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Breuil–Kisin modules and integral p-adic Hodge theory

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Abstract. We construct a category of Breuil–Kisin G_K-modules to classify integral semi-stable Galois representations. Our theory uses Breuil–Kisin modules and Breuil–Kisin–Fargues modules with Galois actions, and can be regarded as the algebraic avatar of the integral p-adic cohomology theories of Bhatt–Morrow–Scholze and Bhatt–Scholze. As a key ingredient, we classify Galois representations that are of finite $E(u)$-height.

Keywords. Breuil–Kisin modules, integral p-adic Hodge theory, (φ, τ)-modules

1. Introduction

1.1. Overview and main theorems

In this paper, we construct a certain "algebraic avatar" of some *integral* p-adic cohomology theories. Let us first fix some notations.

Notation 1.1.1. Let p be a prime. Let k be a perfect field of characteristic p, let $W(k)$ be the ring of Witt vectors, and let $K_0 := W(k)[1/p]$. Let K be a totally ramified finite extension of K_0, let $\mathcal{O}_K$ be the ring of integers, and let $e := [K : K_0]$. Fix an algebraic closure $\overline{K}$ of K and set $G_K := \mathrm{Gal}(\overline{K}/K)$. Let C_p be the p-adic completion of $\overline{K}$, and let $\mathcal{O}_{C_p}$ be the ring of integers. Let v_p be the valuation on C_p such that $v_p(p) = 1$.

The study of p-adic Hodge theory is roughly divided into two closely related directions: in the geometric direction, one studies p-adic cohomology theories and their comparisons; while in the algebraic direction, one studies (semi)linear algebra categories and

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uses them to classify p-adic Galois representations. Let us first recall some foundational theorems.

Theorem 1.1.2 ([19,29,40]). *Let $\mathfrak{X}$ be a proper formal scheme over $\mathcal{O}_K$ with semi-stable reduction. Let $R\Gamma_{\text{log-crys}}$ (resp. $R\Gamma_{\mathrm{dR}}$, $R\Gamma_{\text{ét}}$) denote the log-crystalline (resp. de Rham, étale) cohomology theory. Let $\mathbf{B}_{\mathrm{st}}$ be Fontaine's semi-stable period ring. We have*

$$R\Gamma_{\text{log-crys}}(\mathfrak{X}_k/W(k))[1/p] \otimes_{K_0} K \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)[1/p]; \tag{1.1.1}$$

$$R\Gamma_{\text{log-crys}}(\mathfrak{X}_k/W(k))[1/p] \otimes_{K_0} \mathbf{B}_{\mathrm{st}} \simeq R\Gamma_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)[1/p] \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}}. \tag{1.1.2}$$

Here, when $\mathfrak{X}$ is a scheme, (1.1.1) is proved by Hyodo–Kato, and (1.1.2) (the C_{st}-conjecture of Fontaine–Jannsen) is proved by Tsuji; the general case when $\mathfrak{X}$ is a formal scheme is proved by Colmez–Nizioł. These comparisons imply that the log-crystalline cohomology groups equipped with their natural (φ, N)- and filtration structures give rise to *admissible* filtered (φ, N)-modules (recall that admissible means "coming from Galois representations"). The following theorem of Colmez–Fontaine, which says that "weakly admissible implies admissible", can be regarded as the algebraic avatar of Thm. 1.1.2.

Theorem 1.1.3 ([18]). *The category of* weakly admissible *filtered (φ, N)-modules is equivalent to the category of semi-stable representations of G_K.*

Note that in Thm. 1.1.2, the integral and torsion information in all the cohomology theories are lost in the comparison theorems. Recently, Bhatt–Morrow–Scholze [5,6] and Bhatt–Scholze [7] defined some *integral* p-adic cohomology theories (in the good reduction case) which specialize to all these classical p-adic cohomology theories (and keep all the integral and torsion information). Let us first introduce some notations and definitions.

Notation 1.1.4. Let $\pi \in K$ be a *fixed* uniformizer, and let $E(u) \in W(k)[u]$ be the minimal polynomial of π over K_0. *Fix* a sequence of elements $\pi_n \in \overline{K}$ inductively such that $\pi_0 = \pi$ and $(\pi_{n+1})^p = \pi_n$. Let $K_\infty := \bigcup_{n=1}^{\infty} K(\pi_n)$ and let $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. Let $\mathfrak{S} := W(k)[\![u]\!]$.

Let $R := \mathcal{O}_{C_p}^\flat$ be the tilt of the perfectoid ring $\mathcal{O}_{C_p}$, let $\mathfrak{m}_R$ be its maximal ideal, and let $W(R)$ (also denoted as $\mathbf{A}_{\mathrm{inf}}$) be the ring of Witt vectors. Let $\theta : W(R) \to \mathcal{O}_{C_p}$ be the usual map; let ξ be a generator of the principal ideal $\mathrm{Ker}\,\theta$. Let $\mathrm{Fr}\, R := C_p^\flat$ be the fractional field of R, and let $W(\mathrm{Fr}\, R)$ be the ring of Witt vectors.

Note that $\underline{\pi} := (\pi_n)_{n \geq 0}$ defines an element in R; let $[\underline{\pi}] \in W(R)$ be the Teichmüller lift of $\underline{\pi}$. Then we can define a $W(k)$-linear embedding $\mathfrak{S} \hookrightarrow W(R)$ via $u \mapsto [\underline{\pi}]$; hence $E(u)$ maps to a generator of $\mathrm{Ker}\,\theta$. The Frobenius $\varphi : R \to R$, $x \mapsto x^p$, induces Frobenii (always denoted as φ) on $W(R)$ and $\mathfrak{S}$.

Fix $\mu_n \in \overline{K}$ inductively such that $\mu_0 = 1$, μ_1 is a primitive p-th root of unity and $(\mu_{n+1})^p = \mu_n$. Let $K_{p^\infty} := \bigcup_{n=1}^{\infty} K(\mu_n)$. Let $\underline{\varepsilon} = (1, \mu_1, \mu_2, \ldots) \in R$, and let $[\underline{\varepsilon}] \in W(R)$ be the Teichmüller lift.

Definition 1.1.5. (1) A *Breuil–Kisin module* is a finitely generated $\mathfrak{S}$-module $\mathfrak{M}$ equipped with an $\mathfrak{S}[1/E(u)]$-linear isomorphism

$$\mathfrak{M} \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}[1/E(u)] \to \mathfrak{M}[1/E(u)];$$

it is said to be *of (non-negative) finite* $E(u)$*-height* if under the isomorphism, the image of $\mathfrak{M} \otimes_{\varphi,\mathfrak{S}} \mathfrak{S}$ is contained in $\mathfrak{M}$.

(2) Let $\widehat{\mathfrak{M}}$ be a finitely presented $W(R)$-module such that $\widehat{\mathfrak{M}}[1/p]$ is finite free over $W(R)[1/p]$.

(a) It is called a (non-φ-twisted) *Breuil–Kisin–Fargues module* if it is equipped with a $W(R)[1/\xi]$-linear isomorphism

$$\widehat{\mathfrak{M}} \otimes_{\varphi,W(R)} W(R)[1/\xi] \to \widehat{\mathfrak{M}}[1/\xi].$$

(b) It is called a *φ-twisted Breuil–Kisin–Fargues module* if it is equipped with a $W(R)[1/\varphi(\xi)]$-linear isomorphism

$$\widehat{\mathfrak{M}} \otimes_{\varphi,W(R)} W(R)[1/\varphi(\xi)] \to \widehat{\mathfrak{M}}[1/\varphi(\xi)].$$

Remark 1.1.6. (1) Given a (non-φ-twisted) Breuil–Kisin–Fargues module $\widehat{\mathfrak{M}}$, the tensor product $\widehat{\mathfrak{M}} \otimes_{\varphi,W(R)} W(R)$ is a φ-twisted Breuil–Kisin–Fargues module. As φ is an automorphism on $W(R)$, this induces an equivalence between the relevant categories.

(2) It seems to us that both versions of "Breuil–Kisin–Fargues modules" deserve their merits. Indeed, the φ-twisted version (which is precisely [5, Def. 4.22]) is perhaps the (geometrically) natural version as it naturally appears in cohomology and comparison theorems (see Thm. 1.1.7 below). However, the non-φ-twisted version (which, e.g., is also used in [21, Def. 4.2.1]) has the technical convenience that it is more "parallel" to the Breuil–Kisin modules: for example, one can choose $\xi = E(u)$. In addition, in our *algebraic* study of Breuil–Kisin modules, we also need to embed $\mathfrak{S}$ into various other rings (without φ-twisting), hence the process is more uniform if we use the non-φ-twisted version throughout: this is indeed what we do in this paper (see Def. 1.1.8 and Rem. 1.1.10).

Theorem 1.1.7 ([5–7]). *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$. There exist cohomology theories $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ and $R\Gamma_{\mathbf{A}_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}})$ which are equipped with morphisms φ, such that the cohomology groups are Breuil–Kisin modules and* φ-twisted *Breuil–Kisin–Fargues modules respectively, and such that we have the following comparisons:*

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\varphi,\mathfrak{S}} \mathbf{A}_{\mathrm{inf}} \simeq R\Gamma_{\mathbf{A}_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}}); \tag{1.1.3}$$

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k)); \tag{1.1.4}$$

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}} \mathcal{O}_K \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K); \tag{1.1.5}$$

$$R\Gamma_{\mathbf{A}_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}}) \otimes_{\mathbf{A}_{\mathrm{inf}}} \mathbf{A}_{\mathrm{inf}}[1/\mu] \simeq R\Gamma_{\mathrm{\acute{e}t}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbf{A}_{\mathrm{inf}}[1/\mu]. \tag{1.1.6}$$

Here, the (derived) tensor product in (1.1.3) is via the flat (see [5, Lem. 4.30]) morphism $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S} \hookrightarrow \mathbf{A}_{\mathrm{inf}}$ where the second map is the $W(k)$-linear embedding sending u to $[\underline{\pi}]$; the derived tensor product in (1.1.4) is via $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S} \to W(k)$ where the second map is the $W(k)$-linear map sending u to 0*; the derived tensor product in (1.1.5) is via $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S} \to \mathcal{O}_K$ where the second map is the $W(k)$-linear map sending u to π; and the element μ in (1.1.6) is* $[\underline{\varepsilon}] - 1$.

All the above results were first proved by Bhatt–Morrow–Scholze [5, 6]. Recently, Bhatt–Scholze [7] developed the prismatic cohomologies and reproved all these results; in particular, (1.1.3) can now be regarded as a prismatic base-change theorem. Let us mention that it is natural to expect the existence of a "log"-version of the prismatic site, and hence the log-prismatic cohomologies (e.g., the semi-stable version of $R\Gamma_{\mathbf{A}_{\mathrm{inf}}}$ has already been constructed by Česnavičius–Koshikawa [13]).

The main goal of this paper is to construct the algebraic avatar of these integral p-adic cohomologies (*modulo p-power torsion*) and the comparisons amongst them, which is the following:

Definition 1.1.8. Define $\mathrm{Mod}_{\mathfrak{S},W(R)}^{\varphi,G_K,[-\infty,+\infty]}$ to be the category consisting of triples $(\mathfrak{M},\varphi_{\mathfrak{M}},G_K)$, which we call *Breuil–Kisin G_K-modules*, where

(1) $(\mathfrak{M},\varphi_{\mathfrak{M}})$ is a *finite free* Breuil–Kisin module;

(2) G_K is a continuous $\varphi_{\widehat{\mathfrak{M}}}$-commuting $W(R)$-semilinear G_K-action on the (non-φ-twisted) Breuil–Kisin–Fargues module $\widehat{\mathfrak{M}} := W(R)\otimes_{\mathfrak{S}}\mathfrak{M}$ such that

(a) $\mathfrak{M}\subset\widehat{\mathfrak{M}}^{G_\infty}$ via the embedding $\mathfrak{M}\hookrightarrow\widehat{\mathfrak{M}}$;

(b) $\mathfrak{M}/u\mathfrak{M}\subset(\widehat{\mathfrak{M}}/W(\mathfrak{m}_R)\widehat{\mathfrak{M}})^{G_K}$ via the embedding $\mathfrak{M}/u\mathfrak{M}\hookrightarrow\widehat{\mathfrak{M}}/W(\mathfrak{m}_R)\widehat{\mathfrak{M}}$.

Remark 1.1.9. After choosing a basis of $\mathfrak{M}$, the data in the definition above can be expressed using two matrices (always so if $p>2$ and can be made so if $p=2$), one for φ and one for τ (see Notation 1.4.3 for τ and discussions about $p=2$ case).

Remark 1.1.10. (1) The module $\widehat{\mathfrak{M}}$ together with its G_K-action is called a "Breuil–Kisin–Fargues G_K-module" in [21, Def. 4.2.3]. Here, we call it a "Breuil–Kisin G_K-module" to emphasize the role played by Breuil–Kisin modules in our theory.

(2) If we replace "$\widehat{\mathfrak{M}} := W(R)\otimes_{\mathfrak{S}}\mathfrak{M}$" in Def. 1.1.8 by "$\widehat{\mathfrak{M}} := W(R)\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$", then by Rem. 1.1.6, we get an equivalent "φ-twisted category".

(3) (I thank Peter Scholze for useful discussions on this remark.) One observes that the cohomology theories $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ and $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}})$ in Thm. 1.1.7 "satisfy" conditions in the *φ-twisted version* of Def. 1.1.8 (although the cohomology groups are not necessarily finite free). Namely, there is a φ-commuting G_K-action on $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}})$; via the isomorphism (1.1.3), the image of $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ in $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}})$ is fixed by G_∞, and the image of $R\Gamma_{\mathfrak{S}}(\mathfrak{X})\otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}}W(k)$ is furthermore fixed by G_K. All these follow from the constructions in [5, 6], but they look most natural using the *base change theorem* for prismatic cohomologies in [7]. Indeed, we have (cf. *loc. cit.* for notations concerning prisms and prismatic cohomology),

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}_{\mathcal{O}_{C_p}})\simeq\varphi^*_{\mathbf{A}_{\mathrm{inf}}}R\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{C_p}}/\mathbf{A}_{\mathrm{inf}}),\quad\text{by [7, §17];}\tag{1.1.7}$$

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X})\simeq R\Gamma_{\Delta}(X/\mathfrak{S}),\quad\text{by [7, §15.2].}\tag{1.1.8}$$

(In particular, $R\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{C_p}}/\mathbf{A}_{\mathrm{inf}})$ in [7] gives *non-φ-twisted* Breuil–Kisin–Fargues modules.) Now, the φ-commuting action of G_K on $R\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{C_p}}/\mathbf{A}_{\mathrm{inf}})$ is induced by the G_K-action on the prism $(\mathbf{A}_{\mathrm{inf}},(\xi))\in(\mathcal{O}_{C_p})_{\Delta}$. The morphism $\varphi:\mathfrak{S}\to\mathbf{A}_{\mathrm{inf}}$ induces a morphism of prisms in $(\mathcal{O}_K)_{\Delta}$:

$$(\mathfrak{S}, (E(u))) \to (\mathbf{A}_{\inf}, (\varphi(\xi))); \tag{1.1.9}$$

this morphism, by the base change theorem for prismatic cohomologies, induces an isomorphism

$$R\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\varphi,\mathfrak{S}} \mathbf{A}_{\inf} \simeq \varphi^*_{\mathbf{A}_{\inf}} R\Gamma_{\Delta}(\mathfrak{X}_{\mathcal{O}_{C_p}}/\mathbf{A}_{\inf}), \tag{1.1.10}$$

which then induces a G_K-action on the left hand side. Given $g \in G_\infty$, the composite $\mathfrak{S} \xrightarrow{\varphi} \mathbf{A}_{\inf} \xrightarrow{g} \mathbf{A}_{\inf}$ induces exactly the same morphism in (1.1.9) (since $\mathfrak{S} \subset (\mathbf{A}_{\inf})^{G_\infty}$) and hence exactly the same isomorphism in (1.1.10); this implies that $R\Gamma_{\Delta}(X/\mathfrak{S})$ is fixed by G_∞. Finally, by reduction modulo $W(\mathfrak{m}_R)$, (1.1.9) induces a morphism of prisms in $(\mathcal{O}_K)_{\Delta}$:

$$(W(k), (p)) \to (W(\bar{k}), (p)). \tag{1.1.11}$$

Since $W(k) \subset (W(\bar{k}))^{G_K}$, we deduce again by the prismatic base change theorem that $R\Gamma_{\Delta}(X/\mathfrak{S}) \otimes^{\mathbb{L}}_{\varphi,\mathfrak{S}} W(k)$ is fixed by G_K.

The following is our main theorem, which, in view of Rem. 1.1.10, can be regarded as the algebraic avatar of Thm. 1.1.7.

Theorem 1.1.11 (see Thm. 7.1.7[1]). *The category of Breuil–Kisin G_K-modules is equivalent to the category of G_K-stable $\mathbb{Z}_p$-lattices in semi-stable representations of G_K.*

Remark 1.1.12. (1) We have a crystallinity criterion to tell when a Breuil–Kisin G_K-module comes from a crystalline representation (see Prop. 7.1.10).

(2) Loosely speaking, Thm. 1.1.11 "implies" that the information in (1.1.3) and (1.1.4) is already *enough* to "recover" $R\Gamma_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)$ (modulo torsion). Given a Breuil–Kisin G_K-module as in Def. 1.1.8, let $\varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and let $\varphi^*\widehat{\mathfrak{M}} := W(R) \otimes_{\varphi,W(R)} \widehat{\mathfrak{M}}$. Then we can show (see Prop. 7.1.11)

$$\varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M} \subset (\varphi^*\widehat{\mathfrak{M}}/E(u)\varphi^*\widehat{\mathfrak{M}})^{G_K};$$

this can be regarded as the algebraic avatar of the de Rham comparison (1.1.5).

We now give some historical remarks about (algebraic) integral p-adic Hodge theory, and compare some of the theories.

Remark 1.1.13. (1) In algebraic integral p-adic Hodge theory, we use various (semi)linear objects to classify $\mathbb{Z}_p$-lattices in semi-stable Galois representations. For example, we have Fontaine–Laffaille theory [23], the theory of Wach modules [41, 42] (refined by [2, 16]), and Breuil's theory of strongly divisible S-lattices (conjectured in [11], fully proved in [24, 36] using input from [32]). However, these theories are valid only with certain restrictions on ramification of the base field or on Hodge–Tate weights,

[1] In Thm. 7.1.7, we will stick to representations of non-negative Hodge–Tate weights, as well as Breuil–Kisin modules of non-negative $E(u)$-heights; this makes the writing easier. The general case (as stated in Thm. 1.1.11) can be easily deduced by twisting.

or are valid only for certain crystalline representations. Liu's theory of $(\varphi, \widehat{G})$-modules [37] (with input from Kisin [32]) is so far the only theory that works for *all* integral semi-stable representations. However, unlike our Breuil–Kisin G_K-modules which can be regarded as the algebraic avatar of some cohomology theories, the ring $\widehat{\mathcal{R}}$ (see Appendix B) in Liu's theory is too *implicit*, and it seems hopeless to construct some cohomology theory over it.

(2) Indeed, the author and Tong Liu recently realized that we actually *do not know* if the ring $\widehat{\mathcal{R}}$ is p-adically complete or not: this means that there is a gap in our earlier work on limit of torsion semi-stable Galois representations in [25, 35] (these results are not used in the current paper). Fortunately, this gap can now be (easily) fixed by using the Breuil–Kisin G_K-modules (see Appendix B). Note that the gap arises in the *application* of the theory of $(\varphi, \widehat{G})$-modules (namely, the theory is inadequate for this application), but the theory *per se* remains valid; see also the next item.

(3) In [26], we will show that the theory of Breuil–Kisin G_K-modules *specializes to* (and hence recovers) the theory of $(\varphi, \widehat{G})$-modules (see §7.2 for more remarks). Let us mention here that the proof of our main theorem is independent of the theory of $(\varphi, \widehat{G})$-modules (except some relatively easy results, e.g. Prop. 7.1.4); in particular, we will not use $\widehat{\mathcal{R}}$ anywhere.

We make some speculations about the theory and its possible applications.

Remark 1.1.14. (1) Like all the integral theories listed in Rem. 1.1.13 (1), we would like to use our Breuil–Kisin G_K-modules to study reduction of semi-stable Galois representations as well as the relevant semi-stable Galois deformation rings (these results always play important roles in automorphy lifting theorems). In fact, our result can already at least simplify some of the constructions of the semi-stable substack in Emerton–Gee's stack of (φ, Γ)-modules (see §7.3). In particular, we would like to use our theory to investigate the explicit structures of some semi-stable Galois deformation rings.

(2) In some sense, one can also regard Thm. 1.1.11 as some sort of integral version of the Colmez–Fontaine theorem, particularly because all the modules in both theories are avatars of cohomology theories. It is interesting to speculate if our theory can play some *integral* role in places where the Colmez–Fontaine theorem is used (e.g., in the study of p-adic period domains and period morphisms). We also wonder if there is any connection between our theory and the Fargues–Fontaine curve.

(3) Recently, Bhatt and Scholze [8] established an equivalence between the category of prismatic F-crystals on $(\mathcal{O}_K)_{\Delta}$ and the category of lattices in crystalline representations of G_K. (The semi-stable version is then proved by Du–Liu [20] using the log-prismatic site of Koshikawa [34].) It seems that the *direct* link between their theory and ours is still unclear: in particular, can we directly construct a prismatic F-crystal from a Breuil–Kisin G_K-module, and vice versa? It seems likely explorations about theses questions could shed new light on prismatic crystals.

Let us now sketch the main ideas in the proof of Thm. 1.1.11. Indeed, a key ingredient is the classification of Galois representations which are of finite $E(u)$-height.

Definition 1.1.15. Let T be a finite free $\mathbb{Z}_p$-representation of G_K. It is called *of finite $E(u)$-height* if there exists a finite free Breuil–Kisin module $\mathfrak{M}$ of non-negative finite $E(u)$-height such that there is a G_∞-equivariant isomorphism

$$T|_{G_\infty} \simeq (\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathrm{Fr}\, R))^{\varphi=1}, \tag{1.1.12}$$

where the G_∞-action on the right hand side of (1.1.12) comes from that on $W(\mathrm{Fr}\, R)$.

The following theorem in particular answers positively [37, Question 4.3.1 (2)] by Tong Liu.

Theorem 1.1.16 (= Thm. 6.2.4). *Let $K^{\mathrm{ur}} \subset \overline{K}$ be the maximal unramified extension of K, let*

$$m := 1 + \max\{i \geq 1 : \mu_i \in K^{\mathrm{ur}}\},$$

and let $K_m = K(\pi_{m-1})$. Let T be a finite free $\mathbb{Z}_p$-representation of G_K, and let $V = T[1/p]$. Then T is of finite $E(u)$-height if and only if $V|_{G_{K_m}}$ can be extended to a semi-stable G_K-representation with non-negative Hodge–Tate weights. In particular, if T is of finite $E(u)$-height, then V is potentially semi-stable.

Theorem 1.1.16 is the most difficult part of our paper, and indeed takes the majority of space. In fact, once it is proved, it is then relatively straightforward to prove the main theorem Thm. 1.1.11 by using Thm. 1.1.16 and "comparisons" among various modules.

Thus, we will dedicate the entire §1.2 below to explain the proof of Thm. 1.1.16; before we do so, we list some remarks about this theorem.

Remark 1.1.17. The notion "of finite $E(u)$-height" indeed depends on the choices $\{\pi_n\}_{n\geq 0}$ (as does the embedding $\mathfrak{S} \hookrightarrow W(R)$). If T is of finite height with respect to *all* such choices, then we can use Thm. 1.1.16 to show that V is semi-stable; this intriguing result is due to Gee (see Thm. 7.3.1). Indeed, this result, as we learnt from Gee, is inspired by considerations in the construction of the stack of (semi-stable) (φ, Γ)-modules in [21]; see §7.3 for some more comments.

Remark 1.1.18. In [41], Wach studied some "finite height" (φ, Γ)-modules; it is shown in [41, A.5] that they give rise to de Rham (indeed, potentially crystalline) Galois representations if there is some additional condition on the Lie algebra operator associated to the Γ-action. Note however that the "finite height" condition in *loc. cit.* is of a different type than the one here. Indeed, the analogue of our finite $E(u)$-height condition in the setting of (φ, Γ)-modules should be the "finite q-height" condition, where $q := \frac{(1+T)^p-1}{T}$ is a polynomial (see, e.g., [2, 33]). In fact, we can use similar ideas in the current paper to study finite q-height (φ, Γ)-modules; in parallel, we can also study the finite height (φ, τ)-modules (without $E(u)$ in the play) similarly to [41]. All these will be discussed elsewhere.

Remark 1.1.19. Caruso gave a proof of Thm. 1.1.16 in [12, Thm. 3] (for $p > 2$), which unfortunately contains a rather serious gap. (Indeed, even the statement in *loc. cit.* contains an error, see Rem. 6.2.5.) The gap was first discovered by Yoshiyasu Ozeki, and is discussed in Appendix A. The gap arises when Caruso tries to define a *monodromy operator* on the (φ, τ)-*modules* (cf. below) associated to finite $E(u)$-height Galois representations, using the "truncated log" [12, §3.2.2]; very roughly speaking, Caruso tries to define a log-operator via p-*adic approximation* technique (using p-adic topology on various rings). As will be explained in the next subsection, our approach is *completely different* and has no p-adic approximation technique; in particular, our method uses *overconvergent* (φ, τ)-modules which have been made available only very recently. Indeed, we do not regard our proof as a "fix" to Caruso's proof; see also Rem. 1.2.6 and Rem. 1.2.7.

1.2. Strategy of proof of Thm. 1.1.16

The main tool to prove Thm. 1.1.16 is the theory of *overconvergent* (φ, τ)-*modules*. Let us first give some general remarks about the theory of (φ, τ)-modules. Recall that we already defined $K_\infty = \bigcup_{n=1}^\infty K(\pi_n)$ and $K_{p^\infty} = \bigcup_{n=1}^\infty K(\mu_n)$. Let $L := K_\infty K_{p^\infty}$.

As already mentioned in the last subsection, in the algebraic study of p-adic Hodge theory, we use various linear algebra tools to study p-adic representations of G_K. A key idea in p-adic Hodge theory is to first restrict the Galois representations to some subgroup of G_K. The (φ, τ)-modules used in this paper are constructed by using the subgroup $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$; they are analogues of the more classical (φ, Γ)-modules which are constructed using the subgroup $G_{p^\infty} := \mathrm{Gal}(\overline{K}/K_{p^\infty})$. Here let us only quickly mention that the Γ is the group $\mathrm{Gal}(K_{p^\infty}/K)$, and the τ is a topological generator of the group $\mathrm{Gal}(L/K_{p^\infty})$ (see Notation 1.4.3).

Similar to the (φ, Γ)-modules, the (φ, τ)-modules also classify *all* p-adic representations of G_K. Although these two theories are equivalent, they each have their own technical advantage, and both are indispensable. The (φ, Γ)-modules are perhaps "easier" in the sense that both the φ- and Γ-actions are defined over the same ring; whereas the τ-action in (φ, τ)-modules can only be defined over a much bigger ring. However, the φ-action in (φ, τ)-modules stays tractable even when K has ramification; in contrast, the φ-action in (φ, Γ)-modules becomes quite implicit when K has ramification. This dichotomy becomes much more substantial when we consider *semi-stable* Galois representations: in this situation, there exist very well-behaved Breuil–Kisin modules (also called Kisin modules, or $(\varphi, \widehat{G})$-modules in different contexts) which are a special type of (φ, τ)-modules; in contrast, such special type (φ, Γ)-modules (called Wach modules) exist only if we consider crystalline representations and if $K \subset \bigcup_{n \geq 1} K_0(\mu_n)$ (e.g., when $K = K_0$ is unramified). To save space, we refer the reader to the introduction in [28] for some discussion and comparison of the applications of these two theories in different contexts. Indeed, this paper shows once again that when we consider semi-stable representations, it is fruitful to use the (φ, τ)-modules.

Let us be more precise now. First, recall that the φ-action of a (φ, τ)-module is defined over the field

$$\mathbf{B}_{K_\infty} := \left\{ \sum_{i=-\infty}^{+\infty} a_i u^i : a_i \in K_0, \lim_{i \to -\infty} v_p(a_i) = +\infty, \text{ and } \inf_{i \in \mathbb{Z}} v_p(a_i) > -\infty \right\}. \quad (1.2.1)$$

The τ-action is defined over a bigger field $\widetilde{\mathbf{B}}_L$ which we do not recall here (see §3). Indeed, roughly speaking, a (φ, τ)-module D is a finite free $\mathbf{B}_{K_\infty}$-vector space equipped with certain commuting maps $\varphi : D \to D$ and $\tau : \widetilde{\mathbf{B}}_L \otimes_{\mathbf{B}_{K_\infty}} D \to \widetilde{\mathbf{B}}_L \otimes_{\mathbf{B}_{K_\infty}} D$. By [12, Thm. 1], the (étale) (φ, τ)-modules classify all Galois representations of G_K. One readily observes that a Galois representation is of finite $E(u)$-height as in Def. 1.1.15 if and only if there exists a φ-stable Breuil–Kisin lattice inside the corresponding (φ, τ)-module; note that this property has nothing to do with the τ-action.

To prove Thm. 1.1.16, we need to define a natural monodromy operator on these (φ, τ)-modules. Instead of the p-adic rings mentioned in Rem. 1.1.19, what we propose in the current paper is that one should use certain Fréchet rings (e.g., various Robba-style rings) instead. In fact, we can get a monodromy operator *directly* (no approximation needed) using techniques of *locally analytic vectors*. Furthermore, our monodromy operator will be defined for *all* (rigid-overconvergent, see Thm. 1.2.1 below) (φ, τ)-modules (not just finite $E(u)$-height ones). Before we state the theorem concerning the monodromy operator, let us recall the overconvergence result for (φ, τ)-modules.

Theorem 1.2.1 ([27,28]). *The (φ, τ)-modules (attached to p-adic representations of G_K) are overconvergent. That is (roughly speaking), the φ-action can be defined over the subfield*

$$\mathbf{B}^\dagger_{K_\infty} := \left\{ \sum_{i=-\infty}^{+\infty} a_i u^i \in \mathbf{B}_{K_\infty} : \lim_{i \to -\infty} (v_p(a_i) + i\alpha) = +\infty \text{ for some } \alpha > 0 \right\}; \quad (1.2.2)$$

also, the τ-action can be defined over some subfield $\widetilde{\mathbf{B}}^\dagger_L \subset \widetilde{\mathbf{B}}_L$.

Remark 1.2.2. (1) Thm. 1.2.1 was first conjectured by Caruso [12] (as an analogue of the classical overconvergence theorem for (φ, Γ)-modules by Cherbonnier–Colmez [15]). A first proof (which only works for $K/\mathbb{Q}_p$ a finite extension) was given in a joint work with Liu [27], using a certain "crystalline approximation" technique; later a second proof (which works for all K) was given in a joint work with Poyeton [28], using the idea of locally analytic vectors.

(2) Let us mention that it is the second proof in [28] that will be useful in the current paper. Not only because it works for all K (which is a minor issue), but also more importantly, the idea of locally analytic vectors will be very critically used in the current paper to define the monodromy operator.

Let us introduce the following Robba ring (which contains $\mathbf{B}^\dagger_{K_\infty}$):

$$\mathbf{B}^\dagger_{\mathrm{rig},K_\infty} := \Big\{ f(u) = \sum_{i=-\infty}^{+\infty} a_i u^i : a_i \in K_0,\ f(u) \text{ converges}$$
$$\text{for all } u \in \overline{K} \text{ with } 0 < v_p(u) < \rho(f) \text{ for some } \rho(f) > 0 \Big\}. \quad (1.2.3)$$

Let V be a p-adic Galois representation of G_K, and let $D^\dagger_{K_\infty}(V)$ be the overconvergent (φ, τ)-module associated to V by Thm. 1.2.1. Define

$$D^\dagger_{\mathrm{rig},K_\infty}(V) := \mathbf{B}^\dagger_{\mathrm{rig},K_\infty} \otimes_{\mathbf{B}^\dagger_{K_\infty}} D^\dagger_{K_\infty}(V),$$

which we call the *rigid-overconvergent* (φ, τ)*-module* associated to V; as we will see, it is the natural space which the monodromy operator lives in.

Theorem 1.2.3 (Thm. 4.2.1[2]). *Let* $\nabla_\tau := (\log \tau^{p^n})/p^n$ *for* $n \gg 0$ *be the Lie-algebra operator with respect to the* τ*-action, and define* $N_\nabla := \frac{1}{p\mathfrak{t}} \cdot \nabla_\tau$ *where* $\mathfrak{t}$ *is a certain "normalizing" element (see* §4*). (Note that there might be some modification in certain cases when* $p = 2$*.) Then*

$$N_\nabla(D^\dagger_{\mathrm{rig},K_\infty}(V)) \subset D^\dagger_{\mathrm{rig},K_\infty}(V),$$

so N_∇ *is a well-defined monodromy operator on* $D^\dagger_{\mathrm{rig},K_\infty}(V)$*.*

Remark 1.2.4. (1) For comparison, if we use $D^\dagger_{\mathrm{rig},K_{p^\infty}}(V)$ (denoted as $D^\dagger_{\mathrm{rig},K}(V)$ in [1]) to denote the rigid-overconvergent (φ, Γ)-module associated to the V (which exists by [15]), then one can *easily* define a monodromy operator

$$\nabla_V : D^\dagger_{\mathrm{rig},K_{p^\infty}}(V) \to D^\dagger_{\mathrm{rig},K_{p^\infty}}(V)$$

as in [1, §5.1]. Here ∇_V (notation of [1]) is precisely the Lie-algebra operator associated to the Γ-action.

(2) The difficulty in defining N_∇ for (φ, τ)-modules is that τ (hence ∇_τ) does not act on $D^\dagger_{\mathrm{rig},K_\infty}(V)$ itself (whereas Γ acts directly on $D^\dagger_{\mathrm{rig},K_{p^\infty}}(V)$); the action is defined only when we base change $D^\dagger_{\mathrm{rig},K_\infty}(V)$ over a much bigger ring $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L}$ (see Def. 2.5.3). Fortunately, after dividing ∇_τ by $p\mathfrak{t}$, and using ideas of locally analytic vectors, one gets back to the level of $D^\dagger_{\mathrm{rig},K_\infty}(V)$.

Now, to prove Thm. 1.1.16, via results of Kisin (and some consideration of locally analytic vectors), it suffices to show the following "monodromy descent" result, which we achieve via a "Frobenius regularization" technique.

Proposition 1.2.5 (= Prop. 6.1.1). *Let* $\mathfrak{M}$ *be the finite height Breuil–Kisin lattice inside a* (φ, τ)*-module corresponding to a* G_K*-representation of finite* $E(u)$*-height. Then*

$$N_\nabla(\mathfrak{M}) \subset \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}.$$

Here $\mathcal{O} \subset \mathbf{B}^\dagger_{\mathrm{rig},K_\infty}$ *is the subring consisting of* $f(u)$ *that converge for all* $u \in \overline{K}$ *such that* $0 < v_p(u) \le +\infty$*.*

Remark 1.2.6. Indeed, the "road map" of our proof of Thm. 1.1.16 is roughly the same as in [12]. Namely, one first defines a certain monodromy operator, then one shows that (in the finite $E(u)$-height case) the operator can be defined over the smaller ring $\mathcal{O}$. However, even as Thm. 1.2.3 provides a correct alternative in defining the monodromy operator, the technical details in the latter half of our argument (Prop. 1.2.5, proved in §6.1) are also

[2] Léo Poyeton informed the author that he also obtained Thm. 1.2.3 independently.

completely different from those of Caruso. Indeed, Caruso's argument uses several newly-defined rings (all with p-adic topology), see [12, upper half of p. 2583, Figure 2]; as far as we know, these rings have not been used elsewhere in the literature. In comparison, all the rings we use in §6.1 have already been studied in [28]; in particular, they are all natural analogues of the rings used in (φ, Γ)-module theory, which have been substantially studied since their introduction in, e.g., [1]. Indeed, it seems that our argument is much easier and more natural.

Remark 1.2.7. As discussed in Rem. 1.2.6, we do not know if we can actually *fix* the gap in Caruso's work. That is, we do not know if we can use p-adic approximation technique to fix [12, Prop. 3.7] (see Appendix A); we do not know either if we can use the p-adic argument as in [12] to prove Thm. 1.1.16.

Remark 1.2.8. As a final remark, let us mention that the current paper is (almost completely) independent of [12]. The only exception is that we do use Caruso's category of étale (φ, τ)-modules and its equivalence with the category of p-adic Galois representations (i.e., the content of [12, Thm. 1]); but these are easy consequences of the theory of the field of norms (with respect to the field K_∞), which was already partially developed e.g. in [10, §2]. We refer to Rem. 7.4.1 for some more comments regarding the relation between the current paper and [12].

1.3. Structure of the paper

In §2, we review many period rings in p-adic Hodge theory; in particular, we compute locally analytic vectors in some rings. In §3, we review the theory of (φ, τ)-modules and the overconvergence theorem. In §4, we define the monodromy operator on rigid-overconvergent (φ, τ)-modules. In §5, we review Kisin's theory of $\mathcal{O}$-modules (for semistable representations) and show that the monodromy operator there *coincides* with ours in §4. In §6, when the (φ, τ)-module is of finite $E(u)$-height, we use a Frobenius regularization technique to descend the monodromy operator to $\mathcal{O}$; this implies that the attached representation is potentially semi-stable. In §7, we construct the Breuil–Kisin G_K-modules and prove our main theorem; we also compare our theory with some results of Gee and Liu.

1.4. Some notations and conventions

Notation 1.4.1. Recall that we have already defined

$$K_\infty = \bigcup_{n=1}^{\infty} K(\pi_n), \quad K_{p^\infty} = \bigcup_{n=1}^{\infty} K(\mu_n), \quad L = \bigcup_{n=1}^{\infty} K(\pi_n, \mu_n).$$

Let

$$G_\infty := \mathrm{Gal}(\overline{K}/K_\infty), \quad G_{p^\infty} := \mathrm{Gal}(\overline{K}/K_{p^\infty}), \quad G_L := \mathrm{Gal}(\overline{K}/L), \quad \hat{G} := \mathrm{Gal}(L/K).$$

When Y is a ring with a G_K-action, $X \subset \overline{K}$ is a subfield, we use Y_X to denote the $\mathrm{Gal}(\overline{K}/X)$-invariants of Y; we will use the cases when $X = L, K_\infty$.

1.4.2. Locally analytic vectors. Let us very quickly recall the theory of locally analytic vectors; see [4, §2.1] and [3, §2] for more details. Indeed, almost all the explicit calculations of locally analytic vectors used in this paper are already carried out in [28], hence the reader can refer there for more details.

Recall that a *$\mathbb{Q}_p$-Banach space W* is a $\mathbb{Q}_p$-vector space with a complete non-Archimedean norm $\|\cdot\|$ such that $\|aw\| = \|a\|_p\|w\|$ for all $a \in \mathbb{Q}_p$, $w \in W$, where $\|a\|_p$ is the p-adic norm on $\mathbb{Q}_p$. Recall the multi-index notations: if $\mathbf{c} = (c_1, \ldots, c_d)$ and $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ (here $\mathbb{N} = \mathbb{Z}^{\geq 0}$), then we let $\mathbf{c}^{\mathbf{k}} = c_1^{k_1} \cdot \ldots \cdot c_d^{k_d}$.

Let G be a p-adic Lie group, and let $(W, \|\cdot\|)$ be a $\mathbb{Q}_p$-Banach representation of G. Let H be an open subgroup of G such that there exist coordinates $c_1, \ldots, c_d : H \to \mathbb{Z}_p$ giving rise to an analytic bijection $\mathbf{c} : H \to \mathbb{Z}_p^d$. We say that an element $w \in W$ is an *H-analytic vector* if there exists a sequence $\{w_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^d}$ with $w_{\mathbf{k}} \to 0$ in W such that

$$g(w) = \sum_{\mathbf{k}\in\mathbb{N}^d} \mathbf{c}(g)^{\mathbf{k}} w_{\mathbf{k}}, \quad \forall g \in H.$$

Let $W^{H\text{-an}}$ denote the space of H-analytic vectors. We say that a vector $w \in W$ is *locally analytic* if there exists an open subgroup H as above such that $w \in W^{H\text{-an}}$. Let $W^{G\text{-la}}$ denote the space of such vectors. We have $W^{G\text{-la}} = \bigcup_H W^{H\text{-an}}$ where H runs through open subgroups of G. We can naturally extend these definitions to the case when W is a Fréchet or LF-representation of G, and we use $W^{G\text{-pa}}$ to denote the *pro-analytic* vectors [3, §2].

Notation 1.4.3. Let $\hat{G} = \mathrm{Gal}(L/K)$ be as in Notation 1.4.1, which is a p-adic Lie group of dimension 2. Below we recall the structure of this group.

(1) Recall that:

- if $K_\infty \cap K_{p^\infty} = K$ (always valid when $p > 2$, see [36, Lem. 5.1.2]), then $\mathrm{Gal}(L/K_{p^\infty})$ and $\mathrm{Gal}(L/K_\infty)$ generate $\hat{G}$;
- if $K_\infty \cap K_{p^\infty} \supsetneq K$, then necessarily $p = 2$, and $K_\infty \cap K_{p^\infty} = K(\pi_1)$ (see [37, Prop. 4.1.5]) and $\pm i \notin K(\pi_1)$, and hence $\mathrm{Gal}(L/K_{p^\infty})$ and $\mathrm{Gal}(L/K_\infty)$ generate an open subgroup of $\hat{G}$ of index 2.

Let us mention that when $K_\infty \cap K_{p^\infty} = K(\pi_1)$, some modifications might be needed in some of our arguments, notably with respect to the τ-operator (see (1.4.1) below), and to the N_∇-operator (see §4). (As a side-note, when $p = 2$, by [43, Lem. 2.1] we can always choose *some* $\{\pi_n\}_{n\geq 0}$ so that $K_\infty \cap K_{p^\infty} = K$.)

(2) Note that:

- $\mathrm{Gal}(L/K_{p^\infty}) \simeq \mathbb{Z}_p$, and let $\tau \in \mathrm{Gal}(L/K_{p^\infty})$ be *the* topological generator such that

$$\begin{cases} \tau(\pi_i) = \pi_i\mu_i, \forall i \geq 1, & \text{if } K_\infty \cap K_{p^\infty} = K; \\ \tau(\pi_i) = \pi_i\mu_{i-1} = \pi_i\mu_i^2, \forall i \geq 2, & \text{if } K_\infty \cap K_{p^\infty} = K(\pi_1). \end{cases} \tag{1.4.1}$$

- $\mathrm{Gal}(L/K_\infty)$ ($\subset \mathrm{Gal}(K_{p^\infty}/K) \subset \mathbb{Z}_p^\times$) is not necessarily pro-cyclic when $p = 2$; however, this issue will *never* bother us in this paper.

Notation 1.4.4. We set up some notations with respect to representations of $\hat{G}$.

(1) Given a $\hat{G}$-representation W, we use

$$W^{\tau=1}, \quad W^{\gamma=1}$$

to mean

$$W^{\mathrm{Gal}(L/K_{p^\infty})=1}, \quad W^{\mathrm{Gal}(L/K_\infty)=1}.$$

And we use

$$W^{\tau\text{-la}}, \quad W^{\tau\text{-an}}, \quad W^{\tau_n\text{-an}} \text{ (for } n \geq 1), \quad W^{\gamma\text{-la}}$$

to mean

$$W^{\mathrm{Gal}(L/K_{p^\infty})\text{-la}}, \quad W^{\mathrm{Gal}(L/K_{p^\infty})\text{-an}}, \quad W^{\langle \tau^{p^n}\rangle\text{-an}}, \quad W^{\mathrm{Gal}(L/K_\infty)\text{-la}},$$

where $\langle \tau^{p^n}\rangle \subset \mathrm{Gal}(L/K_{p^\infty})$ is the subgroup topologically generated by τ^{p^n}.

(2) Let $W^{\tau\text{-la},\gamma=1} := W^{\tau\text{-la}} \cap W^{\gamma=1}$. Then by [28, Lem. 3.2.4],

$$W^{\tau\text{-la},\gamma=1} \subset W^{\hat{G}\text{-la}}.$$

Remark 1.4.5. Note that we never define γ to be an element of $\mathrm{Gal}(L/K_\infty)$; although when $p > 2$ (or in general, when $\mathrm{Gal}(L/K_\infty)$ is pro-cyclic), we could have defined it as a topological generator of $\mathrm{Gal}(L/K_\infty)$. In particular, although "$\gamma = 1$" might be slightly ambiguous (but only when $p = 2$), we use the notation for brevity.

1.4.6. Covariant functors, Hodge–Tate weights, Breuil–Kisin heights, and minus signs

- In this paper we will use many categories of modules and functors relating them; we will always use *covariant* functors. This makes the comparisons amongst them easier (i.e., using tensor products rather than Hom's).
- For example, our $D_{\mathrm{st}}(V)$ is defined as $(V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K}$, and hence the Hodge–Tate weight of the cyclotomic character χ_p is -1.
- Indeed, in the main argument of the paper, we will focus on representations with *non-negative* Hodge–Tate weights and Breuil–Kisin modules with *non-negative* $E(u)$-heights. For example, the Breuil–Kisin module associated to χ_p^{-1} has $E(u)$-height 1.
- We will define several *differential operators*, and we always remove minus signs (for convenience) in our choices: see in particular Rem. 2.7.3 for the N-operator and Rem. 4.1.3 for the N_∇-operator.

1.4.7. Some other notations. Throughout this paper, we reserve φ for the Frobenius operator. We sometimes add subscripts to indicate on which object Frobenius is defined. For example, $\varphi_{\mathfrak{M}}$ is the Frobenius defined on $\mathfrak{M}$. We always drop these subscripts if no confusion arises. Given a homomorphism $\varphi : A \to A$ of rings and an A-module M, denote $\varphi^* M := A \otimes_{\varphi,A} M$. We use $\mathrm{Mat}(A)$ to denote the set of matrices with entries in A (the size of the matrix is always obvious from context). Let $\gamma_i(x) := x^i/i!$ be the usual divided power.

2. Rings and locally analytic vectors

In this section, we review some period rings in p-adic Hodge theory. In particular, we compute the locally analytic vectors in some rings. In §2.1, we review some basic period rings; in §2.2, we discuss variations of these rings with respect to extension of fields. In §2.3 and §2.4, we define the rings $\widetilde{\mathbf{B}}^I$, $\mathbf{B}^I$ and study their G_∞-invariants; in §2.5, we study the relation of these rings via locally analytic vectors. In §2.6, we study a locally analytic element $\mathfrak{t}$, which plays a role in the definition of our monodromy operator. In §2.7, we review some log-rings.

2.1. *Some basic period rings*

Let $\widetilde{\mathbf{E}}^+$ be the tilt of $\mathcal{O}_{C_p}$ (denoted as R in Notation 1.1.4), and let $\widetilde{\mathbf{E}} := \mathrm{Fr}\,\widetilde{\mathbf{E}}^+$ be the tilt of C_p. An element of $\widetilde{\mathbf{E}}$ can be uniquely represented by $(x^{(n)})_{n\geq 0}$ where $x^{(n)} \in C_p$ and $(x^{(n+1)})^p = (x^{(n)})$; let $v_{\widetilde{\mathbf{E}}}$ be the usual valuation where $v_{\widetilde{\mathbf{E}}}(x) := v_p(x^{(0)})$. Let

$$\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+), \quad \widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}}), \quad \widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{A}}^+[1/p], \quad \widetilde{\mathbf{B}} := \widetilde{\mathbf{A}}[1/p],$$

where $W(\cdot)$ means the ring of Witt vectors. There is a natural Frobenius operator $x \mapsto x^p$ on $\widetilde{\mathbf{E}}$, which induces natural Frobenius operators (always denoted by φ) on all the rings defined above (and below); there are also natural G_K-actions on the rings defined above induced from that on $\widetilde{\mathbf{E}}$. Note that the G_K-action on $\widetilde{\mathbf{E}}$ is continuous with respect to the $v_{\widetilde{\mathbf{E}}}$-topology (but not the discrete topology); the action on $\widetilde{\mathbf{B}}$ is continuous with respect to the weak topology (but not the strong p-adic topology).

Let $\underline{\pi} := \{\pi_n\}_{n\geq 0} \in \widetilde{\mathbf{E}}^+$. Let $\mathbf{E}^+_{K_\infty} := k[\![\underline{\pi}]\!]$, $\mathbf{E}_{K_\infty} := k((\underline{\pi}))$, and let $\mathbf{E}$ be the separable closure of $\mathbf{E}_{K_\infty}$ in $\widetilde{\mathbf{E}}$.

Let $[\underline{\pi}] \in \widetilde{\mathbf{A}}^+$ be the Teichmüller lift of $\underline{\pi}$. Let $\mathbf{A}^+_{K_\infty} := W(k)[\![u]\!]$ with Frobenius φ extending the arithmetic Frobenius on $W(k)$ and $\varphi(u) = u^p$. There is a $W(k)$-linear Frobenius-equivariant embedding $\mathbf{A}^+_{K_\infty} \hookrightarrow \widetilde{\mathbf{A}}^+$ via $u \mapsto [\underline{\pi}]$. Let $\mathbf{A}_{K_\infty}$ be the p-adic completion of $\mathbf{A}^+_{K_\infty}[1/u]$. Our fixed embedding $\mathbf{A}^+_{K_\infty} \hookrightarrow \widetilde{\mathbf{A}}^+$ determined by $\underline{\pi}$ uniquely extends to a φ-equivariant embedding $\mathbf{A}_{K_\infty} \hookrightarrow \widetilde{\mathbf{A}}$, and we identify $\mathbf{A}_{K_\infty}$ with its image in $\widetilde{\mathbf{A}}$. We note that $\mathbf{A}_{K_\infty}$ is a complete discrete valuation ring with uniformizer p and residue field $\mathbf{E}_{K_\infty}$.

Let $\mathbf{B}_{K_\infty} := \mathbf{A}_{K_\infty}[1/p]$ (which is precisely the field in (1.2.1)). Let $\mathbf{B}$ be the completion for the p-adic norm of the maximal unramified extension of $\mathbf{B}_{K_\infty}$ inside $\widetilde{\mathbf{B}}$, and let $\mathbf{A} \subset \mathbf{B}$ be the ring of integers. Let $\mathbf{A}^+ := \widetilde{\mathbf{A}}^+ \cap \mathbf{A}$. Then

$$(\mathbf{A})^{G_\infty} = \mathbf{A}_{K_\infty}, \quad (\mathbf{B})^{G_\infty} = \mathbf{B}_{K_\infty}, \quad (\mathbf{A}^+)^{G_\infty} = \mathbf{A}^+_{K_\infty}.$$

2.2. *Rings with respect to field extensions*

(The discussions in this subsection will not be used until §3.3.) Note that the rings $\widetilde{\mathbf{E}}^+, \widetilde{\mathbf{A}}^+, \widetilde{\mathbf{A}}$ (and the $\mathbf{B}$-variants) depend only on C_p, in the sense that if we let E be another complete discrete valuation field with perfect residue field where $K \subset E \subset C_p$, then we get exactly the same $\widetilde{\mathbf{E}}^+, \widetilde{\mathbf{A}}^+, \widetilde{\mathbf{A}}$ as if we started with K. However, the rings $\mathbf{E}, \mathbf{A}, \mathbf{B}$, although without subscripts, indeed *depend* on K and K_∞. For example, let $K \subset E \subset C_p$ be as aforementioned, and choose some E_∞ (analogue of K_∞). Then in general we cannot compare the newly constructed $\mathbf{E}, \mathbf{A}, \mathbf{B}$ with the ones constructed using K and K_∞ (as we cannot even compare K_∞ and E_∞; in general we do not even have $K_\infty \subset E_\infty$).

This is very different from the (φ, Γ)-module setting, where once we fix μ_n as in Notation 1.4.1, we can always make some comparison since we always have $K_{p^\infty} \subset E_{p^\infty}$ (and then we can apply the theory of field of norms). This is indeed used e.g. in [15, §II.4].

Fortunately, for our purpose in this paper, we only need to work with certain special case of $K \subset E \subset C_p$, where we can easily make some comparisons.

Notation 2.2.1. Let K'/K be a (not necessarily finite) unramified extension contained in $\overline{K}$, and let $m \geq 0$. Let E be the p-adic completion of $K'(\pi_m)$, and let $E_\infty := \bigcup_{n \geq m} E(\pi_n)$. Let $\underline{\pi_E} := \{\pi_n\}_{n \geq m} \in \widetilde{\mathbf{E}}^+$, and let $u_E \in \widetilde{\mathbf{A}}^+$ be its Teichmüller lift. Then we can analogously construct $\mathbf{E}_{E_\infty}, \mathbf{A}^+_{E_\infty}, \mathbf{A}_{E_\infty}$, and $\mathbf{E}(E), \mathbf{A}^+(E), \mathbf{A}(E)$, as well as the $\mathbf{B}$-variants of these rings. (Here we write $\mathbf{E}^+(E)$ etc. instead of $\mathbf{E}(E, E_\infty)$ etc. for brevity). Then indeed,

$$\mathbf{E} \hookrightarrow \mathbf{E}(E);$$

and the theory of Cohen rings then induces a map $\mathbf{A} \hookrightarrow \mathbf{A}(E)$. Furthermore, we have a natural embedding

$$\mathbf{A}_{K_\infty} \hookrightarrow \mathbf{A}_{E_\infty}$$

using the embedding $W(k) \hookrightarrow W(k')$ (where k' is the residue field of K') and using $u \mapsto u_E^{p^m}$.

2.3. *The rings $\widetilde{\mathbf{B}}^I$ and their G_∞-invariants*

Recall that we defined the element $\underline{\varepsilon} = (1, \mu_1, \mu_2, \ldots) \in \widetilde{\mathbf{E}}^+$ in Notation 1.1.4. Let $\overline{\pi} = \underline{\varepsilon} - 1 \in \widetilde{\mathbf{E}}^+$ (this is not $\underline{\pi}$), and let $[\overline{\pi}] \in \widetilde{\mathbf{A}}^+$ be its Teichmüller lift. When A is a p-adic complete ring, we use $A\{X, Y\}$ to denote the p-adic completion of $A[X, Y]$.

Definition 2.3.1. For $r \in \mathbb{Z}^{\geq 0}[1/p]$, let $\widetilde{\mathbf{A}}^{[r,+\infty]} := \widetilde{\mathbf{A}}^+\{p/[\overline{\pi}]^r\}$, which is a subring of $\widetilde{\mathbf{A}}$. Here, to be rigorous, $\widetilde{\mathbf{A}}^+\{p/[\overline{\pi}]^r\}$ is defined as $\widetilde{\mathbf{A}}^+\{X\}/([\overline{\pi}]^r X - p)$, and similarly for other similar occurrences later; see [1, §2] for more details. Let $\widetilde{\mathbf{B}}^{[r,+\infty]} := \widetilde{\mathbf{A}}^{[r,+\infty]}[1/p] \subset \widetilde{\mathbf{B}}$.

Definition 2.3.2. Suppose $r \in \mathbb{Z}^{\geq 0}[1/p]$, and let $x = \sum_{i \geq i_0} p^i [x_i] \in \widetilde{\mathbf{B}}^{[r,+\infty]}$ $(\subset \widetilde{\mathbf{B}})$. Denote $w_k(x) := \inf_{i \leq k} v_{\widetilde{\mathbf{E}}}(x_i)$. For $s \geq r$ and $s > 0$, let

$$W^{[s,s]}(x) := \inf_{k \geq i_0} \left\{ k + \frac{p-1}{ps} \cdot v_{\widetilde{\mathbf{E}}}(x_k) \right\} = \inf_{k \geq i_0} \left\{ k + \frac{p-1}{ps} \cdot w_k(x) \right\};$$

this is a well-defined valuation (see [17, Prop. 5.4]). For $I \subset [r, +\infty)$ a non-empty closed interval such that $I \neq [0, 0]$, let

$$W^I(x) := \inf_{\alpha \in I, \alpha \neq 0} W^{[\alpha,\alpha]}(x).$$

Remark 2.3.3. We do not define $W^{[0,0]}$ (cf. [28, Rem. 2.1.9]).

Lemma 2.3.4. *Suppose $r \leq s \in \mathbb{Z}^{\geq 0}[1/p]$ and $s > 0$. Then the following holds:*

(1) *If $r > 0$, then $\widetilde{\mathbf{A}}^{[r,+\infty]}$ and $\widetilde{\mathbf{A}}^{[r,+\infty]}[1/[\overline{\pi}]]$ are complete with respect to $W^{[r,r]}$.*

(2) *$W^{[s,s]}(xy) = W^{[s,s]}(x) + W^{[s,s]}(y)$ for all $x, y \in \widetilde{\mathbf{B}}^{[r,+\infty]}$, so $W^{[s,s]}$ is multiplicative.*

(3) *Let $x \in \widetilde{\mathbf{B}}^{[r,+\infty]}$.*
   - (a) *If $r > 0$, then $W^{[r,s]}(x) = \inf\{W^{[r,r]}(x), W^{[s,s]}(x)\}$. In particular, this implies that $W^{[r,s]}$ is submultiplicative.*
   - (b) *$W^{[0,s]}(x) = W^{[s,s]}(x)$.*

Definition 2.3.5. Let $r \in \mathbb{Z}^{\geq 0}[1/p]$.

(1) Suppose $I = [r,s] \subset [r,+\infty)$ is a non-empty closed interval such that $I \neq [0,0]$. Let $\widetilde{\mathbf{A}}^I$ be the completion of $\widetilde{\mathbf{A}}^{[r,+\infty]}$ with respect to W^I. Let $\widetilde{\mathbf{B}}^I := \widetilde{\mathbf{A}}^I[1/p]$.

(2) Let

$$\widetilde{\mathbf{B}}^{[r,+\infty)} := \bigcap_{n \geq 0} \widetilde{\mathbf{B}}^{[r,s_n]}$$

where $s_n \in \mathbb{Z}^{>0}[1/p]$ is any sequence increasing to $+\infty$. We equip $\widetilde{\mathbf{B}}^{[r,+\infty)}$ with its natural Fréchet topology.

Lemma 2.3.6 ([28, Lem. 2.1.10 (4)]). *Let $I = [r,s]$ be a closed interval as above, and let V^I be the p-adic topology on $\widetilde{\mathbf{B}}^I$ defined using $\widetilde{\mathbf{A}}^I$ as ring of integers. Then for any $x \in \widetilde{\mathbf{B}}^I$, we have $V^I(x) = \lfloor W^I(x) \rfloor$.*

Remark 2.3.7. For our purposes (indeed, also in other literature concerning these rings), it is only necessary to study (the explicit structure of) these rings when

$$\inf I, \sup I \in \{0, +\infty, (p-1)p^{\mathbb{Z}}\}.$$

Furthermore, for any interval I such that $\widetilde{\mathbf{A}}^I$ and $\widetilde{\mathbf{B}}^I$ are defined, there is a natural bijection (called Frobenius) $\varphi : \widetilde{\mathbf{A}}^I \simeq \widetilde{\mathbf{A}}^{pI}$ which is valuation-preserving. Hence in practice, it would suffice if we can determine the explicit structure of these rings for

$$I \in \{[r_\ell, r_k], [r_\ell, +\infty], [0, r_k], [0, +\infty]\} \quad \text{with } \ell \leq k \in \mathbb{Z}^{\geq 0},$$

where $r_n := (p-1)p^{n-1}$. The cases of I a general closed interval can be deduced using Frobenius operation; the cases of $I = [r, +\infty)$ can be deduced by taking Fréchet completion.

Convention 2.3.8. Throughout the paper, *all* the intervals I (over $\widetilde{\mathbf{B}}$-rings, $\mathbf{B}$-rings, D-modules, etc.) satisfy

$$\inf I, \sup I \in \{0, +\infty, (p-1)p^{\mathbb{Z}}\}.$$

If they are not closed, then they are of the form $[0, +\infty)$ or $[r, +\infty)$. I is never allowed to be $[0,0]$ (or "$[+\infty, +\infty]$").

Lemma 2.3.9. *We have*

$$\widetilde{\mathbf{A}}^{[0,r_k]} = \widetilde{\mathbf{A}}^{+}\left\{\frac{u^{ep^k}}{p}\right\},$$
$$\widetilde{\mathbf{A}}^{[r_\ell,+\infty]} = \widetilde{\mathbf{A}}^{+}\left\{\frac{p}{u^{ep^\ell}}\right\},$$
$$\widetilde{\mathbf{A}}^{[r_\ell,r_k]} = \widetilde{\mathbf{A}}^{+}\left\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\right\}.$$

Proof. Indeed, these equations are used as definitions in [28, Def. 2.1.1]; these definitions are equivalent to our current Def. 2.3.5 by Lem. 2.3.6. See [28, §2.1] for more details. ■

Proposition 2.3.10 ([28, Prop. 2.1.14]). *Recall that the subscript K_∞ signifies G_∞-invariants. We have*

$$\widetilde{\mathbf{B}}^{[0,r_k]}_{K_\infty} = \widetilde{\mathbf{A}}^{+}_{K_\infty}\left\{\frac{u^{ep^k}}{p}\right\}\left[\frac{1}{p}\right],$$
$$\widetilde{\mathbf{B}}^{[r_\ell,+\infty]}_{K_\infty} = \widetilde{\mathbf{A}}^{+}_{K_\infty}\left\{\frac{p}{u^{ep^\ell}}\right\}\left[\frac{1}{p}\right],$$
$$\widetilde{\mathbf{B}}^{[r_\ell,r_k]}_{K_\infty} = \widetilde{\mathbf{A}}^{+}_{K_\infty}\left\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\right\}\left[\frac{1}{p}\right].$$

2.4. *The rings $\mathbf{B}^I$ and their G_∞-invariants*

Definition 2.4.1. Let $r \in \mathbb{Z}^{\geq 0}[1/p]$.

(1) Let
$$\mathbf{A}^{[r,+\infty]} := \mathbf{A} \cap \widetilde{\mathbf{A}}^{[r,+\infty]}, \quad \mathbf{B}^{[r,+\infty]} := \mathbf{B} \cap \widetilde{\mathbf{B}}^{[r,+\infty]}.$$

(2) Suppose $[r, s] \subset [r, +\infty)$ is a non-empty closed interval such that $s \neq 0$. Let $\mathbf{B}^{[r,s]}$ be the closure of $\mathbf{B}^{[r,+\infty]}$ in $\widetilde{\mathbf{B}}^{[r,s]}$ with respect to $W^{[r,s]}$. Let $\mathbf{A}^{[r,s]} := \mathbf{B}^{[r,s]} \cap \widetilde{\mathbf{A}}^{[r,s]}$, which is the ring of integers in $\mathbf{B}^{[r,s]}$.

(3) Let
$$\mathbf{B}^{[r,+\infty)} := \bigcap_{n\geq 0} \mathbf{B}^{[r,s_n]}$$
where $s_n \in \mathbb{Z}^{>0}[1/p]$ is any sequence increasing to $+\infty$.

Definition 2.4.2. (1) For $r \in \mathbb{Z}^{\geq 0}[1/p]$, let $\mathcal{A}^{[r,+\infty]}(K_0)$ be the ring consisting of infinite series $f = \sum_{k\in\mathbb{Z}} a_k T^k$ with $a_k \in W(k)$ such that f is a holomorphic function on the annulus defined by
$$v_p(T) \in \left(0, \frac{p-1}{ep}\cdot\frac{1}{r}\right].$$
(Note that when $r = 0$, this implies that $a_k = 0$ for all $k < 0$.) Let
$$\mathcal{B}^{[r,+\infty]}(K_0) := \mathcal{A}^{[r,+\infty]}(K_0)[1/p].$$

(2) Suppose $f = \sum_{k\in\mathbb{Z}} a_k T^k \in \mathcal{B}^{[r,+\infty]}(K_0)$.

(a) When $s \geq r$ and $s > 0$, let

$$\mathcal{W}^{[s,s]}(f) := \inf_{k\in\mathbb{Z}} \left\{ v_p(a_k) + \frac{p-1}{ps} \cdot \frac{k}{e} \right\}.$$

(b) For $I \subset [r, +\infty)$ a non-empty closed interval, let

$$\mathcal{W}^I(f) := \inf_{\alpha\in I, \alpha\neq 0} \mathcal{W}^{[\alpha,\alpha]}(f).$$

(3) For $r \leq s \in \mathbb{Z}^{\geq 0}[1/p], s \neq 0$, let $\mathcal{B}^{[r,s]}(K_0)$ be the completion of $\mathcal{B}^{[r,+\infty]}(K_0)$ with respect to $\mathcal{W}^{[r,s]}$. Let $\mathcal{A}^{[r,s]}(K_0)$ be the ring of integers in $\mathcal{B}^{[r,s]}(K_0)$ with respect to $\mathcal{W}^{[r,s]}$.

Lemma 2.4.3. (1) *For $r > 0$, $\mathcal{B}^{[r,+\infty]}(K_0)$ is complete with respect to $\mathcal{W}^{[r,r]}$, and $\mathcal{A}^{[r,+\infty]}(K_0)$ is the ring of integers with respect to this valuation.*

(2) *For $s > 0$, we have $\mathcal{W}^{[0,s]}(x) = \mathcal{W}^{[s,s]}(x)$. Furthermore, $\mathcal{B}^{[0,s]}(K_0)$ is the ring consisting of infinite series $f = \sum_{k\in\mathbb{Z}} a_k T^k$ with $a_k \in K_0$ such that f is a holomorphic function on the closed disk defined by*

$$v_p(T) \in \left[\frac{p-1}{ep} \cdot \frac{1}{s}, +\infty \right].$$

(3) *For $I = [r, s] \subset (0, +\infty)$, we have $\mathcal{W}^I(x) = \inf\{\mathcal{W}^{[r,r]}(x), \mathcal{W}^{[s,s]}(x)\}$. Furthermore, $\mathcal{B}^{[r,s]}(K_0)$ is the ring consisting of infinite series $f = \sum_{k\in\mathbb{Z}} a_k T^k$ with $a_k \in K_0$ such that f is a holomorphic function on the annulus defined by*

$$v_p(T) \in \left[\frac{p-1}{ep} \cdot \frac{1}{s}, \frac{p-1}{ep} \cdot \frac{1}{r} \right].$$

Proof. In [28, Lem. 2.2.5], we stated the results for $[r, s] = [r_\ell, r_k]$; but they are true for general intervals. ∎

Lemma 2.4.4 ([28, Lem. 2.2.7]). *Let $\mathbf{A}^I_{K_\infty}$ be the G_∞-invariants of $\mathbf{A}^I$. The map $f(T) \mapsto f(u)$ induces isometric isomorphisms*

$$\mathcal{A}^{[0,+\infty]}(K_0) \simeq \mathbf{A}^{[0,+\infty]}_{K_\infty};$$
$$\mathcal{A}^{[r,+\infty]}(K_0) \simeq \mathbf{A}^{[r,+\infty]}_{K_\infty}[1/u] \quad \text{when } r > 0;$$
$$\mathcal{A}^I(K_0) \simeq \mathbf{A}^I_{K_\infty} \quad \text{when } I \subset [0, +\infty) \text{ is a closed interval.}$$

We record an easy corollary of the above explicit description of the rings $\mathbf{B}^I_{K_\infty}$.

Corollary 2.4.5. *Let $I \subset [0, +\infty]$ be an interval. Suppose that $x \in \mathbf{B}^I_{K_\infty}$ is such that $\varphi(x) \in \mathbf{B}^I_{K_\infty}$. Then $x \in \mathbf{B}^{I/p}_{K_\infty}$.*

Proof. Indeed, by Lem. 2.4.4, $x = \sum_{i\in\mathbb{Z}} a_i u^i$ with $a_i \in K_0$ satisfying certain convergence conditions related to the interval I as described in Lem. 2.4.3. We have $\varphi(x) = \sum_{i\in\mathbb{Z}} \varphi(a_i)u^{pi}$, hence using the explicit convergence condition, it is easy to see that $\varphi(x) \in \mathbf{B}^I_{K_\infty}$ if and only if $x \in \mathbf{B}^{I/p}_{K_\infty}$. ■

Finally, we write out the explicit structures of some of these rings.

Proposition 2.4.6. [28, Prop. 2.2.10] *We have*

$$\begin{aligned}
\mathbf{A}^{[0,+\infty]}_{K_\infty} &= \mathbf{A}^+_{K_\infty},\\
\mathbf{A}^{[0,r_k]}_{K_\infty} &= \mathbf{A}^+_{K_\infty}\left\{\frac{u^{ep^k}}{p}\right\},\\
\mathbf{A}^{[r_\ell,+\infty]}_{K_\infty} &= \mathbf{A}^+_{K_\infty}\left\{\frac{p}{u^{ep^\ell}}\right\},\\
\mathbf{A}^{[r_\ell,r_k]}_{K_\infty} &= \mathbf{A}^+_{K_\infty}\left\{\frac{p}{u^{ep^\ell}}, \frac{u^{ep^k}}{p}\right\}.
\end{aligned}$$

Remark 2.4.7. Note that $\varphi : \widetilde{\mathbf{B}}^I \to \widetilde{\mathbf{B}}^{pI}$ is always a bijection; however, the map $\varphi : \mathbf{B}^I \to \mathbf{B}^{pI}$ is only an injection. Indeed, $\mathbf{B}/\varphi(\mathbf{B})$ is a degree p field extension. However, one can always find explicit expressions for $\mathbf{B}^I_{K_\infty}$ using Lem. 2.4.3.

2.5. *Locally analytic vectors in rings*

Recall that that we use the subscript L to indicate the $\mathrm{Gal}(\overline{K}/L)$-invariants. Recall for a $\hat{G} = \mathrm{Gal}(L/K)$-representation W, we denote $W^{\tau\text{-la},\gamma=1} := W^{\tau\text{-la}} \cap W^{\gamma=1}$. The following theorem in [28] is the main result concerning calculation of locally analytic vectors in period rings.

Theorem 2.5.1 ([28, Lem. 3.4.2, Thm. 3.4.4]). (1) *For $I = [r_\ell, r_k]$ or $[0, r_k]$, and for each $n \ge 0$, $\varphi^{-n}(u) \in (\widetilde{\mathbf{B}}^I_L)^{\tau_{n+k}\text{-an}}$ (see Notation 1.4.4). So in particular,*

$$u \in (\widetilde{\mathbf{B}}^{[0,r_0]}_L)^{\tau\text{-an}}.$$

(2) *For $I = [r_\ell, r_k]$ or $[0, r_k]$, we have $(\widetilde{\mathbf{A}}^I_L)^{\tau\text{-la},\gamma=1} = \bigcup_{m\ge 0} \varphi^{-m}(\mathbf{A}^{p^m I}_{K_\infty})$.*

(3) *For any $r \ge 0$, $(\widetilde{\mathbf{B}}^{[r,+\infty)}_L)^{\tau\text{-pa},\gamma=1} = \bigcup_{m\ge 0} \varphi^{-m}(\mathbf{B}^{[p^m r,+\infty)}_{K_\infty})$.*

Remark 2.5.2. Let us point out some (fortunately very minor) errors in the proof of the theorem above, all relating to the "τ-issue" in Notation 1.4.3.

(1) Firstly, in [28, Notation 3.2.1], we should always *fix* the τ as we now do in Notation 1.4.3; we are implicitly using the same τ in [28], but only when $K_\infty \cap K_{p^\infty} = K$.

(2) The problem with this τ-issue in concrete computations is that we have

$$\begin{cases} \tau(u) = u[\underline{\varepsilon}] & \text{if } K_\infty \cap K_{p^\infty} = K;\\ \tau(u) = u[\underline{\varepsilon}]^2 & \text{if } K_\infty \cap K_{p^\infty} = K(\pi_1). \end{cases} \tag{2.5.1}$$

(3) In [28, Lem. 3.4.2] and [28, Thm. 3.4.4], if $K_\infty \cap K_{p^\infty} = K(\pi_1)$, then we should change *some* of the a there to $2a$, in the equation above (3.4.2), in (3.4.3), and in the equation below (3.4.8); this is because we now have $\tau(u) = u(1+v)^2$. The changes to $2a$ only make the relevant convergence easier, hence do not change the final results.

Definition 2.5.3. (1) Define the following rings (which are LB spaces):

$$\widetilde{\mathbf{B}}^\dagger := \bigcup_{r\geq 0} \widetilde{\mathbf{B}}^{[r,+\infty]}, \quad \mathbf{B}^\dagger := \bigcup_{r\geq 0} \mathbf{B}^{[r,+\infty]}, \quad \mathbf{B}^\dagger_{K_\infty} := \bigcup_{r\geq 0} \mathbf{B}^{[r,+\infty]}_{K_\infty}.$$

(2) Define the following rings (which are LF spaces):

$$\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} := \bigcup_{r\geq 0} \widetilde{\mathbf{B}}^{[r,+\infty)}, \quad \mathbf{B}^\dagger_{\mathrm{rig}} := \bigcup_{r\geq 0} \mathbf{B}^{[r,+\infty)}, \quad \mathbf{B}^\dagger_{\mathrm{rig},K_\infty} := \bigcup_{r\geq 0} \mathbf{B}^{[r,+\infty)}_{K_\infty}.$$

(3) Define the following notations:

$$\widetilde{\mathbf{B}}^+_{\mathrm{rig}} := \widetilde{\mathbf{B}}^{[0,+\infty)}, \quad \mathbf{B}^+_{\mathrm{rig},K_\infty} := \mathbf{B}^{[0,+\infty)}_{K_\infty}.$$

Note that $\mathbf{B}^\dagger_{K_\infty}$ (resp. $\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}$) is precisely the explicitly defined ring in (1.2.2) (resp. (1.2.3)).

Corollary 2.5.4. *We have*

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1} = \bigcup_{m\geq 0} \varphi^{-m}(\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}), \quad (\widetilde{\mathbf{B}}^+_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1} = \bigcup_{m\geq 0} \varphi^{-m}(\mathbf{B}^+_{\mathrm{rig},K_\infty}).$$

2.6. *The element* $\mathfrak{t}$

In this subsection, we study a locally analytic element $\mathfrak{t}$, which plays a useful role in the definition of our monodromy operators.

Recall we defined $[\underline{\varepsilon}] \in \widetilde{\mathbf{A}}^+$ in Notation 1.1.4. Let $t = \log([\underline{\varepsilon}]) \in \mathbf{B}^+_{\mathrm{cris}}$ be the usual element. Define

$$\lambda := \prod_{n\geq 0} \left(\varphi^n\left(\frac{E(u)}{E(0)}\right)\right) \in \mathbf{B}^{[0,+\infty)}_{K_\infty} \subset \mathbf{B}^+_{\mathrm{cris}}.$$

The equation $\varphi(x) = \frac{pE(u)}{E(0)} \cdot x$ over $\widetilde{\mathbf{A}}^+$ has a solution in $\widetilde{\mathbf{A}}^+ \setminus p\widetilde{\mathbf{A}}^+$, which is unique up to units in $\mathbb{Z}_p$ (see [35, paragraph above Thm. 3.2.2]). By the discussion in [35, Example 5.3.3], there exists a *unique* solution $\mathfrak{t} \in \widetilde{\mathbf{A}}^+$ such that

$$p\lambda\mathfrak{t} = t, \tag{2.6.1}$$

which holds as an equation in $\mathbf{B}^+_{\mathrm{cris}}$. Since $\mathfrak{t} \in \widetilde{\mathbf{A}}^+ \subset \widetilde{\mathbf{B}}^\dagger_L$, and since $\widetilde{\mathbf{B}}^\dagger_L$ is a field [17, Prop. 5.12], there exists some $r(\mathfrak{t}) > 0$ such that $1/\mathfrak{t} \in \widetilde{\mathbf{B}}^{[r(\mathfrak{t}),+\infty]}_L$.

Lemma 2.6.1 ([28, Lem. 5.1.1]). *We have* $\mathfrak{t}, 1/\mathfrak{t} \in (\widetilde{\mathbf{B}}^{[r(\mathfrak{t}),+\infty)}_L)^{\hat{G}\text{-pa}}$.

2.7. *Some "log"-rings*

In this subsection, we introduce some log-rings (corresponding to the "rig"-rings in Def. 2.5.3). We first introduce a convention often used here.

Convention 2.7.1. Let A be a topological ring, and let Y be a variable. Then we always equip $A[Y]$ with the inductive topology using $A[Y] := \bigcup_{n\geq 0}(\bigoplus_{i=0}^{n} A\cdot Y^i)$ where each $A\cdot Y^i$ has the topology induced from that on A.

Choose some $\underline{p} := (p_0, p_1, \dots, p_n, \dots) \in \widetilde{\mathbf{E}}^+$ where $p_0 = p$ and $p_{n+1}^p = p_n$ for all $n \geq 0$. Let X be a formal variable, and define

$$\widetilde{\mathbf{B}}^{\dagger}_{\log} := \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[X].$$

Extend the φ-operator and G_K-action on $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ to $\widetilde{\mathbf{B}}^{\dagger}_{\log}$ such that $\varphi(X) = pX$ and $g(X) = X + c(g)t$ where $c(g)$ is the cocycle such that $g(\underline{p}) = \underline{p}\cdot\underline{\varepsilon}^{c(g)}$; define a $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$-derivation N on $\widetilde{\mathbf{B}}^{\dagger}_{\log}$ such that $N(X) = e$ (Rem. 2.7.3 for this convention). Let

$$\widetilde{\mathbf{B}}^{+}_{\log} := \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[X];$$

it is a subring of $\widetilde{\mathbf{B}}^{\dagger}_{\log}$ which is (φ, G_K, N)-stable.

Proposition 2.7.2 ([17, Prop. 5.15]). *With respect to the choice $\underline{p}$, there exists a φ- and G_K-equivariant map*

$$\log : (\widetilde{\mathbf{B}}^{\dagger})^{*} \to \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[X]$$

which is uniquely determined by the following properties:

(1) $\log xy = \log x + \log y$;

(2) $\log x = \sum_{n=1}^{+\infty} \frac{-(1-x)^n}{n}$ *if the series converges;*

(3) $\log\,[a] = 0$ *if* $a \in \overline{k}$*;*

(4) $\log p = 0$ *and* $\log\,[\underline{p}] = X$.

For this log map, if $x = \sum_{k=k_0}^{+\infty} p^k [x_k]$ *with* $x_{k_0} \neq 0$*, then*

$$N(\log x) = e\cdot v_{\widetilde{\mathbf{E}}}(x_{k_0}). \qquad (2.7.1)$$

Proof. This is exactly [17, Prop. 5.15] except for in (2.7.1): in [17], it is "$N(\log x) = -v_{\widetilde{\mathbf{E}}}(x_{k_0})$." Our (2.7.1) matches the choice $N(X) = e$ (see Rem. 2.7.3 for the reason for this choice): in [17], it is "$N(X) = -1$".

Here, let us sketch the construction of this log map. By (4), it suffices to consider $x \in (\widetilde{\mathbf{B}}^{\dagger})^{*}$ such that $v_p(x) = 0$. Then by (1), it suffices to consider the case when $v_{\widetilde{\mathbf{E}}}(\overline{x}) \in \mathbb{Z}^{\geq 0}$; in this case, it can be uniquely written as

$$x = [\underline{p}^{\alpha}][a]y, \quad \text{where } \alpha \in \mathbb{Z}^{\geq 0},\ a \in \overline{k},\ y \in \widetilde{\mathbf{A}} \cap \widetilde{\mathbf{B}}^{\dagger},\ v_{\widetilde{\mathbf{E}}}(\overline{y} - 1) > 0; \qquad (2.7.2)$$

then we can define

$$\log x = \alpha X + \log y$$

where $\log y \in \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ by [17, Lem. 5.14 (2)]. ∎

Remark 2.7.3. Note that the N-operator here equals $-e \cdot N$ in [17]. We make this choice to have $N(\log u) = 1$. This choice is the same as that in [32]; in particular, we can remove minus signs everywhere for this N-operator. This choice is good for us since $\log u$ is important in the study of (φ, τ)-modules; cf., e.g., Lem. 2.7.7 below. See also Rem. 4.1.3 later where we make some choice to remove minus signs for the N_∇-operator.

Lemma 2.7.4 ([17, Lem. 5.14 (1)]). *Suppose* $x = \sum_{k=0}^{+\infty} p^k [x_k]$ *is a unit in* $\widetilde{\mathbf{A}}^+$ *such that* $v_{\widetilde{\mathbf{E}}}(x_0 - 1) > 0$. *Then* $\sum_{n=1}^{+\infty} \frac{-(1-x)^n}{n}$ *converges in* $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$.

Lemma 2.7.5. *Let* $\beta \in \widetilde{\mathbf{E}}^+$ *with* $\alpha := v_{\widetilde{\mathbf{E}}}(\beta) \neq 0$, *and let* $x := [\beta] \in \widetilde{\mathbf{A}}^+$ *be the Teichmüller lift. Then*

$$\widetilde{\mathbf{B}}^\dagger_{\log} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[\log x], \quad \widetilde{\mathbf{B}}^+_{\log} = \widetilde{\mathbf{B}}^+_{\mathrm{rig}}[\log x].$$

Proof. Write $x = [\underline{p}^\alpha][a]y$ as in (2.7.2). Then y satisfies the condition in Lem. 2.7.4, and hence $\log x = \alpha X + \log y$ with $\log y \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. ∎

Definition 2.7.6. Let $\ell_u := \log u = \log[\underline{\pi}]$, and define

$$\mathbf{B}^\dagger_{\log, K_\infty} := \mathbf{B}^\dagger_{\mathrm{rig}, K_\infty}[\ell_u] \subset \widetilde{\mathbf{B}}^\dagger_{\log}, \quad \mathbf{B}^+_{\log, K_\infty} := \mathbf{B}^+_{\mathrm{rig}, K_\infty}[\ell_u] \subset \widetilde{\mathbf{B}}^+_{\log}.$$

(The containments follow from Lem. 2.7.5.)

Lemma 2.7.7. *Let* $I \subset [0, +\infty)$ *be a closed interval, and let* $k \geq 1$. *Then*

$$\ell_u^k \in \left(\bigoplus_{i=0}^{k} \widetilde{\mathbf{B}}^I_L \cdot \ell_u^i \right)^{\tau\text{-an}, \gamma=1}$$

where $\bigoplus_{i=0}^k \widetilde{\mathbf{B}}^I_L \cdot \ell_u^i$ *is regarded as a Banach space. Hence,* ℓ_u *is always a* τ-analytic *vector* (*not just locally analytic*).

Proof. Note that

$$g(\ell_u) = \ell_u + \sigma(g)t, \quad \forall g \in G_K, \tag{2.7.3}$$

where $\sigma(g) \in \mathbb{Z}_p^\times$ (is the cocycle) such that $g([\underline{\pi}]) = [\underline{\pi}][\underline{\varepsilon}]^{\sigma(g)}$; it is then easy to deduce that ℓ_u is a τ-analytic vector (e.g., using [28, Lem. 3.1.8]). ∎

Corollary 2.7.8.

$$(\widetilde{\mathbf{B}}^\dagger_{\log, L})^{\tau\text{-pa}, \gamma=1} = \bigcup_{m \geq 0} \varphi^{-m}(\mathbf{B}^\dagger_{\log, K_\infty}), \quad (\widetilde{\mathbf{B}}^+_{\log, L})^{\tau\text{-pa}, \gamma=1} = \bigcup_{m \geq 0} \varphi^{-m}(\mathbf{B}^+_{\log, K_\infty}).$$

Proof. This follows from Lem. 2.7.7 and [28, Prop. 3.1.6] (as all ℓ_u^i are analytic vectors). ∎

3. Modules and locally analytic vectors

In this section, we recall the theory of étale (φ, τ)-modules and their overconvergence property. In particular, we discuss the relation between locally analytic vectors and the overconvergence property.

In this section and in §5, we will introduce several categories of *modules with structures*. We will always omit the definition of morphisms for these categories, which are obvious (i.e., module homomorphisms compatible with various structures).

3.1. *Étale* (φ, τ)*-modules*

Definition 3.1.1. Objects in the following are called *étale* φ*-modules*.

(1) Let $\mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$ denote the category of finite free $\mathbf{A}_{K_\infty}$-modules M equipped with a $\varphi_{\mathbf{A}_{K_\infty}}$-semilinear endomorphism $\varphi_M : M \to M$ such that $1 \otimes \varphi : \varphi^* M \to M$ is an isomorphism.

(2) Let $\mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$ denote the category of finite free $\mathbf{B}_{K_\infty}$-modules (indeed, vector spaces) D equipped with a $\varphi_{\mathbf{B}_{K_\infty}}$-semilinear endomorphism $\varphi_D : D \to D$ such that there exists a finite free $\mathbf{A}_{K_\infty}$-lattice M such that $M[1/p] = D$, $\varphi_D(M) \subset M$, and $(M, \varphi_D|_M) \in \mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$.

Definition 3.1.2. Objects in the following are called *étale* (φ, τ)*-modules*.

(1) Let $\mathrm{Mod}^{\varphi,\widehat{G}}_{\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L}$ denote the category of triples $(M, \varphi_M, \widehat{G})$ where
- $(M, \varphi_M) \in \mathrm{Mod}^{\varphi}_{\mathbf{A}_{K_\infty}}$;
- $\widehat{G}$ is a continuous $\varphi_{\hat{M}}$-commuting $\widetilde{\mathbf{A}}_L$-semilinear $\widehat{G}$-action on $\hat{M} := \widetilde{\mathbf{A}}_L \otimes_{\mathbf{A}_{K_\infty}} M$ (here, continuity is with respect to the topology induced by the weak topology on $\widetilde{\mathbf{A}}$);
- regarding M as an $\mathbf{A}_{K_\infty}$-submodule in $\hat{M}$, we have $M \subset \hat{M}^{\mathrm{Gal}(L/K_\infty)}$.

(2) Let $\mathrm{Mod}^{\varphi,\widehat{G}}_{\mathbf{B}_{K_\infty},\widetilde{\mathbf{B}}_L}$ denote the category of triples $(D, \varphi_D, \widehat{G})$ which contains a lattice (in the obvious fashion) $(M, \varphi_M, \widehat{G}) \in \mathrm{Mod}^{\varphi,\widehat{G}}_{\mathbf{A}_{K_\infty},\widetilde{\mathbf{A}}_L}$.

3.1.3. Let $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$) denote the category of finite-dimensional $\mathbb{Q}_p$-vector spaces V with continuous $\mathbb{Q}_p$-linear G_∞ (resp. G_K)-actions.

- For $D \in \mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$, let
$$V(D) := (\widetilde{\mathbf{B}} \otimes_{\mathbf{B}_{K_\infty}} D)^{\varphi=1};$$
then $V(D) \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$. If furthermore $(D, \varphi_D, \widehat{G}) \in \mathrm{Mod}^{\varphi,\widehat{G}}_{\mathbf{B}_{K_\infty},\widetilde{\mathbf{B}}_L}$, then $V(D) \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.
- For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$, let
$$D_{K_\infty}(V) := (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{G_\infty};$$
then $D_{K_\infty}(V) \in \mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$. If furthermore $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, let
$$\widetilde{D}_L(V) := (\widetilde{\mathbf{B}} \otimes_{\mathbb{Q}_p} V)^{G_L};$$

then $\widetilde{D}_L(V) = \widetilde{\mathbf{B}}_L \otimes_{\mathbf{B}_{K_\infty}} D_{K_\infty}(V)$ has a $\widehat{G}$-action, making $(D_{K_\infty}(V), \varphi, \widehat{G})$ an étale (φ, τ)-module.

As already mentioned in Rem. 1.2.8, Thm. 3.1.4 below is the only place where we *use* results from [12].

Theorem 3.1.4. (1) [22, Prop. A 1.2.6] *The functors V and D_{K_∞} induce an exact tensor equivalence between the categories* $\mathrm{Mod}^{\varphi}_{\mathbf{B}_{K_\infty}}$ *and* $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$.

(2) [12, Thm. 1] *The functors V and $(D_{K_\infty}, \widetilde{D}_L)$ induce an exact tensor equivalence between the categories* $\mathrm{Mod}^{\varphi,\widehat{G}}_{\mathbf{B}_{K_\infty}, \widetilde{\mathbf{B}}_L}$ *and* $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

Proof. Note that item (2) was written only for $p > 2$ in [12]. Our Def. 3.1.2 is slightly different from Caruso's definition (although the underlying idea is the same); see the discussion in [27, Rem. 2.1.6]. In particular, item (2) above is valid for all p. ■

3.2. *Overconvergence and locally analytic vectors*

Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Given $I \subset [0, +\infty]$, let

$$D^I_{K_\infty}(V) := (\mathbf{B}^I \otimes_{\mathbb{Q}_p} V)^{G_\infty}, \quad \widetilde{D}^I_L(V) := (\widetilde{\mathbf{B}}^I \otimes_{\mathbb{Q}_p} V)^{G_L}.$$

Definition 3.2.1. Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, and let $\hat{D} = (D_{K_\infty}(V), \varphi, \widehat{G})$ be the étale (φ, τ)-module associated to it. We say that $\hat{D}$ is *overconvergent* if there exists $r > 0$ such that for $I' = [r, +\infty]$,

(1) $D^{I'}_{K_\infty}(V)$ is finite free over $\mathbf{B}^{I'}_{K_\infty}$, and $\mathbf{B}_{K_\infty} \otimes_{\mathbf{B}^{I'}_{K_\infty}} D^{I'}_{K_\infty}(V) = D_{K_\infty}(V)$;

(2) $\widetilde{D}^{I'}_L(V)$ is finite free over $\widetilde{\mathbf{B}}^{I'}_L$, and $\widetilde{\mathbf{B}}_L \otimes_{\widetilde{\mathbf{B}}^{I'}_L} \widetilde{D}^{I'}_L(V) = \widetilde{D}_L(V)$.

Theorem 3.2.2 ([27, 28]). *For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, the associated étale (φ, τ)-module is overconvergent.*

Remark 3.2.3. As we already mentioned in Rem. 1.2.2, a first proof of Thm. 3.2.2 was given in [27] (which only works for $K/\mathbb{Q}_p$ a finite extension), and a second proof was given in [28]; it is the second proof that will be useful for the current paper; see e.g. Cor. 3.2.4 below.

Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of dimension d, and let $D^{I'}_{K_\infty}(V)$ be as in Def. 3.2.1 for $I' = [r(V), +\infty]$ (which exists by Thm. 3.2.2). Let

$$D^{\dagger}_{K_\infty}(V) := D^{I'}_{K_\infty}(V) \otimes_{\mathbf{B}^{I'}_{K_\infty}} \mathbf{B}^{\dagger}_{K_\infty}; \tag{3.2.1}$$

$$D^{\dagger}_{\mathrm{rig},K_\infty}(V) := D^{I'}_{K_\infty}(V) \otimes_{\mathbf{B}^{I'}_{K_\infty}} \mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}. \tag{3.2.2}$$

We call $D^{\dagger}_{K_\infty}(V)$ (resp. $D^{\dagger}_{\mathrm{rig},K_\infty}(V)$) the *overconvergent* (resp. *rigid-overconvergent*) (φ, τ)-*module* associated to V.

Corollary 3.2.4. *The subset $D^{\dagger}_{\mathrm{rig},K_\infty}(V)$ generates the $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1}$-module $((\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{G_L})^{\tau\text{-pa},\gamma=1}$. Indeed,*

$$((\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{G_L})^{\tau\text{-pa},\gamma=1} = D^{\dagger}_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}} (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L})^{\tau\text{-pa},\gamma=1}.$$

Proof. This is extracted from the proof of Thm. 3.2.2 in [28, Thm. 6.2.6]; indeed, it easily follows from [28, (6.2.5)]. ■

3.3. Modules with respect to field extensions

(The discussion here is a continuation of §2.2.) Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. Let E be as in Notation 2.2.1. Then with respect to the G_E-representation $V|_{G_E}$, we can also construct the corresponding (φ, τ)-module and its overconvergent version; denote them as

$$D_{E_\infty}(V|_{G_E}), \quad D^{\dagger}_{E_\infty}(V|_{G_E}), \quad D^{\dagger}_{\mathrm{rig},E_\infty}(V|_{G_E}).$$

These are finite free modules over the rings $\mathbf{B}_{E_\infty}, \mathbf{B}^{\dagger}_{E_\infty}, \mathbf{B}^{\dagger}_{\mathrm{rig},E_\infty}$ respectively, which are constructed analogously to Notation 2.2.1.

Lemma 3.3.1. *We have φ-equivariant isomorphisms*

$$D_{K_\infty}(V) \otimes_{\mathbf{B}_{K_\infty}} \mathbf{B}_{E_\infty} \simeq D_{E_\infty}(V|_{G_E}); \tag{3.3.1}$$

$$D^{\dagger}_{K_\infty}(V) \otimes_{\mathbf{B}^{\dagger}_{K_\infty}} \mathbf{B}^{\dagger}_{E_\infty} \simeq D^{\dagger}_{E_\infty}(V|_{G_E}); \tag{3.3.2}$$

$$D^{\dagger}_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}} \mathbf{B}^{\dagger}_{\mathrm{rig},E_\infty} \simeq D^{\dagger}_{\mathrm{rig},E_\infty}(V|_{G_E}). \tag{3.3.3}$$

Proof. The first two isomorphisms are obvious since both $\mathbf{B}_{E_\infty}$ and $\mathbf{B}^{\dagger}_{E_\infty}$ are fields (see [15, Prop. II.1.6 (1)] in the (φ, Γ)-module setting); the third isomorphism then follows. ■

4. Monodromy operator for (φ, τ)-modules

In this section, we define a natural monodromy operator on rigid-overconvergent (φ, τ)-modules.

Let $\log(\tau^{p^n})$ denote the (formally written) series $(-1) \cdot \sum_{k \geq 1} (1 - \tau^{p^n})^k / k$. Then $\nabla_\tau := \frac{\log(\tau^{p^n})}{p^n}$ for $n \gg 0$ is a well-defined Lie-algebra operator acting on $\widehat{G}$-locally analytic representations.

4.1. Monodromy operator over rings

Recall that by Cor. 2.7.8,

$$(\widetilde{\mathbf{B}}^{\dagger}_{\log,L})^{\tau\text{-pa},\gamma=1} = \bigcup_{m \geq 0} \varphi^{-m}(\mathbf{B}^{\dagger}_{\log,K_\infty}).$$

Hence ∇_τ induces a map

$$\nabla_\tau : \mathbf{B}^\dagger_{\log,K_\infty} \to (\widetilde{\mathbf{B}}^\dagger_{\log,L})^{\hat{G}\text{-pa}}.$$

Recall that by Lem. 2.6.1, we have $1/t \in (\widetilde{\mathbf{B}}_L^{[r(t),+\infty)})^{\hat{G}\text{-pa}}$. We can define an operator

$$N_\nabla : \mathbf{B}^\dagger_{\log,K_\infty} \to (\widetilde{\mathbf{B}}^\dagger_{\log,L})^{\hat{G}\text{-pa}} \tag{4.1.1}$$

by setting

$$N_\nabla := \begin{cases} \frac{1}{pt} \cdot \nabla_\tau & \text{if } K_\infty \cap K_{p^\infty} = K, \\ \frac{1}{p^2 t} \cdot \nabla_\tau = \frac{1}{4t} \cdot \nabla_\tau & \text{if } K_\infty \cap K_{p^\infty} = K(\pi_1) \text{ (see Notation 1.4.3)}. \end{cases} \tag{4.1.2}$$

Remark 4.1.1. The p (resp. p^2) in the denominator of (4.1.2) makes our monodromy operator compatible with earlier theory of Kisin in [32], but *up to a minus sign*, see Rem. 4.1.3 below.

Lemma 4.1.2. *The image of N_∇ in (4.1.1) falls in $\mathbf{B}^\dagger_{\log,K_\infty}$, and hence induces*

$$N_\nabla : \mathbf{B}^\dagger_{\log,K_\infty} \to \mathbf{B}^\dagger_{\log,K_\infty}. \tag{4.1.3}$$

Explicitly, the differential map N_∇ sends $x \in \mathbf{B}^\dagger_{\mathrm{rig},K_\infty}$ to $\lambda \cdot u\frac{d}{du}(x)$, and $N_\nabla(\ell_u) = \lambda$. Furthermore, the rings $\mathbf{B}^+_{\log,K_\infty}$, $\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}$, and $\mathbf{B}^I_{K_\infty}$ (for any $I \subset [0,+\infty)$) are all stable under N_∇.

Proof. Everything follows from easy explicit calculations. For example, if $K_\infty \cap K_{p^\infty} = K$, then $\tau(u) = u[\varepsilon]$, hence we have (using any $n \gg 0$)

$$N_\nabla(u) = \frac{1}{pt} \cdot \frac{-1}{p^n} \cdot \sum_{k\geq 1} \frac{u(1-[\varepsilon]^{p^n})^k}{k} = \frac{1}{pt} \cdot \frac{1}{p^n} \cdot u \cdot (p^n t) = \frac{ut}{pt} = \lambda \cdot u.$$

The fact that $N_\nabla(\ell_u) = \lambda$ follows from a similar computation using (2.7.3). ∎

Remark 4.1.3. Our N_∇ equals $-N_\nabla$ in [32, §1.1.1]. Certainly, this sign change makes no difference for the results in [32] (which we will use later). (Alternatively, we could have added minus signs in (4.1.2) so that everything is strictly compatible with the conventions in [32]; but we prefer to remove the minus signs everywhere; cf. also the choice we made in Rem. 2.7.3.)

4.2. *Monodromy operator over modules*

By Cor. 3.2.4, we have

$$((\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{G_L})^{\hat{G}\text{-pa}} = D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\hat{G}\text{-pa}}. \tag{4.2.1}$$

Hence similarly to §4.1, we have a map (using exactly the same formulae as in (4.1.2))

$$N_\nabla : D^\dagger_{\mathrm{rig},K_\infty}(V) \to D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\hat{G}\text{-pa}}. \tag{4.2.2}$$

Theorem 4.2.1. *The map* (4.2.2) *induces a map*

$$N_\nabla : D^\dagger_{\mathrm{rig},K_\infty}(V) \to D^\dagger_{\mathrm{rig},K_\infty}(V), \tag{4.2.3}$$

so N_∇ is a well-defined operator on $D^\dagger_{\mathrm{rig},K_\infty}(V)$. Furthermore, there exists $r' \geq r(V)$ (see the notation above (3.2.2)*) such that if $I \subset [r', +\infty)$, then*

$$N_\nabla : D^I_{K_\infty}(V) \to D^I_{K_\infty}(V), \tag{4.2.4}$$

where we recall that $D^I_{K_\infty}(V) = D^{[r(V),+\infty]}_{K_\infty}(V) \otimes_{\mathbf{B}^{[r(V),+\infty]}_{K_\infty}} \mathbf{B}^I_{K_\infty}$.

Proof. Note that

$$g\tau g^{-1} = \tau^{\chi_p(g)} \quad \text{for } g \in \mathrm{Gal}(L/K_\infty),$$

where χ_p is the cyclotomic character. Note also that $g(\mathfrak{t}) = \chi_p(g)\mathfrak{t}$ for $g \in \mathrm{Gal}(L/K_\infty)$. Hence

$$gN_\nabla = N_\nabla g \quad \text{for } g \in \mathrm{Gal}(L/K_\infty). \tag{4.2.5}$$

Since $\mathrm{Gal}(L/K_\infty)$ acts trivially on $D^\dagger_{\mathrm{rig},K_\infty}(V)$, it also acts trivially on $N_\nabla(D^\dagger_{\mathrm{rig},K_\infty}(V))$ using (4.2.5). Thus, we have

$$\begin{aligned} N_\nabla(D^\dagger_{\mathrm{rig},K_\infty}(V)) &\subset \big(D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{B^\dagger_{\mathrm{rig},K_\infty}} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\hat{G}\text{-pa}}\big)^{\gamma=1} \\ &= D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\gamma=1,\tau\text{-pa}} \\ &= D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}} \Big(\bigcup_{m\geq 0} \varphi^{-m}(\mathbf{B}^\dagger_{\mathrm{rig},K_\infty})\Big) \quad \text{(by Cor. 2.5.4)}. \end{aligned}$$

Choose a basis $\vec{e}$ of $D^\dagger_{\mathrm{rig},K_\infty}(V)$. Then

$$N_\nabla(\vec{e}) \subset D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}} \varphi^{-m}(\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}) \quad \text{for some } m \gg 0.$$

Recall $\varphi(\mathfrak{t}) = \frac{pE(u)}{E(0)}\mathfrak{t}$. Note that $\varphi\tau = \tau\varphi$ over $D^\dagger_{\mathrm{rig},K_\infty}(V) \otimes_{B^\dagger_{\mathrm{rig},K_\infty}} (\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},L})^{\hat{G}\text{-pa}}$, hence

$$N_\nabla\varphi = \frac{pE(u)}{E(0)}\varphi N_\nabla. \tag{4.2.6}$$

Using (4.2.6), we can easily deduce that

$$N_\nabla(\varphi^m(\vec{e})) \subset D^\dagger_{\mathrm{rig},K_\infty}(V).$$

Note that $\varphi^m(\vec{e})$ is also a basis of $D^\dagger_{\mathrm{rig},K_\infty}(V)$, which concludes the proof of (4.2.3). Finally, we can use $r' = p^m r(V)$ to derive (4.2.4). ■

5. Monodromy operator for semi-stable representations

In this section, we will recall Kisin's definition of a monodromy operator (up to a minus sign, by Rem. 4.1.3) on certain modules associated to semi-stable representations. The main aim of this section is to show that Kisin's monodromy operator *coincides* with the one defined by us in §4 (in the semi-stable case).

In §5.1, we review a result of Cherbonnier on maximal overconvergent submodules and use it to study finite height modules. In §5.2, we review Kisin's construction of $\mathcal{O}$-modules starting from Fontaine's filtered (φ, N)-modules. In §5.3, we prove the main result of this section, namely the coincidence of monodromy operators.

We first recall some ring notations commonly used in [14] and [32]. These notations, unlike the systematic $\mathbf{A}, \mathbf{B}$-notations in §2, are *ad hoc*, but convenient.

Notation 5.0.1. (1) Let $\mathfrak{S} := \mathbf{A}^+_{K_\infty}$.

(2) Let $\mathcal{O}_{\mathcal{E}} := \mathbf{A}_{K_\infty}$ and $\mathcal{O}^\dagger_{\mathcal{E}} := \mathbf{A}^\dagger_{K_\infty} := \mathbf{A}_{K_\infty} \cap \mathbf{B}^\dagger_{K_\infty}$.

(3) Let $\mathcal{O} := \mathbf{B}^{[0,+\infty)}_{K_\infty}$ (denoted as $\mathbf{B}^+_{\mathrm{rig},K_\infty}$ in Def. 2.5.3; also denoted as $\mathcal{O}^{[0,1)}$ in [32]). Explicitly,

$$\mathcal{O} = \left\{ f(u) = \sum_{i=0}^{+\infty} a_i u^i : a_i \in K_0 \text{ and } f(u) \text{ converges}, \forall u \in \mathfrak{m}_{\mathcal{O}_{\overline{K}}} \right\}$$

where $\mathfrak{m}_{\mathcal{O}_{\overline{K}}}$ is the maximal ideal in $\mathcal{O}_{\overline{K}}$; i.e., $\mathcal{O}$ consists of the series that converge on the entire open unit disk. Recall that N_∇ is the operator $u\lambda \frac{d}{du}$ on $\mathcal{O}$.

(4) Let $\mathcal{R}$ be the Robba ring as in [32, §1.3], which is precisely $\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}$ in our Def. 2.5.3; i.e., it consists of the series that converge near the boundary of the open unit disk. Note that

$$\mathcal{O} = \mathbf{B}^{[0,+\infty)}_{K_\infty} \subset \mathbf{B}^\dagger_{\mathrm{rig},K_\infty} = \mathcal{R}.$$

Convention 5.0.2. From now on, we focus on modules of *non-negative* heights, and Galois representations of *non-negative* (Hodge–Tate) weights (as we use covariant functors throughout this paper). Hence, although the notations such as $\mathrm{Mod}^{\varphi,\geq 0}_{\mathfrak{S}}$, $\mathrm{MF}^{\varphi,N,\geq 0}_{K_0}$ etc. in the following might be more rigorous, we use $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$, $\mathrm{MF}^{\varphi,N}_{K_0}$ etc. for short.

5.1. Finite height modules and overconvergent modules

In this subsection, we review a result of Cherbonnier, which says that a finite free étale φ-module (see Def. 3.1.1) always contains a maximal finite free overconvergent submodule (possibly of a smaller rank). If the étale φ-module is of finite height, we show that the $\mathfrak{S}$-module inside it generates the maximal overconvergent submodule.

Recall $\mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$ is the category of finite free étale φ-modules (see Def. 3.1.1). Define $\mathrm{Mod}^{\varphi}_{\mathcal{O}^\dagger_{\mathcal{E}}}$ analogously; indeed, it consists of finite free $\mathcal{O}^\dagger_{\mathcal{E}}$-modules M equipped with a $\varphi_{\mathcal{O}^\dagger_{\mathcal{E}}}$-semilinear $\varphi : M \to M$ such that $1 \otimes \varphi : \varphi^* M \to M$ is an isomorphism.

Definition 5.1.1. (1) Let

$$j^{\dagger *} : \mathrm{Mod}^{\varphi}_{\mathcal{O}^\dagger_{\mathcal{E}}} \to \mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$$

denote the functor $\mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{O}^\dagger_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}}$.

(2) Let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$. Let $F_{\dagger}(\mathcal{M})$ be the set of $\mathcal{O}^{\dagger}_{\mathcal{E}}$-submodules $\mathcal{N} \subset \mathcal{M}$ of finite type such that $\varphi(\mathcal{N}) \subset \mathcal{N}$. Let $j^{\dagger}_{*}(\mathcal{M})$ be the union of all elements in $F_{\dagger}(\mathcal{M})$.

Proposition 5.1.2. *Let* $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$.

(1) *We have* $j^{\dagger}_{*}(\mathcal{M}) \in \mathrm{Mod}^{\varphi}_{\mathcal{O}^{\dagger}_{\mathcal{E}}}$, *and hence* $j^{\dagger}_{*}$ *defines a functor*

$$j^{\dagger}_{*} : \mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}} \to \mathrm{Mod}^{\varphi}_{\mathcal{O}^{\dagger}_{\mathcal{E}}}.$$

Furthermore,

$$\mathrm{rk}_{\mathcal{O}^{\dagger}_{\mathcal{E}}} j^{\dagger}_{*}(\mathcal{M}) \le \mathrm{rk}_{\mathcal{O}_{\mathcal{E}}} \mathcal{M}, \quad j^{\dagger}_{*} \circ j^{\dagger *} \simeq \mathrm{id}.$$

(2) *The functor* $j^{\dagger}_{*}$ *is a right adjoint of* $j^{\dagger *}$*: if* $\mathcal{N}_1 \in \mathrm{Mod}^{\varphi}_{\mathcal{O}^{\dagger}_{\mathcal{E}}}$, $\mathcal{N}_2 \in \mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$, *then*

$$\mathrm{Hom}(\mathcal{N}_1, j^{\dagger}_{*}(\mathcal{N}_2)) = \mathrm{Hom}(j^{\dagger *}(\mathcal{N}_1), \mathcal{N}_2)$$

where Hom *denotes the set of morphisms in each category.*

Proof. All is contained in [14, §3.2, Prop. 2] except the claim that $j^{\dagger}_{*} \circ j^{\dagger *} \simeq \mathrm{id}$, which follows from the fact that $\mathcal{O}^{\dagger}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ is faithfully flat (as noted in [14, beginning of §3.2]). Let us mention that the "φ-operator" in [14, §3.2, Prop. 2] can be *any lift* of the Frobenius $x \mapsto x^p$ on the ring $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}}$ [14, beginning of §2.2]; hence [14, §3.2, Prop. 2] applies (to the different φ-actions) in both the (φ, Γ)-module setting and the (φ, τ)-module setting. ∎

Definition 5.1.3. Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ be the category consisting of $(\mathfrak{M}, \varphi)$ where $\mathfrak{M}$ is a finite free $\mathfrak{S}$-module and $\varphi : \mathfrak{M} \to \mathfrak{M}$ is a $\varphi_{\mathfrak{S}}$-semilinear map such that the $\mathfrak{S}$-linear span of $\varphi(\mathfrak{M})$ contains $E(u)^h\mathfrak{M}$ for some $h \ge 0$. We say that $\mathfrak{M}$ is of $E(u)$*-height* $\le h$. When $\mathfrak{M}$ is of rank d, then

$$T_{\mathfrak{S}}(\mathfrak{M}) := (\mathfrak{M} \otimes_{\mathfrak{S}} \widetilde{\mathbf{A}})^{\varphi=1}$$

is a finite free $\mathbb{Z}_p$-representation of G_∞ of rank d.

Definition 5.1.4. Let $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$. We say T is *of finite* $E(u)$*-height with respect to* $\vec{\pi} = \{\pi_n\}_{n \ge 0}$ if there exists some (hence by [32, Prop. 2.1.12], unique up to isomorphism) $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$ is of finite $E(u)$-height with respect to $\vec{\pi}$ if there exist some G_∞-stable $\mathbb{Z}_p$-lattice T (equivalently, any G_∞-stable lattice by [32, Lem. 2.1.15]) which is so. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is of finite $E(u)$-height with respect to $\vec{\pi}$ if $V|_{G_\infty}$ is so. Throughout the paper, when $E(u)$ and $\vec{\pi}$ are unambiguous, we just say *of finite height* for short.

Remark 5.1.5. Let $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$, and let $\mathcal{M} \in \mathrm{Mod}^{\varphi}_{\mathcal{O}_{\mathcal{E}}}$ be the associated étale φ-module. Then T is of finite height if and only if there is an $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ such that there is a φ-equivariant isomorphism $\mathbf{A}_{K_\infty} \otimes_{\mathfrak{S}} \mathfrak{M} \simeq \mathcal{M}$.

Proposition 5.1.6. *Suppose* $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ *is of finite height, and let* $\mathcal{M}, \mathfrak{M}$ *be as in Rem.* 5.1.5. *Then* $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}^{\dagger}_{\mathcal{E}} \simeq j^{\dagger}_{*}(\mathcal{M})$.

Proof. Clearly $\mathfrak{M} \subset j_*^\dagger(\mathcal{M})$ by Def. 5.1.1. By Prop. 5.1.2 (2),

$$\mathrm{Hom}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}^\dagger, j_*^\dagger(\mathcal{M})) = \mathrm{Hom}(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}, \mathcal{M}). \tag{5.1.1}$$

Using the fact that

$$\mathcal{O}_{\mathcal{E}}^\dagger \cap (\mathcal{O}_{\mathcal{E}})^\times = (\mathcal{O}_{\mathcal{E}}^\dagger)^\times, \tag{5.1.2}$$

one can check that the morphism on the left hand side of (5.1.1) corresponding to the isomorphism on the right hand side has to be an isomorphism itself. See e.g. [14, paragraph above §3.1, Def. 5] for a proof of (5.1.2). ∎

5.2. *Filtered (φ, N)-modules and Kisin's $\mathcal{O}$-modules*

Definition 5.2.1. Let $\mathrm{MF}_{K_0}^{\varphi,N}$ be the category of (*effective*) *filtered* (φ, N)*-modules over* K_0 which consists of finite-dimensional K_0-vector spaces D equipped with

(1) a Frobenius $\varphi : D \to D$ such that $\varphi(ax) = \varphi(a)\varphi(x)$ for all $a \in K_0, x \in D$;

(2) a monodromy $N : D \to D$, which is a K_0-linear map such that $N\varphi = p\varphi N$;

(3) a filtration $(\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}}$ on $D_K = D \otimes_{K_0} K$ by decreasing K-vector subspaces such that $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

Let $\mathrm{MF}_{K_0}^{\varphi,N,\mathrm{wa}}$ denote the usual subcategory of $\mathrm{MF}_{K_0}^{\varphi,N}$ consisting of *weakly admissible* objects.

Definition 5.2.2. Let $\mathrm{Mod}_{\mathcal{O}}^{\varphi,N_\nabla}$ be the category of finite free $\mathcal{O}$-modules M equipped with

(1) a $\varphi_{\mathcal{O}}$-semilinear morphism $\varphi : M \to M$ such that the cokernel of $1 \otimes \varphi : \varphi^* M \to M$ is killed by $E(u)^h$ for some $h \in \mathbb{Z}^{\geq 0}$;

(2) $N_\nabla : M \to M$ is a map such that $N_\nabla(fm) = N_\nabla(f)m + fN_\nabla(m)$ for all $f \in \mathcal{O}$ and $m \in M$, and $N_\nabla\varphi = \frac{pE(u)}{E(0)}\varphi N_\nabla$.

5.2.3. With $D \in \mathrm{MF}_{K_0}^{\varphi,N}$, we can associate an object $M \in \mathrm{Mod}_{\mathcal{O}}^{\varphi,N_\nabla}$ by [32]. The construction is rather complicated, and we only give a very brief sketch. (We want to give a sketch here, since we use it in the construction of the N_∇-operator in this section.)

For $n \geq 0$, let $K_{n+1} = K(\pi_n)$ (hence $K_1 = K$), and let $\widehat{\mathfrak{S}}_n$ be the completion of $K_{n+1} \otimes_{W(k)} \mathfrak{S}$ at the maximal ideal $(u - \pi_n)$; $\widehat{\mathfrak{S}}_n$ is equipped with its $(u - \pi_n)$-adic filtration, which extends to a filtration on the quotient field $\widehat{\mathfrak{S}}_n[1/(u - \pi_n)]$.

There is a natural K_0-linear map $\mathcal{O} \to \widehat{\mathfrak{S}}_n$ determined simply by sending u to u. Recall that we have maps $\varphi : \mathfrak{S} \to \mathfrak{S}$ and $\varphi : \mathcal{O} \to \mathcal{O}$ which extend the absolute Frobenius on $W(k)$ and send u to u^p. Consider $\varphi_W : \mathfrak{S} \to \mathfrak{S}$ and $\varphi_W : \mathcal{O} \to \mathcal{O}$ which act as absolute Frobenius on $W(k)$ and send u to u.

Let $\ell_u = \log u$ as in Def. 2.7.6. We can extend the map $\mathcal{O} \to \widehat{\mathfrak{S}}_n$ to $\mathcal{O}[\ell_u] \to \widehat{\mathfrak{S}}_n$ which sends ℓ_u to

$$\sum_{i=1}^{\infty} (-1)^{i-1} i^{-1} \left(\frac{u - \pi_n}{\pi_n}\right)^i \in \widehat{\mathfrak{S}}_n.$$

Note that $\mathcal{O}[\ell_u]$ is precisely the $\mathbf{B}^+_{\log,K_\infty}$ in Def. 2.7.6; we use the explicit notation $\mathcal{O}[\ell_u]$ for brevity and for easier comparison with Kisin's exposition. By the constructions in §2.7 and Lem. 4.1.2, we can naturally extend φ to $\mathcal{O}[\ell_u]$ by setting $\varphi(\ell_u) = p\ell_u$, and extend N_∇ to $\mathcal{O}[\ell_u]$ by setting $N_\nabla(\ell_u) = \lambda$ (which, by Rem. 4.1.3, differs from Kisin's convention by a minus sign). Finally, write N for the derivation on $\mathcal{O}[\ell_u]$ which acts as $\mathcal{O}$-derivation with respect to the formal variable ℓ_u, i.e., $N(\ell_u) = 1$ (see Rem. 2.7.3 for this convention).

Given $D \in \mathrm{MF}^{\varphi,N}_{K_0}$, write ι_n for the composite map

$$\mathcal{O}[\ell_u] \otimes_{K_0} D \xrightarrow{\varphi_W^{-n}\otimes\varphi^{-n}} \mathcal{O}[\ell_u] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n \otimes_{K_0} D = \widehat{\mathfrak{S}}_n \otimes_K D_K \tag{5.2.1}$$

where the second map is induced from $\mathcal{O}[\ell_u] \to \widehat{\mathfrak{S}}_n$. The composite map extends to

$$\iota_n : \mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D \to \widehat{\mathfrak{S}}_n[1/(u-\pi_n)] \otimes_K D_K. \tag{5.2.2}$$

Now, set

$$M(D) := \left\{x \in (\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D)^{N=0} : \iota_n(x) \in \mathrm{Fil}^0\left(\widehat{\mathfrak{S}}_n[1/(u-\pi_n)] \otimes_K D_K\right), \forall n \geq 1\right\}, \tag{5.2.3}$$

where Fil^0 comes from the tensor product of two filtrations. Then Kisin shows that $M(D)$ is in fact a finite free $\mathcal{O}$-module. The map $\varphi \otimes \varphi$ on $\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D$ induces a map φ on $M(D)$; the map $N_\nabla \otimes 1$ on $\mathcal{O}[\ell_u, 1/\lambda] \otimes_{K_0} D$ induces a map N_∇ on $M(D)$. Kisin shows that this makes $M(D)$ into an object in $\mathrm{Mod}^{\varphi,N_\nabla}_{\mathcal{O}}$.

Conversely, let $M \in \mathrm{Mod}^{\varphi,N_\nabla}_{\mathcal{O}}$. Then one can define $D(M) := M/uM$ with the induced φ, N-structures (where $N := N_\nabla/uN_\nabla$); using a certain *unique* φ-equivariant *section* $\xi : D(M) \to M$ as in [32, Lem. 1.2.6], one can also define a filtration on $D(M) \otimes_{K_0} K$. This gives rise to an object in $\mathrm{MF}^{\varphi,N}_{K_0}$.

Theorem 5.2.4 ([32, Thm. 1.2.15]). *The constructions in* §5.2.3 *induce an equivalence between* $\mathrm{MF}^{\varphi,N}_{K_0}$ *and* $\mathrm{Mod}^{\varphi,N_\nabla}_{\mathcal{O}}$.

Let $\mathrm{Mod}^{\varphi,N_\nabla,0}_{\mathcal{O}}$ be the subcategory of $\mathrm{Mod}^{\varphi,N_\nabla}_{\mathcal{O}}$ consisting of objects M such that $\mathcal{R} \otimes_{\mathcal{O}} M$ is *pure of slope* 0 in the sense of Kedlaya (see [30, 31] or [32, §1.3]).

Theorem 5.2.5 ([32, Thm. 1.3.8]). *The equivalence in Thm.* 5.2.4 *induces an equivalence between* $\mathrm{MF}^{\varphi,N,\mathrm{wa}}_{K_0}$ *and* $\mathrm{Mod}^{\varphi,N_\nabla,0}_{\mathcal{O}}$.

Let $\mathrm{Mod}^{\varphi,N}_{\mathfrak{S}}$ be the category where an object is an $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ together with a K_0-linear map $N : \mathfrak{M}/u\mathfrak{M}[1/p] \to \mathfrak{M}/u\mathfrak{M}[1/p]$ such that $N\varphi = p\varphi N$ over $\mathfrak{M}/u\mathfrak{M}[1/p]$. Let $\mathrm{Mod}^{\varphi,N}_{\mathfrak{S}} \otimes \mathbb{Q}_p$ be its isogeny category.

Theorem 5.2.6. *There exists a fully faithful* $\otimes$*-functor from* $\mathrm{MF}^{\varphi,N,\mathrm{wa}}_{K_0}$ *to* $\mathrm{Mod}^{\varphi,N}_{\mathfrak{S}} \otimes \mathbb{Q}_p$. *Furthermore, suppose* $D \in \mathrm{MF}^{\varphi,N,\mathrm{wa}}_{K_0}$ *maps to* $(\mathfrak{M}, \varphi, N)$. *Then*

(1) *there is a* φ*-equivariant isomorphism*

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O} \simeq M(D); \tag{5.2.4}$$

(2) *there is a canonical G_∞-equivariant isomorphism*

$$T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq V_{\mathrm{st}}(D)|_{G_\infty} \tag{5.2.5}$$

where $V_{\mathrm{st}}(D)$ is the usual covariant *Fontaine functor.*

Proof. Item (1) follows from [32, Lem. 1.3.13, Cor. 1.3.15], and (2) is the covariant version of [32, Prop. 2.1.5]. ■

5.3. Coincidence of monodromy operators

In this subsection, we show that the monodromy operators in Kisin's construction and in our construction *coincide* in the case of semi-stable representations.

Lemma 5.3.1. *Let $D \in \mathrm{MF}_{K_0}^{\varphi,N}$. There is a (φ, N)-equivariant isomorphism*

$$M(D) \otimes_{\mathcal{O}} \mathcal{O}[\ell_u, 1/\lambda] \simeq D \otimes_{K_0} \mathcal{O}[\ell_u, 1/\lambda]. \tag{5.3.1}$$

Proof. Let $\mathcal{D}_0 := (\mathcal{O}[\ell_u] \otimes_{K_0} D)^{N=0}$ (also considered in [32, proof of Lem. 1.2.2]). Solving this differential equation using the fact that N_D is nilpotent (namely, after choosing a basis of D, we get an easy differential equation), we find that $\mathcal{D}_0$ is a finite free $\mathcal{O}$-module of rank d and

$$\mathcal{D}_0 \otimes_{\mathcal{O}} \mathcal{O}[\ell_u] \simeq D \otimes_{K_0} \mathcal{O}[\ell_u]. \tag{5.3.2}$$

Furthermore, by the construction in (5.2.3), we have

$$M(D) \otimes_{\mathcal{O}} \mathcal{O}[1/\lambda] \simeq \mathcal{D}_0 \otimes_{\mathcal{O}} \mathcal{O}[1/\lambda]. \tag{5.3.3}$$

Hence both sides of (5.3.1) are isomorphic to $\mathcal{D}_0 \otimes_{\mathcal{O}} \mathcal{O}[\ell_u, 1/\lambda]$. ■

We record some other comparisons between $M(D)$ and D that will be useful later.

Corollary 5.3.2. *Let $D \in \mathrm{MF}_{K_0}^{\varphi,N}$. We have (φ, N)-equivariant isomorphisms*

$$M(D) \otimes_{\mathcal{O}} \widetilde{\mathbf{B}}^+_{\log}[1/t] \simeq D \otimes_{K_0} \widetilde{\mathbf{B}}^+_{\log}[1/t]; \tag{5.3.4}$$

$$M(D) \otimes_{\mathcal{O}} \widetilde{\mathbf{B}}^\dagger_{\log}[1/t] \simeq D \otimes_{K_0} \widetilde{\mathbf{B}}^\dagger_{\log}[1/t]; \tag{5.3.5}$$

$$M(D) \otimes_{\mathcal{O}} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] \simeq D \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u]. \tag{5.3.6}$$

These isomorphisms induce (compatible) G_K-actions on the left hand sides of these equations.

Proof. The isomorphisms all follow from Lem. 5.3.1 (for (5.3.6), note that λ is a unit in $\widetilde{\mathbf{B}}^{[0,r_0/p]}$). Note that one could change the $[1/t]$ in (5.3.5) to $[1/\lambda]$ since t/λ is a unit in $\widetilde{\mathbf{B}}^\dagger_{\log}$ (see §2.6); but one cannot change $[1/t]$ in (5.3.4) to $[1/\lambda]$. They induce G_K-actions on the left hand sides of these equations, because the right hand sides of these equations are G_K-stable. ■

Proposition 5.3.3. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st},\geq 0}(G_K)$, and let $D = D_{\mathrm{st}}(V)$. Then we have a (φ, N, G_K)-equivariant isomorphism*

$$D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \simeq D^{\dagger}_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]. \tag{5.3.7}$$

Proof. By [1, Prop. 3.4, Prop. 3.5], we have

$$D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \simeq V \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]. \tag{5.3.8}$$

By our overconvergence theorem in §3, we have

$$V \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \simeq D^{\dagger}_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}. \tag{5.3.9}$$

Hence (5.3.7) holds by combining (5.3.8) and (5.3.9). ∎

Theorem 5.3.4. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st},\geq 0}(G_K)$, and let $D = D_{\mathrm{st}}(V)$.*

- *Let $D^{\dagger}_{\mathrm{rig},K_\infty}(V)$ be the rigid-overconvergent (φ,τ)-module attached to V, and let N^{la}_{∇} denote the monodromy operator defined in Thm. 4.2.1.*
- *Let $(M(D), \varphi, N^{\mathrm{Kis}}_{\nabla}) \in \mathrm{Mod}_{\mathcal{O}}^{\varphi, N_\nabla, 0}$ be the module corresponding to D constructed by Kisin. Extend $N^{\mathrm{Kis}}_{\nabla}$ to $M(D) \otimes_{\mathcal{O}} \mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}$ by $N^{\mathrm{Kis}}_{\nabla} \otimes 1 + 1 \otimes N_{\nabla, \mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}}$, which we still denote as $N^{\mathrm{Kis}}_{\nabla}$.*

Then there is a φ-equivariant isomorphism

$$M(D) \otimes_{\mathcal{O}} \mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty} \simeq D^{\dagger}_{\mathrm{rig},K_\infty}(V). \tag{5.3.10}$$

Furthermore, with respect to this isomorphism, we have $N^{\mathrm{Kis}}_{\nabla} = N^{\mathrm{la}}_{\nabla}$.

Proof. By (5.2.4), there is a φ-equivariant isomorphism $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O} \simeq M(D)$, hence it suffices to show that there is a φ-equivariant isomorphism

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathbf{B}^{\dagger}_{K_\infty} \simeq D^{\dagger}_{K_\infty}(V) \tag{5.3.11}$$

where $D^{\dagger}_{K_\infty}(V)$ is the overconvergent φ-module associated to V. The isomorphism in (5.3.11) holds by using Prop. 5.1.6 and (5.2.5).

By using Lem. 2.7.7 for ℓ_u and [28, Lem. 3.1.2 (2)] for $1/\lambda$, we have

$$\mathcal{O}[\ell_u, 1/\lambda] \subset (\widetilde{\mathbf{B}}^{\dagger}_{\log,L}[1/\lambda])^{\tau\text{-pa},\gamma=1}. \tag{5.3.12}$$

By the construction (5.2.3) and use (5.3.12) above, it is clear that

$$M(D) \subset D \otimes_{K_0} \mathcal{O}[\ell_u, 1/\lambda] \subset (D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log,L}[1/\lambda])^{\tau\text{-pa},\gamma=1}. \tag{5.3.13}$$

Hence the $N^{\mathrm{Kis}}_{\nabla}$-operator on $M(D)$, which was defined in an "algebraic" fashion below (5.2.3), is indeed induced by the "locally analytic" N_∇-operator constructed using the locally analytic τ-action on $(D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log,L}[1/\lambda])^{\tau\text{-pa}}$.

By (5.3.7), we have G_K-equivariant isomorphisms

$$D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/\lambda] \simeq D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \simeq D^{\dagger}_{\mathrm{rig},K_\infty}(V) \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]; \quad (5.3.14)$$

here, the first isomorphism follows from the fact that t/λ is a unit in $\widetilde{\mathbf{B}}^{\dagger}_{\log}$ (see §2.6). Thus the G_K-action on $D \otimes_{K_0} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/\lambda]$ is compatible with the G_K-action on the rigid-overconvergent (φ, τ)-module (which induces N_∇^{la}). Hence we must have $N_\nabla^{\mathrm{Kis}} = N_\nabla^{\mathrm{la}}$. ∎

6. Frobenius regularization and finite height representations

In this section, we use our monodromy operator to study finite $E(u)$-height representations. In §6.1, we show that the monodromy operator descends to the ring $\mathcal{O}$ for a finite $E(u)$-height representation; in §6.2, we show such representations are potentially semi-stable. The results will be used in §7 to construct the Breuil–Kisin G_K-modules.

6.1. Frobenius regularization of the monodromy operator

Proposition 6.1.1. *Suppose $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ is of finite $E(u)$-height (with respect to fixed choice of $\vec{\pi} = \{\pi_n\}_{n\geq 0}$), and let $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ be the corresponding Breuil–Kisin module. Let N_∇ be the monodromy operator constructed in Thm. 4.2.1. Then $N_\nabla(\mathfrak{M}) \subset \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$.*

We will use a "Frobenius regularization" technique to prove Prop. 6.1.1. Roughly, by Thm. 4.2.1, we already know that the coefficients of the matrix for N_∇ live near the boundary of the open unit disk; to show that they indeed live on the entire open unit disk in the finite height case, we use the Frobenius operator to extend their range of convergence: this is where we critically use the finite height condition for the Frobenius operator. Indeed, the proof relies on the following key lemma.

Lemma 6.1.2. *Let $h \in \mathbb{R}^{>0}$ and $r \in \mathbb{Z}^{>0}$. For $s > 0$, let $\widetilde{\mathbf{A}}^{[s,+\infty)}$ be the set of $x \in \widetilde{\mathbf{B}}^{[s,+\infty)}$ such that $W^{[s,s]}(x) \geq 0$, and let $\mathbf{A}^{[s,+\infty)} := \widetilde{\mathbf{A}}^{[s,+\infty)} \cap \mathbf{B}^{[s,+\infty)}$. Then*

$$\bigcap_{n\geq 0} p^{-hn}\widetilde{\mathbf{A}}^{[r/p^n,+\infty]} = \widetilde{\mathbf{A}}^{[0,+\infty]} = \widetilde{\mathbf{A}}^{+}, \quad (6.1.1)$$

$$\bigcap_{n\geq 0} p^{-hn}\widetilde{\mathbf{A}}^{[r/p^n,+\infty)} \subset \widetilde{\mathbf{B}}^{[0,+\infty)}, \quad (6.1.2)$$

$$\bigcap_{n\geq 0} p^{-hn}\mathbf{A}_{K_\infty}^{[r/p^n,+\infty]} = \mathbf{A}_{K_\infty}^{[0,+\infty]} = \mathfrak{S}, \quad (6.1.3)$$

$$\bigcap_{n\geq 0} p^{-hn}\mathbf{A}_{K_\infty}^{[r/p^n,+\infty)} \subset \mathbf{B}_{K_\infty}^{[0,+\infty)} = \mathcal{O}. \quad (6.1.4)$$

Proof. Relations (6.1.1) and (6.1.2) come from [1, Lem. 3.1]. We can intersect $\mathbf{B}_{K_\infty}$ with (6.1.1) to get (6.1.3). Finally, (6.1.4) follows from a similar argument to that for

[1, Lem. 3.1]. Indeed, suppose $x \in$ LHS. Then for each $n \geq 0$, we can write $x = a_n + b_n$ with $a_n \in p^{-hn}\mathbf{A}_{K_\infty}^{[r/p^n,+\infty]}$ and $b_n \in \mathbf{B}_{K_\infty}^{[0,+\infty)}$. Then

$$a_n - a_{n+1} \subset (p^{-h(n+1)}\mathbf{A}_{K_\infty}^{[r/p^n,+\infty]}) \cap \mathbf{B}_{K_\infty}^{[0,+\infty)} = p^{-h(n+1)}\mathbf{A}_{K_\infty}^{[0,+\infty]} \subset \mathbf{B}_{K_\infty}^{[0,+\infty)} = \mathcal{O}.$$

By modifying a_{n+1}, we can assume $a_n = a$ for all $n \geq 0$, and hence $a \in \mathfrak{S}$ by (6.1.3). Thus, $x = a + b_0 \in \mathcal{O}$. ■

Remark 6.1.3. We use an example to illustrate the idea of (6.1.4). Consider the element $1/u \in \bigcap_{n\geq 0}\mathbf{B}_{K_\infty}^{[r/p^n,+\infty)}$. It does not belong to the left hand side of (6.1.4) because the valuations $W^{[r/p^n,r/p^n]}(1/u)$ converge to $-\infty$ at an exponential rate, rather than the linear rate on the left hand side of (6.1.4).

By (6.1.4), the proof of Prop. 6.1.1 will rely on some calculations of valuations and ranges of convergence; we first give two lemmas. For the reader's convenience, recall that for $x = \sum_{i\geq i_0} p^i[x_i] \in \widetilde{\mathbf{B}}^{[r,+\infty]}$ and for $s \geq r, s > 0$ we have the formula

$$W^{[s,s]}(x) := \inf_{k\geq i_0}\left\{k + \frac{p-1}{ps}\cdot v_{\widetilde{\mathbf{E}}}(x_k)\right\} = \inf_{k\geq i_0}\left\{k + \frac{p-1}{ps}\cdot w_k(x)\right\}. \tag{6.1.5}$$

Lemma 6.1.4. (1) *Suppose $x \in \widetilde{\mathbf{B}}^+$, then*

$$W^{[r,r]}(x) \geq W^{[s,s]}(x), \quad \forall 0 < r < s < +\infty, \tag{6.1.6}$$

$$W^{[s,s]}(\varphi(x)) \geq W^{[s,s]}(x), \quad \forall s \in (0,+\infty). \tag{6.1.7}$$

(2) *We have*

$$W^{[s,s]}(E(u)) \in (0,1], \quad \forall s \in (0,+\infty).$$

(3) *Recall $\lambda = \prod_{n\geq 0}\varphi^n(\frac{E(u)}{E(0)})$. Let $\ell \in \mathbb{Z}^{\geq 0}$. Then*

$$W^{[s,s]}(\lambda) \geq -\ell, \quad \forall s \in (0, r_\ell].$$

Proof. Item (1) follows from the definitions.

For (2), first recall that $W^{[s,s]}$ is multiplicative. Note that $E(u) = \sum_{i=0}^{e} a_i u^i$ where $a_e = 1$, $p \mid a_i$ for $i > 0$ and $p \parallel a_0$. It is easy to see that $W^{[s,s]}(E(u)) > 0$ for all $s > 0$. When $s \leq r_0$, $W^{[s,s]}(u^e + a_0) = 1$ by (6.1.5), $W^{[s,s]}(\sum_{i=1}^{e-1} a_i u^i) > 1$, and hence $W^{[s,s]}(E(u)) = 1$. Hence $W^{[s,s]}(E(u)) \leq 1$ if $s > r_0$ by (1).

For (3), we have $W^{[s,s]}(\lambda) = \sum_{i\geq 0} W^{[s,s]}(\varphi^i(E(u)/E(0)))$. Using (1), it suffices to treat the case $s = r_\ell$. We have

$$\begin{aligned} W^{[r_\ell,r_\ell]}(\lambda) &= W^{[r_\ell,r_\ell]}(\varphi^\ell(\lambda)) + W^{[r_\ell,r_\ell]}\left(\prod_{i=0}^{\ell-1}\varphi^i\left(\frac{E(u)}{E(0)}\right)\right) \\ &\geq W^{[r_0,r_0]}(\lambda) + W^{[r_\ell,r_\ell]}\left(\prod_{i=0}^{\ell-1}\varphi^i\left(\frac{1}{E(0)}\right)\right) \\ &\geq -\ell \end{aligned}$$

where the last row uses $W^{[r_0,r_0]}(\lambda) > 0$ (note $W^{[r_0,r_0]}(\frac{E(u)}{E(0)}) = 0$ and apply (6.1.7)). ■

Lemma 6.1.5. *Suppose $x \in \mathbf{B}^I_{K_\infty}$. Let $g(u) \in K_0[u]$ be an irreducible polynomial such that $g(u) \notin K_0[u^p] = \varphi(K_0[u])$. Suppose $g(u)^h\varphi(x) \in \mathbf{B}^I_{K_\infty}$ for some $h \geq 1$. Then $x \in \mathbf{B}^{I/p}_{K_\infty}$.*

Proof. We can suppose $\inf I \neq 0$ (otherwise the lemma is trivial). Then we can further suppose $g(u) \neq au$ for some $a \in K_0$ (otherwise the lemma follows easily from Cor. 2.4.5). First we treat the case $h = 1$. Suppose $g = \sum_{i=0}^N b_i u^i$. Let $g_1 := \sum_{p|i} b_i u^i$ (i.e., g_1 contains all the u-powers with p-divisible exponents, including the non-zero constant term) and let $g_2 := \sum_{p\nmid i} b_i u^i$; hence $g_2 \neq 0$ but $g_2 \neq g$. Recall that if we write $x = \sum_{i\in\mathbb{Z}} a_i u^i$ with $a_i \in K_0$, then it satisfies certain convergence conditions with respect to I as described in Lem. 2.4.3; similarly, the expansion of the product $g(u)\varphi(x) = (g_1 + g_2) \cdot (\sum_{i\in\mathbb{Z}} \varphi(a_i)u^{pi})$ satisfies these conditions. However, there is no "intersection" between the two parts $g_1\varphi(x)$ and $g_2\varphi(x)$ as the former part contains exactly all the u-powers with p-divisible exponents. Hence *both* $g_1\varphi(x)$ and $g_2\varphi(x)$ satisfy the aforementioned convergence conditions, and hence *both* $g_1\varphi(x), g_2\varphi(x) \in \mathbf{B}^I_{K_\infty}$. Since g and g_2 are coprime in $K_0[u]$, we must have $\varphi(x) \in \mathbf{B}^I_{K_\infty}$. Thus $x \in \mathbf{B}^{I/p}_{K_\infty}$ by Cor. 2.4.5.

Suppose now $h \geq 2$, and write $g^h = g_1 + g_2$ as above. Similarly we have $g_2\varphi(x) \in \mathbf{B}^I_{K_\infty}$. If $(g^h, g_2) = g^k$ with $k < h$, then $g^k\varphi(x) \in \mathbf{B}^I_{K_\infty}$. This reduces the proof to an induction argument. ∎

Remark 6.1.6. The condition $g(u) \notin K_0[u^p]$ in Lem. 6.1.5 is necessary. For example, suppose $e = p$, let $g(u) = u^p - p$, and let $x = $ "$\frac{1}{u-p}$" $:= \frac{1}{u}(\sum_{i\geq 0}(\frac{p}{u})^i) \in \mathbf{B}^{[r_0,+\infty)}_{K_\infty}$. We have $g(u)\varphi(x) = 1$, but $x \notin \mathbf{B}^{[r_0/p,+\infty)}_{K_\infty}$.

Proof of Proposition 6.1.1. Let $\vec{e}$ be an $\mathfrak{S}$-basis of $\mathfrak{M}$, and suppose

$$\varphi(\vec{e}) = \vec{e}A, \quad N_\nabla(\vec{e}) = \vec{e}M \quad \text{with} \quad A \in \mathrm{Mat}(\mathfrak{S}), M \in \mathrm{Mat}(\mathbf{B}^\dagger_{\mathrm{rig},K_\infty}).$$

Since $N_\nabla\varphi = \frac{p}{E(0)}E(u)\varphi N_\nabla$, we have

$$MA + N_\nabla(A) = \frac{p}{E(0)}E(u)A\varphi(M). \tag{6.1.8}$$

Let $B \in \mathrm{Mat}(\mathfrak{S})$ be such that $AB = E(u)^h \cdot \mathrm{Id}$. Then we get

$$BMA + BN_\nabla(A) = \frac{p}{E(0)}E(u)^{h+1}\varphi(M). \tag{6.1.9}$$

Suppose $N \geq 0$ is maximal such that $E(u) \in K_0[u^{p^N}]$. Denote

$$D_i(u) := \prod_{k=1}^{i} \varphi^{-k}(E(u)), \quad \forall 0 \leq i \leq N,$$

where we always let $D_0(u) := 1$. For brevity, we denote $E := E(u)$ and $D_i := D_i(u)$.

Let

$$\widetilde{M} := D_N^{h+1}M.$$

Then from (6.1.9), we have

$$B\widetilde{M}A + D_N^{h+1}BN_\nabla(A) = \frac{p}{E(0)}\varphi^{-N}(E^{h+1})\varphi(\widetilde{M}). \tag{6.1.10}$$

Let $\ell \gg 0$ be such that

$$\widetilde{M} \in \mathrm{Mat}(\mathbf{B}_{K_\infty}^{[r_\ell,+\infty)}).$$

Note that $A, B, D_i \in \mathrm{Mat}(\mathfrak{S})$ and $N_\nabla(A) \in \lambda \cdot \mathrm{Mat}(\mathfrak{S})$, hence

$$\text{LHS of (6.1.10)} \in \mathrm{Mat}(\mathbf{B}_{K_\infty}^{[r_\ell,+\infty)}).$$

Since $\varphi^{-N}(E)$ satisfies the conditions in Lem. 6.1.5, we can iteratively use the lemma and (6.1.10) to conclude that

$$\widetilde{M} \in \mathrm{Mat}\Big(\bigcap_{n\geq 0} \mathbf{B}_{K_\infty}^{[r_\ell/p^n,+\infty)}\Big). \tag{6.1.11}$$

To show that $M \in \mathrm{Mat}(\mathcal{O})$, we proceed in two steps: we first show that $\widetilde{M} \in \mathrm{Mat}(\mathcal{O})$ using Lem. 6.1.2; then we show that $M \in \mathrm{Mat}(\mathcal{O})$ using a trick of Caruso.

Step 1. Write $r = r_\ell$. Choose $c \gg 0$ (in particular, we ask that $c > \ell$) such that

$$\widetilde{M} \in \mathrm{Mat}(p^{-c}\mathbf{A}_{K_\infty}^{[r,+\infty)}).$$

Then we have

$$\begin{aligned} W^{[r,r]}(\text{LHS of (6.1.10)}) &\geq \min\{W^{[r,r]}(\widetilde{M}), W^{[r,r]}(\lambda)\} && \text{(since } W^{[r,r]} \text{ is multiplicative)}\\ &\geq \min\{-c,-\ell\} && \text{(by Lem. 6.1.4 (3))}\\ &= -c. \end{aligned}$$

By Lem. 6.1.4 (2), we have

$$W^{[r,r]}(\varphi^{-N}(E)) = W^{[p^N r, p^N r]}(E) \leq 1.$$

Hence using (6.1.10), we have

$$W^{[r/p,r/p]}(\widetilde{M}) = W^{[r,r]}(\varphi(\widetilde{M})) \geq -c-(h+1).$$

Iterating the above argument, we find that for all n,

$$W^{[r/p^n,r/p^n]}(\widetilde{M}) \geq -c-n(h+1).$$

By Lem. 2.3.6, we have

$$\widetilde{M} \in \mathrm{Mat}(p^{-c-n(h+1)}\mathbf{A}_{K_\infty}^{[r/p^n,+\infty)}), \quad \forall n.$$

Using (6.1.4) of Lem. 6.1.2, we conclude that $\widetilde{M} \in \mathrm{Mat}(\mathcal{O})$.

Step 2. In this step, we show that $M \in \mathrm{Mat}(\mathcal{O})$. If $N = 0$, then there is nothing to prove. Suppose now $N \geq 1$. Then

$$\widetilde{M} = D_N^{h+1} \cdot M = \varphi^{-N}(E^{h+1}) \cdot D_{N-1}^{h+1} \cdot M \in \mathrm{Mat}(\mathcal{O}). \quad (6.1.12)$$

From (6.1.8), we also have

$$ME^h + N_\nabla(A)B = \frac{p}{E(0)} EA\varphi(M)B,$$

and hence

$$E^{h+1} D_{N-1}^{h+1} \cdot (ME^h + N_\nabla(A)B) = \frac{p}{E(0)} EA\varphi(\widetilde{M})B.$$

So we have

$$E^{h+1} D_{N-1}^{h+1} \cdot ME^h = E^{2h+1} \cdot D_{N-1}^{h+1} \cdot M \in \mathrm{Mat}(\mathcal{O}). \quad (6.1.13)$$

Note that both $\varphi^{-N}(E)$ and E are irreducible in $K_0[u]$ and hence they are coprime. Hence we can use (6.1.12) and (6.1.13) to conclude that

$$D_{N-1}^{h+1} \cdot M \in \mathrm{Mat}(\mathcal{O}).$$

If $N - 1 \geq 1$, we can repeat the above argument. (Note that in the argument of this Step 2, we do not use the fact $\varphi^{-N}(E) \notin K_0[u^p]$; indeed, this condition is used only to deduce (6.1.11)). Hence in the end we must have $M \in \mathrm{Mat}(\mathcal{O})$. ■

Remark 6.1.7. The argument in Step 2 above is taken from [12, p. 2595, paragraph containing (3.15)]; in particular, the use of $D_N(u)$ is inspired by the argument in *loc. cit.* However, the arguments before Step 2 are completely different from those in [12].

6.2. *Potential semistability of finite height representations*

In this subsection, we show that finite height representations are potentially semi-stable; in fact, our result is more precise and stronger. Let us first recall two useful lemmas.

For any $K \subset X \subset \overline{K}$, let

$$m(X) := 1 + \max\{i \geq 1 : \mu_i \in X\}.$$

Recall for each $n \geq 1$, we let $K_n = K(\pi_{n-1})$ (hence $K_1 = K$). Note that

- (for $n \geq 2$) K_n here is K_n in [32], but K_{n-1} in [37, 39].

Lemma 6.2.1. *Let $m := m(K^{\mathrm{ur}})$ where K^{ur} is the maximal unramified extension of K (contained in $\overline{K}$). Suppose $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is semi-stable over K_n for some $n \geq 1$, then V is semi-stable over K_m.*

Proof. This is proved in [39, Rem. 2.5]. Note that this fixes a gap in [37, Thm. 4.2.2], where Liu claims a similar statement using $m := m(K)$ instead. Note that in general, it is possible that $m(K^{\mathrm{ur}}) > m(K)$ [39, §5.4]. Let us mention that this lemma is completely about Galois theory of the fields K_n, and has nothing to do with the $(\varphi, \widehat{G})$-modules of [39]. ■

Lemma 6.2.2. *Let $K \subset K^{(1)} \subset K^{(2)}$ be finite extensions such that $K^{(2)}/K^{(1)}$ is totally ramified. Then the restriction functor from semi-stable $G_{K^{(1)}}$-representations to semi-stable $G_{K^{(2)}}$-representations is fully faithful.*

Proof. This is [39, Lem. 4.11]. (It is basically [12, Prop. 3.4], but it is completely elementary.) ∎

The following definition comes from [39].

Definition 6.2.3. Fix a choice of $\vec{\pi}$. Let $n \geq 1$. We denote by $\mathscr{C}_n(\vec{\pi})$ the category of finite free $\mathbb{Z}_p$-representations T of G_K such that we have an G_{K_n}-equivariant isomorphism

$$T[1/p]|_{G_{K_n}} \simeq W|_{G_{K_n}}$$

for some $W \in \mathrm{Rep}^{\mathrm{st},\geq 0}_{\mathbb{Q}_p}(G_K)$; namely $T[1/p]|_{G_{K_n}}$ is semi-stable and can be extended to a semi-stable G_K-representation.

Theorem 6.2.4. *Let $\mathrm{Rep}^{E(u)\text{-ht}}_{\mathbb{Z}_p}(G_K)$ be the category of finite $E(u)$-height $\mathbb{Z}_p$-representations* (*with respect to a fixed choice of $\vec{\pi}$*). *Then*

$$\mathrm{Rep}^{E(u)\text{-ht}}_{\mathbb{Z}_p}(G_K) = \mathscr{C}_{m(K^{\mathrm{ur}})}(\vec{\pi}).$$

In particular, if $T \in \mathrm{Rep}^{E(u)\text{-ht}}_{\mathbb{Z}_p}(G_K)$, then $T[1/p]|_{G_{K_{m(K^{\mathrm{ur}})}}}$ is semi-stable.

Proof. Suppose $T \in \mathscr{C}_{m(K^{\mathrm{ur}})}(\vec{\pi})$, and let $V = T[1/p]$. Let $U \in \mathrm{Rep}^{\mathrm{st},\geq 0}_{\mathbb{Q}_p}(G_K)$ such that

$$V|_{G_{K_{m(K^{\mathrm{ur}})}}} \simeq U|_{G_{K_{m(K^{\mathrm{ur}})}}}.$$

By Kisin's result, U is of finite $E(u)$-height *with respect to* $\vec{\pi}$, hence so are $U|_{G_\infty} \simeq V|_{G_\infty}$, and hence so are V and T (by Def. 5.1.4). (Note that the semistability of $V|_{G_{K_{m(K^{\mathrm{ur}})}}}$ is *not* enough to guarantee that V is of finite height with respect to $\vec{\pi}$; it only guarantees finite height with respect to $\{\pi_n\}_{n\geq m(K^{\mathrm{ur}})}$.)

Conversely, let V be a finite height representation. By Prop. 6.1.1, we can construct a triple $(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}, \varphi, N_\nabla) \in \mathrm{Mod}^{\varphi, N_\nabla, 0}_{\mathcal{O}}$, which gives us a *semi-stable* G_K-representation W. Here $(\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}, \varphi)$ is pure of slope zero because $(\mathfrak{M} \otimes_{\mathfrak{S}} \mathbf{B}_{K_\infty}, \varphi)$ is part of the étale (φ, τ)-module associated to V.

Let $(D_{K_\infty}(V), \varphi, \widehat{G}_V)$ be the étale (φ, τ)-module associated to V, and let $D^\dagger_{K_\infty}(V)$ (resp. $D^\dagger_{\mathrm{rig},K_\infty}(V)$) be the overconvergent (resp. rigid-overconvergent) module equipped with the induced φ and $\widehat{G}_V$. Let $(D_{K_\infty}(W), \varphi, \widehat{G}_W)$, $D^\dagger_{K_\infty}(W)$, $D^\dagger_{\mathrm{rig},K_\infty}(W)$ be similarly defined. It is clear that

$$\begin{aligned} D_{K_\infty}(V) &= \mathfrak{M} \otimes_{\mathfrak{S}} \mathbf{B}_{K_\infty} = D_{K_\infty}(W), \\ D^\dagger_{K_\infty}(V) &= \mathfrak{M} \otimes_{\mathfrak{S}} \mathbf{B}^\dagger_{K_\infty} = D^\dagger_{K_\infty}(W), \\ D^\dagger_{\mathrm{rig},K_\infty}(V) &= \mathfrak{M} \otimes_{\mathfrak{S}} \mathbf{B}_{\mathrm{rig},K_\infty} = D^\dagger_{\mathrm{rig},K_\infty}(W), \end{aligned}$$

and they carry the same φ-action.

In the following, let $Q \in \{V, W\}$. By the definition of locally analytic actions, the τ_Q-action over $D^{\dagger}_{\mathrm{rig},K_\infty}(Q)$ can be "locally" recovered by

$$\nabla_{\tau,Q} = \begin{cases} p\mathfrak{t} \cdot N^{\mathrm{la}}_{\nabla,Q} & \text{if } K_\infty \cap K_{p^\infty} = K, \\ 4\mathfrak{t} \cdot N^{\mathrm{la}}_{\nabla,Q} & \text{if } K_\infty \cap K_{p^\infty} = K(\pi_1), \end{cases}$$

where $N^{\mathrm{la}}_{\nabla,Q}$ is the monodromy operator defined in Thm. 4.2.1. Indeed, let $\vec{e}$ be an $\mathfrak{S}$-basis of $\mathfrak{M}$. Then there exists some $a \gg 0$ such that

$$\tau_Q^{\alpha}(\vec{e}) = \sum_{i=0}^{\infty} \alpha^i \cdot \frac{(\nabla_{\tau,Q})^i(\vec{e})}{i!}, \quad \forall \alpha \in p^a\mathbb{Z}_p. \tag{6.2.1}$$

Note that *a priori*, the series in (6.2.1) converges to some element in $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K_\infty}} D^{\dagger}_{\mathrm{rig},K_\infty}(Q)$; however since $\vec{e}$ is also a basis for $D_{K_\infty}(Q)$ and $D^{\dagger}_{K_\infty}(Q)$, the limit has to fall in

$$\widetilde{\mathbf{B}}^{\dagger}_L \otimes_{\mathbf{B}^{\dagger}_{K_\infty}} D^{\dagger}_{K_\infty}(Q) \subset \widetilde{\mathbf{B}}_L \otimes_{\mathbf{B}_{K_\infty}} D_{K_\infty}(Q).$$

Now, $N^{\mathrm{la}}_{\nabla,V} = N_\nabla$ by construction, and $N^{\mathrm{la}}_{\nabla,W} = N_\nabla$ by Thm. 5.3.4. Thus we have

$$N^{\mathrm{la}}_{\nabla,V} = N^{\mathrm{la}}_{\nabla,W}.$$

This implies that the $\mathrm{Gal}(L/K_{p^\infty}(\pi_a))$-actions on the two (φ,τ)-modules are the same. Since $K_{p^\infty}(\pi_a) \cap K_\infty \subseteq K(\pi_{a+1})$ (possible equality only when $p = 2$, cf. Notation 1.4.3). Hence we must have

$$V|_{G_{K_{a+2}}} = W|_{G_{K_{a+2}}}.$$

We can always first choose $a \geq m(K^{\mathrm{ur}})$, and hence by Lem. 6.2.1, V is semi-stable over $K_{m(K^{\mathrm{ur}})}$. Thus by Lem. 6.2.2 (using the totally ramified extension $K_{a+2}/K_{m(K^{\mathrm{ur}})}$), we have

$$V|_{G_{K_{m(K^{\mathrm{ur}})}}} = W|_{G_{K_{m(K^{\mathrm{ur}})}}}. \qquad \blacksquare$$

Remark 6.2.5. In [12, Thm. 3], the statement there claims that $\mathrm{Rep}^{E(u)\text{-ht}}_{\mathbb{Z}_p}(G_K)$ is the same as $\mathcal{C}_{m(K)}(\vec{\pi})$. However, by our Thm. 6.2.4, and by the examples in [39, Prop. 3.22 (1)] which show that $\mathcal{C}_{m(K)}(\vec{\pi}) \neq \mathcal{C}_{m(K^{\mathrm{ur}})}(\vec{\pi})$ in general, Caruso's statement is in general false.

7. Breuil–Kisin G_K-modules

In §7.1, we construct the Breuil–Kisin G_K-modules and show that they classify integral semi-stable Galois representations. In §7.2, we discuss the relation between our theory and Liu's $(\varphi, \widehat{G})$-modules (only preliminarily here). In §7.3, we discuss the relation between our theory and some results of Gee and Liu.

7.1. *Breuil–Kisin G_K-modules*

In this subsection, we construct the Breuil–Kisin G_K-modules. Recall that in Notation 1.1.4, we defined the notations R, $W(R)$, $\mathfrak{m}_R$; the ring $W(R)$ is precisely $\widetilde{\mathbf{A}}^+$ of §2, and is also denoted by $\mathbf{A}_{\mathrm{inf}}$ in the literature. Let $\mathrm{Fr}\, R$ be the fraction field of R, and let $W(\mathrm{Fr}\, R)$ be the Witt vectors; this is precisely the $\widetilde{\mathbf{A}}$ of §2.

Definition 7.1.1. Let $\mathrm{Mod}_{\mathfrak{S},W(R)}^{\varphi,G_K}$ be the category of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K)$, which we call (effective) *Breuil–Kisin G_K-modules*, where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi}$;

(2) G_K is a continuous $W(R)$-semilinear G_K-action on $\widehat{\mathfrak{M}} := W(R) \otimes_{\mathfrak{S}} \mathfrak{M}$;

(3) G_K commutes with $\varphi_{\widehat{\mathfrak{M}}}$ on $\widehat{\mathfrak{M}}$;

(4) $\mathfrak{M} \subset \widehat{\mathfrak{M}}^{\mathrm{Gal}(\overline{K}/K_\infty)}$ via the embedding $\mathfrak{M} \hookrightarrow \widehat{\mathfrak{M}}$;

(5) $\mathfrak{M}/u\mathfrak{M} \subset (\widehat{\mathfrak{M}}/W(\mathfrak{m}_R)\widehat{\mathfrak{M}})^{G_K}$ via the embedding $\mathfrak{M}/u\mathfrak{M} \hookrightarrow \widehat{\mathfrak{M}}/W(\mathfrak{m}_R)\widehat{\mathfrak{M}}$.

We record an equivalent condition for Def. 7.1.1 (5).

Lemma 7.1.2. *Let $(\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K)$ be a triple satisfying items (1)–(4) of Def. 7.1.1. Then the condition in Def. 7.1.1 (5) is satisfied if and only if $\widehat{\mathfrak{M}}/W(\mathfrak{m}_R)\widehat{\mathfrak{M}}$ is fixed by $G_{K^{\mathrm{ur}}}$.*

Proof. Necessity is obvious, and we prove sufficiency. If $\widehat{\mathfrak{M}}/W(\mathfrak{m}_R)\widehat{\mathfrak{M}}$ is fixed by $G_{K^{\mathrm{ur}}}$, then so is $\mathfrak{M}/u\mathfrak{M}$. But $\mathfrak{M}/u\mathfrak{M}$ is also fixed by G_{K_∞} by Def. 7.1.1 (4), hence it is fixed by G_K as $K^{\mathrm{ur}} \cap K_\infty = K$. ∎

Definition 7.1.3. Let $\mathrm{wMod}_{\mathfrak{S},W(R)}^{\varphi,G_K}$ denote the category of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K)$ satisfying items (1)–(4) of Def. 7.1.1.

With $\widehat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K)$ as in Def. 7.1.3, we can associate a $\mathbb{Z}_p[G_K]$-module:

$$T_{W(R)}(\widehat{\mathfrak{M}}) := (\widehat{\mathfrak{M}} \otimes_{W(R)} W(\mathrm{Fr}\, R))^{\varphi=1}. \tag{7.1.1}$$

Proposition 7.1.4. *Equation (7.1.1) induces a rank-preserving (i.e., $\mathrm{rk}_{\mathbb{Z}_p} T_{W(R)}(\widehat{\mathfrak{M}}) = \mathrm{rk}_{W(R)} \widehat{\mathfrak{M}}$), exact, and fully faithful functor $T_{W(R)} : \mathrm{wMod}_{\mathfrak{S},W(R)}^{\varphi,G_K} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$.*

Proof. The contravariant version of this proposition (except exactness) is proved in [37, below Rem. 3.1.5]. Note that the proof makes critical use of [37, Lem. 3.1.2, Prop. 3.1.3]; but these results have nothing to do with the ring $\widehat{\mathcal{R}}$ there (and are relatively easy). By the argument in [27, proof of Thm. 2.3.2], the $\mathbb{Z}_p$-dual of our $T_{W(R)}(\widehat{\mathfrak{M}})$ is isomorphic to $\hat{T}(\widehat{\mathfrak{M}})$ of [37]; hence the proposition (except exactness) follows.

It is shown in [32, Cor. 2.1.4] that the contravariant functor $\mathfrak{M} \mapsto \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbf{A})$ is exact (note that $\mathfrak{S}^{\mathrm{ur}}$ there is precisely our $\mathbf{A}$ in §2.1). By the discussions in [27, §2.1, 2.2, 2.3] about covariant versions of various functors of [32], one deduces that the covariant functor $\mathfrak{M} \mapsto (\mathfrak{M} \otimes_{\mathfrak{S}} W(\mathrm{Fr}\, R))^{\varphi=1}$ in Def. 5.1.3 is exact. This implies that $T_{W(R)}$ is exact. ∎

Theorem 7.1.5. *We have equivalences of categories*

$$\mathrm{wMod}^{\varphi, G_K}_{\mathfrak{S}, W(R)} \xrightarrow{T_{W(R)}} \mathrm{Rep}^{E(u)\text{-ht}}_{\mathbb{Z}_p}(G_K) \xrightarrow{=} \mathcal{C}_{m(K^{\mathrm{ur}})}(\vec{\pi}). \tag{7.1.2}$$

Proof. The last equivalence is Thm. 6.2.4. $T_{W(R)}$ is fully faithful by Prop. 7.1.4. Hence it suffices to show that $T_{W(R)}$ is essentially surjective.

Suppose $T \in \mathrm{Rep}^{E(u)\text{-ht}}_{\mathbb{Z}_p}(G_K)$, let $\mathfrak{M}$ be the Breuil–Kisin module attached to T, and let $V = T[1/p]$. By Thm. 6.2.4, $V|_{G_{K_m}}$ is semi-stable where $m = m(K^{\mathrm{ur}})$. Let E be the p-adic completion of $K^{\mathrm{ur}}(\pi_{m-1})$; then $V|_{G_{K^{\mathrm{ur}}(\pi_{m-1})}}$ is a semi-stable representation of G_E. Note that this E satisfies the assumption in Notation 2.2.1. Note furthermore that $K^{\mathrm{ur}}(\pi_{m-1})$ is Galois over K.

Since the G_E-representation is semi-stable, we can construct an $\mathfrak{S}_E$*-Breuil–Kisin module* $\mathfrak{M}_E$ (whose obvious definition is left to the reader). Recall that by Lem. 3.3.1, we have a φ-equivariant isomorphism

$$D_{K_\infty}(V) \otimes_{\mathbf{B}_{K_\infty}} \mathbf{B}_{E_\infty} \simeq D_{E_\infty}(V|_{G_E}). \tag{7.1.3}$$

We claim that

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_E \simeq \mathfrak{M}_E. \tag{7.1.4}$$

To prove the claim, note that under the map $u \mapsto u_E^{p^{m-1}}$ (see Notation 2.2.1), $E(u)$ maps to the minimal polynomial of π_{m-1} over $W(\overline{k})[1/p]$; hence $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_E$ is also an $\mathfrak{S}_E$-Breuil–Kisin module. The isomorphism (7.1.4) follows because both sides give rise to the same G_{E_∞}-representation $T|_{G_{E_\infty}}$.

Now we want to construct a G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S}} W(R)$: indeed, we will show that $\mathfrak{M} \otimes_{\mathfrak{S}} W(R)$ is G_K-stable inside $D^\dagger_{\mathrm{rig}, K_\infty}(V) \otimes \widetilde{\mathbf{B}}^\dagger_{\log}[1/t]$. By the isomorphism (7.1.4), it is natural to try to use the Galois actions related to $\mathfrak{M}_E$. Indeed, we will make use of the following commutative diagram. Here we omit all the subscripts of tensor products for brevity; the G_K's under the arrows signify G_K-equivariances.

$$\begin{array}{ccccccc} V\otimes\widetilde{\mathbf{B}}^\dagger_{\log}[1/t] & \underset{G_K}{\xrightarrow{\simeq}} & D^E_{\mathrm{st}}(V)\otimes\widetilde{\mathbf{B}}^\dagger_{\log}[1/t] & \underset{G_K}{\xrightarrow{\simeq}} & D^\dagger_{\mathrm{rig},E_\infty}(V)\otimes\widetilde{\mathbf{B}}^\dagger_{\log}[1/t] & \underset{G_K}{\xrightarrow{\simeq}} & D^\dagger_{\mathrm{rig},K_\infty}(V)\otimes\widetilde{\mathbf{B}}^\dagger_{\log}[1/t] \\ & & {\scriptstyle i}\uparrow{\scriptstyle G_K} & & \uparrow & & \uparrow \\ & & D^E_{\mathrm{st}}(V)\otimes\widetilde{\mathbf{B}}^+_{\log}[1/t] & \underset{f}{\xrightarrow{\simeq}} & \mathfrak{M}_E\otimes\widetilde{\mathbf{B}}^+_{\log}[1/t] & \underset{g}{\xrightarrow{\simeq}} & \mathfrak{M}\otimes\widetilde{\mathbf{B}}^+_{\log}[1/t] \end{array} \tag{7.1.5}$$

Let us explain the content of this diagram:

(1) Before we even define the objects and maps in the diagram, let us mention that all maps are obviously φ- and N-equivariant; we will hence focus on their Galois equivariance.

(2) We define $D^E_{\mathrm{st}}(V) := (V \otimes \mathbf{B}_{\mathrm{st}})^{G_E}$ (note that G_K acts on it since $K^{\mathrm{ur}}(\pi_{m-1})/K$ is Galois).

(3) By Prop. 5.3.3, we have G_E-equivariant isomorphisms

$$V \otimes \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \simeq D^E_{\mathrm{st}}(V) \otimes \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \simeq D^{\dagger}_{\mathrm{rig},E_\infty}(V) \otimes \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t];$$

they are furthermore G_K-equivariant by the construction of $D^E_{\mathrm{st}}(V)$ and $D^{\dagger}_{\mathrm{rig},E_\infty}(V)$.

(4) The vertical embeddings are just submodules.

(5) The isomorphisms f and g follow from Cor. 5.3.2 and (7.1.4) respectively. Since i is G_K-equivariant (namely, $\widetilde{\mathbf{B}}^{+}_{\log}[1/t]$ is G_K-stable), it follows that $\mathfrak{M}_E \otimes \widetilde{\mathbf{B}}^{+}_{\log}[1/t]$ and $\mathfrak{M} \otimes \widetilde{\mathbf{B}}^{+}_{\log}[1/t]$ are G_K-stable. Thus, the maps f, g, and hence all the vertical embeddings, are indeed G_K-equivariant. (Thus we can put G_K under *all* the arrows in the diagram.)

So now $\mathfrak{M} \otimes \widetilde{\mathbf{B}}^{+}_{\log}[1/t]$ is G_K-stable. By the overconvergence theorem, also $\mathfrak{M} \otimes \widetilde{\mathbf{B}}^{\dagger}$ is G_K-stable. We have

$$\widetilde{\mathbf{B}}^{\dagger} \cap \widetilde{\mathbf{B}}^{+}_{\log} = \widetilde{\mathbf{B}}^{\dagger} \cap \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}} \quad \text{(by taking } N=0 \text{ part)} \tag{7.1.6}$$

$$= W(R)[1/p] \quad \text{(by [1, Lem. 2.18]).} \tag{7.1.7}$$

Hence $\mathfrak{M} \otimes W(R)[1/p][1/t]$ is G_K-stable.

Now fix a basis $\vec{m}$ of $\mathfrak{M}$. Suppose $\varphi(\vec{m}) = \vec{m}A$ and let $B \in \mathrm{Mat}(\mathfrak{S})$ be such that $AB = E(u)^h$ for some $h \geq 0$. For any $g \in G_K$, suppose $g(\vec{m}) = \vec{m}M_g$ where $M_g = t^{-a}M$ with $M \in \mathrm{Mat}(W(R)[1/p])$ for some $a \geq 0$. Since φ and g commute, we have $A\varphi(M) = p^a M g(A)$, and hence

$$\varphi(M)E(u)^h = p^a BMg(A). \tag{7.1.8}$$

For a matrix X defined over $W(R)$, let $v_p(X)$ be the minimum of the p-adic valuation of all its entries. We then have $v_p(\varphi(M)E(u)^h) = v_p(M)$ and hence we must have $a = 0$. This shows that $\mathfrak{M} \otimes W(R)[1/p]$ is G_K-stable. Since $\mathfrak{M} \otimes W(\mathrm{Fr}\, R)$ is also G_K-stable, we finally see that $\mathfrak{M} \otimes W(R)$ is G_K-stable. ■

We introduce some notations before we prove our main theorem.

7.1.6. Let $\nu : W(R) \to W(R)/W(\mathfrak{m}_R) = W(\bar{k})$ be the reduction ring homomorphism. It naturally extends to $\nu : \widetilde{\mathbf{B}}^{[0,r_0/p]} \to W(\bar{k})[1/p]$, e.g., by applying the explicit expression of $\widetilde{\mathbf{B}}^{[0,r_0/p]}$ using Lem. 2.3.9. It then extends to

$$\nu : \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] \to W(\bar{k})[1/p]$$

by setting $\nu(\ell_u) = 0$. The map ν is a φ-equivariant ring homomorphism; it is furthermore G_K-equivariant by using (2.7.3) (and note $\nu(t) = 0$). For any subring $A \subset \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u]$, and a finite free A-module M, the ν-map extends to

$$\nu : M \to W(\bar{k})[1/p] \otimes_A M. \tag{7.1.9}$$

If A is G_K-stable and M is equipped with an A-semilinear G_K-action, then the map ν in (7.1.9) is G_K-equivariant.

The following is our main theorem.

Theorem 7.1.7. *The functor $T_{W(R)}$ induces an equivalence of categories*

$$\mathrm{Mod}^{\varphi,G_K}_{\mathfrak{S},W(R)} \to \mathrm{Rep}^{\mathrm{st},\geq 0}_{\mathbb{Z}_p}(G_K). \tag{7.1.10}$$

Proof. **Part 1.** We first show that if $\widehat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,G_K}_{\mathfrak{S},W(R)}$, then $V := T_{W(R)}(\widehat{\mathfrak{M}})[1/p]$ is semi-stable, and hence $T_{W(R)}$ indeed induces a functor as in (7.1.10). Note that V is of finite $E(u)$-height. Let $E, \mathfrak{S}_E, \mathfrak{M}_E$ etc. be as in the proof of Thm. 7.1.5. Similar to the bottom row of (7.1.5), we have isomorphisms (where $E_0 = W(\bar{k})[1/p]$)

$$D^E_{\mathrm{st}}(V) \otimes_{E_0} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] \simeq \mathfrak{M}_E \otimes_{\mathfrak{S}_E} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] \simeq \mathfrak{M} \otimes_{\mathfrak{S}} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u], \tag{7.1.11}$$

using Cor. 5.3.2 and (7.1.4) respectively; again, they are G_E-equivariant *a priori*, but are indeed G_K-equivariant since $D^E_{\mathrm{st}}(V) \otimes_{E_0} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] \subset D^E_{\mathrm{st}}(V) \otimes_{E_0} \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]$ is G_K-stable. Applying the G_K-equivariant map ν to (7.1.11), we get G_K-equivariant isomorphisms

$$D^E_{\mathrm{st}}(V) \simeq \mathfrak{M}_E/u_E\mathfrak{M}_E \otimes_{W(\bar{k})} W(\bar{k})[1/p] \simeq \mathfrak{M}/u\mathfrak{M} \otimes_{W(k)} W(\bar{k})[1/p]. \tag{7.1.12}$$

Hence $D^E_{\mathrm{st}}(V)$ is fixed by $G_{K^{\mathrm{ur}}}$, and hence V is a semi-stable representation of G_K.

Part 2. Note that $T_{W(R)}$ is fully faithful by Prop. 7.1.4. We now show that $T_{W(R)}$ is essentially surjective. Let $T \in \mathrm{Rep}^{\mathrm{st},\geq 0}_{\mathbb{Z}_p}(G_K)$; it is of finite height by [32]. Hence by Thm. 7.1.5, we can get a unique $(\mathfrak{M}, \varphi, G_K) \in \mathrm{wMod}^{\varphi,G_K}_{\mathfrak{S},W(R)}$. It suffices to show that $\mathfrak{M}/u\mathfrak{M}$ is fixed by G_K. By (5.3.6), we have a G_K-equivariant isomorphism

$$D_{\mathrm{st}}(T[1/p]) \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] \simeq \mathfrak{M} \otimes_{\mathfrak{S}} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u].$$

Applying the ν-map shows that $\mathfrak{M}/u\mathfrak{M} \otimes_{W(k)} W(\bar{k})[1/p]$ is fixed by $G_{K^{\mathrm{ur}}}$, hence we can conclude the proof using Lem. 7.1.2. ■

Remark 7.1.8. By Prop. 7.1.4, the functor $T_{W(R)}$ is exact. However, its quasi-inverse in Thm. 7.1.7 is in general only left exact; see the discussions in [38, Lem. 2.19, Ex. 2.21].

We now give some results related to the theory of Breuil–Kisin G_K-modules. In Prop. 7.1.10, we give a crystallinity criterion; in Prop. 7.1.11, we prove the "algebraic avatar" of the de Rham comparison (1.1.5) (see Rem. 1.1.12). We start with a lemma.

Lemma 7.1.9. *We have* $(\varphi(\mathfrak{t}) \cdot \widetilde{\mathbf{B}}^{[0,r_0]}) \cap W(R) = \varphi(\mathfrak{t}) \cdot W(R)$.

Proof. Let $I^{[1]}W(R) := \{a \in W(R) : \varphi^n(a) \in \operatorname{Ker}\theta, \forall n \geq 0\}$. By the proof of [37, Lem. 3.2.2] (using a result of Fontaine), we have

$$I^{[1]}W(R) = \varphi(\mathfrak{t}) \cdot W(R). \tag{7.1.13}$$

Note that the map $\theta \circ \iota_0 : \widetilde{\mathbf{B}}^{[0,r_0]} \to \mathbf{B}^+_{\mathrm{dR}} \to C_p$ (see [1] for ι_0) induces the θ map on $W(R)$. Thus one easily checks that $(\varphi(\mathfrak{t}) \cdot \widetilde{\mathbf{B}}^{[0,r_0]}) \cap W(R) \subset I^{[1]}W(R)$. ■

Proposition 7.1.10. *Let* $\widehat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi, G_K}_{\mathfrak{S}, W(R)}$. *Then* $V = T_{W(R)}(\widehat{\mathfrak{M}})[1/p]$ *is a crystalline representation if and only if*

$$(\tau - 1)(\mathfrak{M}) \subset \mathfrak{t} W(\mathfrak{m}_R) \otimes_{\mathfrak{S}} \mathfrak{M}. \tag{7.1.14}$$

Proof. **Step 1.** We first show that for any $\widehat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi, G_K}_{\mathfrak{S}, W(R)}$,

$$(\tau - 1)(\mathfrak{M}) \subset \mathfrak{t} W(R) \otimes_{\mathfrak{S}} \mathfrak{M}. \tag{7.1.15}$$

Indeed, by (5.3.1),

$$\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}[\ell_u, 1/\lambda] = D \otimes_{K_0} \mathcal{O}[\ell_u, 1/\lambda] \subset D \otimes_{K_0} \widetilde{\mathbf{B}}^{[0, r_0/p]}[\ell_u]. \tag{7.1.16}$$

Note that by Thm. 2.5.1 (1),

$$\mathcal{O}[1/\lambda] \subset \mathbf{B}_{K_\infty}^{[0, r_0/p]} = (\widetilde{\mathbf{B}}_L^{[0, r_0/p]})^{\tau\text{-an}, \gamma = 1};$$

also by Lem. 2.7.7, ℓ_u^k are τ-*analytic* vectors. Hence for some $k \gg 0$,

$$\begin{aligned}\mathfrak{M} \subset D \otimes_{K_0} \Big(\bigoplus_{i=0}^{k} \mathcal{O}[1/\lambda] \cdot \ell_u^i\Big) &\subset \Big(D_{K_0} \otimes \Big(\bigoplus_{i=0}^{k} \widetilde{\mathbf{B}}_L^{[0, r_0/p]} \cdot \ell_u^i\Big)\Big)^{\tau\text{-an}, \gamma = 1} \\ &= (D \otimes Q)^{\tau\text{-an}, \gamma = 1}\end{aligned} \tag{7.1.17}$$

where for brevity we denote

$$Q := \bigoplus_{i=0}^{k} \widetilde{\mathbf{B}}_L^{[0, r_0/p]} \cdot \ell_u^i.$$

This means that over $\mathfrak{M}$, we have

$$\tau = \sum_{i=0}^{\infty} \frac{\nabla_\tau^i}{i!},$$

where ∇_τ equals $p\mathfrak{t}N_\nabla$ or $p^2\mathfrak{t}N_\nabla$. Since $N_\nabla(\mathfrak{M}) \subset \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}$, the map ∇_τ induces

$$\nabla_\tau : \mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{t} \cdot \mathcal{O}.$$

One easily checks that $\nabla_\tau(\mathfrak{t}^k) \in \mathfrak{t}^{k+1} \cdot \mathcal{O}$; also note $\nabla_\tau(\mathcal{O}) \in \mathfrak{t} \cdot \mathcal{O}$. Hence inductively, we can show that for all $i \geq 1$,

$$\nabla_\tau^i : \mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{t}^i \cdot \mathcal{O}. \tag{7.1.18}$$

Hence if we choose a basis of $\mathfrak{M}$, then for any $m \in \mathfrak{M}$, the coefficient of $(\tau - 1)(m)$ in $\mathfrak{M} \otimes Q$, expressed using that basis, lies in $\mathfrak{t} \cdot \widetilde{\mathbf{B}}^{[0, r_0/p]} \cap W(R) = \mathfrak{t} W(R)$ by Lem. 7.1.9. This finishes the proof of (7.1.15).

Step 2. Suppose now V is crystalline. To show (7.1.14), it suffices to show that

$$\nu\left(\frac{\tau-1}{\mathfrak{t}}(\mathfrak{M})\right)=0. \tag{7.1.19}$$

Here, the expression (7.1.19) (and similar expressions below) means that the image of $\frac{\tau-1}{\mathfrak{t}}(\mathfrak{M})$ is zero under the map

$$\nu:\mathfrak{M}\otimes_{\mathfrak{S}}\widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u]\to\mathfrak{M}\otimes_{\mathfrak{S}}W(\overline{k})[1/p].$$

Note that

$$\frac{\tau-1}{\mathfrak{t}}=\frac{\nabla_\tau}{\mathfrak{t}}+\sum_{i\geq 2}\frac{1}{\mathfrak{t}}\cdot\frac{\nabla_\tau^i}{i!}.$$

Note that $\nu(N_\nabla(\mathfrak{M}))=0$ (since $N_\nabla/uN_\nabla=N_{D_{\mathrm{st}}(V)}=0$ as V is crystalline), hence $\nu(\frac{\nabla_\tau}{\mathfrak{t}}(\mathfrak{M}))=0$. For each $i\geq 2$, $\nu(\frac{1}{\mathfrak{t}}\cdot\frac{\nabla_\tau^i}{i!}(\mathfrak{M}))=0$ by (7.1.18) since $\nu(\mathfrak{t})=0$. This concludes the proof of (7.1.19).

Step 3. Conversely, suppose now (7.1.14) is satisfied. Note that (7.1.13) implies that $\mathfrak{t}W(R)$ and hence $\mathfrak{t}W(\mathfrak{m}_R)$ is G_K-stable. Hence for any $a\geq 0$, we also have

$$(\tau^{p^a}-1)(\mathfrak{M})\subset\mathfrak{t}W(\mathfrak{m}_R)\otimes_{\mathfrak{S}}\mathfrak{M}.$$

Using the definition $N_\nabla=\frac{\nabla_\tau}{p\mathfrak{t}}$ (or $\frac{\nabla_\tau}{p^2\mathfrak{t}}$), one can easily show $\nu(N_\nabla(\mathfrak{M}))=0$ and hence $N_{D_{\mathrm{st}}(V)}=0$. ∎

Proposition 7.1.11. *Let $\widehat{\mathfrak{M}}$ be an object in $\mathrm{Mod}_{\mathfrak{S},W(R)}^{\varphi,G_K}$. Let $\varphi^*\mathfrak{M}:=\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ and $\varphi^*\widehat{\mathfrak{M}}:=W(R)\otimes_{\varphi,W(R)}\widehat{\mathfrak{M}}$. Then*

$$\varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M}\subset(\varphi^*\widehat{\mathfrak{M}}/E(u)\varphi^*\widehat{\mathfrak{M}})^{G_K}. \tag{7.1.20}$$

Proof. By (7.1.15), $\varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M}$ is fixed by τ since $\varphi(\mathfrak{t})\cdot W(R)\subset E(u)\cdot W(R)$. This proves (7.1.20) when $K_\infty\cap K_{p^\infty}=K$. In Thm. 7.3.4, we will show (no circular reasoning here) that the subset $\varphi^*\mathfrak{M}/E(u)\varphi^*\mathfrak{M}$ is indeed independent of the choice of K_∞, and hence (7.1.20) holds in general (as there exists *some* choice such that $K_\infty\cap K_{p^\infty}=K$; see Notation 1.4.3). ∎

7.2. *Specialization to Liu's $(\varphi,\widehat{G})$-modules*

In [26], we will show that using some "specialization" maps, our argument and results recover the results of Liu's theory of $(\varphi,\widehat{G})$-modules. (We only give a quick review of the $(\varphi,\widehat{G})$-modules later in Appendix B, as we do not really use them here.) The proof in [26] makes systematic use of locally analytic vectors, and makes the link between the different theories completely transparent. Here, let us give some hint about what we mean by "specialization"; see [26] for more details.

Recall that if $r_n \in I$, then there are continuous embeddings (see [1])

$$\iota_n : \widetilde{\mathbf{B}}^I \hookrightarrow \mathbf{B}_{\mathrm{dR}}^+,$$

which we call the *de Rham specialization maps*. One easily checks that the image of the embedding $\iota_0 : \widetilde{\mathbf{B}}^{[0,r_1]} \hookrightarrow \mathbf{B}_{\mathrm{dR}}^+$ lands inside $\mathbf{B}_{\mathrm{cris}}^+$. Furthermore, the induced embedding

$$\iota_0 : \widetilde{\mathbf{B}}^{[0,r_1]} \hookrightarrow \mathbf{B}_{\mathrm{cris}}^+ \tag{7.2.1}$$

is *continuous*. The map (7.2.1) also induces the continuous composites

$$\varphi : \widetilde{\mathbf{B}}^{[0,r_0]} \xrightarrow{\varphi} \widetilde{\mathbf{B}}^{[0,r_1]} \xrightarrow{\iota_0} \mathbf{B}_{\mathrm{cris}}^+. \tag{7.2.2}$$

We call the maps in (7.2.1) and (7.2.2) the *crystalline specialization maps*. By adjoining ℓ_u, we also get the continuous *semi-stable specialization maps*

$$\iota_0 : \widetilde{\mathbf{B}}^{[0,r_1]}[\ell_u] \hookrightarrow \mathbf{B}_{\mathrm{st}}^+, \tag{7.2.3}$$

$$\varphi : \widetilde{\mathbf{B}}^{[0,r_0]}[\ell_u] \hookrightarrow \mathbf{B}_{\mathrm{st}}^+. \tag{7.2.4}$$

Note that in (5.3.6), we have (for a semi-stable representation)

$$\mathfrak{M} \otimes_{\mathfrak{S}} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u] = D \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,r_0/p]}[\ell_u];$$

we are using $\widetilde{\mathbf{B}}^{[0,r_0/p]}$ because it has the advantage that λ is a unit in it, and indeed $1/\lambda$ is an τ-*analytic* vector in it. But from (5.3.1), we also have

$$\mathfrak{M} \otimes_{\mathfrak{S}} \widetilde{\mathbf{B}}^{[0,r_0]}[1/\lambda, \ell_u] = D \otimes_{K_0} \widetilde{\mathbf{B}}^{[0,r_0]}[1/\lambda, \ell_u]; \tag{7.2.5}$$

note that these modules are not G_K-stable! However, using the semi-stable specialization map (note that here $\varphi(\lambda)$ is a unit in $\mathbf{B}_{\mathrm{cris}}^+$, but λ is not)

$$\varphi : \widetilde{\mathbf{B}}^{[0,r_0]}[1/\lambda, \ell_u] \to \mathbf{B}_{\mathrm{st}}^+,$$

we get a G_K-equivariant identification

$$\mathfrak{M} \otimes_{\varphi,\mathfrak{S}} \mathbf{B}_{\mathrm{st}}^+ = D \otimes_{\varphi,K_0} \mathbf{B}_{\mathrm{st}}^+. \tag{7.2.6}$$

Let

$$S := \left\{ \sum_{n=0}^{\infty} a_n \frac{E(u)^n}{n!} : a_n \in \mathfrak{S},\ a_n \to 0 \ p\text{-adically} \right\} \subset \mathbf{A}_{\mathrm{cris}}. \tag{7.2.7}$$

Then it is easy to check that elements of S are all τ-*analytic* vectors in $\mathbf{B}_{\mathrm{cris},L}^+ := (\mathbf{B}_{\mathrm{cris}}^+)^{G_L}$. Hence by (7.2.6) and (7.1.17) we have, for $k \gg 0$,

$$\mathfrak{M} \otimes_{\varphi,\mathfrak{S}} S \subset \left(D \otimes_{K_0} \left(\bigoplus_{i=0}^{k} \mathbf{B}_{\mathrm{cris},L}^+ \cdot \ell_u^i \right) \right)^{\tau\text{-an},\gamma=1}. \tag{7.2.8}$$

Hence $\tau = \sum_{i=0}^{\infty} \frac{\nabla_\tau^i}{i!}$ again holds over $\mathfrak{M} \otimes_{\varphi,\mathfrak{S}} S$. Using some results about filtered (φ, N)-modules over S in [9], this easily recovers the Galois action on $\mathfrak{M} \otimes_{\varphi,\mathfrak{S}} \mathbf{B}_{\mathrm{cris}}^+$

as in [36, §5.1]. In particular, rather than "defining" the Galois action in an *ad hoc* fashion as in *loc. cit.* and then showing it is *compatible* with Galois representations (cf. [36, Lem. 5.2.1], our specialization obviously implies the compatibility. Let us mention that this Galois action is indeed one of the key features in Liu's theory of $(\varphi, \widehat{G})$-modules.

7.3. Relation to some results of Gee and Liu

In this subsection, we discuss the relation between our results and some recent results of Toby Gee and Tong Liu.

The statement and idea of the proof of Thm. 7.3.1 below is due to Toby Gee. As we learnt from Gee, this result was inspired by Caruso's result, and was originally used to construct the semi-stable substack inside the stack of étale (φ, Γ)-modules (see [21, Appendix F]). We thank Toby Gee for allowing us to include it here.

Theorem 7.3.1 (Gee). *Let $T \in \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$. Then $T \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st}, \geq 0}(G_K)$* if and only if *$T$ is of finite height with respect to* all *choices of $\vec{\pi}$.*

We first introduce an elementary lemma.

Lemma 7.3.2. *Let $K \subset M_1, M_2$ be finite extensions, and let $M = M_1M_2$. Suppose M/K is Galois and totally ramified, and suppose $M_1 \cap M_2 = K$. Let V be a G_K-representation and suppose it is semi-stable over both M_1 and M_2. Then V is semi-stable over K.*

Proof. It is easy to see that $\mathrm{Gal}(M/K)$ acts trivially on $D_{\mathrm{st}}^M(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_M}$, and hence $D_{\mathrm{st}}^M(V) = D_{\mathrm{st}}^K(V)$. ∎

Proof of Theorem 7.3.1. Necessity is proved in [32]. We prove sufficiency. Let $V := T[1/p]$, let $U := V|_{G_{K^{\mathrm{ur}}}}$, and let $\widehat{K^{\mathrm{ur}}}$ denote the completion of K^{ur}. It suffices to show that the $G_{\widehat{K^{\mathrm{ur}}}}$-representation U is semi-stable. First fix one $\vec{\pi}$. Choose another unifomizer π' of K such that $\pi/\pi' \in \mathcal{O}_{\widehat{K^{\mathrm{ur}}}}^\times \backslash (\mathcal{O}_{\widehat{K^{\mathrm{ur}}}}^\times)^p$, and choose any compatible system $\vec{\pi}' = \{\pi'_n\}_{n \geq 0}$. By Kummer theory, for each $i \geq 1$, the fields $\widehat{K^{\mathrm{ur}}}(\pi_i)$ and $\widehat{K^{\mathrm{ur}}}(\pi'_i)$ are different. Combining this with [37, Lem. 4.1.3], it is easy to show that

$$\widehat{K^{\mathrm{ur}}}(\pi_i) \cap \widehat{K^{\mathrm{ur}}}(\pi'_i) = \widehat{K^{\mathrm{ur}}}, \quad \forall i \geq 1.$$

Let $m := m(K^{\mathrm{ur}})$. Consider the 3-step extensions

$$\widehat{K^{\mathrm{ur}}} \subset \widehat{K^{\mathrm{ur}}}(\pi_{m-1}), \quad \widehat{K^{\mathrm{ur}}}(\pi'_{m-1}) \subset M = \widehat{K^{\mathrm{ur}}}(\pi_{m-1}, \pi'_{m-1}).$$

Since $\mu_{m-1} \in \widehat{K^{\mathrm{ur}}}$, $M/\widehat{K^{\mathrm{ur}}}$ is Galois and totally ramified. By Thm. 6.2.4, V is semi-stable over $K(\pi_{m-1})$, hence U is semi-stable over $\widehat{K^{\mathrm{ur}}}(\pi_{m-1})$; similarly U is semi-stable over $\widehat{K^{\mathrm{ur}}}(\pi'_{m-1})$. Thus U is semi-stable over $\widehat{K^{\mathrm{ur}}}$ by Lem. 7.3.2. ∎

Before the author proved Thm. 6.2.4 which makes Thm. 7.3.1 possible, Gee and Liu proved a weaker result (Thm. 7.3.4 below) which is sufficient for the construction of the semi-stable substack mentioned above. In Def. 7.3.3 below, the notation $\mathcal{C}_{ss}$ comes from "$\mathcal{C}_{d,ss,h}$" in [21, Def. 4.5.1]; here, we omit the rank d and the height h. For each choice

$\vec{\pi} = \{\pi_n\}_{n\geq 0}$, we let $K_{\vec{\pi}} := \bigcup_{n\geq 0} K(\pi_n)$. We can regard $\vec{\pi}$ as an element in R, and let $[\vec{\pi}] \in W(R)$ be its Teichmüller lift. Let $\mathfrak{S}_{\vec{\pi}}$ be the image of $W(k)[\![X]\!] \hookrightarrow W(R), X \mapsto [\vec{\pi}]$. Let $E(X)$ be the minimal polynomial of π_0 over K_0, and let $E_{\vec{\pi}} := E([\vec{\pi}]) \in \mathfrak{S}_{\vec{\pi}}$.

Definition 7.3.3 ([21, Def. F.7]). Let $\mathcal{C}_{ss}(\mathbb{Z}_p)$ be the category consisting of the following data, which are called *Breuil–Kisin–Fargues G_K-modules admitting all descents*:

(1) $\mathfrak{M}^{\inf}$ is a finite free Breuil–Kisin–Fargues module with $W(R)$-semilinear φ-commuting G_K-action.

(2) For each $\vec{\pi}$, $\mathfrak{M}_{\vec{\pi}} \in (\mathfrak{M}^{\inf})^{\mathrm{Gal}(\overline{K}/K_{\vec{\pi}})}$ is a finite free Breuil–Kisin module over $\mathfrak{S}_{\vec{\pi}}$ such that the induced morphism $\mathfrak{M}_{\vec{\pi}} \otimes_{\mathfrak{S}_{\vec{\pi}}} W(R) \to \mathfrak{M}^{\inf}$ is a φ-equivariant isomorphism.

(3) $W(k)$-mod $\mathfrak{M}_{\vec{\pi}}/[\vec{\pi}]\mathfrak{M} \subset W(\bar{k}) \otimes_{W(R)} \mathfrak{M}^{\inf}$ is independent of $\vec{\pi}$.

(4) $\mathcal{O}_K$-mod $\varphi^*\mathfrak{M}_{\vec{\pi}}/E_{\vec{\pi}}\varphi^*\mathfrak{M}_{\vec{\pi}} \subset \mathcal{O}_C \otimes_{W(R)} \varphi^*\mathfrak{M}^{\inf}s$ is independent of $\vec{\pi}$.

Theorem 7.3.4 (Gee–Liu, [21, Thm. F.11]). *The functor* $T_{W(R)}(\mathfrak{M}^{\inf}) := (\mathfrak{M}^{\inf} \otimes W(\mathrm{Fr}\, R))^{\varphi=1}$ *induces an equivalence between* $\mathcal{C}_{ss}(\mathbb{Z}_p)$ *and* $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{st},\geq 0}(G_K)$.

The proof of Gee–Liu makes use of a result of Heng Du (see [21, Prop. F.13]), which shows that conditions (1), (2), and (4) are enough to guarantee that the attached representation is *de Rham*; then they use condition (3) to show semistability. Here, we give a very brief sketch of proof using our results in order to illustrate the relation between the different approaches.

Proof of Theorem 7.3.4. Apparently, given a module in $\mathcal{C}_{ss}(\mathbb{Z}_p)$, the associated representation is semi-stable by Thm. 7.3.1. Conversely, give a semi-stable representation, the $\mathfrak{M}_{\vec{\pi}}$'s have been constructed by Kisin. To verify condition (2) (in Def. 7.3.3), it suffices to check that the various tensor products $\mathfrak{M}_{\vec{\pi}} \otimes_{\mathfrak{S}_{\vec{\pi}}} W(R)$ can all be *identified* inside $V \otimes \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]$. Note $\mathfrak{M}_{\vec{\pi}} \otimes_{\mathfrak{S}_{\vec{\pi}}} \widetilde{\mathbf{B}}^{+}_{\log}[1/t]$ can be identified with $D \otimes \widetilde{\mathbf{B}}^{+}_{\log}[1/t]$, and $\mathfrak{M}_{\vec{\pi}} \otimes \widetilde{\mathbf{B}}^{\dagger}$ can be identified with $V \otimes \widetilde{\mathbf{B}}^{\dagger}$; hence by (7.1.6), $\mathfrak{M}_{\vec{\pi}} \otimes W(R)[1/p][1/t]$ can all be identified with each other. Then one can follow a strategy as below (7.1.6) to show that $\mathfrak{M}_{\vec{\pi}} \otimes W(R)$ can all be identified. (We leave the details to the readers.) For condition (3): note that the independence of $\mathfrak{M}_{\vec{\pi}} \otimes W(R)$ implies the independence of $\mathfrak{M}_{\vec{\pi}}/[\vec{\pi}]\mathfrak{M}_{\vec{\pi}} \otimes W(\bar{k})$, and hence it suffices to show the independence of $\mathfrak{M}_{\vec{\pi}}/[\vec{\pi}]\mathfrak{M}_{\vec{\pi}}[1/p] = M_{\vec{\pi}}/[\vec{\pi}]M_{\vec{\pi}}$ where $M_{\vec{\pi}}$ are the $\mathcal{O}_{\vec{\pi}}$-modules as constructed in §5.2. Then Lem. 5.3.1 implies that $M_{\vec{\pi}}/[\vec{\pi}]M_{\vec{\pi}}$ can all be identified with D (which is well-known to be independent of $\vec{\pi}$). The verification of condition (4) is similar, by using the φ-twist of (5.3.6). ■

7.4. *Independence from Caruso's work*

Remark 7.4.1. (1) The only place in the current paper where we actually *use* results from [12] is in Thm. 3.1.4 (also Def. 3.1.2) which is [12, Thm. 1]. (Lem. 6.2.2 is also from [12], but it is completely elementary.)

(2) Besides those in §1 and in Thm. 3.1.4, the only places where we *make references to* [12] is in Rem. 6.1.7, where we mention some argument from [12].

(3) The results in [12] are not used in any of the cited papers in our bibliography, except [12, Thm. 1], which is used in [27, 28].

(4) Hence, the current paper is independent of [12] (except [12, Thm. 1]).

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Appendix A. A gap in Caruso's work

The statement of [12, Prop. 3.7] is false. Here is a counter-example:

- Let $A = \mathbb{Q}_p$ with $p = 3$, let $m = 4$ and let $\Lambda = p^{-4} \cdot \mathbb{Z}_p$. Let $a = b = 0$.

The mistake in the proof of [12, Prop. 3.7] is rather hidden. Indeed, on [12, p. 2580], Caruso states the formula

$$\log_m(ab) - (\log_m a + \log_m b) \in \sum_{j=1}^{p^m-1} \frac{p^{\ell(p^m-j)+\ell(j)}}{j} \cdot \Lambda.$$

As far as we understand, Caruso is implicitly using that Λ is a *ring* (that is, a $\mathbb{Z}_p$-algebra). But in fact, Λ is only a $\mathbb{Z}_p$-module.

Prop. 3.7 of [12] is a key proposition in that paper. Indeed, it is used to prove Prop. 3.8 and Prop. 3.9 there. These propositions are then repeatedly used in later arguments of that paper.

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Appendix B. Errata for [35] and [25]

In this appendix, using Breuil–Kisin G_K-modules, we fix a gap in the proof of the main theorems in [35] and [25] where we studied the limit of torsion semi-stable representations. Let us point out at the start that the gap arises because we only recently realized that we *do not know* if the ring $\widehat{\mathcal{R}}$ (recalled below) is p-adically complete or not. (The gap is discussed in detail in Step 2 of §B.0.4.)

We first recall the main results of [35] and [25]. Recall that a (finite) p-power torsion representation T of G_K is called *torsion semi-stable* (resp. *crystalline*) *of weight* r if there exists a G_K-stable $\mathbb{Z}_p$-lattice $\widetilde{L}$ in a semi-stable (resp. crystalline) representation with Hodge–Tate weights in $[0, r]$, and such that there exists a G_K-equivariant surjection $\widetilde{L} \twoheadrightarrow T$ (which is called a *loose semi-stable* (resp. *crystalline*) *lift*). The following is the main theorem of [25]:

Theorem B.0.1. *Let T be a finite free $\mathbb{Z}_p$-representation of G_K of rank d. For each $n \geq 1$, suppose $T_n := T/p^n T$ is torsion semi-stable (resp. crystalline) of weight $h(n)$. If*

$$h(n) < \frac{1}{2d} \log_{16} n, \qquad \forall n \gg 0,$$

then $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is semi-stable (resp. crystalline).

When $h(n)$ is a constant, this is precisely the main theorem of [35], which confirms a conjecture of Fontaine.

Unfortunately, there is a gap in the proof of Thm. B.0.1; there is a (practically the same) gap in [35] as well (see Rem. B.0.5 below). We now focus on discussing and fixing the gap in the proof of Thm. B.0.1 (which utilizes a strategy very similar to [35]).

B.0.2. First, we quickly recall the theory of $(\varphi, \widehat{G})$-modules; see [37] for more details. Define a subring inside $\mathbf{B}^+_{\mathrm{cris}}$:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}} : f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},$$

where $t^{\{i\}} = \frac{t^i}{p^{\tilde{q}(i)}\tilde{q}(i)!}$ and $\tilde{q}(i)$ satisfies $i = \tilde{q}(i)(p-1) + r(i)$ with $0 \le r(i) < p-1$. Define

$$\widehat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}.$$

The rings $\mathcal{R}_{K_0}$ and $\widehat{\mathcal{R}}$ are stable under the G_K-action, and the G_K-action factors through $\widehat{G}$. Let $I_+\widehat{\mathcal{R}} = W(\mathfrak{m}_R) \cap \widehat{\mathcal{R}}$. Then $\widehat{\mathcal{R}}/I_+\widehat{\mathcal{R}} \simeq \mathfrak{S}/u\mathfrak{S} = W(k)$.

Definition B.0.3. Let $\mathrm{Mod}^{\varphi,\widehat{G}}_{\mathfrak{S},\widehat{\mathcal{R}}}$ be the category of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \widehat{G})$, which are called $(\varphi, \widehat{G})$-*modules*, where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$;
(2) $\widehat{G}$ is a continuous $\widehat{\mathcal{R}}$-semilinear $\widehat{G}$-action on $\widehat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$;
(3) $\widehat{G}$ commutes with $\varphi_{\widehat{\mathfrak{M}}}$ on $\widehat{\mathfrak{M}}$;
(4) regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\widehat{\mathfrak{M}}$, we have $\mathfrak{M} \subset \widehat{\mathfrak{M}}^{\mathrm{Gal}(L/K_\infty)}$;
(5) $\widehat{G}$ acts on the $W(k)$-module $\widehat{\mathfrak{M}}/I_+\widehat{\mathcal{R}}\widehat{\mathfrak{M}}$ trivially.

Then (the covariant version of) the main theorem of [37] says that the functor $T_{W(R)}(\widehat{\mathfrak{M}}) := (\widehat{\mathfrak{M}} \otimes_{\widehat{\mathcal{R}}} W(\mathrm{Fr}\, R))^{\varphi=1}$ induces an equivalence between $\mathrm{Mod}^{\varphi,\widehat{G}}_{\mathfrak{S},\widehat{\mathcal{R}}}$ and $\mathrm{Rep}^{\mathrm{st},\ge 0}_{\mathbb{Z}_p}(G_K)$.

B.0.4. We now sketch the proof of Thm. B.0.1 given in [25] in two steps; it is in Step 2 that the gap arises.

Step 1. We first show that T is of finite $E(u)$-height. As we need the constructions in our Step 2 (where the gap arises), let us sketch the argument.

Since T_n is torsion semi-stable, one can construct a (not necessarily unique) *p-power torsion* $(\varphi, \widehat{G})$*-module* (the definition of which is obvious) $\widehat{\mathfrak{M}}_n$, simply by projecting down the finite free $(\varphi, \widehat{G})$-module associated to some loose semi-stable lift of T_n.

Note that *a priori*, these $\widehat{\mathfrak{M}}_n$'s for different n have no direct relations. The technical heart in the proof of [25, Thm. 3.1] is that we can *modify* the torsion Breuil–Kisin modules inside these torsion $(\varphi, \widehat{G})$-modules so that in the end we can obtain a *compatible* system of $\mathfrak{M}'_n$ for $n \gg 0$ such that

- $\mathfrak{M}'_n$ is finite free over $\mathfrak{S}_n := \mathfrak{S}/p^n\mathfrak{S}$;

- letting $\mathcal{O}_{\mathcal{E}_n} := \mathcal{O}_{\mathcal{E}}/p^n\mathcal{O}_{\mathcal{E}}$, and letting M_n be the torsion étale φ-module associated to $T_n|_{G_\infty}$, which is a finite free $\mathcal{O}_{\mathcal{E}_n}$-module, we have $\mathfrak{M}'_n \otimes_{\mathfrak{S}_n} \mathcal{O}_{\mathcal{E}_n} \simeq M_n$, i.e., $\mathfrak{M}'_n$ is a "Breuil–Kisin model" of M_n;
- $\mathfrak{M}'_{n+1}/p^n = \mathfrak{M}'_n$.

Hence we can form the inverse limit $\widetilde{\mathfrak{M}} := \varprojlim_{n\gg 0} \mathfrak{M}'_n$, which is a finite free $\mathfrak{S}$-module. Using some techniques related to the Weierstrass preparation theorem, we can in fact show that $\widetilde{\mathfrak{M}}$ is of finite height (this is easy when $h(n)$ is constant, but more difficult in the general case). Furthermore, we obviously have $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$.

Step 2. Now it remains to show that the $\hat{G}$-action on $\widetilde{\mathfrak{M}} \otimes_{\varphi,\mathfrak{S}} \widehat{\mathcal{R}}$ is stable, and hence T indeed comes from a $(\varphi, \hat{G})$-module and hence is semi-stable.

This is where the *gap* arises. Indeed, we know that the $\hat{G}$-action on $\mathfrak{M}'_n \otimes_{\varphi,\mathfrak{S}} \widehat{\mathcal{R}}$ is stable, because it comes from projection of the $\hat{G}$-action of a finite free $(\varphi, \hat{G})$-module. However, unfortunately, we only recently realized that we *do not know* whether $\widehat{\mathcal{R}}$ is p-adically complete or not! Hence we cannot directly conclude that $\widetilde{\mathfrak{M}} \otimes_{\varphi,\mathfrak{S}} \widehat{\mathcal{R}}$ is $\hat{G}$-stable! Indeed, recall that $\widehat{\mathcal{R}} = W(R) \cap \mathcal{R}_{K_0}$. In fact, it is not even clear *which* "p-adic topology" we should use here: should it be the one induced from $\mathbf{B}^+_{\mathrm{cris}}$, or the one induced from $W(R)$? For either of these choices, it is very difficult to actually compute the p-adic valuations of elements in $\widehat{\mathcal{R}}$.

Remark B.0.5. Note that in the proof of [35, Prop. 6.1.1], it is implicitly assumed that $\mathcal{R}_{K_0} \cap \mathbf{A}_{\mathrm{cris}}$ is p-adically complete; again, it is not clear how to actually prove this: the difficulty is the same as for $\widehat{\mathcal{R}}$.

B.0.6. Fixing the gap using Breuil–Kisin G_K-modules. To fix the gap, we can simply replace all the mentions of "$(\varphi, \hat{G})$-modules" above by "Breuil–Kisin G_K-modules", all the $\widehat{\mathcal{R}}$ by $W(R)$, and all the $\hat{G}$ by G_K. Since $W(R)$ is p-adically complete, we readily conclude that $\widetilde{\mathfrak{M}} \otimes_{\mathfrak{S}} W(R)$ is G_K-stable! Furthermore, in each torsion level for $n \gg 0$, we know $\mathfrak{M}'_n/u\mathfrak{M}'_n$ is fixed by G_K (again because the G_K-action comes from that on a finite free Breuil–Kisin G_K-module); hence $\widetilde{\mathfrak{M}}/u\widetilde{\mathfrak{M}}$ is also fixed by G_K, simply because $W(k)$ is p-adically complete. Hence we have shown that T indeed comes from a finite free Breuil–Kisin G_K-module, and hence is semi-stable. If all the T_n are furthermore torsion crystalline, then the torsion version of the condition in Prop. 7.1.10 holds, and hence the condition also holds for $\widetilde{\mathfrak{M}}$, again because all the rings (and ideals) in Prop. 7.1.10 are p-adically complete.

Remark B.0.7. As we can observe from the above paragraph, in order to fix the gap in [25, 35], it suffices to use some "integral p-adic linear-algebra category" where all the rings involved are p-adic complete. Thus, one can also fix the gap in [25, 35] using Thm. 7.3.1 or Thm. 7.3.4; we leave the details to the reader.

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