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Fourier integrator for periodic NLS: low regularity estimates via discrete Bourgain spaces

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Abstract. In this paper, we propose a new scheme for the integration of the periodic nonlinear Schrödinger equation and rigorously prove convergence rates at low regularity. The new integrator has decisive advantages over standard schemes at low regularity. In particular, it is able to handle initial data in H^s for $0 < s \leq 1$. The key feature of the integrator is its ability to distinguish between low and medium frequencies in the solution and to treat them differently in the discretization. This new approach requires a well-balanced filtering procedure which is carried out in Fourier space. The convergence analysis of the proposed scheme is based on discrete (in time) Bourgain space estimates which we introduce in this paper. A numerical experiment illustrates the superiority of the new integrator over standard schemes for rough initial data.

Keywords. Numerical analysis of nonlinear Schrödinger equations, discrete Bourgain spaces, low regularity, error analysis

1. Introduction

We consider the cubic periodic Schrödinger equation (NLS)

$$i\partial_t u = -\partial_x^2 u + |u|^2 u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}, \tag{1.1}$$

which, together with its full space counterpart, has been extensively studied in the literature. In the last decades, Strichartz estimates and Bourgain spaces allowed various authors to establish well-posedness results for dispersive equations in low regularity spaces (see [2, 3, 20, 21]). The numerical theory of dispersive PDEs, on the other hand, is still restricted to smooth solutions, in general. In the case of the nonlinear Schrödinger equation (1.1) this stems from the following two reasons:

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(A) Standard time stepping techniques, e.g., splitting methods [13] or exponential integrators [6], are based on freezing the free Schrödinger flow $S(t) = \mathrm{e}^{it\partial_x^2}$ during a step of size τ. Such freezing techniques, related to Taylor series expansion of the linear flow, however, produce derivatives in the local error terms restricting the approximation property to smooth solutions. More precisely, for first-order methods, the expansion of the free flow $S(t+\xi) = S(t) + \mathcal{O}(\tau\partial_x^2)$, $0 < \xi \le \tau$, requires the boundedness of (at least) two additional derivatives, while higher-order approximations increase the regularity requirements by two more derivatives for each additional order.

(B) Standard stability arguments in addition require smooth Sobolev spaces. Indeed, they rely on classical product estimates

$$\|fg\|_{H^s} \le C\|f\|_{H^s}\|g\|_{H^s}, \quad s > 1/2,$$

to handle the nonlinear terms in the error analysis. This restricts the global error analysis to smooth Sobolev spaces H^s with $s > 1/2$ leaving out the important class of L^2 spaces.

The standard local error structure introduced by the Schrödinger operator, i.e., the loss of two derivatives, together with a standard stability argument thus restricts global first-order convergence to $H^{2+1/2+\varepsilon}$ solutions (for any $\varepsilon > 0$). Using a refined global error analysis, by first proving fractional convergence of the scheme in a suitable higher-order Sobolev space (which implies a priori the boundedness of the numerical solution in this space [13]), allows one to obtain stability in L^2 for $H^{1/2+\varepsilon}$ solutions. However, due to the standard local error structure $\mathcal{O}(\tau^2\partial_x^2)$, the first-order convergence rate is nevertheless only retained for H^2 solutions. The latter is not only a technical formality. The order reduction in the case of nonsmooth solutions is also observed numerically (see, e.g., the examples in [10, 16] and Fig. 1 in Section 9 below). Very little is known on how to overcome this problem.

Recently, the first obstacle (A) could be overcome partly by developing specifically tailored schemes which optimize the structure of the local error approximation. This has been achieved by employing Fourier based techniques that are able to discretize the central oscillations in an efficient and correct way (see [7, 16, 17, 19]). The second obstacle (B), on the other hand, is much harder to circumvent. The control of nonlinear terms in PDEs is an ongoing challenge in (computational) mathematics at large, and in contrast to the parabolic setting no pointwise smoothing can be expected for dispersive PDEs. On the continuous level, however, important space time estimates featuring a gain in integrability can be used to extend well-posedness results to lower regularity spaces H^s with $s < 1/2$. On the full space, the Strichartz estimates

$$\|\mathrm{e}^{it\partial_x^2}u_0\|_{L_t^q L_x^r} \le c_{q,r}\|u_0\|_2 \quad \text{for } 2 \le q,\ r \le \infty,\ 2/q + 1/r = 1/2, \tag{1.2}$$

can be used. In the periodic setting, though waves do not disperse, one can gain integrability by using Bourgain spaces (we shall give the definition of these spaces in Section 2).

For (1.1) on the torus, the crucial estimate used in the analysis is

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{T})} \leq C\|u\|_{X^{0,3/8}} \tag{1.3}$$

which leads to global well-posedness for initial data in L^2. We refer for example to [2, 3, 20, 21].

The natural question therefore arises: To what extent can we inherit this subtle smoothing property on a discrete level? The critical issue here is twofold: the estimates (1.2) and (1.3) are not pointwise in time and moreover their gain lies in integrability and not regularity. Discrete versions of these estimates are therefore delicate to reproduce. At the same time they are essential for establishing numerical stability in the same space where we have stability of the PDE. While discrete Strichartz-type estimates were successfully employed on the full space $\mathbb{R}^d$ (see, e.g., [8, 9, 15]) a global low regularity analysis on bounded domains $\Omega \subset \mathbb{R}^d$ remains an open problem. The step from the full space to the bounded setting is – as in the continuous setting – nontrivial due to the loss of dispersion. Strichartz estimates are weaker on bounded domains as the solution cannot "disperse" to infinity in space. Nevertheless bounded domains are computationally very interesting as spatial discretizations of nonlinear PDEs are in general restricted to truncated domains.

In this work we introduce discrete Bourgain spaces for the periodic Schrödinger equation (1.1). This will allow us to break standard stability restrictions on the torus. In particular, we establish a discrete version of (1.3) permitting L^2 error estimates in H^s also for $s \leq 1/2$. Note that the resulting stability analysis can be extended to various other numerical schemes, e.g., splitting methods. For the discretization of (1.1) we propose a new twice-filtered Fourier based technique that correctly discretizes the central oscillations of the problem. In particular, it involves a frequency localization through a filter Π_K (see (1.8)) which projects on frequencies $|k| \leq K$ and which will allow us to optimize the total (time and frequency) discretization error. This novel discretization approach can be applied to a larger class of dispersive equations. For simplicity, however, we restrict our attention here to the cubic Schrödinger equation. The precise form of the proposed scheme is given in (1.11) below. Employing the particular structure of its local error in combination with the stability bounds that result from our discrete Bourgain type estimates we can prove the following global error estimate.

Theorem 1.1. *For every $T > 0$ and $u_0 \in H^{s_0}$, $0 \leq s_0 \leq 1$, denote by $u \in \mathcal{C}([0,T], H^{s_0})$ the exact solution of* (1.1) *with initial datum u_0, and by u_τ^n the sequence defined by the scheme* (1.11) *below. Then we have the following error estimates:*

(i) *for $K = \tau^{-1/2}$ and $0 < s_0 \leq 1/4$, there exist $\tau_0 > 0$ and $C_T > 0$ such that for every step size $\tau \in (0, \tau_0]$,*

$$\|u_\tau^n - u(t_n)\|_{L^2} \leq C_T \tau^{s_0/2}, \quad 0 \leq n\tau \leq T; \tag{1.4}$$

(ii) *for $1/4 < s_0 \leq 1/2$, and any $\varepsilon > 0$ such that $1/4 + \varepsilon < s_0$, with the choice $K = \tau^{-\frac{s_0+1/8-\varepsilon/2}{s_0+1/2}}$, there exist $\tau_0 > 0$ and $C_T > 0$ such that for every step size $\tau \in (0, \tau_0]$,*

$$\|u_\tau^n - u(t_n)\|_{L^2} \leq C_T \tau^{s_0\left(1-\frac{1}{2s_0+1}(3/4+\varepsilon)\right)}, \quad 0 \leq n\tau \leq T; \tag{1.5}$$

(iii) *for* $1/2 < s_0 \le 1$ *and* $\varepsilon \in (0, 1/4)$, *with the choice* $K = \tau^{-1+\frac{1}{s_0}(1/8+\varepsilon/2)}$, *there exist* $\tau_0 > 0$ *and* $C_T > 0$ *such that for every step size* $\tau \in (0, \tau_0]$,

$$\|u_\tau^n - u(t_n)\|_{L^2} \le C_T \tau^{s_0-(1/8+\varepsilon/2)}, \quad 0 \le n\tau \le T. \tag{1.6}$$

In case (i), the error estimate we obtain for our new Fourier based discretization is not better than the one we would expect for standard schemes (based on classical Taylor series expansion techniques). The interesting feature, however, is that the analysis we develop is able to provide an error estimate even for data at this low level of regularity. So far error estimates (even with arbitrarily low order of convergence) were restricted to solutions (at least) in H^s, $s > 1/2$. We note that our analysis can be employed for a large class of schemes, e.g., splitting methods or exponential integrators.

In cases (ii) and (iii), we observe that we get a better estimate than $\tau^{s_0/2}$ which is the one we would expect for standard numerical schemes with a loss of two derivatives in the local error (cf. (A)). Observe, for example, that for $s_0 = 1$, we get an error estimate of order $\tau^{7/8}$, which is much better than the standard $\tau^{1/2}$. Note that it is even (slightly) better than the convergence order $\tau^{5/6}$ which we obtained with the help of discrete Strichartz estimates on the full space in [15], though dispersive effects are much weaker in the periodic case. This comes from our improved Fourier based discretization with the use of the two different filters $\Pi_{\tau^{-1/2}}$ and Π_K. The favourable error behaviour is numerically underlined in Fig. 1 (see Section 9 below).

It seems also possible to extend the analysis developed in this paper to higher dimensions and more general nonlinearities. A large part of the framework that we introduce can be readily extended; the central task would be to establish the corresponding discrete counterpart of the continuous Bourgain estimates given in [2] in the various cases, as done here in Lemma 3.6.

The main idea in our discretization is the following. Instead of attacking (1.1) directly, we discretize the projected equation

$$\begin{aligned} i\partial_t u^K = &-\partial_x^2 u^K + 2\Pi_K(\Pi_{\tau^{-1/2}} u^K\, \Pi_{\tau^{-1/2}} \overline{u}^K\, \Pi_{K+} u^K) \\ &+ \Pi_K(\Pi_{\tau^{-1/2}} u^K\, \Pi_{\tau^{-1/2}} u^K\, \Pi_K \overline{u}^K), \end{aligned} \tag{1.7}$$

where the projection operator Π_L for $L > 0$ is defined by the Fourier multiplier

$$\Pi_L = \chi^2\left(\frac{-i\partial_x}{L}\right) = \overline{\Pi}_L, \tag{1.8}$$

and where Π_{K+} projects on the intermediate frequencies $\tau^{-1/2} \le |k| \le K$, i.e.,

$$\Pi_{K+} = \Pi_K - \Pi_{\tau^{-1/2}}. \tag{1.9}$$

Here χ is a smooth nonnegative even function which is 1 on $[-1, 1]$ and supported in $[-2, 2]$. The number $K \ge 1$ is considered as a parameter that will later depend on the step size τ. Note that the projection operator Π_K in Fourier space reads

$$\widehat{\Pi_K \phi_\ell} = \widehat{\phi_\ell}\, \chi^2\left(\frac{\ell}{K}\right), \quad \ell \in \mathbb{Z}.$$

Splitting methods with numerical filters have been successfully introduced in [1] for nonlinear Schrödinger equations in the semiclassical regime with attractive interaction to numerically suppress the modulation instability.

Here, the relation between K and τ can be seen as a CFL-type condition linking the time discretization parameter to the highest frequency in the system. We optimize this relation in such a way that the optimal rate of convergence is achieved for a given regularity; see Theorem 1.1.

The reason why we base our discretization on (1.7) is twofold. First, we consider

$$i\partial_t v^K = -\partial_x^2 v^K + \Pi_K(|\Pi_K v^K|^2 \Pi_K v^K), \quad v^K(0) = \Pi_K u_0, \tag{1.10}$$

as an intermediate problem for the single-filtered equation where all high frequencies are truncated. The difference between solutions of (1.1) and (1.10) is estimated in Corollary 2.6 and easy to control. Second, we refine the truncated model (1.10) by considering a second projection $\Pi_{\tau^{-1/2}}$ to low frequencies. Roughly speaking, each function with frequencies below K is then decomposed into two parts: low frequencies for which $|k| \le \tau^{-1/2}$ and the remaining intermediate frequencies. Since the original problem is cubic, these two projections lead to six terms in total. For our discretization, we only consider those terms in which two of the factors are of low frequencies. This motivates us to consider the twice-filtered equation (1.7) as an approximation to (1.1).

The discretization of the twice-filtered Schrödinger equation (1.7) is carried out in such a way that the terms with intermediate frequencies $\tau^{-1/2} < |k| \le K$ are treated *exactly*, while the lower order terms with frequencies $|k| \le \tau^{-1/2}$ are approximated in a suitable manner. This approach allows *low regularity approximation* of solutions of (1.1). Motivated by our previous work [11], we thus propose the following numerical scheme:

$$\begin{aligned} u_\tau^{n+1} &=: \Phi_\tau^K(u_\tau^n) \\ &= \mathrm{e}^{i\tau\partial_x^2} u_\tau^n - 2i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \mathcal{J}_1^\tau(\Pi_{\tau^{-1/2}}\overline{u}_\tau^n, \Pi_{K+}u_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n) \\ &\quad - i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \mathcal{J}_2^\tau(\Pi_K\overline{u}_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n), \\ u_\tau^0 &= \Pi_K u(0), \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} \mathcal{J}_1^\tau(v_1, v_2, v_3) = {}& \frac{i}{2}\mathrm{e}^{-i\tau\partial_x^2}[(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}v_2)\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}(v_1v_3)] \\ & - \frac{i}{2}(\partial_x^{-1}v_2)\partial_x^{-1}(v_1v_3) + \tau\widehat{(v_2)}_0 v_1v_3 + \tau(v_2 - \widehat{(v_2)}_0)\widehat{(v_1v_3)}_0, \end{aligned} \tag{1.12}$$

$$\begin{aligned} \mathcal{J}_2^\tau(v_1, v_2, v_3) = {}& \frac{i}{2}\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}[(\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}v_1)(\mathrm{e}^{i\tau\partial_x^2}v_2v_3)] \\ & - \frac{i}{2}\partial_x^{-1}(v_2v_3\partial_x^{-1}v_1) + \tau\widehat{(v_1v_2v_3)}_0 + \tau\widehat{(v_1)}_0\big(v_2v_3 - \widehat{(v_2v_3)}_0\big). \end{aligned} \tag{1.13}$$

Here, for any function $f \in L^2(\mathbb{T})$ we define the operator ∂_x^{-1} by

$$\partial_x^{-1} f(x) = \sum_{k \neq 0} (ik)^{-1} \widehat{f_k} \mathrm{e}^{ikx}.$$

Note that u_τ^n in (1.11) is considered as an approximation to the exact solution of the nonlinear Schrödinger equation (1.1) at time $t_n = n\tau$.

Outline of the paper. The paper is organized as follows. In Section 2, we recall the main steps of the analysis of the Cauchy problem for (1.1) and we use them to estimate the difference between the exact solution of (1.1) and the solution of the projected equation (1.7). In particular, we prove that

$$\sup_{[0,T]} \|u - u^K\|_{L^2} \leq C_T \left(\frac{1}{K^{s_0}} + \tau^{s_0} \right); \tag{1.14}$$

see Corollary 2.6 and Proposition 2.7.

In Section 3, we introduce a notion of discrete Bourgain spaces for sequences $(u_n)_n \in (L^2(\mathbb{T}))^{\mathbb{N}}$ and prove their main properties. The crucial property for the error analysis is the following L^4 estimate:

$$\|\Pi_K u_n\|_{l^4_\tau L^4} \leq C(K\tau^{1/2})^{1/2} \|u_n\|_{X_\tau^{0,3/8}},$$

which holds uniformly for $K \geq \tau^{-1/2}$ and $0 < \tau \leq 1$. This estimate is proven in Section 8. The l^p_τ norm for vector valued sequences is defined in (3.21). From this property, we see that the choice $K = \tau^{-1/2}$ allows one to get an estimate without loss similar to the continuous case (1.3). Nevertheless, such a choice of K yields a rather bad space discretization error (1.14). We shall thus optimize K by taking it of the form $K = \tau^{-\alpha/2}$ for $\alpha \in [1, 2]$ to get the best possible total error.

In Section 4, we establish embedding estimates between discrete and continuous Bourgain spaces.

In Section 5, we analyze the local error of our scheme, and in Section 6 we provide global error estimates. Finally, in Section 7, we prove the main error estimate of Theorem 1.1.

We conclude in Section 9 with numerical experiments underlying the favourable error behaviour of the new scheme for rough data.

Notations. We close this section with some notation that will be used throughout the paper. For two expressions a and b, we write $a \lesssim b$ whenever $a \leq Cb$ with some constant $C > 0$, uniformly in $\tau \in (0, 1]$ and $K \geq 1$. We further write $a \sim b$ if $b \lesssim a \lesssim b$. When we want to emphasize that C depends on an additional parameter γ, we write $a \lesssim_\gamma b$. In cases where a certain estimate holds for all parameters $p > q$ (with constants that might blow up for p tending to q) we say that this estimate holds for q_+. In the same way, we use the notation q_- to indicate the number $q - \varepsilon$ for any (fixed) $\varepsilon > 0$. Further, we denote $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

2. Cauchy problem for (1.1)

Let us recall the definition of Bourgain spaces. A tempered distribution $u(t,x)$ on $\mathbb{R}\times\mathbb{T}$ belongs to the *Bourgain space* $X^{s,b}$ if the following norm is finite:

$$\|u\|_{X^{s,b}} = \Big(\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} (1+|k|\Big)^{2s} (1+|\sigma+k^2|)^{2b} |\widetilde{u}(\sigma,k)|^2 \,\mathrm{d}\sigma)^{1/2}, \tag{2.1}$$

where $\widetilde{u}$ is the space-time Fourier transform of u:

$$\widetilde{u}(\sigma,k) = \int_{\mathbb{R}\times\mathbb{T}} \mathrm{e}^{-i\sigma t - ikx} u(t,x)\,\mathrm{d}t\,\mathrm{d}x.$$

We shall also use a localized version of this space: $u \in X^{s,b}(I)$ where $I\subset\mathbb{R}$ is an open interval if $\|u\|_{X^{s,b}(I)} < \infty$, where

$$\|u\|_{X^{s,b}(I)} = \inf\{\|\overline{u}\|_{X^{s,b}} : \overline{u}|_I = u\}.$$

When $I=(0,T)$ we will often simply use the notation $X^{s,b}(T)$.

We shall now recall well-known properties of these spaces. For details, we refer for example to [2], and the books [12,21].

Lemma 2.1. *For $\eta\in\mathcal{C}_c^\infty(\mathbb{R})$, we have*

$$\|\eta(t)\mathrm{e}^{it\partial_x^2} f\|_{X^{s,b}} \lesssim_{\eta,b} \|f\|_{H^s}, \quad s\in\mathbb{R},\ b\in\mathbb{R},\ f\in H^s(\mathbb{T}), \tag{2.2}$$

$$\|\eta(t)u\|_{X^{s,b}} \lesssim_{\eta,b} \|u\|_{X^{s,b}}, \quad s\in\mathbb{R},\ b\in\mathbb{R}, \tag{2.3}$$

$$\|\eta(t/T)u\|_{X^{s,b'}} \lesssim_{\eta,b,b'} T^{b-b'}\|u\|_{X^{s,b}}, \quad s\in\mathbb{R}, -1/2<b'\le b<1/2,\ 0<T\le 1, \tag{2.4}$$

$$\Big\|\eta(t)\int_{-\infty}^{t} \mathrm{e}^{i(t-s)\Delta}F(s)\,\mathrm{d}s\Big\|_{X^{s,b}} \lesssim_{\eta,b} \|F\|_{X^{s,b-1}}, \quad s\in\mathbb{R},\ b>1/2, \tag{2.5}$$

$$\|u\|_{L^\infty(\mathbb{R},H^s)} \lesssim_b \|u\|_{X^{s,b}}, \quad b>1/2,\ s\in\mathbb{R}. \tag{2.6}$$

We actually have the continuous embedding $X^{s,b}\subset\mathcal{C}(\mathbb{R},H^s)$ for $b>1/2$. Note that we shall discuss below an extension of the definition of the Bourgain spaces and of this lemma to a discrete setting suitable for the analysis of numerical schemes and give the proofs in this discrete setting.

The crucial estimate for the analysis of the cubic NLS on the torus $\mathbb{T}$ is the following:

Lemma 2.2. *There exists a constant $C>0$ such that for every $u\in X^{0,3/8}$, we have the estimate*

$$\|u\|_{L^4(\mathbb{R}\times\mathbb{T})} \le C\|u\|_{X^{0,3/8}}.$$

Again, we refer to [21, Proposition 2.13] for the proof. Note that, by duality, we also obtain

$$\|u\|_{X^{0,-3/8}} \lesssim \|u\|_{L^{4/3}(\mathbb{R}\times\mathbb{T})}.$$

By combining the two estimates with Hölder, this further implies that

$$\|uvw\|_{X^{0,-3/8}} \lesssim \|u\|_{X^{0,3/8}} \|v\|_{X^{0,3/8}} \|w\|_{X^{0,3/8}}. \tag{2.7}$$

For (1.1), we have the following global well-posedness result.

Theorem 2.3. *For every $T > 0$ and $u_0 \in L^2$, there exists a unique solution u of (1.1) such that $u \in \mathcal{C}([0,T], L^2) \cap X^{0,b}(T)$ for any $b \in (1/2, 5/8)$. Moreover, if $u_0 \in H^{s_0}$ for some $s_0 > 0$, then $u \in \mathcal{C}([0,T], H^{s_0}) \cap X^{s_0,b}(T)$.*

Proof. Let us recall the main steps of the proof. The existence is proven by a fixed point argument for the following truncated problem:

$$v \mapsto F(v)$$

where

$$F(v)(t) = \eta(t)\, \mathrm{e}^{it\partial_x^2} u_0 - i\eta(t) \int_0^t \mathrm{e}^{i(t-s)\partial_x^2} \big(\eta(s/\delta)|v(s)|^2 v(s)\big)\, \mathrm{d}s, \tag{2.8}$$

with $\eta \in [0,1]$ a smooth compactly supported function which is equal to 1 on $[-1,1]$ and supported in $[-2,2]$. For $|t| \le \delta \le 1/2$, a fixed point of F gives a solution of the original Cauchy problem, denoted by u.

Thanks to Lemma 2.1, there exists $C > 0$ which does not depend on u_0 such that

$$\|\eta(t)\, \mathrm{e}^{it\partial_x^2} u_0\|_{X^{0,b}} \le C \|u_0\|_{L^2}.$$

Moreover, by using Lemma 2.1 and (2.7), we can estimate the Duhamel term by

$$\begin{aligned} \left\|\eta(t) \int_0^t \mathrm{e}^{i(t-s)\Delta} \left(\eta\left(\frac{s}{\delta}\right)|v(s)|^2 v(s)\right) \mathrm{d}s\right\|_{X^{0,b}} &\le C \left\|\eta\left(\frac{s}{\delta}\right)|v(s)|^2 v(s)\right\|_{X^{0,b-1}} \\ &\le C\delta^{\varepsilon_0} \big\||v(s)|^2 v(s)\big\|_{X^{0,-3/8}} \le C\delta^{\varepsilon_0} \|v\|^3_{X^{0,3/8}} \le C\delta^{\varepsilon_0} \|v\|^3_{X^{0,b}}, \end{aligned}$$

where $C > 0$ is again a generic constant and $\varepsilon_0 = 5/8 - b > 0$ by the choice of b. Therefore, we have obtained

$$\|F(v)\|_{X^{0,b}} \le C \|u_0\|_{L^2} + C\delta^{\varepsilon_0} \|v\|^3_{X^{0,b}}.$$

In a similar way, we show that if v_1 and v_2 are such that $\|v_1\|_{X^{0,b}} \le R$, $\|v_2\|_{X^{0,b}} \le R$, then

$$\|F(v_1) - F(v_2)\|_{X^{0,b}} \le 4C\delta^{\varepsilon_0} R^2 \|v_1 - v_2\|_{X^{0,b}}.$$

Consequently, by taking $R = 2C\|u_0\|_{L^2}$, we find that there exists $\delta > 0$ sufficiently small that depends only on $\|u_0\|_{L^2}$ such that F is a contraction on the closed ball $B(0,R)$ of $X^{0,b}$. This proves the existence of a fixed point v for F and hence the existence of a solution u of (1.1) on $[0,\delta]$. By using Lemma 2.1, we actually see that $u \in \mathcal{C}([0,\delta], L^2)$. Since for $s \ge 0$,

$$\|F(v)\|_{X^{s,b}} \le C \|u_0\|_{H^s} + C\delta^{\varepsilon_0} \|v\|^2_{X^{0,b}} \|v\|_{X^{s,b}},$$

we also see that if u_0 is in H^s then $u \in X^{s,b}([0,\delta])$. Since the L^2 norm is conserved for (1.1), we can reiterate the construction on $[\delta, 2\delta], \dots$ to get a global solution. Moreover, since δ depends only on the L^2 norm of u_0, we deduce that if u_0 is in H^s, $s \geq 0$, then $u \in X^{s,b}(T)$ and thus $u \in \mathcal{C}([0,T], H^s)$ for every T. ∎

Let us now consider v^K that solves the frequency truncated equation

$$i\partial_t v^K = -\partial_x^2 v^K + \Pi_K(|\Pi_K v^K|^2 \Pi_K v^K), \quad v^K(0) = \Pi_K u_0. \tag{2.9}$$

As in Theorem 2.3, we can easily get

Proposition 2.4. *For $u_0 \in H^{s_0}$, $s_0 \geq 0$, and $K \geq 1$, there exists a unique solution v^K of (2.9) such that $v^K \in X^{s_0,b}(T)$ for $b \in (1/2, 5/8)$ and every $T > 0$. Moreover, for every $T > 0$, there exists M_T such that for every $K \geq 1$, we have the estimate*

$$\|v^K\|_{X^{s_0,b}(T)} \leq M_T.$$

We shall not detail the proof of this proposition that follows exactly the lines of the proof of Theorem 2.3.

Remark 2.5. Since $\Pi_{2K}\Pi_K = \Pi_K$, we see that $\Pi_{2K} v^K$ solves the same equation (2.9) with the same initial data. Hence, by uniqueness,

$$\Pi_{2K} v^K(t) = v^K(t) \quad \text{for all } t \in [0,T].$$

We can also easily get the following corollary.

Corollary 2.6. *For $u_0 \in H^{s_0}$, $s_0 \geq 0$, and every $T > 0$, there exists $C_T > 0$ such that for every $K \geq 1$ we have the estimate*

$$\|u - v^K\|_{X^{0,b}(T)} \leq C_T K^{-s_0},$$

where b is as in Theorem 2.3.

Proof. For appropriately chosen $\delta > 0$ and η as in (2.8), we observe that on $[0,\delta]$, v^K is the restriction of $V^K \in X^{s_0,3/8}(\mathbb{R})$ that solves

$$V^K(t) = \eta(t) e^{it\partial_x^2} v^K(0) - i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left(\eta\left(\frac{s}{\delta}\right) \Pi_K\big(|\Pi_K V^K(s)|^2 \Pi_K V^K(s)\big)\right) ds.$$

Consequently, denoting by $U \in X^{s_0,3/8}(\mathbb{R})$ the fixed point of F such that on $[0,\delta]$, $U = u$, we obtain

$$\begin{aligned}
U(t) - V^K(t) &= \eta(t) e^{it\partial_x^2}(1 - \Pi_K) u_0 \\
&- i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left(\eta\left(\frac{s}{\delta}\right) \Pi_K\big(|U(s)|^2 U(s) - |\Pi_K U(s)|^2 \Pi_K U(s)\big)\right) ds \\
&- i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left(\eta\left(\frac{s}{\delta}\right) \Pi_K\big(|\Pi_K U(s)|^2 \Pi_K U(s) - |\Pi_K V^K(s)|^2 \Pi_K V^K(s)\big)\right) ds \\
&- i\eta(t) \int_0^t e^{i(t-s)\partial_x^2} \left(\eta\left(\frac{s}{\delta}\right) (1 - \Pi_K)\big(|U(s)|^2 U(s)\big)\right) ds.
\end{aligned}$$

Now, let us fix M_T independent of $K \geq 1$ such that

$$\|V^K\|_{X^{s_0,b}} + \|U\|_{X^{s_0,b}} \leq M_T.$$

Note that for every f,

$$\|f - \Pi_K f\|_{X^{0,b}} \lesssim \frac{1}{K^{s_0}} \|f\|_{X^{s_0,b}}.$$

By employing the same estimates as before, we thus obtain

$$\begin{aligned}
\|U - V^K\|_{X^{0,b}} &\lesssim \frac{1}{K^{s_0}} \|u_0\|_{H^{s_0}} + \delta^{\varepsilon_0} \|U - \Pi_K U\|_{X^{0,b}} \|U\|^2_{X^{0,b}} \\
&\quad + \delta^{\varepsilon_0} \|U - V^K\|_{X^{0,b}} (\|U\|^2_{X^{0,b}} + \|V^K\|^2_{X^{0,b}}) \\
&\quad + \delta^{\varepsilon_0} \frac{1}{K^{s_0}} \left\| |U|^2 U \right\|_{X^{s_0,b-1}} \\
&\lesssim \frac{1}{K^{s_0}} (\|u_0\|_{H^{s_0}} + M_T^3) + \delta^{\varepsilon_0} M_T^2 \|U - V^K\|_{X^{0,b}}.
\end{aligned}$$

For δ sufficiently small, this yields the desired estimate. We can then iterate in order to get the estimate on $[0, T]$. ■

Instead of performing directly a time discretization of equation (2.9), it will be convenient for our analysis to study a slightly modified equation. Let u^K be the solution of

$$\begin{aligned}
&i \partial_t u^K \\
&= -\partial_x^2 u^K + 2\Pi_K(\Pi_{\tau^{-1/2}} u^K \Pi_{\tau^{-1/2}} \overline{u}^K \Pi_{K+} u^K) + \Pi_K(\Pi_{\tau^{-1/2}} u^K \Pi_{\tau^{-1/2}} u^K \Pi_K \overline{u}^K)
\end{aligned}$$

(cf. (1.7)), again with the initial data $u^K(0) = \Pi_K u_0$. Note that the difference between this truncated equation and (2.9) is in the trilinear terms, where we can always project at least two factors on frequencies less than $\tau^{-1/2}$. Further, note that u^K depends also on τ though we do not explicitly mention it in order to keep reasonable notation. However, we will henceforth link K and τ by the relation

$$K = \tau^{-\alpha/2}, \quad \alpha \geq 1,$$

with the (optimal) value of α still to be determined.

Again, we have existence and uniqueness of the solution.

Proposition 2.7. *For $u_0 \in H^{s_0}$, $s_0 \geq 0$, and $K \geq 1$, there exists a unique solution u^K of (1.7) such that $u^K \in X^{s_0,b}(T)$ (with b as in Theorem 2.3) for every $T > 0$. Moreover, for every $T > 0$, there exists M_T such that for every $K \geq 1$, we have the estimate*

$$\|u^K\|_{X^{s_0,b}(T)} \leq M_T.$$

Further, uniformly for $K \geq 1$,

$$\|u^K - v^K\|_{X^{0,b}(T)} \leq C_T \tau^{s_0}.$$

Observe that by combining the last estimate with the estimate of Corollary 2.6, we actually get

$$\|u - u^K\|_{X^{0,b}(T)} \le C_T \tau^{s_0 \alpha/2} \tag{2.10}$$

for $K = \tau^{-\alpha/2}$ and α such that $1 \le \alpha \le 2$.

Proof of Proposition 2.7. The proof of the first part follows again the lines of the proof of Theorem 2.3. Let us explain how to prove the error estimate. Let us denote by $G_{\tau,K}(u^K)$ the nonlinear term on the right-hand side of (1.7). We first observe that we can write

$$\Pi_K(|\Pi_K v^K|^2 \Pi_K v^K) = G_{\tau,K}(v^K) + R_K(v^K),$$

where the remainder is a sum

$$R_K(v^K) = \sum_{(i_1,i_2,i_3)} \Pi_K(Q_{i_1} v^K Q_{i_2} \overline{v^K} Q_{i_3} v^K)$$

and where Q_i can be $\Pi_{\tau^{-1/2}}$ or $(1 - \Pi_{\tau^{-1/2}})\Pi_K$ and at least two different Q_i are $(1 - \Pi_{\tau^{-1/2}})\Pi_K$. Let us again denote by U^K and V^K the fixed points of the corresponding Duhamel term extended by η such that uniformly for $K \ge 1$, we have

$$\|V^K\|_{X^{s_0,3/8}} + \|U^K\|_{X^{s_0,3/8}} \le M_T.$$

Then

$$\begin{aligned} U^K(t) - V^K(t) = &-i\eta(t) \int_0^t \mathrm{e}^{i(t-s)\Delta} \left(\eta\left(\frac{s}{\delta}\right) \left(G_{\tau,K}(U^K(s)) - G_{\tau,K}(V^K(s))\right) \right) \mathrm{d}s \\ &- i\eta(t) \int_0^t \mathrm{e}^{i(t-s)\Delta} \left(\eta\left(\frac{s}{\delta}\right) R_K(V^K(s)) \right) \mathrm{d}s. \end{aligned}$$

By using the properties of Bourgain spaces and the estimate

$$\|f - \Pi_{\tau^{-1/2}} f\|_{X^{0,b}} \lesssim \tau^{s_0/2} \|f\|_{X^{s_0,b}}, \quad \forall f \in X^{s_0,b},$$

we obtain

$$\begin{aligned} \|U^K - V^K\|_{X^{0,b}} &\lesssim \delta^{1/4} \|U^K - V^K\|_{X^{0,b}} M_T^2 + \|V^K - \Pi_{\tau^{-1/2}} V^K\|_{X^{0,b}}^2 \|V^K\|_{X^{0,b}} \\ &\lesssim \delta^{1/4} \|U^K - V^K\|_{X^{0,b}} M_T^2 + \tau^{s_0} M_T^3 \end{aligned}$$

and we can conclude the proof as before. ∎

We shall need the following corollary about the propagation of higher regularity with respect to the b parameter.

Corollary 2.8. *Let $b \in (5/8, 1]$ and assume that $s_0 > 0$. Then for every $0 \le s_0' < s_0$ and every $T > 0$, uniformly in τ,*

$$\|u^K\|_{X^{s_0',b}(T)} \le M_T.$$

Proof. Since u^K solves (1.7), we have

$$u^K(t) = \mathrm{e}^{it\partial_x^2}\Pi_K u_0 + \int_0^t \mathrm{e}^{i(t-s)\partial_x^2} F(u^K(s))\,\mathrm{d}s,$$

where we set for short

$$iF(u^K) = 2\Pi_K(\Pi_{\tau^{-1/2}}u^K\Pi_{\tau^{-1/2}}\overline{u}^K\Pi_{K+}u^K) + \Pi_K(\Pi_{\tau^{-1/2}}u^K\Pi_{\tau^{-1/2}}u^K\Pi_K\overline{u}^K).$$

Let $u^{K,\eta}$ denote the solution of the truncated Duhamel equation

$$u^{K,\eta}(t) = \eta(t)\mathrm{e}^{it\partial_x^2}u_0 + \eta(t)\int_0^t \mathrm{e}^{i(t-s)\partial_x^2}\eta(s)F(u^{K,\eta}(s))\,\mathrm{d}s, \tag{2.11}$$

which belongs to the global Bourgain space $X^{s_0,b}(\mathbb{R}\times\mathbb{T})$ for $b \in (1/2, 5/8)$ as established in the previous proposition. From the same estimates as before, we obtain

$$\|u^{K,\eta}\|_{X^{s_0',b}} \lesssim \|u_0\|_{H^{s_0'}} + \|F(u^{K,\eta})\|_{X^{s_0',b-1}}.$$

In order to estimate $F(u^{K,\eta})$, we first just use $b-1 \le 0$ so that

$$\|F(u^{K,\eta})\|_{X^{s_0',b-1}} \lesssim \|\langle\partial_x\rangle^{s_0'}F(u^{K,\eta})\|_{L^2(\mathbb{R}\times\mathbb{T})},$$

where $\langle\partial_x\rangle^s$ stands for the Fourier multiplier $\langle k\rangle^s$. Then, by using the generalized Leibniz rule (which reads, see for example [14],

$$\|\langle\partial_x\rangle^s(fg)\|_{L^p} \lesssim \|\langle\partial_x\rangle^s f\|_{L^{p_1}}\|g\|_{L^{q_1}} + \|f\|_{L^{p_2}}\|\langle\partial_x\rangle^s g\|_{L^{q_2}} \tag{2.12}$$

for every $s > 0$ and all $p \in (1,\infty)$, $p_1, p_2, q_1, q_2 \in (1,+\infty]$ such that $p^{-1} = p_1^{-1} + q_1^{-1} = p_2^{-1} + q_2^{-1}$) we get

$$\|F(u^{K,\eta})\|_{X^{s_0',b-1}} \lesssim \|\langle\partial_x\rangle^{s_0'}u^{K,\eta}\|^3_{L^6(\mathbb{R}\times\mathbb{T})}. \tag{2.13}$$

To conclude, we use another $X^{s,b}$ space estimate due to Bourgain [2]: for every $\varepsilon > 0$ and $b' > 1/2$, we have the continuous embedding $X^{\varepsilon,b'} \subset L^6(\mathbb{R}\times\mathbb{T})$, that is to say, for every $f \in X^{\varepsilon,b'}$,

$$\|f\|_{L^6(\mathbb{R}\times\mathbb{T})} \lesssim \|f\|_{X^{\varepsilon,b'}}.$$

By using this last estimate in (2.13), we thus get

$$\|F(u^{K,\eta})\|_{X^{s_0',b-1}} \lesssim \|u^{K,\eta}\|^3_{X^{s_0'+\varepsilon,b'}}.$$

Since we can choose $b' < 5/8$ and $\varepsilon > 0$ such that $s_0' + \varepsilon \le s_0$, the right-hand side is already controlled thanks to Proposition 2.7. This ends the proof. ■

3. Discrete Bourgain spaces

For a sequence $(u_n)_{n\in\mathbb{Z}}$, we shall define its Fourier transform as

$$\mathcal{F}_\tau(u_n)(\sigma) = \tau\sum_{m\in\mathbb{Z}} u_m \mathrm{e}^{im\tau\sigma}. \tag{3.1}$$

This defines a periodic function on $[-\pi/\tau, \pi/\tau]$ and we have the inverse Fourier transform formula

$$u_m = \frac{1}{2\pi} \int_{-\pi/\tau}^{\pi/\tau} \mathcal{F}_\tau(u_n)(\sigma)\, \mathrm{e}^{-im\tau\sigma}\, \mathrm{d}\sigma.$$

With these definitions the Parseval identity reads

$$\|u_n\|_{l_\tau^2} = \|\mathcal{F}_\tau(u_n)\|_{L^2(-\pi/\tau,\pi/\tau)},$$

where the norms are defined by

$$\|u_n\|_{l_\tau^2}^2 = \tau \sum_{n\in\mathbb{Z}} |u_n|^2, \quad \|\mathcal{F}_\tau(u_n)\|_{L^2(-\pi/\tau,\pi/\tau)}^2 = \frac{1}{2\pi} \int_{-\pi/\tau}^{\pi/\tau} |\mathcal{F}_\tau(u_n)(\sigma)|^2\, \mathrm{d}\sigma.$$

In this section, we write L^2 instead of $L^2(-\pi/\tau, \pi/\tau)$ for short. We stress that this is not the standard way of normalizing the Fourier series.

We then define in a natural way Sobolev spaces H_τ^b of sequences $(u_n)_{n\in\mathbb{Z}}$ by

$$\|u_n\|_{H_\tau^b} = \|\langle d_\tau(\sigma)\rangle^b \mathcal{F}_\tau(u_n)\|_{L^2},$$

with $d_\tau(\sigma) = \frac{\mathrm{e}^{i\tau\sigma}-1}{\tau}$ so that we have equivalent norms

$$\|u_n\|_{H_\tau^b} = \|\langle D_\tau\rangle^b u_n\|_{l_\tau^2},$$

where the operator D_τ is defined by $(D_\tau(u_n))_n = (\frac{u_{n-1}-u_n}{\tau})_n$ since by definition of the Fourier transform,

$$\mathcal{F}_\tau(D_\tau u_n)(\sigma) = d_\tau(\sigma)\mathcal{F}_\tau(u_n)(\sigma).$$

Note that d_τ is $2\pi/\tau$-periodic and uniformly in τ, we have $|d_\tau(\sigma)| \sim |\sigma|$ for $|\tau\sigma| \le \pi$.

For sequences of functions $(u_n(x))_{n\in\mathbb{Z}}$, we define the Fourier transform $\widetilde{u_n}(\sigma, k)$ by

$$\mathcal{F}_{\tau,x}(u_n)(\sigma, k) = \widetilde{u_n}(\sigma, k) = \tau \sum_{m\in\mathbb{Z}} \widehat{u_m}(k)\, \mathrm{e}^{im\tau\sigma}, \quad \widehat{u_m}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_m(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x.$$

Parseval's identity then reads

$$\|\widetilde{u_n}\|_{L^2 l^2} = \|u_n\|_{l_\tau^2 L^2}, \tag{3.2}$$

where

$$\|\widetilde{u_n}\|_{L^2 l^2}^2 = \int_{-\pi/\tau}^{\pi/\tau} \sum_{k\in\mathbb{Z}} |\widetilde{u_n}(\sigma, k)|^2\, \mathrm{d}\sigma, \quad \|u_n\|_{l_\tau^2 L^2}^2 = \tau \sum_{m\in\mathbb{Z}} \int_{-\pi}^{\pi} |u_m(x)|^2\, \mathrm{d}x.$$

We then define the discrete Bourgain spaces $X_\tau^{s,b}$ for $s \ge 0$, $b \in \mathbb{R}$, $\tau > 0$ by

$$\|u_n\|_{X_\tau^{s,b}} = \|\mathrm{e}^{-in\tau\partial_x^2} u_n\|_{H_\tau^b H^s} = \|\langle D_\tau\rangle^b \langle\partial_x\rangle^s (\mathrm{e}^{-in\tau\partial_x^2} u_n)\|_{l_\tau^2 L^2}. \tag{3.3}$$

As in the continuous case, we obtain the following properties.

Lemma 3.1. *With the above definition, we have*

$$\|u_n\|_{X_\tau^{s,b}} \sim \|\langle k\rangle^s \langle d_\tau(\sigma-k^2)\rangle^b \widetilde{u_n}(\sigma,k)\|_{L^2 l^2}. \tag{3.4}$$

Moreover, for $s \in \mathbb{R}$ and $b > 1/2$, we have $X_\tau^{s,b} \subset l_\tau^\infty H^s$:

$$\|u_n\|_{l_\tau^\infty H^s} \lesssim_b \|u_n\|_{X_\tau^{s,b}}. \tag{3.5}$$

The weight $d_\tau(\sigma-k^2)$ obviously vanishes if $\tau(\sigma-k^2) = 2m\pi$ for $m \in \mathbb{Z}$. For a localized function such that k is constrained to $|k| \lesssim \tau^{-1/2}$ this weight will behave as in the continuous case with only a cancellation when $\sigma = k^2$. For larger frequencies, however, there are additional cancellations that will create some loss in product estimates.

Note that the seemingly different behaviour that we have here in the discrete case compared with the definition (2.1) in the continuous case comes from our definition (3.1) of the discrete Fourier transform. Let us recall that in the continuous case

$$\|u\|_{X^{s,b}} = \|\overline{u}\|_{X^{s,b}_{\sigma=k^2}},$$

where $\|\overline{u}\|_{X^{s,b}_{\sigma=k^2}} = \|\langle k\rangle^s\langle\sigma-k^2\rangle^b\tilde{u}\|_{L^2(\mathbb{R}\times\mathbb{Z})}$ so that we can easily deduce the properties of $X^{s,b}_{\sigma=k^2}$ from the properties of $X^{s,b}$.

Proof of Lemma 3.1. Let us set $f_n(x) = \mathrm{e}^{-in\tau\partial_x^2}u_n(x)$. From the definition of $\mathcal{F}_\tau$, we get

$$\widetilde{f_n}(\sigma,k) = \tau\sum_{m\in\mathbb{Z}}\widehat{u_m}(k)\,\mathrm{e}^{im\tau(\sigma+k^2)}$$

so that

$$\widetilde{f_n}(\sigma,k) = \widetilde{u_n}(\sigma+k^2,k). \tag{3.6}$$

Therefore,

$$\|u_n\|^2_{X_\tau^{s,b}} = \sum_{k\in\mathbb{Z}}\int_{-\pi/\tau}^{\pi/\tau}\langle d_\tau(\sigma)\rangle^{2b}\langle k\rangle^{2s}|\widetilde{u_n}(\sigma+k^2,k)|^2\,\mathrm{d}\sigma$$

and the result follows by a change of variables.

To prove the embedding (3.5), it suffices to prove that

$$\|f_n\|_{l_\tau^\infty H^s} \lesssim \|f_n\|_{H_\tau^b H^s}.$$

Since

$$\widehat{f_n}(k) = \int_{-\pi/\tau}^{\pi/\tau}\widetilde{f_m}(\sigma,k)\mathrm{e}^{-in\tau\sigma}\,\mathrm{d}\sigma,$$

from Cauchy–Schwarz we get

$$|\widehat{f_n}(k)| \lesssim \left(\int_{-\pi/\tau}^{\pi/\tau}\frac{1}{\langle d_\tau(\sigma)\rangle^{2b}}\,\mathrm{d}\sigma\right)^{1/2}\|\langle d_\tau(\sigma)\rangle^b\widetilde{f_m}(\sigma,k)\|_{L^2}.$$

The result then follows by multiplying the above inequality by $\langle k\rangle^s$ and taking the L^2 norm with respect to k. ∎

Remark 3.2. From Lemma 3.1, we can make the following useful observation:

$$\sup_{\delta\in[-4,4]} \|e^{in\tau\delta\partial_x^2} u_n\|_{X_\tau^{s,b}} \lesssim_b \|u_n\|_{X_\tau^{s,b}}. \tag{3.7}$$

Note that this follows at once from $|d_\tau(\sigma - k^2 + \delta)| \lesssim \langle d_\tau(\sigma - k^2)\rangle$.

Remark 3.3. Since $\langle d_\tau(\sigma)\rangle \lesssim 1/\tau$, the discrete spaces satisfy the embedding

$$\|u_n\|_{X_\tau^{0,b}} \lesssim \frac{1}{\tau^{b-b'}} \|u_n\|_{X_\tau^{0,b'}}, \quad b \geq b'. \tag{3.8}$$

Indeed, from the above observation we find that the inequality is true for $b \geq 0$, $b' = 0$. Next, by interpolation we obtain the case $b \geq b' \geq 0$. The case $0 \geq b \geq b'$ then follows by duality, and the general case by composition.

We shall now establish the counterpart of Lemma 2.1 at the discrete level.

Lemma 3.4. *For $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\tau \in (0, 1]$, we have*

$$\|\eta(n\tau)e^{in\tau\partial_x^2} f\|_{X_\tau^{s,b}} \lesssim_{\eta,b} \|f\|_{H^s}, \qquad s\in\mathbb{R},\ b\in\mathbb{R},\ f\in H^s, \tag{3.9}$$

$$\|\eta(n\tau)u_n\|_{X_\tau^{s,b}} \lesssim_{\eta,b} \|u_n\|_{X_\tau^{s,b}}, \qquad s\in\mathbb{R},\ b\in\mathbb{R},\ u_n\in X_\tau^{s,b}, \tag{3.10}$$

$$\left\|\eta\left(\frac{n\tau}{T}\right)u_n\right\|_{X_\tau^{s,b'}} \lesssim_{\eta,b,b'} T^{b-b'}\|u_n\|_{X_\tau^{s,b}}, \qquad s\in\mathbb{R}, -1/2 < b' \leq b < 1/2,\ 0 < T = N\tau \leq 1,\ N \geq 1. \tag{3.11}$$

In addition, for

$$U_n(x) = \eta(n\tau)\tau \sum_{m=0}^{n} e^{i(n-m)\tau\partial_x^2} u_m(x),$$

we have

$$\|U_n\|_{X_\tau^{s,b}} \lesssim_{\eta,b} \|u_n\|_{X_\tau^{s,b-1}}, \quad s\in\mathbb{R},\ b > 1/2. \tag{3.12}$$

We stress that all the above estimates are uniform in τ.

Proof. We begin with (3.9). Let us set $u_n(x) = \eta(n\tau)e^{in\tau\partial_x^2} f(x)$. We first observe that

$$\widetilde{u_n}(\sigma, k) = \mathcal{F}_\tau(\eta(n\tau))(\sigma - k^2)\hat{f}(k).$$

The function $g(\sigma) = \mathcal{F}_\tau(\eta(n\tau))(\sigma)$ is fast decreasing in the sense that

$$|d_\tau(\sigma)^L g(\sigma)| \lesssim 1, \tag{3.13}$$

where the estimate is uniform in τ and σ for every integer $L \geq 1$. Indeed,

$$d_\tau(\sigma)g(\sigma) = \tau \sum_{n\in\mathbb{Z}} \frac{\eta((n-1)\tau) - \eta(n\tau)}{\tau} e^{in\tau\sigma}$$

and therefore

$$|d_\tau(\sigma)g(\sigma)| \lesssim 1 \tag{3.14}$$

by the smoothness of η. We easily get the boundedness of higher powers by induction. The estimate then follows easily from Lemma 3.1.

Let us prove (3.10). We recall that

$$\eta(n\tau) = \frac{1}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} g(\sigma)\,\mathrm{e}^{-in\tau\sigma}\,\mathrm{d}\sigma.$$

We deduce from (3.13) that for every $L \geq 1$, there exists $C > 0$ such that for every $\tau \in (0, 1]$ and σ with $\tau\sigma \in [-\pi, \pi]$,

$$|g(\sigma)| \leq C/\langle\sigma\rangle^{L}. \tag{3.15}$$

This yields, by using the fast decay of $g(\sigma)$,

$$\|\eta(n\tau)u_n\|_{X_\tau^{s,b}} \lesssim \int_{-\pi/\tau}^{\pi/\tau} \frac{1}{\langle\sigma_0\rangle^{L}}\|u_n\,\mathrm{e}^{-in\tau\sigma_0}\|_{X_\tau^{s,b}}\,\mathrm{d}\sigma_0.$$

Next, since $\mathscr{F}_{\tau,x}(u_n\,\mathrm{e}^{-in\tau\sigma_0})(\sigma, k) = \widetilde{u_n}(\sigma-\sigma_0, k)$ and

$$\langle d_\tau(\sigma-\sigma_0-k^2)\rangle^{b} \lesssim \langle\sigma_0\rangle^{|b|}\langle d_\tau(\sigma-k^2)\rangle^{b},$$

we get

$$\|\eta(n\tau)u_n\|_{X_\tau^{s,b}} \leq \int_{-\pi/\tau}^{\pi/\tau} \frac{1}{\langle\sigma_0\rangle^{L-|b|}}\,\mathrm{d}\sigma_0\,\|u_n\|_{X_\tau^{s,b}} \lesssim \|u_n\|_{X_\tau^{s,b}}$$

by choosing L sufficiently large.

We turn to the proof of (3.11). We follow the steps of the proof of the continuous case in [21]. We observe that by composition it suffices to handle the cases $0 \leq b' \leq b$ or $b' \leq b \leq 0$. By duality, it suffices then to establish the inequality in the case $0 \leq b' \leq b$. By standard interpolation, we have

$$\left\|\eta\left(\frac{n\tau}{T}\right)u_n\right\|_{X_\tau^{s,b'}} \leq \left\|\eta\left(\frac{n\tau}{T}\right)u_n\right\|_{X_\tau^{s,0}}^{1-b'/b}\left\|\eta\left(\frac{n\tau}{T}\right)u_n\right\|_{X_\tau^{s,b}}^{b'/b}.$$

It thus suffices to prove that

$$\left\|\eta\left(\frac{n\tau}{T}\right)u_n\right\|_{X_\tau^{s,b}} \lesssim \|u_n\|_{X_\tau^{s,b}} \tag{3.16}$$

and

$$\left\|\eta\left(\frac{n\tau}{T}\right)u_n\right\|_{X_\tau^{s,0}} \leq T^{b}\|u_n\|_{X_\tau^{s,b}} \tag{3.17}$$

for $b < 1/2$, where the estimates are uniform for $T \in (0, 1]$. We start with the first estimate. Note that we cannot use (3.10) directly to get an estimate uniform in T. Let us set $f_n = \mathrm{e}^{-in\tau\partial_x^2}u_n$ and $U_n = \eta(n\tau/T)f_n$. We want to estimate

$$\left\|\eta\left(\frac{n\tau}{T}\right)f_n\right\|_{H_\tau^b H^s} = \|U_n\|_{H_\tau^b H^s}.$$

We have

$$\widetilde{U_m}(\sigma,k) = \frac{1}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} g_T(\sigma-\sigma')\,\widetilde{f_m}(\sigma',k)\,\mathrm{d}\sigma',$$

where we have set

$$g_T(\sigma) = \tau\sum_n \eta\left(\frac{n\tau}{T}\right)\mathrm{e}^{in\tau\sigma}.$$

Using the same argument as above, we observe that for every $L \geq 0$,

$$|g_T(\sigma)| \lesssim_L \frac{T}{\langle T\sigma\rangle^L}. \tag{3.18}$$

In particular, this yields

$$\begin{aligned} &\|g_T\|_{L^1(-\pi/\tau,\pi/\tau)} \lesssim 1, \quad \|\langle\sigma\rangle^b g_T\|_{L^2(-\pi/\tau,\pi/\tau)} \lesssim T^{1/2-b}, \\ &\|\langle\sigma\rangle^b g_T\|_{L^1(-\pi/\tau,\pi/\tau)} \lesssim T^{-b}. \end{aligned} \tag{3.19}$$

We can first write, by using Young's inequality for convolutions,

$$\begin{aligned} \|\langle d_\tau(\sigma)\rangle^b \widetilde{U_m}(\sigma,k)\|_{L^2} \lesssim{}& \|g_T\|_{L^1}\|\langle d_\tau\rangle^b \widetilde{f_m}(\cdot,k)\|_{L^2} \\ &+ \left\| \int_{-\pi/\tau}^{\pi/\tau} |\langle d_\tau(\sigma-\sigma')\rangle^b g_T(\sigma-\sigma')|\,|\widetilde{f_m}(\sigma',k)|\,\mathrm{d}\sigma' \right\|_{L^2}. \end{aligned}$$

To estimate the last integral, we split $\widetilde{f_m}(\sigma',k) = \widetilde{f_m}(\sigma',k)1_{|\sigma'T|\leq 1} + \widetilde{f_m}(\sigma',k)1_{|\sigma'T|\geq 1}$. For the first contribution, we write

$$\begin{aligned} &\left\| \int_{-\pi/\tau}^{\pi/\tau} |\langle d_\tau(\sigma-\sigma')\rangle^b g_T(\sigma-\sigma')|\,|\widetilde{f_m}(\sigma',k)|1_{|\sigma'T|\leq 1}\,\mathrm{d}\sigma' \right\|_{L^2} \\ &\qquad \lesssim \|\langle\sigma\rangle^b g_T\|_{L^2(-\pi/\tau,\pi/\tau)}\|1_{|\sigma T|\leq 1}\widetilde{f_m}(\sigma,k)\|_{L^1} \\ &\qquad \lesssim T^{1/2-b}T^{b-1/2}\|\widetilde{f_m}(\cdot,k)\|_{L^2}, \end{aligned}$$

where we have used Cauchy–Schwarz to get the last estimate. For the second contribution, we use

$$\begin{aligned} &\left\| \int_{-\pi/\tau}^{\pi/\tau} |\langle d_\tau(\sigma-\sigma')\rangle^b g_T(\sigma-\sigma')|\,|\widetilde{f_m}(\sigma',k)|1_{|\sigma'T|\geq 1}\,\mathrm{d}\sigma' \right\|_{L^2} \\ &\qquad \lesssim \|\langle\sigma\rangle^b g_T\|_{L^1(-\pi/\tau,\pi/\tau)}\|1_{|\sigma T|\geq 1}\widetilde{f_m}(\sigma,k)\|_{L^2} \\ &\qquad \lesssim T^{-b}\|1_{|\sigma T|\geq 1}\widetilde{f_m}(\cdot,k)\|_{L^2} \lesssim \|\langle\sigma\rangle^b\widetilde{f_m}(\cdot,k)\|_{L^2}, \end{aligned}$$

where we have used $T^{-b} \lesssim \langle\sigma\rangle^b$ for $|\sigma T| \geq 1$. We have thus obtained

$$\|\langle d_\tau\rangle^b \widetilde{U_m}(\cdot,k)\|_{L^2} \lesssim \|\langle d_\tau\rangle^b \widetilde{f_m}(\cdot,k)\|_{L^2}.$$

It suffices to multiply by $\langle k\rangle^s$ and to take the L^2 norm in k to get (3.16).

We next prove (3.17). Again, it suffices to prove that

$$\|U_n\|_{l^2_\tau H^s} \lesssim T^b \|f_n\|_{H^b_\tau H^s}.$$

To establish this estimate, we split $f_n = f_{n,1} + f_{n,2}$ with

$$\widetilde{f_{m,1}}(\sigma, k) = \widetilde{f_m}(\sigma, k) 1_{|T\sigma|\geq 1}, \quad \widetilde{f_{m,2}}(\sigma, k) = \widetilde{f_m}(\sigma, k) 1_{|T\sigma|\leq 1}.$$

For the first part, we readily deduce from the definition of the norm that

$$\|f_{n,1}\|_{l^2_\tau H^s} \lesssim T^b \|f_n\|_{H^b_\tau, H^s}$$

since $1 \lesssim T^b |\sigma|^b$ on the support of integration. Since η is bounded,

$$\left\| \eta\left(\frac{n\tau}{T}\right) f_{n,1} \right\|_{l^2_\tau H^s} \lesssim T^b \|f_n\|_{H^b_\tau H^s}.$$

For the other part, we use

$$\widehat{f_{n,2}}(k) = \int_{-\pi/\tau}^{\pi/\tau} \mathrm{e}^{-in\tau\sigma} |\sigma|^{-b} 1_{T|\sigma|\leq 1} |\sigma|^b \widetilde{f_m}(\sigma, k) \,\mathrm{d}\sigma.$$

This yields, by Cauchy–Schwarz,

$$|\widehat{f_{n,2}}(k)|^2 \lesssim T^{2b-1} \left(\int_{-\pi/\tau}^{\pi/\tau} \langle d_\tau(\sigma) \rangle^{2b} |\widetilde{f_m}(\sigma, k)|^2 \,\mathrm{d}\sigma \right)$$

and therefore, for every n,

$$\|f_{n,2}\|_{H^s}^2 = \|\langle k \rangle^s \widehat{f_{n,2}}\|_{l^2}^2 \lesssim T^{2b-1} \|f_m\|_{l^2_\tau H^s}^2.$$

This yields

$$\begin{aligned} \left\| \eta\left(\frac{n\tau}{T}\right) f_{n,2} \right\|_{l^2_\tau H^s}^2 &= \tau \sum_n \eta\left(\frac{n\tau}{T}\right)^2 \|f_{n,2}\|_{H^s}^2 \\ &\lesssim \tau \sum_n \eta\left(\frac{n\tau}{T}\right)^2 T^{2b-1} \|f_m\|_{l^2_\tau H^s}^2 \lesssim T T^{2b-1} \|f_m\|_{l^2_\tau H^s}^2 \end{aligned}$$

and we get (3.17), which concludes the proof of (3.11).

We finally prove (3.12). Let us set

$$F_n(x) = \mathrm{e}^{-in\tau\partial_x^2} U_n(x), \quad f_n(x) = \mathrm{e}^{-in\tau\partial_x^2} u_n(x)$$

so that

$$F_n(x) = \eta(n\tau)\tau \sum_{m=0}^{n} f_m.$$

It suffices to prove that

$$\|F_n\|_{H^b_\tau H^s} \lesssim \|f_n\|_{H^{b-1}_\tau H^s}.$$

We shall only prove the estimate for $s = 0$; the general case follows by applying $\langle \partial_x \rangle^s$. Let us use the function $g(\sigma) = \mathcal{F}_\tau(\eta(n\tau))(\sigma)$ as above. By direct computation,

$$\widehat{F_n}(k) = \eta(n\tau)\tau \int_{-\pi/\tau}^{\pi/\tau} \widetilde{f_m}(\sigma_0, k) \frac{1 - \mathrm{e}^{-i(n+1)\tau\sigma_0}}{1 - \mathrm{e}^{-i\tau\sigma_0}} \, \mathrm{d}\sigma_0$$

and therefore

$$\widetilde{F_m}(\sigma, k) = \int_{-\pi/\tau}^{\pi/\tau} \frac{\mathrm{e}^{i\tau\sigma_0}}{d_\tau(\sigma_0)} \widetilde{f_m}(\sigma_0, k)\big(g(\sigma) - \mathrm{e}^{-i\tau\sigma_0} g(\sigma - \sigma_0)\big) \, \mathrm{d}\sigma_0.$$

We then split

$$\widetilde{F_m}(\sigma, k) = \widetilde{F_{m,1}}(\sigma, k) + \widetilde{F_{m,2}}(\sigma, k),$$

where we replace $\widetilde{f_m}$ by $\widetilde{f_m}(\sigma_0, k)1_{|\sigma_0| \geq 1}$ in $\widetilde{F_{m,1}}(\sigma, k)$, and $\widetilde{f_m}$ by $\widetilde{f_m}(\sigma_0, k)1_{|\sigma_0| \leq 1}$ in $\widetilde{F_{m,2}}(\sigma, k)$. By using the fast decay (3.18) of g, this yields

$$\begin{aligned} \langle d_\tau \rangle^b |\widetilde{F_{m,1}}(\sigma, k)| &\lesssim \langle \sigma \rangle^{b-L} \Big(\int_{-\pi/\tau}^{\pi/\tau} \frac{1}{\langle d_\tau(\sigma_0) \rangle^{2b}} \, \mathrm{d}\sigma_0 \Big)^{1/2} \| \langle d_\tau \rangle^{b-1} \widetilde{f_m}(\cdot, k) \|_{L^2} \\ &\quad + \int_{-\pi/\tau}^{\pi/\tau} \langle d_\tau(\sigma_0) \rangle^{b-1} |\widetilde{f_m}(\sigma_0, k)| \langle \sigma - \sigma_0 \rangle^{b-L} \, \mathrm{d}\sigma_0. \end{aligned}$$

Therefore, by taking the L^2 norm in σ, and by using Young's inequality for convolutions for the second term, we obtain, since $b > 1/2$,

$$\| \langle d_\sigma \rangle^b \widetilde{F_{m,1}}(\cdot, k) \|_{L^2} \lesssim \| \widehat{f_m}(k) \|_{H_\tau^{b-1}}.$$

To estimate $\widetilde{F_{m,2}}$, we observe that for $|\sigma_0| \leq 1$, we can use Taylor's formula to get

$$\left| \frac{\langle d_\tau(\sigma_0) \rangle^b}{d_\tau(\sigma_0)} \big(g(\sigma) - \mathrm{e}^{-i\tau\sigma_0} g(\sigma - \sigma_0) \big) \right| \lesssim \frac{1}{\langle \sigma - \sigma_0 \rangle^L}.$$

The estimate then follows from the same arguments.

We have thus proven that

$$\| \langle d_\sigma \rangle^b \widetilde{F_m}(\cdot, k) \|_{L^2} \lesssim \| \widehat{f_m}(k) \|_{H_\tau^{b-1}}.$$

To conclude the proof, it suffices to take the L^2 norm with respect to k. ∎

Remark 3.5. Note that in the proof of (3.10), we have also established a useful time translation invariance property of discrete Bourgain spaces:

$$\sup_{\delta \in [-4,4]} \| \mathrm{e}^{in\tau\delta} u_n \|_{X_\tau^{s,b}} \lesssim_b \| u_n \|_{X_\tau^{s,b}}. \tag{3.20}$$

To end this section we shall study the discrete counterpart of Lemma 2.2 which is crucial for the analysis of nonlinear problems.

In the discrete setting, for a sequence $(u_n) \in l^p(\mathbb{Z}, X)$ with X a normed space, we use the norm

$$\| u_n \|_{l_\tau^p(X)} = \Big(\tau \sum_{n \in \mathbb{Z}} \| u_n \|_X^p \Big)^{1/p}. \tag{3.21}$$

Lemma 3.6. *For $K \geq \tau^{-1/2}$, we have*

$$\|\Pi_K u_n\|_{l_\tau^4 L^4} \lesssim (K\tau^{1/2})^{1/2} \|u_n\|_{X_\tau^{0,3/8}}. \tag{3.22}$$

The above inequality is important for our paper, but understanding its proof requires tools not yet introduced and is not necessary to continue reading the paper. Therefore, we postpone the proof to Section 8.

By duality, we also deduce from (3.22) that

$$\|\Pi_K u_n\|_{X_\tau^{0,-3/8}} \lesssim (K\tau^{1/2})^{1/2} \|u_n\|_{l_\tau^{4/3} L^{4/3}}. \tag{3.23}$$

As a consequence, we obtain the following crucial product estimates for sequences u_n, v_n, and w_n.

Corollary 3.7. *We have the following product estimate:*

$$\|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{X_\tau^{0,-3/8}} \lesssim K\tau^{1/2} \|u_n\|_{X_\tau^{0,3/8}} \|v_n\|_{X_\tau^{0,3/8}} \|w_n\|_{X_\tau^{0,3/8}}. \tag{3.24}$$

Moreover, for any $s_1 > 1/4$,

$$\|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{X_\tau^{0,-3/8}} \lesssim (K\tau^{1/2})^{1/2} \|u_n\|_{X_\tau^{0,3/8}} \|v_n\|_{X_\tau^{0,3/8}} \|w_n\|_{l_\tau^4 H^{s_1}}, \tag{3.25}$$

$$\|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{X_\tau^{0,-3/8}} \lesssim \|u_n\|_{X_\tau^{s_1,3/8}} \|v_n\|_{X_\tau^{s_1,3/8}} \|w_n\|_{l_\tau^\infty L^2}, \tag{3.26}$$

and for $s_2 > 1/2$,

$$\|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{X_\tau^{0,-3/8}} \lesssim \|u_n\|_{X_\tau^{0,3/8}} \|\Pi_{\tau^{-1/2}} v_n\|_{l_\tau^4 H^{s_1}} \|w_n\|_{l_\tau^\infty H^{s_2}}, \tag{3.27}$$

$$\|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{X_\tau^{0,-3/8}} \lesssim \|u_n\|_{X_\tau^{0,3/8}} \|v_n\|_{X_\tau^{0,3/8}} \|w_n\|_{l_\tau^\infty H^{s_2}}. \tag{3.28}$$

Note that (3.26) is of particular interest if the two lower frequency factors have at least $1/4$ regularity. Then we do not need the factor $K\tau^{1/2}$ which is large if $\alpha > 1$ (recall that $K = \tau^{-\alpha/2}$). This will be useful to prove the stability of the scheme for $\alpha > 1$. The estimate (3.25) will turn out to be useful to optimize the convergence rate of the scheme when s_0 is large enough.

Proof of Corollary 3.7. We start by proving (3.24). We first deduce from the estimate (3.23) that

$$\|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{X_\tau^{0,-3/8}} \lesssim (K\tau^{1/2})^{1/2} \|\Pi_K(\Pi_{\tau^{-1/2}} u_n \Pi_{\tau^{-1/2}} v_n \Pi_K w_n)\|_{l_\tau^{4/3} L^{4/3}}.$$

From the continuity of Π_K on L^p and the Hölder inequality, we next get

$$\begin{aligned}&\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{X_\tau^{0,-3/8}}\\&\quad\lesssim (K\tau^{1/2})^{1/2}\|\Pi_{\tau^{-1/2}}u_n\|_{l_\tau^4L^4}\|\Pi_{\tau^{-1/2}}v_n\|_{l_\tau^4L^4}\|\Pi_K w_n\|_{l_\tau^4L^4}. \qquad (3.29)\end{aligned}$$

By using again (3.22), we thus find

$$\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{X_\tau^{0,-3/8}}\lesssim K\tau^{1/2}\|u_n\|_{X_\tau^{0,3/8}}\|v_n\|_{X_\tau^{0,3/8}}\|w_n\|_{X_\tau^{0,3/8}}.$$

This proves (3.24).

For the proof of (3.25), we use again (3.29). However, we only estimate $\|\Pi_{\tau^{-1/2}}u_n\|_{l_\tau^4L^4}$ and $\|\Pi_{\tau^{-1/2}}v_n\|_{l_\tau^4L^4}$ with the help of (3.22). For the last term, we use the Sobolev embedding $H^{s_1}\subset L^4$ to get

$$\|\Pi_K w_n\|_{l_\tau^4L^4}\lesssim\|\Pi_K w_n\|_{l_\tau^4H^{s_1}}.$$

To get (3.26), we just use

$$\begin{aligned}&\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{X_\tau^{0,-3/8}}\\&\quad\lesssim\|\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n\|_{X_\tau^{0,0}} \qquad (3.30)\end{aligned}$$

and employ Hölder's inequality to get

$$\begin{aligned}&\|\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n\|_{X_\tau^{0,-3/8}}\\&\quad\lesssim\|\Pi_{\tau^{-1/2}}u_n\|_{l_\tau^4L^\infty}\|\Pi_{\tau^{-1/2}}v_n\|_{l_\tau^4L^\infty}\|\Pi_K w_n\|_{l_\tau^\infty L^2}.\end{aligned}$$

We then use (3.5) to write

$$\|\Pi_K w_n\|_{l_\tau^\infty L^2}\lesssim\|w_n\|_{X_\tau^{0,b}},$$

and the Sobolev embedding $W^{s_1,4}\subset L^\infty$ and (3.22) to obtain

$$\|\Pi_{\tau^{-1/2}}u_n\|_{l_\tau^4L^\infty}\lesssim\|u_n\|_{X_\tau^{s_1,3/8}},\quad \|\Pi_{\tau^{-1/2}}v_n\|_{l_\tau^4L^\infty}\lesssim\|v_n\|_{X_\tau^{s_1,3/8}}.$$

This concludes the proof of (3.26).

For (3.27) and (3.28), we just use (3.30) again, and then the inequality

$$\|\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n\|_{X^{0,0}}\lesssim\|\Pi_{\tau^{-1/2}}u_n\|_{l_\tau^4L^4}\|\Pi_{\tau^{-1/2}}v_n\|_{l_\tau^4L^4}\|\Pi_K w_n\|_{l_\tau^\infty L^\infty}.$$

To conclude the proof, we use Lemma 3.6 and the Sobolev embedding $H^{s_2}\subset L^\infty$ or Lemma 3.6 and the Sobolev embeddings $H^{s_2}\subset L^\infty$, $H^{s_1}\subset L^4$. ∎

Remark 3.8. Another version intermediate between (3.25) and (3.26) will also be useful. We have

$$\begin{aligned}&\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{X_\tau^{0,-3/8}}\\&\quad\lesssim (K\tau^{1/2})^{1/2}\|u_n\|_{X_\tau^{0,3/8}}\|v_n\|_{X_\tau^{s_1,3/8}}\|w_n\|_{l_\tau^4L^2} \qquad (3.31)\end{aligned}$$

with $s_1 > 1/4$. Indeed, we first use (3.23) to get

$$\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{X_\tau^{0,-3/8}} \leq (K\tau^{1/2})^{1/2}\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{l_\tau^{4/3}L^{4/3}}$$

and we employ Hölder's inequality to get

$$\|\Pi_K(\Pi_{\tau^{-1/2}}u_n\Pi_{\tau^{-1/2}}v_n\Pi_K w_n)\|_{l_\tau^{4/3}L^{4/3}} \leq \|\Pi_K w_n\|_{l_\tau^4 L^2}\|\Pi_{\tau^{-1/2}}v_n\|_{l_\tau^4 L^\infty}\|\Pi_{\tau^{-1/2}}u_n\|_{l_\tau^4 L^4}.$$

We conclude the proof by using the Sobolev embedding $W^{s_1,4} \subset L^\infty$ and (3.22).

4. Estimates of the exact solution in discrete Bourgain spaces

In this section, we shall prove that the sequence $u^K(t_n)$ is an element of $X_\tau^{s,b}$ for suitable s. It will be convenient to use the following general lemma.

Lemma 4.1. *For any $b \geq 0$, $b' > 1/2$ and $s \in \mathbb{R}$, consider a sequence of functions $(u_n(x))_{n\in\mathbb{Z}}$ of the form $u_n(x) = u(n\tau, x)$. Then*

$$\|u_n\|_{X_\tau^{s,b}} \lesssim_b \|u\|_{X^{s,b+b'}}.$$

Proof. By setting $f = \mathrm{e}^{-it\partial_x^2}u$ and $f_n(x) = f(n\tau, x)$, it suffices to prove that

$$\|f_n\|_{H_\tau^b L^2} \lesssim \|f\|_{H^{b+b'}L^2},$$

extension to general s being straightforward. Since by definition

$$\widetilde{f_m}(\sigma, k) = \tau\sum_{n\in\mathbb{Z}}\widetilde{f}(n\tau, k)\,\mathrm{e}^{in\tau\sigma},$$

Poisson's summation formula implies that

$$\widetilde{f_n}(\sigma, k) = \sum_{m\in\mathbb{Z}}\widetilde{f}\Big(\sigma + \frac{2\pi}{\tau}m, k\Big).$$

Therefore,

$$\langle d_\tau(\sigma)\rangle^b\widetilde{f_n}(\sigma, k) = \sum_{m\in\mathbb{Z}}\Big\langle d_\tau\Big(\sigma + \frac{2\pi}{\tau}m\Big)\Big\rangle^b\widetilde{f}\Big(\sigma + \frac{2\pi}{\tau}m, k\Big),$$

since d_τ is also a $2\pi/\tau$-periodic function. Since $|d_\tau(\sigma)| \lesssim \langle\sigma\rangle$, this yields, by Cauchy–Schwarz,

$$\begin{aligned}|\langle d_\tau(\sigma)\rangle^b\widetilde{f_n}(\sigma, k)|^2 &\lesssim \sum_\mu\frac{1}{\langle\sigma + \frac{2\pi}{\tau}\mu\rangle^{2b'}}\sum_{m\in\mathbb{Z}}\Big\langle\sigma + \frac{2\pi}{\tau}m\Big\rangle^{2b+2b'}\Big|\widetilde{f}\Big(\sigma + \frac{2\pi}{\tau}m, k\Big)\Big|^2\\ &\lesssim \sum_{m\in\mathbb{Z}}\Big\langle\sigma + \frac{2\pi}{\tau}m\Big\rangle^{2b+2b'}\Big|\widetilde{f}\Big(\sigma + \frac{2\pi}{\tau}m, k\Big)\Big|^2\end{aligned}$$

since $2b' > 1$. By integrating with respect to σ, we obtain

$$\|\langle d_\tau\rangle^b \widetilde{f}_n(\cdot,k)\|_{L^2(-\pi/\tau,\pi/\tau)} \lesssim \|\langle\sigma\rangle^{b+b'} \widetilde{f}(\cdot,k)\|^2_{L^2(\mathbb{R})}.$$

We finish the proof by summing over k. ∎

As a consequence of the previous lemma, we obtain the following result.

Proposition 4.2. *Let u^K be the solution of (1.7) and define $u_n^K(x) = u^K(n\tau + t', x)$. Assume that $u_0 \in H^{s_0}$, $s_0 > 0$. Then, for every s_1 such that $0 \le s_1 < s_0$,*

$$\sup_{t'\in[0,4\tau]} \|\eta(n\tau)u_n^K\|_{X_\tau^{s_1,3/8}} \le C_T.$$

Proof. It suffices to combine Lemma 4.1 and Corollary 2.8 by taking b' arbitrarily close to $1/2$. ∎

5. Local error of time discretization

In this section we analyse the time discretization error which is introduced when discretizing the twice-filtered Schrödinger equation (1.7) with the scheme (1.11).

Setting

$$\begin{aligned}\mathcal{A} &= \{\kappa = (\kappa_1,\kappa_2,\kappa_3) : \kappa_1,\kappa_2,\kappa_3 \in \{\tau^{-1/2}, K^+\}, \exists i \neq j : \kappa_i = \kappa_j = \tau^{-1/2}\}\\ &= \{(\tau^{-1/2},\tau^{-1/2},\tau^{-1/2}), (\tau^{-1/2},\tau^{-1/2},K^+),\\ &\qquad (\tau^{-1/2},K^+,\tau^{-1/2}), (K^+,\tau^{-1/2},\tau^{-1/2})\}\end{aligned} \tag{5.1}$$

allows us to express the filtered Schrödinger equation (1.7) as follows:

$$i\partial_t u^K = -\partial_x^2 u^K + \sum_{\kappa\in\mathcal{A}} \Pi_K(\Pi_{\kappa_1}u^K \Pi_{\kappa_2}\overline{u}^K \Pi_{\kappa_3}u^K) \tag{5.2}$$

and Duhamel's formula (with step size τ) takes the form

$$u^K(t_n+\tau) = \mathrm{e}^{i\tau\partial_x^2}u^K(t_n) - i\Pi_K \mathrm{e}^{i\tau\partial_x^2}\sum_{\kappa\in\mathcal{A}} \mathcal{T}_\kappa(u^K)(\tau,t_n), \tag{5.3}$$

where

$$\mathcal{T}_\kappa(u^K)(\tau,t_n) = \int_0^\tau \mathrm{e}^{-is\partial_x^2}\big(\Pi_{\kappa_1}u^K(t_n+s)\Pi_{\kappa_2}\overline{u}^K(t_n+s)\Pi_{\kappa_3}u^K(t_n+s)\big)\,\mathrm{d}s. \tag{5.4}$$

Henceforth, we will use the following notation:

$$V_{\kappa_\ell}^K(s,t) = \mathrm{e}^{is\partial_x^2}\Pi_{\kappa_\ell}u^K(t), \quad W_{\kappa_\ell}^K(s,t) = \mathrm{e}^{is\partial_x^2}\Pi_{\kappa_\ell}\Pi_K\sum_{\sigma\in\mathcal{A}}\mathcal{T}_\sigma(u^K)(s,t).$$

Iterating Duhamel's formula (5.3), i.e., plugging the expansion

$$\Pi_{\kappa_\ell}u^K(t_n+s) = V_{\kappa_\ell}^K(s,t_n) - iW_{\kappa_\ell}^K(s,t_n)$$

into (5.3), yields the representation

$$u^K(t_n+\tau) = \mathrm{e}^{i\tau\partial_x^2} u^K(t_n) - i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \sum_{\kappa\in\mathcal{A}} \int_0^\tau \mathrm{e}^{-is\partial_x^2}[V_{\kappa_1}^K(s,t_n)\overline{V}_{\kappa_2}^K(s,t_n)V_{\kappa_3}^K(s,t_n)]\,\mathrm{d}s$$
$$- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \sum_{\kappa\in\mathcal{A}} E_\kappa(\tau,t_n) \tag{5.5}$$

with the remainder

$$E_\kappa(\tau,t_n) = \int_0^\tau \mathrm{e}^{-is\partial_x^2}(E_{\kappa,1}+E_{\kappa,2}+E_{\kappa,3}+E_{\kappa,4}+E_{\kappa,5}+E_{\kappa,6}+E_{\kappa,7})(s,t_n)\,\mathrm{d}s \tag{5.6}$$

defined by

$$\begin{aligned}
E_{\kappa,1}(s,t_n) &= i V_{\kappa_1}^K(s,t_n)\overline{W}_{\kappa_2}^K(s,t_n)V_{\kappa_3}^K(s,t_n),\\
E_{\kappa,2}(s,t_n) &= -i V_{\kappa_1}^K(s,t_n)\overline{V}_{\kappa_2}^K(s,t_n)W_{\kappa_3}^K(s,t_n),\\
E_{\kappa,3}(s,t_n) &= -i W_{\kappa_1}^K(s,t_n)\overline{V}_{\kappa_2}^K(s,t_n)V_{\kappa_3}^K(s,t_n),\\
E_{\kappa,4}(s,t_n) &= W_{\kappa_1}^K(s,t_n)\overline{W}_{\kappa_2}^K(s,t_n)V_{\kappa_3}^K(s,t_n),\\
E_{\kappa,5}(s,t_n) &= -W_{\kappa_1}^K(s,t_n)\overline{V}_{\kappa_2}^K(s,t_n)W_{\kappa_3}^K(s,t_n),\\
E_{\kappa,6}(s,t_n) &= V_{\kappa_1}^K(s,t_n)\overline{W}_{\kappa_2}^K(s,t_n)W_{\kappa_3}^K(s,t_n),\\
E_{\kappa,7}(s,t_n) &= -i W_{\kappa_1}^K(s,t_n)\overline{W}_{\kappa_2}^K(s,t_n)W_{\kappa_3}^K(s,t_n).
\end{aligned} \tag{5.7}$$

It remains to analyse the error introduced by the time discretization of the integrals in (5.5), where the discretization is carried out in such a way that the *dominant* terms in (5.5), i.e., the intermediate frequency terms $\tau^{-1/2} < |k| \le K$, are treated *exactly*, while the lower-order frequency terms $|k| \le \tau^{-1/2}$ are approximated in a suitable manner.

Lemma 5.1. *For sufficiently smooth functions v, w,*

$$\int_0^\tau \mathrm{e}^{-is\partial_x^2}[(\mathrm{e}^{is\partial_x^2}v)|\mathrm{e}^{is\partial_x^2}w|^2]\,\mathrm{d}s = \mathcal{J}_1^\tau(\overline{w},v,w) + R_1(\overline{w},v,w) \tag{5.8}$$

with $\mathcal{J}_1^\tau$ defined in (1.12) and the remainder given by

$$R_1(v_1,v_2,v_3) = -2i\int_0^\tau \mathrm{e}^{-is\partial_x^2}\Bigg[(\mathrm{e}^{is\partial_x^2}v_2)\int_0^s \mathrm{e}^{i(s-s_1)\partial_x^2}[(\mathrm{e}^{-is_1\partial_x^2}\partial_x^2 v_1)(\mathrm{e}^{is_1\partial_x^2}v_3)$$
$$+ (\mathrm{e}^{-is_1\partial_x^2}\partial_x v_1)(\mathrm{e}^{is_1\partial_x^2}\partial_x v_3)]\,\mathrm{d}s_1\Bigg]\,\mathrm{d}s. \tag{5.9}$$

Proof. The proof is in two steps. First we will show that in fact

$$\mathcal{J}_1^\tau(\overline{w},v,w) = \int_0^\tau \mathrm{e}^{-is\partial_x^2}[(\mathrm{e}^{is\partial_x^2}v)\mathrm{e}^{is\partial_x^2}|w|^2]\,\mathrm{d}s. \tag{5.10}$$

The Fourier expansion of the above integral together with the relation

$$(-k_1+k_2+k_3)^2 - (k_2^2 + (-k_1+k_3)^2) = 2k_2(-k_1+k_3)$$

yields

$$
\begin{aligned}
&\int_0^\tau \mathrm{e}^{-is\partial_x^2}[(\mathrm{e}^{is\partial_x^2}v)\mathrm{e}^{is\partial_x^2}|w|^2]\,\mathrm{d}s\\
&= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=-k_1+k_2+k_3}} \overline{\hat{w}}_{k_1}\hat{v}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\int_0^\tau \mathrm{e}^{is(-k_1+k_2+k_3)^2}\mathrm{e}^{-is(k_2^2+(-k_1+k_3)^2)}\,\mathrm{d}s\\
&= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=-k_1+k_2+k_3}} \overline{\hat{w}}_{k_1}\hat{v}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\int_0^\tau \mathrm{e}^{2isk_2(-k_1+k_3)}\,\mathrm{d}s\\
&= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k_2\neq 0,\ k_1\neq k_3\\ k=-k_1+k_2+k_3}} \overline{\hat{w}}_{k_1}\hat{v}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\int_0^\tau \mathrm{e}^{2isk_2(-k_1+k_3)}\,\mathrm{d}s + \tau\hat{v}_0|w|^2 + \tau\widehat{(|w|^2)}_0(v-\hat{v}_0)\\
&= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k_2\neq 0,\ k_1\neq k_3\\ k=-k_1+k_2+k_3}} \overline{\hat{w}}_{k_1}\hat{v}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\frac{\mathrm{e}^{2i\tau k_2(-k_1+k_3)}-1}{2ik_2(-k_1+k_3)} + \tau\hat{v}_0|w|^2 + \tau\widehat{(|w|^2)}_0(v-\hat{v}_0)\\
&= \frac{i}{2}\mathrm{e}^{-i\tau\partial_x^2}[(\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}v)\mathrm{e}^{i\tau\partial_x^2}\partial_x^{-1}|w|^2] - \frac{i}{2}(\partial_x^{-1}v)\partial_x^{-1}|w|^2 + \tau\hat{v}_0|w|^2\\
&\quad + \tau\widehat{(|w|^2)}_0(v-\hat{v}_0),
\end{aligned}
$$

which implies (5.10). Thanks to (5.10) we can furthermore conclude by (5.8) that

$$
R_1(\overline{w},v,w) = \int_0^\tau \mathrm{e}^{-is\partial_x^2}\big[(\mathrm{e}^{is\partial_x^2}v)[|\mathrm{e}^{is\partial_x^2}w|^2 - \mathrm{e}^{is\partial_x^2}|w|^2]\big]\,\mathrm{d}s. \tag{5.11}
$$

We note that

$$
\begin{aligned}
&-2i\int_0^s \mathrm{e}^{i(s-s_1)\partial_x^2}[(\mathrm{e}^{-is_1\partial_x^2}\partial_x^2\overline{w})(\mathrm{e}^{is_1\partial_x^2}w) + |\mathrm{e}^{is_1\partial_x^2}\partial_x w|^2]\,\mathrm{d}s_1\\
&= -2i\sum_{\ell_1,\ell_2\in\mathbb{Z}} \overline{\hat{w}}_{\ell_1}\hat{w}_{\ell_2}\mathrm{e}^{i(-\ell_1+\ell_2)x}\mathrm{e}^{-is(-\ell_1+\ell_2)^2}\\
&\qquad\qquad \times\int_0^s \mathrm{e}^{is_1(-\ell_1+\ell_2)^2}\mathrm{e}^{is_1(\ell_1^2-\ell_2^2)}(-\ell_1^2+\ell_1\ell_2)\,\mathrm{d}s_1\\
&= \sum_{\ell_1,\ell_2\in\mathbb{Z}} \overline{\hat{w}}_{\ell_1}\hat{w}_{\ell_2}\mathrm{e}^{i(-\ell_1+\ell_2)x}\mathrm{e}^{-is(-\ell_1+\ell_2)^2}(\mathrm{e}^{is(-\ell_1+\ell_2)^2}\mathrm{e}^{is(\ell_1^2-\ell_2^2)}-1)\\
&= \sum_{\ell_1,\ell_2\in\mathbb{Z}} \overline{\hat{w}}_{\ell_1}\hat{w}_{\ell_2}\mathrm{e}^{i(-\ell_1+\ell_2)x}(\mathrm{e}^{is(\ell_1^2-\ell_2^2)} - \mathrm{e}^{-is(-\ell_1+\ell_2)^2}) = |\mathrm{e}^{is\partial_x^2}w|^2 - \mathrm{e}^{is\partial_x^2}|w|^2.
\end{aligned}
$$

Plugging the above relation into (5.11) yields (5.9). This concludes the proof. ∎

Lemma 5.2. *For sufficiently smooth functions v, w,*

$$
\int_0^\tau \mathrm{e}^{-is\partial_x^2}[(\mathrm{e}^{-is\partial_x^2}\overline{v})(\mathrm{e}^{is\partial_x^2}w)^2]\,\mathrm{d}s = \mathcal{J}_2^\tau(\overline{v},w,w) + R_2(\overline{v},w,w) \tag{5.12}
$$

with $\mathcal{J}_2^\tau$ defined in (1.13) *and the remainder given by*

$$
\begin{aligned}
R_2(v_1,v_2,v_3) = -2i\int_0^\tau \mathrm{e}^{-is\partial_x^2}\Big[&(\mathrm{e}^{-is\partial_x^2}v_1)\\
&\times\int_0^s \mathrm{e}^{i(s-s_1)\partial_x^2}(\mathrm{e}^{is_1\partial_x^2}\partial_x v_2)(\mathrm{e}^{is_1\partial_x^2}\partial_x v_3)\,\mathrm{d}s_1\Big]\,\mathrm{d}s.
\end{aligned} \tag{5.13}
$$

Proof. Again we prove the assertion in two steps. First we show that in fact

$$\mathcal{J}_2^\tau(\overline{v}, w, w) = \int_0^\tau \mathrm{e}^{-is\partial_x^2}[(\mathrm{e}^{-is\partial_x^2}\overline{v})(\mathrm{e}^{is\partial_x^2}w^2)]\,\mathrm{d}s. \tag{5.14}$$

This follows by Fourier expansion of the integral together with the relation

$$(-k_1+k_2+k_3)^2 + k_1^2 - (k_2+k_3)^2 = -2k_1(-k_1+k_2+k_3),$$

which implies that

$$\begin{aligned}
&\int_0^\tau \mathrm{e}^{-is\partial_x^2}[(\mathrm{e}^{-is\partial_x^2}\overline{v})(\mathrm{e}^{is\partial_x^2}w^2)]\,\mathrm{d}s\\
&\quad= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=-k_1+k_2+k_3}} \hat{\overline{v}}_{k_1}\hat{w}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\int_0^\tau \mathrm{e}^{is(-k_1+k_2+k_3)^2}\mathrm{e}^{is(k_1^2-(k_2+k_3)^2)}\,\mathrm{d}s\\
&\quad= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=-k_1+k_2+k_3}} \hat{\overline{v}}_{k_1}\hat{w}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\int_0^\tau \mathrm{e}^{-2isk_1k}\,\mathrm{d}s\\
&\quad= \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=-k_1+k_2+k_3}} \hat{\overline{v}}_{k_1}\hat{w}_{k_2}\hat{w}_{k_3}\mathrm{e}^{ikx}\frac{\mathrm{e}^{-2ik_1k\tau}-1}{-2ik_1k}\\
&\quad= \frac{i}{2}\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}[(\mathrm{e}^{-i\tau\partial_x^2}\partial_x^{-1}\overline{v})(\mathrm{e}^{i\tau\partial_x^2}w^2)] - \frac{i}{2}\partial_x^{-1}(w^2\partial_x^{-1}\overline{v}) + \tau\widehat{(\overline{v}w^2)}_0\\
&\qquad+ \tau\hat{\overline{v}}_0\big(w^2 - \widehat{(w^2)}_0\big).
\end{aligned} \tag{5.15}$$

Thanks to (5.14) we can furthermore conclude by (5.12) that

$$R_2(\overline{v}, w, w) = \int_0^\tau \mathrm{e}^{-is\partial_x^2}\big[(\mathrm{e}^{-is\partial_x^2}\overline{v})[(\mathrm{e}^{is\partial_x^2}w)^2 - \mathrm{e}^{is\partial_x^2}w^2]\big]\,\mathrm{d}s. \tag{5.16}$$

We note that

$$\begin{aligned}
&-2i\int_0^s \mathrm{e}^{i(s-s_1)\partial_x^2}(\mathrm{e}^{is_1\partial_x^2}\partial_x w)^2\,\mathrm{d}s_1\\
&\quad= 2i\sum_{\ell_1,\ell_2\in\mathbb{Z}} \hat{w}_{\ell_1}\hat{w}_{\ell_2}\mathrm{e}^{i(\ell_1+\ell_2)x}\mathrm{e}^{-is(\ell_1+\ell_2)^2}\int_0^s \mathrm{e}^{is_1(\ell_1+\ell_2)^2}\mathrm{e}^{-is_1(\ell_1^2+\ell_2^2)}\ell_1\ell_2\,\mathrm{d}s_1\\
&\quad= 2i\sum_{\ell_1,\ell_2\in\mathbb{Z}} \hat{w}_{\ell_1}\hat{w}_{\ell_2}\mathrm{e}^{i(\ell_1+\ell_2)x}\mathrm{e}^{-is(\ell_1+\ell_2)^2}\int_0^s \mathrm{e}^{2is_1\ell_1\ell_2}\ell_1\ell_2\,\mathrm{d}s_1\\
&\quad= \sum_{\ell_1,\ell_2\in\mathbb{Z}} \hat{w}_{\ell_1}\hat{w}_{\ell_2}\mathrm{e}^{i(\ell_1+\ell_2)x}\mathrm{e}^{-is(\ell_1+\ell_2)^2}(\mathrm{e}^{is(\ell_1+\ell_2)^2}\mathrm{e}^{-is(\ell_1^2+\ell_2^2)}-1)\\
&\quad= (\mathrm{e}^{is\partial_x^2}w)^2 - \mathrm{e}^{is\partial_x^2}w^2.
\end{aligned}$$

Plugging the above relation into (5.16) proves the assertion. ■

Lemma 5.3 (Local error). *The local error*

$$\mathcal{E}(\tau, t_n) := u^K(t_n + \tau) - \Phi_\tau^K(u^K(t_n))$$

of the time discretization scheme (1.11) *applied to the filtered Schrödinger equation* (1.7) *equals*

$$\begin{aligned}\mathcal{E}(\tau, t_n) = &-2i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} R_1\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K^+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} R_2\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \sum_{\kappa\in\mathcal{A}} E_\kappa(\tau, t_n),\end{aligned} \tag{5.17}$$

where $E_\kappa(\tau, t_n)$ *is defined in* (5.6) *and the remainders* R_1 *and* R_2 *are given in* (5.9) *and* (5.13)*, respectively.*

Proof. The assertion follows by the expansion of the exact solution $u^K(t_n + \tau)$ given in (5.5) together with Lemmas 5.1 and 5.2.

More precisely, employing Lemma 5.1 to approximate the integral arising for $\kappa_1 = K^+$ or $\kappa_3 = K^+$ in (5.5) and Lemma 5.2 to approximate the integrals arising for $\kappa_2 = K^+$ or $\kappa_1 = \kappa_2 = \kappa_3 = \tau^{-1/2}$ in (5.5) yields

$$\begin{aligned}u^K(t_n+\tau) = &\,\mathrm{e}^{i\tau\partial_x^2}u^K(t_n) - 2i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \mathcal{J}_1^\tau\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K^+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \mathcal{J}_2^\tau\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &- 2i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} R_1\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K^+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} R_2\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2} \sum_{\kappa\in\mathcal{A}} E_\kappa(\tau, t_n).\end{aligned} \tag{5.18}$$

The assertion thus follows by taking the difference of the expansion of the exact solution given in (5.18) and the numerical flow defined in (1.11). ∎

6. Global error analysis

Let $e^{n+1} = u^K(t_{n+1}) - u_\tau^{n+1}$ denote the time discretization error, i.e., the difference between the numerical solution $u_\tau^{n+1} = \Phi_\tau^K(u_\tau^n)$ defined in (1.11) and the exact solution of the filtered Schrödinger equation (1.7). Inserting a zero in terms of $\pm\Phi_\tau^K(u^K(t_n))$, i.e., using

$$e^{n+1} = u^K(t_{n+1}) - \Phi_\tau^K(u^K(t_n)) + \Phi_\tau^K(u^K(t_n)) - \Phi_\tau^K u_\tau^n,$$

we obtain, by the definition of the numerical flow Φ_τ^K in (1.11),

$$\begin{aligned}e^{n+1} = &\,\mathrm{e}^{i\tau\partial_x^2}e^n - 2i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2}\big[\mathcal{J}_1^\tau\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K^+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &\qquad - \mathcal{J}_1^\tau(\Pi_{\tau^{-1/2}}\overline{u}_\tau^n, \Pi_{K^+}u_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n)\big]\\ &- i\,\Pi_K \mathrm{e}^{i\tau\partial_x^2}\big[\mathcal{J}_2^\tau\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\\ &\qquad - \mathcal{J}_2^\tau(\Pi_K\overline{u}_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n)\big]\\ &+ \mathcal{E}(\tau, t_n),\end{aligned} \tag{6.1}$$

where $\mathcal{J}_1^\tau$ and $\mathcal{J}_2^\tau$ are defined in (1.12) and (1.13) and the local error (5.17) is given in Lemma 5.3.

By solving the above recursion, we find that for $0 \leq n \leq N_1 = \lfloor T_1/\tau \rfloor$ with $T_1 \leq T$, the global error e^n satisfies

$$e^n = \tau\eta(t_n) \sum_{k=0}^{n-1} \mathrm{e}^{i(n-k)\tau\partial_x^2} \eta\left(\frac{k\tau}{T_1}\right) \Pi_K G_k + \mathcal{R}_{1,n} + \mathcal{R}_{2,n}, \tag{6.2}$$

where

$$\begin{aligned} G_n = \frac{-2i}{\tau} \big[& \mathcal{J}_1^\tau\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ & - \mathcal{J}_1^\tau(\Pi_{\tau^{-1/2}}\overline{u}_\tau^n, \Pi_{K+}u_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n)\big] \\ - \frac{i}{\tau}\big[& \mathcal{J}_2^\tau\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ & - \mathcal{J}_2^\tau(\Pi_K\overline{u}_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n, \Pi_{\tau^{-1/2}}u_\tau^n)\big], \end{aligned} \tag{6.3}$$

and

$$\mathcal{R}_{i,n} = \tau\eta(t_n) \sum_{k=0}^{n-1} \mathrm{e}^{i(n-k)\tau\partial_x^2} \eta(t_k)\Pi_K \mathcal{F}_i(t_k), \quad i = 1, 2,$$

with

$$\begin{aligned} \mathcal{F}_1(t_n) = \frac{1}{\tau}\Big(& -2i\,\Pi_K R_1\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ & - i\,\Pi_K R_2\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\Big), \end{aligned} \tag{6.4}$$

$$\mathcal{F}_2(t_n) = -\frac{i}{\tau}\Pi_K \sum_{\kappa\in\mathcal{A}} E_\kappa(\tau, t_n). \tag{6.5}$$

Note that E_κ is defined in (5.6) and R_1, R_2 in (5.9) and (5.13). We have introduced the truncation function η in order to work with global Bourgain spaces. As before, we will assume that u_τ^n and u^K are globally defined, though they coincide with the actual solutions of the scheme and the PDE on a finite interval of time. We will choose T_1 sufficiently small later.

We shall first estimate $\mathcal{R}_{1,n}$, which gives the dominant contribution to the error.

Lemma 6.1. *Let $s_0 \in (0, 1]$ and $b \in (1/2, 5/8)$. For $s_0 > 0$,*

$$\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} \leq C_T K\tau^{1/2}\tau^{(s_0)-}. \tag{6.6}$$

Moreover, if $s_0 > 1/4$, then

$$\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} \leq C_T (K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+}, \tag{6.7}$$

and if $s_0 > 1/2$, then

$$\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} \leq C_T \tau^{s_0-(1/8)+}. \tag{6.8}$$

Proof. By using again (3.11) and (3.12), we get

$$\|\mathcal{R}_{1,n}\|_{X^{0,b}_\tau} \lesssim \|\mathcal{F}_1(t_n)\|_{X^{0,-3/8}_\tau}.$$

By using (6.4) this amounts to estimating

$$I_1 = \frac{1}{\tau}\big\|R_1\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\big\|_{X^{0,-3/8}_\tau},$$
$$I_2 = \frac{1}{\tau}\big\|R_2\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big)\big\|_{X^{0,-3/8}_\tau}.$$

We first prove (6.6). We start with the estimate of I_2. We use (5.13), (3.24) and Remark 3.2 to obtain

$$I_2 \lesssim K\tau^{1/2}\|u^K(t_n)\|_{X^{0,3/8}_\tau}\|\tau^{1/2}\partial_x\Pi_{\tau^{-1/2}}u^K(t_n)\|^2_{X^{0,3/8}_\tau}.$$

By using Proposition 4.2, this yields

$$I_2 \lesssim K\tau^{1/2}\|u^K\|_{X^{s_0,b}}\|\tau^{1/2}\partial_x\Pi_{\tau^{-1/2}}u^K\|^2_{X^{s_2,b}}$$

with $s_0 > s_2 > 0$, s_2 arbitrarily small and $b \in (7/8, 1)$. Consequently, from the frequency localization, we find that

$$I_2 \lesssim \tau^{s_0-s_2}(K\tau^{1/2})\|u^K\|_{X^{s_0,b}}\|\Pi_{\tau^{-1/2}}u^K\|^2_{X^{s_0,b}} \leq C_T\tau^{s_0-s_2}(K\tau^{1/2}),$$

where we use Corollary 2.8 for the last estimate.

It remains to estimate I_1. By using the definition (5.9) and the same arguments, we get

$$I_1 \lesssim K\tau^{1/2}\|(\tau^{1/2}\partial_x)^2\Pi_{\tau^{-1/2}}u\|_{X^{s_2,b}}\|\Pi_{K+}u^K\|_{X^{s_2,b}}\|u^K\|_{X^{s_2,b}}$$
$$+ K\tau^{1/2}\|u^K\|_{X^{s_0,b}}\|\tau^{1/2}\partial_x\Pi_{\tau^{-1/2}}u^K\|^2_{X^{s_2,b}}$$

again with $s_0 > s_2 > 0$ and s_2 arbitrarily small. The second term is similar to the one before. For the first term, by using frequency localization, in particular the fact that $\tau^{1/2}|\xi| \geq 1$ on the support of Π_{K+}, we then obtain

$$\|(\tau^{1/2}\partial_x)^2\Pi_{\tau^{-1/2}}u\|_{X^{s_2,b}} \lesssim \tau^{(s_0-s_2)/2}\|u\|_{X^{s_0,b}},$$
$$\|\Pi_{K+}u^K\|_{X^{s_2,b}} \lesssim \tau^{(s_0-s_2)/2}\|u^K\|_{X^{s_0,b}}.$$

This also yields

$$I_1 \leq C_T\tau^{s_0-s_2}\,(K\tau^{1/2}),$$

which concludes the proof of (6.6).

To prove (6.7), we follow the same lines, but we use (3.25) instead of (3.24) since $s_0 > 1/4$. This yields

$$\|\mathcal{R}_{1,n}\|_{X^{0,b}_\tau} \lesssim (K\tau^{1/2})^{1/2}\|u^K(t_n)\|_{l^4_\tau H^{(1/4)+}}\|\tau^{1/2}\partial_x\Pi_{\tau^{-1/2}}u^K(t_n)\|^2_{X^{0,3/8}_\tau}$$
$$+ (K\tau^{1/2})^{1/2}\|(\tau^{1/2}\partial_x)^2\Pi_{\tau^{-1/2}}u^K(t_n)\|_{X^{0,3/8}_\tau}\|\Pi_{K+}u^K(t_n)\|_{l^4_\tau H^{(1/4)+}}$$
$$\|\Pi_{\tau^{-1/2}}u^K(t_n)\|_{X^{0,3/8}_\tau}.$$

By using again the same estimates as above, it only remains to estimate $\|u^K(t_n)\|_{l^4_\tau H^{(1/4)+}}$ and $\|\Pi_{K+}u^K(t_n)\|_{l^4_\tau H^{(\frac{1}{4})+}}$. We can just use

$$\|u^K(t_n)\|_{l^4_\tau H^{(1/4)+}} \lesssim T^{1/4}\|u^K\|_{L^\infty_T H^{s_0}} \lesssim T^{1/4}\|u^K\|_{X^{s_0,b}}, \quad b > 1/2,$$

and, by frequency localization for $|\xi| \geq \tau^{-1/2}$,

$$\|\Pi_{K+}u^K(t_n)\|_{l^4_\tau H^{(1/4)+}} \lesssim_T \tau^{\frac{1}{2}(s_0-(1/4)+)}\|u^K\|_{L^\infty_T H^{s_0}}. \tag{6.9}$$

This yields (6.7).

Finally, to get (6.8), we follow the same lines but we now use (3.27) and (3.28) . This yields

$$\begin{aligned}\|\mathcal{R}_{1,n}\|_{X^{0,b}_\tau} &\lesssim \|\tau^{1/2}\partial_x\Pi_{\tau^{-1/2}}u^K(t_n)\|^2_{X^{0,3/8}_\tau}\|u^K(t_n)\|_{l^\infty_\tau H^{(1/2)+}} \\ &\quad + \|(\tau^{1/2}\partial_x)^2\Pi_{\tau^{-1/2}}u^K(t_n)\|_{X^{0,3/8}_\tau}\|\Pi_{K+}u^K(t_n)\|_{l^4_\tau H^{(1/4)+}}\|u^K(t_n)\|_{l^\infty_\tau H^{(1/2)+}}.\end{aligned}$$

We then use the same estimates, in particular (6.9) and the fact that

$$\|u^K(t_n)\|_{l^\infty_\tau H^{(1/2)+}} \leq \|u^K\|_{L^\infty_T H^{(1/2)+}}.$$

This ends the proof. ∎

We shall next estimate $\mathcal{R}_{2,n}$.

Lemma 6.2. *For $s_0 > 0$ and $b \in (1/2, 5/8)$,*

$$\|\mathcal{R}_{2,n}\|_{X^{0,b}_\tau} \leq C_T\big(\tau^{1/4}(K\tau^{1/2})^2 + \tau^{1/2}(K\tau^{1/2})^3 + \tau^{3/2}(K\tau^{1/2})^4\big). \tag{6.10}$$

Moreover if $s_0 > 1/4$, then

$$\|\mathcal{R}_{2,n}\|_{X^{0,b}_\tau} \leq C_T(K\tau^{1/2})^{1/2}\tau^{5/8}, \tag{6.11}$$

and if $s_0 > 1/2$, then

$$\|\mathcal{R}_{2,n}\|_{X^{0,b}_\tau} \leq C_T\tau. \tag{6.12}$$

Proof. We first use (3.12) and Remark 3.2 to estimate

$$\|\mathcal{R}_{2,n}\|_{X^{0,b}_\tau} \lesssim \|\mathcal{F}_2(t_n)\|_{X^{0,-3/8}_\tau}.$$

Next, by using (5.6) and the product estimate (3.24), we get

$$\begin{aligned}\|\mathcal{F}_{2,n}\|_{X^{0,-3/8}_\tau} &\lesssim K\tau^{1/2}\sum_{\kappa\in\mathcal{A}}\big(\|u^K(t_n)\|^2_{X^{0,3/8}}\|\mathcal{T}_\kappa(u^K(t_n))\|_{X^{0,3/8}_\tau} \\ &\quad + \|u^K(t_n)\|_{X^{0,3/8}_\tau}\|\mathcal{T}_\kappa(u^K(t_n))\|^2_{X^{0,3/8}_\tau} + \|\mathcal{T}_\kappa(u^K(t_n))\|^3_{X^{0,3/8}_\tau}\big).\end{aligned}$$

Next, by using (5.4), we get

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \lesssim \tau \sup_{t'\in[0,\tau]} \sum_{\kappa\in\mathcal{A}} \|\Pi_{\kappa_1} u^K(t_n+t')\Pi_{\kappa_2}\overline{u}^K(t_n+t')\Pi_{\kappa_3}\overline{u}^K(t_n+t')\|_{X_\tau^{0,3/8}}.$$

By using (3.8), we thus obtain

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \lesssim \tau^{1/4} \sup_{t'\in[0,\tau]} \sum_{\kappa\in\mathcal{A}} \|\Pi_{\kappa_1} u^K(t_n+t')\Pi_{\kappa_2}\overline{u}^K(t_n+t')\Pi_{\kappa_3}\overline{u}^K(t_n+t')\|_{X_\tau^{0,-3/8}}.$$

Consequently, by using the product estimate (3.24) again, we find that

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \lesssim \tau^{1/4}(K\tau^{1/2}) \sup_{t'\in[0,\tau]} \|u^K(t_n+t')\|^3_{X_\tau^{0,3/8}}.$$

Then, by Proposition 4.2,

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \lesssim \tau^{1/4}(K\tau^{1/2})C_T$$

and hence

$$\|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} \lesssim K\tau^{1/2}\big(\tau^{1/4}(K\tau^{1/2}) + (\tau^{1/4}(K\tau^{1/2}))^2 + (\tau^{1/4}(K\tau^{1/2}))^3\big)C_T.$$

This proves (6.10).

To get (6.11), we use (3.23) and (3.25) to get

$$\begin{aligned}
\|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} &\lesssim \|\mathcal{F}_2(t_n)\|_{X_\tau^{0,-3/8}} \\
&\lesssim (K\tau^{1/2})^{1/2} \sum_{\kappa\in\mathcal{A}} \big(\|u^K(t_n)\|^2_{X_\tau^{0,3/8}}\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}} \\
&\qquad + \|u^K(t_n)\|_{X_\tau^{0,3/8}}\|u^K(t_n)\|_{l_\tau^4 H^{(1/4)+}}\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \\
&\qquad + \|\mathcal{T}_\kappa(u^K(t_n))\|^2_{X_\tau^{0,3/8}}\|u^K(t_n)\|_{l_\tau^4 H^{(1/4)+}} \\
&\qquad + \|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}}\|u^K(t_n)\|_{X_\tau^{0,3/8}}\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}} \\
&\qquad + \|\mathcal{T}_\kappa(u^K(t_n))\|^2_{X_\tau^{0,3/8}}\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}}\big).
\end{aligned}$$

We can again use (3.8) and the trivial estimate

$$\|u^K(t_n)\|_{l_\tau^4 H^{(1/4)+}} \lesssim T^{1/4}\|u^K\|_{L^\infty H^{(1/4)+}}$$

so that it only remains to estimate $\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}}$ and $\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}}$. For the first one, we use again (3.8) to write

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \lesssim \tau^{5/8} \sup_{t'\in[0,\tau]} \sum_{\kappa\in\mathcal{A}} \|\Pi_{\kappa_1} u^K(t_n+t')\Pi_{\kappa_2}\overline{u}^K(t_n+t')\Pi_{\kappa_3}u^K(t_n+t')\|_{X_\tau^{0,0}}.$$

Next, from Hölder's inequality and the Sobolev embedding $W^{(1/4)+,4} \subset L^\infty$, we get

$$\begin{aligned}&\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}}\\&\qquad\lesssim \tau^{5/8} \sup_{t'\in[0,\tau]} \|\Pi_{\tau^{-1/2}} u^K(t_n+t')\|^2_{l^4_\tau W^{(1/4)+,4}} \|\Pi_K u^K(t_n+t')\|_{l^\infty_\tau L^2}\end{aligned}$$

and hence by using (3.22), Proposition 4.2 and (3.7) we get, for $s_0 > 1/4$,

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{X_\tau^{0,3/8}} \lesssim \tau^{5/8}\|u^K\|^3_{X^{s_0,b}} \lesssim C_T\tau^{5/8}.$$

Next, we estimate $\|\mathcal{T}_\kappa(u^K(t_n))\|_{l^4_\tau H^{(1/4)+}}$. We begin with

$$\begin{aligned}&\|\mathcal{T}_\kappa(u^K(t_n))\|_{l^4_\tau H^{(1/4)+}}\\&\lesssim \tau \sup_{t'\in[0,\tau]} \sum_{\kappa\in\mathcal{A}} \|\Pi_{\kappa_1} u^K(t_n+t')\Pi_{\kappa_2}\overline{u}^K(t_n+t')\Pi_{\kappa_3}u^K(t_n+t')\|_{l^4_\tau H^{(1/4)+}}.\end{aligned} \tag{6.13}$$

Then we observe that for all sequences (u_n), (v_n), (w_n), and $s>0$, we have

$$\begin{aligned}&\|\Pi_{K+}u_n\Pi_{\tau^{-1/2}}v_n\Pi_{\tau^{-1/2}}w_n\|_{H^s}\\&\qquad\lesssim \|\Pi_{K+}u_n\|_{H^s}\|\Pi_{\tau^{-1/2}}v_n\|_{L^\infty}\|\Pi_{\tau^{-1/2}}w_n\|_{L^\infty}.\end{aligned} \tag{6.14}$$

Indeed, by using the generalized Leibniz rule (2.12), we have

$$\begin{aligned}\|\Pi_{K+}u_n\Pi_{\tau^{-1/2}}v_n\Pi_{\tau^{-1/2}}w_n\|_{H^s} &\lesssim \|\Pi_{K+}u_n\|_{H^s}\|\Pi_{\tau^{-1/2}}v_n\|_{L^\infty}\|\Pi_{\tau^{-1/2}}w_n\|_{L^\infty}\\&+\|\Pi_{K+}u_n\|_{L^2}\|\langle\partial_x\rangle^s\Pi_{\tau^{-1/2}}v_n\|_{L^\infty}\|\Pi_{\tau^{-1/2}}w_n\|_{L^\infty}\\&+\|\Pi_{K+}u_n\|_{L^2}\|\Pi_{\tau^{-1/2}}v_n\|_{L^\infty}\|\langle\partial_x\rangle^s\Pi_{\tau^{-1/2}}w_n\|_{L^\infty}.\end{aligned}$$

By frequency localization, we observe that

$$\begin{aligned}&\|\Pi_{K+}u_n\|_{L^2}\lesssim \tau^{s/2}\|u_n\|_{H^s},\quad \|\langle\partial_x\rangle^s\Pi_{\tau^{-1/2}}v_n\|_{L^\infty}\lesssim 1/\tau^{s/2}\|v_n\|_{L^\infty},\\&\|\langle\partial_x\rangle^s\Pi_{\tau^{-1/2}}w_n\|_{L^\infty}\lesssim \frac{1}{\tau^{s/2}}\|w_n\|_{L^\infty},\end{aligned}$$

and hence (6.14) follows. We thus deduce from (6.13) and (6.14) that

$$\begin{aligned}&\|\mathcal{T}_\kappa(u^K(t_n))\|_{l^4_\tau H^{(1/4)+}}\\&\quad\lesssim \sup_{t'\in[0,\tau]} \tau\|u^K(t_n+t')\|_{l^\infty_\tau H^{(1/4)+}}\|\Pi_{\tau^{-1/2}}u^K(t_n+t')\|^2_{l^8_\tau W^{(1/4)+,4}}\\&\quad\lesssim \sup_{t'\in[0,\tau]} \tau^{3/4} \sup_{t'\in[0,\tau]} \tau\|u^K(t_n+t')\|_{l^\infty_\tau H^{(1/4)+}}\|\Pi_{\tau^{-1/2}}u^K(t_n+t')\|^2_{l^4_\tau W^{(1/4)+,4}}.\end{aligned}$$

Hence by using (3.22), Proposition 4.2 and (3.7) again we finally get

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{l^4_\tau H^{(1/4)+}} \lesssim C_T\tau^{3/4}$$

if $s_0 > 1/4$. We thus deduce (6.11).

It remains to prove (6.12). We now use (3.27) and (3.28) to get

$$\begin{aligned}\|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} &\lesssim \|\mathcal{F}_2(t_n)\|_{X_\tau^{0,-3/8}}\\ &\lesssim \|u^K(t_n)\|^2_{X_\tau^{0,3/8}} \|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^\infty H^{(1/2)+}}\\ &\quad + \|u^K(t_n)\|_{X_\tau^{0,3/8}} \|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}} \|u^K(t_n)\|_{l_\tau^\infty H^{(1/2)+}}\\ &\quad + \|u^K(t_n)\|_{X_\tau^{0,3/8}} \|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}} \|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^\infty H^{(1/2)+}}\\ &\quad + \|\mathcal{T}_\kappa(u^K(t_n))\|^2_{l_\tau^4 H^{(1/4)+}} \|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^\infty H^{(1/2)+}}.\end{aligned}$$

Note that for the last term in the above right-hand side, we have used the fact that in the estimate (3.27), we can replace in the right-hand side the norm $\|u_n\|_{X_\tau^{0,3/8}}$ by the norm $\|u_n\|_{l_\tau^4 H^{(1/4)+}}$ by using the Sobolev embedding in space instead of the Bourgain estimate (3.22). Since we have the obvious estimate $\|u^K(t_n)\|_{l_\tau^\infty H^{(1/2)+}} \lesssim \|u^K\|_{L_T^\infty H^{(1/2)+}}$ and since

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^4 H^{(1/4)+}} \lesssim T^{1/4} \|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^\infty H^{(1/2)+}},$$

it only remains to estimate $\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^\infty H^{(1/2)+}}$. From standard product estimates, since $H^{(1/2)+}$ is an algebra, we get

$$\|\mathcal{T}_\kappa(u^K(t_n))\|_{l_\tau^\infty H^{(1/2)+}} \lesssim \tau \|u^K(t_n)\|^3_{l_\tau^\infty H^{(1/2)+}} \lesssim C_T \tau.$$

This concludes the proof. ∎

7. Proof of Theorem 1.1

We first observe that thanks to (2.10), the triangle inequality yields

$$\|u(t_n) - u_\tau^n\|_{L^2} \le C_T \tau^{s_0\alpha/2} + \|u^K(t_n) - u_\tau^n\|_{L^2} \le C_T \tau^{s_0\alpha/2} + \|e^n\|_{l_\tau^\infty L^2}, \tag{7.1}$$

where e^n solves (6.2). To get the error estimates of Theorem 1.1, it thus suffices to estimate $\|e^n\|_{X_\tau^{0,b}}$ for some $b \in (1/2, 5/8)$ thanks to (3.5). Note that there are two parts in the total error, the space discretization part above and the time discretization error on the right-hand side of (6.2), which is estimated in Lemmas 6.1 and 6.2. We shall optimize the total error by choosing the best possible α as regularity allows.

We first prove (1.4). For very rough data, when $0 < s_0 \le 1/4$, we need the estimate (3.22) without loss. This forces us to choose $K = \tau^{-1/2}$, hence $\alpha = 1$ without allowing us to optimize the error. We thus deduce from Lemmas 6.1 and 6.2 that

$$\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} + \|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} \le C_T(\tau^{s_0-0+} + \tau^{1/4}) \le C_T \tau^{s_0-0+}. \tag{7.2}$$

Next, we decompose

$$G_n = L_n - Q_n + C_n \tag{7.3}$$

with

$$\begin{aligned} L_n = &-\frac{2i}{\tau}\big[\mathcal{J}_1^r\big(\Pi_{\tau^{-1/2}}\overline{e}^n, \Pi_{K+}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ &\quad + \mathcal{J}_1^r\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K+}e^n, \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ &\quad + \mathcal{J}_1^r\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K+}u^K(t_n), \Pi_{\tau^{-1/2}}e^n\big)\big] \\ &-\frac{i}{\tau}\big[\mathcal{J}_2^r\big(\Pi_K\overline{e}^n, \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ &\quad + \mathcal{J}_2^r\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}e^n, \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ &\quad + \mathcal{J}_2^r\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}u_K(t_n), \Pi_{\tau^{-1/2}}e^n\big)\big], \end{aligned} \tag{7.4}$$

$$\begin{aligned} Q_n = &-\frac{2i}{\tau}\big[\mathcal{J}_1^r\big(\Pi_{\tau^{-1/2}}\overline{e}^n, \Pi_{K+}e^n, \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ &\quad + \mathcal{J}_1^r\big(\Pi_{\tau^{-1/2}}\overline{e}^n, \Pi_{K+}u^K(t_n), \Pi_{\tau^{-1/2}}e^n\big) \\ &\quad + \mathcal{J}_1^r\big(\Pi_{\tau^{-1/2}}\overline{u}^K(t_n), \Pi_{K+}e^n, \Pi_{\tau^{-1/2}}e^n\big)\big] \\ &-\frac{i}{\tau}\big[\mathcal{J}_2^r\big(\Pi_K\overline{e}^n, \Pi_{\tau^{-1/2}}e^n, \Pi_{\tau^{-1/2}}u^K(t_n)\big) \\ &\quad + \mathcal{J}_2^r\big(\Pi_K\overline{e}^n, \Pi_{\tau^{-1/2}}u^K(t_n), \Pi_{\tau^{-1/2}}e^n\big) \\ &\quad + \mathcal{J}_2^r\big(\Pi_K\overline{u}^K(t_n), \Pi_{\tau^{-1/2}}e^n, \Pi_{\tau^{-1/2}}u^K(t_n)\big)\big], \end{aligned} \tag{7.5}$$

and

$$C_n = \frac{1}{\tau}\big[-2i\,\mathcal{J}_1^r(\Pi_{\tau^{-1/2}}\overline{e}^n, \Pi_{K+}e^n, \Pi_{\tau^{-1/2}}e^n) - i\,\mathcal{J}_2^r(\Pi_K\overline{e}^n, \Pi_{\tau^{-1/2}}e^n, \Pi_{\tau^{-1/2}}e^n)\big]. \tag{7.6}$$

By using Lemma 3.4 and (7.2), we deduce from (6.2) that

$$\|e^n\|_{X_\tau^{0,b}} \le C_T T_1^{\varepsilon_0}\|G_n\|_{X_\tau^{0,-3/8}} + C_T\tau^{s_0-(0)+}, \quad n\tau \le T_1,$$

where $\varepsilon_0 = 5/8 - b > 0$. Next, we have

$$\|G_n\|_{X_\tau^{0,-3/8}} \le \|L_n\|_{X_\tau^{0,-3/8}} + \|Q_n\|_{X_\tau^{0,-3/8}} + \|C_n\|_{X_\tau^{0,-3/8}}.$$

To estimate the right-hand side, we use the equivalent definitions (5.10), (5.14) and again (3.24) and (3.20) (we recall that for this case we choose $K\tau^{1/2} = 1$). This yields

$$\|e^n\|_{X_\tau^{0,b}} \le C_T T_1^{\varepsilon_0}(\|e^n\|_{X_\tau^{0,b}} + \|e^n\|_{X_\tau^{0,b}}^2 + \|e^n\|_{X_\tau^{0,b}}^3) + C_T\tau^{s_0-(0)+}.$$

By choosing T_1 sufficiently small we thus get

$$\|e^n\|_{X_\tau^{0,b}} \le C_T\tau^{s_0-(0)+}.$$

This proves the desired estimate (1.4) for $0 \le n \le N_1 = T_1/\tau$. We can then iterate the argument on $T_1/\tau \le n \le 2T_1/\tau$ and so on to get the final estimate. We thus finally deduce from (7.1) that

$$\|u(t_n) - u_\tau^n\|_{L^2} \le C_T(\tau^{s_0/2} + \tau^{s_0-(0)+}),$$

which means that for every $0 < \varepsilon < s_0$, we have for some C_T (depending on ε) the estimate

$$\|u(t_n) - u_\tau^n\|_{L^2} \le C_T(\tau^{s_0/2} + \tau^{s_0-\varepsilon}).$$

Since we can always choose ε small enough so that $s_0 - \varepsilon > s_0/2$, we get (1.4).

We next prove (1.5). We follow the same lines, but we can now optimize the total error. From Lemmas 6.1 and 6.2, we get

$$\begin{aligned}\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} + \|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} &\le C_T\big((K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+} + (K\tau^{1/2})^{1/2}\tau^{5/8}\big)\\ &\le C_T(K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+}.\end{aligned}$$

We now choose K such that $(K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+} = 1/K^{s_0}$, which gives

$$K = \tau^{-\alpha/2} = \tau^{-\frac{s_0+(1/8)-}{s_0+1/2}}, \quad \alpha = 2\frac{s_0+(1/8)_-}{s_0+1/2} = 2\bigg(1 - \frac{1}{2s_0+1}\Big(\frac{3}{4}\Big)_+\bigg). \tag{7.7}$$

Note that $\alpha \in [1,2]$ since $1/4 < s_0 \le 1/2$, and further

$$\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} + \|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} \le C_T\tau^{s_0(1-\frac{1}{2s_0+1}(3/4)+)}. \tag{7.8}$$

By using Lemma 3.4, from (6.2) we get

$$\begin{aligned}\|e^n\|_{X_\tau^{0,b}} &\le C_T T_1^{\varepsilon_0}(\|L_n\|_{X_\tau^{0,-3/8}} + \|Q_n\|_{X_\tau^{0,-3/8}} + \|C_n\|_{X_\tau^{0,-3/8}})\\ &\quad + C_T\tau^{s_0(1-\frac{1}{2s_0+1}(3/4)+)}.\end{aligned} \tag{7.9}$$

To estimate L_n, we use the product estimates (3.25), (3.26) to get

$$\|L_n\|_{X_\tau^{0,-3/8}} \lesssim \|u^n\|^2_{X_\tau^{(1/4)+,3/8}}\|e^n\|_{l_\tau^\infty L^2} + \|e^n\|_{X_\tau^{0,3/8}}\|u^n\|_{X_\tau^{0,3/8}}\|u^n\|_{X_\tau^{(1/4)+,3/8}}$$

and hence by using again Proposition 4.2 and (3.5), we obtain

$$\|L_n\|_{X_\tau^{0,-3/8}} \lesssim C_T\|e^n\|_{X_\tau^{0,b}}. \tag{7.10}$$

To estimate C_n, we use again (3.24) and (3.20). This yields

$$\|C_n\|_{X_\tau^{0,-3/8}} \le C_T K\tau^{1/2}\|e^n\|^3_{X_\tau^{0,3/8}}. \tag{7.11}$$

To estimate Q_n, we use (3.25) and (3.31) and again Proposition 4.2. This yields

$$\|Q_n\|_{X_\tau^{0,-3/8}} \le C_T(K\tau^{1/2})^{1/2}\|e^n\|^2_{X_\tau^{0,b}} \tag{7.12}$$

since $s_0 > 1/4$. By setting $Y = \|e^n\|_{X_\tau^{0,b}}/\tau^{s_0(1-\frac{1}{2s_0+1}(3/4)+)}$, we deduce from the above estimates and (7.9) that

$$\begin{aligned}Y \le C_T T_1^{\varepsilon_0}\Big(Y &+ (K\tau^{1/2})^{1/2}\tau^{s_0(1-1/2s_0+1(3/4)+)}Y^2\\ &+ \big((K\tau^{1/2})^{1/2}\tau^{s_0(1-\frac{1}{2s_0+1}(3/4)+)}\big)^2Y^3\Big) + C_T.\end{aligned}$$

We can then check that with the choice (7.7), for $1/4 < s_0 \le 1/2$, the exponent β of

$$\tau^\beta = (K\tau^{1/2})^{1/2}\tau^{s_0(1-\frac{1}{2s_0+1}(3/4)+)}$$

is positive. Hence, we can argue as before to get (1.5).

It remains to prove (1.6). From Lemmas 6.1 and 6.2, we now get

$$\|\mathcal{R}_{1,n}\|_{X_\tau^{0,b}} + \|\mathcal{R}_{2,n}\|_{X_\tau^{0,b}} \le C_T\tau^{s_0-(1/8)+}.$$

We thus choose K such that $\tau^{s_0-(1/8)+} = 1/K^{s_0}$ in order to optimize the total error. We find

$$K = \tau^{\frac{1}{s_0}(1/8)+-1}, \quad \alpha = 2 - \frac{1}{s_0}\left(\frac{1}{4}\right)_+. \tag{7.13}$$

We can use (7.10)–(7.12) to infer from (6.2) that

$$\|e^n\|_{X_\tau^{0,b}} \le C_T T_1^{\varepsilon_0}\big(\|e^n\|_{X_\tau^{0,b}} + (K\tau^{1/2})^{1/2}\|e^n\|_{X_\tau^{0,b}}^2 + K\tau^{1/2}\|e^n\|_{X_\tau^{0,b}}^3\big) + C_T\tau^{s_0-(1/8)+}.$$

Again, by setting $Y = \|e^n\|_{X_\tau^{0,b}}/\tau^{s_0-(1/8)+}$, we get

$$Y \le C_T T_1^{\varepsilon_0}\Big(Y + (K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+}Y^2 + \big((K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+}\big)^2 Y^3\Big) + C_T$$

and we conclude the proof as before, by observing that the exponent of τ in

$$(K\tau^{1/2})^{1/2}\tau^{s_0-(1/8)+}$$

is positive with the choice (7.13).

8. Proof of Lemma 3.6

We have to prove (3.22). For this purpose, we adapt the proof in [21] (which is attributed to N. Tzvetkov). We first observe that

$$\|\Pi_K u_n\|_{l_\tau^4 L^4}^2 = \|(\Pi_K u_n)^2\|_{l_\tau^2 L^2}. \tag{8.1}$$

By the definition of the $X_\tau^{0,b}$ norm and by setting $f_n = \mathrm{e}^{-in\tau\partial_x^2}\Pi_K u_n$, it is equivalent to prove that

$$\|(\mathrm{e}^{in\tau\partial_x^2} f_n)^2\|_{l_\tau^2 L^2} \lesssim K\tau^{1/2}\|f_n\|_{H_\tau^{3/8}L^2}^2.$$

By using the space-time Fourier transform we shall decompose $\widetilde{f_m}(\sigma, k)$ by using a Littlewood–Paley decomposition with respect to σ. Note that since $\sigma \in [-\pi/\tau, \pi/\tau]$, there are actually a finite number of terms. We write

$$f_n = \sum_{l\ge 0} f_{n,l}$$

where $\widetilde{f_{m,l}}(\cdot,k)$ is supported in $2^{l-1} \leq \langle\sigma\rangle \leq 2^{l+1}$ for every k. By symmetry and the triangle inequality, it is sufficient to prove that

$$\sum_{p\leq q} \|e^{in\tau\partial_x^2} f_{n,p}\, e^{in\tau\partial_x^2} f_{n,q}\|_{l_\tau^2 L^2} \lesssim K\tau^{1/2}\|f_n\|^2_{H_\tau^{3/8}L^2}.$$

We shall actually prove that there exists $\varepsilon > 0$ (we shall see that we can take $\varepsilon = 1/8$) such that for all p, q with $p \leq q$,

$$\|e^{in\tau\partial_x^2} f_{n,p}\, e^{in\tau\partial_x^2} f_{n,q}\|_{l_\tau^2 L^2} \lesssim K\tau^{1/2}\, 2^{\varepsilon(p-q)}(2^{3p/8}\|f_{n,p}\|_{l_\tau^2 L^2})(2^{3q/8}\|f_{n,q}\|_{l_\tau^2 L^2}). \tag{8.2}$$

Once this inequality is proven, the result follows easily. Indeed, let us set $a_m = 2^{3m/8}\|f_{n,m}\|_{l_\tau^2 L^2}$, $b_m = 2^{\varepsilon m}1_{m\leq 0}$. By Parseval, we have $|a_m| \lesssim \|f_{n,m}\|_{H_\tau^{3/8}L^2}$ and

$$\|a_m\|_{l^2} \lesssim \|f_n\|_{H_\tau^{3/8}L^2}.$$

Moreover, assuming that (8.2) is proven we obtain

$$\sum_{p\leq q} \|e^{in\tau\partial_x^2} f_{n,p} e^{in\tau\partial_x^2} f_{n,q}\|_{l_\tau^2 L^2} \lesssim K\tau^{1/2}\|(b * a)_m a_m)\|_{l^1} \lesssim K\tau^{1/2}\|a_m\|^2_{l^2}$$

from Cauchy–Schwarz and Young's inequality for sequences (observe that $b \in l^1$), which is the desired estimate.

We shall now prove (8.2). From Parseval and by using (3.6), we have

$$\|e^{in\tau\partial_x^2} f_{n,p}\, e^{in\tau\partial_x^2} f_{n,q}\|^2_{l_\tau^2 L^2} = \sum_k \int_{-\pi/\tau}^{\pi/\tau} \Big| \sum_{k_1+k_2=k} \int_{\sigma_1+\sigma_2=\sigma} \widetilde{f_{n,p}}(\sigma_1 - k_1^2, k_1)\widetilde{f_{n,q}}(\sigma_2 - k_2^2, k_2)\, d\sigma_1 \Big|^2 d\sigma.$$

Now let us notice that we have a nontrivial contribution if $\sigma_1 - k_1^2$ is in the support of $\widetilde{f_{n,p}}(\cdot, k_1)$ and $\sigma_2 - k_2^2$ in the support of $\widetilde{f_{n,q}}(\cdot, k_2)$. By periodicity in the σ variable, this means that there exist m_1, m_2 in $\mathbb{Z}$ such that

$$\left|\sigma_1 - k_1^2 - \frac{2m_1\pi}{\tau}\right| \lesssim 2^p, \quad \left|\sigma_2 - k_2^2 - \frac{2m_2\pi}{\tau}\right| \lesssim 2^q.$$

In other words, $\sigma_1 - k_1^2 \in E_p$, $\sigma_2 - k_2^2 \in E_q$ where $E_l = \bigcup_{|m|\leq N}[2m\pi/\tau - 2^l, 2m\pi/\tau + 2^l]$. Note that since the frequencies k_1^2, k_2^2 are smaller than K^2, we can take $N \lesssim \tau K^2$.

By using Cauchy–Schwarz again, we thus get

$$\|e^{in\tau\partial_x^2} f_{n,p}\, e^{in\tau\partial_x^2} f_{n,q}\|^2_{l_\tau^2 L^2} \lesssim M_{p,q}\|\widetilde{f_{n,p}}\|^2_{L^2 l^2}\|\widetilde{f_{n,q}}\|^2_{L^2 l^2}, \tag{8.3}$$

where

$$M_{p,q} = \sup_{k,\sigma} \sum_{k_1+k_2=k} \int_{\sigma_1+\sigma_2=\sigma,\, \sigma_1-k_1^2\in E_p,\, \sigma_2-k_2^2\in E_q} d\sigma_1.$$

To estimate $M_{p,q}$, we observe that only $\sigma \in k_1^2 + k_2^2 + E_p + E_q \subset k_1^2 + k_2^2 + 2E_q$ gives a nonzero contribution and that the integral is bounded by a constant times 2^p. Since $k_1 + k_2 = k$, we have

$$k_1^2 + k_2^2 = \tfrac{1}{2}(k^2 + (k_1 - k_2)^2)$$

and hence

$$(k_1 - k_2)^2 \in 2\sigma - k^2 - 4E_q.$$

Therefore, $k_1 - k_2$ is constrained to intervals of length $\lesssim 2^{q/2}$ and there are at most $2\tau K^2$ such intervals. As a consequence,

$$M_{p,q} \lesssim \tau K^2 2^{q/2} 2^p = \tau K^2 2^{(p-q)/4} 2^{3p/4} 2^{3q/4}.$$

Taking the square root, we thus deduce (8.2) from (8.3). This concludes the proof of (3.22).

9. Numerical experiment

In this section we illustrate our main result (Theorem 1.1) on the L^2 error estimate by a numerical experiment. For this purpose, we solve the periodic Schrödinger equation (1.1) with initial value

$$u(0) = f_{H^1} + \frac{2\sin x}{2 - \cos x}$$

on the torus. Here f_{H^1} is a randomized H^1 function normalized in L^2 (see [11] for details on the construction of f_{H^1}). We compare our new integrator (1.11) with the previously introduced single-filtered Fourier based method [15, 16] and two standard integration schemes for periodic Schrödinger equations: a Lie splitting and exponential integrator method (see, e.g., [4, 13]). For the latter, we employ a standard Fourier pseudospectral method for the discretization in space and we choose as largest Fourier mode $K = 2^{10}$ (i.e., the spatial mesh size $\Delta x = 0.0061$). On the other hand, for our twice-filtered Fourier based integrator, we have to use the relation $K = \tau^{-\alpha/2}$ with α given in (7.13), as we are in case (iii) of Theorem 1.1 (recall that $s_0 = 1$). This results in $K \approx \tau^{-5/6}$.

We observe from the experiment that the new twice-filtered Fourier integrator is convergent of order 1 for rough solutions in H^1, whereas the standard discretization techniques as well as our previously introduced single-filtered Fourier based method all suffer from order reduction; see Figure 1. In particular, the numerically obtained order for the exponential integrator is reduced down to 0.3, whereas the results for the standard Lie splitting scheme are highly irregular. Both integrators are thus unreliable and inefficient for such low regularity initial data. The single-filtered Fourier based integrator shows a more regular error behaviour for the example considered, but its order is reduced to 3/4. The only method that is able to appropriately integrate the low regularity problem under consideration is the twice-filtered Fourier based scheme (1.11) proposed in this paper. For smooth solutions the new twice-filtered Fourier integrator performs similarly to the Lie

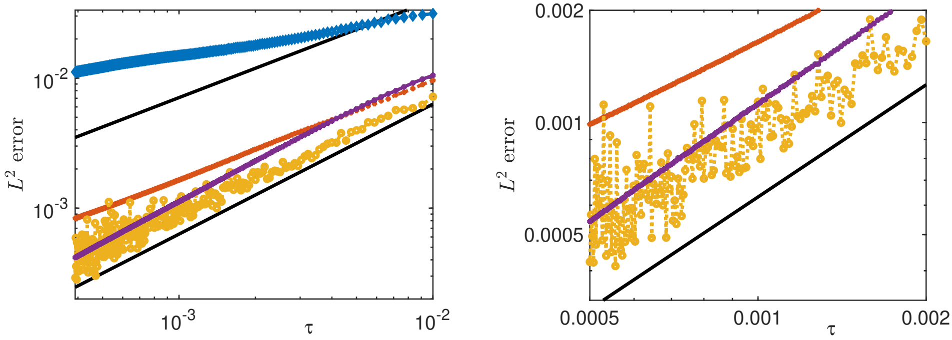

Fig. 1. L^2 error of the new twice-filtered Fourier based scheme (1.11) (purple), the Lie splitting scheme (yellow), the exponential integrator (blue), and the original single-filtered Fourier based scheme (red) proposed in [15,16]. Left: the slope of the (black) reference lines is 1 and $3/4$, respectively. Right: zoom into the region of the left lower corner; the slope of the (black) reference line is 1.

splitting and the exponential integrator. More precisely, for initial values at least in H^2 all three schemes converge with first-order accuracy τ; see, e.g., [5] for the analysis of the Lie splitting method for H^2 data.

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