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Savage surfaces

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Abstract. Let G be the topological fundamental group of a given nonsingular complex projective surface. We prove that the Chern slopes $c_1^2(S)/c_2(S)$ of minimal nonsingular surfaces of general type S with $\pi_1(S) \simeq G$ are dense in the interval $[1,3]$.

Keywords. Fundamental group, Chern numbers, geography of surfaces of general type

1. Introduction

By the Lefschetz hyperplane theorem, we know that the fundamental group of any nonsingular projective variety is the fundamental group of some nonsingular projective surface. There are lots of groups that are fundamental groups of varieties. Serre proved, for example, that any finite group is realizable [32]. For singular surfaces we know that every finitely presented group is possible as fundamental group by [21] (reducible surfaces) and [20] (irreducible surfaces), but there are of course various restrictions in the case of nonsingular projective surfaces. (See the survey [3] for more on that topic, and see the book [1] for Kähler manifolds.) A natural geographical question is: *Are there any constraints on the Chern slope of surfaces of general type after we fix the fundamental group?* In more generality, this question has been studied for 4-manifolds (see [22]) with a particular focus on symplectic 4-manifolds (see e.g. [4, 5, 16, 28]). For example, Park [28] showed that the set of Chern slopes c_1^2/c_2 of minimal symplectic 4-manifolds S with $\pi_1(S) \simeq G$ is dense in the interval $[0,3]$, for any fixed finitely presented group G.

For complex surfaces, we know that simply connected surfaces of general type have Chern slopes dense in $[1/5,3]$ (see [9, 29–31, 35]), which is the largest possible interval by the Noether inequality $1/5(c_2-36) \leq c_1^2$ and the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 3c_2$ (cf. [6]). (See [36] for an analogous geographical result for surfaces in positive characteristic.) In general, however, it is known that for low slopes we do have some

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constraints on the possible fundamental groups. For instance, from [24] we deduce that if S is a surface of general type with $c_1^2(S) < \frac{1}{3}c_2(S)$ and $\pi_1(S)$ finite, then the order of $\pi_1(S)$ is at most 9. We would also like to mention Reid's conjecture: The fundamental group of a surface with $c_1^2 < \frac{1}{2}c_2$ is either finite or commensurable with the fundamental group of a compact Riemann surface (see [6, p. 294] for details). Pardini's proof [27] of the Severi inequality together with Xiao's result [37, Thm. 1] give evidence for this conjecture at the level of étale fundamental groups.

On the other hand, we remark that a similar question for pairs (c_1^2, c_2) has much stronger constraints. By Gieseker [15], there are only finitely many possibilities of π_1 for a given pair. A concrete example: It is expected that for numerical Godeaux surfaces (i.e., $c_1^2 = 1$, $p_g = q = 0$) the fundamental group belongs to the set $\{1, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/5\}$. Also, on the Bogomolov–Miyaoka–Yau line $\pi_1(S)$ is an infinite group (since those surfaces are ball quotients by results of Miyaoka [26, Prop. 2.1.1] and Yau [38]), and, on the opposite side, on the Noether line we have only simply connected surfaces by the classification of Horikawa [18, 19].

In this article we prove the following.

Theorem 1.1. *Let G be the* (*topological*) *fundamental group of a nonsingular complex projective surface. Then the Chern slopes $c_1^2(S)/c_2(S)$ of minimal nonsingular surfaces S of general type with $\pi_1(S)$ isomorphic to G are dense in the interval* $[1, 3]$.

In this way, any finite group G densely populates the wide zone $[1, 3]$. At the other extreme, the same happens when G is the π_1 of a smooth compact ball quotient. To put our theorem in perspective with this particular case, we point out that recently Stover and Toledo have constructed an explicit surface whose fundamental group is isomorphic to the π_1 of a fake projective plane, but the surface is not a fake projective plane [34, Theorem 1.4].[1] In complete generality, it has already been known that the isomorphism class of the fundamental group does not determine the biholomorphic class of the surface (see e.g. [8, Cor. 1.3]).[2] Theorem 1.1 goes beyond, proving density of Chern slopes in $[1, 3]$ for any fixed fundamental group.

The method of proving the theorem is very different from the one used in [31, Thm. 6.3] for trivial π_1, but we do consider as key input the extremal simply connected surfaces constructed in that paper. An observation here is that the π_1 trivial surfaces constructed by Persson [29] do not work for our method, and they cannot work since, if they do, then some of them would violate Mendes-Lopes–Pardini's theorem mentioned above for low Chern slopes. Chen surfaces in [9, Thm. 1] do not work for our method either.

[1] In the literature, one can find the misconception [39, Thm. 4 (b)] about Siu's rigidity theorem of [33]. The problem is that two surfaces with the same π_1 need not have the same homotopy type, even in the case of ball quotients (e.g. the surface in [34, Thm. 1.4] has $\pi_2 \neq \{0\}$). Stover and Toledo go further on this issue, showing that if, in addition, $\pi_2 = \{0\}$, then such a surface must be biholomorphic to a ball quotient [34, Thm. 6.1].

[2] One could also construct plenty of such situations via cyclic covers of surfaces branched along ample divisors by using the main result of [10].

We now explain roughly the idea of the proof together with the central ingredients. Let Y be a minimal nonsingular projective surface with $\pi_1(Y) \simeq G$, let $r \in [1,3]$, and let $\{X_p\}$ be a sequence of simply connected surfaces as in [31, Thm. 6.3], so that $c_1^2(X_p)/c_2(X_p)$ approaches r as $p \to \infty$. Let Γ_p, B be very ample divisors in X_p and Y respectively, and consider the very ample divisor $\Gamma_p + B$ in $X_p \times Y$. As in [8, Section 1], one obtains a surface S_p from the intersection of two general sections in $|\Gamma_p + B|$ so that $\pi_1(S_p) \simeq G$ (Lefschetz hyperplane theorem), but it is not possible to have the result for $c_1^2(S_p)/c_2(S_p)$ since we have no control on Γ_p. On the other hand, an appropriate Γ_p to control $c_1^2(S_p)/c_2(S_p)$ might not even be ample, so we may not have $\pi_1(S_p) \simeq G$, or even an S_p to start with. To overcome both difficulties, we consider a convenient Γ_p which works for $c_1^2(S_p)/c_2(S_p)$ and it is also a lef (*Lefschetz effettivamente funziona*) line bundle, as introduced by de Cataldo and Migliorini [12]. It turns out that such a Γ_p allows us to prove existence of S_p as above which, by a generalization of the Lefschetz hyperplane theorem due to Goresky and MacPherson [17, Part II, Theorem 1.1], satisfy $\pi_1(S_p) \simeq G$. These surfaces are used to prove the claim on density of Chern slopes in $[1,3]$. We also show that it is not possible to improve the lower bound 1 by using modifications of the surfaces X_p.

We finish the paper with two conjectures in relation to geography of Chern slopes for surfaces with ample canonical class, and for Brody hyperbolic surfaces, which might be proved by using the same techniques as in this paper.

2. Semi-small morphisms, lef line bundles, Bertini and Lefschetz type theorems

Throughout this paper the ground field is $\mathbb{C}$. For a given line bundle M and integer n, the line bundle $M^{\otimes n}$ will be denoted by either nM or M^n. The following definition can be found in several places, e.g. [17, p. 151], [25, Def. 4.1] or [12, Def. 2.1.1].

Definition 2.1. Let X, Y be irreducible varieties. For a proper surjective morphism $f: X \to Y$, we define

$$Y_f^k = \{y \in Y \mid \dim f^{-1}(y) = k\}.$$

We say that f is *semi-small* if $\dim(Y_f^k) + 2k \leq \dim(X)$ for every $k \geq 0$. (Note that $\dim(\emptyset) = -\infty$.) If no confusion can arise, the subscript f will be suppressed.

We note that for a semi-small morphism we have $\dim(X) = \dim(Y)$.

Lemma 2.2. *Let X, Y be surfaces. If $f: X \to Y$ is a proper surjective morphism, then f is semi-small.*

Proof. It is clear that $\dim(Y^1) = 0$ and $\dim(Y^0) = 2$, since f is surjective. Then the inequality $\dim(Y^k) + 2k \leq \dim(X)$ holds for any $k \geq 0$. ∎

Proposition 2.3. *Let $f: X \to Y$ and $g: Z \to W$ be two semi-small morphisms. Then the product morphism $f \times g: X \times Z \to Y \times W$ is semi-small.*

Proof. Let $n = \dim(X)$ and $m = \dim(Z)$. Since f and g are semi-small, we have $\dim(Y^k) \leq n - 2k$ for any $k \geq 0$, $\dim(Z^l) \leq m - 2l$ for any $l \geq 0$, and $\dim(Y^0) = n$, $\dim(W^0) = m$. We also have $(Y \times W)^q = \bigcup_{i+j=q} Y^i \times W^j$, and so

$$\dim (Y \times W)^q \leq \max_{i+j=q} \dim(Y^i \times W^j) \leq n + m - 2i - 2j = n + m - 2q.$$

Hence $f \times g$ is semi-small. ■

Proposition 2.4. *Let X, Y, Z be nonsingular projective varieties. Assume that $f: X \to Y$ is semi-small, and that $g: Y \to Z$ is finite morphism. Then $h = g \circ f: X \to Z$ is semi-small.*

Proof. Since g is a finite, we have $Z_h^k = g(Y_f^k)$ for each $k \geq 0$, and so $\dim(Z_h^k) = \dim(Y_f^k)$. Thus $\dim(Z_h^k) + 2k \leq \dim(X)$, and so h is semi-small. ■

Definition 2.5 ([12, Def. 2.3]). Let X be a nonsingular projective variety, and let M be a line bundle on X. We say that M is *lef* if there exists $n > 0$ such that $|nM|$ is generated by global sections, and the morphism $\psi_{|nM|}$ associated to $|nM|$ is semi-small onto its image. The *exponent* of M is the smallest n such that M is lef. We denote it by $\exp(M)$.

If L is an ample line bundle, then L is lef. If L is very ample, then $\exp(L) = 1$. Next we record a corollary of Proposition 2.4 which will be used later.

Proposition 2.6. *Let $f: X \to Y$ be semi-small between nonsingular projective varieties, and let L be very ample on Y. Then $f^*(L)$ is lef with* $\exp(f^*(L)) = 1$.

A useful Bertini type theorem for lef line bundles is the following (see [12, Prop. 2.1.7] or [25, Lemma 4.3]).

Proposition 2.7. *Let X be a nonsingular projective variety of dimension at least* 2. *Let M be a lef line bundle on X. Assume that M is globally generated and with* $\exp(M) = e$. *Then any generic member $Y \in |M|$ is a nonsingular projective variety, and the restriction $M|_Y$ is lef on Y with* $\exp(M|_Y) \leq e$.

We now state a Lefschetz type theorem relevant for the computation of the fundamental group, which is due to Goresky and MacPherson [17], and was conjectured by Deligne [13]. For comparison, we mention the usual Lefschetz theorem for ample line bundles (see e.g. [23, Thm. 3.1.21]).

Theorem 2.8 (Lefschetz theorem for homotopy groups). *Let X be a nonsingular projective variety of dimension n. Let $\iota: A \to X$ be the inclusion of an effective ample divisor A. Then the induced homomorphism*

$$\iota^*: \pi_i(A) \to \pi_i(X)$$

is bijective if $i \leq n - 2$, and surjective if $i = n - 1$.

Theorem 2.9. *Let X be a nonsingular projective variety of dimension n. Suppose that $f: X \to \mathbb{P}^N$ is a proper morphism, and let H be a linear subspace of codimension c. Define $\phi(k) := \dim((\mathbb{P}^N \setminus H)^k_f)$. Then the induced homorphism*

$$\pi_i(f^{-1}(H)) \to \pi_i(X)$$

is an isomorphism if $i < \hat{n}$, and it is surjective if $i = \hat{n}$, where

$$\hat{n} = n - 1 - \sup_k\Big(2k - n + \phi(k) + \inf_k(\phi(k), c-1)\Big).$$

Proof. This is [17, Part II, Thm. 1.1, pp. 150–151], under the hypothesis that f is proper. ∎

Corollary 2.10. *If H in Theorem 2.9 is a hyperplane and $f: X \to \mathbb{P}^N$ is semi-small into its image, then*

$$\pi_i(f^{-1}(H)) \simeq \pi_i(X) \quad \text{for } i < n-1.$$

Proof. The case $f(X) \subset H$ is trivial. In the computation of $\hat{n}$ we can ignore the values $\phi(k) = -\infty$. Then we compute $\hat{n} = n - 1$ since $\dim(f(X)) = n$, the codimension of H is $c = 1$, and we have the inequality $\phi(k) \leq \dim((f(X))^k_f)$. The last inequality is because $(\mathbb{P}^N \setminus H)^k_f = (f(X) \setminus (f(X) \cap H))^k_f \subset (f(X))^k_f$. ∎

Corollary 2.11. *Let X be a nonsingular projective variety with $\dim(X) \geq 3$. Let M be a lef line bundle on X with $\exp(M) = 1$. If $E \in |M|$, then $\pi_1(E) \simeq \pi_1(X)$.*

Corollary 2.12. *Let X be a nonsingular projective variety with $\dim(X) \geq 4$. Let M be a lef line bundle with $\exp(M) = 1$. Then a generic member $E \in |M|$ is a nonsingular projective variety, and $M_E := M|_E$ is lef. Moreover, if $F \in |M_E|$, then $\pi_1(F) \simeq \pi_1(X)$.*

Proof. The first part is just Proposition 2.7. If $F \in M_E$, then by Corollary 2.11 we find that $\pi_1(F) \simeq \pi_1(E) \simeq \pi_1(X)$. ∎

3. RU surfaces

In this section we recall the construction and certain properties of some surfaces X_p of general type from [31, Section 6]. These surfaces are key in the main result of this paper. We will follow the conventions in [31]. In particular, an *arrangement of curves* is a collection $\{C_1, \ldots, C_r\}$ of curves on a nonsingular surface. A *k-point* of an arrangement of curves is a point of it locally of the form $(0,0) \in \{(x - \xi_1 y)\cdots(x - \xi_k y) = 0\} \subset \mathbb{C}^2_{x,y}$ for some $\xi_i \neq \xi_j$.

Let $p \geq 5$ be a prime number, and let $\alpha, \beta > 0$ be integers. Let $n = 3\alpha p$. Let $\tau: H \to \mathbb{P}^2$ be the blow-up at the twelve 3-points of the dual Hesse arrangement of nine lines $(x^3 - y^3)(y^3 - z^3)(x^3 - z^3) = 0$ in $\mathbb{P}^2$. As in [31, Sections 3 and 5], we will consider the following diagram of varieties and morphisms (here $i \in \{0, 1, \infty, \zeta\}$):

$$\begin{array}{ccccccc} Y_n & \xrightarrow{\sigma_n} & Z_n & \xrightarrow{\varphi_n} & H & \xrightarrow{\tau} & \mathbb{P}^2 \\ & & & & \downarrow{\scriptstyle \pi_i'} & & \\ & & & & \mathbb{P}^1 & & \end{array}$$

where σ_n and φ_n are further blow-ups at special points of an arrangement of curves $\mathcal{H}_n'$ (see below for precise definitions), and the four π_i' are the four elliptic fibrations defined through the four choices of triples of three concurrent lines from the dual Hesse arrangement. Hence each π_i' has precisely three singular fibers, which correspond to the three sets of three concurrent lines.

The three singular fibers of π_i' are denoted by $F_{i,1}$, $F_{i,2}$, $F_{i,3}$. Each $F_{i,j}$ consists of four $\mathbb{P}^1$'s: one central curve $N_{i,j}$ with multiplicity 3, and three reduced curves transversal to $N_{i,j}$ at one point each. We write $N_i = N_{i,1} + N_{i,2} + N_{i,3}$. Let M be the nine $\mathbb{P}^1$'s from the lines of the dual Hesse arrangement, and let N be the twelve exceptional $\mathbb{P}^1$'s from its twelve 3-points. We have $N = \sum_{i=0,1,\zeta,\infty} N_i$, and

$$F_{i,1} + F_{i,2} + F_{i,3} = M + 3N_i.$$

We now consider the very special arrangement of $\frac{4n^2-12}{3}$ elliptic curves

$$\mathcal{H}_n' := \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_\infty + \mathcal{E}_\zeta$$

in H, where each $\mathcal{E}_i$ is the special arrangement of $\frac{n^2-3}{3}$ elliptic curves defined in [31, Section 3]. All curves in $\mathcal{E}_i$ are fibers of π_i'.

For $i \in \{0, 1, \infty, \zeta\}$, let $\mathcal{E}_i'$ be $\beta^2 p^2$ general fibers of π_i' (defined also in [31, Section 3]), and let $\mathcal{A}_{2d} = L_1 + \cdots + L_{2d}$ be the strict transform in H of an arrangement of $2d$ general lines in $\mathbb{P}^2$, where $3 \leq 2d \leq p$.

We define $a_0 = a_1 = b_i = 1$ for $1 \leq i \leq d$, and $a_\infty = a_\zeta = b_i = p-1$ for $d+1 \leq i \leq 2d$. Then

$$\mathcal{O}_H\Big(\sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i' + \sum_{i=0,1,\zeta,\infty} a_i(F_{i,1} + F_{i,2} + F_{i,3}) + \sum_{i=1}^{2d} 3b_iL_i\Big)$$

is isomorphic to $\mathcal{L}_0^p$, where

$$\mathcal{L}_0 := \mathcal{O}_H\Big(3p(3\alpha^2 + \beta^2)\Big(\sum_{i=0,1,\zeta,\infty} a_iF_i\Big) + 3dL\Big),$$

and all symbols have been defined in [31, Section 5].

Let $\varphi_n \colon Z_n \to H$ be the blow-up of H at all the $\frac{(n^2-3)(n^2-9)}{3}$ 4-points in $\mathcal{H}_n'$. For each i, we denote the strict transforms of $\mathcal{E}_i$, $\mathcal{E}_i'$, L_j, $F_{i,j}$ in Z_n by the same symbols. Then

$$\mathcal{O}_{Z_n}\Big(\sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i' + \sum_{i=0,1,\zeta,\infty} a_i(F_{i,1} + F_{i,2} + F_{i,3}) + \sum_{i=1}^{2d} 3b_iL_i\Big)$$

is $\mathcal{L}_1^p$ where $\mathcal{L}_1 := \varphi_n^*(\mathcal{L}_0) \otimes \mathcal{O}_{Z_n}(-6E)$, and E is the exceptional divisor of φ_n.

Finally, let $\sigma_n : Y_n \to Z_n$ be the blow-up at all the $4(n^2-3)$ 3-points in $\mathcal{H}'_n$. Again, we denote the strict transforms of $\mathcal{E}_i$, $\mathcal{E}'_i$, L_j, $F_{i,j}$, M, N_i, N in Y_n by the same symbols. Then

$$\mathcal{O}_{Y_n}\Big(\sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} 3a_i N_i + \sum_{i=1}^{2d} 3b_i L_i\Big) \simeq \mathcal{L}^p,$$

where $\mathcal{L} := \sigma_n^*(\mathcal{L}_1) \otimes \mathcal{O}_{Y_n}(-2M - 6G)$.

With this data, we construct a p-th root cover X_p of Y_n branch along

$$A := \sum_{i=0,1,\zeta,\infty} \mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} \mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} N_i + \sum_{i=1}^{2d} L_i.$$

(See e.g. [35, Section 1] for details on p-th root covers.) Let

$$f : X_p \to Y_n$$

be the corresponding morphism for the p-th root cover, as in [31, Section 5]. The nonsingular projective surface X_p is simply connected [31, Prop. 6.1], and minimal [31, Prop. 6.2].

Let us write

$$A = \sum_j \nu_j A_j = \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}_i + \sum_{i=0,1,\zeta,\infty} 3a_i\mathcal{E}'_i + \sum_{i=0,1,\zeta,\infty} 3a_i N_i + \sum_{i=1}^{2d} 3b_i L_i,$$

where A_j are the irreducible curves in A. Hence ν_j is equal to either $3a_i$ or $3b_k$ for some i, k. The arrangement A has only 2-points, and the number of these points is (see [31, Section 6])

$$t_2 = 108\alpha^2\beta^2 p^4 + 18\beta^4 p^4 + 72d\alpha^2 p^2 - 25d + 24d\beta^2 p^2 + 2d^2.$$

By [31, Prop. 4.1], the log Chern numbers of A are

$$\bar{c}_1^2 = n^4 + 2t_2 - 10d - 48 \quad \text{and} \quad \bar{c}_2 = \frac{n^4}{3} + t_2 - 4d - 12.$$

We recall that the log Chern numbers of A are the Chern numbers of the rank 2 locally free sheaf $\Omega^1_{Y_n}(\log(A))^\vee$. As in [31, Section 5], the Chern numbers of X_p are

$$c_1^2(X_p) = p\bar{c}_1^2 - 2\Big(t_2 + 2\sum_j (g(A_j) - 1)\Big) + \frac{1}{p}\sum_j A_j^2 - \sum_{i<j} c(q_{i,j}, p) A_i \cdot A_j,$$

$$c_2(X_p) = p\bar{c}_2 - \Big(t_2 + 2\sum_j (g(A_j) - 1)\Big) + \sum_{i<j} l(q_{i,j}, p) A_i \cdot A_j,$$

where $0 < q_{i,j} < p$ with $\nu_i + q_{i,j}\nu_j \equiv 0 \pmod p$,

$$c(q_{i,j}, p) := 12 s(q_{i,j}, p) + l(q_{i,j}, p),$$

and $s(q_{i,j}, p)$ and $l(q_{i,j}, p)$ are the Dedekind sum and the length of the Hirzebruch–Jung continued fraction associated to the pair $(q_{i,j}, p)$ respectively (see [31, Def. 5.2]).

For the particular multiplicities $a_0 = a_1 = b_i = 1$ for $1 \le i \le d$ and $a_\infty = a_\zeta = b_i = p-1$ for $d+1 \le i \le 2d$ we chose, we have to consider only the numbers $c(p-1, p) = \frac{2p-2}{p}$ and $c(1, p) = \frac{p^2-2p+2}{p}$, and $l(p-1, p) = p-1$ and $l(1, p) = 1$. Therefore,

$$\sum_{i<j} c(q_{i,j}, 4p) A_i \cdot A_j = \frac{2p-2}{p} t_{2,1} + \frac{p^2-2p+2}{p} t_{2,2},$$
$$\sum_{i<j} l(q_{i,j}, 4p) A_i \cdot A_j = (p-1) t_{2,1} + t_{2,2},$$

where $t_{2,1}$ and $t_{2,2}$ are the number of 2-points corresponding to the singularities $\frac{1}{p}(1, p-1)$ and $\frac{1}{p}(1,1)$ respectively. Hence

$$t_{2,1} = 6\beta^4 p^4 + 36\alpha^2\beta^2 p^4 + 36d\alpha^2 p^2 - 13d + 12d\beta^2 p^2 + d^2,$$
$$t_{2,2} = 12\beta^4 p^4 + 72\alpha^2\beta^2 p^4 + 36d\alpha^2 p^2 - 12d + 12d\beta^2 p^2 + d^2.$$

By plugging in the formulas for Chern numbers, we obtain

$$c_1^2(X_p) = (81\alpha^4 + 144\alpha^2\beta^2 + 24\beta^4) p^5 + \text{l.o.t.},$$
$$c_2(X_p) = (27\alpha^4 + 144\alpha^2\beta^2 + 24\beta^4) p^5 + \text{l.o.t.},$$

where l.o.t. (lower order terms) is a Laurent polynomial in p of degree less than 5. In this way, we obtain

$$\lim_{p\to\infty} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{27x^4 + 48x^2 + 8}{9x^4 + 48x^2 + 8} =: \lambda(x),$$

where $x := \alpha/\beta$. We note that $\lambda([0, \infty^+]) = [1, 3]$. This allows us to prove the following theorem (see [31, Thm. 6.3]).

Theorem 3.1. *For any number $r \in [1, 3]$, there are simply connected minimal surfaces of general type X with $c_1^2(X)/c_2(X)$ arbitrarily close to r.*

Proposition 3.2. *Let $\Gamma_p := f^*(L)$, where L is the pull-back in Y of a general line in $\mathbb{P}^2$. Then $\Gamma_p^2 = p$ and $\Gamma_p \cdot K_{X_p} = -3p + (p-1)(2d + 36\alpha^2 p^2 - 12 + 12\beta^2 p^2)$.*

Proof. As f is a generically finite morphism of degree p, we have $\Gamma_p^2 = p$. Let us consider L generic, so that $f^*(L)$ is a nonsingular projective curve. We note that $L \cdot N_i = 0$ for all i, $L \cdot \sum_{i=1}^{2d} L_i = 2d$, $L \cdot \sum_{i=0,1,\zeta,\infty} \mathcal{E}_i = 36\alpha^2 p^2 - 12$, and $L \cdot \sum_{i=0,1,\zeta,\infty} \mathcal{E}_i' = 12\beta^2 p^2$. Therefore, the morphism $f_{\Gamma_p} \colon \Gamma_p \to L = \mathbb{P}^1$ is totally ramified at $2d + 36\alpha^2 p^2 - 12 + 12\beta^2 p^2$ points, and so, by the Riemann–Hurwitz formula and adjunction, we obtain the desired equality for $\Gamma_p \cdot K_{X_p}$. ∎

We finish this section with a proof that the best lower bound for Chern slopes in this construction is indeed 1. That is, we will prove that for any choice of the multiplicities $a_0, a_1, a_\zeta, a_\infty, b_i$, the limit of the Chern slopes of the surfaces X_p is always at least 1. Hence the choice of multiplicities that we considered gives an optimal lower bound.

As was shown above, we first note that the values of the b_i's do not contribute to the asymptotic final result. We also point out that it is enough to have either $\sum_{i=0,1,\zeta,\infty} a_i = p$ or $\sum_{i=0,1,\zeta,\infty} a_i = 2p$ by considering $0 < a_i < p$ and multiplying by units modulo p. In fact, we can and do take $a_0 = 1$, $a_1 = a$, $a_\zeta = b$, and $a_\infty = c$ with $1 + a + b + c = mp$ for m equal to either 1 or 2.

Through the formulas obtained above, we can compute

$$\lim_{x\to 0} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{12 - \frac{1}{p}C}{6 + \frac{1}{p}L},$$

where

$$C := c(-a, p) + c(-b, p) + c(-c, p) + c(-ba^{-1}, p) + c(-ca^{-1}, p) + c(-cb^{-1}, p),$$
$$L := l(-a, p) + l(-b, p) + l(-c, p) + l(-ba^{-1}, p) + l(-ca^{-1}, p) + l(-cb^{-1}, p),$$

and all the q's in these expressions are taken modulo p with $0 < q < p$. For example, for general a, b, c one can prove that C/p and L/p tend to 0 as $p \to \infty$, and so the limit of the Chern slopes is 2 (see [35] for this general behavior).

Since $c(q, p) = 12s(q, p) + l(q, p)$, it is enough to show that

$$6S + L \le 3p + 3 - 6/p \quad \text{for any } p,$$

where

$$S := s(-a, p) + s(-b, p) + s(-c, p) + s(-ba^{-1}, p) + s(-ca^{-1}, p) + s(-cb^{-1}, p).$$

Indeed, that inequality ensures that $3 \le 12 - C/p$, and the inequality is equivalent to

$$\frac{6 + \frac{1}{p}L}{12 - \frac{1}{p}C} - \frac{\frac{6}{p} - \frac{12}{p^2}}{12 - \frac{1}{p}C} \le 1,$$

and so we obtain $1 \le \lim_{p\to\infty} \frac{12-\frac{1}{p}C}{6+\frac{1}{p}L}$.

The proof of the inequality will use the following numerical lemma.

Lemma 3.3. *Let $0 < q < p$ be coprime integers, and let $p/q = [e_1, \ldots, e_l]$ be the associated Hirzebruch–Jung continued fraction. Then $\sum_{i=1}^{l}(e_i - 1) \le p - 1$.*

Proof. We use induction on p. Suppose the statement is true for all coprime pairs (q', p') with $p' < p$. We write $p/q = [e_1, \ldots, e_l]$. Then $e_1 = [p/q] + 1$, and $q/r = [e_2, \ldots, e_l]$ with (r, q) coprime and $q < p$. Hence

$$\sum_{i=1}^{l}(e_i - 1) = [p/q] + \sum_{i=2}^{l}(e_i - 1) \le [p/q] + q - 1$$

by the induction hypothesis. Therefore, we should prove that $[p/q] + q \le p$. Let $q \ne 1$ (otherwise we are done). Let $1 \le r < q$ be the unique integer such that $[p/q]q + r = p$. Then $[p/q] + q \le p$ is equivalent to $\frac{q-r}{q-1} + q \le p$. But $\frac{q-r}{q-1} \le 1$ if $r \ge 1$, and $q + 1 \le p$. ∎

Proposition 3.4. *We have* $6S + L \leq 3p + 3 - 6/p$.

Proof. Let $0 < q < p$ be integers where p is a prime number. Then (see e.g. [35, Example 3.5])

$$12s(q,p) = \frac{q+q^{-1}}{p} + \sum_{i=1}^{l}(e_i - 3)$$

where $p/q = [e_1, \dots, e_l]$ and q^{-1} is the integer between 0 and p such that $qq^{-1} \equiv 1 \pmod{p}$. Hence $6s(q,p) + l = \frac{q+q^{-1}}{2p} + \frac{1}{2}\sum_{i=1}^{l}(e_i - 1)$. We note that always $\frac{q+q^{-1}}{2p} \leq \frac{p-1}{p}$. We now apply this equality for each of the six terms in S and in L, and use Lemma 3.3 to conclude that

$$6S + L \leq 3p - 3 + 6\frac{p-1}{p} = 3p + 3 - \frac{6}{p}. \qquad \blacksquare$$

4. Key construction and density theorem

In this section, we generalize the construction used in [8, Section 1] in the context of lef line bundles, which will be used for the main theorem.

Proposition 4.1. *Let X and Y be nonsingular projective surfaces. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the usual projections. Let Γ and B be lef line bundles on X and Y respectively. Assume that* $\exp(\Gamma) = \exp(B) = 1$. *Then $p^*(\Gamma) \otimes q^*(B)$ is a lef line bundle on $X \times Y$ of exponent* 1.

Proof. This is elementary; we sketch an argument. Let $M := p^*(\Gamma) \otimes q^*(B)$. Let $s_0, \dots, s_l$ be a basis of $H^0(X, \Gamma)$, and $t_0, \dots, t_b$ a basis of $H^0(Y, B)$. Since $H^0(X,\Gamma) \otimes H^0(Y,B) \simeq H^0(X \times Y, M)$ (see e.g. [7, Fact III.22, i]), then M is generated by the global sections $s_i t_j$ with $0 \leq i \leq l$ and $0 \leq j \leq b$. The morphism $\psi_{|M|}\colon X \times Y \to \mathbb{P}(|M|)$ is $\Sigma_{l,b} \circ (\psi_{|\Gamma|} \times \psi_{|B|})$, where $\Sigma_{l,b}$ is the Segre embedding. Therefore $\psi_{|M|}$ is semi-small into its image as $\psi_{|\Gamma|} \times \psi_{|B|}$ is semi-small by Proposition 2.3. It follows that M is lef and $\exp(M) = 1$. $\blacksquare$

Theorem 4.2. *Let X and Y be nonsingular projective surfaces with nef canonical class, and $K_X^2 > 0$. Let B be a very ample line bundle on Y, and let Γ be a lef line bundle on X with* $\exp(\Gamma) = 1$. *Then there exists a nonsingular projective surface $S \subset X \times Y$ with the following properties:*

(1) $\pi_1(S) \simeq \pi_1(X) \times \pi_1(Y)$.

(2) *The morphisms $p|_S\colon S \to X$ and $q|_S\colon S \to Y$ have degrees* $\deg(p|_S) = B^2$ *and* $\deg(q|_S) = \Gamma^2$.

(3) *We have*

$$c_1^2(S) = c_1^2(X)B^2 + c_1^2(Y)\Gamma^2 + 8c(\Gamma, B) - 4\Gamma^2 B^2,$$
$$c_2(S) = c_2(X)B^2 + c_2(Y)\Gamma^2 + 4c(\Gamma, B) + 4\Gamma^2 B^2,$$

where

$$c(\Gamma, B) = \tfrac{7}{2}\Gamma^2 B^2 + \tfrac{3}{2}(\Gamma \cdot K_X)B^2 + \tfrac{3}{2}(B \cdot K_Y)\Gamma^2 + \tfrac{1}{2}(\Gamma \cdot K_X)(B \cdot K_Y).$$

(4) *K_S is big and nef.*

Proof. We first construct a surface $S \subset X \times Y$ which satisfies (1) and (2). Let $M := p^*(\Gamma) \otimes q^*(B)$. Then, by Proposition 4.1, M is lef with $\exp(M) = 1$. We take general sections E, E' of M, and we define $S := E \cap E'$. We note that this intersection is nonempty and nonsingular by Bertini's theorem since M is base point free and has enough sections. By Proposition 2.7, E is a nonsingular projective variety and $M|_E$ is lef with $\exp(M|_E) = 1$. Since $S = E'|_E$ is smooth, we know by [12, Prop. 2.1.5] that $H^0(S, \mathbb{Z}) \simeq H^0(E, \mathbb{Z}) = \mathbb{Z}$, and so S is a nonsingular projective surface. Moreover, by Corollary 2.12, we have $\pi_1(S) \simeq \pi_1(X) \times \pi_1(Y)$. We also see that the degree of $p|_S$ is $((p^*(\Gamma) \otimes q^*(B))|_Y)^2 = B^2$. Similarly the morphism $q|_S$ has degree Γ^2.

Now we prove (3). By the adjunction formula applied twice, and since $K_{X\times Y} \sim p^*(K_X) + q^*(K_Y)$, we get

$$K_S \sim p|_S^*(K_X + 2\Gamma) + q|_S^*(K_Y + 2B).$$

We note that given nonsingular curves C, C' in X, Y respectively, we have

$$p|_S^*(C) \cdot q|_S^*(C') = p^*(C) \cdot q^*(C') \cdot E \cdot E' = (C \times C') \cdot M^2 = M|_{C\times C'}^2,$$

and therefore $p|_S^*(C) \cdot q|_S^*(C') = 2(\Gamma \cdot C)(B \cdot C')$. This extends to the intersection $p|_S^*(D) \cdot q|_S^*(D')$ for any divisors D, D' in X, Y respectively, and so

$$\begin{aligned} K_S^2 &= \big(p|_S^*(K_X + 2\Gamma) + q|_S^*(K_Y + 2B)\big)^2 \\ &= B^2(K_X + 2\Gamma)^2 + \Gamma^2(K_Y + 2B)^2 \\ &\quad + 4((K_X + 2\Gamma)\cdot\Gamma)((K_Y + 2B)\cdot B) \\ &= K_X^2 B^2 + K_Y^2 \Gamma^2 + 24\Gamma^2 B^2 + 12\big((\Gamma \cdot K_X)B^2 + (B \cdot K_Y)\Gamma^2\big) \\ &\quad + 4(\Gamma \cdot K_X)(B \cdot K_Y). \end{aligned}$$

To calculate $\chi(S)$, we use the following exact Koszul complex. Since S is a complete intersection of two sections of M, and $X \times Y$ is nonsingular, we have an exact sequence (see e.g. [14, pp. 76–77])

$$0 \to \mathcal{O}_{X\times Y}(-2M) \to \mathcal{O}_{X\times Y}^{\oplus 2}(-M) \to \mathcal{O}_{X\times Y} \to \mathcal{O}_S \to 0.$$

By the additivity of the Euler characteristic and the Künneth formula (see e.g. [11, Thm. 17.23])

$$H^n(X \times Y, M) = \bigoplus_{i+j=n} H^i(X, \Gamma) \otimes H^j(Y, B),$$

we obtain

$$
\begin{aligned}
\chi(\mathcal{O}_S) &= \chi(\mathcal{O}_{X\times Y}) + \chi(\mathcal{O}_{X\times Y}(-2\Gamma - 2B)) - 2\chi(\mathcal{O}_{X\times Y}(-\Gamma - B)) \\
&= \chi(\mathcal{O}_X)\chi(\mathcal{O}_Y) + \chi(\mathcal{O}_X(-2\Gamma))\chi(\mathcal{O}_Y(-2B)) - 2\chi(\mathcal{O}_X(-\Gamma))\chi(\mathcal{O}_Y(-B)) \\
&= \chi(\mathcal{O}_X)\chi(\mathcal{O}_Y) \\
&\quad + \big(\chi(\mathcal{O}_X) + \tfrac{1}{2}(4\Gamma^2 + 2(\Gamma\cdot K_X))\big)\big(\chi(\mathcal{O}_Y) + \tfrac{1}{2}(4B^2 + 2(B\cdot K_Y))\big) \\
&\quad - 2\big(\chi(\mathcal{O}_X) + \tfrac{1}{2}(\Gamma^2 + \Gamma\cdot K_X)\big)\big(\chi(\mathcal{O}_Y) + \tfrac{1}{2}(B^2 + B\cdot K_Y)\big) \\
&= \chi(\mathcal{O}_X)B^2 + \chi(\mathcal{O}_Y)\Gamma^2 + c(\Gamma, B),
\end{aligned}
$$

where

$$
c(\Gamma, B) = \tfrac{7}{2}\Gamma^2B^2 + \tfrac{3}{2}(\Gamma\cdot K_X)B^2 + \tfrac{3}{2}(B\cdot K_Y)\Gamma^2 + \tfrac{1}{2}(\Gamma\cdot K_X)(B\cdot K_Y).
$$

Finally, we show (4). Let C be an irreducible curve on S. Let $a = \deg p|_C = a$ and $b = \deg q|_C$. Then, by the projection formula for generically finite morphisms, we have

$$
\begin{aligned}
C\cdot K_S &= C\cdot p|_S^*(K_X + 2\Gamma) + C\cdot q|_S^*(K_Y + 2B) \\
&= ap(C)\cdot(K_X + 2\Gamma) + bq(C)\cdot(K_Y + 2B).
\end{aligned}
$$

We note that K_X, K_Y, and Γ are nef, and B is very ample, and so $C\cdot K_S \geq 0$. Using the formula we just got for K_S^2 to show part (3), we obtain $K_S^2 > 0$. ∎

We now present our main result, which puts together all the ingredients elaborated until now.

Theorem 4.3. *Let Y be a nonsingular projective surface with K_Y nef, and let $r \in [1, 3]$ be a real number. Then there are minimal nonsingular projective surfaces S with $c_1^2(S)/c_2(S)$ arbitrarily close to r, and $\pi_1(S) \simeq \pi_1(Y)$.*

Proof. Let X_p be the collection of simply connected surfaces described in Section 3. Let Γ_p be the line bundle defined in Proposition 3.2. For any p the bundle Γ_p is lef by Proposition 2.6. (We note that Γ_p is not ample because of the resolution of singularities involved in the construction of the surfaces X_p.) Let B be a very ample divisor on Y. Note that we satisfy all the hypotheses in Theorem 4.2 with $X := X_p$ and $\Gamma := \Gamma_p$. Therefore, there are surfaces $S_p := S$ such that all the conclusions in Theorem 4.2 hold. In particular, $\pi_1(S_p) \simeq \pi_1(Y)$.

The formulas in Theorem 4.2 (3) are

$$
\begin{aligned}
c_1^2(S_p) &= c_1^2(X_p)B^2 + c_1^2(Y)\Gamma_p^2 + 8c(\Gamma_p, B) - 4\Gamma_p^2B^2, \\
c_2(S_p) &= c_2(X_p)B^2 + c_2(Y)\Gamma_p^2 + 4c(\Gamma_p, B) + 4\Gamma_p^2B^2,
\end{aligned}
$$

where $c(\Gamma_p, B)$ is as in Theorem 4.2.

By Proposition 3.2 we find that $\Gamma_p^2 = p$ and $\Gamma_p.K_{X_p}$ is a polynomial in p of degree 3. Thus $c(\Gamma_p, B)$ is a polynomial in p of degree 3. By Section 3, the invariants $c_1^2(X_p)$ and $c_2(X_p)$ are Laurent polynomials in p of degree 5. Therefore, by the formulas above,

$$
\lim_{p\to\infty} \frac{c_1^2(S_p)}{c_2(S_p)} = \lim_{p\to\infty} \frac{c_1^2(X_p)}{c_2(X_p)} = \frac{27x^4 + 48x^2 + 8}{9x^4 + 48x^2 + 8} =: \lambda(x)
$$

where $x := \alpha/\beta$, as in Section 3. In this way, just as in [31, Thm. 6.3], we obtain the desired surfaces $S = S_p$ with $c_1^2(S)/c_2(S)$ arbitrarily close to r. ∎

Corollary 4.4. *Let G be the fundamental group of a nonsingular projective surface. Then the Chern slopes $c_1^2(S)/c_2(S)$ of nonsingular projective surfaces S with $\pi_1(S) \simeq G$ are dense in* $[1, 3]$.

Proof. Since π_1 is invariant under birational transformations between nonsingular projective surfaces, it is enough to consider surfaces with no (-1)-curves. If G is the fundamental group of $\mathbb{P}^1 \times C$, where C is a nonsingular projective curve, then, for example, we can take as Y a surface in [31, Corollary 6.4] to apply Theorem 4.3. Otherwise, we have a nonruled surface with nef canonical class, and we can use Theorem 4.3 directly. ∎

As remarked in the introduction, the previous corollary covers the fundamental group G of any nonsingular projective variety by means of the usual Lefschetz hyperplane theorem.

One may be tempted to use the result of Persson [29] on density of Chern slopes of simply connected minimal surfaces of general type in $[1/5, 2]$ as an input in Theorem 4.3, but the strategy does not work. It is not clear in that case how to find a suitable line bundle Γ_p which makes things work. On top of that, and as mentioned in the introduction, this cannot work in full generality since, for example, from [24] one can deduce that if S is a surface of general type with $c_1^2(S) < \frac{1}{3}c_2(S)$ and $\pi_1(S)$ finite, then the order of $\pi_1(S)$ is at most 9. In this way, the question of "freedom" of fundamental groups remains open for the interval $[1/3, 1]$.

We finish with two conjectures in relation to geography of Chern slopes for surfaces with ample canonical class, and for Brody hyperbolic surfaces. We could directly show that these conjectures hold if we could prove that the projection

$$q|_{S_p} : S_p \to Y$$

is a finite morphism. In other words, we need to show that we can construct S_p such that $q|_{S_p}$ does not contract any curve. This depends on the line bundles Γ_p. Catanese [8, Lemma 1.1] proves that $q|_{S_p}$ is a finite morphism if Γ_p is very ample.

Before stating the conjectures, we note that in [31] it is proved that the Chern slopes c_1^2/c_2 of simply connected minimal surfaces of general type are dense in $[1, 3]$, but the canonical class for each of the constructed surfaces was not ample, because of the presence of arbitrarily many (-2)-curves.

Conjecture 4.5. *Let G be the* (*topological*) *fundamental group of a nonsingular complex projective surface. Then the Chern slopes $c_1^2(S)/c_2(S)$ of minimal nonsingular surfaces S of general type with $\pi_1(S)$ isomorphic to G and ample canonical class are dense in* $[1, 3]$.

Conjecture 4.6. *Let Y be a Brody hyperbolic nonsingular projective surface. Then the Chern slopes of hyperbolic nonsingular projective surfaces S with $\pi_1(S)$ isomorphic to $\pi_1(Y)$ are dense in* $[1, 3]$.

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