in \mathbb{S}$, then $f = g$ on Ω.*

Proof. By assumption, we have $\Omega_I \neq \emptyset$ so that Corollary 3.9 (i) implies Ω_I is a non-empty domain in $\mathbb{C}_I$. Therefore, using the Splitting Lemma and the identity principle for classical holomorphic functions of a complex variable, we deduce that f and g coincide on Ω_I.

If $\Omega_{\mathbb{R}} = \emptyset$, then $\Omega = \Omega_I$ due to Proposition 3.8 (i) so that $f = g$ on Ω.

Otherwise, we have $\Omega_{\mathbb{R}} \neq \emptyset$. By Corollary 3.9 (i), Ω_J is a domain in $\mathbb{C}_J$ for all $J \in \mathbb{S}$. Since $f = g$ on $\Omega_{\mathbb{R}}$ $(\subset \Omega_I)$, it follows that $f = g$ on Ω_J for any $J \in \mathbb{S}$. Consequently, $f = g$ on $\Omega = \bigcup_{J\in\mathbb{S}} \Omega_J$. ■

Now we can give the proof of the identity principle for st-domains.

Proof of Theorem 2.10. We consider the set

$$A := \{x \in \Omega : \exists V \in \tau_s(\Omega) : x \in V \text{ and } f = g \text{ on } V\}.$$

By definition, A is a slice-open set in Ω.

Next, we show that A is nonempty. Due to Proposition 3.5, there exists a real-connected st-domain U that contains the accumulation point p and $U \subset \Omega$. It follows from Lemma 4.1 applied to U that $f = g$ on U. This means that $p \in A$, so that A is nonempty.

Finally, we claim that $\Omega \setminus A$ is slice-open. From this claim and the fact that Ω is slice-connected, we conclude that $A = \Omega$, so that $f = g$ on Ω.

It remains to prove the claim. Let $q \in \Omega \setminus A$ be arbitrary. From Proposition 3.5, there exists a real-connected st-domain V containing q with $V \subset \Omega$. We already know that both A and V are slice-open, hence so is $A \cap V$.

If $A \cap V \neq \emptyset$, then $A \cap V$ is a non-empty slice-open set. Since $f = g$ on $A \cap V$, it follows from Lemma 4.1 that $f = g$ on V. This means that $q \in A$, a contradiction.

Therefore, $A \cap V = \emptyset$. This implies that q is a slice-interior point of $\Omega \setminus A$. Hence $\Omega \setminus A$ is slice-open. This proves the claim and finishes the proof. ■

5. Slice functions

Slice functions play a fundamental role in the theory of slice regular functions. The related stem function theory for slice analysis has been established in the case of real alternative $*$-algebras [16]. See [18] for recent developments.

In this section, we give several equivalent characterizations of slice functions. For convenience, we consider slice functions on an arbitrary domain of definition.

We remark that our definition of the slice function is different from the classical one.

Definition 5.1. Let Ω be an arbitrary set in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is called a *slice function* if there is a function $F : \mathbb{R}^2 \to \mathbb{H}^{2\times 1}$ such that

$$f(x+yI) = (1, I)F(x, y) \tag{5.1}$$

for any $x + yI \in \Omega$ such that $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $y \geq 0$.

The function F is referred to as an *upper stem function* of the slice function f.

We note that we are not requiring, at this stage, any condition on F and since it is defined in $\mathbb{R}^2$, for $x + Iy \notin \Omega$, we set $F(x, y) = (0, 0)^T$. Set

$$\mathbb{S}^2_* := \{(I, J) \in \mathbb{S}^2 : I \neq J\}.$$

For any $(J, K) \in \mathbb{S}^2_*$ we have the noteworthy identity

$$(J-K)^{-1}J = -K(J-K)^{-1}. \tag{5.2}$$

From this, it is easy to check that

$$\begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} = \begin{pmatrix} (J-K)^{-1}J & (K-J)^{-1}K \\ (J-K)^{-1} & (K-J)^{-1} \end{pmatrix}. \tag{5.3}$$

Proposition 5.2. *For any function $f : \Omega \to \mathbb{H}$ with $\Omega \subset \mathbb{H}$, the following statements are equivalent:*

(i) *f is a slice function.*

(ii) *There exists a function $F : \mathbb{R}^2 \to \mathbb{H}^{2\times 1}$ such that*

$$f(x+yI) = (1, I)F(x, y) \tag{5.4}$$

for any $x + yI \in \Omega$ with $x, y \in \mathbb{R}$ and any $I \in \mathbb{S}$.

(iii) *If $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $(J, K) \in \mathbb{S}^2_*$ with $x + yL \in \Omega$ for $L = I, J, K$, then*

$$f(x+yI) = (1, I)\begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} \begin{pmatrix} f(x+yJ) \\ f(x+yK) \end{pmatrix}. \tag{5.5}$$

(iv) *If $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $(J, K) \in \mathbb{S}^2_*$ with $x + yL \in \Omega$ for $L = I, J, K$, then*

$$\begin{aligned} f(x+yI) = {} & (J-K)^{-1}[Jf(x+yJ) - Kf(x+yK)] \\ & + I(J-K)^{-1}[f(x+yJ) - f(x+yK)]. \end{aligned} \tag{5.6}$$

(v) *If $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $(J, K) \in \mathbb{S}^2_*$ with $x + yL \in \Omega$ for $L = I, J, K$, then*

$$f(x+yI) = (I-K)(J-K)^{-1}f(x+yJ) + (I-J)(K-J)^{-1}f(x+yK). \tag{5.7}$$

Proof. It follows from (5.2) and (5.3) that assertions (iii)–(v) are equivalent.

(i)⇒(ii). If f is a slice function, then there is a function $G = (G_1, G_2)^T : \mathbb{R}^2 \to \mathbb{H}^{2\times 1}$ such that

$$f(x + yI) = (1, I)G(x, y)$$

for any $x + yI \in \Omega$ with $x, y \in \mathbb{R}$, $I \in \mathbb{S}$, and $y \geq 0$.

Hence we can take a function $F : \mathbb{R}^2 \to \mathbb{H}^{2\times 1}$ defined by

$$F(x, y) := \begin{cases} (G_1, G_2)^T(x, y), & y \geq 0, \\ (G_1, -G_2)^T(x, -y), & y < 0. \end{cases}$$

Direct calculation shows that (5.4) holds.

(ii)⇒(iii). According to (5.4), we have

$$\begin{pmatrix} f(x + yJ) \\ f(x + yK) \end{pmatrix} = \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix} F(x, y)$$

for any $x, y \in \mathbb{R}$ and $(J, K) \in \mathbb{S}^2_*$. This implies that

$$F(x, y) = \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} \begin{pmatrix} f(x + yJ) \\ f(x + yK) \end{pmatrix}. \tag{5.8}$$

Combining this with (5.4), we deduce that (5.5) holds.

(iii)⇒(i). We consider the sets

$$\begin{aligned} \mathcal{A} &:= \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |(x + y\mathbb{S}) \cap \Omega| = 1\}, \\ \mathcal{B} &:= \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } |(x + y\mathbb{S}) \cap \Omega| > 1\}, \end{aligned}$$

where $|S|$ denotes the cardinality of the set S.

If $(x, y) \in \mathcal{B}$, then there are at least two distinct points in the set $(x + y\mathbb{S}) \cap \Omega$. Therefore, the axiom of choice shows that we can choose $(J_{x,y}, K_{x,y}) \in \mathbb{S}^2_*$ such that

$$x + yJ_{x,y}, x + yK_{x,y} \in (x + y\mathbb{S}) \cap \Omega$$

for any $(x, y) \in \mathcal{B}$.

From this, we can construct a function $G : \mathcal{B} \to \mathbb{H}^{2\times 1}$ defined by

$$G(x, y) := \begin{pmatrix} 1 & J_{x,y} \\ 1 & K_{x,y} \end{pmatrix}^{-1} \begin{pmatrix} f(x + yJ_{x,y}) \\ f(x + yK_{x,y}) \end{pmatrix}.$$

Finally, we can define the desired function $F : \mathbb{R}^2 \to \mathbb{H}^{2\times 1}$ via

$$F(x, y) := \begin{cases} (f(x + yI_{x,y}), 0)^T, & (x, y) \in \mathcal{A}, \\ G(x, y), & (x, y) \in \mathcal{B}, \\ (0, 0)^T, & \text{otherwise}, \end{cases}$$

where $I_{x,y}$ is the unique imaginary unit $I \in \mathbb{S}$ such that $x + yI \in \Omega$ for $(x, y) \in \mathcal{A}$. It is easy to check that F satisfies (5.1), so that f is a slice function. ∎

We remark that the form in (5.7) is also in [16, Proposition 6] for the related class of functions.

Remark 5.3. By Proposition 5.2, the classical representation formula in [4, Theorem 3.2] can be interpreted in the formalism of slice functions. That is, any slice regular function defined on an axially symmetric s-domain is a slice function.

6. Extension theorem

In [4, Theorem 4.2], the extension theorem is generalized from balls centered on the real axis to axially symmetric s-domains. In this section, we consider a further generalization to not necessarily axially symmetric st-domains.

For any $\mathbb{I} = (I_1, I_2) \in \mathbb{S}^2_*$, we set

$$\tau[\mathbb{I}] := \{(U, V) : U \in \tau(\mathbb{C}_{I_1}) \text{ and } V \in \tau(\mathbb{C}_{I_2})\}.$$

Associated with

$$\mathbb{U} = (U_1, U_2) \in \tau[\mathbb{I}] \quad \text{with} \quad \mathbb{I} = (I_1, I_2) \in \mathbb{S}^2_*,$$

we introduce the following three sets:

$$\begin{aligned} \mathbb{U}_s^+ &:= (U_1 \cap \mathbb{C}^+_{I_1}) \sqcup (U_2 \cap \mathbb{C}^+_{I_2}) \sqcup (U_1 \cap U_2 \cap \mathbb{R}), \\ \mathbb{U}_s^\Delta &:= \{x + y\mathbb{S} : (x + yI_1, x + yI_2) \in \mathbb{U},\ y \in \mathbb{R},\ y \geq 0\}, \\ \mathbb{U}_s^{+\Delta} &:= \mathbb{U}_s^+ \cup \mathbb{U}_s^\Delta. \end{aligned}$$

Sometimes we also replace $\mathbb{U}_s^{+\Delta}$ by $\mathbb{U}_{s,\mathbb{I}}^{+\Delta}$ to emphasize its dependence on $\mathbb{I}$.

Lemma 6.1. *Let $\mathbb{I} \in \mathbb{S}^2_*$ be fixed. Then $\mathbb{U}_s^{+\Delta}$ is slice-open.*

Proof. We need to show that any $q \in \mathbb{U}_s^{+\Delta}$ is a slice-interior point of $\mathbb{U}_s^{+\Delta}$.

Case 1: $q \in \mathbb{U}_s^{+\Delta} \setminus \mathbb{R}$. If $q \in \mathbb{U}_s^+ \setminus \mathbb{R}$, then q is an interior point of $U_1 \cap \mathbb{C}^+_{I_1}$ or $U_2 \cap \mathbb{C}^+_{I_2}$. Hence q is a slice-interior point of $\mathbb{U}_s^+$ and of $\mathbb{U}_s^{+\Delta}$.

If $q \in \mathbb{U}_s^\Delta \setminus \mathbb{R}$, it can be expressed as $q = x + yJ$ for some $J \in \mathbb{S}$, $x, y \in \mathbb{R}$ with $y > 0$. By definition of $\mathbb{U}_s^\Delta$,

$$x + yI_1 \in U_1 \cap \mathbb{C}^+_{I_1}, \quad x + yI_2 \in U_2 \cap \mathbb{C}^+_{I_2}.$$

Hence there exists an $r \in \mathbb{R}_+$ such that

$$B_{I_1}(x + yI_1, r) \subset U_1 \cap \mathbb{C}^+_{I_1}, \quad B_{I_2}(x + yI_2, r) \subset U_2 \cap \mathbb{C}^+_{I_2}.$$

This means $B_J(x + yJ, r) \subset \mathbb{U}_s^\Delta$, so that q is a slice-interior point of $\mathbb{U}_s^{+\Delta}$.

Case 2: $q \in \mathbb{U}_s^{+\Delta} \cap \mathbb{R}$. It is easy to check that

$$\mathbb{U}_s^{+\Delta} \cap \mathbb{R} = U_1 \cap U_2 \cap \mathbb{R}.$$

Since $q \in \mathbb{U}_s^{+\Delta} \cap \mathbb{R}$, there exists an $r \in \mathbb{R}_+$ such that

$$B_{I_1}(q, r) \subset U_1, \quad B_{I_2}(q, r) \subset U_2,$$

which implies, by definition, that $B_J(q, r) \subset \mathbb{U}_s^{\Delta}$ for any $J \in \mathbb{S}$. Hence $B(q, r) \subset \mathbb{U}_s^{+\Delta}$. ■

Theorem 6.2. *Let $\mathbb{I} \in \mathbb{S}_*^2$ and $\mathbb{U} = (U_1, U_2) \in \tau[\mathbb{I}]$. If $f : U_1 \cup U_2 \to \mathbb{H}$ is a function such that $f|_{U_1}$ and $f|_{U_2}$ are both holomorphic, then $f|_{\mathbb{U}_s^+}$ admits a slice regular extension $\widetilde{f}$ over $\mathbb{U}_s^{+\Delta}$.*

Moreover, if W is an st-domain such that

$$W \subseteq \mathbb{U}_s^{+\Delta}, \quad W \cap \mathbb{U}_s^+ \neq \emptyset,$$

then $\widetilde{f}|_W$ is a slice function and it is the unique slice regular extension of $f|_{W \cap \mathbb{U}_s^+}$ over W.

Proof. Define a function $g : \mathbb{U}_s^{\Delta} \to \mathbb{H}$ by

$$\begin{aligned} g(x + yJ) := {} & (J - I_2)(I_1 - I_2)^{-1} f(x + yI_1) \\ & + (J - I_1)(I_2 - I_1)^{-1} f(x + yI_2) \end{aligned} \tag{6.1}$$

for any $J \in \mathbb{S}$, $x, y \in \mathbb{R}$ with $y \geq 0$ and $x + yI_\lambda \in U_\lambda$, $\lambda = 1, 2$.

By direct calculation (see the proof of [4, Theorem 3.2]), we find that g is slice regular on $\mathbb{U}_s^{\Delta}$ and $g = f$ on $\mathbb{U}_s^{\Delta} \cap \mathbb{U}_s^+$. Hence the function $\widetilde{f} : \mathbb{U}_s^{+\Delta} \to \mathbb{H}$ defined by

$$\widetilde{f} := \begin{cases} g & \text{on } \mathbb{U}_s^{\Delta}, \\ f & \text{on } \mathbb{U}_s^{+}, \end{cases} \tag{6.2}$$

is a slice regular extension of $f|_{\mathbb{U}_s^+}$.

If $h : W \to \mathbb{H}$ is a slice regular extension of $f|_{W \cap \mathbb{U}_s^+}$, then

$$h = f = \widetilde{f} \quad \text{on } W \cap \mathbb{U}_s^+,$$

so that the Identity Principle 2.10 implies $h = \widetilde{f}|_W$. Consequently, $\widetilde{f}|_W$ is the unique slice regular extension of $f|_{W \cap \mathbb{U}_s^+}$ over W.

By (5.3), (6.2) and direct calculations, we rewrite (6.1) as

$$\widetilde{f}(x + yJ) = (1, J)\mathcal{F}_{x,y}$$

for any $x, y \in \mathbb{R}$ and $J \in \mathbb{S}$ with $y \geq 0$ and $x + yK \in W$, $K = J, I_1, I_2$, where

$$\mathcal{F}_{x,y} = \begin{pmatrix} 1 & I_1 \\ 1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} f(x + yI_1) \\ f(x + yI_2) \end{pmatrix}.$$

Now we can introduce a function $G : \mathbb{R}^2 \to \mathbb{H}$ defined by

$$G(x, y) := \begin{cases} \mathcal{F}_{x,y}, & x + yI_1, x + yI_2 \in W, \\ (f(x + yI_1), 0)^T, & x + yI_1 \in W \text{ and } x + yI_2 \notin W, \\ (f(x + yI_2), 0)^T, & x + yI_1 \notin W \text{ and } x + yI_2 \in W, \\ (0, 0)^T, & \text{otherwise.} \end{cases}$$

It is easy to show that G is an upper stem function of $f|_W$. This means that $\widetilde{f}$ is slice on W by definition. ∎

Corollary 6.3. *If $f : B_I(q, r) \to \mathbb{H}$ is a holomorphic function with $I \in \mathbb{S}$, $q \in \mathbb{C}_I$ and $r \in \mathbb{R}_+$, then it can be uniquely extended to a slice regular function over the σ-ball $\Sigma(q, r)$.*

Proof. Case 1: $B_I(q,r) \cap \mathbb{R} = \emptyset$. In this case, we have $B_I(q,r) = \Sigma(q,r)$ so that $f = \widetilde{f}$ is the unique slice regular extension of itself.

Case 2: $B_I(q, r) \cap \mathbb{R} \neq \emptyset$. Now we take

$$\mathbb{I} := (I, -I) \in \mathbb{S}_*^2, \quad \mathbb{U} := (B_I(q, r), B_I(q, r)) \in \tau(\mathbb{I}).$$

It is easy to see $\mathbb{U}_s^{+\Delta} = \Sigma(q, r)$, which is an st-domain. By Proposition 6.2, f admits a unique slice regular extension $\widetilde{f}$ over $\Sigma(q, r)$. ∎

7. Path-slice functions and representation formula

In this section we extend the representation formula from axially symmetric domains to non-axially-symmetric domains. To this end, we introduce the new notion of path-slice functions. It turns out that any slice regular function on a slice-open set is path-slice (see Theorem 7.4). We can also prove the representation formula for path-slice functions.

We denote by $\mathscr{P}(\mathbb{C})$ the set of paths $\gamma : [0, 1] \to \mathbb{C}$ with initial point $\gamma(0)$ in $\mathbb{R}$ and we consider its subset

$$\mathscr{P}(\mathbb{C}^+) := \{\gamma \in \mathscr{P}(\mathbb{C}) : \gamma(0, 1] \subset \mathbb{C}^+\}.$$

Definition 7.1. A function $f : \Omega \to \mathbb{H}$ with $\Omega \subset \mathbb{H}$ is called a *path-slice function* if for any $\gamma \in \mathscr{P}(\mathbb{C})$, there is a function $F_\gamma : [0, 1] \to \mathbb{H}^{2\times 1}$ such that

$$f \circ \gamma^I = (1, I)F_\gamma \tag{7.1}$$

for any $I \in \mathbb{S}$ with $\gamma^I \subset \Omega$.

We call $\{F_\gamma\}_{\gamma \in \mathscr{P}(\mathbb{C})}$ a (path-)*stem system* of the path-slice function f.

Obviously, if $\Omega_{\mathbb{R}} = \emptyset$, then for each $\gamma \in \mathcal{P}(\mathbb{C})$, there is no $I \in \mathbb{S}$ such that $\gamma^I \subset \Omega$. Thus, by definition, every function $f : \Omega \to \mathbb{H}$ is path-slice.

Now, we provide equivalent characterizations for path-slice functions.

Proposition 7.2. *For any function $f : \Omega \to \mathbb{H}$ with $\Omega \subset \mathbb{H}$, the following statements are equivalent:*

(i) *f is a path-slice function.*

(ii) *For any $\gamma \in \mathscr{P}(\mathbb{C})$, there is an element $q_\gamma \in \mathbb{H}^{2\times 1}$ such that*

$$f \circ \gamma^I(1) = (1, I)q_\gamma \tag{7.2}$$

for any $I \in \mathbb{S}$ with $\gamma^I \subset \Omega$.

(iii) *For any $\gamma \in \mathscr{P}(\mathbb{C}^+)$, there is an element $p_\gamma \in \mathbb{H}^{2\times 1}$ such that*

$$f \circ \gamma^I(1) = (1, I)p_\gamma \tag{7.3}$$

for any $I \in \mathbb{S}$ with $\gamma^I \subset \Omega$.

(iv) *For any $\gamma \in \mathscr{P}(\mathbb{C})$ and $I, J, K \in \mathbb{S}$ with $J \neq K$ and $\gamma^I, \gamma^J, \gamma^K \subset \Omega$, we have*

$$f \circ \gamma^I = (1, I)\begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1}\begin{pmatrix} f \circ \gamma^J \\ f \circ \gamma^K \end{pmatrix}. \tag{7.4}$$

(v) *For any $\gamma \in \mathscr{P}(\mathbb{C})$ and $I, J, K \in \mathbb{S}$ with $J \neq K$ and $\gamma^I, \gamma^J, \gamma^K \subset \Omega$, we have*

$$f \circ \gamma^I = (J - K)^{-1}(Jf \circ \gamma^J - Kf \circ \gamma^K) + I(J - K)^{-1}(f \circ \gamma^J - f \circ \gamma^K).$$

(vi) *For any $\gamma \in \mathscr{P}(\mathbb{C})$ and $I, J, K \in \mathbb{S}$ with $J \neq K$ and $\gamma^I, \gamma^J, \gamma^K \subset \Omega$, we have*

$$f \circ \gamma^I = (I - K)(J - K)^{-1} f \circ \gamma^J + (I - J)(K - J)^{-1} f \circ \gamma^K.$$

Proof. From (5.2) and (5.3), one can deduce that assertions (iv)–(vi) are equivalent.

(i)⇒(iv). Suppose that f is a path-slice function and let $\{F_\gamma\}_{\gamma\in\mathscr{P}(\mathbb{C})}$ be its stem system. By (7.1) it follows that

$$\begin{pmatrix} f \circ \gamma^J \\ f \circ \gamma^K \end{pmatrix} = \begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix} F_\gamma \tag{7.5}$$

for any $\gamma \in \mathscr{P}(\mathbb{C})$ and $I, J, K \in \mathbb{S}$ with $J \neq K$ and $\gamma^I, \gamma^J, \gamma^K \subset \Omega$. It follows from (7.1) and (7.5) that (7.4) holds.

(iv)⇒(iii). Suppose (iv) holds. We consider the two sets

$$\mathscr{A} := \{\gamma \in \mathscr{P}(\mathbb{C}^+) : |\{I \in \mathbb{S} : \gamma^I \subset \Omega\}| = 1\}, \tag{7.6}$$

$$\mathscr{B} := \{\gamma \in \mathscr{P}(\mathbb{C}^+) : |\{I \in \mathbb{S} : \gamma^I \subset \Omega\}| > 1\}. \tag{7.7}$$

By the axiom of choice, there is $(J_\gamma, K_\gamma) \in \mathbb{S}^2_*$ such that $\gamma^{J_\gamma}, \gamma^{K_\gamma} \subset \Omega$ for any $\gamma \in \mathscr{B}$. We denote by I_γ the unique imaginary unit I in $\mathbb{S}$ such that $\gamma^I \subset \Omega$ for any $\gamma \in \mathscr{A}$.

For any $\gamma \in \mathscr{P}(\mathbb{C}^+)$, we set

$$p_\gamma := \begin{cases} (f \circ \gamma^{I_\gamma},\ 0)^T, & \gamma \in \mathscr{A}, \\ \begin{pmatrix} 1 & J_\gamma \\ 1 & K_\gamma \end{pmatrix}^{-1}\begin{pmatrix} f \circ \gamma^{J_\gamma}(1) \\ f \circ \gamma^{K_\gamma}(1) \end{pmatrix}, & \gamma \in \mathscr{B}, \\ (0,\ 0)^T, & \text{otherwise}, \end{cases}$$

It is immediate to verify that (7.3) holds.

(iii)⇒(ii). Let $\gamma \in \mathscr{P}(\mathbb{C})$ be arbitrary. We define

$$s := \max\{t \in [0,1] : \gamma(t) \in \mathbb{R}\}$$

and define the path $\delta : [0,1] \to \mathbb{C}$ by

$$\delta(t) := \begin{cases} \gamma(1), & \gamma(1) \in \mathbb{R}, \\ \gamma((1-s)t+s), & \gamma(1) \in \mathbb{C}^+, \\ \overline{\gamma((1-s)t+s)}, & \text{otherwise}. \end{cases}$$

By construction $\delta \in \mathscr{P}(\mathbb{C}^+)$, and moreover, if $\gamma^I \subset \Omega$ for some $I \in \mathbb{S}$, then $\delta^{\epsilon I} \subset \Omega$, where

$$\epsilon := \begin{cases} 1, & \gamma(1) \in \mathbb{C}^+, \\ -1, & \text{otherwise}. \end{cases} \tag{7.8}$$

We take

$$q_\gamma := \begin{cases} (p_{\delta,1}, \epsilon p_{\delta,2})^T, & |\{I \in \mathbb{S} : \gamma^I \subset \Omega\}| \geq 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $p_\delta = (p_{\delta,1}, p_{\delta,2})^T \in \mathbb{H}^{2\times 1}$ is an element satisfying (7.3), i.e.,

$$f \circ \delta^I(1) = (1, I)p_\delta$$

for any $I \in \mathbb{S}$ with $\delta^I \subset \Omega$. Obviously, q_γ satisfies (7.2) so that (ii) holds.

(ii)⇒(i). Let $\gamma \in \mathscr{P}(\mathbb{C})$ be arbitrary and fix a point $t \in [0,1]$. We consider the path $\delta : [0,1] \to \mathbb{C}$ defined by $\delta(s) := \gamma(ts)$. Then δ is a path from $\gamma(0)$ to $\gamma(t)$ such that $\delta \in \mathscr{P}(\mathbb{C})$.

Let q_δ be an element satisfying (7.2), i.e.,

$$f \circ \delta^I(1) = (1, I)q_\delta$$

for any $I \in \mathbb{S}$ with $\delta^I \subset \Omega$.

Now we can define a function $F_\gamma : [0,1] \to \mathbb{H}^{2\times 1}$ via

$$F_\gamma(t) := \begin{cases} q_\delta, & \exists I \in \mathbb{S} : \gamma^I \subset \Omega, \\ (0,\ 0)^T, & \text{otherwise}. \end{cases}$$

We remark that, by construction, the path δ depends on the parameter t.

It is direct to verify that $f \circ \gamma^I = (1, I)F_\gamma$for any $I \in \mathbb{S}$ with $\gamma^I \subset \Omega$. This means that f is path-slice since γ is arbitrary. ∎

Proposition 7.3. *Every slice function is a path-slice function.*

Proof. If f is a slice function, then (5.4) holds. If we set $\gamma^I(t) := x(t) + y(t)I$, it is clear that (7.2) follows from (5.4). This implies that f is path-slice. ∎

Theorem 7.4. *Every slice regular function on a slice-open set is path-slice.*

Proof. Let Ω be a slice-open set and $f:\Omega\to\mathbb{H}$ be a slice regular function. We show that f is path-slice. To this end, by Proposition 7.2 we only need to verify (7.3), namely we need to choose $p_\gamma\in\mathbb{H}^{2\times 1}$ such that

$$f\circ\gamma^I(1)=(1,I)p_\gamma$$

for any $\gamma\in\mathscr{P}(\mathbb{C}^+)$ and $I\in\mathbb{S}$ with $\gamma^I\subset\Omega$. We have to treat three cases.

Case 1: Let $\mathscr{B}$ be as in (7.7) and $\gamma\in\mathscr{B}$. In virtue of (7.7), there exist $J,K\in\mathbb{S}$ such that

$$J\neq K,\quad \gamma^J,\gamma^K\subset\Omega.$$

Take U_J and U_K such that

$$\gamma^J\subset U_J,\quad \gamma^K\subset U_K,$$

and U_J is a domain in Ω_J and U_K is a domain in Ω_K.

Let us set $\mathbb{J}=(J,K)$ and $\mathbb{U}=(U_J,U_K)$. We consider the function

$$g=f|_{U_J\cup U_K}.$$

This function satisfies the conditions in the Extension Theorem 6.2. Therefore, $g|_{\mathbb{U}_s^+}$ has a slice regular extension $\widetilde{g}$ over the slice-connected component W of $\mathbb{U}_{s,\mathbb{J}}^{+\Delta}\cap\Omega$ containing $\gamma(0)$. By the Identity Principle (see Theorem 2.10), we have $f=\widetilde{g}$ on W. Since $\widetilde{g}$ is slice on W, it follows that f is slice on W.

Recall that $\gamma\in\mathscr{P}(\mathbb{C}^+)$. By construction we have

$$\gamma^J,\gamma^K\subset\mathbb{U}_{s,\mathbb{J}}^{+\Delta}.$$

This implies that for any $L\in\mathbb{S}$,

$$\gamma^L\subset\mathbb{U}_{s,\mathbb{J}}^{+\Delta}.$$

Then for any $I\in\mathbb{S}$ with $\gamma^I\subset\Omega$,

$$\gamma^I\subset\mathbb{U}_{s,\mathbb{J}}^{+\Delta}\cap\Omega.$$

Since $\gamma^I(0)\in W$ and W is a slice-connected component of $\mathbb{U}_{s,\mathbb{J}}^{+\Delta}\cap\Omega$, we thus conclude

$$\gamma^I\subset W.$$

Due to the fact that f is slice on W, Proposition 5.2 (ii) implies that there exists a function $F_\gamma:\mathbb{R}^2\to\mathbb{H}^{2\times 1}$ such that

$$f(x_\gamma+y_\gamma I)=(1,I)F_\gamma(x_\gamma,y_\gamma) \tag{7.9}$$

for any $I\in\mathbb{S}$ with $\gamma^I\subset\Omega$, where we have written $\gamma(1)=x_\gamma+y_\gamma i$ for some $x_\gamma,y_\gamma\in\mathbb{R}$. Finally, we set

$$p_\gamma:=F_\gamma(x_\gamma,y_\gamma),\quad \gamma\in\mathscr{B}. \tag{7.10}$$

Case 2: Let $\mathcal{A}$ be as in (7.6) and $\gamma \in \mathcal{A}$. In this case, we take

$$p_\gamma := (f \circ \gamma^{I_\gamma}(1), 0)^T, \quad \gamma \in \mathcal{A}, \tag{7.11}$$

where I_γ is the unique imaginary unit $I \in \mathbb{S}$ such that $\gamma^I \subset \Omega$.

Case 3: Let $\gamma \notin \mathcal{A} \cup \mathcal{B}$. In this case, we take $p_\gamma := (0, 0)^T$.

With the choice of p_γ above, it is clear that p_γ satisfies (7.3) as desired. ∎

Proof of Theorem 2.11. It is a direct consequence of Proposition 7.2 and Theorem 7.4. ∎

Remark 7.5. A slice regular function on a slice-open set is not necessarily a slice function. To provide an example, let us fix $J \in \mathbb{S}$ and consider $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ defined by

$$f(q) = \begin{cases} 0, & q \in \mathbb{C}_J \setminus \mathbb{R}, \\ 1, & \text{otherwise}. \end{cases}$$

Then f is a slice regular function defined on the slice-open set $\mathbb{H} \setminus \mathbb{R}$ but it is not a slice function. Indeed, f is a constant in each slice $\mathbb{C}_I$, $I \in \mathbb{C}_I$, so that f is a slice regular function. Suppose that f is a slice function. If $x + yI \in \mathbb{H} \setminus \mathbb{R}$ with $I \neq \pm J$, then $f(x + yJ) = f(x - yJ) = 0$ and we would have

$$f(x + yI) = (1, I) \begin{pmatrix} 1 & J \\ 1 & -J \end{pmatrix}^{-1} \begin{pmatrix} f(x + yJ) \\ f(x - yJ) \end{pmatrix} = 0.$$

However, $x + yI \notin \mathbb{C}_J \setminus \mathbb{R}$, so that $f(x + yI) = 1$, which is a contradiction.

Proposition 7.6. *The set of slice functions and the set of path-slice functions on an axially symmetric slice-path-connected set which intersects with* $\mathbb{R}$ *contain the same elements.*

Proof. Let Ω be an axially symmetric slice-path-connected set with $\Omega_{\mathbb{R}} \neq \emptyset$. According to Proposition 7.3, we just need to prove that any path-slice function on Ω is slice. Let $f : \Omega \to \mathbb{H}$ be a path-slice function and let us prove that f is slice. Since Ω is slice-path-connected, for any $z \in \Omega_{\mathbb{R}}$ and $q \in \Omega$ there is a slice-path α in Ω from z to q.

We write $q = x + yI$ for some $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$. Since

$$\mathbb{H} \setminus \mathbb{C}_I = \bigcup_{J \in \mathbb{S} \setminus \{\pm I\}} \mathbb{C}_J^+$$

is slice-open, the preimage $\alpha^{-1}(\mathbb{C}_I)$ is closed in $[0, 1]$. Let $[t, 1]$ for some $t \in [0, 1]$ be the connected component of $\alpha^{-1}(\mathbb{C}_I)$ containing 1.

We consider the path $\gamma : [0, 1] \to \mathbb{C}$ defined by

$$\gamma(s) := \mathcal{P}_I^{-1} \circ \alpha(t + (1 - t)s).$$

It is clear that γ is in $\mathscr{P}(\mathbb{C})$ and γ^I is a path from $\alpha(t)$ to q.

Since Ω is axially symmetric, we have $\gamma^J \subset \Omega$ for any $J \in \mathbb{S}$. Since

$$\gamma^J(1) = x + yJ, \quad \forall J \in \mathbb{S},$$

it follows from Proposition 7.2 (iv) that

$$f(x+yL) = (1, L)\begin{pmatrix} 1 & J \\ 1 & K \end{pmatrix}^{-1} \begin{pmatrix} f(x+yJ) \\ f(x+yK) \end{pmatrix}$$

for any $L, J, K \in \mathbb{S}$ with $J \neq K$. This implies that f is slice by Proposition 5.2 (iii). ■

Remark 7.7. We note that the set of axially symmetric slice-path-connected open sets with intersection with $\mathbb{R}$ coincide with the set of axially symmetric s-domains. In fact, it is immediate that any axially symmetric s-domain Ω is an axially symmetric slice-path-connected open set. To prove the converse, we consider an axially symmetric slice-path-connected open set Ω with $\Omega_{\mathbb{R}} \neq \emptyset$, and we show that it is an axially symmetric s-domain. First of all, we note that for any $I \in \mathbb{S}$ and any $p, q \in \Omega_{\mathbb{R}}$, there is a path in Ω_I from p to q. We define the map $\mathscr{P} : \mathbb{H} \to \mathbb{C}$ by

$$\mathscr{P}(x+yJ) := \begin{cases} x, & y = 0, \\ x + |y|i, & y \neq 0. \end{cases}$$

Since Ω is slice-path-connected, there is a slice-path γ from p to q. Then $[\mathscr{P}(\gamma)]^I$ is a path in Ω_I from p to q. Let now $p \in \Omega_I$ and $q \in \Omega_{\mathbb{R}}$, one may show that there is a path γ in $\mathbb{C}_I$ from p to q by reasoning as in the proof of Proposition 3.8 (ii), and similarly, for any $p, q \in \Omega_I$, there is a path in $\mathbb{C}_I$ from p to q. Hence Ω_I is a domain and so Ω is an axially symmetric s-domain.

Remark 7.8. Suppose that Ω in Theorem 2.11 is an axially symmetric s-domain in $\mathbb{H}$. For any $q = x + yI \in \Omega$, there exists a point $p \in \Omega_{\mathbb{R}}$ and a path γ in $\mathbb{C}$ such that γ^I is a path from p to q. Since Ω is axially symmetric, we know that, for all $K \in \mathbb{S}$, $\gamma^K \subset \Omega$ and $\gamma^K(1) = x + yK$. By Theorem 2.11, we have

$$f(x+yI) = (I-K)(J-K)^{-1} f(x+yJ) + (I-J)(K-J)^{-1} f(x+yK) \tag{7.12}$$

for any $J, K \in \mathbb{S}$ with $J \neq K$. This means that Theorem 2.11 recovers the classical representation formula [4, Theorem 3.2].

8. Counterexample on non-axially-symmetric domains

In this section, we give an example to illustrate that the classical representation formula may not hold for non-axially-symmetric domains.

Let $s \in [0, 1]$ be fixed. Define a ray $\gamma_s : [0, 1) \to \mathbb{C}$ by

$$\gamma_s(t) := \frac{i}{2} + \frac{t}{1-t} e^{i(\pi/4 + s\pi/2)}.$$

Geometrically, the ray starts from $i/2$ to ∞ and the angle between the ray and the positive real axis is $\pi/4 + s\pi/2$.

For any continuous function $\varphi : \mathbb{S} \to [0,1]$, we define a continuous function $F : \mathbb{S} \times [0,1) \to \mathbb{H}$ by

$$F(I,s) = \mathcal{P}_I \circ \gamma_{\varphi(I)}(s).$$

The complement of the image of F is denoted by

$$\Omega_\varphi := \mathbb{H} \setminus F(\mathbb{S} \times [0,1)).$$

Proposition 8.1. *The set Ω_φ is an s-domain and*

$$\Omega_\varphi \cap \mathbb{S} = \mathbb{S} \setminus \varphi^{-1}(1/2).$$

Proof. (i) For any $I \in \mathbb{S}$, we denote

$$\gamma_\varphi[I] := \mathcal{P}_I \circ \gamma_{\varphi(I)}([0,1)).$$

Then $\gamma_\varphi[I]$ is an image of a ray in $\mathbb{C}_I$ from $I/2$ to ∞. And the angle between the ray and the positive real axis is

$$\pi/4 + \varphi(I)\pi/2.$$

By definition,

$$F(\mathbb{S} \times [0,1)) = \bigcup_{I \in \mathbb{S}} \gamma_\varphi[I]$$

and

$$\Omega_\varphi = \mathbb{H} \setminus \bigcup_{I \in \mathbb{S}} \gamma_\varphi[I]. \tag{8.1}$$

Since

$$\mathcal{P}_I \circ \gamma_{\varphi(I)}(t) := \frac{I}{2} + \frac{t}{1-t} e^{\varphi(I)\pi I/2 + \pi I/4}, \quad \forall t \in [0,1),$$

we have

$$\left[\bigcup_{K \in \mathbb{S}} B\left(\frac{K}{2}, \frac{t}{1-t}\right)\right] \cap \mathcal{P}_I \circ \gamma_{\varphi(I)}[0,1) = \mathcal{P}_I \circ \gamma_{\varphi(I)}[0,t)$$

for any $t \in (0,1)$ and $I \in \mathbb{S}$. Taking the union over all $I \in \mathbb{S}$, we get

$$\left[\bigcup_{K \in \mathbb{S}} B\left(\frac{K}{2}, \frac{t}{1-t}\right)\right] \cap F(\mathbb{S} \times [0,1)) = F(\mathbb{S} \times [0,t)). \tag{8.2}$$

Denote

$$A_t := \left[\bigcup_{K \in \mathbb{S}} B\left(\frac{K}{2}, \frac{t}{1-t}\right)\right] \cap (\mathbb{H} \setminus F(\mathbb{S} \times [0,t])). \tag{8.3}$$

Since F is continuous, the set $F(\mathbb{S} \times [0,t])$ is compact, so that A_t is open.

By (8.2) and (8.3), we have

$$A_t = \left[\bigcup_{K \in \mathbb{S}} B\left(\frac{K}{2}, \frac{t}{1-t}\right)\right] \cap [\mathbb{H} \setminus F(\mathbb{S} \times [0,1))] = \left[\bigcup_{K \in \mathbb{S}} B\left(\frac{K}{2}, \frac{t}{1-t}\right)\right] \cap \Omega_\varphi$$

and

$$\bigcup_{t\in(0,1)} A_t = \bigcup_{t\in(0,1)}\left(\left[\bigcup_{K\in\mathbb{S}} B\left(\frac{K}{2},\frac{t}{1-t}\right)\right]\cap\Omega_\varphi\right)$$
$$=\left(\bigcup_{t\in(0,1)}\left[\bigcup_{K\in\mathbb{S}} B\left(\frac{K}{2},\frac{t}{1-t}\right)\right]\right)\cap\Omega_\varphi=\mathbb{H}\cap\Omega_\varphi=\Omega_\varphi.$$

This means that Ω_φ is open.

Note that $\Omega_\varphi\cap\mathbb{C}_I$ is $\mathbb{C}_I$ with two rays deleted. One is emitting from $I/2$ lying in the upper space $\mathbb{C}_I^+$, while the other one emitting from $-I/2$ lying in the lower space $\mathbb{C}_I^- := \mathbb{C}_{-I}^+$. Therefore, $(\Omega_\varphi)_I$ is a domain in $\mathbb{C}_I$ and path-connected. And since $\Omega_\varphi\cap\mathbb{R}=\mathbb{R}$, Ω_φ is an s-domain.

(ii) Note that for any $I\in\mathbb{S}$, $I\in\gamma_\varphi[I]$ if and only if $\varphi(I)=1/2$. It follows that

$$\Omega_\varphi\cap\mathbb{S}=\mathbb{S}\setminus\varphi^{-1}(1/2). \qquad \blacksquare$$

Let us now fix $J\in\mathbb{S}$. The classical theory of holomorphic functions shows that the function

$$\Psi(z)=\sqrt{2z-J},\quad \forall z\in J/2+\mathbb{R}_+, \tag{8.4}$$

admits a unique holomorphic extension Ψ_s over $\mathbb{C}_J\setminus(\gamma_s[J]\cup\gamma_s[-J])$, where $\gamma_s[J] := \mathcal{P}_J\circ\gamma_s([0,1))$ for any $s\in[0,1]$.

Remark 8.2. The function Ψ_s has the following properties.

(i) For any $s,t\in[0,1]$,

$$\Psi_s|_{\mathbb{R}} = \Psi_t|_{\mathbb{R}}\,.$$

(ii) For any $s\in[0,1]$, we have

$$\Psi_s(-J)=\sqrt{3}\,e^{-J\pi/4}$$

and

$$\Psi_s(J)=\begin{cases}-e^{J\pi/4}, & s\in[0,1/2),\\ e^{J\pi/4}, & s\in(1/2,1].\end{cases}$$

(iii) For any $s\in[0,1]$, denote $\alpha:=\pi/2+s\pi/2$. Then for any $\lambda\in\mathbb{R}_+$

$$\lim_{\theta\to\alpha^-}\Psi_s(J/2+\lambda e^{\theta J})=\sqrt{\lambda}\,e^{\alpha J/2},\qquad \lim_{\theta\to\alpha^+}\Psi_s(J/2+\lambda e^{\theta J})=-\sqrt{\lambda}\,e^{\alpha J/2}.$$

This implies that Ψ_s cannot be extended continuously to any point in $\gamma_s[J]\setminus\{J/2\}$.

Proposition 8.3. *Let $\varphi:\mathbb{S}\to[0,1]$ be a continuous function and Ψ be as in (8.4). Then the function $\Psi_\varphi:\Omega_\varphi\to\mathbb{H}$ defined by*

$$\Psi_\varphi(x+yI):=\frac{1-IJ}{2}\Psi_{\varphi(I)}(x+yJ)+\frac{1+IJ}{2}\Psi_{\varphi(I)}(x-yJ), \tag{8.5}$$

for $y\geq 0$, is the unique slice regular extension of Ψ over Ω_φ. In particular,

$$(\Psi_\varphi)_J=\Psi_{\varphi(J)}.$$

Proof. By direct calculation, $(\Psi_\varphi)_I$ is a holomorphic extension of $\Psi_{\varphi(I)}|_{\mathbb{R}}$. And Ψ_φ is well-defined by Remark 8.2 (i). It is clear that Ψ_φ is the unique slice regular extension of Ψ. ■

Proposition 8.4. *Formula* (7.12) *does not hold, in general, for slice regular functions defined on non-axially-symmetric domains.*

Proof. To show that the statement does not hold in general, we provide a counterexample. Let us recall that $J \in \mathbb{S}$ is fixed in (8.4) and in this proof. Let $\varphi(K) = \frac{1}{2}|K - J|$. Then (using the above notations)

$$\Psi_\varphi(I) \neq \frac{1 - IJ}{2}\Psi_\varphi(J) + \frac{1 + IJ}{2}\Psi_\varphi(-J)$$

for each $I \in \mathbb{S}$ with $1/2 < \varphi(I) < 1$. In fact, since $\varphi(-J) = 1$, we have $I \neq -J$ and $(1 - IJ) \neq 0$. By Remark 8.2 (ii),

$$\Psi_\varphi(J) = -\Psi_{\varphi(I)}(J) \neq 0 \quad \text{and} \quad \Psi_\varphi(-J) = \Psi_{\varphi(I)}(-J).$$

From (8.5) we obtain

$$\Psi_\varphi(I) - \left[\frac{1 - IJ}{2}\Psi_\varphi(J) + \frac{1 + IJ}{2}\Psi_\varphi(-J)\right] = (1 - IJ)\Psi_{\varphi(I)}(J) \neq 0. \qquad ■$$

Definition 8.5. Let $\Omega \subset \mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is called *slice-Euclidean continuous* if for any $U \in \tau(\mathbb{H})$, the preimage $f^{-1}(U)$ is slice-open. In other words, $f : (\Omega, \tau_s(\mathbb{H})) \to (\mathbb{H}, \tau(\mathbb{H}))$ is continuous.

Proposition 8.6. *Let $\Omega \subset \mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is slice-Euclidean continuous if and only if for any $I \in \mathbb{S}$, f_I is continuous.*

Proof. Let $I \in \mathbb{S}$ and $U \in \tau(\mathbb{H})$. If f_I is continuous, then $(f_I)^{-1}(U)$ is open in $\mathbb{C}_I$. Hence

$$f^{-1}(U) = \bigcup_{I \in \mathbb{S}} (f_I)^{-1}(U)$$

is slice-open.

Conversely, if $f : \Omega \to \mathbb{H}$ is slice-Euclidean continuous, then for any $U \in \tau(\mathbb{H})$ we have $f^{-1}(U) \in \tau_s(\mathbb{H})$. This means $(f_I)^{-1}(U)$ is open in $\mathbb{C}_I$ for any $I \in \mathbb{S}$. Therefore, f_I is continuous. ■

Proposition 8.7. *Every slice regular function is slice-Euclidean continuous.*

Proof. This follows directly from Proposition 8.6. ■

Proposition 8.8. *Let Ψ_φ be as in* (8.5)*. For any continuous function $\varphi : \mathbb{S} \to [0, 1]$, there is a unique slice regular extension $\widetilde{\Psi}_\varphi$ of Ψ_φ on*

$$\widetilde{\Omega}_\varphi := \Omega_\varphi \cup \gamma_\varphi[-J].$$

Moreover, $\widetilde{\Psi}_\varphi$ cannot be extended slice-Euclidean continuously to any point in $(\mathbb{H} \setminus \widetilde{\Omega}_\varphi) \cup (\mathbb{H} \setminus \frac{1}{2}\mathbb{S})$.

Proof. Note that $\Psi_\varphi(q) = \sqrt{2q - J}$ for any $q \in J/2 + \mathbb{R}_+$. From complex analysis, we know that Ψ_φ can be extended slice regularly to $\gamma_\varphi[-J]$. This extension, denoted by $\widetilde{\Psi}_\varphi$, is unique by the Identity Principle 2.10. For any $\lambda \in \mathbb{R}_+$, we have

$$\lim_{\theta\to\beta} \Psi_\varphi(-J/2 + \lambda e^{-J\theta}) = \widetilde{\Psi}_\varphi(-J/2 + \lambda e^{-J\beta}), \tag{8.6}$$

where

$$\beta := \pi/2 + \varphi(-J)\pi/2.$$

For any $I \in \mathbb{S} \setminus \{-J\}$, denote

$$\alpha := \pi/2 + \varphi(I)\pi/2.$$

It follows from (8.5) and (8.6) that for any $\lambda \in \mathbb{R}_+$,

$$\lim_{\theta\to\alpha-} \widetilde{\Psi}_\varphi\left(\frac{I}{2}+\lambda e^{I\theta}\right) = \frac{1-IJ}{2} \lim_{\theta\to\alpha-} \Psi_{\varphi(I)}\left(\frac{J}{2}+\lambda e^{J\theta}\right) + \frac{1+IJ}{2}\widetilde{\Psi}_\varphi\left(\frac{-J}{2}+\lambda e^{-J\alpha}\right),$$
$$\lim_{\theta\to\alpha+} \widetilde{\Psi}_\varphi\left(\frac{I}{2}+\lambda e^{I\theta}\right) = \frac{1-IJ}{2} \lim_{\theta\to\alpha+} \Psi_{\varphi(I)}\left(\frac{J}{2}+\lambda e^{J\theta}\right) + \frac{1+IJ}{2}\widetilde{\Psi}_\varphi\left(\frac{-J}{2}+\lambda e^{-J\alpha}\right).$$

By Remark 8.2 (iii), we find

$$\lim_{\theta\to\alpha-} \widetilde{\Psi}_\varphi(I/2 + \lambda e^{I\theta}) \neq \lim_{\theta\to\alpha+} \widetilde{\Psi}_\varphi(I/2 + \lambda e^{I\theta}).$$

According to Proposition 8.6, $\widetilde{\Psi}_\varphi$ cannot be extended slice-Euclidean continuously to any point in $\gamma_\varphi[I] \setminus \{I/2\}$. Since

$$\bigcup_{I\in\mathbb{S}\setminus\{-J\}} (\gamma_\varphi[I] \setminus \{I/2\}) = (\mathbb{H} \setminus \widetilde{\Omega}_\varphi) \cup \left(\mathbb{H} \setminus \tfrac{1}{2}\mathbb{S}\right),$$

it follows that $\widetilde{\Psi}_\varphi$ cannot be extended slice-Euclidean continuously to any point in $(\mathbb{H} \setminus \widetilde{\Omega}_\varphi) \cup (\mathbb{H} \setminus \frac{1}{2}\mathbb{S})$. ∎

Proposition 8.9. *$\widetilde{\Psi}_\varphi$ cannot be slice regularly extended to any st-domain strictly containing $\widetilde{\Omega}_\varphi$.*

Proof. This is a direct consequence of Propositions 8.7 and 8.8. ∎

Remark 8.10. Notice that Ω_φ is not axially symmetric when φ is not constant. Moreover, the only axially symmetric st-open set including Ω_φ is $\mathbb{H}$, since

$$\bigcup_{x+yI\in\Omega_\varphi} x + y\mathbb{S} = \mathbb{H}.$$

By Remark 3.12 and Proposition 8.9, Ψ_φ cannot be slice regularly extended to any axially symmetric s-domain in $\mathbb{H}$ when $\varphi(K) = \frac{1}{2}|K - J|$ for $K \in \mathbb{S}$.

We now provide an example of a slice regular function defined on a slice-open set $\Omega \in \tau_s \setminus \tau_\sigma$. The fact that this is indeed an example as required can be seen following the reasonings in this section.

Example 8.11. For each $s \in (0, 2]$, define

$$W_s := \{x + yi : x = 0 \text{ and } s/8 \le y \le 1/2\} \cup \{x + yi : x \le 0 \text{ and } y = s/8\}$$

and $W_0 := \emptyset$. Fix $I \in \mathbb{S}$. Define

$$\Omega := \bigcup_{J \in \mathbb{S}} [\mathbb{C}_J^+ \setminus \mathcal{P}_J(W_{|J+I|})] \cup \mathbb{R}.$$

Then $\Omega \in \tau_s \setminus \tau_\sigma$. The function Ψ defined in (8.4) can be extended to a slice regular function Ψ' on Ω. And Ψ' cannot be extended slice regularly to any slice-open set strictly containing Ω.

9. Domains of slice regularity

In this section, we consider domains of slice regularity for slice regular functions, analogous to domains of holomorphy of holomorphic functions. It turns out that the σ-balls and axially symmetric slice-open sets are domains of slice regularity.

In contrast to complex analysis of one variable, a slice-open set may fail to be a domain of slice regularity.

We also give a property of domains of slice regularity (see Proposition 9.7).

Definition 9.1. A slice-open set $\Omega \subset \mathbb{H}$ is called a *domain of slice regularity* if there are no slice-open sets Ω_1 and Ω_2 in $\mathbb{H}$ with the following properties:

(i) $\emptyset \neq \Omega_1 \subset \Omega_2 \cap \Omega$.

(ii) Ω_2 is slice-connected and not contained in Ω.

(iii) For any slice regular function f on Ω, there is a slice regular function $\widetilde{f}$ on Ω_2 such that $f = \widetilde{f}$ in Ω_1.

Moreover, if there are slice-open sets $\Omega, \Omega_1, \Omega_2$ satisfying (i)–(iii), then we call $(\Omega, \Omega_1, \Omega_2)$ a *slice-triple*.

In a similar way, we give the following:

Definition 9.2. Let Ω be a slice-open set, $I \in \mathbb{S}$ and U_1, U_2 be open sets in $\mathbb{C}_I$. Then (Ω, U_1, U_2) is called an *I-triple* if

(i) $\emptyset \neq U_1 \subset U_2 \cap \Omega_I$.

(ii) U_2 is connected in $\mathbb{C}_I$ and not contained in Ω_I.

(iii) For any slice regular function f on Ω, there is a holomorphic function $\widetilde{f} : U_2 \to \mathbb{H}$ such that $f = \widetilde{f}$ in U_1.

Lemma 9.3. *Let U be a slice-open set and Ω be an st-domain with $U \subsetneq \Omega$. Then $\Omega \cap \partial_I U_I \neq \emptyset$ for some $I \in \mathbb{S}$, where $\partial_I U_I$ is the boundary of U_I in $\mathbb{C}_I$.*

Proof. Suppose that $\Omega \cap \partial_I U_I = \emptyset$ for each $I \in \mathbb{S}$. Since Ω_I and $\mathbb{C}_I \setminus (\partial_I U_I \cup U_I)$ are open in $\mathbb{C}_I$, so is

$$\begin{aligned}[\Omega \cap (\mathbb{H} \setminus U)]_I &= \Omega_I \cap (\mathbb{C}_I \setminus U_I) = \Omega_I \setminus (\Omega_I \cap U_I) \\ &= \Omega_I \setminus [\Omega_I \cap (\partial_I U_I \cup U_I)] = \Omega_I \cap [\mathbb{C}_I \setminus (\partial_I U_I \cup U_I)].\end{aligned}$$

By definition, $\Omega \cap (\mathbb{H} \setminus U)$ is slice-open. Hence Ω is the disjoint union of the nonempty slice-open sets $\Omega \cap (\mathbb{H} \setminus U)$ and $\Omega \cap U$. Hence Ω is not slice-connected, a contradiction. ∎

Proposition 9.4. *A slice-open set $\Omega \subset \mathbb{H}$ is a domain of slice regularity if and only if for any $I \in \mathbb{S}$ there are no open sets U_1 and U_2 in $\mathbb{C}_I$ such that (Ω, U_1, U_2) is an I-triple.*

Proof. Let Ω be a domain of slice regularity and suppose, towards a contradiction, that there exists an I-triple (Ω, U_1, U_2) for $I \in \mathbb{S}$.

Let $V \subset U_1$ be a nonempty domain in $\mathbb{C}_I$ such that $V \cap \mathbb{R} = \emptyset$, and choose $J \in \{\pm I\}$ such that $V \subset \mathbb{C}_J^+$. It is clear that V is an st-domain and, by definition, (Ω, V, U_2) is a J-triple. Let $f : \Omega \to \mathbb{H}$ be a slice regular function. By Theorem 6.2, where we take $\mathbb{I} := (J, -J)$ and $\mathbb{U} := (U_2, U_2)$, we deduce that $f|_V$ can be extended to a slice regular function $\widetilde{f}$ on a slice-open set $\mathbb{U}_{s,\mathbb{I}}^{+\Delta} \supset U_2$. Let $\widetilde{V}$ be the slice-connected component of $\mathbb{U}_{s,\mathbb{I}}^{+\Delta}$ containing V. Since $U_2 \supset V$ is connected in $\mathbb{C}_J$, we have $\widetilde{V} \supset U_2$. Since $U_2 \not\subseteq \Omega_J$, we have $(\widetilde{V})_J \not\subseteq \Omega_J$, and so $\widetilde{V} \not\subseteq \Omega$. Thus $\emptyset \neq V \subset \widetilde{V} \cap \Omega$, $\widetilde{V}$ is slice-connected and not contained in Ω. Moreover, for any slice regular function f on Ω, there is a slice regular function $\widetilde{f}|_{\widetilde{V}}$ on $\widetilde{V}$ such that $f = \widetilde{f}$ on V.

We conclude that $(\Omega, V, \widetilde{V})$ is a slice-triple, and Ω is not a domain of slice regularity, which is a contradiction.

Now we prove the converse, i.e. a slice-open set Ω is a domain of slice regularity if for each $I \in \mathbb{S}$ there are no open sets U_1 and U_2 in $\mathbb{C}_I$ such that (Ω, U_1, U_2) is an I-triple. So suppose that Ω is not a domain of slice regularity. Then there are slice-open sets Ω_1, Ω_2 such that $(\Omega, \Omega_1, \Omega_2)$ is a slice-triple. Let U be a slice-connected component of $\Omega \cap \Omega_2$ with $U \cap \Omega_1 \neq \emptyset$. By the Identity Principle 2.10, (Ω, U, Ω_2) is also a slice-triple.

We claim that for any $I \in \mathbb{S}$, U_I is a union of some connected components of $\Omega_I \cap (\Omega_2)_I$ in $\mathbb{C}_I$. (This follows from the general fact that if Σ is a slice-open set and U is a slice-connected component of Σ, then for each $I \in \mathbb{S}$, U_I is a union of some connected components of Σ_I.) Then for any $I \in \mathbb{S}$,

$$\partial_I U_I \subset \partial_I((\Omega_2)_I \cap \Omega_I) \subset \partial_I((\Omega_2)_I) \cup \partial_I(\Omega_I).$$

Since $(\Omega_2)_I \cap \partial_I((\Omega_2)_I) = \emptyset$, we have

$$(\Omega_2)_I \cap \partial_I U_I \subset \partial_I \Omega_I. \tag{9.1}$$

By Lemma 9.3, $(\Omega_2)_J \cap \partial_J U_J \neq \emptyset$ for some $J \in \mathbb{S}$. Let $p \in (\Omega_2)_J \cap \partial_J U_J$. By (9.1), we have $p \in \partial_J \Omega_J$, and so $p \notin \Omega_J$. Let Ω_3 be the connected component of $(\Omega_2)_J$ containing p in $\mathbb{C}_J$, and $U' := U_J \cap \Omega_3$. Since $p \in \partial_J U_J$ and p is an interior point of Ω_3 in $\mathbb{C}_J$, we have $p \in \partial_J U'$ and thus $U' \neq \emptyset$. Hence

(i) $\emptyset \neq U' \subset \Omega_3 \cap \Omega_J$.

(ii) Ω_3 is connected in $\mathbb{C}_J$ and not contained in Ω_J (by $p \in \Omega_3$ and $p \notin \Omega_J$).

(iii) Since (Ω, U, Ω_2) is a slice-triple, for any slice regular function f on Ω, there is a slice regular function $f' : \Omega_2 \to \mathbb{H}$ such that $f = f'$ in U. Since $\Omega_3 \subset \Omega_2$ and $U' \subset U$, $f'|_{\Omega_3}$ is a holomorphic function such that $f = f'|_{\Omega_3}$ on U'. This implies that (Ω, U', Ω_3) is a J-triple, a contradiction, and the assertion follows. ∎

Proposition 9.5. *Any axially symmetric slice-open set is a domain of slice regularity.*

Proof. Suppose that an axially symmetric slice-open set Ω is not a domain of slice regularity. By Proposition 9.4, there is an I-triple (Ω, U_1, U_2) for some $I \in \mathbb{S}$. Using the fact that Ω is axially symmetric, and Theorem 6.2 where we set $\mathbb{I} = (I, -I)$, $\mathbb{U} = (\Omega_I, \Omega_I)$, we deduce that any holomorphic function $f : \Omega_I \to \mathbb{C}_I$ can be extended to a slice regular function $\widetilde{f}$ defined on Ω. Since (Ω, U_1, U_2) is an I-triple, the function $f|_{U_1} = \widetilde{f}|_{U_1}$ can be extended to a holomorphic function $\breve{f} : U_2 \to \mathbb{H}$. By the Splitting Lemma 2.5 and the Identity Principle in complex analysis, $\breve{f}$ is a $\mathbb{C}_I$-valued holomorphic function. Thus, for any holomorphic function $f : \Omega_I \to \mathbb{C}_I$, there is a holomorphic function $\breve{f} : U_2 \to \mathbb{C}_I$ such that $f = \breve{f}$ on U_1. We conclude that Ω_I is not a domain of holomorphy in $\mathbb{C}_I$, which is a contradiction. ∎

Proposition 9.6. *Any σ-ball is a domain of slice regularity.*

Proof. Let $p \in \mathbb{H}$ and $r \in \mathbb{R}_+$, let $\Sigma(p, r)$ be the σ-ball with center at p and with radius r. Consider the function $f : \Sigma(p, r) \to \mathbb{H}$ defined by

$$f(q) = \sum_{n \in \mathbb{N}} \left(\frac{q - p}{r} \right)^{*2^n}.$$

We know that $p \in \mathbb{C}_K$ for some $K \in \mathbb{S}$, and from classical complex analysis arguments, $f_K : \Sigma_K(p, r) \to \mathbb{C}_K$ does not extend to a holomorphic function near any point of the boundary of $\Sigma_K(p, r) := \Sigma(p, r) \cap \mathbb{C}_K$. If $\Sigma(p, r)$ is not a domain of slice regularity, then by Proposition 9.4, there is an I-triple $(\Sigma(p, r), U_1, U_2)$ for some $I \in \mathbb{S}$. Let U_1' be a connected component of $\Sigma(p, r) \cap U_2$ in $\mathbb{C}_I$ with $U_1' \cap U_1 \neq \emptyset$. Then $(\Sigma(p, r), U_1', U_2)$ is also an I-triple and $U_2 \cap \partial_I U_1' \subset \partial_I(\Sigma_I(p, r))$. If $p \in \mathbb{C}_I$, the holomorphic function $f_I : \Sigma_I(p, r) \to \mathbb{C}_I$ can be extended to a holomorphic function near a point of the boundary of $\Sigma_I(p, r)$, which is a contradiction.

Otherwise, if $p \notin \mathbb{C}_I$, then $p \in \mathbb{C}_J^+$ for some $J \in \mathbb{S} \setminus \{\pm I\}$. Take $z = x + yL \in U_2 \cap \partial_I U_1'$ with $y > 0$ and $L \in \{\pm I\}$. Then $x + yJ \in \Sigma_J(p, r)$ and $x - yJ \in \partial_J(\Sigma_J(p, r))$. There is $r_1 \in \mathbb{R}_+$ such that

$$B_L(x + yL, r_1) \subset U_2 \quad \text{and} \quad B_J(x + yJ, r_1) \subset \Sigma(p, r) \cap \mathbb{C}_J^+.$$

Using Theorem 6.2 where we set

$$\mathbb{I} = (L, J) \quad \text{and} \quad \mathbb{U} := (\Sigma_L(p,r) \cup B_L(x+yL, r_1), \Sigma_J(p,r)),$$

we conclude that the holomorphic function $f_J : \Sigma_J(p,r) \to \mathbb{C}_J$ can be extended to a holomorphic function on $B_J(x-yJ, r_1) \cup \Sigma_J(p,r)$ near the point $x - yJ \in \partial_J(\Sigma_J(p,r))$, which is a contradiction. Therefore, $\Sigma(p,r)$ is a domain of slice regularity. ∎

Proposition 9.7. *Let $I \in \mathbb{S}$ and let Ω be a domain of slice regularity. If $\gamma \in \mathscr{P}(\mathbb{C})$ and $(J,K) \in \mathbb{S}^2_*$ with $\gamma^J, \gamma^K \subset \Omega$, then $\gamma^I \subset \Omega$ for any $I \in \mathbb{S}$.*

Proof. For contradiction, suppose that $\gamma^I \not\subset \Omega$ for some $I \in \mathbb{S}$. Since γ^I is a slice-path, $(\gamma^I)^{-1}(\Omega)$ is open in $[0,1]$. Set

$$t := \min\{s \in [0,1] : \gamma^I(s) \notin \Omega\}.$$

By assumption, we have

$$z_J := \gamma^J(t) \in \Omega, \quad z_K := \gamma^K(t) \in \Omega,$$

so that

$$B_J(z_J, r) \subset \Omega, \quad B_K(z_K, r) \subset \Omega$$

for some $r \in \mathbb{R}_+$.

Since γ^I is continuous in $\mathbb{C}_I$, there is $t' \in [0,t)$ such that $\gamma^I(t') \in B_I(z_I, r)$, where $z_I := \gamma^I(t)$.

For any slice regular function f on Ω, define a function $g : B_I(z_I, r) \to \mathbb{H}$ by

$$g(x+yI) = (I-K)(J-K)^{-1} f(x+yJ) + (I-J)(K-J)^{-1} f(x+yK) \quad (9.2)$$

for any $x, y \in \mathbb{R}$ with $x + yJ \in B_J(z_J, r)$.

By direct calculation (see [4, proof of Theorem 3.2]), g is holomorphic. Note that $\gamma^I(t') \in \Omega_I \cap B_I(z_I, r)$. By the Representation Formula 2.11 and (9.2), $f = g$ on $\Omega_I \cap B_I(z_I, r) \cap \gamma^I[0,t)$. Then, according to the Identity Principle in complex analysis and Splitting Lemma 2.5, we have

$$g(x+yI) = f(x+yI)$$

for each point $x + yI$ in the connected component of $\Omega_I \cap B_I(z_I, r)$ containing $\gamma^I(t')$, i.e. $g = f$ near $\gamma^I(t')$.

By Corollary 6.3, there is a unique slice regular extension $\widetilde{g}$ on $\Omega_1 := \Sigma(z_I, r)$ of $f|_{B_I(z_I,r)}$. Since $\Sigma(z_I, r)$ and Ω are slice-open, it follows that $\Sigma(z_I, r) \cap \Omega$ is slice-open. Hence the slice-connected component Ω_2 of $\Sigma(z_I, r) \cap \Omega$ containing $\gamma^I(t')$ is an st-domain. By the Identity Principle 2.10, $f = g$ on Ω_2.

It is easy to check that Ω, Ω_1 and Ω_2 satisfy (i)–(iii) in Definition 9.1. Hence Ω is not a domain of slice regularity, which is a contradiction. ∎

10. Final remarks

Let $\widetilde{\Omega}_\varphi$ and $\widetilde{\Psi}_\varphi$ be defined as in Proposition 8.8. Proposition 8.9 implies that $\widetilde{\Psi}_\varphi$ cannot be slice regularly extended to a larger st-domain. However, according to Proposition 9.7, $\widetilde{\Omega}_\varphi$ is not a domain of slice regularity when φ is not constant. This suggests establishing an analogue of the theory of Riemann domains for quaternions and characterize the domain of existence of $\widetilde{\Psi}_\varphi$ which is an analogue of a Riemann domain. Since the slice topology is not Euclidean near $\mathbb{R}$, we cannot consider quaternionic manifolds along the lines used in this paper. Instead, orbifolds over $(\mathbb{H}, \tau_s)$ could be considered.

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