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# Convergence in Wasserstein distance for empirical measures of Dirichlet diffusion processes on manifolds

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**Abstract.** Let $M$ be a $d$-dimensional connected compact Riemannian manifold with boundary $\partial M$, let $V \in C^2(M)$ be such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure, and let $X_t$ be the diffusion process generated by $L := \Delta + \nabla V$ with $\tau := \inf\{t \ge 0 : X_t \in \partial M\}$. Consider the empirical measure $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s$ under the condition $t < \tau$ for the diffusion process. If $d \le 3$, then for any initial distribution not fully supported on $\partial M$,

$$c\sum_{m=1}^{\infty}\frac{2}{(\lambda_m-\lambda_0)^2} \le \liminf_{t\to\infty}\inf_{T\ge t} t\mathbb{E}[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] \le \limsup_{t\to\infty}\sup_{T\ge t} t\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] \le \sum_{m=1}^{\infty}\frac{2}{(\lambda_m-\lambda_0)^2}$$

holds for some constant $c \in (0,1]$ with $c = 1$ when $\partial M$ is convex, where $\mu_0 := \phi_0^2\mu$ for the first Dirichlet eigenfunction $\phi_0$ of $L$, $\{\lambda_m\}_{m\ge 0}$ are the Dirichlet eigenvalues of $-L$ listed in increasing order counting multiplicities, and the upper bound is finite if and only if $d \le 3$. When $d = 4$, $\sup_{T\ge t}\mathbb{E}[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau]$ decays on the order of $t^{-1}\log t$, while for $d \ge 5$ it behaves like $t^{-2/(d-2)}$, as $t\to\infty$.

**Keywords.** Conditional empirical measure, Dirichlet diffusion process, Wasserstein distance, eigenvalues, eigenfunctions

## 1. Introduction

Let $M$ be a $d$-dimensional connected compact Riemannian manifold with a smooth boundary $\partial M$. Let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on $M$, where $\mathrm{d}x$ is the Riemannian volume measure. Let $X_t$ be the diffusion process generated by $L := \Delta + \nabla V$ with hitting time

$$\tau := \inf\{t \ge 0 : X_t \in \partial M\}.$$

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Denote by $\mathscr{P}$ the set of all probability measures on $M$, and let $\mathbb{E}^\nu$ be the expectation taken for the diffusion process with initial distribution $\nu \in \mathscr{P}$. We consider the empirical measure

$$\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s, \quad t > 0,$$

under the condition that $t < \tau$. Since $\tau = 0$ when $X_0 \in \partial M$, to ensure $\mathbb{P}^\nu(\tau > t) > 0$, where $\mathbb{P}^\nu$ is the probability taken for the diffusion process with initial distribution $\nu$, we only consider

$$\nu \in \mathscr{P}_0 := \{\nu \in \mathscr{P} : \ \nu(M^\circ) > 0\}, \quad M^\circ := M \setminus \partial M.$$

Let $\{\lambda_m\}_{m\ge 0}$ be all Dirichlet eigenfunctions of $-L$ on $M$, listed in increasing order counting multiplicities. Let $\{\phi_m\}_{m\ge 0}$ be the associated unit Dirichlet eigenfunctions, i.e. $L\phi_m = -\lambda_m\phi_m, \phi_m|_{\partial M} = 0$ and $\{\phi_m\}_{m\ge0}$ is an orthonormal basis of $L^2(\mu)$. Moreover, we take $\phi_0|_{M^\circ} > 0$ as $\phi_0$ is non-zero in $M^\circ$. It is well known (see [5]) that $\lambda_0 > 0$ and

$$\|\phi_m\|_\infty \le \alpha_0\sqrt{m}, \quad \alpha_0^{-1}m^{2/d} \le \lambda_m - \lambda_0 \le \alpha_0 m^{2/d}, \quad m \ge 1, \tag{1.1}$$

for some constant $\alpha_0 > 1$.

Let $\mu_0 = \phi_0^2\mu$. We investigate the convergence rate of $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,t<\tau]$ as $t\to\infty$, where $\mathbb{W}_2$ is the $L^2$-Wasserstein distance induced by the Riemannian metric $\rho$. In general, for any $p \ge 1$,

$$\mathbb{W}_p(\mu_1,\mu_2) := \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)}\bigg(\int_{M\times M}\rho(x,y)^p\,\pi(\mathrm{d}x,\mathrm{d}y)\bigg)^{1/p}, \quad \mu_1,\mu_2 \in \mathscr{P},$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the set of all probability measures on $M\times M$ with marginal distributions $\mu_1$ and $\mu_2$, and $\rho(x,y)$ is the Riemannian distance between $x$ and $y$, i.e. the length of the shortest curve on $M$ linking $x$ and $y$.

Recently, the convergence rate under $\mathbb{W}_2$ has been characterized in [24] for the empirical measures of the $L$-diffusion processes without boundary (i.e. $\partial M = \emptyset$) or with a reflecting boundary. Moreover, the convergence of $\mathbb{W}_2(\mu_t^\nu,\mu_0)$ for the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu(\mu_t\,|\,t<\tau), \quad t > 0,$$

is investigated in [20]. Compared with $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,t<\tau]$, in $\mu_t^\nu$ the conditional expectation is inside the Wasserstein distance. According to [20], $\mathbb{W}_2(\mu_t^\nu,\mu_0)^2$ behaves like $t^{-2}$, whereas the following result says that $\mathbb{E}[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,t<\tau]$ decays at a slower rate, which coincides with the rate of $\mathbb{E}[\mathbb{W}_2(\hat\mu_t,\mu)^2]$ given by [24, Theorems 1.1, 1.2], where $\hat\mu_t$ is the empirical measure of the reflecting diffusion process generated by $L$. See also [21] for the study of diffusion processes on non-compact manifolds, [22] for semilinear SPDEs, and [23] for subordinated diffusions.

**Theorem 1.1.** *Let $\{\lambda_m\}_{m\ge0}$ be the Dirichlet eigenvalues of $-L$ listed in increasing order counting multiplicities. Then for any $\nu\in\mathscr{P}_0$, the following assertions hold.*

(1) *In general,*

$$\limsup_{t\to\infty} t \sup_{T\geq t} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] \leq \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}, \tag{1.2}$$

*and there exists a constant* $c>0$ *such that*

$$\liminf_{t\to\infty} t \inf_{T\geq t} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] \geq c \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}. \tag{1.3}$$

*If* $\partial M$ *is convex and* $d\leq 3$, *then*

$$\lim_{t\to\infty} \sup_{T\geq t} \left| t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] - \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}\right| = 0.$$

(2) *When* $d=4$, *there exists a constant* $c>0$ *such that*

$$\sup_{T\geq t} \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] \leq ct^{-1}\log t, \quad t\geq 2. \tag{1.4}$$

(3) *When* $d\geq 5$, *there exist a constant* $c>1$ *such that*

$$\begin{aligned} c^{-1}t^{-\frac{2}{d-2}} &\leq \mathbb{E}^\nu[\mathbb{W}_1(\mu_t,\mu_0)^2 \,|\, T<\tau] \leq \mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2 \,|\, T<\tau] \\ &\leq ct^{-\frac{2}{d-2}}, \quad T\geq t\geq 2. \end{aligned}$$

Let $X_t^0$ be the diffusion process generated by $L_0 := L + 2\nabla \log \phi_0$ in $M^\circ$. It is easy to see that for any initial distribution supported on $M^\circ$ and any $t_0>0$, the law of $\{X_t^0 : t\in[0,t_0]\}$ is the weak limit of the conditional distribution of $\{X_t : t\in[0,t_0]\}$ given $T<\tau$ as $T\to\infty$. Indeed, for $T>t_0$ and $s\in[0,t_0]$, let $\psi_s = P_s^D 1$ and let $\{X_t^T : t\in[0,T)\}$ be the diffusion process on $M^\circ$ generated by $\Delta + 2\nabla\log\psi_{T-t}$, $t\in[0,T)$. It is easy to see that for any $f\in C_0^\infty(M^\circ)$ and $t\in(0,t_0]$, the process

$$M_s := \frac{P_{t-s}^D(f\psi_{T-t})}{P_{t-s}^D\psi_{T-t}}(X_s^T), \quad s\in[0,t],$$

is a martingale, so that

$$\mathbb{E}^x[f(X_t)\,|\,T<\tau] = \frac{P_t^D(f\psi_{T-t})}{P_t^D\psi_{T-t}}(x) = \mathbb{E}^x M_0 = \mathbb{E}^x M_t = \mathbb{E}^x[f(X_t^T)], \quad x\in M^\circ.$$

By the Markov property, this implies that the law of $\{X_t^T : t\in[0,t_0]\}$ coincides with the conditional law of $\{X_t : t\in[0,t_0]\}$ given $T<\tau$. Since

$$\lim_{T\to\infty} \nabla\log\psi_{T-t} = \nabla\log\phi_0$$

locally uniformly on $M^\circ\times[0,t_0]$, as $T\to\infty$ the law of $\{X_t^T : t\in[0,t_0]\}$ converges weakly to the law of $\{X_t^0 : t\in[0,t_0]\}$. In conclusion, the conditional distribution of $\{X_t : t\in[0,t_0]\}$ given $T<\tau$ converges weakly to the law of $\{X_t^0 : t\in[0,t_0]\}$ as $T\to\infty$. Therefore, the following is a direct consequence of Theorem 1.1.

**Corollary 1.2.** *Let* $\mu_t^0 = \frac{1}{t}\int_0^t \delta_{X_s^0}\,\mathrm{d}s$. *Let* $\nu \in \mathscr{P}_0$ *with* $\nu(M^\circ) = 1$.

(1) *In general,*

$$\limsup_{t\to\infty} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^0,\mu_0)^2] \le \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2},$$

*and there exists a constant* $c>0$ *such that*

$$\liminf_{t\to\infty} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^0,\mu_0)^2] \ge c\sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}.$$

*If* $\partial M$ *is convex, then*

$$\lim_{t\to\infty} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^0,\mu_0)^2] = \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}.$$

(2) *When* $d=4$, *there exists a constant* $c>0$ *such that*

$$\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^0,\mu_0)^2] \le ct^{-1}\log t, \quad t\ge 2.$$

(3) *When* $d\ge 5$, *there exists a constant* $c>1$ *such that*

$$c^{-1}t^{-\frac{2}{d-2}} \le \mathbb{E}^\nu[\mathbb{W}_2(\mu_t^0,\mu_0)^2] \le ct^{-\frac{2}{d-2}}, \quad t\ge 2.$$

In the next section, we first recall some facts on the Dirichlet semigroup and the diffusion semigroup $P_t^0$ generated by $L_0 := L + 2\nabla\log\phi_0$, and then establish the Bismut derivative formula for $P_t^0$ which will be used to estimate the lower bound of $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,t<\tau)$. With these preparations, in Sections 3 and 4 we prove Propositions 3.1 and 4.1, which imply Theorem 1.1.

## 2. Some preparations

As in [24], we first recall some well known facts on the Dirichlet semigroup; see for instances [5, 6, 12, 19]. Let $\rho_\partial$ be the Riemannian distance function to the boundary $\partial M$. Then $\phi_0^{-1}\rho_\partial$ is bounded such that

$$\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty, \quad p\in[1,3). \tag{2.1}$$

The Dirichlet heat kernel has the representation

$$p_t^D(x,y) = \sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\phi_m(x)\phi_m(y), \quad t>0,\ x,y\in M. \tag{2.2}$$

Let $\mathbb{E}^x$ denote the expectation for the $L$-diffusion process starting at $x$. Then the Dirichlet diffusion semigroup generated by $L$ is given by

$$\begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t)1_{\{t<\tau\}}] = \int_M p_t^D(x,y)f(y)\,\mu(\mathrm{d}y) \\ &= \sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\mu(\phi_m f)\phi_m(x), \quad t>0,\ f\in L^2(\mu). \end{aligned} \tag{2.3}$$

Consequently,

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t}\mathbb{P}^\nu(t<\tau)=\lim_{t\to\infty}\mathrm{e}^{\lambda_0 t}\nu(P_t^D 1)=\mu(\phi_0)\nu(\phi_0),\quad \nu\in\mathscr{P}_0. \tag{2.4}$$

Moreover, there exists a constant $c>0$ such that

$$\|P_t^D\|_{L^p(\mu)\to L^q(\mu)} := \sup_{\mu(|f|^p)\le 1}\|P_t^D f\|_{L^q(\mu)}\le c\mathrm{e}^{-\lambda_0 t}(1\wedge t)^{-\frac{d(q-p)}{2pq}},\quad t>0,\, q\ge p\ge 1. \tag{2.5}$$

Indeed, the Sobolev inequality implies

$$\|P_t^D\|_{L^1(\mu)\to L^\infty(\mu)}\le c(t\wedge 1)^{-d/2},\quad t>0,$$

for some constant $c>0$, which together with

$$\|P_t^D\|_{L^2(\mu)}\le \mathrm{e}^{-\lambda_0 t},\quad t\ge 0,$$

and the interpolation theorem (see for instance [6]) implies (2.5).

On the other hand, let

$$L_0=L+2\nabla\log\phi_0.$$

Noting that $L_0 f=\phi_0^{-1}L(f\phi_0)+\lambda_0 f$, $L_0$ is a self-adjoint operator in $L^2(\mu_0)$ and the associated semigroup $P_t^0:=\mathrm{e}^{tL_0}$ satisfies

$$P_t^0 f=\mathrm{e}^{\lambda_0 t}\phi_0^{-1}P_t^D(f\phi_0),\quad f\in L^2(\mu_0),\, t\ge 0. \tag{2.6}$$

So, $\{\phi_0^{-1}\phi_m\}_{m\ge 0}$ is an eigenbasis of $L_0$ in $L^2(\mu_0)$ with

$$L_0(\phi_m\phi_0^{-1})=-(\lambda_m-\lambda_0)\phi_m\phi_0^{-1},\quad P_t^0(\phi_m\phi_0^{-1})=\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1},\quad m\ge 0,\, t\ge 0. \tag{2.7}$$

Consequently,

$$P_t^0 f=\sum_{m=0}^\infty \mu_0(f\phi_m\phi_0^{-1})\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1},\quad f\in L^2(\mu_0), \tag{2.8}$$

and the heat kernel of $P_t^0$ with respect to $\mu_0$ is given by

$$p_t^0(x,y)=\sum_{m=0}^\infty(\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y)\mathrm{e}^{-(\lambda_m-\lambda_0)t},\quad x,y\in M,\, t>0. \tag{2.9}$$

By the intrinsic ultracontractivity (see for instance [13]), there exists a constant $\alpha_1\ge 1$ such that

$$\|P_t^0-\mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)} := \sup_{\mu_0(|f|)\le 1}\|P_t^0 f-\mu_0(f)\|_\infty\le \frac{\alpha_1\mathrm{e}^{-(\lambda_1-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{2}}},\quad t>0. \tag{2.10}$$

Combining this with the semigroup property and the contraction of $P_t^0$ in $L^p(\mu)$ for any $p \geq 1$, we find a constant $\alpha_2 \geq 1$ such that

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(|f|^p)\leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \leq \alpha_2 \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \geq 0,\ p \geq 1. \tag{2.11}$$

By the interpolation theorem, (2.10) and (2.11) imply that for some constant $\alpha_3 > 0$,

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)} \leq \alpha_3 \mathrm{e}^{-(\lambda_1-\lambda_0)t}\{1 \wedge t\}^{-\frac{(d+2)(q-p)}{2pq}}, \quad t > 0, \infty \geq q > p \geq 1. \tag{2.12}$$

By this and (2.7), there exists a constant $\alpha_4 > 0$ such that

$$\|\phi_m \phi_0^{-1}\|_\infty \leq \alpha_4 m^{\frac{d+2}{2d}}, \quad m \geq 1. \tag{2.13}$$

In the remainder of this section, we establish the Bismut derivative formula for $P_t^0$, which is not covered by existing results due to the singularity of $\nabla \log \phi_0$ included in $L_0$: $|\nabla \log \phi_0| \sim \rho_\partial^{-1}$ and $\|\mathrm{Hess}_{\log \phi_0}\| \sim \rho_\partial^{-2}$ around the boundary, where Hess is the Hessian tensor. Let $X_t^0$ be the diffusion process generated by $L_0$, which solves the following Itô SDE on $M^\circ$ (see [8]):

$$\mathrm{d}^I X_t^0 = \nabla(V + 2\log \phi_0)(X_t^0)\mathrm{d}t + \sqrt{2}\, U_t \mathrm{d}B_t, \tag{2.14}$$

where $\mathrm{d}^I$ is Itô's differential, $B_t$ is the $d$-dimensional Brownian motion, and $U_t \in O_{X_t^0}(M)$ is the horizontal lift of $X_t^0$ to the frame bundle $O(M)$. Let Ric and Hess be the Ricci curvature and the Hessian tensor on $M$ respectively. Then the Bakry–Émery curvature of $L_0$ is given by

$$\mathrm{Ric}_{L_0} := \mathrm{Ric} - \mathrm{Hess}_{V+2\log \phi_0}.$$

Let $\mathrm{Ric}^\#_{L^0}(U_t) \in \mathbb{R}^d \otimes \mathbb{R}^d$ be defined by

$$\langle \mathrm{Ric}^\#_{L^0}(U_t)a, b\rangle_{\mathbb{R}^d} = \mathrm{Ric}_{L_0}(U_t a, U_t b), \quad a, b \in \mathbb{R}^d.$$

We consider the following ODE on $\mathbb{R}^d \otimes \mathbb{R}^d$:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t = -\mathrm{Ric}^\#_{L^0}(U_t) Q_t, \quad Q_0 = I, \tag{2.15}$$

where $I$ is the identity matrix.

**Lemma 2.1.** *For any $\varepsilon > 0$, there exist constants $\delta_1, \delta_2 > 0$ such that*

$$\mathbb{E}^x[\mathrm{e}^{\delta_1 \int_0^t \phi_0(X_s^0)^{-2}\,\mathrm{d}s}] \leq \delta_2 \phi_0(x)^{-\varepsilon} \mathrm{e}^{\delta_2 t}, \quad t \geq 0, x \in M^\circ. \tag{2.16}$$

*Consequently:*

(1) *For any $\varepsilon > 0$ and $p > 1$, there exists a constant $\kappa > 0$ such that*

$$|\nabla P_t^0 f(x)|^2 \leq \kappa \phi_0(x)^{-\varepsilon} \mathrm{e}^{\kappa t} \{P_t^0 |\nabla f|^{2p}(x)\}^{1/p}, \quad f \in C_b^1(M).$$

(2) *For any $\varepsilon > 0$ and $p \geq 1$, there exists a constant $\kappa > 0$ such that for any stopping time $\tau'$,*

$$\mathbb{E}^x[\|Q_{t\wedge\tau'}\|^p] \leq \kappa\phi_0(x)^{-\varepsilon}\mathrm{e}^{\kappa t}, \quad t \geq 0.$$

*Proof.* Since $L\phi_0 = -\lambda_0\phi_0$, $\phi_0 > 0$ in $M^\circ$, $\|\phi_0\|_\infty < \infty$ and $|\nabla\phi_0|$ is strictly positive in a neighborhood of $\partial M$, we find constants $c_1, c_2 > 0$ such that

$$L_0 \log\phi_0^{-1} = -\phi_0^{-1}L\phi_0 + \phi_0^{-2}|\nabla\phi_0|^2 - 2\phi_0^{-2}|\nabla\phi_0|^2 \leq c_1 - c_2\phi_0^{-2}.$$

So, by (2.14) and Itô's formula, we obtain

$$\mathrm{d}\log\phi_0(X_t^0)^{-1} \leq \{c_1 - c_2\phi_0(X_t^0)^{-2}\}\mathrm{d}t + \sqrt{2}\,\langle\nabla\log\phi_0(X_t^0)^{-1}, U_t\mathrm{d}B_t\rangle.$$

This implies

$$\begin{aligned}\mathbb{E}^x\int_0^t[\phi_0(X_s^0)^{-2}]\,\mathrm{d}s &\leq \frac{1}{c_2}(c_1 t + \log\phi_0(x)^{-1} + \mathbb{E}\log\phi_0(X_t^0))\\ &\leq \frac{1}{c_2}(c_1 t + \log\phi_0(x)^{-1} + \log\|\phi_0\|_\infty) \leq ct + c\log(1+\phi_0(x)^{-1}), \quad t \geq 0, \end{aligned} \tag{2.17}$$

for some constant $c > 0$, and for any constant $\delta > 0$,

$$\begin{aligned}\mathbb{E}^x\big[\mathrm{e}^{\delta c_2\int_0^t\phi_0(X_s^0)^{-2}\}\,\mathrm{d}s}\big] &\leq \mathbb{E}^x\big[\mathrm{e}^{\delta\log\phi_0(x)^{-1}+\delta\log\phi_0(X_t^0)+c_1\delta t-\delta\sqrt{2}\int_0^t\langle\nabla\log\phi_0(X_s^0),U_s\,\mathrm{d}B_s\rangle}\big]\\ &\leq \mathrm{e}^{c_1\delta t}\phi_0(x)^{-\delta}\|\phi_0\|_\infty^\delta(\mathbb{E}^x[\mathrm{e}^{4\delta^2\int_0^t|\nabla\log\phi_0|^2(X_s^0)\,\mathrm{d}s}])^{1/2}.\end{aligned}$$

Letting $c_3 = 4\|\nabla\phi_0\|_\infty^2$ and taking $\delta \in (0, c_2/c_3]$, we derive

$$\mathbb{E}^x\big[\mathrm{e}^{\delta c_2\int_0^t\phi_0(X_s^0)^{-2}\}\mathrm{d}s}\big] \leq \mathrm{e}^{2c_1\delta t}\phi_0(x)^{-2\delta}, \quad \delta \in (0, c_2/c_3].$$

This implies (2.16) for $\varepsilon \in (0, 2c_2/c_3]$. Since for $\varepsilon > 2c_2/c_3$ we have

$$\phi_0^{-\varepsilon} \geq \phi_0^{-2c_2/c_3}\|\phi_0\|_\infty^{\varepsilon-2c_2/c_3},$$

(2.16) also holds for $\varepsilon > 2c_2/c_3$. Below we prove assertions (1) and (2).

Since $V \in C_b^2(M)$, $\phi_0 \in C_b^2(M)$ with $\phi_0 > 0$ in $M^\circ$, and

$$-\mathrm{Hess}_{\log\phi_0} = -\frac{\mathrm{Hess}_{\phi_0}}{\phi_0} + \frac{\nabla\phi_0\otimes\nabla\phi_0}{\phi_0^2} \geq -\frac{\mathrm{Hess}_{\phi_0}}{\phi_0},$$

we find a constant $\alpha_1 > 0$ such that

$$\mathrm{Ric}_{L_0}(U,U) \geq -\alpha_1\phi_0(x)^{-1}|U|^2, \quad x \in M^\circ,\ U \in T_xM. \tag{2.18}$$

By (2.14), (2.18), and the formulas of Itô and Bochner, for fixed $t > 0$ this implies

$$\begin{aligned}&\mathrm{d}|\nabla P_{t-s}^0 f|^2(X_s^0)\\ &= \{L_0|\nabla P_{t-s}^0 f|^2(X_s^0) - 2\langle\nabla P_{t-s}^0 f, \nabla L_0 P_{t-s}^0 f\rangle\}\mathrm{d}s + \sqrt{2}\,\langle\nabla|\nabla P_{t-s}^0 f|^2(X_s^0), U_s\mathrm{d}B_s\rangle\\ &\geq 2\mathrm{Ric}_{L^0}(\nabla P_{t-s}^0 f, \nabla P_{t-s}^0 f)(X_s^0)\mathrm{d}s + \sqrt{2}\,\langle\nabla|\nabla P_{t-s}^0 f|^2(X_s^0), U_s\mathrm{d}B_s\rangle\\ &\geq -2\alpha_1\{\phi_0^{-1}|\nabla P_{t-s}^0 f|^2\}(X_s^0)\mathrm{d}s + \sqrt{2}\,\langle\nabla|\nabla P_{t-s}^0 f|^2(X_s), U_s\mathrm{d}B_s\rangle\mathrm{d}s.\end{aligned}$$

Then by a Gronwall type inequality, we obtain

$$\begin{aligned}|\nabla P_t^0 f(x)|^2 &= \mathbb{E}^x|\nabla P_t^0 f|^2(X_0^0) \le \mathbb{E}^x[|\nabla f|^2(X_t^0)\mathrm{e}^{2\int_0^t 2\alpha_1\phi_0(X_u^0)^{-1}\,\mathrm{d}u}]\\ &\le \big\{\mathbb{E}^x\big[\mathrm{e}^{\frac{2\alpha_1 p}{p-1}\int_0^t \phi_0(X_u^0)^{-1}\mathrm{d}u}\big]\big\}^{\frac{p-1}{p}}\{P_t^0|\nabla f|^{2p}(x)\}^{1/p}.\end{aligned}$$

Noting that

$$\frac{2\alpha_1 p}{p-1}\int_0^t \phi_0(X_s^0)^{-1}\,\mathrm{d}s \le \delta_1\int_0^t \phi_0(X_s^0)^{-2}\,\mathrm{d}s + \frac{\alpha_1^2 p^2 t}{\delta_1(p-1)^2}, \quad \delta_1 > 0, \tag{2.19}$$

by combining this with (2.16), we prove (1).

Next, by (2.15) and (2.18), we obtain

$$\|Q_{t\wedge\tau'}\| \le \mathrm{e}^{\alpha_1\int_0^t \phi(X_s^0)^{-1}\mathrm{d}s}, \quad t \ge 0.$$

This together with (2.16) and (2.19) implies (2). ∎

**Lemma 2.2.** *For any $t > 0$ and $\gamma \in C^1([0,t])$ with $\gamma(0) = 0$ and $\gamma(t) = 1$, we have*

$$\nabla P_t^0 f(x) = \mathbb{E}^x\bigg[f(X_t^0)\int_0^t \gamma'(s)Q_s^*\,\mathrm{d}B_s\bigg], \quad x \in M^\circ,\ f \in \mathscr{B}_b(M^\circ), \tag{2.20}$$

*where $\mathscr{B}_b(M^\circ)$ is the set of all bounded measurable functions on $M^\circ$. Consequently, for any $\varepsilon > 0$ and $p > 1$, there exists a constant $c > 0$ such that*

$$|\nabla P_t^0 f| \le \frac{c\phi_0^{-\varepsilon}}{\sqrt{1\wedge t}}(P_t^0|f|^p)^{1/p}, \quad t > 0,\ f \in \mathscr{B}_b(M^\circ). \tag{2.21}$$

*Proof.* Since (2.21) follows from (2.20) with $\gamma(s) := \frac{t-s}{t}$ and Lemma 2.1 (2), it suffices to prove the Bismut formula (2.20). By an approximation argument, we only need to prove the formula for $f \in C_b^1(M)$. The proof is standard by Elworthy–Li's martingale argument [7]; see also [14]. By $\|\nabla f\|_\infty < \infty$ and Lemma 2.1 (1) for $\varepsilon = 1/4$, we find a constant $c_1 > 0$ such that

$$|\nabla P_s^0 f|(x) \le c_1\phi_0(x)^{-1/4}, \quad s \in [0,t],\ x \in M^\circ. \tag{2.22}$$

Next, since $L\phi_0 = -\lambda_0\phi_0$ implies $L_0\phi_0^{-1} = \lambda_0\phi_0^{-1}$, by Itô's formula we obtain

$$\mathbb{E}^x[\phi_0(X_{t\wedge\tau_n}^0)^{-1}] \le \phi_0(x)^{-1}\mathrm{e}^{\lambda_0 t}, \quad t \ge 0,\ n \ge 1, \tag{2.23}$$

where $\tau_n := \inf\{t \ge 0 : \phi_0(X_s^0) \le 1/n\} \uparrow \infty$ as $n \uparrow \infty$ by noting that the process $X_t^0$ is non-explosive in $M^\circ$.

Moreover, by Itô's formula, for any $a \in \mathbb{R}^d$ we have

$$\begin{aligned}&\mathrm{d}\langle\nabla P_{t-s}^0 f(X_s^0), U_sQ_sa\rangle = \sqrt{2}\,\mathrm{Hess}_{P_{t-s}f}(U_s\mathrm{d}B_s, U_sQ_sa)(X_s^0),\\ &\mathrm{d}P_{t-s}^0 f(X_s^0) = \sqrt{2}\,\langle\nabla P_{t-s}^0 f(X_s^0), U_s\mathrm{d}B_s\rangle, \quad s \in [0,t].\end{aligned}$$

Due to the integration by parts formula, this and $\gamma(0) = 0$ imply

$$
\begin{aligned}
&-\frac{1}{\sqrt{2}}\mathbb{E}^x\bigg[f(X^0_{t\wedge\tau_n})\int_0^{t\wedge\tau_n}\gamma'(s)\langle Q_s a, \mathrm{d}B_s\rangle\bigg]\\
&= \mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\langle\nabla P^0_{t-s}f(X^0_s), U_s Q_s a\rangle\,\mathrm{d}(1-\gamma)(s)\bigg]\\
&= \mathbb{E}\big[(1-\gamma)(t\wedge\tau_n)\langle\nabla P^0_{t-t\wedge\tau_n}f(X^0_{t\wedge\tau_n}), Q_{t\wedge\tau_n}a\rangle\big] - \langle\nabla P^0_t f(x), U_0 a\rangle\\
&\quad - \mathbb{E}\bigg[\int_0^{t\wedge\tau_n}(1-\gamma)(s)\,\mathrm{d}\langle\nabla P^0_{t-s}f(X^0_s), U_s Q_s a\rangle\bigg]\\
&= \mathbb{E}\big[(1-\gamma)(t\wedge\tau_n)\langle\nabla P^0_{t-t\wedge\tau_n}f(X^0_{t\wedge\tau_n}), Q_{t\wedge\tau_n}a\rangle\big] - \langle\nabla P_t f(x), U_0 a\rangle, \quad n\ge 1.
\end{aligned}
\tag{2.24}
$$

Since $\gamma$ is bounded with $\gamma(t) = 1$ such that $(1-\gamma)(t\wedge\tau_n)\to 0$ as $n\to\infty$, and (2.22), (2.23) and Lemma 2.1(2) imply

$$
\begin{aligned}
&\sup_{n\ge 1}\mathbb{E}^x[\langle\nabla P^0_{t-t\wedge\tau_n}f(X^0_{t\wedge\tau_n}), Q_{t\wedge\tau_n}a\rangle^2]\\
&\qquad\qquad \le c_1\sup_{n\ge 1}(\mathbb{E}[\phi_0(X^0_{t\wedge\tau_n})^{-1}])^{1/2}(\mathbb{E}^x\|Q_{t\wedge\tau_n}\|^4)^{1/2} < \infty,
\end{aligned}
$$

by the dominated convergence theorem we may take $n\to\infty$ in (2.24) to derive (2.20). ∎

## 3. Upper bound estimates

In this section we prove the following result which includes upper bound estimates in Theorem 1.1.

**Proposition 3.1.** *Let $\nu\in\mathscr{P}_0$.*

(1) (1.2) *holds.*

(2) *When $d = 4$, there exists a constant $c > 0$ such that* (1.4) *holds.*

(3) *When $d\ge 5$, there exists a constant $c > 0$ such that*

$$
\sup_{T\ge t}\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,T<\tau]\le ct^{-\frac{2}{d-2}}, \quad t\ge 2.
$$

The main tool in the study of the upper bound estimate is the following inequality due to [1] (see also [24, Lemma 2.3]): for any probability density $g\in L^2(\mu_0)$,

$$
\mathbb{W}_2(g\mu_0,\mu_0)^2\le\int_M\frac{|\nabla L_0^{-1}(g-1)|^2}{\mathscr{M}(g,1)}\,\mathrm{d}\mu_0, \tag{3.1}
$$

where $\mathscr{M}(a,b) := \frac{a-b}{\log a-\log b}1_{\{a\wedge b>0\}}$. To apply this inequality, as in [24], we first modify $\mu_t$ to $\mu_{t,r} := \mu_t P^0_r$ for some $r>0$, where for a probability measure $\nu$ on $M^\circ$, $\nu P^0_r$ is the law of the $L_0$-diffusion process $X_r^0$ with initial distribution $\nu$. Obviously, by (2.9) we have

$$\begin{aligned}\rho_{t,r} &:= \frac{\mathrm{d}\mu_{t,r}}{\mathrm{d}\mu_0} = \frac{1}{t}\int_0^t p_r^0(X_s,\cdot)\,\mathrm{d}s = 1 + \sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)r}\psi_m(t)\phi_m\phi_0^{-1},\\ \psi_m(t) &:= \frac{1}{t}\int_0^t \{\phi_m\phi_0^{-1}\}(X_s)\,\mathrm{d}s,\end{aligned} \tag{3.2}$$

which are well-defined on the event $\{t < \tau\}$.

**Lemma 3.2.** (1) *If $d \le 3$ and $\nu = h\mu$ with $h\phi_0^{-1} \in L^p(\mu_0)$ for some $p > \frac{d+2}{2}$, then there exists a constant $c > 0$ such that*

$$\sup_{T\ge t}\left|t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] - 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\right|$$
$$\le ct^{-1}(r^{-\frac{(d-2)^+}{2}} + 1_{\{d=2\}}\log r^{-1}),\quad r\in(0,1],\ t\ge 1.$$

(2) *If $d \ge 4$ and $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$, then for any $k > \frac{d-4}{6}$, there exists a constant $c > 0$ such that*

$$\sup_{T\ge t} t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau]$$
$$\le c\{r^{-\frac{d-4}{2}} + 1_{\{d=4\}}\log r^{-1} + t^{-1}r^{-k}\},\quad t\ge 1,\ r\in(0,1).$$

*Proof.* By the Markov property, (2.6) and (2.3), we have

$$\begin{aligned}\mathbb{E}^x[f(X_s)1_{\{T<\tau\}}] &= \mathbb{E}^x[1_{\{s<\tau\}}f(X_s)\mathbb{E}^{X_s}1_{\{T-s<\tau\}}]\\ &= P_s^D\{fP_{T-s}^D 1\}(x) = \mathrm{e}^{-\lambda_0 T}(\phi_0 P_s^0\{fP_{T-s}^0\phi_0^{-1}\})(x),\quad s<T.\end{aligned} \tag{3.3}$$

For the same reason, and noting that $\mathbb{E}^\nu = \int_M \mathbb{E}^x\,\nu(\mathrm{d}x)$, we derive

$$\begin{aligned}\mathbb{E}^\nu[f(X_{s_1})f(X_{s_2}))1_{\{T<\tau\}}] &= \int_M \mathbb{E}^x[1_{\{s_1<\tau\}}f(X_{s_1})\mathbb{E}^{X_{s_1}}\{f(X_{s_2-s_1})1_{\{T-s_1<\tau\}}\}]\,\nu(\mathrm{d}x)\\ &= \mathrm{e}^{-\lambda_0 T}\nu\big(\phi_0 P_{s_1}^0[fP_{s_2-s_1}^0\{fP_{T-s_2}^0\phi_0^{-1}\}]\big),\quad s_1<s_2<T.\end{aligned}$$

In particular, the formula with $f = 1$ yields

$$\mathbb{P}^\nu(T<\tau) = \mathrm{e}^{-\lambda_0 T}\nu(\phi_0 P_T^0\phi_0^{-1}). \tag{3.4}$$

Since $\{\frac{\nabla(\phi_m\phi_0^{-1})}{\sqrt{\lambda_m-\lambda_0}}\}_{m\ge1}$ is othornormal in $L^2(\mu_0)$, we have

$$\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2) = \sum_{m=1}^\infty \frac{\psi_m(t)^2}{(\lambda_m-\lambda_0)\mathrm{e}^{2(\lambda_m-\lambda_0)r}}.$$

Combining these with (3.2), we obtain

$$t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] = \sum_{m=1}^\infty \frac{t\mathbb{E}^\nu[\psi_m(t)^2\,|\,T<\tau]}{\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)}.$$

Noting that

$$\mathbb{E}^\nu(\xi\,|\,T<\tau) := \frac{\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}]}{\mathbb{P}^\nu(T<\tau)}$$

for an integrable random variable $\xi$, by (2.7) and the symmetry of $P_t^0$ in $L^2(\mu_0)$, we deduce from this for $\nu = h\mu$ that

$$\begin{aligned}
&t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau]\\
&\quad= \sum_{m=1}^\infty \frac{2\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathbb{E}^\nu[1_{\{T<\tau\}}(\phi_m\phi_0^{-1})(X_{s_1})(\phi_m\phi_0^{-1})(X_{s_2})]\,\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\nu(\phi_0 P_T^0\phi_0^{-1})}\\
&\quad= \sum_{m=1}^\infty \frac{2\int_0^t \mathrm{d}s_1 \int_{s_1}^t \nu\big(\phi_0^{-1}P_{s_1}^0\{\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\}\big)\,\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\nu(\phi_0 P_T^0\phi_0^{-1})}\\
&\quad= \sum_{m=1}^\infty \frac{2\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\big)\,\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\mu_0(\phi_0^{-1}P_T^0(h\phi_0^{-1}))}.
\end{aligned} \tag{3.5}$$

Since $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ and $\|h\phi_0^{-1}\|_{L^1(\mu_0)} = \mu(h\phi_0) \le \|\phi_0\|_\infty < \infty$, by (2.12) we find a constant $c_1 > 0$ such that

$$\begin{aligned}
&|\mu_0(\phi_0^{-1}P_T^0(h\phi_0^{-1})) - \mu(\phi_0)\nu(\phi_0)| \le \|\phi_0^{-1}(P_T^0-\mu_0)(h\phi_0^{-1})\|_{L^1(\mu_0)}\\
&\qquad\le \|P_T^0-\mu_0\|_{L^1(\mu_0)\to L^2(\mu_0)}\|h\phi_0^{-1}\|_{L^1(\mu_0)} \le c_1\mathrm{e}^{-(\lambda_1-\lambda_0)T}, \quad T\ge 1.
\end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned}
&\mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\big)\\
&\qquad= \nu(\phi_0)\mu(\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)} + J_1(s_1,s_2) + J_2(s_1,s_2) + J_3(s_1,s_2),
\end{aligned} \tag{3.7}$$

where, due to (2.7),

$$\begin{aligned}
J_1(s_1,s_2) &:= \mu_0\big(\{P_{s_1}^0(h\phi_0^{-1}) - \mu(h\phi_0)\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}(P_{T-s_2}^0\phi_0^{-1}-\mu(\phi_0))]\big),\\
J_2(s_1,s_2) &:= \mu(\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mu_0\big(\{P_{s_1}^0(h\phi_0^{-1}) - \mu(h\phi_0)\}\{\phi_m\phi_0^{-1}\}^2\big),\\
J_3(s_1,s_2) &:= \mu(h\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mu_0\big(\{\phi_m\phi_0^{-1}\}^2\{P_{T-s_2}^0\phi_0^{-1}-\mu(\phi_0)\}\big).
\end{aligned}$$

By (3.5)–(3.7) and

$$\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\,\mathrm{d}s_2 = \frac{t}{\lambda_m-\lambda_0} - \frac{1-\mathrm{e}^{-(\lambda_m-\lambda_0)t}}{(\lambda_m-\lambda_0)^2},$$

we find a constant $\kappa > 0$ such that

$$\sup_{T\geq t}\left|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau\big]-2\sum_{m=1}^\infty\frac{e^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\right|$$
$$\leq\frac{\kappa}{t}\sum_{m=1}^\infty\left(\frac{e^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}+\frac{e^{-2(\lambda_m-\lambda_0)r}}{\lambda_m-\lambda_0}\int_0^t ds_1\int_{s_1}^t|J_1+J_2+J_3|(s_2,s_2)\,ds_2\right),\quad t\geq 1. \tag{3.8}$$

Below we prove assertions (1) and (2).

(1) Let $d\leq 3$. Since $\|h\phi_0^{-1}\|_{L^p(\mu_0)}<\infty$, $\|\phi_0^{-1}\|_{L^\theta(\mu_0)}<\infty$ for $\theta<3$ due to (2.1), $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)}=1$, by (2.12), for any $\theta\in(5/2,3)$, we find constants $c_1,c_2>0$ such that

$$\begin{aligned}|J_1(s_1,s_2)|&\leq c_1\|P^0_{s_1}-\mu_0\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|\cdot\|\phi_m\phi_0^{-1}(P^0_{T-s_2}\phi_0^{-1}-\mu(\phi_0))\|_{L^2(\mu_0)}\\&\leq c_1\|P^0_{s_1}-\mu_0\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|P^0_{T-s_2}-\mu_0\|_{L^\theta(\mu_0)\to L^\infty(\mu_0)}\\&\leq c_2e^{-(\lambda_1-\lambda_0)(s_1+T-s_2)}(1\wedge s_1)^{-\frac{d+2}{2p}}\{1\wedge(T-s_2)\}^{-\frac{d+2}{2\theta}},\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}&|(J_2+J_3)(s_1,s_2)|\\&\leq c_1e^{-(\lambda_m-\lambda_0)(s_2-s_1)}(\|P^0_{s_1}-\mu_0\|_{p\to\infty}+\|P^0_{T-s_2}-\mu_0\|_{L^\theta(\mu_0)\to L^\infty(\mu_0)})\\&\leq c_2e^{-(\lambda_m-\lambda_0)(s_2-s_1)}(\{1\wedge s_1\}^{-\frac{d+2}{2p}}e^{-(\lambda_1-\lambda_0)s_1}+\{1\wedge(T-s_2)\}^{-\frac{d+2}{2\theta}}e^{-(\lambda_1-\lambda_0)(t-s_2)}).\end{aligned}\tag{3.10}$$

Since $\theta>\frac{5}{2}$ and $p>\frac{d+2}{2}$ imply $\frac{d+2}{2\theta}\vee\frac{d+2}{2p}<1$ for $d\leq 3$, by (3.9) and (3.10) we find a constant $c>0$ such that

$$\int_0^t ds_1\int_{s_1}^t|J_1+J_2+J_3|(s_1,s_2)\,ds_2\leq c,\quad T\geq t\geq 1,\ m\geq 1.$$

Combining this with (3.8) and (1.1), we find constants $c_3,c_4,c_5,c_6>0$ such that

$$\begin{aligned}&\sup_{T\geq t}\left|t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau]-\sum_{m=0}^\infty\frac{e^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\right|\\&\qquad\leq\frac{c_3}{t}\sum_{m=1}^\infty\frac{e^{-2(\lambda_m-\lambda_0)r}}{\lambda_m-\lambda_0}\\&\qquad\leq\frac{c_4}{t}\int_1^\infty s^{-2/d}e^{-c_5s^{2/d}r}ds\leq c_6t^{-1}(r^{-\frac{(d-2)^+}{2}}+1_{\{d=2\}}\log r^{-1}),\quad t\geq 1.\end{aligned}$$

(2) Let $d\geq 4$. Let $\nu=h\mu$ with $|h\phi_0^{-1}|\leq C$ for a constant $C>0$, we have

$$\mathbb{E}^\nu=\int_{\mathbb{R}^d}\mathbb{E}^x\,\nu(dx)\leq C\mu(\phi_0)^{-1}\mathbb{E}^{\nu_0}.\tag{3.11}$$

On the other hand, by (3.4), we see that

$$\frac{\mathbb{P}^{\nu}(t<\tau)}{\mathbb{P}^{\nu_0}(t<\tau)} = \frac{\nu(\phi_0 P_t^0 \phi_0^{-1})}{\nu_0(\phi_0 P_t^0 \phi_0^{-1})} > 0$$

is continuous in $t \geq 0$, and it converges to $\frac{\nu(\phi_0)}{\nu_0(\phi_0)} \in (0,\infty)$ due to (2.4). So, there exists a constant $c>0$ such that

$$\mathbb{P}^{\nu}(t<\tau) \geq c\mathbb{P}^{\nu_0}(t<\tau), \quad t \geq 0.$$

Combining this with (3.11), we find a constant $K>0$ such that

$$\mathbb{E}^{\nu}(\cdot\,|\,t<\tau) = \frac{\mathbb{E}^{\nu}(1_{t<\tau}\cdot)}{\mathbb{P}^{\nu}(t<\tau)} \leq \frac{K\mathbb{E}^{\nu_0}(1_{t<\tau}\cdot)}{\mathbb{P}^{\nu_0}(t<\tau)} = K\mathbb{E}^{\nu_0}(\cdot\,|\,t<\tau), \quad t \geq 0. \tag{3.12}$$

So, it suffices to prove (2) for $\nu=\nu_0$. In this case, $h\phi_0^{-1}=\mu(\phi_0)^{-1}$ is a constant, so that $J_1=J_2=0$. By (2.12), $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)}=1$ and $\|\phi_0\|_{L^\theta(\mu_0)}<\infty$ for $\theta\in(1,3)$, we find a constant $c_1>0$ such that

$$\begin{aligned}
|J_2(s_1,s_2)| &\leq c_1 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)} \| |(P_{r/2}^0-\mu_0)(\phi_m\phi_0^{-1})|^2 (P_{T-s_2}^0-\mu_0)\phi_0^{-1}\|_{L^1(\mu_0)} \\
&\leq c_1 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)} \\
&\quad \cdot \|P_{r/2}^0-\mu_0\|^2_{L^{2p}(\mu_0)\to L^2(\mu_0)} \|P_{T-s_2}^0-\mu_0\|_{L^\theta(\mu_0)\to L^{\frac{p}{p-1}}(\mu_0)} \|\phi_0^{-1}\|_{L^\theta(\mu_0)} \\
&\quad c_2 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)-(\lambda_1-\lambda_0)(T-s_2)} r^{-\frac{(d+2)(p-1)}{2p}} [1\wedge(T-s_2)]^{-\frac{(d+2)[\theta-(\theta-1)p]}{2\theta p}}, \\
&\qquad p\in[1,\theta/(\theta-1)].
\end{aligned}$$

Let $p_0 := \frac{3(d+2)}{2d+10}$. Since

$$\begin{aligned}
&\lim_{p\downarrow p_0} \frac{(d+2)(p-1)}{2p} = \frac{d-4}{6}, \\
&\lim_{\theta\uparrow 3} \frac{(d+2)[\theta-(\theta-1)p]}{2\theta p} = \frac{(d+2)(3-2p)}{6p} \in (0,1) \quad \text{if } p\in(p_0,3/2),
\end{aligned}$$

for any $k>\frac{d-4}{6}$ there exist $\theta\in(1,3)$ and $p\in(p_0,\frac{\theta}{\theta-1})$ such that

$$\frac{(d+2)(p-1)}{2p} \leq k, \quad \varepsilon := \frac{(d+2)[\theta-(\theta-1)p]}{2\theta p} < 1.$$

Thus,

$$|J_3(s_1,s_2)| \leq c_2 \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1-r)-(\lambda_1-\lambda_0)(T-s_2)} r^{-k} [1\wedge(T-s_2)]^{-\varepsilon}.$$

Since $J_1=J_2=0$, this implies

$$\int_0^t \mathrm{d}s_1 \int_{s_1}^t |J_1+J_2+J_3|(s_1,s_2)\,\mathrm{d}s_2 \leq \frac{c_3 \mathrm{e}^{(\lambda_m-\lambda_0)r} r^{-k}}{\lambda_m-\lambda_0}, \quad t\geq 1,\ r\in(0,1),$$

for some constant $c_3 > 0$. Substituting into (3.8) for $\nu = \nu_0$, we find a constant $c_4 > 0$ such that

$$\sup_{T\ge t} t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] \\ \le 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} + \frac{c_4 r^{-k}}{t}\sum_{m=1}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},$$

which implies the desired estimate since by (1.1), the inequality

$$\sum_{m=1}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} \le c_5\int_1^\infty s^{-1/d}\mathrm{e}^{-c_5 s^{2/d}r}\mathrm{d}r \le c_6(r^{-\frac{d-4}{2}} + 1_{\{d=4\}}\log r^{-1})$$

holds for some constants $c_5, c_6 > 0$ and all $r\in(0,1)$. ∎

**Lemma 3.3.** *There exists a constant $c>0$ such that for any $t>0$ and non-negative random variable $\xi\in\sigma(X_s : s\le t)$,*

$$\sup_{T\ge t}\mathbb{E}^\nu[\xi\,|\,T<\tau] \le c\mathbb{E}^\nu[\xi\,|\,t<\tau], \quad t\ge 1, \nu\in\mathscr{P}_0.$$

*Proof.* By the Markov property, (2.5) for $p=q=\infty$ and (2.4), we find constants $c_1, c_2 > 0$ such that

$$\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}] = \mathbb{E}^\nu[\xi 1_{\{t<\tau\}} P_{T-t}^D 1(X_t)] \le c_1\mathrm{e}^{-\lambda_0(T-t)}\mathbb{E}^\nu[\xi 1_{\{t<\tau\}}],$$
$$\mathbb{P}^\nu(T<\tau) \ge c_2\mathbb{P}^\nu(t<\tau)\mathrm{e}^{-(T-t)\lambda_0}, \quad T\ge t\ge 1.$$

Then

$$\mathbb{E}^\nu[\xi\,|\,T<\tau] = \frac{\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}]}{\mathbb{P}^\nu(T<\tau)} \le \frac{c_1\mathbb{E}^\nu[\xi 1_{\{t<\tau\}}]}{c_2\mathbb{P}^\nu(t<\tau)} = \frac{c_1}{c_2}\mathbb{E}^\nu[\xi\,|\,t<\tau].$$ ∎

**Lemma 3.4.** *Let $d\le 3$ and denote $\nu_0 = \frac{\phi_0}{\mu(\phi_0)}\mu$. For any $\varepsilon\in(\frac{d}{4}\vee\frac{d^2}{2d+4}, 1)$ there exists a constant $c>0$ such that*

$$\sup_{T\ge t}\mathbb{E}^{\nu_0}[|\rho_{t,r}(y)-1|^2\,|\,T<\tau] \le c\phi_0(y)^{-2}t^{-1}r^{-\varepsilon}, \quad t\ge 1,\ r\in(0,1],\ y\in M^\circ.$$

*Proof.* By Lemma 3.3, it suffices to prove the conclusion with $T=t$ replacing $T\ge t$. For fixed $y\in M^\circ$, let $f = p_r^0(\cdot,y)-1$. We have

$$\rho_{t,r}(y) - 1 = \frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s.$$

Then

$$\mathbb{E}^{\nu_0}[|\rho_{t,r}(y)-1|^2 1_{\{t<\tau\}}] = \frac{2}{t^2}\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathbb{E}^{\nu_0}[1_{\{t<\tau\}}f(X_{s_1})f(X_{s_2})]\,\mathrm{d}s_2. \tag{3.13}$$

By (3.3), $\mu_0(f) = 0$, and the symmetry of $P_t^0$ in $L^2(\mu_0)$, we obtain

$$\begin{aligned}
I &:= \mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}[1_{\{t<\tau\}} f(X_{s_1}) f(X_{s_2})] = \mu(\phi_0)^{-1}\mu_0\big(P_{s_1}^0\{f P_{s_2-s_1}^0 (f P_{t-s_2}^0 \phi_0^{-1})\}\big)\\
&= \mu(\phi_0)^{-1}\mu_0\big(f P_{s_2-s_1}^0 (f P_{t-s_2}^0 \phi_0^{-1})\big) = \mu(\phi_0)^{-1}\mu_0(\{f P_{t-s_2}^0 \phi_0^{-1}\} P_{s_2-s_1}^0 f)\\
&= \mu(\phi_0)^{-1}\mu_0\big(\{f P_{t-s_2}^0 \phi_0^{-1}\}\{P_{s_2-s_1}^0 - \mu_0\} f\big).
\end{aligned} \tag{3.14}$$

Taking $q \in (\frac{5}{2}, 3)$ so that $\varepsilon_1 := \frac{d+2}{2q} < 1$ for $d \le 3$ and $\|\phi_0^{-1}\|_{L^q(\mu_0)} < \infty$ due to (2.1), for any $p \in (1, 2]$ we deduce from this and (2.12) that

$$\begin{aligned}
\mu(\phi_0) I &\le \|f\|_{L^p(\mu_0)} \|P_{t-s_2}^0 \phi_0^{-1}\|_{L^\infty(\mu_0)} \|(P_{s_2-s_1}^0 - \mu_0) f\|_{L^{\frac{p}{p-1}}(\mu_0)}\\
&\le \|f\|_{L^p(\mu_0)} \|P_{t-s_2}^0\|_{L^q(\mu_0)\to L^\infty(\mu_0)} \|\phi_0^{-1}\|_{L^q(\mu_0)} \|P_{s_2-s_1}^0 - \mu_0\|_{L^2(\mu_0)\to L^{\frac{p}{p-1}}(\mu_0)}\\
&\quad \cdot \|f\|_{L^2(\mu_0)}\\
&\le c_1 \|f\|_{L^p(\mu_0)} \|f\|_{L^2(\mu_0)} \{1 \wedge (t-s_2)\}^{-\varepsilon_1} \{1 \wedge (s_2 - s_1)\}^{-\frac{(d+2)(2-p)}{2p}} \mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)}
\end{aligned} \tag{3.15}$$

for some constant $c_1 > 0$. Since $f = p_r^0(\cdot, y) - 1$ and $\inf \phi_0^{-1} > 0$, by (2.5) and (2.6) we find constants $\beta_1, \beta_2 > 0$ such that

$$\begin{aligned}
&\|f\|_{L^p(\mu_0)} \le 1 + \|p_r^0(\cdot, y)\|_{L^p(\mu_0)} \le 1 + \mathrm{e}^{r\lambda_0}\phi_0(y)^{-1} \|\phi_0^{-1} p_r^D(\cdot, y)\|_{L^p(\mu_0)}\\
&\le 1 + \beta_1 \phi_0(y)^{-1} \|\phi_0\|_\infty^{\frac{2-p}{p}} \|p_r^D(\cdot, y)\|_{L^p(\mu)} \le \beta_2 \phi_0(y)^{-1} r^{-\frac{d(p-1)}{2p}}, \quad r \in (0, 1],\ p \in [1, 2].
\end{aligned}$$

Combining this with (3.15) we find a constant $c_2 > 0$ such that

$$I \le c_2 \phi_0(y)^{-2} r^{-\frac{d(p-1)}{2p} - \frac{d}{4}} \{1 \wedge (t - s_2)\}^{-\varepsilon_1} \{1 \wedge (s_2 - s_1)\}^{-\frac{(d+2)(2-p)}{2p}} \mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)},$$
$$p \in (1, 2].$$

Letting $p_0 := 1 \vee \frac{2(d+2)}{d+6}$ and taking $p > p_0$ such that

$$\varepsilon_2 := \frac{(d+2)(2-p)}{4p} \le \frac{5(2-p)}{4p} < 1,$$

we arrive at

$$I \le c_2 \phi_0(y)^{-2} r^{-\frac{d(p-1)}{2p} - \frac{d}{4}} \{1 \wedge (t - s_2)\}^{-\varepsilon_1} \{1 \wedge (s_2 - s_1)\}^{-\varepsilon_2} \mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)}$$

for some constants $\varepsilon_1, \varepsilon_2 \in (0, 1)$. Combining this with (3.13), we obtain

$$\mathbb{E}^{\nu_0}[|\rho_{t,r}(y) - 1|^2 \,|\, t < \tau] \le c\phi_0(y)^{-2} t^{-1} r^{-\frac{d(p-1)}{2p} - \frac{d}{4}}, \quad t \ge 1.$$

Noting that

$$\lim_{p \downarrow p_0} \left\{ \frac{d(p-1)}{2p} + \frac{d}{4} \right\} = \frac{d}{4} \vee \frac{d^2}{2d+4} < 1 \quad \text{for } d \le 3,$$

for any $\varepsilon \in (\frac{d}{4} \vee \frac{d^2}{2d+4}, 1)$ there exists $p > p_0$ such that $\frac{d}{4} \vee \frac{d^2}{2d+4} \le \varepsilon$. Therefore, the proof is finished. ∎

**Lemma 3.5.** *Let $d \le 3$ and denote $\psi_m(t) = \frac{1}{t}\int_0^t (\phi_m\phi_0^{-1})(X_s)\,\mathrm{d}s$. Then there exists a constant $c > 0$ such that for any $p \in [1,2]$,*

$$\sup_{T\ge t} \mathbb{E}^{\nu_0}[|\psi_m(t)|^{2p} \,|\, t < \tau] \le c m^{\frac{p(d+4)-d-8}{2d}} t^{-p}, \quad t \ge 1,\ m \ge 1,\ r \in (0,1).$$

*Proof.* By Lemma 3.3, it suffices to prove the conclusion with $T = t$ replacing $T \ge t$. By Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}^{\nu_0}[|\psi_m(t)|^{2p} \,|\, T < \tau] &= \mathbb{E}^{\nu_0}[|\psi_m(t)|^{4-2p}|\psi_m(t)|^{4p-4} \,|\, T < \tau]\\ &\le \{\mathbb{E}^{\nu_0}[|\psi_m(t)|^2 \,|\, T < \tau]\}^{2-p}\{\mathbb{E}^{\nu_0}[|\psi_m(t)|^4 \,|\, T < \tau]\}^{p-1}.\end{aligned}$$

Combining this with (2.4), it suffices to find a constant $c > 0$ such that

$$\mathbb{E}^{\nu_0}[|\psi_m(t)|^2 1_{\{t<\tau\}}] \le \frac{c\mathrm{e}^{-\lambda_0 t}}{t m^{2/d}}, \quad t \ge 1,\ r \in (0,1), \tag{3.16}$$

$$\mathbb{E}^{\nu_0}[|\psi_m(t)|^4 1_{\{t<\tau\}}] \le c\sqrt{m}\,\mathrm{e}^{-\lambda_0 t} t^{-2}, \quad t \ge 1,\ r \in (0,1). \tag{3.17}$$

(a) *Proof of* (3.16). Let $\hat\phi_m = \phi_m\phi_0^{-1}$. We have

$$\mathbb{E}^{\nu_0}[|\psi_m(t)|^2 1_{\{t<\tau\}}] = \frac{2}{t^2}\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathbb{E}^{\nu_0}[1_{\{t<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})]\,\mathrm{d}s_2. \tag{3.18}$$

By (2.7), (3.3), $\mu_0(|\hat\phi_m|^2) = 1$, and the symmetry of $P_t^0$ in $L^2(\mu_0)$, we find a constant $c_1 > 0$ such that

$$\begin{aligned}&\mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}[1_{\{T<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})] = \nu_0\big(\phi_0 P_{s_1}^0\{\hat\phi_m P_{s_2-s_1}^0(\hat\phi_m P_{t-s_2}^0\phi_0^{-1})\}\big)\\ &\quad= \frac{1}{\mu(\phi_0)}\mu_0\big(\hat\phi_m P_{s_2-s_1}^0(\hat\phi_m P_{t-s_2}^0\phi_0^{-1})\big) = \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}}{\mu(\phi_0)}\mu_0(|\hat\phi_m|^2 P_{t-s_2}^0\phi_0^{-1})\\ &\quad\le c_1\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|P_{t-s_2}\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|\phi_0^{-1}\|_{L^p(\mu_0)}, \quad p > 1.\end{aligned}$$

Since $d \le 3$, we may take $p \in (1,3)$ such that $\varepsilon := \frac{d+2}{2q} < 1$ and $\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty$ due to (2.1), so that this and (2.12) imply

$$\mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}[1_{\{t<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})] \le c_2\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\{1 \wedge (t-s_2)\}^{-\varepsilon}$$

for some constant $c_3 > 0$. Therefore, (3.16) follows from (3.18) and (1.1).

(b) *Proof of* (3.17). For any $s > 0$ we have

$$\begin{aligned}&s^4\mathbb{E}^{\nu_0}[|\psi_m(s)|^4 1_{\{s<\tau\}}]\\ &= 24\int_0^s \mathrm{d}s_1\int_{s_1}^s \mathrm{d}s_2\int_{s_2}^s \mathrm{d}s_3\int_{s_3}^s \mathbb{E}^{\nu_0}[1_{\{s<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\hat\phi_m(X_{s_3})\hat\phi_m(X_{s_4})]\,\mathrm{d}s_4\\ &= 24\int_0^s \mathrm{d}s_1\int_{s_1}^s \mathrm{d}s_2\int_{s_2}^s \mathrm{d}s_3\int_{s_3}^s \mathbb{E}^{\nu_0}[1_{\{s_3<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})g_s(s_3,s_4)]\,\mathrm{d}s_4,\end{aligned} \tag{3.19}$$

where

$$g_s(s_3,s_4) := \mathbb{E}^{\nu_0}[1_{\{s<\tau\}}\hat\phi_m(X_{s_3})\hat\phi_m(X_{s_4}) \,|\, X_r : r \le s_3].$$

By (3.3) and the Markov property,

$$
\begin{aligned}
g_s(s_3, s_4) &= \hat{\phi}_m(X_{s_3})\mathbb{E}^{X_{s_3}}[1_{\{s-s_3<\tau\}}\hat{\phi}_m(X_{s_4-s_3})] \\
&= \mathrm{e}^{-\lambda_0(s-s_3)}\{\hat{\phi}_m\phi_0 P^0_{s_4-s_3}(\hat{\phi}_m P^0_{s-s_4}\phi_0^{-1})\}(X_{s_3}), \quad 0 < s_3 < s_4 \le s. \qquad (3.20)
\end{aligned}
$$

By Fubini's theorem and Schwarz's inequality, (3.19) and (3.20) yield

$$
\begin{aligned}
I(s) &:= s^4\mathrm{e}^{\lambda_0 s}\mathbb{E}^{\nu_0}[|\psi_m(s)|^4 1_{\{s<\tau\}}] \\
&= 12\mathrm{e}^{\lambda_0 s}\int_0^s \mathrm{d}r_1 \int_{r_1}^s \mathbb{E}^{\nu_0}\left[1_{\{r_1<\tau\}}g_s(r_1, r_2)\left|\int_0^{r_1}\hat{\phi}_m(X_r)\,\mathrm{d}r\right|^2\right]\mathrm{d}r_2 \\
&\le 12\sup_{r\in[0,s]}\sqrt{I(r)}\int_0^s \mathrm{d}r_1\int_{r_1}^s \{\mathrm{e}^{2\lambda_0 s-\lambda_0 r_1}\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}g_s(r_1, r_2)^2]\}^{1/2}\,\mathrm{d}r_2.
\end{aligned}
$$

Consequently,

$$
I(t) \le \sup_{s\in[0,t]} I(s) \le \left(12\sup_{s\in[0,t]}\int_0^s \mathrm{d}r_1\int_{r_1}^s \{\mathrm{e}^{\lambda_0(2s-r_1)}\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}g_s(r_1, r_2)^2]\}^{1/2}\,\mathrm{d}r_2\right)^2. \qquad (3.21)
$$

On the other hand, by the definition of $\nu_0$, (3.3), (3.20) and the fact that $\mu_0$ is $P_t^0$-invariant, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|g_s(r_1, r_2)|^2] \\
&\le \frac{\mathrm{e}^{-2\lambda_0(s-r_1)-\lambda_0 r_1}}{\mu(\phi_0)}\mu_0\big(P^0_{r_1}\{\phi_0^{-1}|\hat{\phi}_m\phi_0 P^0_{r_2-r_1}(\hat{\phi}_m P^0_{s-r_2}\phi_0^{-1})|^2\}\big) \\
&= \frac{\mathrm{e}^{-\lambda_0(2s-r_1)}}{\mu(\phi_0)}\mu_0\big(\phi_0|\hat{\phi}_m P^0_{r_2-r_1}(\hat{\phi}_m P^0_{s-r_2}\phi_0^{-1})|^2\big) \\
&\le \frac{2\mathrm{e}^{-\lambda_0(2s-r_1)}}{\mu(\phi_0)}\mu_0\big(\phi_0\{|\hat{\phi}_m(P^0_{r_2-r_1}\hat{\phi}_m)\mu(\phi_0)|^2 + |\hat{\phi}_m P^0_{r_2-r_1}(\hat{\phi}_m[P^0_{s-r_2}-\mu_0]\phi_0^{-1})|^2\}\big). \qquad (3.22)
\end{aligned}
$$

Then, by (3.20), (2.7), (3.3), $\mu_0(|\hat{\phi}_m|^2) = 1$, and noting that $\mu_0$ is $P_t^0$-invariant, we find a constant $c_1 > 0$ such that

$$
\begin{aligned}
&\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|g_s(r_1, r_2)|^2] \\
&\quad \le 2\mathrm{e}^{-\lambda_0(2s-r_1)-(\lambda_m-\lambda_0)(r_2-r_1)}\|\phi_m\|_\infty\|\phi_0\|_\infty\mu_0(|\hat{\phi}_m|\,|P_{(r_r-r_1)/2}\hat{\phi}_m|^2) \\
&\qquad + \frac{2\mathrm{e}^{-\lambda_0(2s-r_1)}\|\phi_m\|_\infty}{\mu(\phi_0)}\mu_0\big(|\hat{\phi}_m|\cdot\big|P^0_{r_2-r_1}(\hat{\phi}_m(P^0_{s-r_2}-\mu_0)\phi_0^{-1})\big|^2\big) \\
&\quad \le c_1\mathrm{e}^{-\lambda_0(2s-r_1)}\{\mathrm{e}^{-2(\lambda_m-\lambda_0)(r_2-r_1)}\|\phi_m\|_\infty\|P_{(r_2-r_1)/2}-\mu_0\|^2_{L^2(\mu_0)\to L^4(\mu_0)} \\
&\qquad + \|\phi_m\|_\infty\|P^0_{r_2-r_1}(\hat{\phi}_m[P^0_{s-r_2}-\mu_0]\phi_0^{-1})\|^2_{L^4(\mu_0)}\}.
\end{aligned}
$$

By (1.1), (2.12), $\|\hat{\phi}_m\|_{L^2(\mu_0)} = 1$, $\|\phi_0^{-1}\|_{L^q(\mu_0)} < \infty$ and $\varepsilon := \frac{d+2}{8} \vee \frac{d+2}{2q} < 1$ for $q \in (\frac{5}{2}, 3)$ due to (2.1) and $d \le 3$, we find a constant $c_2 > 0$ such that

$$
\|\phi_m\|_\infty\|P_{(r_2-r_1)/2}-\mu_0\|^2_{L^2(\mu_0)\to L^4(\mu_0)} \le c_2\sqrt{m}\{1\wedge(r_2-r_1)\}^{-d/4},
$$

and

$$\begin{aligned}
&\|\phi_m\|_\infty \|P^0_{r_2-r_1}(\hat\phi_m[P^0_{s-r_2}-\mu_0]\phi_0^{-1})\|^2_{L^4(\mu_0)}\\
&\quad \le \|\phi_m\|_\infty \|P^0_{r_2-r_1}\|^2_{L^2(\mu_0)\to L^4(\mu_0)}\|\hat\phi_m\|^2_{L^2(\mu_0)}\|(P^0_{s-r_2}-\mu_0)\phi_0^{-1}\|^2_{L^\infty(\mu_0)}\\
&\quad \le \|\phi_m\|_\infty \|P^0_{r_2-r_1}\|^2_{L^2(\mu_0)\to L^4(\mu_0)}\|P^0_{s-r_2}-\mu_0\|^2_{L^q(\mu_0)\to L^\infty(\mu_0)}\|\phi_0^{-1}\|^2_{L^q(\mu_0)}\\
&\quad \le c_2\sqrt{m}\,\mathrm{e}^{-2(\lambda_1-\lambda_0)(s-r_2)}\{1\wedge(r_2-r_1)\}^{-2\varepsilon}\{1\wedge(s-r_2)\}^{-2\varepsilon}.
\end{aligned}$$

Therefore, there exist constants $c_3>0$ and $\varepsilon\in(0,1)$ such that

$$\begin{aligned}
\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2] &\le c_3\mathrm{e}^{-\lambda_0(2s-r_1)-(\lambda_m-\lambda_0)(r_2-r_1)}\sqrt{m}\{1\wedge(r_2-r_1)\}^{-d/4}\\
&\quad + c_3\sqrt{m}\,\mathrm{e}^{-\lambda_0(2s-r_1)-2(\lambda_1-\lambda_0)(s-r_2)}\{1\wedge(r_2-r_1)\}^{-2\varepsilon}\{1\wedge(t-r_2)\}^{-2\varepsilon}.
\end{aligned}$$

Combining this with (3.21) and the definition of $I(t)$, we prove (3.17) for some constant $c>0$, and hence finish the proof. ■

**Lemma 3.6.** *Let $d\le 3$. Then for any $p\in(1,\frac{3d+16}{5d+8}\wedge\frac{d+2}{d+1})$, there exists a constant $c>0$ such that*

$$\sup_{r>0,T\ge t}\mathbb{E}^{\nu_0}[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p})\,|\,T<\tau]\le ct^{-p},\quad t\ge 1.$$

*Proof.* By Lemma 3.3, it suffices to prove the conclusion with $T=t$ replacing $T\ge t$. Let $p\in(1,\frac{3d+16}{5d+8}\wedge\frac{d+2}{d+1})$. Note that $p>1$ implies

$$\frac{p}{2p-1}\in(0,1), \tag{3.23}$$

while $p<\frac{3d+16}{5d+8}\wedge\frac{d+2}{d+1}$ implies

$$\frac{(d+2)(2p-2)}{4}+\frac{d(p-1)}{2}+\Big(\frac{p(d+4)+d}{4}-2\Big)^+<1,$$

hence there exists $\varepsilon\in(0,1)$ such that

$$\frac{(d+2)(2p-2+\varepsilon)}{4}+\frac{d(p-1)}{2}+\Big(\frac{p(d+4)+d}{4}-2\Big)^+<1. \tag{3.24}$$

By (2.21) and $L_0^{-1}=-\int_0^\infty P_s^0\,\mathrm{d}s$, and applying Hölder's inequality, we find a constant $c_1,c_2>0$ such that

$$\begin{aligned}
\int_M|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p}\,\mathrm{d}\mu_0 &\le \int_M\Big(\int_0^\infty|\nabla P_s^0(\rho_{t,r}-1)|\,\mathrm{d}s\Big)^{2p}\mathrm{d}\mu_0\\
&\le c_1\int_M\Big(\int_0^\infty\frac{1}{\sqrt{1\wedge s}}\{P^0_{s/4}|P^0_{3s/4}(\rho_{t,r}-1)|^p\}^{1/p}\,\mathrm{d}s\Big)^{2p}\phi_0^{-\varepsilon}\,\mathrm{d}\mu_0\\
&\le c_1\Big(\int_0^\infty(1\wedge s)^{-\frac{p}{2p-1}}\mathrm{e}^{-\frac{2p\theta s}{2p-1}}\,\mathrm{d}s\Big)^{\frac{2p-1}{2p}}\\
&\quad\times\int_0^\infty\mathrm{e}^{\theta s}\mu_0(\phi_0^{-\varepsilon}\{P^0_{s/4}|P^0_{3s/4}(\rho_{t,r}-1)|^p\}^2)\,\mathrm{d}s,\quad\theta>0.
\end{aligned}\tag{3.25}$$

By (3.23), we obtain

$$\int_0^\infty (1\wedge s)^{-\frac{p}{2p-1}} \mathrm{e}^{-\frac{2p\theta s}{2p-1}}\,\mathrm{d}s < \infty, \quad \theta > 0. \tag{3.26}$$

Moreover, since $\|\phi_0^{-\varepsilon}\|_{L^{2\varepsilon^{-1}}(\mu_0)} = 1$ and $\mu_0(\rho_{t,r}-1)=0$, and $P_t^0$ is contractive in $L^p(\mu_0)$ for $p\ge 1$, by (2.12) and Hölder's inequality we find a constant $c_2>0$ such that

$$\begin{aligned}
\mu_0(\phi_0^{-\varepsilon}\{P^0_{s/4}|P^0_{3s/4}(\rho_{t,r}-1)|^p\}^2) &\le \|P^0_{s/4}|P^0_{3s/4}(\rho_{t,r}-1)|^p\|^2_{L^{\frac{4}{2-\varepsilon}}(\mu_0)}\|\phi_0^{-\varepsilon}\|_{L^{2\varepsilon^{-1}}(\mu_0)}\\
&\le \|P^0_{s/4}\|^2_{L^{\frac{4}{2-\varepsilon}}(\mu_0)}\|(P^0_{s/2}-\mu_0)(P^0_{s/4}\rho_{t,r}-1)\|^{2p}_{L^{\frac{4p}{2-\varepsilon}}(\mu_0)}\\
&\le \|P^0_{s/2}-\mu_0\|^{2p}_{L^2(\mu_0)\to L^{\frac{4p}{2-\varepsilon}}(\mu_0)}\|P^0_{s/4}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\\
&\le c_2(1\wedge s)^{-\frac{(d+2)(2p-2+\varepsilon)}{4}}\mathrm{e}^{-(\lambda_1-\lambda_0)ps}\|P^0_{s/4}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}, \quad s>0.
\end{aligned}$$

Combining this with (3.26), we find a function $c:(0,\infty)\to(0,\infty)$ such that

$$\begin{aligned}
&\mathbb{E}^{\nu_0}[1_{\{t<\tau\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p})]\\
&\quad\le c(\theta)\int_0^\infty \mathrm{e}^{\theta s}(1\wedge s)^{-\frac{(d+2)(2p-2+\varepsilon)}{4}}\mathrm{e}^{-(\lambda_1-\lambda_0)ps}\mathbb{E}^{\nu_0}[1_{\{t<\tau\}}\|P^0_{s/4}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}]\,\mathrm{d}s,\\
&\qquad\qquad \theta>0.
\end{aligned}\tag{3.27}$$

By (2.7), (3.2) and Hölder's inequality, we obtain

$$\begin{aligned}
\|P^0_{s/4}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)} &= \Big(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}|\psi_m(t)|^2\Big)^p\\
&\le \Big(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}\Big)^{p-1}\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}|\psi_m(t)|^{2p}.
\end{aligned}$$

Noting that (1.1) implies

$$\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)} \le \alpha_1\int_1^\infty \mathrm{e}^{-\alpha_2(2r+s/2)t^{2/d}}\,\mathrm{d}t \le \alpha_3(1\wedge s)^{-d/2}$$

for some constants $\alpha_1,\alpha_2,\alpha_3>0$, we derive

$$\begin{aligned}
&\mathbb{E}^{\nu_0}[\|P^0_{s/4}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\,|\,t<\tau]\\
&\qquad\le c_3(1\wedge s)^{-\frac{d(p-1)}{2}}\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(2r+s/2)}\mathbb{E}^{\nu_0}[|\psi_m(t)|^{2p}\,|\,t<\tau]
\end{aligned}$$

for some constant $c_3>0$. Combining this with Lemma 3.5 and (1.1), we find constants $c_4,c_5,c_6,c_7>0$ such that

$$\begin{aligned}\mathbb{E}^{\nu_0}[\|P^0_{s/4}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\,|\,t<\tau] &\le c_4 t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}}\int_1^\infty \mathrm{e}^{-c_5 s u^{2/d}} u^{\frac{p(d+4)-d-8}{2d}}\,\mathrm{d}u\\ &\le c_6 t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}} s^{2-\frac{p(d+4)+d}{4}}\int_s^\infty t^{\frac{p(d+4)+d}{4}-3}\mathrm{e}^{-t}\,\mathrm{d}t\\ &\le c_7 t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}-(\frac{p(d+4)+d}{4}-2)^+}\log(2+s^{-1}),\end{aligned}$$

where the term $\log(2+s^{-1})$ comes when $\frac{p(d+4)+d}{4}-3=-1$. This together with (3.24) and (3.27) for $\theta\in(0,\lambda_1-\lambda_0)$ implies the desired estimate. ■

**Lemma 3.7.** *Let $d\le 3$. If $r_t=t^{-\alpha}$ for some $\alpha\in(1,\frac{4}{d}\wedge\frac{2d+4}{d^2})$, then $\rho_{t,r_t,r_t}:=(1-r_t)\rho_{t,r_t}+r_t$ satisfies*

$$\lim_{t\to\infty}\sup_{T\ge t}\mathbb{E}^{\nu_0}[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\,|\,T<\tau]=0,\quad q\ge 1.$$

*Proof.* By Lemma 3.3, it suffices to prove the conclusion with $T=t$ replacing $T\ge t$. For the same reason leading to (3.16) in [24], for any $\eta\in(0,1)$ and $y\in M$ we have

$$\begin{aligned}&\mathbb{E}^{\nu_0}\big[|\mathscr{M}(\rho_{t,r_t,r_t}(y),1)^{-1}-1|^q\,\big|\,t<\tau\big]\\ &\qquad\le\bigg|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\bigg|^q+\mathbb{P}^{\nu_0}(|\rho_{t,r_t}(y)-1|>\eta\,|\,t<\tau).\end{aligned}$$

Combining this with Lemma 3.4 we find constants $c>0$ and $\varepsilon\in(0,\alpha^{-1})$ such that

$$\mathbb{E}^{\nu_0}\big[|\mathscr{M}(\rho_{t,r_t,r_t}(y),1)^{-1}-1|^q\,\big|\,t<\tau\big]\le\bigg|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\bigg|^q+c\eta^{-1}\phi_0(y)^{-2}t^{-1+\alpha\varepsilon}.$$

Since $\mu_0(\phi_0^{-2})=1$, we obtain

$$\begin{aligned}&\mathbb{E}^{\nu_0}\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\,\big|\,t<\tau\big]\\ &\qquad\le\bigg|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\bigg|^q+c\eta^{-1}t^{-1+\alpha\varepsilon},\quad \eta\in(0,1),t\ge 1.\end{aligned}$$

Noting that $\alpha\varepsilon<1$, by letting first $t\to\infty$ and then $\eta\to 0$ we finish the proof. ■

**Lemma 3.8.** *Let $\mu_{t,r,r}=(1+\rho_{t,r,r})\mu_0$, where $\rho_{t,r,r}:=(1-r)\rho_{t,r}+r$ for $r\in(0,1]$. Assume that $\nu=h\mu$ with $h\phi_0^{-1}\in L^p(\mu_0)$ for some $p>1$. Then there exists a constant $c>0$ such that*

$$\sup_{T\ge t}\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r,r},\mu_t)^2\,|\,T<\tau]\le cr,\quad t>0,r\in(0,1].$$

*Proof.* By Lemma 3.3, it suffices to prove the conclusion with $T=t$ replacing $T\ge t$. Firstly, letting $D$ be the diameter of $M$, we have

$$\begin{aligned}\mathbb{W}_2(\mu,\nu)^2 &:=\inf_{\pi\in\mathscr{C}(\mu,\nu)}\int_{M\times M}\rho(x,y)^2\,\pi(\mathrm{d}x,\mathrm{d}y)\\ &\le D^2\inf_{\pi\in\mathscr{C}(\mu,\nu)}\pi(\{(x,y):x\ne y\})=\frac{1}{2}D^2\|\mu-\nu\|^2_{\mathrm{var}},\end{aligned}\tag{3.28}$$

where $\|\mu - \nu\|_{\mathrm{var}} := \sup_{\|f\|_\infty \le 1} |\mu(f) - \nu(f)|$ is the total variation norm. Then

$$\begin{aligned}\mathbb{W}_2(\mu_{t,r,r}, \mu_{t,r})^2 &\le D^2 \|\mu_{t,r,r} - \mu_{t,r}\|_{\mathrm{var}} \\ &= D^2 \mu_0(|\rho_{t,r,r} - \rho_{t,r}|) \le 2D^2 r, \quad r \in (0,1].\end{aligned} \tag{3.29}$$

Next, by the definition of $\mu_{t,r}$, we have

$$\pi(\mathrm{d}x, \mathrm{d}y) := \mu_t(\mathrm{d}x) P_r^0(x, \mathrm{d}y) \in \mathscr{C}(\mu_t, \mu_{t,r}),$$

where $P_r^0(x, \cdot)$ is the distribution of $X_r^0$ starting at $x$. So,

$$\mathbb{W}_2(\mu_t, \mu_{t,r})^2 \le \int_M \mathbb{E}^x[\rho(x, X_r^0)^2]\, \mu_t(\mathrm{d}x). \tag{3.30}$$

Moreover, by Itô's formula and $L_0 = L + 2\nabla \log \phi_0$, we find a constant $c_1 > 0$ such that

$$\mathrm{d}\rho(x, X_r^0)^2 = L_0 \rho(x, \cdot)^2 (X_r^0)\, \mathrm{d}r + \mathrm{d}M_r \le \{c_1 + c_1 \phi_0^{-1}(X_r^0)\}\, \mathrm{d}r + \mathrm{d}M_r$$

for some martingale $M_r$. Combining this with (2.17), and noting that $\log(1 + \phi_0^{-1}) \ge \log(1 + \|\phi_0\|_\infty^{-1}) > 0$, we find a constant $c_2 > 0$ such that

$$\begin{aligned}\mathbb{W}_2(\mu_t, \mu_{t,r})^2 &\le c_1 r + c_1 \int_M \left( \mathbb{E}^x \int_0^r \phi_0^{-1}(X_s^0)\, \mathrm{d}s \right) \mu_t(\mathrm{d}x) \\ &\le c_2 r \mu_t(\log(1 + \phi_0^{-1})) = \frac{c_2 r}{t} \int_0^t \log\{1 + \phi_0^{-1}(X_s)\}\, \mathrm{d}s, \quad r \in (0,1].\end{aligned}$$

Combining this with (3.29), (3.3), $\|P_t^0\|_{L^p(\mu_0)} = 1$ for $t \ge 0$ and $p \ge 1$, and noting that

$$\inf_{t \ge 0} \mu_0(h \phi_0^{-1} P_t^0 \phi_0^{-1}) > 0,$$

we find constants $c_3, c_4 > 0$ such that

$$\begin{aligned}&\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r,r}, \mu_t)^2 \,|\, t < \tau] \\ &\quad = \frac{\mathbb{E}^\nu[1_{\{t<\tau\}} \mathbb{W}_2(\mu_{t,r,r}, \mu_t)^2]}{\mathbb{P}^\nu(t < \tau)} \\ &\quad \le \frac{c_3 r}{t \mu_0(h \phi_0^{-1} P_t^0 \phi_0^{-1})} \int_0^t \mu_0(h \phi_0^{-1} P_s^0 \log\{1 + \phi_0^{-1}\})\, \mathrm{d}s \\ &\quad \le c_3 r \|h \phi_0^{-1}\|_{L^p(\mu_0)} \|\log(1 + \phi_0^{-1})\|_{L^{\frac{p}{p-1}}(\mu_0)} \le c_4 r, \quad r \in (0,1].\end{aligned} \tag{3.31}$$

Combining this with (3.29) we finish the proof. ∎

We are now ready to prove the main result in this section.

*Proof of Proposition* 3.1 (1). Since the upper bound is infinite for $d \ge 4$, it suffices to consider $d \le 3$.

(a) We first assume that $\nu = h\mu$ with $h \le C\phi_0$ for some constant $C > 0$. Let $\mu_{t,r_t,r_t} = \{(1-r_t)\rho_{t,r_t} + r_t\}\mu_0$ with $r_t = t^{-\alpha}$ for some $\alpha \in (1, \frac{4}{d} \wedge \frac{2d+4}{d^2})$. By Lemma 3.8 and the triangle inequality for $\mathbb{W}_2$, there exists a constant $c_1 > 0$ such that for any $t \ge 1$,

$$\begin{aligned}&\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,t<\tau]\\&\quad\le(1+\varepsilon)\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t,r_t},\mu_0)^2\,|\,t<\tau]+c_1(1+\varepsilon^{-1})t^{-\alpha},\quad \varepsilon>0.\end{aligned}\tag{3.32}$$

On the other hand, (3.1) implies

$$\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r_t,r_t},\mu_0)^2\,|\,t<\tau]\le\mathbb{E}^\nu\bigg[\int_M\frac{|\nabla L_0^{-1}(\rho_{t,r_t}-1)|^2}{\mathscr{M}(\rho_{t,r_t,r_t},1)}\,\mathrm{d}\mu_0\,\bigg|\,t<\tau\bigg]\le I_1+I_2,\tag{3.33}$$

where

$$\begin{aligned}I_1&:=\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t}-1)|^2)\,\big|\,t<\tau\big],\\I_2&:=\mathbb{E}^\nu\big[\mu_0\big(|\nabla L_0^{-1}(\rho_{t,r_t}-1)|^2|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|\big)\,\big|\,t<\tau\big].\end{aligned}$$

By Lemma 3.2 (1), we have

$$\limsup_{t\to\infty} tI_1\le\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2},$$

while by Lemma 3.6 with $p \in (1, \frac{3d+16}{5d+8} \wedge \frac{d+2}{d+1})$, Lemma 3.7 for $q = \frac{p}{p-1}$, and (3.12), we derive

$$\begin{aligned}\limsup_{t\to\infty} tI_2&\le\limsup_{t\to\infty} t\big(\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t}-1)|^{2p})\,\big|\,t<\tau\big]\big)^{1/p}\\&\quad\times\big(\mathbb{E}^\nu\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\,\big|\,t<\tau\big]\big)^{1/q}\\&=0.\end{aligned}$$

Combining these with (3.33) and (3.32) where $\alpha > 1$, we prove (1.2).

(b) In general, for any $t \ge 2$ and $\varepsilon \in (0,1)$, we consider

$$\mu_t^\varepsilon:=\frac{1}{t-\varepsilon}\int_\varepsilon^t\delta_{X_s}\,\mathrm{d}s.$$

By (3.28), we find a constant $c_1 > 0$ such that

$$\begin{aligned}&\mathbb{W}_2(\mu_t^\varepsilon,\mu_t)^2\le D^2\|\mu_t-\mu_t^\varepsilon\|_{\mathrm{var}}\\&\quad\le D^2\int_\varepsilon^t\Big(\frac{1}{t-\varepsilon}-\frac{1}{t}\Big)\mathrm{d}s+\frac{D^2}{t}\int_0^\varepsilon\mathrm{d}s\le c_1\varepsilon t^{-1},\quad t\ge2,\ \varepsilon\in(0,1).\end{aligned}\tag{3.34}$$

On the other hand, by the Markov property we obtain

$$\begin{aligned}\mathbb{E}^\nu[1_{\{t<\tau\}}\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2]&=\mathbb{E}^\nu[1_{\{\varepsilon<\tau\}}\mathbb{E}^{X_\varepsilon}(1_{\{t-\varepsilon<\tau\}}\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2)]\\&=\mathbb{P}^\nu(\varepsilon<\tau)\mathbb{E}^{\nu_\varepsilon}[1_{\{t-\varepsilon<\tau\}}\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2]\\&=\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)\mathbb{P}^\nu(\varepsilon<\tau)\mathbb{E}^{\nu_\varepsilon}[\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2\,|\,t-\varepsilon<\tau],\end{aligned}$$

where $\nu_\varepsilon = h_\varepsilon \mu$ with

$$h_\varepsilon(y) := \frac{1}{\mathbb{P}^\nu(\varepsilon < \tau)} \int_M p_\varepsilon^D(x,y)\,\nu(\mathrm{d}x) \le c(\varepsilon,\nu)\phi_0(y)$$

for some constant $c(\varepsilon,\nu) > 0$. Moreover, by (2.2), (2.4) and $\nu_\varepsilon = h_\varepsilon\mu$, we have

$$\lim_{t\to\infty} \frac{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon < \tau)\mathbb{P}^\nu(\varepsilon<\tau)}{\mathbb{P}^\nu(t<\tau)} = 1.$$

So, (a) implies

$$\begin{aligned}
&\limsup_{t\to\infty} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2\,|\,t<\tau] \\
&\quad = \limsup_{t\to\infty} \frac{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)\mathbb{P}^\nu(\varepsilon<\tau)}{\mathbb{P}^\nu(t<\tau)} t\mathbb{E}^{\nu_\varepsilon}[\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2\,|\,t-\varepsilon<\tau] \\
&\quad \le \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}.
\end{aligned}$$

Combining this with (3.34), we arrive at

$$\begin{aligned}
&\limsup_{t\to\infty} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,t<\tau] \\
&\quad \le (1+\varepsilon^{1/2}) \limsup_{t\to\infty} t\mathbb{E}^\nu[\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2\,|\,t<\tau] + c_1\varepsilon(1+\varepsilon^{-1/2}) \\
&\quad \le (1+\varepsilon^{1/2}) \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2} + c_1\varepsilon(1+\varepsilon^{-1/2}), \quad \varepsilon\in(0,1).
\end{aligned}$$

By letting $\varepsilon\to 0$, we derive (1.2). ∎

*Proof of Proposition* 3.1 (2, 3). Let $d\ge 4$. By (3.34), it suffices to prove the desired estimates for $\mu_t^1$ replacing $\mu_t$. Therefore, we may and do assume $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. By Lemma 3.2 (2) and the following inequality due to [11, Theorem 2] for $p=2$:

$$\mathbb{W}_2(f\mu_0,\mu_0)^2 \le 4\mu_0(|\nabla L_0^{-1}(f-1)|^2), \quad f\mu_0\in\mathscr{P}_0,$$

for any $k > \frac{d-4}{6}$ we find a constant $c>0$ such that

$$\begin{aligned}
&t\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r,r},\mu_0)^2\,|\,T<\tau] \\
&\quad \le c\{r^{-\frac{d-4}{2}} + 1_{\{d=4\}}\log r^{-1} + t^{-1}r^{-k}\}, \quad T\ge t\ge 1, r\in(0,1).
\end{aligned}$$

Combining this with Lemma 3.8, we find a function $c:(\frac{d-4}{6},\infty)\to(0,\infty)$ such that

$$\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2\,|\,T<\tau] \le c(k)\{t^{-1}r^{-\frac{d-4}{2}} + t^{-1}1_{\{d=4\}}\log r^{-1} + t^{-2}r^{-k} + r\} \tag{3.35}$$

for $T\ge t\ge 1$, $r\in(0,1)$, $k>(d-4)/6$.

(a) Let $d=4$. We take for instance $k=\frac{1}{2}$ and $r=t^{-1}$ for $t>1$, so that (3.35) implies (1.4) for some constant $c>0$.

(b) When $d\ge 5$, we take for instance $k=\frac{d-4}{4}$ and $r=t^{-\frac{2}{d-2}}$ for $t>1$. Then (3.35) implies the inequality in (3). ∎

## 4. Lower bound estimate

This section is devoted to the proof of the following result, which together with Proposition 3.1 implies Theorem 1.1.

**Proposition 4.1.** *Let $\nu \in \mathscr{P}_0$. There exists a constant $c > 0$ such that (1.3) holds, and when $\partial M$ is convex it holds for $c = 1$. Moreover, when $d \geq 5$, there exists a constant $c' > 0$ such that*

$$\inf_{T \geq t} t\mathbb{E}[\mathbb{W}_2(\mu_t, \mu_0) \,|\, T < \tau] \geq c' t^{-\frac{2}{d-2}}, \quad t \geq 1. \tag{4.1}$$

To estimate the Wasserstein distance from below, we use the idea of [1] to construct a pair of functions in Kantorovich's dual formula, which leads to the following lemma.

**Lemma 4.2.** *There exists a constant $c > 0$ such that*

$$\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 \geq \mu_0(|\nabla L_0^{-1}(\rho_{t,r} - 1)|^2) - c\|\rho_{t,r} - 1\|_\infty^{7/3}(1 + \|\rho_{t,r} - 1\|_\infty^{1/3}), \quad t, r > 0.$$

*Proof.* Let $f = L_0^{-1}(\rho_{t,r} - 1)$, and take

$$\varphi_\theta^\varepsilon = -\varepsilon \log P_{\varepsilon\theta/2}^0 \mathrm{e}^{-\varepsilon^{-1} f}, \quad \theta \in [0, 1], \, \varepsilon > 0.$$

We have $\varphi_0^\varepsilon = f$ and by [24, Lemma 2.9],

$$\begin{aligned}
&\varphi_1^\varepsilon(y) - f(x) \leq \tfrac{1}{2}\{\rho(x, y)^2 + \varepsilon\|(L_0 f)^+\|_\infty + c_1\varepsilon^{1/2}\|\nabla f\|_\infty^2\}, \\
&\mu_0(f - \varphi_1^\varepsilon) \leq \tfrac{1}{2}\mu_0(|\nabla f|^2) + c_1\varepsilon^{-1}\|\nabla f\|_\infty^4.
\end{aligned}$$

Since $L_0 f = \rho_{t,r} - 1$, this and the integration by parts formula imply

$$\begin{aligned}
&\tfrac{1}{2}\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 + \varepsilon\|\rho_{t,r} - 1\|_\infty + c_1\varepsilon^{1/2}\|\nabla f\|_\infty^2 \geq \mu_0(\varphi_1^\varepsilon) - \mu_{t,r}(f) \\
&\quad = \mu_0(\varphi_1^\varepsilon - f) - \mu_0(f L_0 f) \geq \tfrac{1}{2}\mu_0(|\nabla f|^2) - c_1\varepsilon^{-1}\|\nabla f\|_\infty^4, \quad \varepsilon > 0.
\end{aligned} \tag{4.2}$$

Next, by Lemma 2.2 for $p = \infty$ and (2.11), we find constants $c_2, c_3, c_4 > 0$ such that

$$\begin{aligned}
\|\nabla f\|_\infty &= \|\nabla L_0^{-1}(\rho_{t,r} - 1)\|_\infty \leq \int_0^\infty \|\nabla P_s^0(\rho_{t,r} - 1)\|_\infty \, \mathrm{d}s \\
&\leq c_2 \int_0^\infty (1 + s^{-1/2})\|P_{s/2}^0(\rho_{t,r} - 1)\|_\infty \, \mathrm{d}s \\
&\leq c_3\|\rho_{t.r} - 1\|_\infty \int_0^\infty (1 + s^{-1/2})\mathrm{e}^{-(\lambda_1 - \lambda_0)s/2} \, \mathrm{d}s \leq c_4\|\rho_{t.r} - 1\|_\infty.
\end{aligned}$$

Combining this with (4.2) we find a constant $c_5 > 0$ such that for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 \\
&\quad \geq \mu_0(|\nabla L_0^{-1}(\rho_{t,r} - 1)|^2) - c_5\{\varepsilon\|\rho_{t,r} - 1\|_\infty + \varepsilon^{1/2}\|\rho_{t,r} - 1\|_\infty^2 + \varepsilon^{-1}\|\rho_{t,r} - 1\|_\infty^4\}.
\end{aligned}$$

By taking $\varepsilon = \|\rho_{t,r} - 1\|_\infty^{4/3}$ we finish the proof. ∎

By Lemma 4.2, to derive a sharp lower bound of $\mathbb{W}_2(\mu_{t,r},\mu_0)^2$, we need to estimate $\|\rho_{t,r}-1\|_\infty$ and $\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau]$, which is done in the following three lemmas.

**Lemma 4.3.** *For any $r>0$ and $\nu=h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$, there exists a constant $c(r)>0$ such that*

$$\sup_{T\geq t}\mathbb{E}^\nu[\|\rho_{t,r}-1\|_\infty^4\,|\,T<\tau]\leq c(r)t^{-2},\quad t\geq 1.$$

*Proof.* By Lemma 3.3 and (3.12), it suffices to prove this for $\nu=\nu_0$ and $T=t$ replacing $T\geq t$, i.e. for a constant $c(r)>0$ we have

$$\mathbb{E}^{\nu_0}[\|\rho_{t,r}-1\|_\infty^4\,|\,t<\tau]\leq c(r)t^{-2},\quad t\geq 1. \tag{4.3}$$

By (3.22), (2.7), (2.11), and $\|\phi_0^{-1}\|_{L^2(\mu_0)}=1$, we find a constant $c_1>0$ such that

$$\begin{aligned}&\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2]\\&\leq c_1\mathrm{e}^{-\lambda_0(2s-\lambda_1)}\|\hat\phi_m\|_\infty^4\{\mathrm{e}^{-(\lambda_m-\lambda_0)(r_2-r_1)}+\mathrm{e}^{-(\lambda_1-\lambda_0)(s-r_2)}\},\quad s>r_2>r_1>0.\end{aligned}$$

By (3.21) and $\mathbb{P}^{\nu_0}(t<\tau)\geq c_0\mathrm{e}^{-\lambda_0 t}$ for some constant $c_0>0$ and all $t\geq 1$, this implies

$$\mathbb{E}^{\nu_0}[|\psi_m(t)|^4\,|\,t<\tau]:=\frac{\mathbb{E}^{\nu_0}[|\psi_m(t)|^4 1_{\{t<\tau\}}]}{\mathbb{P}^{\nu_0}(t<\tau)}\leq c_2\|\hat\phi_m\|_\infty^4 t^{-2},\quad m\geq 1,\ t>1,$$

for some constant $c_2>0$. Combining this with (3.2) gives

$$\begin{aligned}&\mathbb{E}^{\nu_0}[\|\rho_{t,r}-1\|_\infty^4\,|\,t<\tau]\\&\quad\leq\Big(\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat\phi_m\|_\infty^{4/3}\Big)^3\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}e^{\lambda_0 t}\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|\psi_m(t)|^4]\\&\quad\leq\Big(\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat\phi_m\|_\infty^{4/3}\Big)^3 c_2t^{-2}\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat\phi_m\|_\infty^4.\end{aligned}$$

By (1.1) and (2.13), this implies (4.3) for some constant $c(r)>0$. ∎

**Lemma 4.4.** *Let $\nu=h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$. Then for any $r>0$ there exists a constant $c(r)>0$ such that*

$$\sup_{T\geq t}\bigg|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau\big]-2\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\bigg|\leq\frac{c(r)}{t},\quad t\geq 1.$$

*Proof.* Let $\{J_i:i=1,2,3\}$ be as in (3.7). By (2.11), (2.13), and $\|\hat\phi_m\|_{L^2(\mu_0)}=1$, we find a constant $c_1>0$ such that for any $T\geq t\geq s_2\geq s_1>0$,

$$\begin{aligned}|J_1(s_1,s_2)|&\leq\|h\phi_0^{-1}\|_\infty\|P_{s_1}^0-\mu_0\|_{L^\infty(\mu_0)}\|\phi_m\phi_0^{-1}\|_\infty^2\|P_{T-s_2}^0-\mu_0\|_{L^1(\mu_0)}\|\phi_0^{-1}\|_{L^1(\mu_0)}\\&\leq c_1\|\phi_m\phi_0^{-1}\|_\infty^2\mathrm{e}^{-(\lambda_1-\lambda_0)(t+s_1-s_2)},\\|J_2(s_1,s_2)|&\leq\|\phi_0\|_\infty\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|h\phi_0^{-1}\|_\infty\|P_{s_1}^0-\mu_0\|_{L^\infty(\mu_0)}\\&\leq c_1\mathrm{e}^{-(\lambda_1-\lambda_0)s_2},\end{aligned}$$

$$
\begin{aligned}
|J_3(s_1,s_2)| &\le \|\phi_0\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)} \|\phi_m\phi_0^{-1}\|_\infty^2 \|P^0_{T-s_2}-\mu_0\|_{L^1(\mu_0)} \|\phi_0^{-1}\|_{L^1(\mu_0)} \\
&\le c_1 \|\phi_m\phi_0^{-1}\|_\infty^2 \mathrm{e}^{-(\lambda_1-\lambda_0)(t-s_1)}.
\end{aligned}
$$

Substituting these into (3.8) and applying (1.1) and (2.13), we find a constant $c(r) > 0$ such that the desired estimate holds. ■

**Lemma 4.5.** *Let $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. Then for any $r > 0$ and $p \ge 2$, there exists a constant $c(r,p) > 0$ such that*

$$
\|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p}\|_{L^{2p}(\mu_0)} \le c(r,p), \quad t > 0.
$$

*Proof.* Since $\rho_{t,r} = \frac{1}{t}\int_0^t p_r^0(X_s,\cdot)\,\mathrm{d}s$, we have $\mu_0(\rho_{t,r}) = 1$ and $\|\rho_{t,r}\|_\infty \le \|p_r^0\|_\infty < \infty$. Then by (2.11) and $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, we find a constant $c_1(r) > 0$ such that

$$
\begin{aligned}
\mu_0\big(\phi_0^{-1}\{P^0_{s/4}|P^0_{3s/4}(\rho_{t,r}-1)|^p\}^2\big) &\le \|\phi_0^{-1}\|_{L^2(\mu_0)} \|(P^0_{3s/4}-\mu_0)\rho_{t,r}\|^{2p}_{L^{4p}(\mu_0)} \\
&\le \|P^0_{3s/4}-\mu_0\|^{2p}_{L^{4p}(\mu_0)} \|\rho_{t,r}\|^{2p}_\infty \le c_1(r)\mathrm{e}^{-3(\lambda_1-\lambda_0)s}.
\end{aligned}
$$

Combining this with (3.25) for $\varepsilon = 1$ and $\theta \in (0, \frac{1}{\lambda_1-\lambda_0})$, we finish the proof. ■

Finally, since $\mu_{t,r} = \mu_t P_r^0$, to derive a lower bound of $\mathbb{W}_2(\mu_t,\mu_0)$ from that of $\mathbb{W}_2(\mu_{t,r},\mu_0)$, we present the following result.

**Lemma 4.6.** *There exist constants $K_1, K_2 > 0$ such that for any probability measures $\mu_1, \mu_2$ on $M^\circ$,*

$$
\mathbb{W}_2(\mu_1 P_t^0, \mu_2 P_t^0) \le K_1 \mathrm{e}^{K_2 t}\, \mathbb{W}_2(\mu_1,\mu_2), \quad t \ge 0. \tag{4.4}
$$

*When $\partial M$ is convex, this estimate holds for $K_1 = 1$.*

*Proof.* When $\partial M$ is convex, by [20, Lemma 2.16] there exists a constant $K$ such that

$$
\mathrm{Ric} - \mathrm{Hess}_{V+2\log\phi_0} \ge -K,
$$

so that the desired estimate holds for $K_1 = 1$ and $K_2 = K$ (see [16]).

In general, following the line of [18], we make the boundary convex by using a conformal change of metric. Let $N$ be the inward normal unit vector field on $\partial M$. Then the second fundamental form of $\partial M$ is a two-tensor on the tangent space of $\partial M$ defined by

$$
\mathbb{I}(X,Y) := -\langle \nabla_X N, Y\rangle, \quad X, Y \in T\partial M.
$$

Since $M$ is compact, we find a function $f \in C_b^\infty(M)$ such that $f \ge 1$, $N$ is parallel to $\nabla f$ on $\partial M$, and $N\log f|_{\partial M} + \mathbb{I}(u,u) \ge 0$ on $\partial M$ for any $u \in T\partial M$ with $|u| = 1$. By [18, Lemma 2.1] or [19, Theorem 1.2.5], $\partial M$ is convex under the metric

$$
\langle\cdot,\cdot\rangle' = f^{-2}\langle\cdot,\cdot\rangle.
$$

Let $\Delta'$, $\nabla'$ and $\mathrm{Hess}'$ be the Laplacian, gradient and Hessian in the new metric $\langle\cdot,\cdot\rangle'$. We have $\nabla' = f^2\nabla$ and (see [15, (2.2)])

$$L_0 = f^{-2}\Delta' + f^{-2}\nabla'\{V + 2\log\phi_0 + (d-2)f^{-1}\}.$$

Then the $L_0$-diffusion process $X_t^0$ with $X_0^0$ having distribution $\mu_1$ can be constructed by solving the following Itô SDE on $M^\circ$ with metric $\langle\cdot,\cdot\rangle'$ (see [2]):

$$\begin{aligned}\mathrm{d}^I X_t^0 &= \{f^{-2}\nabla'(V + 2\log\phi_0 + (d-2)f^{-1})\}(X_t^0)\,\mathrm{d}t\\ &\quad + \sqrt{2}\,f^{-1}(X_t^0)U_t\mathrm{d}B_t,\end{aligned}\tag{4.5}$$

where $B_t$ is the $d$-dimensional Brownian motion, and $U_t$ is the horizontal lift of $X_t^0$ to the frame bundle $O'(M)$ with respect to the metric $\langle\cdot,\cdot\rangle'$.

Let $Y_0^0$ be a random variable independent of $B_t$ with distribution $\mu_2$ such that

$$\mathbb{W}_2(\mu_1,\mu_2)^2 = \mathbb{E}[\rho(X_0^0, Y_0^0)^2].\tag{4.6}$$

For any $x, y \in M^\circ$, let $P'_{x,y} : T_xM \to T_yM$ be the parallel transport along the minimal geodesic from $x$ to $y$ induced by the metric $\langle\cdot,\cdot\rangle'$, which is contained in $M^\circ$ by convexity. Consider the coupling by parallel displacement

$$\begin{aligned}\mathrm{d}^I Y_t^0 &= \{f^{-2}\nabla'(V + 2\log\phi_0 + (d-2)f^{-1})\}(Y_t^0)\mathrm{d}t\\ &\quad + \sqrt{2}\,f^{-1}(Y_t^0)P'_{X_t^0,Y_t^0}U_t\mathrm{d}B_t.\end{aligned}\tag{4.7}$$

As explained in [2, Section 3], we may assume that $(M^\circ, \langle\cdot,\cdot\rangle')$ does not have cut-locus such that $P'_{x,y}$ is a smooth map, which ensures the existence and uniqueness of $Y_t^0$. Since the distributions of $X_0^0$ and $Y_0^0$ are $\mu_1, \mu_2$ respectively, the law of $(X_t^0, Y_t^0)$ is in the class $\mathscr{C}(\mu_1 P_t^0, \mu_2 P_t^0)$, so that

$$\mathbb{W}_2(\mu_1 P_t^0, \mu_2 P_t^0)^2 \le \mathbb{E}[\rho(X_t^0, Y_t^0)^2], \quad t \ge 0.\tag{4.8}$$

Let $\rho'(x,y)$ be the Riemannian distance between $x$ and $y$ induced by $\langle\cdot,\cdot\rangle' := f^{-2}\langle\cdot,\cdot\rangle$. As $1 \le f \in C_b^\infty(M)$ we have

$$\|f\|_\infty^{-1}\rho \le \rho' \le \rho.\tag{4.9}$$

Since except the term $f^{-2}\nabla'\log\phi_0$, all coefficients in the SDEs are in $C_b^\infty(M)$, by Itô's formula there exists a constant $K$ such that

$$\mathrm{d}\rho'(X_t^0, Y_t^0)^2 \le \{K\rho'(X_t^0, Y_t^0)^2 + I\}\,\mathrm{d}t + \mathrm{d}M_t,\tag{4.10}$$

where $M_t$ is a martingale and

$$I := \langle (f^{-2}\nabla'\log\phi_0)(\gamma_1), \dot\gamma_1\rangle' - \langle (f^{-2}\nabla'\log\phi_0)(\gamma_0), \dot\gamma_0\rangle',$$

where $\gamma : [0,1] \to M$ is the minimal geodesic from $X_t^0$ to $Y_t^0$ induced by the metric $\langle\cdot,\cdot\rangle'$, which is contained in $M^\circ$ by convexity, we obtain

$$
\begin{aligned}
I &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\langle (f^{-2}\nabla' \log\phi_0)(\gamma_s), \dot\gamma_s\rangle' \,\mathrm{d}s \\
&= \int_0^1 \bigg\{ \frac{f^{-2}(\gamma_s)\mathrm{Hess}'_{\phi_0}(\dot\gamma_s,\dot\gamma_s) + \langle \nabla' f^{-2}(\gamma_s), \dot\gamma_s\rangle' \langle \nabla'\phi_0(\gamma_s),\dot\gamma_s\rangle'}{\phi_0(\gamma_s)} \\
&\qquad\qquad - \frac{\{\langle \nabla'\phi_0(\gamma_s),\dot\gamma_s\rangle'\}^2}{(f^2\phi_0^2)(\gamma_s)}\bigg\}\,\mathrm{d}s \\
&\le \int_0^1 \bigg\{ (\phi_0^{-1} f^{-2})(\gamma_s)\mathrm{Hess}'_{\phi_0}(\dot\gamma_s,\dot\gamma_s) + \frac{f^2}{4}[\langle \nabla' f^{-2}(\gamma_s),\dot\gamma_s\rangle']^2\bigg\}\,\mathrm{d}s \le C\rho'(X_t^0,Y_t^0)^2
\end{aligned}
$$

for some constant $C>0$, where the last step is due to $\langle\dot\gamma_s,\dot\gamma_s\rangle' = \rho'(X_t^0,Y_t^0)^2$, $1\le f\in C_b^\infty(M)$, and that by the proof of [20, Lemma 2.1] the convexity of $\partial M$ under $\langle\cdot,\cdot\rangle'$ implies $\mathrm{Hess}'_{\phi_0}\le c\phi_0$ for some constant $c>0$. This and (4.10) yield

$$
\mathbb{E}[\rho'(X_t^0,Y_t^0)^2] \le \mathbb{E}[\rho'(X_0^0,Y_0^0)^2]\mathrm{e}^{(K+C)t}, \quad t\ge 0.
$$

Combining this with (4.6) and (4.9), we prove (4.4) for some constants $K_1,K_2>0$. ∎

We are now ready to prove the main result in this section.

*Proof of Proposition* 4.1. (a) According to (3.34), it suffices to prove the assertion for $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$. Let $r>0$ be fixed. By Lemma 4.2, we obtain

$$
\begin{aligned}
t\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r},\mu_0)^2 \,|\, T<\tau] &\ge t\mathbb{E}^\nu[1_{\{\|\rho_{t,r}-1\|_\infty\le\varepsilon\}}\mathbb{W}_2(\mu_{t,r},\mu_0)^2 \,|\, T<\tau] \\
&\ge t\mathbb{E}^\nu[1_{\{\|\rho_{t,r}-1\|_\infty\le\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] - c\varepsilon^2 \\
&\ge t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] - c\varepsilon^2 \\
&\quad - t\mathbb{E}^\nu[1_{\{\|\rho_{t,r}-1\|_\infty>\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau], \quad \varepsilon>0, T\ge t. \qquad (4.11)
\end{aligned}
$$

By Lemma 4.3 and Lemma 4.5 with $p=3$, we find some constants $c_1,c_2>0$ such that

$$
\begin{aligned}
&t\mathbb{E}^\nu[1_{\{\|\rho_{t,r}-1\|_\infty>\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] \\
&\qquad \le c_1 t\{\mathbb{P}^\nu(\|\rho_{t,r}-1\|_\infty>\varepsilon\,|\,T<\tau)\}^{2/3} \\
&\qquad \le c_1 t\varepsilon^{-8/3}\{\mathbb{E}^\nu(\|\rho_{t,r}-1\|_\infty^4\,|\,T<\tau)\}^{2/3} \\
&\qquad \le c_2\varepsilon^{-8/3}t^{-1/3}, \quad T\ge t.
\end{aligned}
$$

Combining this with (4.11) and Lemma 4.4, we find a constant $c_3>0$ such that

$$
\begin{aligned}
t\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\,|\,T<\tau] &\ge t\mathbb{E}^\nu[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\,|\,T<\tau] - \varepsilon_t \\
&\ge 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} - \varepsilon_t - c_3 t^{-1}, \quad T\ge t\ge 1,
\end{aligned}
$$

where

$$
\varepsilon_t := \inf_{\varepsilon>0}\{c\varepsilon^2 + c_2\varepsilon^{-8/3}t^{-1/3}\}\to 0 \quad \text{as } t\to\infty.
$$

Therefore,

$$\liminf_{t\to\infty}\inf_{T\geq t} t\mathbb{E}^{\nu}[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\,|\,T<\tau]\}\geq 2\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},\qquad r>0.$$

Combining this with Lemma 4.6, we derive

$$\liminf_{t\to\infty}\inf_{T\geq t} t\mathbb{E}^{\nu}[\mathbb{W}_2(\mu_{t},\mu_0)^2\,|\,T<\tau]\}\geq 2K_1^{-1}\mathrm{e}^{-K_1 r}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},\qquad r>0.$$

Letting $r\to 0$ we prove (1.3) for $c=K_1^{-1}$. By Lemma 4.6, we may take $c=1$ when $\partial M$ is convex.

(b) The second assertion can be proved as in [24, Section 4.2]. For any $t\geq 1$ and $N\in\mathbb{N}$, let $\bar\mu_N:=\frac{1}{N}\sum_{i=1}^{N}\delta_{X_{t_i}}$, where $t_i:=\frac{(i-1)t}{N}$, $1\leq i\leq N+1$. [10, Proposition 4.2] (see also [9, Corollary 12.14]) implies that for some constant $c_0>0$,

$$\mathbb{W}_1(\bar\mu_N,\mu_0)^2\geq c_0 N^{-2/d},\qquad N\in\mathbb{N},\ t\geq 1. \tag{4.12}$$

Write

$$\mu_t=\frac{1}{N}\sum_{i=1}^{N}\frac{N}{t}\int_{t_i}^{t_{i+1}}\delta_{X_s}\,\mathrm{d}s.$$

By the convexity of $\mathbb{W}_2^2$, which follows from the Kantorovich dual formula, we have

$$\mathbb{W}_2(\bar\mu_N,\mu_t)^2\leq\frac{1}{N}\sum_{i=1}^{N}\frac{N}{t}\int_{t_i}^{t_{i+1}}\mathbb{W}_2(\delta_{X_{t_i}},\delta_{X_s})^2\,\mathrm{d}s=\frac{1}{t}\sum_{i=1}^{N}\int_{t_i}^{t_{i+1}}\rho(X_{t_i},X_s)^2\,\mathrm{d}s. \tag{4.13}$$

On the other hand, by the Markov property,

$$\mathbb{E}^{\nu}[\rho(X_{t_i},X_s)^2 1_{\{T<\tau\}}]=\mathbb{E}^{\nu}[1_{\{t_i<\tau\}}P^D_{s-t_i}\{\rho(X_{t_i},\cdot)^2P^D_{T-s}1\}(X_{t_i})]. \tag{4.14}$$

Since $P_t^D 1\leq c_1\mathrm{e}^{-\lambda_0 t}$ for some constant $c_1>0$ and all $t\geq 0$, (2.6) implies

$$\begin{aligned}P^D_{s-t_i}\{\rho(x,\cdot)^2P^D_{T-s}1\}(x)&\leq c_1\mathrm{e}^{-\lambda_0(T-s)}P^D_{s-t_i}\rho(x,\cdot)^2(x)\\&\leq c_1\mathrm{e}^{-\lambda_0(T-s)}\phi_0(x)P^0_{s-t_i}\{\rho(x,\cdot)^2\phi_0^{-1}\}(x).\end{aligned} \tag{4.15}$$

It is easy to see that

$$L_0\{\rho(x,\cdot)^2\phi_0^{-1}\}\leq c_2\phi_0^{-2}$$

on $M^\circ$ for some constant $c_2>0$. So, by (2.17), we find a constant $c_3>0$ such that

$$P^0_{s-t_i}\{\rho(x,\cdot)^2\phi_0^{-1}\}(x)\leq c_2\mathbb{E}^x\int_0^{s-t_i}\phi_0(X_r)^{-2}\,\mathrm{d}r\leq c_3(s-t_i)\log(1+\phi_0(x)^{-1}).$$

Combining this with (4.14) and (4.15), and using $P_t^D 1\leq c_1\mathrm{e}^{-\lambda_0 t}$ observed above, we find a constant $c_5>0$ such that

$$\begin{aligned}\mathbb{E}^{\nu}[\rho(X_{t_i},X_s)^2 1_{\{T<\tau\}}]&\leq c_4\mathrm{e}^{-\lambda_0 T}\nu(\log(1+\phi_0^{-1}))(s-t_i)\\&\leq c_4\|h\phi_0^{-1}\|_\infty\mu(\phi_0\log(1+\log\phi_0^{-1}))(s-t_i)\mathrm{e}^{-\lambda_0 T}\leq c_5(s-t_i)\mathrm{e}^{-\lambda_0 T},\quad s\geq t_i.\end{aligned}$$

Since $\mathbb{P}^\nu(T < \tau) \geq c_0 \mathrm{e}^{-\lambda_0 T}$ for some constant $c_0 > 0$ and all $T \geq 1$, we find a constant $c > 0$ such that

$$\mathbb{E}^\nu[\rho(X_{t_i}, X_s)^2 \,|\, T < \tau] \leq c(s - t_i), \quad s \geq t_i.$$

Combining this with (4.12) and (4.13), we find a constant $c_6 > 0$ such that

$$\mathbb{E}^\nu[\mathbb{W}_1(\mu_t, \mu_0)^2 \,|\, T < \tau] \geq \frac{c_1}{2} N^{-2/d} - c_6 t N^{-1}, \quad T \geq t.$$

Taking $N = \sup\{i \in \mathbb{N} : i \leq \alpha t^{\frac{d}{d-2}}\}$ for some $\alpha > 0$, we derive

$$t^{\frac{2}{d-2}} \inf_{T \geq t} \mathbb{E}^\nu[\mathbb{W}_1(\mu_0, \mu_t)^2 \,|\, T < \tau] \geq \frac{c_2}{2\alpha^{2/d}} - \frac{2c'}{\alpha}, \quad t \geq 1.$$

Therefore,

$$t^{\frac{2}{d-2}} \inf_{T \geq t} \mathbb{E}^\nu[\mathbb{W}_1(\mu_0, \mu_t)^2 \,|\, T < \tau] \geq \sup_{\alpha > 0} \left( \frac{c_2}{2\alpha^{2/d}} - \frac{2c'}{\alpha} \right) > 0, \quad t \geq 1. \qquad \blacksquare$$

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