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An anisotropic monotonicity formula, with applications to some segregation problems

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Abstract. We prove an Alt–Caffarelli–Friedman montonicity formula for pairs of functions solving elliptic equations driven by different ellipticity matrices in their positivity sets. As an application, we derive Liouville-type theorems for subsolutions of some elliptic systems, and we analyze segregation phenomena for systems of equations where the diffusion of each density is described by a different operator.

Keywords. Alt–Caffarelli–Friedman monotonicity formula, anisotropic free-boundary problems, strong competition, segregation, Liouville-type theorems

1. Introduction

The Alt–Caffarelli–Friedman (ACF) monotonicity formula is a cornerstone in the theory of free-boundary problems with two or more phases. In its original formulation [1], it establishes that if $u, v \in H^1_{\mathrm{loc}}(B_R) \cap C(B_R)$ are nonnegative, continuous, subharmonic functions with disjoint positivity sets, i.e.

$$u, v \geq 0, \quad -\Delta u \leq 0, \quad -\Delta v \leq 0, \quad u \cdot v \equiv 0 \quad \text{in } B_R \subset \mathbb{R}^N,$$

then the functional

$$r \mapsto J(u, v, x_0, r) = \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla u|^2}{|x - x_0|^{N-2}}\, dx \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x - x_0|^{N-2}}\, dx \tag{1.1}$$

is nondecreasing for $0 < r < \operatorname{dist}(x_0, \partial B_R)$. Here and in the rest of the paper $B_r(x_0)$ (resp. $S_r(x_0) = \partial B_r(x_0)$) denotes the Euclidean ball (resp. sphere) of center x_0 and radius $r > 0$, and we simply write B_r and S_r if $x_0 = 0$.

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The monotonicity formula was introduced in [1], as the key tool to prove the optimal Lipschitz regularity of solutions to a two-phase problem, and since then has been successfully applied in a number of different contexts. Several generalizations of the ACF formula are now available, tailored to deal with elliptic or parabolic equations with variable coefficients [7, 10], and also equations with right hand side [8, 19, 23, 38]; in the latter case, one obtains the so-called almost monotonicity formula. Moreover, a counterpart of the ACF formula is also available for the fractional Laplacian [34–36] and for the p-Laplacian [18]. A common feature of all these contributions is that different phases satisfy equations driven by the same operator, as in the original ACF result.

In this paper we address the case when, on the contrary, u and v satisfy equations involving different uniformly elliptic operators. Only recently have some related free boundary problems been investigated in the literature. In [2], Andersson and Mikayelyan prove partial regularity of the zero set of weak solutions to a quasilinear divergence problem at the jump. Kim, Lee and Shahgholian [20, 21] are concerned with the regularity of solutions and of the nodal set to equations with a jump of conductivity. Moreover, Caffarelli, De Silva and Savin [4] deal with a two-phase anisotropic problem in dimension 2, and prove the Lipschitz regularity of the solutions; finally we quote [6], where the regularity of interfaces of a Pucci type segregation problem is investigated.

In [20], the authors focus on the problem

$$-\operatorname{div}(a_+(x)\nabla u) \le 0, \quad -\operatorname{div}(a_-(x)\nabla v) \le 0 \quad \text{in } B_R$$

for different *scalar* positive functions $a_\pm$. The fact that $a_\pm$ are different scalar functions makes the problem asymmetric, but essentially isotropic, and indeed the authors obtained a perturbed monotonicity formula for the same functional J defined in (1.1). In contrast, we deal with a truly anisotropic two-phase problem, assuming that $\operatorname{div}(A_1\nabla u) \ge 0$ and $\operatorname{div}(A_2\nabla v) \ge 0$ for two positive definite symmetric $N \times N$ matrices A_1, A_2 with constant coefficients. This makes our setting somehow similar to that of [2], where the authors consider weak solutions to $\operatorname{div}(B(w)\nabla w) = 0$, where $B(w) = (A - \mathrm{Id})\chi_{\{w>0\}} + \mathrm{Id}$. As far as we know, the following is the first monotonicity formula of ACF type specifically tailored for the anisotropic case. After some transformations, we can always assume that $A_1 = A$ is diagonal, with lowest eigenvalue equal to 1, and A_2 is the identity (see the proof of Theorem 3.1 below for more details), and we obtain the following result.

Theorem 1.1. *Let $N \ge 2$, let $A \neq \mathrm{Id}$ be an $N \times N$ diagonal matrix with diagonal entries*

$$1 = a_1 \le \cdots \le a_N,$$

and let

$$\Gamma_A(x) := \left(\sum_{i=1}^N \frac{x_i^2}{a_i} \right)^{(2-N)/2}. \tag{1.2}$$

Let $u, v \in H^1_{\mathrm{loc}}(B_R)$ be such that

$$u, v \ge 0, \quad -\operatorname{div}(A\nabla u) \le 0, \quad -\Delta v \le 0, \quad u \cdot v \equiv 0 \quad \textit{in } B_R \subset \mathbb{R}^N.$$

There exists an exponent $\nu_{A,N} \in (0,2)$ *depending on* A *and on* N *such that the functional*

$$r \mapsto J(u,v,x_0,r) = \frac{1}{r^{2\nu_{A,N}}} \int_{B_r(x_0)} \langle A\nabla u, \nabla u\rangle \Gamma_A(x-x_0)\,dx \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x-x_0|^{N-2}}\,dx$$

is nondecreasing for $0 < r < \operatorname{dist}(x_0, \partial B_R)$, $x_0 \in B_R$.

The exponent $\nu_{A,N}$ is explicitly given as the solution of an optimal partition problem, involving eigenvalues of Dirichlet forms on the unit sphere $\mathbb{S}^{N-1}$, as in the original ACF formula. While in the isotropic case $A = \mathrm{Id}$ the optimal value is known to be 2, in the anisotropic case $A \neq \mathrm{Id}$ we shall show that such a spectral optimal value $\nu_{A,N}$ is always smaller than 2 (Lemma 2.4). One may still wonder whether or not it is possible to replace $\nu_{A,N}$ with 2, in the monotonicity formula, with a strategy different to ours, thus improving Theorem 1.1. It is worth noting that the answer is negative in general: the optimal exponent in the anisotropic monotonicity formula is strictly smaller than 2, at least for suitable choices of A. This marks a striking difference with the symmetric-isotropic case, and we refer to Remark 3.3 for a detailed discussion of this point.

One of the difficulties in the proof of Theorem 1.1 is that the natural domains of integration for integrals involving Γ_A are the ellipsoids $\mathcal{E}_r(x_0) := \{|A^{-1/2}(x-x_0)| < r\}$ rather than Euclidean balls. However, using different domains of integration for the two factors of J makes it impossible to reduce the proof of the monotonicity formula to an optimal partition problem, since $\partial\mathcal{E}_r(x_0)$ and $\partial B_r(x_0)$ do not coincide. In order to overcome this obstruction, we introduce suitable weights in the various integrations by parts, in analogy with the approach used in [22] to prove an Almgren monotonicity formula for variable coefficients operators by avoiding the use of radial deformations or Riemannian metric considerations.

Finally, we mention that the possibility of proving a monotonicity formula without assuming the continuity of the phases has already been considered in the literature (for instance in [38]).

Remark 1.2. As already observed, the condition $\nu < 2$ is necessary to prove the monotonicity of the functional $J(u,v,x_0,r)$ with respect to r, for general A. Still, in the present setting one may try to prove the mere boundedness of

$$r \mapsto \frac{1}{r^{2\nu}} \int_{B_r(x_0)} \langle A\nabla u, \nabla u\rangle \Gamma_A(x-x_0)\,dx \int_{B_r(x_0)} \frac{|\nabla v|^2}{|x-x_0|^{N-2}}\,dx$$

for a larger range of ν, in the spirit of the Caffarelli–Jerison–Kenig almost monotonicity formula [8]. It is worth remarking that also such a weaker result cannot hold with the exponent $\nu = 2$ in general. A counterexample to the boundedness for some choices of A is provided by the pair (u,v) in Proposition 3.7.

Applications to segregation problems. The asymptotic analysis of phase separation in reaction-diffusion systems with multiple phases is a relevant field of application of the ACF monotonicity formula, as highlighted in the recent literature, starting from [12, 13]. In particular, the ACF monotonicity formula can be applied to prove a priori bounds of the solutions, independent of the singular perturbation parameter. Typical examples of such singularly perturbed systems fit under the comprehensive model

$$-\Delta u_i = f_i(x, u_i) - \beta g_i(u_1, \dots, u_k) \quad \text{in } \Omega \subset \mathbb{R}^N,$$

where the elliptic operator $-\Delta$ and the functions $\beta g_i \geq 0$ describe, respectively, the diffusion process and the interaction between the densities, and can assume different shapes according to the underlying phenomena. The parameter $\beta > 0$ describes the strength of the competition, and one is particularly interested in understanding the behavior of solutions in the singular limit $\beta \to +\infty$, which is the limit of strong competition leading to total segregation. The following particular cases have been widely investigated in light of their relevance both from the mathematical point of view, and from the physical/biological one:

(i) the *Lotka–Volterra quadratic interaction* $g_i(u_1, \dots, u_k) = u_i \sum_{j \neq i} b_{ij} u_j$; see [9, 13, 15, 31, 33, 37] and references therein;

(ii) the *variational cubic interaction* $g_i(u_1, \dots, u_k) = u_i \sum_{j \neq i} b_{ij} u_j^2$ with $b_{ij} = b_{ji}$ (it possesses a gradient structure since $g_i = \partial_{u_i} G$, where $G(u_1, \dots, u_k) = \frac{1}{2} \sum_{i<j} b_{ij} u_i^2 u_j^2$); see [11, 12, 14, 16, 24, 27, 31–33] and references therein.

In addition, we mention [26, 39] and [17, 35, 36] for analogous studies in fully nonlinear or nonlocal contexts; [5, 28] for long-range interaction models; and [40] for partial results involving a wider class of interaction terms.

Most of these results concern doubly-symmetric settings, in the sense that there is a symmetry both in the interaction terms ($b_{ij} = b_{ji}$) and in the diffusion processes governing the spread of the components (all the equations are driven by the same operator). To our knowledge, asymmetric problems have been studied only in [13, 37] (in the case of Lotka–Volterra interactions with $b_{ij} \neq b_{ji}$), and in [40] (very general, possibly asymmetric, interaction in dimension $N = 2$). In particular, nothing has been known if each density u_i is driven by a different operator L_i, and in what follows we describe our main results in this framework. We shall treat separately both the Lotka–Volterra quadratic interactions, and the variational cubic ones.

Lotka–Volterra quadratic interactions. Let $N, k \geq 2$ be integers, and let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. We consider the system

$$\begin{cases} L_i u_i = \beta u_i \sum_{j \neq i} b_{ij} u_j, & u_i > 0 \quad \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega, \end{cases} \qquad i = 1, \dots, k. \tag{1.3}$$

The operators L_i are of type $L_i = \operatorname{div}(A_i \nabla(\cdot))$, where $A_1, \dots, A_k$ are positive definite symmetric matrices with constant coefficients. The coefficients b_{ij} are positive, so that the system is competitive, and not necessarily symmetric. Regarding the boundary data φ_i, we suppose that they are the restriction on $\partial\Omega$ of $C^{1,\gamma}(\overline{\Omega})$ functions, for some $\gamma \in (0, 1)$, with the property that $\varphi_i \cdot \varphi_j \equiv 0$ in $\overline{\Omega}$.

Theorem 1.3. *Let* $\mathbf{u}_\beta = (u_{1,\beta}, \dots, u_{k,\beta})$ *be a solution of* (1.3) *for fixed* $\beta > 1$. *There exists* $\bar{\nu} \in (0, 2)$ *depending only on* $A_1, \dots, A_k$ *and on* N *such that the following holds:*

For any $\alpha \in (0, \bar{\nu}/2)$, there exists $C > 0$ independent of β such that $\|\mathbf{u}_\beta\|_{C^{0,\alpha}(\overline{\Omega})} \leq C$. Moreover, as $\beta \to +\infty$, up to a subsequence we have

$$\mathbf{u}_\beta \to \mathbf{u} \quad \text{in } C^{0,\alpha}(\overline{\Omega}) \text{ and in } H^1_{\text{loc}}(\Omega), \text{ for every } \alpha \in (0, \bar{\nu}/2),$$

and the limit $\mathbf{u}$ is a vector of nonnegative functions satisfying

$$\begin{cases} L_i u_i = 0 & \text{in } \{u_i > 0\},\ i = 1, \dots, k, \\ u_i \cdot u_j \equiv 0 & \text{in } \Omega \text{ for every } i \neq j, \\ u_i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

Variational cubic interactions. Let $N, k \geq 2$ be integers, and let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. We consider the system

$$\begin{cases} -L_i u_i = f_{i,\beta}(x, u_i) - \beta u_i \sum_{j \neq i} b_{ij} u_j^2, \quad u_i > 0 & \text{in } \Omega, \\ u_{i,\beta} = 0 & \text{on } \partial\Omega, \end{cases} \quad i = 1, \dots, k. \tag{1.4}$$

As in the Lotka–Volterra case, we assume that $L_i = \operatorname{div}(A_i \nabla(\cdot))$, with A_i positive definite, symmetric, with constant coefficients. Moreover, we assume that the functions $f_{i,\beta} : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous, and that the coupling coefficients are positive and symmetric: $b_{ij} = b_{ji} > 0$, so that the system has a variational structure.

Theorem 1.4. *Let $\mathbf{u}_\beta = (u_{1,\beta}, \dots, u_{k,\beta})$ be a solution of (1.3) at fixed $\beta > 1$. Suppose that $\{\mathbf{u}_\beta : \beta > 1\}$ is uniformly bounded in $L^\infty(\Omega)$, and that $f_{i,\beta}$ maps bounded sets of $\Omega \times \mathbb{R}$ into bounded sets of $\mathbb{R}$, uniformly with respect to β. Then there exists $\bar{\nu} \in (0, 2)$ depending only on $A_1, \dots, A_k$ and on N such that the following holds: For every $\alpha \in (0, \bar{\nu}/2)$ there exists $C > 0$ independent of β such that $\|\mathbf{u}_\beta\|_{C^{0,\alpha}(\overline{\Omega})} \leq C$. Moreover, up to a subsequence,*

$$\mathbf{u}_\beta \to \mathbf{u} \quad \text{in } C^{0,\alpha}(\overline{\Omega}) \text{ and in } H^1(\Omega), \text{ for every } \alpha \in (0, \bar{\nu}/2).$$

If $f_{i,\beta} \to f_i$ locally uniformly as $\beta \to +\infty$, then the limit function $\mathbf{u}$ satisfies

$$\begin{cases} -L_i u_i = f_i(x, u_i) & \text{in } \{u_i > 0\},\ i = 1, \dots, k, \\ u_i \cdot u_j \equiv 0 & \text{in } \Omega \text{ for every } i \neq j, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and the domain variation formula

$$2 \int_\Omega \sum_i \left(\langle dY A_i \nabla u_i, \nabla u_i \rangle - f_i(x, u_i) \langle \nabla u_i, Y \rangle \right) - \int_\Omega \operatorname{div} Y \sum_i \langle A_i \nabla u_i, \nabla u_i \rangle = 0. \tag{1.5}$$

Theorems 1.3 and 1.4 can be considered as the perfect anisotropic counterpart of the main results in [13] and [24]. The value $\bar{\nu}$ is given explicitly as the minimum of a finite number of optimal exponents appearing in Theorem 1.1 for different choices of A. In particular, given $A_1, \dots, A_k$, the value $\bar{\nu}$ is the same for both Theorems 1.3 and 1.4. The proofs of these results follow the blow-up strategy developed in [13, 24]. In these contexts, the ACF monotonicity formula is crucially employed to obtain some Liouville-type theorems for the limit configuration in the blow-up.

The results here are not stated in the broader setting, and some extensions could be proved by combining the method presented here with others already used in the literature. For instance, it would not be difficult to add a nonlinear term $f_{i,\beta}$ in (1.3), or to obtain local interior estimates under no regularity or boundedness assumptions on Ω. We refer the interested reader to [27] for further generalizations. We have preferred to treat the prototypical problems (1.3) and (1.4), in analogy with [13, 24], in order to emphasize the main differences and difficulties which one has to face when passing from the isotropic setting to the anisotropic one, without inessential technicalities.

Remark 1.5. In the setting of Theorem 1.3, the existence of $\mathbf{u}_\beta$ can be proved by using Leray–Schauder degree theory as in [13, Theorem 2.1], or fixed point arguments as in [5, Theorem 4.1]. Regarding Theorem 1.4, the existence of $\mathbf{u}_\beta$ can be proved by variational methods (minimization or min-max), under different assumptions on $f_{i,\beta}$.

It is by now well known that the assumption that $\{\mathbf{u}_\beta\}$ is uniformly bounded in $L^\infty(\Omega)$ in Theorem 1.4 is natural and very mild. For instance, it is satisfied by a family of solutions sharing the same variational characterization, at each $\beta > 1$ fixed. In Theorem 1.3, such an assumption is implicit, since it follows from the sign of $\mathbf{u}_\beta$, the subharmonicity, and the boundary conditions.

Once Theorems 1.3 and 1.4 are proven, it is natural to investigate the free-boundary problem arising in the limit: that is, to understand the regularity of the limit configuration $\mathbf{u}$ and of the associated nodal set $\Gamma = \{u_i = 0 \text{ for every } i\}$. From this point of view, the local symmetric case is essentially understood as a consequence of the results in [9, 11, 14, 24, 33]: $\mathbf{u}$ is Lipschitz continuous, and Γ is the union of $C^{1,\alpha}$-hypersurfaces of dimension $N-1$, up to a singular set of dimension $N-2$. Moreover, in a neighborhood of each point x_0 on the regular part of Γ precisely two components of $\mathbf{u}$ are different from 0, and their difference is smooth (reflection law). The anisotropic case offers a number of challenges, and will be the object of future investigations. Here we only address a simplified setting, and in particular a 2-component Lotka–Volterra system, in order to understand the type of result we shall look at. We recall that for systems of two components it is always possible to suppose that $A_2 = \mathrm{Id}$, and that $A_1 = A$ is a diagonal matrix with lowest eigenvalue equal to 1. Thus, we define $\nu_{A,N}$ as in Theorem 1.1. Moreover, if necessary replacing u_1 with $(a_{21}/a_{12})u_1$, we can assume symmetry of the coupling coefficients, $a_{12} = a_{21}$.

Theorem 1.6. *In the previous setting, let* $\mathbf{u} = (u, v)$ *be a limit profile for solutions to* (1.3), *given by Theorem* 1.3. *Then* $w = u - v$ *is a weak solution of the quasi-linear equation*

$$\operatorname{div}(B(w)\nabla w) = 0 \quad \text{in } \Omega, \tag{1.6}$$

with $B(w) = (A - \mathrm{Id})\chi_{\{w>0\}} + \mathrm{Id}$. *Further, w is α-Hölder continuous for every exponent $\alpha \in (0, \nu_{A,N}/2)$. Moreover, $\mu = \Delta w^-$ is a positive and locally finite measure with support in $\{w = 0\}$, and has σ-finite $(N-1)$-dimensional Hausdorff measure. Furthermore, for μ-a.e. $x \in \{w = 0\}$ there exists $r > 0$ such that $\{w = 0\} \cap B_r(x)$ is a $C^{1,\alpha}$ graph.*

The theorem follows directly from the convergence in Theorem 1.3 and the main result in [2] concerning the nodal set of solutions of equations like (1.6). The regularity theory both for the solutions to (1.6), and for their nodal set, seems to be a difficult task. Up to our knowledge, it is only known that weak solutions are Hölder continuous with some exponent. Concerning the nodal set, the only available results are those in [2].

Remark 1.7. From equation (1.6), it is not difficult to deduce that limits of the Lotka–Volterra system (1.3) with two components satisfy the free-boundary condition

$$|\nabla u|\langle A\nu, \nu\rangle = |\nabla v|$$

on the regular part of $\{u = 0 = v\}$; indeed, if $\omega \subset\subset \Omega$ and $\gamma = \{u = 0 = v\} \cap \omega$ is C^1, then

$$0 = \int_{\omega\cap\{u>0\}} \langle A\nabla u, \nabla\varphi\rangle - \int_{\omega\cap\{v>0\}} \langle \nabla v, \nabla\varphi\rangle = \int_\gamma \varphi(\langle A\nabla u, \nu\rangle - \langle \nabla v, \nu_2\rangle)$$

for every $\varphi \in C_c^\infty(\omega)$, with $\nu = -\nu_2 = \frac{\nabla v}{|\nabla v|} = -\frac{\nabla u}{|\nabla u|}$.

Instead, under mild additional assumptions on the nonlinear terms $f_{i,\beta}$, limits of gradient-type systems (1.4) with two components satisfy the free-boundary condition

$$|\nabla u|^2\langle A\nu, \nu\rangle = |\nabla v|^2$$

on the regular part of $\{u = 0 = v\}$. This follows directly from the domain variation formula (1.5), by reasoning as in [16, Proposition 2.1]. Therefore, the limit classes for problems (1.3) and (1.4) do not coincide. This is another interesting difference from the analogous symmetric problems, where the limit profiles can be studied in a unified way, and in particular the free-boundary condition reads $|\nabla u| = |\nabla v|$ for both Lotka–Volterra and variational interactions (we refer to [33, Section 8] for more details).

Structure of the paper. In Section 2 we prove the anisotropic monotonicity formula and some variants concerning nonsegregated solutions of some competitive systems. In Section 3 we deduce various Liouville-type theorems. Such theorems will be used in Sections 4 and 5, which contain the proofs of Theorems 1.3, 1.6 and 1.4.

2. Anisotropic Alt–Caffarelli–Friedman monotonicity formula

Let A be a positive definite $N \times N$ diagonal matrix with constant coefficients, with lowest eigenvalue 1:

$$A := \mathrm{diag}(a_1, \dots, a_N) \quad \text{with } 1 = a_1 \le \cdots \le a_N. \tag{2.1}$$

We first introduce the basic notation which will be used throughout.

- Γ_A denotes the function defined in (1.2). Notice that $\Gamma_A \equiv 1$ in dimension $N = 2$, while for $N \geq 3$ it is (a multiple of) the fundamental solution of $\operatorname{div}(A\nabla\cdot)$ (for the explicit expression of Γ_A, we refer to [3, Chapter 5, p. 214]).
- As in [22], we define

$$\mu(x) := \left\langle A\frac{x}{|x|}, \frac{x}{|x|} \right\rangle, \quad \text{so} \quad 1 \leq \mu \leq a_N, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product.
- Let ν be the outer unit vector on a sphere $S_r(x_0)$. We consider the tangential gradient (computed with respect to the scalar product induced by A)

$$\nabla_\theta^A \varphi := \nabla\varphi - \frac{\langle A\nabla\varphi, \nu\rangle}{\langle A\nu, \nu\rangle}\nu = \nabla\varphi - \frac{\langle A\nabla\varphi, \nu\rangle}{\mu(x - x_0)}\nu. \tag{2.3}$$

In this way, the gradient can be split into its normal and tangential parts as usual:

$$\langle A\nabla\varphi, \nabla\varphi\rangle = \langle A\nabla_\theta^A\varphi, \nabla_\theta^A\varphi\rangle + \frac{\langle A\nabla\varphi, \nu\rangle^2}{\mu(x - x_0)}; \tag{2.4}$$

notice that, in case $A = \mathrm{Id}$, this identity boils down to $|\nabla\varphi|^2 = |\nabla_\theta\varphi|^2 + (\partial_\nu\varphi)^2$.
- For $u \in H^1(\mathbb{S}^{N-1})$, we consider the optimal value

$$\lambda(A, u) := \inf\left\{ \frac{\int_{\mathbb{S}^{N-1}} \langle A\nabla_\theta^A\varphi, \nabla_\theta^A\varphi\rangle\, d\sigma}{\int_{\mathbb{S}^{N-1}} \varphi^2\mu\, d\sigma} : \begin{array}{l} \varphi \in H^1(\mathbb{S}^{N-1} \setminus \{0\}) \text{ and} \\ \mathcal{H}^{N-1}(\{\varphi \neq 0\} \cap \{u = 0\}) = 0 \end{array} \right\},$$

where $d\sigma = d\sigma_x$ and $\mathcal{H}^{N-1}$ stands for the usual $(N-1)$-dimensional Hausdorff measure. Notice that if u is also continuous, then $\lambda(\mathrm{Id}, u)$ is the first eigenvalue of the Laplace–Beltrami operator with homogeneous Dirichlet boundary condition on the open set $\{\xi \in \mathbb{S}^{N-1} : u(\xi) > 0\}$.
- For $u \in H^1(S_r(x_0))$, we set $u_{x_0,r}(\xi) = u(x_0 + r\xi) \in H^1(S_1) \simeq H^1(\mathbb{S}^{N-1})$.
- We define $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \frac{N-2}{2}.$$

- For $N \geq 3$ and $\delta > 0$, we define $\phi_\delta : [0, +\infty) \to (0, +\infty)$ and $\Phi_\delta : \mathbb{R}^N \to (0, +\infty)$ by

$$\phi_\delta(r) = \begin{cases} \frac{N}{2}\delta^{2-N} + \frac{2-N}{2}\delta^{-N}r^2 & \text{if } 0 \leq r \leq \delta, \\ r^{2-N} & \text{if } r > \delta, \end{cases} \qquad \Phi_\delta(x) = \phi_\delta(|x|). \tag{2.5}$$

Then Φ_δ is a C^1 positive superharmonic function in $\mathbb{R}^N$. Therefore, $\Phi_{A,\delta}(x) = \Phi_\delta(A^{-1/2}x)$ is in turn a C^1 positive function in $\mathbb{R}^N$, with the properties that $\operatorname{div}(A\nabla\Phi_{A,\delta}) \leq 0$, and $\Phi_{A,\delta} = \Gamma_A$ in the set $\mathcal{E}_\delta^c = \{|A^{-1/2}x| \geq \delta\}$. The set $\mathcal{E}_r := \{|A^{-1/2}x| < r\}$ is an ellipsoid, and since $a_1 = 1$,

$$B_r \subset \mathcal{E}_r \subset B_{ra_N^{1/2}}.$$

Remark 2.1. It is convenient to observe that

$$\frac{a_2}{a_N}\int_\omega |\nabla_\theta\varphi|^2\,d\sigma \le \int_\omega \langle A\nabla_\theta^A\varphi, \nabla_\theta^A\varphi\rangle\,d\sigma \le a_N a_{N-1}\int_\omega |\nabla_\theta\varphi|^2\,d\sigma$$

for any $\varphi \in H^1(\omega)$ any $\omega \subset \mathbb{S}^{N-1}$, so that $\int_\omega \langle A\nabla_\theta^A\varphi, \nabla_\theta^A\varphi\rangle$ is a seminorm in $H^1(\omega)$, equivalent to the standard one $\int_\omega |\nabla_\theta\varphi|^2$. The above inequality can be easily checked:

$$\begin{aligned}\int_\omega \langle A\nabla_\theta^A\varphi, \nabla_\theta^A\varphi\rangle &= \int_\omega\left(\langle A\nabla\varphi, \nabla\varphi\rangle - \frac{\langle A\nabla\varphi, \nu\rangle^2}{\langle A\nu, \nu\rangle}\right)\\ &= \int_\omega\left(\sum_i a_i\varphi_{x_i}^2 - \frac{(\sum_i a_i\varphi_{x_i}x_i)^2}{\sum_i a_i x_i^2}\right)\\ &= \int_\omega \frac{1}{\sum_i a_i x_i^2}\sum_{i<j} a_i a_j(\varphi_{x_i}x_j - \varphi_{x_j}x_i)^2,\end{aligned}$$

and similarly

$$\int_\omega |\nabla_\theta\varphi|^2 = \int_\omega \sum_{i<j}(\varphi_{x_i}x_j - \varphi_{x_j}x_i)^2.$$

2.1. Monotonicity formula in dimension $N \ge 3$

For $N \ge 3$, we define

$$I_A(u, x_0, r) = \int_{B_r(x_0)} \langle A\nabla u, \nabla u\rangle\Gamma_A(x - x_0)\,dx. \tag{2.6}$$

Lemma 2.2. *Let $N \ge 3$, and let $u \in H^1_{\mathrm{loc}}(B_R)$ be nonnegative and such that* $\operatorname{div}(A\nabla u) \ge 0$ *in B_R. Then, for almost every $r > 0$ such that $B_r(x_0) \subset\subset B_R$, we have*

$$I_A(u, x_0, r) \le \frac{a_N^{N/2} r}{2\gamma(\lambda(A, u_{x_0,r}))}\int_{S_r(x_0)} \langle A\nabla u, \nabla u\rangle\Gamma_A(x - x_0)\,d\sigma.$$

Proof. In order to simplify notation, we consider $x_0 = 0$, and we often omit the dependence on x_0 and A for most of the quantities.

Let u_ε be a mollification of u, which still satisfies $\operatorname{div}(A\nabla u_\varepsilon) \ge 0$ and $u_\varepsilon \ge 0$. By using the coarea formula, it is not difficult to check that

(i) for almost every $r \in (0, R)$ the restrictions of u and of $\partial_{x_i}u$ $(i = 1, \ldots, k)$ to S_r are well defined, are in $L^2(S_r)$, and $u|_{S_r} \in H^1(S_r)$;

(ii) for almost every $r \in (0, R)$ the restrictions of u_ε and of $\partial_{x_i}u_\varepsilon$ to S_r strongly converge to those of u and of $\partial_{x_i}u$ in $L^2(S_r)$ as $\varepsilon \to 0^+$.

We consider $r \in (0, R)$ such that both (i) and (ii) hold, and prove the lemma for those r. Let $\delta > 0$ be such that

$$\{|A^{-1/2}x| \le \delta\} \subset\subset B_r.$$

In this way, we have $\Phi_{A,\delta} = \Gamma_A$ in a neighborhood of S_r. We also recall that $\mu = \langle A\nu, \nu\rangle$ on S_r. By testing the equation for u_ε with $u_\varepsilon\Phi_{A,\delta}$ in B_r, and recalling that

$\mathrm{div}(A\nabla\Phi_{A,\delta}) \le 0$, we obtain

$$\int_{B_r} \langle A\nabla u_\varepsilon, \nabla u_\varepsilon\rangle \Phi_{A,\delta} \le \int_{S_r} \Gamma_A u_\varepsilon \langle A\nabla u_\varepsilon, \nu\rangle - \frac{1}{2}\int_{B_r} \langle \nabla(u_\varepsilon^2), A\nabla\Phi_{A,\delta}\rangle$$
$$\le \int_{S_r} \Gamma_A u_\varepsilon \langle A\nabla u_\varepsilon, \nu\rangle - \frac{1}{2}\int_{S_r} u_\varepsilon^2 \langle A\nabla\Gamma_A, \nu\rangle.$$

By taking the limit first as $\varepsilon \to 0^+$ and then as $\delta \to 0^+$ (thanks to (i) and (ii)), we deduce that

$$I(u,r) \le \int_{S_r} u\Gamma_A \langle A\nabla u, \nu\rangle - \int_{S_r} \frac{u^2}{2}\langle A\nabla\Gamma_A, \nu\rangle. \tag{2.7}$$

Now, on S_r we have

$$-\langle A\nabla\Gamma_A, \nu\rangle = (N-2)\Big(\sum_i \frac{x_i^2}{a_i}\Big)^{-N/2}|x| \le (N-2)\frac{a_N^{N/2}}{r^{N-1}}.$$

Thus, recalling also that $\mu = |x|^{-2}\sum_i a_i x_i^2 \ge a_1 = 1$, we have

$$-\int_{S_r} \frac{u^2}{2}\langle A\nabla\Gamma_A, \nu\rangle \le \frac{(N-2)a_N^{N/2}}{2r^{N-1}}\int_{S_r} u^2\mu. \tag{2.8}$$

Similarly,

$$\int_{S_r} \Gamma_A u \langle A\nabla u, \nu\rangle \le \frac{a_N^{(N-2)/2}}{r^{N-2}}\int_{S_r} |u|\,|\langle A\nabla u, \nu\rangle|$$
$$\le \frac{a_N^{N/2}}{r^{N-2}}\left[\frac{\alpha}{2r}\int_{S_r} u^2\mu + \frac{r}{2\alpha}\int_{S_r}\frac{\langle A\nabla u, \nu\rangle^2}{\mu}\right] \tag{2.9}$$

with $\alpha > 0$ to be conveniently chosen later (in the last step, we have used $a_N \ge a_1 = 1$). By combining (2.7)–(2.9), we obtain

$$I(u,r) \le \frac{a_N^{N/2}}{2r^{N-3}}\left[\frac{N-2+\alpha}{r^2}\int_{S_r} u^2\mu + \frac{1}{\alpha}\int_{S_r}\frac{\langle A\nabla u, \nu\rangle^2}{\mu}\right]$$
$$\le \frac{a_N^{N/2}}{2r^{N-3}}\left[\frac{N-2+\alpha}{\lambda(A,u_r)}\int_{S_r} \langle A\nabla_\theta^A u, \nabla_\theta^A u\rangle + \frac{1}{\alpha}\int_{S_r}\frac{\langle A\nabla u, \nu\rangle^2}{\mu}\right],$$

where we have used the definition of $\lambda(A,u_r)$. We now choose $\alpha > 0$ in order to perfectly balance the coefficients: that is, we impose

$$\frac{N-2+\alpha}{\lambda(A,u_r)} = \frac{1}{\alpha}, \quad \text{so} \quad \alpha = \gamma(\lambda(A,u_r)).$$

In this way we deduce that

$$I(u,r) \le \frac{a_N^{N/2}}{2\gamma(\lambda(A,u_r))r^{N-3}}\left[\int_{S_r} \langle A\nabla_\theta^A u, \nabla_\theta^A u\rangle + \int_{S_r}\frac{\langle A\nabla u, \nu\rangle^2}{\mu}\right]$$
$$= \frac{a_N^{N/2} r^{3-N}}{2\gamma(\lambda(A,u_r))}\int_{S_r}\langle A\nabla u, \nabla u\rangle \le \frac{a_N^{N/2} r}{2\gamma(\lambda(A,u_r))}\int_{S_r}\langle A\nabla u, \nabla u\rangle\Gamma_A,$$

where we have used (2.4) and the fact that $\Gamma_A(x) \ge a_1^{(N-2)/2}|x|^{2-N} = r^{2-N}$ on S_r. ∎

Remark 2.3. The lemma is still valid for $A = \mathrm{Id}$. Of course, in that case we have $a_1 = a_N = 1$, and $\Gamma_A(x) = |x|^{2-N}$.

Motivated by Lemma 2.2, we now study the following asymmetric optimal partition problem:

$$\nu_{A,N} := \inf \left\{ a_N^{-N/2} \gamma(\lambda(A,u)) + \gamma(\lambda(\mathrm{Id}, v)) : \begin{array}{c} u, v \in H^1(\mathbb{S}^{N-1}) \\ \int_{\mathbb{S}^{N-1}} u^2 v^2 = 0 \end{array} \right\}, \tag{2.10}$$

with the convention that $\lambda(A,u) = +\infty$ if $u \equiv 0$ on $\mathbb{S}^{N-1}$ (this gives some continuity to $\lambda(A,\cdot)$, since if $\mathcal{H}^{N-1}(\{u_n > 0\}) \to 0$, then $\lambda(A,u_n) \to +\infty$ by the Sobolev inequality). The infimum is nonnegative, since we are minimizing the sum of two nonnegative quantities. We also recall that, in the symmetric case $A = \mathrm{Id}$, it is known that $\nu_{\mathrm{Id},N} = 2$ (Friedland–Hayman inequality[1]), and the optimal value is reached if and only if u and v are 1-homogeneous functions supported on disjoint half-spherical caps (see [25, Chapter 2] and references therein for more details). In contrast to the symmetric case, we are not able to characterize $\nu_{A,N}$ or classify the optimizers. We are only able to exclude that $\nu_{A,N} = 0$, and to bound $\nu_{A,N}$ from above.

Lemma 2.4. *Let* $A \neq \mathrm{Id}$ *be a matrix as in* (2.1). *Then* $0 < \nu_{A,N} < 2$.

Proof. We first prove that $\nu_{A,N} > 0$. Suppose $\nu_{A,N} = 0$, and let (u_n, v_n) be a minimizing sequence. By definition of γ, this implies that there exist $(u_n, v_n) \in H^1(\mathbb{S}^{N-1}) \times H^1(\mathbb{S}^{N-1})$ such that

$$\int_{\mathbb{S}^{N-1}} \langle A\nabla_\theta^A u_n, \nabla_\theta^A u_n \rangle \to 0, \quad \int_{\mathbb{S}^{N-1}} |\nabla_\theta v_n|^2 \to 0,$$
$$\int_{\mathbb{S}^{N-1}} u_n^2 \mu \equiv 1, \quad \int_{\mathbb{S}^{N-1}} v_n^2 \equiv 1, \quad \int_{\mathbb{S}^{N-1}} u_n^2 v_n^2 \equiv 0.$$

Recalling Remark 2.1, we deduce that up to a subsequence, $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{S}^{N-1})$, with strong convergence in $L^2(\mathbb{S}^{N-1})$, and almost everywhere in $\mathbb{S}^{N-1}$. Therefore

$$\int_{\mathbb{S}^{N-1}} \langle A\nabla_\theta^A u, \nabla_\theta^A u \rangle = 0, \quad \int_{\mathbb{S}^{N-1}} |\nabla_\theta v|^2 = 0,$$
$$\int_{S^{N-1}} u^2 \mu \equiv 1, \quad \int_{\mathbb{S}^{N-1}} v^2 \equiv 1, \quad \int_{\mathbb{S}^{N-1}} u^2 v^2 \equiv 0.$$

But then necessarily u and v are positive constants on $\mathbb{S}^{N-1}$ with disjoint positivity sets, which is clearly impossible.

[1] The Friedland–Hayman inequality is usually stated in a slightly different form, involving partitions of the sphere into disjoint open sets. However, in light of the condition $\int_{\mathbb{S}^{N-1}} u^2 v^2 = 0$ in the definition of $\nu_{A,N}$, it is not difficult to check that the inequality is equivalent to the fact that $\nu_{\mathrm{Id},N} = 2$.

Now we show that $\nu_{A,N} < 2$ for $A \neq \mathrm{Id}$ as in (2.1). To this end we choose $u = x_1^+$ as a test function for $\lambda(A, x_1^+)$:

$$\frac{\int_{\omega_+}\left(\langle A\nabla x_1, \nabla x_1\rangle - \frac{\langle A\nabla x_1,\nu\rangle^2}{\langle A\nu,\nu\rangle}\right) d\sigma}{\int_{\omega_+} x_1^2\langle A\nu,\nu\rangle\, d\sigma} = \frac{1}{\int_{\omega_+} x_1^2(x_1^2+\sum_{i>1} a_i x_i^2)\, d\sigma} \times \int_{\omega_+} \frac{\sum_{i>1} a_i x_i^2}{x_1^2+\sum_{i>1} a_i x_i^2}\, d\sigma,$$

where ω_+ denotes the half-spherical cap $\{x_1 > 0\} \cap \mathbb{S}^{N-1}$ (recall that $a_1 = 1$, and $\nu = x$ on $\mathbb{S}^{N-1}$). This proves that

$$\lambda(A, x_1^+) \leq \frac{\int_{\omega_+} \frac{\sum_{i>1} a_i x_i^2}{x_1^2+\sum_{i>1} a_i x_i^2}\, d\sigma}{\int_{\omega_+} x_1^2(x_1^2+\sum_{i>1} a_i x_i^2)\, d\sigma} =: \frac{\varphi(a_2,\dots,a_N)}{\psi(a_2,\dots,a_N)}, \tag{2.11}$$

and we aim to show that

$$\lambda(A, x_1^+) < N-1 = \lambda(\mathrm{Id}, x_1^+) \tag{2.12}$$

for every matrix $A \neq \mathrm{Id}$ as in (2.1). This amounts to showing that the right hand side in (2.11) is strictly smaller than $N-1$ if $A \neq \mathrm{Id}$, which in turn is equivalent to proving that

$$\Phi(a_2,\dots,a_N) := \varphi(a_2,\dots,a_N) - (N-1)\psi(a_2,\dots,a_N) < 0$$

for every $(a_2,\dots,a_N) \in [1,+\infty)^{N-1}$ with at least one component $a_j > 1$. Firstly, it is immediate to check that $\Phi(1,\dots,1) = 0$. Moreover,

$$\frac{\partial\Phi(a_2,\dots,a_N)}{\partial a_k} = \int_{\omega_+} \frac{x_1^2 x_k^2}{(x_1^2+\sum_{i>1} a_i x_i^2)^2}\, d\sigma - (N-1)\int_{\omega_+} x_1^2 x_k^2\, d\sigma < (2-N)\int_{\omega} x_1^2 x_k^2\, d\sigma \leq 0,$$

where the strict inequality follows from the fact that $a_j > 1$ for some j. This means that Φ is decreasing in each of its variables in $[1,+\infty)^{N-1}$, and hence $\Phi(a_2,\dots,a_N) < \Phi(1,\dots,1) = 0$ for every $(a_2,\dots,a_N) \in [1,+\infty)^{N-1}$, with at least one component $a_j > 1$. Claim (2.12) follows.

At this point we proceed with the estimate for $\nu_{A,N}$, by taking the admissible competitor $(u,v) = (x_1^+, x_1^-)$. Since $\lambda(\mathrm{Id}, x_1^-) = N-1$, by (2.12) we have

$$\nu_{A,N} \leq a_N^{-N/2}\gamma(\lambda(A,x_1^+)) + \gamma(\lambda(\mathrm{Id},x_1^-)) < \gamma(N-1) + \gamma(N-1) = 2,$$

which is the desired upper bound. ■

Proof of Theorem 1.1. To simplify notation we do not stress the dependence of the various functionals on u, v and x_0. It is standard to check that I_A and I_{Id} are absolutely continuous functions for $0 < r < \rho = \mathrm{dist}(x_0, \partial B_R)$, and hence a.e. $r \in (0,\rho)$ is a

Lebesgue point of J. Moreover, for a.e. $r \in (0, \rho)$ the restrictions $u|_{S_r(x_0)}$, $\partial_{x_i} u|_{S_r(x_0)}$ and $v|_{S_r(x_0)}$, $\partial_{x_i} v|_{S_r(x_0)}$ are in $L^2(S_r(x_0))$. We compute the derivative of J with respect to the radius, denoted by J', at any point r for which both the above properties are satisfied and in addition Lemma 2.2 holds, and verify that $J'(r) \geq 0$.

We suppose that both $I_A(r) > 0$ and $I_{\mathrm{Id}}(r) > 0$, otherwise the fact that $J'(r) \geq 0$ follows simply from the nonnegativity of J.

Let $u_{x_0,r}(\cdot) = u(x_0 + r\,\cdot)$, and $v_{x_0,r}(\cdot) = v(x_0 + r\,\cdot)$. By assumption $\int_{\mathbb{S}^{N-1}} u_{x_0,r}^2 v_{x_0,r}^2 = 0$, and by Lemma 2.2 (see also Remark 2.3) we have

$$\begin{aligned}
\frac{J'(r)}{J(r)} &= \frac{I_A'(r)}{I_A(r)} + \frac{I_{\mathrm{Id}}'(r)}{I_{\mathrm{Id}}(r)} - \frac{2\nu_{A,N}}{r} \\
&= \frac{\int_{S_r(x_0)} \langle A\nabla u, \nabla u\rangle \Gamma_A}{\int_{B_r(x_0)} \langle A\nabla u, \nabla u\rangle \Gamma_A} + \frac{\int_{S_r(x_0)} |\nabla v|^2 |x|^{2-N}}{\int_{B_r(x_0)} |\nabla v|^2 |x|^{2-N}} - \frac{2\nu_{A,N}}{r} \\
&\geq \frac{2}{r}\big(a_N^{-N/2} \gamma(\lambda(A, u_{x_0,r})) + \gamma(\lambda(\mathrm{Id}, v_{x_0,r})) - \nu_{A,N}\big) \geq 0,
\end{aligned}$$

where the last inequality follows from the very definition of $\nu_{A,N}$. ■

Remark 2.5. We carried out the proof of the monotonicity formulas for disjointly supported nonnegative subsolutions of $\mathrm{div}(A_1 \nabla u) \geq 0$ and $\mathrm{div}(A_2 \nabla v) \geq 0$ with $A_1 = A$, $A_2 = \mathrm{Id}$, with A diagonal. As already mentioned in the introduction, this is not restrictive since we can always reduce to this case with a change of variables. However, one may also proceed directly with A_1 and A_2, defining $\nu(A_1, A_2)$, and try to choose a change of coordinates maximizing the corresponding exponent $\nu(B^t A_1 B, B^t A_2 B)$. We stress that, in any case, the results in Section 3 imply that the optimal value is again smaller than 2 in general (see in particular Remark 3.3). For this reason, we have decided not to pursue this strategy.

2.2. *Monotonicity formula in dimension $N = 2$*

The 2-dimensional case is easier than the higher-dimensional one, since it is not necessary to work with the fundamental solution Γ_A. As a consequence, the optimal partition problem defining the exponent in the monotonicity formula is slightly different.

In dimension $N = 2$ we modify the definition of I_A as

$$I_A(u, x_0, r) = \int_{B_r(x_0)} \langle A\nabla u, \nabla u\rangle \, dx. \tag{2.13}$$

As a consequence, Lemma 2.2 is simplified as follows.

Lemma 2.6. *Let $N = 2$, and let $u \in H^1_{\mathrm{loc}}(B_R)$ be nonnegative and such that* $\mathrm{div}(A\nabla u) \geq 0$ *in B_R. Then, for almost every $r > 0$ such that $B_r(x_0) \subset\subset B_R$, we have*

$$I_A(u, x_0, r) \leq \frac{r}{2\sqrt{\lambda(A, u_{x_0,r})}} \int_{S_r(x_0)} \langle A\nabla u, \nabla u\rangle \, d\sigma.$$

Proof. Let $x_0 = 0$ to simplify notation. We test the inequality for u against u, and integrate by parts:

$$\begin{aligned}\int_{B_r} \langle A\nabla u, \nabla u\rangle &\le \int_{B_r} \operatorname{div}(uA\nabla u) = \int_{S_r} u\langle A\nabla u, \nu\rangle \\ &\le \frac{\sqrt{\lambda(A,u_r)}}{2r}\int_{S_r} u^2\mu + \frac{r}{2\sqrt{\lambda(A,u_r)}}\int_{S_r}\frac{\langle A\nabla u,\nu\rangle^2}{\mu} \\ &\le \frac{r}{2\sqrt{\lambda(A,u_r)}}\left[\int_{S_r}\langle A\nabla_\theta^A u, \nabla_\theta^A u\rangle + \int_{S_r}\frac{\langle A\nabla u,\nu\rangle^2}{\mu}\right] \\ &= \frac{r}{2\sqrt{\lambda(A,u_r)}}\int_{S_r}\langle A\nabla u, \nabla u\rangle,\end{aligned}$$

which is precisely the desired inequality. ■

We slightly modify the definition of $\nu_{A,2}$ according to the previous lemma:

$$\nu_{A,2} := \inf\left\{\sqrt{\lambda(A,u)} + \sqrt{\lambda(\mathrm{Id},v)} : \begin{array}{l} u, v \in H^1(\mathbb{S}^{N-1}), \\ \int_{\mathbb{S}^{N-1}} u^2v^2 = 0. \end{array}\right\}. \tag{2.14}$$

Exactly as in Lemma 2.4, one can show that $0 < \nu_{A,2} < 2$ whenever $A \neq \mathrm{Id}$. At this point one can proceed as in the higher-dimensional case, and prove Theorem 1.1 with the value $\nu_{A,2}$.

2.3. *Perturbed monotonicity formula*

In this subsection we generalize the previous monotonicity formulae in order to deal with nonsegregated subsolutions of a class of elliptic systems. In the symmetric case $A = \mathrm{Id}$, results of this kind are obtained in [13, 24, 29]. We focus only on $N \ge 3$ (as already observed, the case $N = 2$ is a bit simpler) and consider systems of two inequalities such as

$$\begin{cases} -\operatorname{div}(A\nabla u) + u^q g_1(x,v) \le 0 \\ -\Delta v + v^p g_2(x,u) \le 0 \\ u, v \ge 0 \end{cases} \quad \text{in } \mathbb{R}^N, \text{ with } p, q \ge 1, \tag{2.15}$$

under the following assumptions on the continuous functions $g_1, g_2 : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$:

(H1) $\bar g_i(t) := \inf_{x\in\mathbb{R}^N} g_i(x,t)$ is a continuous function of $t \ge 0$, with the property that $\bar g_i(t) > 0$ for any $t > 0$, and $\bar g_i(0) = 0$. Even more, we suppose that $g_i(x,0) = 0$ for every $x \in \mathbb{R}^N$.

(H2) For every $x \in \mathbb{R}^N$, $g_i(x,\cdot)$ is nondecreasing on $[0,+\infty)$.

A prototypical example is

$$g_i(x,t) = \sum_{j=1}^m b_j(x)t^{p_j} \quad \text{with } \inf_{\mathbb{R}^N} b_j > 0 \text{ and } p_j > 0.$$

For (u, v) solving (2.15), $x_0 \in \mathbb{R}^N$ and $r > 0$, we use the notation

$$\begin{aligned} I_1(u,v,x_0,r) &= \int_{B_r(x_0)} \big(\langle A\nabla u, \nabla u\rangle + u^{q+1} g_1(x,v)\big)\Gamma_A(x-x_0)\,dx, \\ I_2(u,v,x_0,r) &= \int_{B_r(x_0)} \big(|\nabla v|^2 + v^{p+1} g_2(x,u)\big)|x-x_0|^{2-N}\,dx. \end{aligned} \tag{2.16}$$

Theorem 2.7 (Perturbed montonicity formula). *Let $(u, v) \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfy* (2.15) *with $g_1, g_2 : \mathbb{R}^N \times [0, +\infty) \to \mathbb{R}$ continuous and satisfying* (H1) *and* (H2). *For any $\varepsilon > 0$ there exist $x_0 \in \mathbb{R}^N$ and $\bar r = \bar r(u, v, \varepsilon) > 0$ such that the function*

$$r \mapsto J(u,v,x_0,r) = \frac{1}{r^{2(\nu_{A,N}-\varepsilon)}} I_1(u,v,x_0,r) I_2(u,v,x_0,r)$$

is nondecreasing for $r > \bar r$.

For the proof, we start with an estimate similar to the one in Lemma 2.2. We introduce

$$\begin{aligned} \Lambda_1(x_0,r) &= \frac{r^2 \int_{S_r(x_0)} (\langle A\nabla^A_\theta u, \nabla^A_\theta u\rangle + u^{q+1} g_1(x,v))\,d\sigma}{\int_{S_r(x_0)} u^2 \mu\,d\sigma}, \\ \Lambda_2(x_0,r) &= \frac{r^2 \int_{S_r(x_0)} (|\nabla_\theta v|^2 + v^{p+1} g_2(x,u))\,d\sigma}{\int_{S_r(x_0)} v^2\,d\sigma}. \end{aligned}$$

Lemma 2.8. *In the above setting, for every $x_0 \in \mathbb{R}^N$ and $r > 0$,*

$$I_1(u,v,x_0,r) \le \frac{a_N^{N/2} r}{2\gamma(\Lambda_1(x_0,r))} \int_{S_r(x_0)} \big(\langle A\nabla u, \nabla u\rangle + u^{q+1} g_1(x,v)\big)\Gamma_A(x-x_0)\,d\sigma_x.$$

Proof. Without loss of generality, we consider $x_0 = 0$, and omit the dependence on A of all the quantities. Let $r > 0$ be such that (i) in the proof of Lemma 2.2 holds; almost every $r \in (0, R)$ is admissible. Recalling the definition of $\Phi_{A,\delta}$ (see (2.5)), we take $\delta > 0$ such that $\{|A^{-1/2}x| \le \delta\} \subset\subset B_r$. By multiplying the inequality for u with $u\Phi_{A,\delta}$, and proceeding as in Lemma 2.2, we obtain

$$\begin{aligned} \int_{B_r} (\langle A\nabla u, \nabla u\rangle + u^{q+1})\Phi_\delta &\le \int_{S_r} \Phi_\delta u\langle A\nabla u, \nu\rangle - \frac{1}{2}\int_{B_r} \langle \nabla(u^2), A\nabla\Phi_\delta\rangle \\ &\le \int_{S_r} \Big(\Gamma u\langle A\nabla u, \nu\rangle - \frac{u^2}{2}\langle A\nabla\Gamma, \nu\rangle\Big). \end{aligned}$$

By taking the limits as $\delta \to 0^+$, we infer that

$$\int_{B_r} \big(\langle A\nabla u, \nabla u\rangle + u^{q+1} g_1(x,v)\big)\Gamma \le \int_{S_r} \Big(\Gamma u\langle A\nabla u, \nu\rangle - \frac{u^2}{2}\langle A\nabla\Gamma, \nu\rangle\Big).$$

At this point we proceed exactly as in Lemma 2.2, simply replacing $\lambda(A, u_r)$ with $\Lambda_1(0, r)$. ∎

We also need a suitable variant of the mean value inequality for A-subharmonic functions.

Lemma 2.9. *Let $u \in C(\mathbb{R}^N) \cap H^1_{\mathrm{loc}}(\mathbb{R}^N)$ be a nonnegative function such that* $\operatorname{div}(A\nabla u) \geq 0$ *in $\mathbb{R}^N$. Then there exists $C > 0$ depending on A and N such that*

$$\frac{1}{r^{N-1}} \int_{S_r} u^2 \mu \geq C u^2(0) \quad \textit{for almost every } r > 0.$$

Proof. Let $\tilde{u}(x) = u(A^{1/2}x)$; we have $\Delta(\tilde{u}^2) \geq 0$ in $\mathbb{R}^N$, and then the mean value inequality yields

$$u^2(0) = \tilde{u}^2(0) \leq \frac{a_N^{N/2}}{|B_1| r^N} \int_{B_{a_N^{-1/2} r}} \tilde{u}^2 = \frac{a_N^{N/2} \det A^{-1/2}}{|B_1| r^N} \int_{\{|A^{-1/2}x| < a_N^{-1/2} r\}} u^2$$

for every $r > 0$. Now the ellipsoid $\{|A^{-1/2}x| < a_N^{-1/2} r\}$ is contained in the ball B_r, so that

$$u^2(0) \leq \frac{a_N^{N/2} \det A^{-1/2}}{|B_1| r^N} \int_{B_r} u^2, \quad \forall r > 0. \tag{2.17}$$

In order to obtain a similar estimate for the boundary integral, we observe that

$$\begin{aligned} &\int_{B_r} \operatorname{div}(A\nabla(u^2))(r^2 - |x|^2) \\ &\qquad = 2 \int_{B_r} \big(u \operatorname{div}(A\nabla u) + \langle A\nabla u, \nabla u\rangle\big)(r^2 - |x|^2) \geq 0. \end{aligned} \tag{2.18}$$

On the other hand,

$$\begin{aligned} \int_{B_r} \operatorname{div}(A\nabla(u^2))(r^2 - |x|^2) &= 2 \int_{B_r} \langle A\nabla(u^2), x\rangle = 2 \int_{B_r} \langle \nabla(u^2), Ax\rangle \\ &= 2r \int_{S_r} u^2 \mu - 2 \sum_i a_i \int_{B_r} u^2 \end{aligned}$$

for almost every $r > 0$. The conclusion follows directly from (2.17) and (2.18). ∎

Proof of Theorem 2.7. The proof is similar to the one of [29, Lemma 5.2] (see also [13, Lemma 7.3], [24, Lemma 2.5]). If $u \cdot v \equiv 0$ in $\mathbb{R}^N$, then we directly apply Theorem 1.1. Thus, we can suppose that there exists $x_0 \in \mathbb{R}^N$ with both $u(x_0) > 0$ and $v(x_0) > 0$. Without loss of generality, we suppose that $x_0 = 0$. By continuity, we deduce that $u \cdot v > 0$ in a neighborhood of x_0, and hence both $I_1(u, v, x_0, r) \neq 0$ and $I_2(u, v, x_0, r) \neq 0$ for every $r > 0$. Let now $r > 0$ be such that $u|_{S_r}$ and $\partial_{x_i} u|_{S_r}$ are in $L^2(S_r)$, and assume moreover r is a Lebesgue point for J; almost every $r > 0$ is admissible. As in the proof of Theorem 1.1, thanks to Lemma 2.8 we have

$$\frac{J'(r)}{J(r)} \geq \frac{2}{r}\big(a_N^{-N/2} \gamma(\Lambda_1(0, r)) + \gamma(\Lambda_2(0, r)) - (\nu_{A,N} - \varepsilon)\big),$$

and the conclusion follows if we show that the right hand side is nonnegative for r sufficiently large. Suppose that this is not true; then there exists $r_n \to +\infty$ such that

$$a_N^{-N/2}\gamma(\Lambda_1(0,r_n)) + \gamma(\Lambda_2(0,r_n)) < \nu_{A,N} - \varepsilon, \tag{2.19}$$

and in particular $\{\Lambda_i(0,r_n)\}$ $(i = 1,2)$ are bounded sequences. Let

$$u_n(x) = \frac{u(r_n x)}{\left(\frac{1}{r_n^{N-1}}\int_{S_{r_n}} u^2\mu\right)^{1/2}}, \quad v_n(x) = \frac{v(r_n x)}{\left(\frac{1}{r_n^{N-1}}\int_{S_{r_n}} v^2\right)^{1/2}}.$$

We have

$$\int_{S_1} \langle A\nabla_\theta^A u_n, \nabla_\theta^A u_n\rangle \le \Lambda_1(0,r_n), \quad \int_{S_1} |\nabla_\theta v_n|^2 \le \Lambda_2(0,r_n),$$

so that $\{u_n\}$ and $\{v_n\}$ are bounded in $H^1(S_1)$, and moreover

$$\begin{aligned}
&\int_{S_1} u_n^{q+1}\bar{g}_1\left(\left(\frac{1}{r_n^{N-1}}\int_{S_{r_n}} v^2\right)^{1/2} v_n\right)\\
&\qquad\le \frac{r_n^2\int_{S_{r_n}} u^{q+1}g_1(x,v)}{\int_{S_{r_n}} u^2\mu}\cdot\frac{1}{r_n^2\left(\frac{1}{r_n^{N-1}}\int_{S_{r_n}} u^2\mu\right)^{(q-1)/2}}\\
&\qquad\le \frac{\Lambda_1(0,r_n)}{r_n^2\left(\frac{1}{r_n^{N-1}}\int_{S_{r_n}} u^2\mu\right)^{(q-1)/2}} \to 0
\end{aligned}$$

as $n \to \infty$, where we have used assumption (H1) and Lemma 2.9. Therefore, we deduce that up to a subsequence $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in $H^1(S_1)$, strongly in $L^2(S_1)$, and almost everywhere, where $\tilde{u}\cdot\tilde{v} \equiv 0$ on S_1: indeed, since v is subharmonic with $v(0) > 0$,

$$\left(\frac{1}{r^{N-1}}\int_{S_{r_n}} v^2\right)^{1/2} \ge Cv(0) =: \delta > 0,$$

and hence by the Fatou lemma and the assumptions on g_1,

$$\begin{aligned}
\int_{S_1} \tilde{u}^{q+1}\bar{g}_1(\delta\tilde{v}) &\le \liminf_{n\to\infty}\int_{S_1} u_n^{q+1}\bar{g}_1(\delta v_n)\\
&\le \liminf_{n\to\infty}\int_{S_1} u_n^{q+1}\bar{g}_1\left(\left(\frac{1}{r_n^{N-1}}\int_{S_{r_n}} v^2\right)^{1/2} v_n\right) = 0,
\end{aligned}$$

so that at each point of S_1 one of $\tilde{u}$ and $\tilde{v}$ must vanish, that is, $\tilde{u}\cdot\tilde{v} \equiv 0$ on S_1.

Coming back to (2.19), by definitions of $\nu_{A,N}$ and γ we obtain

$$\begin{aligned}
\nu_{A,N} &\le a_N^{-N/2}\gamma(\lambda(A,\tilde{u})) + \gamma(\lambda(\mathrm{Id},\tilde{v}))\\
&\le \liminf_{n\to\infty}\left(a_N^{-N/2}\gamma\left(\int_{S_1}\langle A\nabla_\theta^A u_n, \nabla_\theta^A u_n\rangle\right) + \gamma\left(\int_{S_1}|\nabla_\theta v_n|^2\right)\right)\\
&\le \liminf_{n\to\infty}\left(a_N^{-N/2}\gamma(\Lambda_1(0,r_n)) + \gamma(\Lambda_2(0,r_n))\right) \le \nu_{A,N} - \varepsilon,
\end{aligned}$$

which is a contradiction. ∎

3. Liouville-type theorems

In analogy with the symmetric case $A_i = \mathrm{Id}$, the validity of an ACF monotonicity formula allows us to obtain some nonexistence results, both for disjointly supported subsolutions of different linear equations, and for solutions of certain elliptic systems.

3.1. Liouville theorem for disjointly supported functions

In this framework, our main result is the following.

Theorem 3.1. *Let $k, N \geq 2$ be positive integers, and, for $i = 1, \dots, k$, let $u_i \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be nonnegative functions such that*

$$u_i \cdot u_j \equiv 0 \quad \text{in } \mathbb{R}^N \text{ if } i \neq j, \quad -\mathrm{div}(A_i \nabla u_i) \leq 0 \quad \text{in } \mathbb{R}^N,$$

where A_i are positive definite symmetric matrices with constant coefficients. There exists an exponent $\bar{\nu} \in (0, 2)$ depending on N and on $A_1, \dots, A_k$ such that the following hold: Suppose that for every $i = 1, \dots, k$ the functions u_i grow at most like $|x|^{\alpha_i}$, namely

$$|u_i(x)| \leq C(1 + |x|^{\alpha_i}) \quad \text{for every } |x| \in \mathbb{R}^N, \text{ for some } C > 0,$$

with

$$\alpha_i > 0 \quad \text{for every } i, \quad \alpha_i + \alpha_j < \bar{\nu} \quad \text{for every } i \neq j. \tag{3.1}$$

Then $k - 1$ of the functions u_i are identically 0.

In particular, condition (3.1) is satisfied if $\alpha_i = \alpha \in (0, \bar{\nu}/2)$ for every $i = 1, \dots, k$, which gives the counterpart of [13, Proposition 7.2] in the anisotropic framework.

Remark 3.2. We will prove the theorems with a value of $\bar{\nu}$ explicitly given in terms of a finite number of optimal partition problems of type (2.10). In particular, if $k = 2$, $A_1 = A$ and $A_2 = \mathrm{Id}$, then $\bar{\nu} = \nu_{A,N}$.

Remark 3.3. Once Theorem 3.1 is proven, one can introduce the optimal exponent for the Liouville theorem in the following way. First, given $k \geq 2$ and positive definite symmetric matrices $A_1, \dots, A_k$, we define

$$\mathcal{S}_{\nu,N} := \left\{ (u_1, \dots, u_k) \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N) : \begin{array}{l} (u_1, \dots, u_k) \text{ satisfies all the assumptions of} \\ \text{Theorem 3.1, } u_i \text{ grows at most like } |x|^{\alpha_i} \\ \text{with } \alpha_i + \alpha_j < \nu \text{ for every } i \neq j, \\ \text{and at least two components are nontrivial} \end{array} \right\},$$

and then we set

$$\nu_{\mathrm{Liou},N} := \inf\{\nu > 0 : \mathcal{S}_{\nu,N} \neq \emptyset\}. \tag{3.2}$$

Theorem 3.1 implies that $\nu_{\mathrm{Liou},N} \geq \bar{\nu}$.

When $A_i = \mathrm{Id}$ for every i, it follows from [13, Proposition 7.2] that $\nu_{\mathrm{Liou},N} \geq 2$; moreover, $\nu_{\mathrm{Liou},N} \leq 2$ since $(x_1^+, x_1^-, 0, \ldots, 0)$ is a k-uple of nontrivial Lipschitz subharmonic functions with disjoint positivity sets. Hence in the isotropic case $A_i = \mathrm{Id}$ there is a perfect matching between the optimal threshold in the Liouville theorem and the optimal exponent in the ACF monotonicity formula: both are equal to 2.

In the asymmetric case, it is an open problem to establish whether the equality holds or $\nu_{\mathrm{Liou},N} > \bar{\nu}$ is possible, at least for some choices of A_i and N. In particular, even if $\bar{\nu} < 2$ by Lemma 2.4, this does not imply that $\nu_{\mathrm{Liou,N}} < 2$ as well. However, we shall directly prove that in general $\nu_{\mathrm{Liou},N} < 2$. To this end we construct two nontrivial homogeneous functions u, v of degrees α_1 and α_2, with $\alpha_1 + \alpha_2 < 2$, satisfying all the assumptions of Theorem 3.1 for different matrices A_1 and $A_2 = \mathrm{Id}$. In dimension $N \geq 3$, the existence of such functions was already pointed out in [4, p. 479], and one can take u and v with the same degree of homogeneity; instead, in dimension $N = 2$, the degrees have to be different. We shall present the examples in detail in Section 3.3.

The examples are relevant, since if Theorem 1.1 were valid with $\nu_{A,N}$ replaced by 2, then we would able to prove nonexistence as in Theorem 3.1 for all $\alpha_i + \alpha_j < 2$, deducing that $\nu_{\mathrm{Liou},N} \geq 2$. But, as discussed above, this is not true. Therefore, the fact that the optimal exponent in Theorem 1.1 is smaller than 2 is a natural peculiarity of the anisotropic case, and not a limitation of our proof.

Proof of Theorem 3.1. Let us consider the pair u_1, u_2. Since A_2 is positive definite and symmetric, there exist an orthogonal matrix O and a diagonal positive definite matrix D such that $O^t A_2 O = D$. By defining $\bar{u}_i(x) = u_i(OD^{1/2}x)$, it is not difficult to check that $\bar{u}_1$ and $\bar{u}_2$ grow at most like $|x|^{\alpha_1}$ and $|x|^{\alpha_2}$ respectively at infinity, $\bar{u}_1 \cdot \bar{u}_2 \equiv 0$, and

$$-\operatorname{div}(\bar{A}_1 \nabla \bar{u}_1) \leq 0 \quad \text{and} \quad -\Delta \bar{u}_2 \leq 0 \quad \text{in } \mathbb{R}^N,$$

where $\bar{A}_1$ is again a positive definite symmetric matrix. Since $\bar{A}_1$ is positive definite and symmetric, there exist an orthogonal matrix M and a diagonal positive definite matrix $\hat{A}_1$ such that $M^t \bar{A}_1 M = \hat{A}_1$. Without loss of generality, we can suppose that the diagonal elements of $\hat{A}_1$ appear in increasing order on the diagonal, so that $\hat{a} := (\hat{A}_1)_{11}$ is the lowest eigenvalue of $\hat{A}_1$. Let now $u(x) = \bar{u}_1(\hat{a}^{1/2}Mx)$, $v(x) = \bar{u}_2(\hat{a}^{1/2}Mx)$, and $A = (\hat{a}^{-1})\hat{A}_1$. Then u and v grow at most like $|x|^{\alpha_1}$ and $|x|^{\alpha_2}$ respectively; moreover, $u \cdot v \equiv 0$, and

$$-\operatorname{div}(A\nabla u) \leq 0 \quad \text{and} \quad -\Delta v \leq 0 \quad \text{in } \mathbb{R}^N, \tag{3.3}$$

where A is a diagonal matrix as in (2.1). In particular, we notice that $\nu_{12} := \nu_{A,N} \in (0, 2)$ is a well defined value given by the optimal partition problem (2.10).

The above procedure can be carried out for any pair (u_i, u_j) with $i \neq j$ (actually, by construction $\nu_{ij} = \nu_{ji}$), yielding a finite number of ACF exponents $\nu_{ij} \in (0, 2)$. We take

$$\bar{\nu} := \min\{\nu_{ij} : i \neq j\}, \tag{3.4}$$

and prove the theorem for this choice of $\bar{\nu}$.

Suppose for contradiction that two components, say u_1 and u_2, are both nontrivial and satisfy all the assumptions of the theorem. The previous argument shows that there exist two nontrivial nonnegative functions $u, v \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ with disjoint positivity sets, growing at most like $|x|^{\alpha_1}$ and $|x|^{\alpha_2}$ respectively, and satisfying (3.3). Notice that $\alpha_1 + \alpha_2 < \nu_{12} = \nu_{A,N}$. By using the asymmetric monotonicity formula we show that this provides a contradiction, following the same strategy as originally developed in [13, Proposition 7.2].

The segregation condition $u \cdot v \equiv 0$ implies that there exist $x_0 \in \mathbb{R}^N$ and $\bar r > 0$ sufficiently large such that $u(x_0) = v(x_0) = 0$, and both u and v are nonconstant in $B_{\bar r}(x_0)$. In particular, $J(u, v, x_0, \bar r) > 0$, so that, by the monotonicity formula in Theorem 1.1,

$$I_A(u, x_0, r) I_{\mathrm{Id}}(v, x_0, r) \ge C r^{2\nu_{A,N}} \quad \text{for } r > \bar r. \tag{3.5}$$

Let now $r > \bar r$, and consider a radial smooth cut-off function η such that $0 \le \eta \le 1$, $\eta = 1$ in $B_r(x_0)$, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2r}(x_0)$, and $|\nabla \eta| \le C/r$. Let also $\delta > 0$ be such that $\{|A^{1/2}x| \le \delta\} \subset\subset B_r$. By testing the inequality satisfied by u with $\eta^2 \Phi_{A,\delta}(x - x_0) u$ (with $\Phi_{A,\delta}$ defined in (2.5)), we obtain

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \eta^2 \Phi_{A,\delta}(x-x_0) \langle A\nabla u, \nabla u\rangle \\
&\quad \le - \int_{B_{2r}(x_0)} \big(2\eta u \Phi_{A,\delta}(x-x_0)\langle A\nabla u, \nabla \eta\rangle + \eta^2 u \langle A\nabla u, \nabla \Phi_{A,\delta}(x-x_0)\rangle\big) \\
&\quad \le \int_{B_{2r}(x_0)} \big(\tfrac12 \eta^2 \Phi_{A,\delta}(x-x_0)\langle A\nabla u, \nabla u\rangle + 2\Phi_{A,\delta}(x-x_0) u^2 \langle A\nabla \eta, \nabla \eta\rangle\big) \\
&\qquad - \int_{B_{2r}(x_0)} \eta^2 u \langle A\nabla u, \nabla \Phi_{A,\delta}(x-x_0)\rangle.
\end{aligned} \tag{3.6}$$

In order to deal with the last term, we recall that $\mathrm{div}(A\nabla \Phi_{A,\delta}) \le 0$ in $\mathbb{R}^N$, whence

$$\begin{aligned}
0 &\le \int_{B_{2r}(x_0)} \tfrac12 \langle A\nabla \Phi_{A,\delta}(x-x_0), \nabla(u^2\eta^2)\rangle \\
&= \int_{B_{2r}(x_0)} \big(\eta u^2 \langle A\nabla \Phi_{A,\delta}(x-x_0), \nabla \eta\rangle + \eta^2 u \langle A\nabla u, \nabla \Phi_{A,\delta}(x-x_0)\rangle\big).
\end{aligned}$$

Hence (3.6) yields

$$\begin{aligned}
&\int_{B_r(x_0)} \Phi_{A,\delta}(x-x_0)\langle A\nabla u, \nabla u\rangle \\
&\qquad \le \int_{B_{2r}(x_0)\setminus B_r(x_0)} \big(4\Gamma_A(x-x_0) u^2 \langle A\nabla\eta, \nabla\eta\rangle + 2\eta u^2 \langle A\nabla \eta, \nabla \Gamma_A(x-x_0)\rangle\big),
\end{aligned}$$

where we have used the fact that $\nabla \eta \equiv 0$ in $B_r(x_0)$, and $\Gamma_A = \Phi_{A,\delta}$ outside $B_r(x_0)$. By taking the limit as $\delta \to 0^+$, thanks to the Fatou lemma and the growth condition on u we infer that

$$I_A(u, x_0, r) \le \frac{C}{r^2} \int_r^{2r} \frac{\rho^{2\alpha_1}}{\rho^{N-2}} \rho^{N-1}\, d\rho + \frac{1}{r} \int_r^{2r} \frac{\rho^{2\alpha_1}}{\rho^{N-1}} \rho^{N-1}\, d\rho \le C r^{2\alpha_1}.$$

In the same way, by testing the inequality satisfied by v with $\eta^2\Phi_{\mathrm{Id},\delta}(x-x_0)v$, one can show that

$$I_{\mathrm{Id}}(v,x_0,r) \le Cr^{2\alpha_2}.$$

By combining the inequalities with (3.5), we finally conclude that for $r>\bar{r}$,

$$C_1 r^{2\nu_{A,N}} \le I_A(u,x_0,r) I_{\mathrm{Id}}(v,x_0,r) \le C_2 r^{2(\alpha_1+\alpha_2)},$$

which is a contradiction for large r since $\alpha_1+\alpha_2<\nu_{A,N}$. ∎

3.2. Liouville theorem for subsolutions and solutions to certain elliptic systems

Our first goal is to prove nonexistence of nontrivial nonnegative subsolutions for a system with two components.

Theorem 3.4. *Let $N\ge 2$, and let $u,v\in H^1_{\mathrm{loc}}(\mathbb{R}^N)\cap C(\mathbb{R}^N)$ satisfy* (2.15) *in $\mathbb{R}^N$, under assumptions* (H1) *and* (H2) *on the coupling terms g_1 and g_2. Assume moreover that u and v grow at most like $|x|^{\alpha_1}$ and $|x|^{\alpha_2}$ respectively, with*

$$\alpha_1,\alpha_2>0 \quad \textit{and} \quad \alpha_1+\alpha_2<\nu_{A,N}.$$

Then u or v vanishes identically.

Proof. Suppose for contradiction that neither u nor v vanishes identically. Let $0<\varepsilon<\nu_{A,N}-(\alpha_1+\alpha_2)$. Then, by Theorem 2.7, there exist $x_0\in\mathbb{R}^N$ and $C,\bar{r}>0$ such that

$$I_1(u,v,x_0,r) I_2(u,v,x_0,r) \ge Cr^{2(\nu_{A,N}-\varepsilon)} \tag{3.7}$$

for $r>\bar{r}$. On the other hand, let η be a cut-off function as in the proof of Theorem 3.1, and let $\Phi_{A,\delta}$ be defined in (2.5) with $\delta>0$ such that $\{|A^{1/2}x|<\delta\}\subset B_{\bar{r}}$. By testing the inequality satisfied by u (resp. v) with $\eta^2\Phi_{A,\delta}(x-x_0)u$ (resp. $\eta^2\Phi_\delta(x-x_0)v$), we obtain

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \big(\langle A\nabla u,\nabla u\rangle + u^{q+1}g_1(x,v)\big)\Phi_{A,\delta}(x-x_0)\\
&\quad\le \int_{B_{2r}(x_0)} \tfrac{1}{2}\eta^2\Phi_{A,\delta}(x-x_0)\langle A\nabla u,\nabla u\rangle\\
&\qquad+\int_{B_{2r}(x_0)} \big(2\Phi_{A,\delta}(x-x_0)u^2\langle A\nabla\eta,\nabla\eta\rangle - \eta^2 u\langle A\nabla u,\nabla\Phi_{A,\delta}(x-x_0)\rangle\big).
\end{aligned}$$

As in the proof of Theorem 3.1, it is not difficult to deduce that

$$I_1(u,v,x_0,r)\le Cr^{2\alpha_1}$$

for $r>\bar{r}$. In the same way one can estimate $I_2(u,v,x_0,r)$, obtaining a contradiction with (3.7) since $\alpha_1+\alpha_2<\nu_{A,N}-\varepsilon$. ∎

As an application, we present a general Liouville theorem for possibly sign-changing solutions of some elliptic systems with arbitrarily many components. To state our results in full generality, we introduce some notation. Let $k, N \geq 2$ be integers. For an arbitrary $m \leq k$, we say that a vector $\mathbf{b} = (b_0, \dots, b_m) \in \mathbb{N}^{m+1}$ is an *m-decomposition* of k if

$$0 = b_0 < b_1 < \cdots < b_{m-1} < b_m = k;$$

given an m-decomposition $\mathbf{b}$ of k, we set, for $h = 1, \dots, k$,

$$\begin{aligned} I_h &:= \big\{i \in \{1, \dots, d\} : b_{h-1} < i \leq b_h\big\}, \\ \mathcal{K}_1 &:= \{(i, j) \in I_h^2 \text{ for some } h = 1, \dots, m, \text{ with } i \neq j\}, \\ \mathcal{K}_2 &:= \{(i, j) \in I_{h_1} \times I_{h_2} \text{ with } h_1 \neq h_2\}. \end{aligned} \tag{3.8}$$

Let now $\mathbf{u} = (u_1, \dots, u_k) \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfy

$$-\operatorname{div}(A_i \nabla u_i) = -\sum_{\substack{j=1 \\ j \neq i}}^{k} u_i |u_i|^{p_{ij}-1} g_{ij}(x, |u_j|) \quad \text{in } \mathbb{R}^N, \quad i = 1, \dots, d, \tag{3.9}$$

under the following assumptions on the data:

(G1) A_i are positive definite symmetric matrices with constant coefficients;

(G2) $p_{ij} > 0$ for every $i \neq j$, and $p_{ij} \geq 1$ for every $(i, j) \in \mathcal{K}_2$;

(G3) $g_{ij} \equiv 0$ for $(i, j) \in \mathcal{K}_1$, and g_{ij} satisfies assumptions (H1) and (H2) in Theorem 2.7 for every $(i, j) \in \mathcal{K}_2$.

The term $-u_i |u_i|^{p_{ij}-1} g_{ij}(x, |u_j|)$ describes the interaction between u_i and u_j. By introducing an m-decomposition of k, we have divided the components of $\mathbf{u}$ into m groups: $\{u_i : i \in I_1\}, \dots, \{u_i : i \in I_m\}$. Assumption (G3) means that u_i and u_j do not interact ($g_{ij} = 0$) if $(i, j) \in \mathcal{K}_1$, i.e. if u_i and u_j are in the same group; instead, they interact in a competitive way ($g_{ij} > 0$) if $(i, j) \in \mathcal{K}_2$, i.e. if u_i and u_j are in different groups.

Theorem 3.5. *In the above setting, let $\bar{\nu} \in (0, 2)$ be given by Theorem 3.1. Suppose that each function u_i grows at most like $|x|^{\alpha_i}$, where*

$$\alpha_i > 0 \quad \textit{for every } i, \quad \alpha_i + \alpha_j < \bar{\nu} \quad \textit{for every } (i, j) \in \mathcal{K}_2.$$

Then there exists $\ell \in \{1, \dots, m\}$ such that $u_i \equiv 0$ for every $i \in I_h$ with $h \neq \ell$, and u_i is constant for $i \in I_\ell$.

Remark 3.6. A similar Liouville theorem was proved in [27] for a specific choice of g_{ij}. The validity of Theorem 3.5 allows us to extend the validity of Theorems 1.3 and 1.4 in cases when competition takes place among groups of components, as in [27]. We do not insist on this point for the sake of simplicity.

Proof of Theorem 3.5. We show that it is possible to apply Theorem 3.4 to any couple (u_i, u_j) where $i \in I_h$, $j \in I_k$, with $h \neq k$. Then it is necessary that $m-1$ groups of components vanish identically, and the components of the last group are constants (by (G3), they are harmonic and globally Hölder continuous in $\mathbb{R}^N$).

Suppose first that u_i and u_j are also nonnegative. Then

$$\begin{cases} -\operatorname{div}(A_i \nabla u_i) \leq -u_i^{p_{ij}} g_{ij}(x, u_j) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(A_j \nabla u_j) \leq -u_j^{p_{ji}} g_{ji}(x, u_i) & \text{in } \mathbb{R}^N. \end{cases}$$

As in the proof of Theorem 3.1, we may suppose that $A_j = \mathrm{Id}$ and $A_i = A$ is diagonal as in (2.1). Thus, $\nu_{A,N}$ is well defined as in (2.10), and recalling the definition (3.4) of $\bar{\nu}$, we have $2\alpha < \nu_{A,N}$. Therefore, u_i or u_j must vanish identically by Theorem 3.4.

If instead the components can change sign, recalling the assumptions on g_{ij} we have

$$\begin{cases} -\operatorname{div}(A_i \nabla u_i^+) \leq -(u_i^+)^{p_{ij}} g_{ij}(x, u_j^+) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(A_j \nabla u_j^+) \leq -(u_j^+)^{p_{ji}} g_{ji}(x, u_i^+) & \text{in } \mathbb{R}^N, \end{cases}$$

and analogous systems are satisfied by (u_i^+, u_j^-), (u_i^-, u_j^+), (u_i^-, u_j^-). In each case, it is possible to suppose that $A_j = \mathrm{Id}$ and A_i is diagonal as in (2.1). Thus, by applying Theorem 3.4 to all the possible pairs, we deduce that u_i or u_j vanishes identically. ∎

3.3. *Upper estimate on $\nu_{\mathrm{Liou},N}$*

In this section we show that, at least for a suitable choice of A_1 and A_2, the optimal value $\nu_{\mathrm{Liou},N}$ defined in (3.2) is strictly less than 2. This follows directly from the following:

Proposition 3.7. *Let $N \geq 2$. There exists a positive definite diagonal matrix A with constant coefficients, two disjoint open cones $\mathcal{C}_1, \mathcal{C}_2$ of $\mathbb{R}^N$, and two nonnegative and nontrivial homogeneous functions u and v in $H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, of degrees $\alpha_1 > 0$ and $\alpha_2 > 0$ with $\alpha_1 + \alpha_2 < 2$, such that*

$$\operatorname{div}(A \nabla u) = 0 \quad \text{in } \mathcal{C}_1 = \{u > 0\}, \qquad \Delta v = 0 \quad \text{in } \mathcal{C}_2 = \{v > 0\}.$$

Moreover, if $N \geq 3$ we can construct u and v with $\alpha_1 = \alpha_2$.

Proof of Proposition 3.7 *in dimension $N = 2$.* Let $\phi_1, \phi_2 \in (0, \pi/2)$, $\omega_1 = (-\phi_1, \phi_1)$, and $\omega_2 = (\phi_2, 2\pi - \phi_2)$. We consider the eigenvalue problems on the circle

$$\begin{cases} -\varphi'' = \lambda \varphi, \quad \varphi > 0 & \text{in } \omega_1, \\ \varphi = 0 & \text{on } \partial\omega_1, \end{cases} \qquad \begin{cases} -\psi'' = \mu \psi, \quad \psi > 0 & \text{in } \omega_2, \\ \psi = 0 & \text{on } \partial\omega_2. \end{cases}$$

The problems can be explicitly solved, deducing in particular that $\lambda = (\pi/(2\phi_1))^2$, $\mu = (\pi/(2(\pi - \phi_2)))^2$. Let φ_1 and ψ_1 denote the corresponding normalized eigenfunctions, and let $\alpha_1 = \sqrt{\lambda}$ and $\alpha_2 = \sqrt{\mu}$; it is well known that $w = r^{\alpha_1}\varphi_1$ and $v = r^{\alpha_2}\psi_1$ are homogeneous harmonic functions in the cones $\mathcal{D}_1, \mathcal{C}_2$ generated by ω_1 and ω_2, respectively. Notice that $\alpha_1 > 1$ can be made arbitrarily close to 1 by taking ϕ_1 close to $\pi/2$.

Similarly, $\alpha_2 < 1$ can be made close to $1/2$ by taking ϕ_2 close to 0. In particular, for any $0 < \varepsilon < 1/2$ we can take ϕ_1 and ϕ_2 such that

$$\alpha_1 > 1, \quad \alpha_2 < 1, \quad \alpha_1 + \alpha_2 < 3/2 + \varepsilon < 2.$$

Now, for $b \in (0, 1)$, let

$$B = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}, \quad A = B^{-1},$$

and $u(x) = w(B^{1/2}x)$. Then $\mathrm{div}(A\nabla u) = 0$ in $\mathcal{C}_1 = \{x \in \mathbb{R}^2 : B^{1/2}x \in \mathcal{D}_1\} = B^{-1/2}\mathcal{D}_1$, $u = 0$ on $\partial\mathcal{C}_1$, and it is homogeneous of degree λ. Now $\mathcal{C}_1$ is a cone, generated by a set $\omega' \subset \mathbb{S}^1$, and it is not difficult to check that if b is sufficiently small, then $\omega' \subset \mathbb{S}^1 \setminus \omega_2$. Therefore u and v provide the desired example. ∎

Remark 3.8. Notice that, up to exchanging the roles of the variables x_1 and x_2, the matrix A satisfies the structural assumptions (2.1), i.e. it is a diagonal matrix with lower entry equal to 1.

It is interesting that in the previous example u is superlinear and v is sublinear. This means in particular that even if we take b such that $\partial\omega' = \partial\omega_2$, then u and v cannot satisfy a free-boundary condition of the type

$$\partial_\nu u = G(\partial_\nu v, \nu) \quad \text{on } \partial\omega', \text{ with } G \text{ increasing in the first variable.}$$

This is in accordance with the main result in [4], which implies in particular that in dimension $N = 2$ one cannot construct an example where u and v have the same degree of homogeneity less than 1.

Now we consider the case $N \geq 3$. Of course, the two-dimensional example can also be considered in higher dimensions. We think however that it is interesting to produce an example where u and v have the same degree (which is not possible in dimension $N = 2$). The idea of the construction was suggested to us by Daniela De Silva in a personal communication. We start with a preliminary result concerning an eigenvalue problem on the unit sphere $\mathbb{S}^2$. We parametrize the sphere with spherical coordinates $(\varphi, \theta) \in [0, \pi] \times [-\pi, \pi]$ (φ is the polar angle, θ is the azimuthal angle).

Lemma 3.9. *For $\alpha \in (\pi/2, \pi)$, $\beta \in (0, \pi/2)$, let*

$$\omega = \{(\varphi, \theta) \in (\pi/2 - \beta, \pi/2 + \beta) \times (-\alpha, \alpha)\}.$$

There exist α and β such that the first eigenvalue of the problem

$$\begin{cases} -\Delta_{\mathbb{S}^2} u = \lambda u & \text{in } \omega, \\ u = 0 & \text{on } \partial\omega \end{cases}$$

is strictly smaller than 2.

Proof. We separate variables by letting $u(\varphi,\theta) = v(\theta)w(\varphi)$, plug this ansatz in the differential equation for u, and search for a positive solution. The differential equation reads

$$\frac{\sin\varphi\,(\sin\varphi\, w')'}{w} + \lambda \sin^2\varphi = -\frac{v''}{v}.$$

Hence there exists $c \in \mathbb{R}$ such that $v'' + cv = 0$, which together with $v > 0$ and the boundary conditions $v(-\alpha) = 0 = v(\alpha)$ implies that $c = m^2 = (\pi/(2\alpha))^2$ and $v(\theta) = \cos(m\theta)$ (up to a multiplicative constant). At this point we come back to the boundary value problem for w; by changing variable $s = \cos\varphi$, we obtain

$$\begin{cases} -((1-s^2)\bar{w}')' + \frac{m^2}{1-s^2}\bar{w} = \lambda\bar{w} & \text{in } (-\rho,\rho), \\ \bar{w}(-\rho) = 0 = \bar{w}(\rho), \end{cases} \tag{3.10}$$

where $\rho = \cos(\pi/2-\beta) \in (0,1)$ and $\bar{w}(s) = w(\varphi(s))$. This is a typical Sturm–Liouville problem with strictly positive potential $m^2/(1-s^2)$, and hence the existence of a first positive eigenvalue λ_1, together with a first positive normalized eigenfunction w_1, is guaranteed. We need an upper bound on λ_1, and this can be obtained from the variational characterization

$$\lambda_1 = \inf_{\varphi\in H_0^1(-\rho,\rho)\setminus\{0\}} Q_{\rho,m}(\varphi) = \inf_{\varphi\in H_0^1(-\rho,\rho)\setminus\{0\}} \frac{\int_{-\rho}^{\rho}\left((1-s^2)(\varphi')^2 + \frac{m^2}{1-s^2}\varphi^2\right)}{\int_{-\rho}^{\rho}\varphi^2}.$$

By choosing the test function $\psi(s) = \cos(\frac{\pi s}{2\rho})$, we infer that

$$\lambda_1 \le Q_{\rho,m}(\psi) = \frac{\pi^2}{4\rho^2}\int_{-1}^{1}(1-\rho^2t^2)\sin^2\left(\frac{\pi}{2}t\right)dt + m^2\int_{-1}^{1}\frac{\cos^2\left(\frac{\pi}{2}t\right)}{1-\rho^2t^2}\,dt.$$

The right hand side is continuous with respect to $(\rho,m) \in (0,1]\times\mathbb{R}^+$. By taking $\alpha \simeq \pi$ and $\beta \simeq \pi/2$, we can make m arbitrarily close to $1/2$, and ρ arbitrarily close to 1. This means that for such a choice of α and β we have

$$\lambda_1 \le Q_{\rho,m}(\psi) \simeq Q_{1,1/2}(\psi) \approx \frac{\pi^2}{4}\cdot 0.47 + \frac{1}{4}\cdot 1.22 \approx 1.47 < 2,$$

which is the desired result. ■

Proof of Proposition 3.7 in dimension $N \ge 3$. The main idea is to show the existence of a domain ω' on the sphere $\mathbb{S}^2$ that contains more than half of a great circle such that, for suitable $\mu \in (0,1)$ and a positive definite symmetric constant matrix A, the solution of $\operatorname{div}(A\nabla w) = 0$ which vanishes on the cone generated by $\partial\omega'$ has homogeneity μ. If $N = 3$ we can take two complementary domains with this property (for instance those separated by the white line of a typical tennis ball).

We now present the details. Let us consider the half great circle $\gamma_1 = \{x \in \mathbb{S}^2 : x_3 = 0, x_2 > 0\}$, and let $\omega = \{(\varphi,\theta) \in [\pi/2-\beta, \pi/2+\beta]\times[-\delta, \pi+\delta]\}$, where $\delta \in (0,\pi/2)$ is such that $\pi + 2\delta = 2\alpha$, with α and β given by Lemma 3.9. Then the first eigenvalue of the Laplace–Beltrami operator on ω, with homogeneous Dirichlet boundary condition, is smaller than 2, and this implies that the positive harmonic function w in the cone $\mathcal{D}_1$ generated by ω, vanishing on $\partial\omega$, has homogeneity $\mu < 1$. Now, for $b \in (0, 1)$, we consider the diagonal matrices

$$B = \begin{pmatrix} b^2 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = B^{-1},$$

and let $u(x) = w(B^{1/2}x)$. Then $\mathrm{div}(A\nabla u) = 0$ in $\mathcal{C}_1 = \{x \in \mathbb{R}^3 : B^{1/2}x \in \mathcal{D}_1\}$, $u = 0$ on $\partial\mathcal{C}_1$, and it is homogeneous of degree μ. Now, $\mathcal{C}_1$ is a cone, generated by a set $\omega' \subset \mathbb{S}^2$. It is not difficult to check that the set ω' can be included in an arbitrarily small neighborhood of γ_1, by taking b sufficiently small. Now we consider a second band ω_2 of the same type as ω, but surrounding the half great circle $\gamma_2 = \{x \in \mathbb{S}^2 : x_1 = 0,\ x_2 < 0\}$. We fix b so small that $\omega_2 \cap \omega' = \emptyset$, and notice that by Lemma 3.9 the positive harmonic function v in the cone $\mathcal{C}_2$ generated by ω_2, vanishing on $\partial\omega_2$, is homogeneous of degree $\mu < 1$. Thus the pair (u, v) fulfills all the requirements of the theorem (with a matrix A satisfying the structural assumptions (2.1), up to exchanging the coordinates). ∎

4. Spatial segregation of competitive systems: Lotka–Volterra interaction

In this section we prove Theorems 1.3 and 1.6, by following the blow-up method used in [13, Theorem 4]. Before entering the proof, we observe that each $\mathbf{u}_\beta$ is C^1 up to the boundary, and $\{\mathbf{u}_\beta\}$ is uniformly bounded in $L^\infty(\Omega)$, since each $u_{i,\beta}$ is L_i-subharmonic and the boundary data are fixed. Notice also that we can define $\bar\nu = \bar\nu(N, A_1, \dots, A_k) \in (0, 1)$ as in Theorem 3.1.

Lemma 4.1. *Let $w \in H^1(B_{2r}) \cap C(\overline{B_{2r}})$ be a positive subsolution to*

$$-\mathrm{div}(A\nabla w) \le -Mw + \delta \quad \text{in } B_{2r},$$

with $M > 0$, $\delta \ge 0$, and A positive definite, symmetric, with constant coefficients. Then there exist $C, c > 0$ such that

$$\sup_{x \in B_r} w(x) \le C\|w\|_{L^\infty(B_{2r})} e^{-cr\sqrt{M}} + \delta/M.$$

Proof. Let $x_0 \in B_r$. The function $\bar w := w/\|w\|_{L^\infty(B_r(x_0))}$ is a positive subsolution to

$$-\mathrm{div}(A\nabla \bar w) \le -M\bar w + \frac{\delta}{\|w\|_{L^\infty(B_r(x_0))}}, \quad \bar w \le 1 \quad \text{in } B_r(x_0).$$

Let Λ be the maximal eigenvalue of A. Then, as observed in [5, Lemma 5.2], the function

$$z(x) = \sum_{i=1}^N \cosh(\sqrt{M}\, x_i/\Lambda)$$

is a supersolution of $\operatorname{div}(A\nabla z) \le Mz$ in B_r, and satisfies

$$z(x) \ge Ce^{c\sqrt{M}r} \quad \text{for every } x \in S_r$$

for suitable $c, C > 0$ depending on A and N. Let us consider

$$\bar{z}(x) := \frac{z(x - x_0)}{Ce^{c\sqrt{M}r}} + \frac{\delta}{M\|w\|_{L^\infty(B_r(x_0))}}.$$

We have

$$-\operatorname{div}(A\nabla\bar{z}) \ge -M\bar{z} + \frac{\delta}{\|w\|_{L^\infty(B_r(x_0))}} \quad \text{in } B_r(x_0),$$

with $\bar{z} \ge 1$ on $S_r(x_0)$. Then the comparison principle yields

$$w(x_0) \le C\|w\|_{L^\infty(B_r(x_0))} z(0)e^{-c\sqrt{M}r} + \delta/M \le C\|w\|_{L^\infty(B_{2r})}e^{-c\sqrt{M}r} + \delta/M,$$

and we obtain the conclusion by taking the supremum over $x_0 \in B_r$. ■

Lemma 4.2. *Let A be a positive definite symmetric matrix with constant coefficients. Suppose that w is globally α-Hölder continuous in Ω for some $\alpha \in (0, 1)$.*

(i) *If* $\operatorname{div}(A\nabla w) = 0$ *in* $\Omega = \mathbb{R}^N$*, then w is constant.*

(ii) *If* $\operatorname{div}(A\nabla w) = 0$ *in a half-space Ω, and w is constant on the boundary, then it is constant.*

Here and in what follows we say that a function w is *globally α-Hölder continuous* in Ω if its α-Hölder seminorm $[w]_{C^{0,\alpha}(\overline{\Omega})}$ is bounded; notice that we do not ask that $w \in L^\infty(\Omega)$.

Proof. (i) After a rotation and a scaling, we obtain a harmonic function $\tilde{w}$ in $\mathbb{R}^N$, still globally α-Hölder continuous, thus constant by the Liouville theorem.

(ii) After a rotation and a scaling, we obtain a harmonic function $\tilde{w}$ in a half-space, constant on the boundary of the half-space. We can then extend it in a symmetric way to obtain a harmonic function in the whole space $\mathbb{R}^N$, still globally α-Hölder continuous, and hence constant. ■

We now address the proof of Theorem 1.3. Let $\alpha \in (0, \bar{\nu}/2)$, and suppose for contradiction that $\{\mathbf{u}_\beta\}$ is not bounded in $C^{0,\alpha}(\overline{\Omega})$, that is, there exists a sequence $\beta \to +\infty$ such that

$$L_\beta := \sup_i \sup_{\substack{x \ne y \\ x,y \in \overline{\Omega}}} \frac{|u_{i,\beta}(x) - u_{i,\beta}(y)|}{|x - y|^\alpha} \to +\infty.$$

Since, for each β fixed, $\mathbf{u}_\beta$ is of class $C^{0,\alpha'}(\overline{\Omega})$ with $\alpha' > \alpha$, we can assume without loss of generality that L_β is attained by $u_{1,\beta}$ at the pair (x_β, y_β). The uniform boundedness in $L^\infty(\Omega)$ yields

$$|x_\beta - y_\beta|^\alpha = \frac{|u_{1,\beta}(x_\beta) - u_{1,\beta}(y_\beta)|}{L_\beta} \le \frac{2\|u_{1,\beta}\|_{L^\infty(\Omega)}}{L_\beta} \to 0.$$

We consider the following blow-up of $\mathbf{u}_\beta$ with center x_β, with $r_\beta \to 0^+$ to be chosen later:

$$\mathbf{v}_\beta(x) := \frac{1}{L_\beta r_\beta^\alpha}\mathbf{u}_\beta(x_\beta + r_\beta x), \quad x \in \Omega_\beta := \frac{\Omega - x_\beta}{r_\beta}.$$

According to the behavior of $\mathrm{dist}(x_\beta, \partial\Omega_\beta)$, and by the regularity of $\partial\Omega$, either Ω_β exhausts $\mathbb{R}^N$ as $\beta \to +\infty$, or Ω_β tends to a half-space. In both cases, we denote the limit domain by Ω_∞.

Plainly, $\mathbf{v}_\beta$ is a positive solution to

$$\begin{cases} L_i v_{i,\beta} = M_\beta v_{i,\beta} \sum_{j\neq i} b_{ij} v_{j,\beta} & \text{in } \Omega_\beta, \\ v_{i,\beta} = \varphi_{i,\beta} & \text{on } \partial\Omega_\beta, \end{cases}$$

where $M_\beta = L_\beta r_\beta^{2+\alpha}\beta$, and $\varphi_{i,\beta}$ is defined by scaling the boundary datum φ_i. Furthermore, for all $\beta > 1$,

$$\max_i \max_{\substack{x\neq y \\ x,y\in\overline{\Omega}}} \frac{|v_{i,\beta}(x) - v_{i,\beta}(y)|}{|x-y|^\alpha} = \frac{\left|v_{1,\beta}(0) - v_{1,\beta}\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{x_\beta - y_\beta}{r_\beta}\right|^\alpha} = 1.$$

The next lemma will be useful in order to deal with the case when the scaled domains converge to a half-space.

Lemma 4.3. *Suppose that Ω_β tends to a half-space Ω_∞. Then one can extend $\mathbf{v}_\beta$ outside Ω_β in a Lipschitz fashion in such a way that:*

(i) *If $\{\mathbf{v}_\beta(0)\}$ is bounded, then $\mathbf{v}_\beta \to \mathbf{v}$ in $C^{0,\alpha'}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $0 < \alpha' < \alpha$, up to a subsequence; moreover, the limit function $\mathbf{v}$ attains a constant value on the boundary $\partial\Omega_\infty$, and at most one component of $\mathbf{v}$ is different from 0 in $\mathbb{R}^N$.*

(ii) *If $\{\mathbf{v}_\beta(0)\}$ is unbounded, then $\tilde{\mathbf{v}}_\beta(x) := \mathbf{v}_\beta(x) - \mathbf{v}_\beta(0)$ converges to $\tilde{\mathbf{v}}$ in $C^{0,\alpha'}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $0 < \alpha' < \alpha$, up to a subsequence; moreover, the limit function $\tilde{\mathbf{v}}$ has a constant value on $\partial\Omega_\infty$.*

Proof. (i) Let ϕ_i be the harmonic extension of φ_i over Ω, which is in $C^{1,\gamma}(\overline{\Omega})$. By the comparison principle, $0 \le u_{i,\beta} \le \phi_i$ for every β. Now, thanks to the Kirszbraun theorem, we can extend the functions φ_i over the whole space $\mathbb{R}^N$ in a Lipschitz fashion, preserving their Lipschitz constants. The extended function will still be denoted by φ_i. We also extend ϕ_i and $u_{i,\beta}$ over $\mathbb{R}^N$, by letting them equal φ_i on Ω^c. Let $\varphi_{i,\beta}$ and $\phi_{i,\beta}$ be given by scaling ϕ_i and φ_i in the same way as $u_{i,\beta}$. Then $v_{i,\beta}$, $\varphi_{i,\beta}$ and $\phi_{i,\beta}$ are defined everywhere, and $\varphi_{i,\beta} = \phi_{i,\beta} = v_{i,\beta}$ in Ω_β^c. Plainly:

(i) $v_{i,\beta}$ is locally α-Hölder continuous in $\mathbb{R}^N$, with α-Hölder seminorm $[v_{i,\beta}]_{C^{0,\alpha}(K)}$ uniformly bounded with respect to β, for any compact set $K \subset \mathbb{R}^N$.

(ii) $\varphi_{i,\beta}$ and $\phi_{i,\beta}$ are locally Lipschitz continuous in $\mathbb{R}^N$, with Lipschitz seminorms $[\varphi_{i,\beta}]_{C^{0,1}(K)}$ and $[\phi_{i,\beta}]_{C^{0,1}(K)}$ uniformly bounded with respect to β, for any compact set $K \subset \mathbb{R}^N$.

Thus, since $\{v_{i,\beta}(0)\}$ is bounded, up to a subsequence $v_{i,\beta} \to v_i$ locally uniformly in $\mathbb{R}^N$. But $v_{i,\beta} = \varphi_{i,\beta} = \phi_{i,\beta}$ in Ω_β^c, so $\varphi_{i,\beta} \to \varphi_{i,\infty}$ and $\phi_{i,\beta} \to \phi_{i,\infty}$ locally uniformly in Ω_∞^c. In turn, by uniform Lipschitz continuity, we infer that $\varphi_{i,\beta} \to \varphi_{i,\infty}$ and $\phi_{i,\beta} \to \phi_{i,\infty}$ locally uniformly in the whole of $\mathbb{R}^N$. The local uniform convergence entails $\varphi_{i,\infty} \cdot \varphi_{j,\infty} \equiv 0$ in $\overline{\Omega_\infty}$. Moreover $0 \le v_i \le \phi_{i,\infty}$ in Ω_∞.

Now we show that both $\varphi_{i,\infty}$ and $\phi_{i,\infty}$ are constant in $\mathbb{R}^N$, and since they coincide in Ω_∞^c, they actually coincide everywhere. This is a consequence of the fact that $\varphi_{i,\infty}$ and $\phi_{i,\infty}$ are obtained as limits of scaling of a fixed Lipschitz continuous function, so that if $x \neq y$ then

$$\frac{|\varphi_{i,\beta}(x) - \varphi_{i,\beta}(y)|}{|x-y|^\alpha} = \frac{|\varphi_i(x_\beta + r_\beta x) - \varphi_i(x_\beta + r_\beta y)|}{L_\beta r_\beta^\alpha |x-y|^\alpha} \le \frac{[\varphi_i]_{C^{0,1}(\mathbb{R}^N)} r_\beta^{1-\alpha}}{L_\beta} |x-y|^{1-\alpha},$$

and the right hand side tends to 0 locally uniformly in $\mathbb{R}^N$. The very same argument proves that also $\phi_{i,\infty}$ is constant.

To sum up, so far we have shown that the extended functions $v_{i,\beta}, \varphi_{i,\beta}, \phi_{i,\beta}$ converge locally uniformly in $\mathbb{R}^N$, coincide in Ω_β^c, and $\varphi_{i,\infty} = \phi_{i,\infty}$ are constants in $\mathbb{R}^N$. Recalling the segregation condition $\varphi_{i,\infty} \cdot \varphi_{j,\infty} \equiv 0$ in Ω_∞, and hence also in $\mathbb{R}^N$, we deduce that at most one component $\phi_{i,\infty}$ can be different from 0. But then, since $0 \le v_i \le \phi_{i,\infty}$ in Ω_∞, at most one component v_i is different from 0 in Ω_∞. And finally, since $v_i = \varphi_{i,\infty}$ in Ω_∞^c, we conclude that v_i is constant on $\partial\Omega_\infty$.

The proof of (ii) is analogous. ∎

Lemma 4.4. *Let $r_\beta \to 0^+$ be such that*

(i) *there exists $R' > 0$ such that $|x_\beta - y_\beta| \le R' r_\beta$;*

(ii) $M_\beta \not\to 0$.

Then $\{\mathbf{v}_\beta(0)\}$ is bounded in β.

Proof. The proof is analogous to the one of [13, Lemma 6.1] (see also [24, Lemma 3.4] for more details), and hence we only sketch it. Suppose for contradiction that along a subsequence $v_{h,\beta}(0) \to +\infty$ for some index h, and let $R > R'$. By assumption (ii) and the global Hölder bound, we have

$$I_\beta := M_\beta \inf_{B_{2R} \cap \Omega_\beta} v_{h,\beta} \to +\infty.$$

Now we can argue as in [13, Lemma 6.1], by using Lemma 4.1 instead of [13, Lemma 4.4], to deduce that for every $R > R'$,

$$\|v_{i,\beta}\|_{L^\infty(B_R \cap \Omega_\beta)} \to 0 \quad \forall i \neq h, \quad \|L_i v_{i,\beta}\|_{L^\infty(B_R \cap \Omega_\beta)} \to 0 \quad \forall i,$$

as $\beta \to +\infty$. Let then $\tilde{\mathbf{v}}_\beta(x) := \mathbf{v}_\beta(x) - \mathbf{v}_\beta(0)$. The above discussion shows that $\tilde{\mathbf{v}}_\beta \to \tilde{\mathbf{v}}$ locally uniformly in $\mathbb{R}^N$, where $\tilde{\mathbf{v}}$ is globally α-Hölder continuous in Ω_∞, and $\tilde{v}_i \equiv 0$ for $i \neq h$ (in case Ω_∞ is a half-space, we can use Lemma 4.3). The uniform convergence of the A_i-Laplacians implies that actually $\tilde{\mathbf{v}}_\beta \to \tilde{\mathbf{v}}$ in $C^1_{\text{loc}}(\Omega_\infty)$. We claim that $\tilde{v}_1$ is not constant. To prove this, we recall that, by assumption (i), $(y_\beta - x_\beta)/r_\beta$ converges to a limit z up to a subsequence. If $z = 0$, then by boundedness in C^1_{loc},

$$1 = \frac{\left|v_{1,\beta}(0) - v_{1,\beta}\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\alpha} = \frac{\left|\tilde v_{1,\beta}(0) - \tilde v_{1,\beta}\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^\alpha} \le C\left|\frac{y_\beta - x_\beta}{r_\beta}\right|^{1-\alpha} \to 0,$$

a contradiction. Then $z \neq 0$, and $|\tilde v_1(0) - \tilde v_1(z)| = |z|^\alpha$, so that $\tilde v_1$ is a nonconstant A_1-harmonic function in Ω_∞, globally α-Hölder continuous. If Ω_∞ is a half-space, by Lemma 4.3 we can also see that $\tilde v_1$ is constant on $\partial\Omega_\infty$. Therefore Lemma 4.2 provides a contradiction both for $\Omega_\infty = \mathbb{R}^N$, and for Ω_∞ a half-space. ■

Lemma 4.5. *We have*

$$\limsup_{\beta\to+\infty} \beta L_\beta |x_\beta - y_\beta|^{2+\alpha} = +\infty.$$

Proof. Suppose $\beta L_\beta |x_\beta - y_\beta|^{2+\alpha}$ is bounded. Then, by choosing

$$r_\beta = (\beta L_\beta)^{-\frac{1}{2+\alpha}},$$

we see that $M_\beta = 1$, and $|x_\beta - y_\beta| \le R' r_\beta$ for a constant $R' > 0$. Hence $\{\mathbf{v}_\beta(0)\}$ is bounded by Lemma 4.4, and by uniform Hölder continuity $\mathbf{v}_\beta \to \mathbf{v}$ locally uniformly in Ω_∞, up to a subsequence. In addition, if Ω_∞ is a half-space, we know by Lemma 4.3 that each component of $\mathbf{v}$ except possibly one vanishes, and the remaining one is constant on $\partial\Omega_\infty$. Moreover, since $M_\beta = 1$, we find that $L_i v_{i,\beta}$ converges locally uniformly, and hence $\mathbf{v}_\beta \to \mathbf{v}$ in $C^1_{\mathrm{loc}}(\Omega_\infty)$, with $\mathbf{v}$ globally α-Hölder continuous in Ω_∞, and

$$-\operatorname{div}(A_i \nabla v_i) = -v_i \sum_{j\neq i} b_{ij} v_j, \quad v_i \ge 0 \quad \text{in } \Omega_\infty.$$

In fact, either $v_i > 0$ or $v_i \equiv 0$ in Ω_∞, by the strong maximum principle. Finally, as in the last part of the proof of Lemma 4.4, we also deduce that v_1 is nonconstant in Ω_∞.

Let $\Omega_\infty = \mathbb{R}^N$. Then, since $2\alpha < \bar\nu$, by Theorem 3.5[2] we infer that $\mathbf{v}$ is constant, a contradiction.

If instead Ω_∞ is a half-space, then by Lemma 4.3 we know that at most one component v_i does not vanish identically. Since v_1 is nonconstant, we infer that $v_i \equiv 0$ for every $i \neq 1$, and v_1 is a nonconstant A_1-harmonic function in a half-space, globally α-Hölder continuous, which attains a constant boundary datum on $\partial\Omega_\infty$. This contradicta Lemma 4.2. ■

At this point we fix the choice of r_β and complete the contradiction argument.

[2] In the present case, each I_h is a singleton, and the assumptions on the coupling terms g_{ij} are satisfied since $b_{ij} > 0$. Notice also that the α-Hölder continuity implies that $v_1, \dots, v_k$ grow at most like $|x|^\alpha$ at infinity.

Conclusion of the proof of Theorem 1.3. Let

$$r_\beta = |x_\beta - y_\beta|.$$

By Lemma 4.5, we have $M_\beta \to +\infty$. Thus the assumptions of Lemma 4.4 are satisfied, and we deduce that $\mathbf{v}_\beta \to \mathbf{v}$ locally uniformly in $\mathbb{R}^N$, with $\mathbf{v}$ globally α-Hölder continuous (if Ω_∞ is a half-space, we consider the extension of $\mathbf{v}_\beta$ defined in Lemma 4.3; in this case, $\mathbf{v}$ is constant on $\partial\Omega_\infty$ and has at most one nontrivial component). Furthermore, there exists $z \in \partial B_1 \cap \overline{\Omega_\infty}$ such that $|v_1(z) - v_1(0)| = 1$, and hence v_1 is nonconstant.

Now, let $r > 0$ and x_0 be such that $B_{2r}(x_0) \subset\subset \Omega_\infty$, and let η be a smooth cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_r(x_0)$, and $\eta \equiv 0$ in $B_{2r}(x_0)^c$. By testing the equation for $v_{i,\beta}$ with η, we deduce that

$$M_\beta \int_{B_r(x_0)} v_{i,\beta} \sum_{j \neq i} b_{ij} v_{j,\beta} \le \int_{B_{2r}(x_0)} \operatorname{div}(A_i \nabla \eta) v_{i,\beta} \le C, \tag{4.1}$$

since $\{v_{i,\beta}\}$ is locally bounded in L^∞. But $M_\beta \to +\infty$, so that $v_i \cdot v_j \equiv 0$ in Ω_∞.

Moreover, by testing the equation for $v_{i,\beta}$ with $v_{i,\beta}\eta^2$, we obtain

$$\begin{aligned} \int_{B_r(x_0)} &\langle A_i \nabla v_{i,\beta}, \nabla v_{i,\beta} \rangle \\ &\le 4 \int_{B_{2r}(x_0)} \langle A_i \nabla \eta, \nabla \eta \rangle v_{i,\beta}^2 + 2M_\beta \int_{B_{2r}(x_0)} v_{i,\beta}^2 \sum_{j \neq i} b_{ij} v_{j,\beta} \\ &\le \|v_{i,\beta}\|_{L^\infty(B_{2r(x_0)})} \left(C + M_\beta \int_{B_{2r}(x_0)} v_{i,\beta} \sum_{j \neq i} b_{ij} v_{j,\beta} \right) \le C, \end{aligned} \tag{4.2}$$

where we have used (4.1). That is, $\{\mathbf{v}_\beta\}$ is locally bounded in $H^1_{\mathrm{loc}}(\Omega_\infty)$ and hence, up to a further subsequence, $\mathbf{v}_\beta \rightharpoonup \mathbf{v}$ weakly in $H^1_{\mathrm{loc}}(\Omega_\infty)$. Since $-\operatorname{div}(A_i \nabla v_{i,\beta}) \le 0$ in Ω_β, if we take the weak limit we infer that

$$v_i \cdot v_j \equiv 0 \quad \text{if } i \neq j, \quad -\operatorname{div}(A_i \nabla v_i) \le 0 \quad \text{in } \Omega_\infty.$$

If Ω_∞ is a half-space, Lemma 4.3 also implies that all components of $\mathbf{v}$ but v_1 must vanish identically. If instead $\Omega = \mathbb{R}^N$, the same conclusion follows from Theorem 3.1, since $2\alpha < \bar{\nu}$. In any case, for every β,

$$-\operatorname{div}(A_1 \nabla v_{1,\beta}) + \sum_{j=2}^k \frac{b_{1j}}{b_{j1}} \operatorname{div}(A_j \nabla v_{j,\beta}) = M_\beta \sum_{j=2}^k \sum_{h \neq 1,j} \frac{b_{1j} b_{jh}}{b_{j1}} v_{j,\beta} v_{h,\beta} \ge 0 \quad \text{in } \Omega_\beta$$

in the weak sense. By passing to the weak limit, and recalling that $v_j \equiv 0$ in $\mathbb{R}^N$ for $j \neq 1$, we deduce that

$$-\operatorname{div}(A_1 \nabla v_1) \ge 0 \quad \text{in } \Omega_\infty \tag{4.3}$$

in the weak sense. But then $\operatorname{div}(A_1 \nabla v_1) = 0$ in Ω_∞, and Lemma 4.2 gives a contradiction with the fact that v_1 is α-Hölder continuous and nonconstant in $\mathbb{R}^N$. This contradiction finally shows that $\{\mathbf{u}_\beta\}$ is bounded in $C^{0,\alpha}(\overline{\Omega})$, as desired. Now we proceed with the second part of the theorem.

Clearly, up to a subsequence $\mathbf{u}_\beta \to \mathbf{u}$ in $C^{0,\alpha}(\overline{\Omega})$ for every $\alpha \in (0, \bar\nu/2)$. As in (4.1) and (4.2) one can check that $\mathbf{u}_\beta \rightharpoonup \mathbf{u}$ weakly in $H^1_{\text{loc}}(\Omega)$, and that the limit function is segregated: $u_i \cdot u_j \equiv 0$ for $i \neq j$. Moreover, by testing the equation for $u_{i,\beta}$ with $(u_{i,\beta} - u)\eta$, where $\eta \in C^\infty_c(\Omega)$ is an arbitrary cut-off function, we deduce that

$$\begin{aligned}
&\int_\Omega \eta \langle A_i \nabla u_{i,\beta}, \nabla(u_{i,\beta} - u)\rangle \\
&\quad = -\int_\Omega (u_{i,\beta} - u_i)\langle A_i \nabla u_{i,\beta}, \nabla \eta\rangle - \beta \int_\Omega \eta u_{i,\beta} \sum_{j\neq i} b_{ij} u_{j,\beta}(u_{i,\beta} - u_{j,\beta}) \\
&\quad \le C \|u_{i,\beta} - u_i\|_{L^\infty(\Omega)} \left(\|\nabla u_{i,\beta}\|_{L^2(\operatorname{supp}\eta)} + \beta \int_\Omega u_{i,\beta} \sum_{j\neq i} b_{ij} u_{j,\beta} \right) \to 0
\end{aligned}$$

as $\beta \to \infty$, by uniform convergence and local boundedness in H^1. Therefore, by weak convergence,

$$\begin{aligned}
0 &= \lim_{\beta\to+\infty} \int_\Omega \eta \langle A_i \nabla u_{i,\beta}, \nabla(u_{i,\beta} - u)\rangle \\
&= \lim_{\beta\to+\infty} \int_\Omega \eta \big(\langle A_i \nabla u_{i,\beta}, \nabla u_{i,\beta}\rangle - \langle A_i \nabla u_i, \nabla u_i\rangle \big),
\end{aligned}$$

which gives the strong convergence $\mathbf{u}_\beta \to \mathbf{u}$ in $H^1_{\text{loc}}(\Omega)$. Finally, to show that u_i is A_i-harmonic in $\{u_i > 0\}$, we proceed as for (4.3), proving that $\operatorname{div}(A\nabla u_i) \le 0$ in $\{u_i > 0\}$. But by weak convergence $\operatorname{div}(A_i \nabla u_i) \ge 0$ in Ω, and the conclusion follows. ∎

Proof of Theorem 1.6. Recall that $k = 2$, and we have reduced to the case when $A_1 = A$ is diagonal, with lowest eigenvalue 1, $A_2 = \text{Id}$, and $a_{12} = a_{21}$. We use the notation $(u_{1,\beta}, u_{2,\beta}) = (u_\beta, v_\beta)$. Let us consider $w_\beta = u_\beta - v_\beta$. We know that, up to a subsequence, $w_\beta \to w = u - v$ in $C^{0,\alpha}(\overline{\Omega}) \cap H^1_{\text{loc}}(\Omega)$ for every $0 < \alpha < \bar\nu/2 = \nu_{A,N}/2$. On the other hand, since $\operatorname{div}(A_i \nabla u_\beta) = \Delta v_\beta$, we have

$$\int_\Omega (\langle A\nabla u_\beta, \nabla\varphi\rangle - \langle \nabla v_\beta, \nabla \varphi\rangle) = 0$$

for every $\varphi \in C^1_c(\Omega)$. By taking the limit, by weak convergence in H^1 we deduce that $w = u - v$ is a weak solution of the quasi-linear equation (1.6). The rest of the theorem follows directly from the main result in [2]. ∎

5. Spatial segregation of competitive systems: variational interaction

In this section we prove Theorem 1.4. Notice that each $\mathbf{u}_\beta$ is a vector of positive functions in Ω, of class $C^{1,\gamma}(\overline{\Omega})$. Moreover, we can define $\bar\nu = \bar\nu(N, A_1, \dots, A_k) \in (0, 2)$ as in Theorem 3.1.

Now, the first part of the proof of Theorem 1.4 rests upon the same contradiction argument used for Theorem 1.3, with the obvious modifications related to the different structure of the system, and to the different boundary conditions. We only give a sketchy summary, referring for the details to the previous section and to [24,27] (of course, we use Theorems 3.1 and 3.5 and Lemma 4.2 instead of the corresponding "symmetric results" when necessary).

Let $\alpha \in (0, \bar{\nu}/2)$, and suppose for contradiction that $\{\mathbf{u}_\beta\}$ is not bounded in $C^{0,\alpha}(\overline{\Omega})$, so there exists a sequence $\beta \to +\infty$ such that

$$L_\beta := \sup_i \sup_{\substack{x \neq y \\ x,y \in \overline{\Omega}}} \frac{|u_{i,\beta}(x) - u_{i,\beta}(y)|}{|x-y|^\alpha} \to +\infty.$$

We can assume that M_β is attained by $u_{1,\beta}$ at the pair (x_β, y_β), with $|x_\beta - y_\beta| \to 0$. Then we introduce the following blow-up of $\mathbf{u}_\beta$ with center x_β, and $r_\beta \to 0^+$ to be chosen later:

$$\mathbf{v}_\beta(x) := \frac{1}{L_\beta r_\beta^\alpha} \mathbf{u}_\beta(x_\beta + r_\beta x), \quad x \in \Omega_\beta := \frac{\Omega - x_\beta}{r_\beta}.$$

The scaled domains Ω_β can either exhaust $\mathbb{R}^N$, or tend to a half-space. In both cases, we denote the limit domain by Ω_∞. The function $\mathbf{v}_\beta$ is a positive solution to

$$\begin{cases} -L_i v_{i,\beta} = g_{i,\beta}(x, v_{i,\beta}) - M_\beta v_{i,\beta} \sum_{j \neq i} b_{ij} v_{j,\beta}^2 & \text{in } \Omega_\beta, \\ v_{i,\beta} = 0 & \text{on } \partial\Omega_\beta, \end{cases}$$

where $M_\beta = L_\beta^2 r_\beta^{2+2\alpha} \beta$, and

$$\begin{aligned} g_{i,\beta}(x, v_{i,\beta}(x)) &= \frac{r_\beta^{2-\alpha}}{L_\beta} f_{i,\beta}\big(x_\beta + r_\beta x, L_\beta r_\beta^\alpha v_{i,\beta}(x)\big) \\ &= \frac{r_\beta^{2-\alpha}}{L_\beta} f_{i,\beta}\big(x_\beta + r_\beta x, u_{i,\beta}(x_\beta + r_\beta x)\big). \end{aligned}$$

Notice that $\|g_{i,\beta}(\cdot, v_{i,\beta}(\cdot))\|_{L^\infty(\Omega_\beta)} \to 0$ as $\beta \to +\infty$, thanks to the assumptions on $f_{i,\beta}$ and the upper bound on $\|u_{i,\beta}\|_{L^\infty(\Omega)}$. Moreover, for every β,

$$\max_i \max_{\substack{x \neq y \\ x,y \in \overline{\Omega}}} \frac{|v_{i,\beta}(x) - v_{i,\beta}(y)|}{|x-y|^\alpha} = \frac{\left|v_{1,\beta}(0) - v_{1,\beta}\left(\frac{y_\beta - x_\beta}{r_\beta}\right)\right|}{\left|\frac{x_\beta - y_\beta}{r_\beta}\right|^\alpha} = 1.$$

Lemma 5.1. *Suppose that Ω_β tends to a half-space Ω_∞. Then one can extend $\mathbf{v}_\beta$ outside Ω_β in a Lipschitz fashion in such a way that:*

(i) *If $\{\mathbf{v}_\beta(0)\}$ is bounded, then $\mathbf{v}_\beta \to \mathbf{v}$ in $C^{0,\alpha'}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $0 < \alpha' < \alpha$, up to a subsequence; moreover, the limit function $\mathbf{v}$ has constant value 0 on $\partial\Omega_\infty$.*

(ii) *If $\{\mathbf{v}_\beta(0)\}$ is unbounded, then $\Omega_\infty = \mathbb{R}^N$.*

Proof. (i) This is very similar to point (i) of Lemma 4.3, once we extend $\mathbf{u}_\beta$ as equal to 0 outside Ω.

(ii) Let $R > 0$ be arbitrary. If $v_{i,\beta}(0) \to +\infty$ along a subsequence, then by uniform Hölder estimates,

$$\inf_{B_R} v_{i,\beta} \geq v_{i,\beta}(0) - CR^\alpha \to +\infty.$$

But $v_{i,\beta} \equiv 0$ in Ω_β^c, and hence $B_R(0) \subset \Omega_\beta$ eventually. ■

With the help of this lemma, and by following [24, Section 3] (see also Section 4), it is not difficult to prove

Lemma 5.2. *Let $r_\beta \to 0^+$ be such that*

(i) *there exists $R' > 0$ such that $|x_\beta - y_\beta| \leq R' r_\beta$;*

(ii) $M_\beta \not\to 0$.

Then $\{\mathbf{v}_\beta(0)\}$ is bounded in β.

Lemma 5.3. *We have*

$$\limsup_{\beta\to+\infty} \beta L_\beta^2 |x_\beta - y_\beta|^{2+2\alpha} = +\infty.$$

At this point we fix the choice of r_β as in Section 4, and analyze the asymptotic behavior of $\mathbf{v}_\beta$.

Lemma 5.4. *Let*

$$r_\beta = |x_\beta - y_\beta|.$$

There exists $\mathbf{v}$, globally α-Hölder continuous with $[\mathbf{v}]_{C^{0,\alpha}(\mathbb{R}^N)} = 1$, such that, as $\beta \to +\infty$, the following hold up to a subsequence:

(i) *$\Omega_\infty = \mathbb{R}^N$, and $\mathbf{v}_\beta \to \mathbf{v}$ in $C^{0,\alpha'}_{\mathrm{loc}}(\mathbb{R}^N)$;*

(ii) *$\int_{B_r(x_0)} M_\beta v_{i,\beta}^2 v_{j,\beta}^2 \to 0$ for any $r > 0$ and $x_0 \in \mathbb{R}^N$;*

(iii) *$\mathbf{v}_\beta \to \mathbf{v}$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.*

For the proof, we refer to [24, Lemmas 3.6 and 3.7] (see also the conclusion of the proof of Theorem 1.3). The properties of the limit profile are collected in the next statement.

Lemma 5.5. *Let $\mathbf{v}$ be the limit function defined in Lemma 5.4. Then*

(i) *$v_i \equiv 0$ in $\mathbb{R}^N$ for every $i \neq 1$;*

(ii) *v_1 is nonconstant, and $\mathrm{div}(A_1 \nabla v_1) = 0$ in $\{v_1 > 0\}$;*

(iii) *$\{v_1 = 0\} \neq \emptyset$ and $\{v_1 > 0\}$ is connected.*

Proof. By Lemma 5.4 we know that $v_i \cdot v_j \equiv 0$ in $\mathbb{R}^N$. Moreover, by H^1_{loc} convergence and recalling that $\|g_{i,\beta}(\cdot, v_{i,\beta}(\cdot))\|_{L^\infty(\Omega_\beta)} \to 0$, we deduce that v_i is A_i-subharmonic for every i. Since $\mathbf{v}$ is α-Hölder with $\alpha < \bar{\nu}/2$, Theorem 3.1 implies that only one component of $\mathbf{v}$ does not vanish identically. But by uniform convergence $\max_{x\in\partial B_1(0)} |v_1(x) - v_1(0)| = 1$, so that $v_i \equiv 0$ in $\mathbb{R}^N$ for every $i \neq 1$, and v_1 is nonconstant. The fact that v_1

is harmonic in the open set $\{v_1 > 0\}$ can be checked as in [24, Lemma 3.7], by using Lemma 4.1 instead of [24, Lemma 3.1]. This completes the proof of (i) and (ii).

Suppose now for contradiction that $\{v_1 = 0\}$ is empty; then v_1 would be a positive globally α-Hölder A_1-harmonic nonconstant function, contradicting Lemma 4.2.

Finally, suppose for contradiction that $\{v_1 > 0\}$ is disconnected, and let ω_1 and ω_2 be two of its connected components. Then the functions $w_1 = v_1\chi_{\omega_1}$ and $w_2 = v_1\chi_{\omega_2}$ are nontrivial and satisfy the assumptions of Theorem 3.1, a contradiction again. ∎

From now on we shall mainly focus on the component v_1, the only one which survived in the limit process. Therefore, in order to simplify some expressions below, we perform a change of coordinates as in the proof of Theorem 3.1 in order to have $A_1 = \mathrm{Id}$, and hence v_1 is harmonic in its positivity set.

Remark 5.6. In the conclusion of the proof of Theorem 1.3, we considered the difference between the differential equations of the component $v_{1,\beta}$ and the others, by taking the weak limit. This gave us the inequality $\operatorname{div}(A_1\nabla v_1) \geq 0$ in $\mathbb{R}^N$, leading to the A_1-harmonicity of v_1, which finally provided a contradiction. When we deal with system (1.4), this strategy fails, due to the lack of symmetry in the exponents of the competition terms. By following [24], one may be tempted to consider an Almgren frequency function $N_\beta(\mathbf{v}_\beta, x_0, r)$ associated with $\mathbf{v}_\beta$, compute its derivative, and then pass to the limit in β in order to derive a monotonicity formula for the frequency function of the limit problem (see [24, Section 3.2]). However, in the present setting this strategy fails, due to the lack of symmetry in the diffusion operators. This lack of symmetry creates several complications in the derivation of a good expression for the derivative of $N_\beta(\mathbf{v}_\beta, x_0, r)$, complications which we could not overcome. We shall therefore argue in a different way. First, by the variational structure of the problem (this requires $b_{ij} = b_{ji}$), we derive a domain variation formula for $\mathbf{v}_\beta$. Then we pass to the limit in β. The properties collected in Lemmas 5.4 and 5.5 at this level allow us to obtain the validity of a domain variation formula for the only nontrivial component v_1, in the whole of $\mathbb{R}^N$ (and not only in the interior of its support). In this way, even if we cannot establish a monotonicity formula for the Almgren frequency function associated with $\mathbf{v}_\beta$, we can still recover a monotonicity formula for the component v_1 of the limit profile. This is sufficient for our purposes.

Lemma 5.7. *Let $Y \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Then*

$$2\int_{\Omega_\beta}\Big(\sum_i \langle dY A_i \nabla v_{i,\beta}, \nabla v_{i,\beta}\rangle - \sum_i g_{i,\beta}(x, v_{i,\beta})\langle \nabla v_{i,\beta}, Y\rangle\Big) - \int_{\Omega_\beta} \operatorname{div} Y \Big(\sum_i \langle A_i \nabla v_{i,\beta}, \nabla v_{i,\beta}\rangle + \beta \sum_{i<j} b_{ij} v_{i,\beta}^2 v_{j,\beta}^2\Big) = 0 \tag{5.1}$$

for every β sufficiently large, and

$$\int_{\mathbb{R}^N} (2\langle dY \nabla v_1, \nabla v_1\rangle - \operatorname{div} Y\, |\nabla v_1|^2) = 0. \tag{5.2}$$

Proof. By multiplying the equation for $v_{i,\beta}$ with $\langle \nabla v_{i,\beta}, Y\rangle$ and integrating, we deduce that

$$\int_{\Omega_\beta} \Big(\langle A_i \nabla v_{i,\beta}, \nabla(\langle \nabla v_{i,\beta}, Y\rangle)\rangle + \beta v_{i,\beta} \sum_{j\neq i} b_{ij} v_{j,\beta}^2 \langle \nabla v_{i,\beta}, Y\rangle - g_{i,\beta}(x, v_{i,\beta})\langle \nabla v_{i,\beta}, Y\rangle \Big) = 0.$$

With lengthy but elementary computations, it is not difficult to check that

$$\langle A_i \nabla v_{i,\beta}, \nabla(\langle \nabla v_{i,\beta}, Y\rangle)\rangle = \tfrac{1}{2}\langle Y, \nabla(\langle A_i \nabla v_{i,\beta}, \nabla v_{i,\beta}\rangle)\rangle + \langle dY A_i \nabla v_{i,\beta}, \nabla v_{i,\beta}\rangle,$$

whence, by integrating by parts, we deduce that

$$\int_{\Omega_\beta} \langle dY A_i \nabla v_{i,\beta}, \nabla v_{i,\beta}\rangle - g_{i,\beta}(x, v_{i,\beta})\langle \nabla v_{i,\beta}, Y\rangle \\ - \int_{\Omega_\beta} \Big(\tfrac{1}{2} \operatorname{div} Y \, \langle A \nabla v_{i,\beta}, \nabla v_{i,\beta}\rangle - \beta v_{i,\beta} \sum_{j\neq i} b_{ij} v_{j,\beta}^2 \langle \nabla v_{i,\beta}, Y\rangle \Big) = 0.$$

We sum over i from 1 to k and integrate by parts once again to obtain (5.1).

Moreover, by taking the limit as $\beta \to +\infty$, and recalling the properties listed in Lemmas 5.4 and 5.5, and the fact that $\|g_{i,\beta}(\cdot, v_{i,\beta}(\cdot))\|_{L^\infty(\Omega_\beta)} \to 0$, we also obtain (5.2). ■

Conclusion of the proof of Theorem 1.4. From the second formula in Lemma 5.7, we infer that

$$\int_{S_r(x_0)} |\nabla v_1|^2 = \frac{N-2}{r} \int_{B_r} |\nabla v_1|^2 + 2 \int_{S_r(x_0)} (\partial_\nu v_1)^2$$

for every $x_0 \in \mathbb{R}^N$ and almost every $r > 0$ (we refer to [30, second part of the proof of Lemma 2.11] for the details). Furthermore, we have

$$\frac{d}{dr}\left(\int_{S_r(x_0)} v_1^2 \right) = \frac{N-1}{r} \int_{S_r(x_0)} v_1^2 + 2 \int_{S_r(x_0)} v_1 \partial_\nu v_1$$

(see [30, Lemma 2.8] for the details). Therefore, by introducing the Almgren frequency function

$$N(x_0, r) = \frac{r \int_{B_r(x_0)} |\nabla v_1|^2}{\int_{S_r(x_0)} v_1^2},$$

it is standard to prove that $N(x_0, \cdot)$ is nondecreasing, and it is constantly c if and only v_1 is c-homogeneous. At this point we can proceed exactly as in [24, end of the proof of Theorem 1.3] or [30, conclusion of the proof of Proposition 2.1, p. 278] to deduce that $\{v_1 = 0\}$ is a linear subspace of dimension at most $N-2$, and in particular has local capacity 0. But then, since $v \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$, we infer that v_1 is harmonic everywhere, is nonconstant, and is globally α-Hölder continuous for some $\alpha \in (0,1)$, contrary to the Liouville theorem. This completes the proof of the boundedness of $\{\mathbf{u}_\beta\}$ in $C^{0,\alpha}(\overline{\Omega})$. The rest of the assertion of Theorem 1.4 follows as in [24] (for the domain variation formula (1.5), one can argue as in Lemma 5.7 with the functions $\mathbf{u}_\beta$, and then take the limit). ■

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